

A note of the Gorenstein symmetry conjecture (II)

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Abstract

We prove the Gorenstein symmetry conjecture for every finite-dimensional split square-zero extension of an algebra derived equivalent, over the ground field, to a finite-dimensional hereditary algebra.

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1 Introduction

Self-injective algebras and algebras of finite global dimension have finite injective dimension on both sides. The Gorenstein symmetry conjecture asks whether finiteness on one side alone forces finiteness on the other. It is recorded among the homological conjectures for Artin algebras by Auslander, Reiten and Smalø [1, p. 410, Conjecture (13)]. For a finite-dimensional algebra A over a field k , the assertion is

$$\mathrm{id}(A_A) < \infty \iff \mathrm{id}({}_A A) < \infty,$$

where id denotes injective dimension and the subscripts specify the side of the regular module.

Derived decompositions provide reductions of this conjecture. Han and Qin prove its invariance under derived equivalence [3, Corollary 7] and a transfer theorem for 2-recollements [3, Proposition 5]. A 2-recollement has four layers of triangle functors, every three consecutive layers forming a recollement. Trivial extensions provide a different setting: their homological dimensions can be studied through a single derived tensor endofunctor.

A split square-zero extension of an algebra R by a k -central bimodule M is the algebra $R \ltimes M$ on $R \oplus M$ with multiplication $(r, m)(s, n) = (rs, rn + ms)$. Minamoto and Yamaura describe its homological dimensions by iterated derived tensor and Hom functors [5]. Their later criterion identifies the Iwanaga–Gorenstein property with an invertible restriction of the derived tensor functor to an admissible subcategory and a uniformly vanishing power on its right perpendicular

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[6]. An Iwanaga–Gorenstein algebra has finite injective dimension on both sides. Starting from one finite side, the remaining task is to construct that subcategory and its adjoints.

Kostas also studies Gorensteinness of cleft extensions under perfectness and nilpotence conditions on the associated tensor functor [4]. Here the tensor functor may retain a nonzero invertible part. Write $\text{mod } R$ for finite right R -modules and $\text{D}^b(\text{mod } R)$ for their bounded derived category. An algebra is hereditary if its global dimension is at most one. We prove the following theorem in Section 4.

Theorem 1.1. *Let k be any field and let R be a nonzero finite-dimensional k -algebra admitting a k -linear triangle equivalence*

$$\text{D}^b(\text{mod } R) \simeq \text{D}^b(\text{mod } H)$$

with a finite-dimensional hereditary k -algebra H . For every finite-dimensional k -central R -bimodule M , the algebra $A = R \ltimes M$ satisfies

$$\text{id}(A_A) < \infty \iff \text{id}({}_A A) < \infty.$$

The correspondence between thick subcategories of a hereditary derived category and wide subcategories of its module category is classical; see [2, Theorem 6.4.1]. We prove a stabilization theorem for a k -linear exact endofunctor F of $\text{D}^b(\text{mod } H)$. If F preserves all morphisms from a sufficiently deep thick image to every ambient object, its thick images stabilize. The stabilized image has both inclusion adjoints, and a single power of F annihilates its right perpendicular. The proof uses finite-length hearts, relative Serre duality and their Ext graphs, then applies Minamoto–Yamaura over R and M .

2 Notation and conventions

Set-up 2.1. The ground field k is arbitrary. A bimodule is k -central when its two scalar actions coincide. Algebras are nonzero finite-dimensional associative unital k -algebras, and bimodules are finite-dimensional and k -central. Unless a side is displayed, modules are right modules. The category $\text{mod } R$ consists of finite-dimensional right R -modules, and $\text{D}^b(\text{mod } R)$ is its bounded derived category with cohomological grading: $d_X^n : X^n \rightarrow X^{n+1}$, $H^n(X) = \ker d_X^n / \text{im } d_X^{n-1}$, and $X[1]^n = X^{n+1}$ with differential $-d_X^{n+1}$. We write $D = \text{Hom}_k(-, k)$ and $\text{per}(R) = K^b(\text{proj } R)$ for the bounded homotopy category of finite projective right modules. Derived tensor and derived Hom are computed by projective or injective resolutions in the corresponding variables.

For an object or a collection of objects in a triangulated category, $\text{thick}(-)$ is the smallest full triangulated subcategory containing it and closed under direct summands. All subcategories are understood to be full and closed under isomorphism. For such a subcategory $\mathcal{T}$ of a triangulated category $\mathcal{K}$, its right perpendicular is

$$\mathcal{T}^\perp = \{Y \in \mathcal{K} : \text{Hom}_{\mathcal{K}}(X, Y[r]) = 0 \text{ for every } X \in \mathcal{T}, r \in \mathbb{Z}\}.$$

The subcategory is admissible if its inclusion has both a left and a right adjoint. The kernel of a functor consists of the objects it sends to zero. An exact functor between triangulated categories means a triangle functor.

A wide subcategory of a module category is closed under kernels, cokernels, and extensions. A length category is an abelian category in which every object has a finite composition series. For a length heart with finitely many simple isomorphism classes, its directed Ext graph has

those classes as vertices and an arrow $S \rightarrow T$ precisely when $\text{Ext}^1(S, T) \neq 0$; loops are allowed in this definition. For any endofunctor F , write F^d for its d -fold composite for integers $d \geq 0$, with $F^0 = \text{Id}$.

For a finite-dimensional algebra R and a R -bimodule M , the ordinary trivial extension $A = R \ltimes M$ is the vector space $R \oplus M$ with multiplication

$$(r, m)(s, n) = (rs, rn + ms).$$

Thus M is a square-zero ideal and the quotient map to R has an algebra section. An algebra is Iwanaga–Gorenstein if its left and right regular modules both have finite injective dimension. Gorenstein symmetry is the assertion that finiteness on either one of these two sides implies finiteness on the other.

The radical of an object of a length category is the intersection of its maximal subobjects, its top is the quotient by its radical, and its socle is the sum of its simple subobjects. These are taken in the indicated heart when it is a subcategory of $\text{mod } H$. A projective cover is an epimorphism from a projective object with superfluous kernel; a subobject is superfluous if its sum with a proper subobject remains proper. An injective envelope is an embedding into an injective object with essential image, meaning an image meeting every nonzero subobject.

