

MAXIMAL SUBLATTICES OF CONVEX GEOMETRIES WITH UNBOUNDED COMPLEMENT MINIMA

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ABSTRACT. For every integer $r \geq 3$, we construct a finite atomistic convex geometry of convex dimension exactly three having a maximal proper sublattice whose complement has exactly r minimal elements and exactly two maximal elements. The lattice has $(r+2)^2$ elements and the sublattice has $(r+1)(r+2)/2+3$ elements. At $r = 3$ the construction is the 25-element example of Bhattacharjee. Thus the number of minimal elements in such a complement is unbounded even when the convex dimension is fixed at three, in contrast to the common-minimum theorem in convex dimension two.

1. INTRODUCTION

The complement of a maximal proper sublattice of a finite distributive lattice is an interval [7, 4]; the same conclusion holds for bounded lattices in the technical lattice-theoretic sense [1]. For finite convex geometries, a different picture emerges. Adaricheva, Mata, Silberger and Zamojska-Dzienio [2, Theorem 6.1] showed that in convex dimension two the complement is either an interval or the union of two intervals with a common minimum. They also gave examples with arbitrarily many *maximal* complement elements as the convex dimension increases.

Bhattacharjee [3, Theorem B] subsequently constructed a convex geometry of convex dimension three on six points, with 25 closed sets, admitting a maximal sublattice whose complement has three minimal elements. We extend his three-order construction to every integer $r \geq 3$, obtaining r minimal complement elements but only two maximal ones. The contrast shows that these two extremal counts can behave quite differently at fixed convex dimension.

A *convex geometry* on a finite ground set E is a closure operator cl with $\text{cl}(\emptyset) = \emptyset$ satisfying the anti-exchange property: for every closed $A \subseteq E$ and distinct $x, y \notin A$,

$$x \in \text{cl}(A \cup \{y\}) \implies y \notin \text{cl}(A \cup \{x\}).$$

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Its lattice L of closed sets is ordered by inclusion, with $A \wedge B = A \cap B$ and $A \vee B = \text{cl}(A \cup B)$. A sublattice is *maximal proper* if it is proper and adjoining any element outside it generates the whole lattice under meets and joins. The *convex dimension* $\text{cdim}(L)$ is the least number of linear orders of E whose initial segments generate the closed sets by intersections [5, 6]. We write $\text{Min}(C)$ and $\text{Max}(C)$ for the minimal and maximal elements of a subposet C .

2. THE FAMILY AND ITS MAIN PROPERTIES

Fix $r \geq 3$, put $m = r - 2$, and set

$$E_r = \{a, u, v, d, f, y_1, \dots, y_m\}, \quad Y = \{y_1, \dots, y_m\}.$$

Consider the following three linear orders on E_r :

- (1) $a < u < v < d < y_1 < \dots < y_m < f$,
- (2) $f < v < y_1 < \dots < y_m < u < d < a$,
- (3) $d < y_m < \dots < y_1 < u < v < a < f$.

Let L_r consist of all intersections of initial segments of these orders, including empty and full initial segments, and define

$$(4) \quad M_r = \{\emptyset, \{a\}, \{f\}\} \cup \{X \in L_r : d \in X\}, \quad C_r = L_r \setminus M_r.$$

Theorem 1. *For every integer $r \geq 3$, the following hold.*

- (1) L_r is the lattice of an atomistic convex geometry of convex dimension exactly three.
- (2) M_r is a maximal proper sublattice of L_r .
- (3) The minimal and maximal elements of its complement are

$$\begin{aligned} \text{Min}(C_r) &= \{\{u\}, \{v\}, \{y_1\}, \dots, \{y_m\}\}, \\ \text{Max}(C_r) &= \{\{a, u, v\}, \{f, u, v\} \cup Y\}. \end{aligned}$$

In particular, $|\text{Min}(C_r)| = r$ and $|\text{Max}(C_r)| = 2$.