3 Stabilization in a hereditary derived category

Throughout this section, H is a finite-dimensional hereditary k -algebra and $\mathcal{K} = \text{D}^b(\text{mod } H)$. We prove stabilization for successive thick images of an exact endofunctor.

For an essentially small triangulated category $\mathcal{C}$, write $K_0(\mathcal{C})$ for its Grothendieck group, with the relations $[Y] = [X] + [Z]$ for triangles $X \rightarrow Y \rightarrow Z \rightarrow X[1]$. For an abelian category the relations instead come from short exact sequences. On a full triangulated subcategory of $\mathcal{K}$, the Euler pairing is the bilinear form

$$\chi([X], [Y]) = \sum_{r \in \mathbb{Z}} (-1)^r \dim_k \text{Hom}_{\mathcal{K}}(X, Y[r]).$$

The sum is finite. We use the same notation for the induced real bilinear forms and for their restrictions. Connected components of a directed graph mean components after forgetting the directions.

The following lemma records hereditary splitting and the resulting finite hearts.

Lemma 3.1. *Let H be a finite-dimensional hereditary k -algebra, let $\mathcal{K} = \text{D}^b(\text{mod } H)$, and let $C \in \mathcal{K}$. Set $\mathcal{T} = \text{thick}_{\mathcal{K}}(C)$ and $\mathcal{W} = \mathcal{T} \cap \text{mod } H$. Then the following hold.*

- (i) *The standard bounded t -structure on $\mathcal{K}$ restricts to $\mathcal{T}$, with heart $\mathcal{W}$. This heart is a wide length subcategory of $\text{mod } H$ and has finitely many simple objects up to isomorphism.*
- (ii) *For $X, Y \in \mathcal{W}$, the natural map $\text{Ext}_{\mathcal{W}}^1(X, Y) \rightarrow \text{Ext}_H^1(X, Y)$ is an isomorphism. Every subobject in $\mathcal{W}$ of a projective object of $\mathcal{W}$ is projective, and every quotient in $\mathcal{W}$ of an injective object of $\mathcal{W}$ is injective.*
- (iii) *Inclusion identifies $K_0(\mathcal{W})$ with $K_0(\mathcal{T})$. If $S_1, \dots, S_t$ represent the simple objects of $\mathcal{W}$, put*

$$\Delta_i = \text{End}_H(S_i), \quad d_i = \dim_k \Delta_i, \quad h_{ij} = \dim_k \text{Ext}_H^1(S_i, S_j).$$

Here δ_{ij} is 1 if $i = j$ and 0 otherwise. The classes $[S_i]$ form a basis of $K_0(\mathcal{T})$, the matrix of its Euler pairing is $(\delta_{ij}d_i - h_{ij})$, and $h_{ij} > 0$ implies $h_{ij} \geq \max(d_i, d_j)$.

Proof. Write $H^q(X)$ for the q th cohomology module of X . Heredity gives a splitting

$$X \simeq \bigoplus_{q \in \mathbb{Z}} H^q(X)[-q] \quad (X \in \mathcal{K}). \quad (3.1)$$

Indeed, in the truncation triangle at the largest nonzero cohomological degree q , its connecting morphism has components in $\text{Ext}_H^{q-p+1}(H^q(X), H^p(X))$ with $p < q$. These groups vanish, and induction gives the splitting. Thus $\mathcal{T}$ is closed under cohomology and standard truncations. Kernels and cokernels of maps in $\mathcal{W}$ are cohomology objects of their cones; extensions are middle terms of triangles. This proves that $\mathcal{W}$ is wide and is the inherited heart. A strict chain of its subobjects is a strict chain of finite-dimensional H -submodules, so its objects have finite length.

Take the finite set of $\mathcal{W}$ -composition factors of all $H^q(C)$. The objects of $\mathcal{W}$ with composition factors in this set form a Serre subcategory of $\mathcal{W}$. The cohomology long exact sequence shows that objects of $\mathcal{T}$ whose cohomology lies in this Serre subcategory form a thick subcategory. It contains C , and hence equals $\mathcal{T}$. Every simple object of $\mathcal{W}$ therefore belongs to the chosen finite set.

Fullness and extension closure identify the two groups of extensions of degree one. If $0 \rightarrow X \rightarrow P \rightarrow Y \rightarrow 0$ is exact in $\mathcal{W}$ and P is projective there, then, for $Z \in \mathcal{W}$, the ambient exact sequence gives

$$\text{Ext}_H^1(P, Z) \longrightarrow \text{Ext}_H^1(X, Z) \longrightarrow \text{Ext}_H^2(Y, Z).$$

The outer groups vanish. Thus X is projective in $\mathcal{W}$. If $0 \rightarrow X \rightarrow I \rightarrow Y \rightarrow 0$ is exact with I injective in $\mathcal{W}$, the exact sequence

$$\text{Ext}_H^1(Z, I) \longrightarrow \text{Ext}_H^1(Z, Y) \longrightarrow \text{Ext}_H^2(Z, X)$$

proves that Y is injective in $\mathcal{W}$.

The map $[X] \mapsto \sum_q (-1)^q [H^q(X)]$ is inverse to the map on Grothendieck groups induced by inclusion. The simple classes freely generate the Grothendieck group of a length category, by composition multiplicities. Distinct simple objects have no nonzero morphisms, and each Δ_i is a finite-dimensional division ring. This gives the Euler matrix. Finally, $\text{Ext}_H^1(S_i, S_j)$ is a right Δ_i -vector space by pullback and a left Δ_j -vector space by pushout. A nonzero such space has dimension over k at least the dimension of either division ring. $\square$

The next lemma describes the part of a heart on which a nonnegative additive function can be supported if it vanishes on nonprojective indecomposable objects. For an object X of a length category, we write $[X : S_i]$ for the multiplicity of S_i in a composition series.