- (4) *Their cardinalities are*

$$(5) \quad |L_r| = (r+2)^2, \quad |M_r| = \frac{(r+1)(r+2)}{2} + 3, \quad |C_r| = \frac{r(r+5)}{2}.$$

For $r = 3$ we have $m = 1$. Substituting $u = b$, $v = c$ and $y_1 = e$ in (1)–(3) gives the three orders of [3, Section 4, Theorem B] exactly; the resulting L_3 and M_3 are the same 25-element geometry and its specified maximal sublattice.

3. PROOF OF THE THEOREM

3.1. Closure and convex dimension. For $S \neq \emptyset$ let $P_i(S)$ denote the smallest initial segment of the i th order containing S . Then the closure associated with L_r is

$$(6) \quad \text{cl}(S) = P_1(S) \cap P_2(S) \cap P_3(S), \quad \text{cl}(\emptyset) = \emptyset.$$

Indeed, each $P_i(S)$ is closed, while every closed set containing S contains the displayed intersection. In particular, cl is a closure operator with $\text{cl}(\emptyset) = \emptyset$.

If A is closed and $x \notin A$, there is an order in which x follows every element of A ; otherwise x would belong to the intersection in (6). If also $y \notin A$ and $x \in \text{cl}(A \cup \{y\})$, then in that order y must follow x . Consequently $y \notin \text{cl}(A \cup \{x\})$. This proves anti-exchange.

Every pair of distinct ground-set points occurs in both relative orders among (1)–(3). For the y_i this follows by reversing their order; a and f occur at opposite ends of two orders; and the relative positions of u, v, d and the y_i also reverse between the displayed orders. Hence $\text{cl}(\{x\}) = \{x\}$ for every $x \in E_r$. Every closed set is the join of its singleton subsets, so L_r is atomistic.

The given orders show $\text{cdim}(L_r) \leq 3$. To prove equality, consider

$$(7) \quad P = \{a, u, v\}, \quad Q = \{f, v\} \cup Y, \quad R = \{d\} \cup Y.$$

These are pairwise incomparable closed sets. The respective unique upper covers in L_r are

$$P \cup \{d\}, \quad Q \cup \{u\}, \quad R \cup \{u\}.$$

Each displayed extension is closed, as is seen from (6). The points outside P are $\{d, f\} \cup Y$, and adjoining any of them to P forces d . Likewise, the points outside Q are $\{u, d, a\}$, each of which forces u when adjoined to Q ; the points outside R are $\{u, v, a, f\}$, and each forces u when adjoined to R . Thus any strictly larger closed set contains the indicated extension, which is consequently the unique upper cover. The three sets in (7) are therefore meet-irreducible.

In any representation of a finite closed-set lattice by intersections of initial segments, a meet-irreducible closed set must itself be an initial segment of one of the representing orders: it equals the intersection of the least such segments containing it, and cannot be the meet of strictly larger closed sets. Initial segments of a single order form a chain. Since P, Q, R are incomparable, at least three orders are needed. Therefore

$$\text{cdim}(L_r) = 3.$$

3.2. Maximality of the sublattice. The family of closed sets containing d is closed under meets and joins. Meets with either $\{a\}$ or $\{f\}$ remain in $\{\emptyset, \{a\}, \{f\}\}$, and joins with such a singleton still contain d . Finally,

$$(8) \quad \{a\} \vee \{f\} = E_r.$$

Thus M_r is a sublattice, and it is proper since $\{u\} \notin M_r$.

The following closed sets belong to M_r for $1 \leq j \leq m$:

$$\{d, u\}, \quad \{d, u, v\}, \quad \{d, y_j\}.$$

If $X \in C_r$, then $d \notin X$, and X is neither empty nor either exceptional singleton $\{a\}, \{f\}$. Nor can X consist of $\{a, f\}$, because (8) forces d into every closed set containing both points. Thus X contains at least one of

$u, v, y_1, \dots, y_m$. Meeting X with $\{d, u\}$ if $u \in X$, with $\{d, u, v\}$ if $u \notin X$ and $v \in X$, or with $\{d, y_j\}$ for some $y_j \in X$ otherwise, produces one of the singletons $\{u\}, \{v\}, \{y_j\}$.