Lemma 3.2. *Let H be a finite-dimensional hereditary k -algebra and let $\mathcal{W} \subseteq \text{mod } H$ be a wide length subcategory with finitely many simple objects up to isomorphism. Choose representatives $S_1, \dots, S_t$ and put*

$$d_i = \dim_k \text{End}_H(S_i), \quad h_{ij} = \dim_k \text{Ext}_H^1(S_i, S_j).$$

Let $\lambda : K_0(\mathcal{W}) \rightarrow \mathbb{R}$ be additive, nonnegative on the simple classes, and zero on every nonprojective indecomposable object of $\mathcal{W}$. If $\lambda([S_i]) > 0$, then the connected component of S_i in the directed Ext graph is an oriented line, possibly a single vertex. Every edge $S_u \rightarrow S_v$ of this component satisfies $h_{uv} = d_u = d_v$. The Euler quadratic form on the real span of its simple classes is positive definite.

Proof. All radicals, tops, and socles in this proof are taken in $\mathcal{W}$.

A bound for indecomposable projectives. Let P be an indecomposable projective object of $\mathcal{W}$. Its top is a nonzero semisimple object. If that top had a nontrivial direct-sum projection, projectivity would lift the projection along $P \rightarrow \text{top } P$ to an endomorphism of P . The Fitting decomposition of a finite-length indecomposable object says that an endomorphism is either invertible or nilpotent: the stabilized kernel and image of a sufficiently large power give the decomposition. Neither possibility induces a nontrivial projection on the top. Thus $\text{top } P$ is simple.

The radical of a finite-length object is superfluous: if a proper subobject N satisfied $N + \text{rad } P = P$, a maximal subobject containing N would also contain $\text{rad } P$, a contradiction. Consequently $P \rightarrow \text{top } P$ is a projective cover. If two indecomposable projectives have isomorphic tops, a lift of this isomorphism is surjective by superfluity. It splits by projectivity of the target and is an isomorphism by indecomposability of the source. There are therefore at most t isomorphism classes of indecomposable projectives, and their dimensions over k have a common bound. There is at least one such object here, since $\lambda([S_i]) > 0$ forces S_i to be projective.

An essential injective extension. Start with $J_0 = S_i$. For each $\ell \geq 0$ and each v , set $\Delta_v = \text{End}_H(S_v)$ and choose a right Δ_v -basis $\xi_{v,1}, \dots, \xi_{v,e_{v,\ell}}$ of $\text{Ext}_H^1(S_v, J_\ell)$, where the action is pullback. All the integers $e_{v,\ell}$ are finite. Let $V_\ell = \bigoplus_v S_v^{\oplus e_{v,\ell}}$ and take the extension

$$0 \longrightarrow J_\ell \longrightarrow J_{\ell+1} \longrightarrow V_\ell \longrightarrow 0 \quad (3.2)$$

whose class under $\text{Ext}_H^1(V_\ell, J_\ell) \cong \bigoplus_v \text{Ext}_H^1(S_v, J_\ell)^{\oplus e_{v,\ell}}$ has components $\xi_{v,j}$. Its middle term belongs to $\mathcal{W}$ by extension closure. For every v , the connecting map is the right Δ_v -linear isomorphism

$$\text{Hom}_H(S_v, V_\ell) \longrightarrow \text{Ext}_H^1(S_v, J_\ell), \quad (\alpha_1, \dots, \alpha_{e_{v,\ell}}) \longmapsto \sum_j \xi_{v,j} \alpha_j.$$

The Hom exact sequence therefore identifies $\text{Hom}(S_v, J_\ell)$ with $\text{Hom}(S_v, J_{\ell+1})$. Every simple subobject of $J_{\ell+1}$ lies in J_ℓ , so induction gives $\text{soc } J_\ell = S_i$. Every nonzero subobject of a finite-length object contains a simple subobject. Thus this socle is essential, meaning that it meets every nonzero subobject, and J_ℓ is indecomposable.

Since λ is nonnegative on all composition factors and S_i occurs in J_ℓ , we have $\lambda([J_\ell]) > 0$. Hence each J_ℓ is projective. If all V_ℓ were nonzero, (3.2) would make the dimensions of the J_ℓ strictly increase, contradicting the bound for indecomposable projectives. At some stage $V_\ell = 0$, so $\text{Ext}_H^1(S_v, J_\ell) = 0$ for every v . Induction on composition length, using the long exact Ext sequence in the first variable, gives $\text{Ext}_H^1(X, J_\ell) = 0$ for every $X \in \mathcal{W}$. Write $I_i = J_\ell$. It is an injective envelope of S_i , since its socle is essential, and it is also an indecomposable projective. Let S_j be its simple top and write $I_i \simeq P_j$.

Projective and injective supports. Whenever the projective cover P_u of S_u exists, its radical is projective. This follows from the ambient Ext exact sequence, as in Lemma 3.1(ii). Decompose this radical into indecomposable projectives. Applying $\text{Hom}_H(-, S_v)$ to $0 \rightarrow \text{rad } P_u \rightarrow P_u \rightarrow S_u \rightarrow 0$ gives

$$\text{Hom}_H(\text{rad } P_u, S_v) \simeq \text{Ext}_H^1(S_u, S_v).$$

Here $\text{Hom}(S_u, S_v) \rightarrow \text{Hom}(P_u, S_v)$ is an isomorphism because every map to a simple object factors through the top. Since $\text{Hom}(P_v, S_v)$ has dimension d_v , we obtain

$$\text{rad } P_u \simeq \bigoplus_{v: h_{uv} > 0} P_v^{\oplus m_{uv}}, \quad m_{uv} = h_{uv}/d_v. \quad (3.3)$$

In particular, every cover displayed on the right exists. Along an edge $u \rightarrow v$ reachable from u , the dimension of P_v is strictly smaller than that of P_u . Iterating this observation shows that all vertices reachable from u have projective covers and that their graph has no directed cycle. Each nonzero m_{uv} is a positive integer, since the displayed Hom space is a vector space over $\text{End}(S_v)$. Composition multiplicities satisfy $[P_u : S_v] = \delta_{uv} + \sum_w m_{uw} [P_w : S_v]$. Iterating this recurrence in the finite acyclic graph gives

$$[P_u : S_v] = \sum_{\gamma: u \rightsquigarrow v} \prod_{(p \rightarrow q) \in \gamma} m_{pq}.$$

The sum runs over directed paths, including the empty path when $u = v$; the empty product is one. Thus the support of P_u , namely the vertices with positive composition multiplicity, is exactly the set reachable from u .

The injective recurrence uses the other division-ring multiplicity. An indecomposable injective has simple socle: a projection on a decomposable socle would extend to an endomorphism, contradicting the same Fitting alternative. Its socle is essential by finite length. Two injectives with the same simple socle are isomorphic, since an extension of the identity on the socle is injective by essentiality, splits by injectivity of the source, and is surjective by indecomposability of the target. Write I_v for the resulting injective envelope whenever it exists. Its quotient by S_v is injective, by the ambient Ext exact sequence. Applying $\text{Hom}_H(S_u, -)$ gives

$$\text{Hom}_H(S_u, I_v/S_v) \simeq \text{Ext}_H^1(S_u, S_v),$$

because every map from a simple object to I_v lands in its socle. Decomposition into indecomposable injectives now yields

$$I_v/S_v \simeq \bigoplus_{u: h_{uv} > 0} I_u^{\oplus n_{uv}}, \quad n_{uv} = h_{uv}/d_u. \quad (3.4)$$

Dimensions strictly decrease when this recurrence is followed backwards along an edge. All vertices from which v is reachable have injective envelopes, and their graph has no directed cycle. The recurrence $[I_v : S_u] = \delta_{uv} + \sum_w n_{wv} [I_w : S_u]$ gives

$$[I_v : S_u] = \sum_{\gamma: u \rightsquigarrow v} \prod_{(p \rightarrow q) \in \gamma} n_{pq}.$$

Its support is the set of vertices from which v is reachable.

Apply these formulas to $P_j \simeq I_i$. Their common support is both the set reachable from j and the set from which i is reachable. The projective path formula gives $[P_j : S_j] = 1$. Thus $[I_i : S_j] = 1$, and the injective formula forces exactly one path from j to i . Every support vertex lies on this path. Any additional edge inside the support would give a second path or a directed cycle. An edge leaving the support would make its endpoint reachable from j ; an edge entering it would make its starting point reach i . Both contradict the description of the support. Hence it is an entire connected component and is an oriented line.

The injective path formula also gives $[I_i : S_i] = 1$. The two equalities $[I_i : S_j] = [P_j : S_i] = 1$ show that the product of the n -multiplicities along the unique path and the product of its m -multiplicities are both one. Each multiplicity is a positive integer, so all equal one. By (3.3) and (3.4), every edge satisfies $h_{uv} = d_u = d_v$. If the line has s vertices, in their path order, let $d > 0$ be their common division-ring dimension. Its Euler quadratic form is

$$d \left(\sum_{v=1}^s x_v^2 - \sum_{v=1}^{s-1} x_v x_{v+1} \right) = \frac{d}{2} \left(x_1^2 + x_s^2 + \sum_{v=1}^{s-1} (x_v - x_{v+1})^2 \right).$$

This is positive for every nonzero real vector, including when $s = 1$. □

Once the Ext graph is acyclic, a full self-embedding is already surjective. We will use this both on components and on the full thick subcategory.

Lemma 3.3. *Let H be a finite-dimensional hereditary k -algebra, put $\mathcal{K} = \mathbf{D}^b(\text{mod } H)$, and let $\mathcal{T} = \text{thick}_{\mathcal{K}}(C)$ for some $C \in \mathcal{K}$. Suppose that the directed Ext graph of $\mathcal{W} = \mathcal{T} \cap \text{mod } H$ is acyclic. Then every fully faithful k -linear exact functor $J : \mathcal{T} \rightarrow \mathcal{T}$ is an equivalence.*

Proof. By Lemma 3.1, the heart has finitely many simple objects. If there are none, $\mathcal{T} = 0$ and the assertion holds. Otherwise number them $S_1, \dots, S_t$ so that every Ext arrow points from a smaller index to a larger one. Heredity and acyclicity give

$$\text{Hom}_{\mathcal{K}}(S_i, S_i[r]) = 0 \quad (r \neq 0), \quad \text{Hom}_{\mathcal{K}}(S_j, S_i[r]) = 0 \quad (j > i, r \in \mathbb{Z}).$$

The Euler matrix on these simple classes is upper triangular, with positive diagonal entries $\dim_k \text{End}(S_i)$. It is therefore nonsingular.

Set $E_i = J(S_i)$ and $\Delta_i = \text{End}_{\mathcal{K}}(E_i)$. Full faithfulness preserves the displayed vanishings, the division rings, and the Euler matrix. Consequently $[E_1], \dots, [E_t]$ are linearly independent and form a basis of $K_0(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. We show that they generate $\mathcal{T}$ by successively removing morphisms from $E_t, E_{t-1}, \dots, E_1$.

For $X \in \mathcal{T}$ and a fixed index i , the groups $\text{Hom}_{\mathcal{K}}(E_i, X[q])$ are finite right Δ_i -vector spaces, by precomposition, and only finitely many are nonzero. Choose bases and let e_q be their dimensions over Δ_i . The basis morphisms define an evaluation triangle

$$\bigoplus_q E_i[-q]^{\oplus e_q} \longrightarrow X \longrightarrow X' \longrightarrow \left(\bigoplus_q E_i[-q]^{\oplus e_q} \right) [1].$$

The first map induces an isomorphism on $\text{Hom}_{\mathcal{K}}(E_i, -[p])$ for every p : only the summands with $q = p$ contribute, and their images are the chosen basis. Thus $\text{Hom}_{\mathcal{K}}(E_i, X'[p]) = 0$ for every p . If X is already right orthogonal to E_j for $j > i$, then X' remains so, since all morphisms from these E_j to shifts of E_i vanish.