The remaining missing singletons can all be obtained from any one of these. Directly from (6),

$$\begin{aligned} (\{v\} \vee \{a\}) \wedge \{d, u\} &= \{u\}, \\ (\{y_j\} \vee \{f\}) \wedge \{d, u, v\} &= \{v\}, \\ (\{u\} \vee \{f\}) \wedge \{d, y_j\} &= \{y_j\} \quad (1 \leq j \leq m). \end{aligned}$$

Since $m \geq 1$, these identities propagate from any missing singleton to all of them. The sublattice already contains $\{a\}, \{d\}$ and $\{f\}$, so adjoining any $X \in C_r$ generates every singleton and hence every closed set. This proves maximality. It also proves that the minimal members of C_r are exactly the r stated singletons.

3.3. The two maximal members of the complement. Every member of C_r avoids d and cannot contain both a and f . If it contains a , the first order and the exclusion of d force it to lie in $\{a, u, v\}$. Otherwise it lies in

$$E_r \setminus \{a, d\} = \{f, u, v\} \cup Y.$$

Both of these bounding sets are closed by (6), belong to C_r and are incomparable. They are consequently precisely the two maximal elements of C_r .

3.4. Enumeration of the closed sets. Write $Y_0 = \emptyset$, $Y_j = \{y_1, \dots, y_j\}$ for $1 \leq j \leq m$, and $Y_{i:j} = \{y_i, \dots, y_j\}$ for $1 \leq i \leq j \leq m$. We partition the closed sets according to their membership of a, d, f . The following list is exhaustive; indices j range from 0 to m and $1 \leq i \leq j \leq m$ where an interval $Y_{i:j}$ occurs.

Closed sets avoiding d :

$$\begin{array}{ll} \emptyset, & Y_{i:j}, \quad \{u\} \cup Y_j, \quad \{v\} \cup Y_j, \quad \{u, v\} \cup Y_j & (\text{neither } a \text{ nor } f), \\ \{a\}, & \{a, u\}, \quad \{a, u, v\} & (a, \text{ not } f), \\ \{f\}, & \{f, v\} \cup Y_j, \quad \{f, u, v\} \cup Y & (f, \text{ not } a). \end{array}$$

Closed sets containing d :

$$\begin{array}{ll} \{d\}, & \{d\} \cup Y_{i:j}, \quad \{d, u\} \cup Y_j, \quad \{d, u, v\} \cup Y_j & (\text{neither } a \text{ nor } f), \\ \{a, d, u, v\} & \cup Y_j & (a, \text{ not } f), \\ E_r \setminus \{a\} & & (f, \text{ not } a), \\ E_r & & (\text{both } a \text{ and } f). \end{array}$$

To see that the list is exhaustive, first note that $\emptyset$ is included. For each nonempty closed set X , let $I_k = P_k(X)$ be the least initial segment of the k th order containing X , so that $X = I_1 \cap I_2 \cap I_3$. If $a \in X$, the second prefix is full and the third contains every point except possibly f . When

$f \notin X$, we have $X = I_1 \setminus \{f\}$, which is a proper prefix of the first order. Its endpoint lies before d or at $d, y_1, \dots, y_m$, giving exactly the listed sets containing a but not f . Similarly, if $f \in X$ and $a \notin X$, the first and third prefixes are full and X is a proper prefix of the second order. This gives the listed f -sets. If both a and f belong to X , all three prefixes are full and $X = E_r$.