Starting at $i = t$ and proceeding downwards leaves an object X_0 right orthogonal to every E_i in every degree. Each $H^q(X_0)$ is a shifted direct summand by (3.1), belongs to $\mathcal{W}$, and has Euler pairing zero with all $[E_i]$ on the left. Nonsingularity of the Euler form and the basis property give $[H^q(X_0)] = 0$ in $K_0(\mathcal{W}) \otimes \mathbb{R}$. Its composition multiplicities are nonnegative, so $H^q(X_0) = 0$. Hence $X_0 = 0$, and the evaluation triangles put X in $\text{thick}(E_1, \dots, E_t)$.

The essential image of J is thick. Fullness lifts morphisms whose cones are needed, and full faithfulness lifts idempotents, which split in $\mathcal{T}$. Since the image contains all E_i , it is all of $\mathcal{T}$. $\square$

Theorem 3.4. *Let H be a finite-dimensional hereditary k -algebra, put $\mathcal{K} = \mathbf{D}^b(\text{mod } H)$, and let $F : \mathcal{K} \rightarrow \mathcal{K}$ be a k -linear exact functor with a left adjoint L and a right adjoint G . Set $\mathcal{T}_d = \text{thick}_{\mathcal{K}}(F^d H)$ for $d \geq 0$. Suppose that for some integer $a \geq 1$ the map induced by F is an isomorphism*

$$\text{Hom}_{\mathcal{K}}(X, Y[r]) \longrightarrow \text{Hom}_{\mathcal{K}}(FX, FY[r])$$

for every $X \in \mathcal{T}_a$, every $Y \in \mathcal{K}$, and every $r \in \mathbb{Z}$. Then $\mathcal{T}_a = \mathcal{T}_{a+1}$. For every $b \geq a$, the inherited heart $\mathcal{W}_b = \mathcal{T}_b \cap \text{mod } H$ is a length category with finitely many simple objects up to isomorphism and an acyclic directed Ext graph. The restriction $F|_{\mathcal{T}_a}$ is an autoequivalence, the inclusion $\mathcal{T}_a \hookrightarrow \mathcal{K}$ has both adjoints, and

$$\mathcal{T}_a^{\perp} = \ker F^a, \quad F(\mathcal{T}_a^{\perp}) \subseteq \mathcal{T}_a^{\perp}.$$

The hypothesis controls morphisms to every ambient target. This will give a nondegenerate numerical subspace and relative Serre duality before admissibility has been established.

Proof. Finite hearts and numerical detection. Since $\mathcal{K} = \text{thick}_{\mathcal{K}}(H)$ and exact functors preserve finite thick constructions, $F^d \mathcal{K} \subseteq \mathcal{T}_d$ for every d . In particular $FH \in \text{thick}(H)$ implies $\mathcal{T}_{d+1} \subseteq \mathcal{T}_d$. For $b \geq a$, the hypothesis makes $F|_{\mathcal{T}_b}$ fully faithful. Its essential image is thick by lifting cones and split idempotents, and equals $\mathcal{T}_{b+1}$ because it contains $F^{b+1}H$ and is contained in its thick closure. Thus

$$F : \mathcal{T}_b \xrightarrow{\sim} \mathcal{T}_{b+1} \quad (b \geq a). \quad (3.5)$$

Lemma 3.1 applies to each $\mathcal{T}_d$. Write Q_d for the directed Ext graph of $\mathcal{W}_d$; the zero heart has the empty graph.

Let n be the number of simple H -modules, let $V_0 = K_0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{R}$, and let $B : V_0 \rightarrow V_0$ be the linear map induced by F . The ambient Euler pairing is nondegenerate. Indeed, choose simple representatives $T_1, \dots, T_n$ and their indecomposable projective covers $P_1, \dots, P_n$. Then

$$\chi([P_i], [T_j]) = \delta_{ij} \dim_k \text{End}_H(T_i).$$

This shows that the n projective classes are linearly independent, hence form a basis of V_0 , and proves nondegeneracy.

For $b \geq a$, let V_b be the subspace of V_0 spanned by the classes of all objects of $\mathcal{T}_b$, and put $U = \text{im } B^n$. The hypothesis implies

$$\chi(v, x) = \chi(Bv, Bx) \quad (v \in V_b, x \in V_0). \quad (3.6)$$

We have $BV_b \subseteq V_b$. If $Bv = 0$ for $v \in V_b$, this identity and ambient nondegeneracy give $v = 0$. Finite dimensionality therefore gives $BV_b = V_b$. Since $F^b P_i \in \mathcal{T}_b$, also $\text{im } B^b \subseteq V_b$. For $N \geq \max(n, b)$, images of powers of B have stabilized, and hence

$$U = \text{im } B^N \subseteq V_b = B^N V_b \subseteq \text{im } B^N = U.$$

Thus $V_b = U$ for every $b \geq a$.

The finite-dimensional Fitting decomposition is $V_0 = U \oplus \ker B^N$ for $N \geq n$. Iterating (3.6) gives $\chi(U, \ker B^N) = 0$. A vector of U left orthogonal to U is therefore left orthogonal to V_0 and is zero. The restriction $\chi|_U$ is consequently nondegenerate on both sides. We obtain the detection property

$$\left. \begin{array}{l} c \geq b \geq a, \quad X \in \mathcal{T}_b, \\ \text{Hom}_{\mathcal{K}}(Z, X[r]) = 0 \quad (Z \in \mathcal{T}_c, r \in \mathbb{Z}) \end{array} \right\} \implies X = 0. \quad (3.7)$$

To prove it, each cohomology module $H^q(X)$ is a shifted summand of X , so has the same orthogonality. Its ambient class belongs to U and pairs to zero on the left with the classes of $\mathcal{T}_c$, which span U . It is zero by nondegeneracy. A module has a nonnegative composition vector in the ambient simple basis, so a zero class forces the module to be zero. All cohomology modules of X therefore vanish.

Relative Serre duality and a nonnegative function. The category $\mathcal{K}$ has a Serre autoequivalence $\mathbb{S}$. Here it can be constructed directly: on finite projectives the Nakayama functor is $P \mapsto D \text{Hom}_H(P, H)$. The functor $\text{Hom}_H(-, H)$ is a duality between finite right and left projectives, by evaluation into the double dual, and D is a duality from finite left projectives to finite right injectives. Their composite is therefore an equivalence, which extends to the bounded homotopy

categories. Both categories model $\mathcal{K}$, since H has finite global dimension. The evaluation identity for finite projectives, extended to bounded complexes, gives the natural isomorphism

$$D \operatorname{Hom}_{\mathcal{K}}(X, Z) \simeq \operatorname{Hom}_{\mathcal{K}}(Z, \mathbb{S}X).$$

Fix $b \geq a$ and choose $c \geq 2b$. For $X \in \mathcal{T}_c$, choose $X_0 \in \mathcal{T}_b$ with $X \simeq F^{c-b}X_0$, using (3.5), and define $R_X = F^{c-b}\mathbb{S}X_0$. This object lies in $\mathcal{T}_b$, since $F^{c-b}\mathcal{K} \subseteq \mathcal{T}_{c-b} \subseteq \mathcal{T}_b$. For $Z \in \mathcal{T}_c$, choose $Z_0 \in \mathcal{T}_b$ with $Z \simeq F^{c-b}Z_0$. Iteration of the mixed Hom isomorphisms gives natural isomorphisms

$$\begin{aligned} D \operatorname{Hom}_{\mathcal{K}}(X, Z) &\simeq D \operatorname{Hom}_{\mathcal{K}}(X_0, Z_0) \\ &\simeq \operatorname{Hom}_{\mathcal{K}}(Z_0, \mathbb{S}X_0) \simeq \operatorname{Hom}_{\mathcal{K}}(Z, R_X). \end{aligned} \quad (3.8)$$

The sources remain in $\mathcal{T}_b$ throughout the iteration. A choice of quasi-inverse for $F^{c-b} : \mathcal{T}_b \rightarrow \mathcal{T}_c$ makes these isomorphisms natural in Z , including shifts.

Suppose that X is nonzero and indecomposable. Its endomorphism ring is local by the finite-dimensional Fitting decomposition. Yoneda and vector-space duality show that the functor $D \operatorname{Hom}_{\mathcal{T}_c}(X, -)$ also has local endomorphism ring, and hence is indecomposable. If R_X had two nonzero direct summands, (3.7) would make the restricted Hom functor of each summand nonzero on $\mathcal{T}_c$. This would decompose the functor in (3.8). Thus R_X is indecomposable; it is nonzero because its Hom from X is dual to $\operatorname{End}_{\mathcal{K}}(X)$.

Now let $X \in \mathcal{W}_c$ be indecomposable. By (3.1), write $R_X = Y[s]$ with a nonzero module $Y \in \mathcal{W}_b$ and $s \in \mathbb{Z}$. The nonvanishing of $\operatorname{Hom}_{\mathcal{K}}(X, Y[s])$ forces $s = 0$ or $s = 1$. If $s = 0$, then for every $Z \in \mathcal{W}_c$,

$$D \operatorname{Ext}_H^1(X, Z) \simeq \operatorname{Hom}_{\mathcal{K}}(Z[1], Y) = 0.$$

Thus X is projective in $\mathcal{W}_c$. Every nonprojective indecomposable $X \in \mathcal{W}_c$ therefore has $R_X = Y[1]$. For every $r \in \mathbb{Z}$, (3.8) gives $\dim_k \operatorname{Hom}(X, Z[r]) = \dim_k \operatorname{Hom}(Z, Y[1-r])$. In the alternating sum, substituting $q = 1 - r$ changes the sign and gives

$$\chi([X], [Z]) = -\chi([Z], [Y]) \quad (Z \in \mathcal{T}_c). \quad (3.9)$$

Number the simples of $\mathcal{W}_b$ as $S_1, \dots, S_t$, and use d_i and h_{ij} as in Lemma 3.1. Let $C_b = (\delta_{ij}d_i - h_{ij})$ be their Euler matrix. Let $w \in \mathbb{R}^t$ have entry one at each vertex lying on a directed cycle of Q_b , including a loop, and entry zero elsewhere. Every such vertex has an incoming and an outgoing edge within this set. The inequality $h_{ij} \geq \max(d_i, d_j)$ on an edge implies

$$C_b w \leq 0, \quad C_b^\top w \leq 0$$

coordinatewise; outside the set the diagonal contributes zero. Let $z \in U$ be the ambient class of the direct sum of the simple objects selected by w , and define $\lambda([X]) = -\chi([X], z)$ for $X \in \mathcal{W}_b$. This is an additive function nonnegative on all such modules.

For nonprojective indecomposable $X \in \mathcal{W}_c$, take Y from (3.9). The classes of $\mathcal{T}_c$ span U , so the equality extends linearly to $[Z] = z$. If $x, y \in \mathbb{Z}_{\geq 0}^t$ are the $\mathcal{W}_b$ -composition vectors of X, Y , it becomes

$$x^\top C_b w = -w^\top C_b y.$$

The left side is nonpositive and the right side nonnegative. Both are zero, so $\lambda([X]) = 0$. If Q_b has a directed cycle, z is the nonzero ambient class of a module. Nondegeneracy of $\chi|_U$ and the spanning of U by the $\mathcal{W}_c$ -simple classes show that λ is nonzero on one of these classes. Its nonnegativity makes that value strictly positive.

Orthogonal components and absence of cycles. For any of the finite hearts under consideration, positive definiteness of its Euler quadratic form excludes a directed cycle. Indeed, let I be the vertices of a directed cycle without repeated vertices except the endpoint, and let u be their indicator vector. Then

$$\chi(u, u) = \sum_{i \in I} d_i - \sum_{i, j \in I} h_{ij} \leq 0. \quad (3.10)$$

The selected edge leaving i contributes at least d_i ; all additional Ext terms only decrease the expression. This also covers a loop.