It remains to take $a, f \notin X$. For a nonempty X , the first and second prefixes are then proper. As both orders list the y -points in increasing index order, their intersection with Y is an initial block Y_j , possibly empty. The third prefix meets Y in a final block, since its y -points occur in reverse order. Consequently $X \cap Y$ is a consecutive interval $Y_{i:j}$, or an initial block Y_j if the third prefix reaches u and hence contains all of Y . If neither u nor v belongs to X , this gives $\emptyset$ or $Y_{i:j}$ when $d \notin X$, and $\{d\}$ or $\{d\} \cup Y_{i:j}$ when $d \in X$. If u or v belongs to X , the third prefix contains all of Y , so its y -part is Y_j . When $d \notin X$, any of the three nonempty patterns $\{u\}$, $\{v\}$ and $\{u, v\}$ can accompany Y_j . When $d \in X$, the occurrence of v also forces u : the third order puts u before v , and the first two prefixes already contain u because they contain d . The only patterns are therefore $\{d, u\} \cup Y_j$ and $\{d, u, v\} \cup Y_j$.

Conversely, every set in the list is an intersection of the three prefixes. For instance, $Y_{i:j}$ is obtained by ending the prefixes at y_j, y_j, y_i , respectively, and $\{d\} \cup Y_{i:j}$ by ending them at y_j, d, y_i . The remaining u, v, d patterns follow by ending the first prefix at d when $j = 0$ or at y_j when $j \geq 1$, and choosing the second and third endpoints among v, y_j, u, d as appropriate. The sets containing exactly one of a and f are the proper prefixes already described. This proves the classification.

There are $m(m+1)/2$ nonempty Y -intervals. Counting the three lines of d -free sets gives

$$\begin{aligned} \#\{X \in L_r : d \notin X\} &= \left(1 + \frac{m(m+1)}{2} + 3(m+1)\right) + 3 + (m+3) \\ &= \frac{(m+4)(m+5)}{2}. \end{aligned}$$

The four lines of d -containing sets give

$$\begin{aligned} \#\{X \in L_r : d \in X\} &= \left(1 + \frac{m(m+1)}{2} + 2(m+1)\right) + (m+1) + 1 + 1 \\ &= \frac{(m+3)(m+4)}{2}. \end{aligned}$$

Adding and substituting $m = r - 2$ yields $|L_r| = (r+2)^2$. By (4), M_r consists of the latter d -containing sets together with the three distinct sets $\emptyset, \{a\}, \{f\}$. Thus

$$|M_r| = \frac{(r+1)(r+2)}{2} + 3, \quad |C_r| = \frac{r(r+5)}{2},$$

completing the proof of Theorem 1.

4. CONSEQUENCES

Corollary 2. *There is no bound, as a function of convex dimension alone, on the number of minimal elements in the complement of a maximal proper sublattice of a finite atomistic convex geometry. This already occurs at convex dimension three.*

Proof. Theorem 1 provides r minimal complement elements for every $r \geq 3$, while the convex dimension remains three. $\square$

The threshold of three is sharp for this phenomenon: by [2, Theorem 6.1], at convex dimension two the complement always has a unique minimum. Since convex-geometry lattices are join-semidistributive, the two-maxima count is optimal whenever the complement has multiple minima. Indeed, a unique maximal element in a finite complement would be greatest, forcing the complement to be an interval with a unique minimum [2, Theorem 4.5]. The complements here remain order-convex as subsets of L_r : if $X \subseteq Z \subseteq W$ with $X, W \in C_r$, then $d \notin Z$, and Z contains a point of $\{u, v\} \cup Y$ because X does. Consequently $Z \in C_r$.

These examples do not address the stronger semidistributive-lattice setting. Indeed, with $x = \{u\}$, $y = \{a\}$ and $z = \{v\}$, one has

$$x \wedge y = x \wedge z = \emptyset, \quad x \wedge (y \vee z) = \{u\}.$$

Thus L_r fails meet-semidistributivity, although it is the lattice of a convex geometry.

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