Let Q_b have ν connected components. For each component, take the objects of $\mathcal{W}_b$ whose composition factors lie in it, and let $\mathcal{K}_i$ be the corresponding full subcategory of $\mathcal{T}_b$ whose cohomology lies there. Objects supported on distinct components have zero Hom and zero Ext of degree one. This follows by induction on their composition lengths from the assertion for simple objects and the ambient long exact sequences. Extensions across components split. Induction on length therefore decomposes each heart object into its component parts, and (3.1) gives an orthogonal decomposition

$$\mathcal{T}_b = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_\nu.$$

Here orthogonality means vanishing of all morphisms between distinct factors in all degrees. Each factor is thick and is generated by the finite sum of its simple objects. It has no further nonzero orthogonal decomposition: each simple is indecomposable, a nonzero Ext edge forces its endpoints into the same factor, and the simples generate.

Let $U_i \subseteq V_0$ be the ambient span of the classes of $\mathcal{K}_i$. It is nonzero because it contains a nonzero module class. The U_i are pairwise Euler-orthogonal in both directions and sum to U . Their sum is direct. In fact, a relation $\sum_i u_i = 0$ implies that u_j pairs to zero with U_j , by pairing the relation with U_j . It already pairs to zero with every other U_i , so nondegeneracy on U gives $u_j = 0$. The same argument proves that each $\chi|_{U_i}$ is nondegenerate.

Decompose $\mathcal{T}_c$ into the analogous factors $\mathcal{K}'_1, \dots, \mathcal{K}'_\nu$. There are exactly ν of them, because the equivalence $F^{c-b} : \mathcal{T}_b \rightarrow \mathcal{T}_c$ preserves the indecomposable orthogonal factors just characterized. Write σ for its bijection of factor indices, so that F^{c-b} restricts to an equivalence $\mathcal{K}_i \rightarrow \mathcal{K}'_{\sigma(i)}$.

The actual inclusion $\mathcal{W}_c \subseteq \mathcal{W}_b$ gives another map of components. A $\mathcal{W}_c$ -simple is indecomposable as a H -module: its endomorphism ring is a division ring and idempotent decompositions would lie in the wide subcategory. It therefore lies in one component of $\mathcal{W}_b$. An Ext edge in Q_c can only join two simples lying in the same component of Q_b . Denote the resulting map of component indices by ρ , so $\mathcal{K}'_j \subseteq \mathcal{K}_{\rho(j)}$. If ρ omitted a component, all classes of $\mathcal{T}_c$ would lie in a proper direct sum of the U_i . They instead span U , so ρ is surjective. Both component sets have ν elements, hence ρ is a bijection. Thus $\pi = \rho \circ \sigma$ is a permutation, and F^{c-b} , regarded as an endofunctor of $\mathcal{T}_b$, restricts to fully faithful functors

$$\mathcal{K}_i \longrightarrow \mathcal{K}_{\pi(i)}.$$

Consider a component $\mathcal{K}'_j$ whose Euler quadratic form is positive definite. For $i = \sigma^{-1}(j)$, the equivalence to this component makes the Euler form of $\mathcal{K}_i$ positive definite. By (3.10), its heart has acyclic Ext graph. Iterate F^{c-b} around the finite π -cycle through i . This gives a fully faithful exact endofunctor of $\mathcal{K}_i$, which is an equivalence by Lemma 3.3. The last functor in this composition is essentially surjective because its image contains the image of the composition. It is therefore an equivalence. Composing with its inverse makes the preceding composition an equivalence, and repeating proves that every functor around this cycle is an equivalence. In particular the target $\mathcal{K}_{\pi(i)} = \mathcal{K}_{\rho(j)}$ has positive definite Euler form and acyclic Ext graph.

Suppose now that Q_b has a directed cycle. The function λ constructed above restricts to a nonnegative additive function on $K_0(\mathcal{W}_c)$, vanishes on every nonprojective indecomposable, and is positive on some simple object. By Lemma 3.2, that simple belongs to a component $\mathcal{K}'_j$ with positive definite Euler form. The preceding paragraph puts it, under actual inclusion, in an acyclic component $\mathcal{K}_{\rho(j)}$ of Q_b . But z is supported on vertices lying on cycles of Q_b , and hence has no component in $\mathcal{K}_{\rho(j)}$. Orthogonality of the components forces λ to vanish on every module of this factor, contradicting its positive value on the chosen simple. Thus Q_b is acyclic for every $b \geq a$.

Stabilization and the adjoints. Apply Lemma 3.3 to $F|_{\mathcal{T}_a}$, viewed as a fully faithful endofunctor of $\mathcal{T}_a$. Its image is $\mathcal{T}_{a+1}$ by (3.5). Hence $\mathcal{T}_a = \mathcal{T}_{a+1}$ and $E = F|_{\mathcal{T}_a}$ is an autoequivalence. The finite-heart conclusion came from Lemma 3.1, and the component argument proves the asserted acyclicity for every $b \geq a$.

Let $\iota : \mathcal{T}_a \hookrightarrow \mathcal{K}$ be inclusion. For $Z \in \mathcal{T}_a$ and $X \in \mathcal{K}$, iterating the hypothesis a times gives natural isomorphisms

$$\mathrm{Hom}_{\mathcal{K}}(\iota Z, X) \simeq \mathrm{Hom}_{\mathcal{K}}(F^a Z, F^a X) \simeq \mathrm{Hom}_{\mathcal{T}_a}(Z, E^{-a} F^a X).$$

The object $F^a X$ lies in $\mathcal{T}_a$, so $\iota^!(X) = E^{-a} F^a X$ defines a right adjoint of ι . Complete its counit to a triangle

$$\iota^! X \longrightarrow X \longrightarrow O_X \longrightarrow \iota^! X[1].$$

Adjunction and full faithfulness of ι imply that the first map induces an isomorphism on Hom from every object of $\mathcal{T}_a$ in every degree. Thus $O_X \in \mathcal{T}_a^\perp$. If X is left orthogonal in all degrees to $\mathcal{T}_a^\perp$, then $X \rightarrow O_X$ is zero. Applying $\mathrm{Hom}_{\mathcal{K}}(X, -)$ to the triangle lifts 1_X to a map $X \rightarrow \iota^! X$, exhibiting X as a retract of $\iota^! X$. This proves

$$\{X \in \mathcal{K} : \mathrm{Hom}_{\mathcal{K}}(X, O[q]) = 0 \text{ for all } O \in \mathcal{T}_a^\perp, q \in \mathbb{Z}\} = \mathcal{T}_a.$$

By adjunction, $X \in \mathcal{T}_a^\perp$ if and only if $\iota^! X = 0$: in the forward direction one may take $Z = \iota^! X$ to make its identity vanish. Since E is an equivalence, this is equivalent to $F^a X = 0$. Consequently $\mathcal{T}_a^\perp = \ker F^a$. For such X , $F^a(FX) = F(F^a X) = 0$, which proves $F(\mathcal{T}_a^\perp) \subseteq \mathcal{T}_a^\perp$.

Finally L^a is left adjoint to F^a . For $O \in \mathcal{T}_a^\perp$ and every $q \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{K}}(L^a X, O[q]) \simeq \mathrm{Hom}_{\mathcal{K}}(X, F^a O[q]) = 0.$$

The preceding characterization of the left perpendicular therefore places $L^a X$ in $\mathcal{T}_a$. For $Y \in \mathcal{T}_a$, we obtain natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}_a}(E^a L^a X, Y) &\simeq \mathrm{Hom}_{\mathcal{T}_a}(L^a X, E^{-a} Y) \\ &\simeq \mathrm{Hom}_{\mathcal{K}}(X, F^a(E^{-a} Y)) \simeq \mathrm{Hom}_{\mathcal{K}}(X, \iota Y). \end{aligned}$$

Thus $\iota^*(X) = E^a L^a X$ is a left adjoint of ι . When $\mathcal{T}_a = 0$, the same formulas give the zero inclusion adjoints. $\square$

4 Ordinary trivial extensions

We retain the conventions of Section 2. We apply Theorem 3.4 to the derived tensor functor of an ordinary bimodule. The two results of Minamoto–Yamaura below connect this functor to the injective dimensions of the trivial extension.

The first is the finite-global-dimension specialization of [5, Theorem 5.10 and Corollary 2.10].

Proposition 4.1 (Minamoto–Yamaura). *Let R have finite global dimension and let M be a R -bimodule. On $\mathrm{D}^b(\mathrm{mod} R)$ put $F_R = - \otimes_R^{\mathbf{L}} M$ and $G_R = \mathbf{R}\mathrm{Hom}_R(M, -)$, and write $\eta : \mathrm{Id} \rightarrow G_R F_R$ for their adjunction unit. If $\mathrm{id}((R \ltimes M)_{R \ltimes M}) < \infty$, then $G_R^a(\eta_R)$ is an isomorphism for all sufficiently large integers a .*

Here the component η_R is the derived left-action map

$$R \longrightarrow \mathbf{R}\mathrm{Hom}_R(M, M), \quad r \longmapsto (m \longmapsto rm).$$

The second result is [6, Definition 4.3 and Theorem 4.10, (2) $\Rightarrow$ (1)]. Finite global dimension makes R Iwanaga–Gorenstein and identifies $\mathrm{per}(R)$ with $\mathrm{D}^b(\mathrm{mod} R)$; it also gives finite projective dimension to M on each side, as required there.

Proposition 4.2 (Minamoto–Yamaura). *Let R have finite global dimension, let M be a R -bimodule, and put $F_R = - \otimes_R^{\mathbf{L}} M$ on $\mathrm{D}^b(\mathrm{mod} R)$. Suppose that an admissible thick subcategory $\mathcal{T}$ has the following properties: F_R restricts to an autoequivalence of $\mathcal{T}$, preserves $\mathcal{T}^\perp$, and $F_R^a(\mathcal{T}^\perp) = 0$ for some integer $a \geq 1$. Then $R \ltimes M$ is Iwanaga–Gorenstein.*

We apply these propositions over R , using Theorem 3.4 after transport to the hereditary category.

Proof of Theorem 1.1. Finite global dimension of the base. Put $\mathcal{D} = \mathrm{D}^b(\mathrm{mod} R)$ and $\mathcal{K} = \mathrm{D}^b(\mathrm{mod} H)$, and fix a k -linear triangle equivalence $\Psi : \mathcal{D} \rightarrow \mathcal{K}$. Let $S_1, \dots, S_t$ represent the simple right R -modules. There are integers $u \leq v$ such that the cohomology of each ΨS_i is concentrated in degrees $u, \dots, v$. Heredity gives the decomposition

$$\Psi S_i \simeq \bigoplus_{p=u}^v H^p(\Psi S_i)[-p].$$

Indeed, the connecting morphisms of the truncation triangles lie in extension groups of degree at least two and hence vanish. Between the summands in degrees p and q , a morphism to a shift by j lies in an extension group over H of degree $j + p - q$. Consequently, if $N = v - u + 1$, then

$$\mathrm{Ext}_R^j(S_i, S_\ell) \cong \mathrm{Hom}_{\mathcal{K}}(\Psi S_i, \Psi S_\ell[j]) = 0 \quad (j > N, 1 \leq i, \ell \leq t).$$

Induction on composition length in each variable gives $\mathrm{Ext}_R^{N+1}(U, V) = 0$ for all finite-dimensional right modules U, V . To see that this bounds projective dimensions, take the first N steps of a resolution of U by finite projectives, and let Z be its N th syzygy. Dimension shifting gives $\mathrm{Ext}_R^1(Z, V) = 0$ for all such V . In particular, the kernel of a finite projective cover of Z gives a split cover, so Z is projective. Hence every finite right module has projective dimension at most N . Duality gives

$$\mathrm{Ext}_{R^{\mathrm{op}}}^j(U', V') \cong \mathrm{Ext}_R^j(DV', DU')$$

for finite left R -modules U', V' . The same argument bounds their projective dimensions by N . Thus R has finite global dimension on both sides, and $\mathcal{D} = \mathrm{per}(R) = \mathrm{thick}(R)$.

Both adjoints of the tensor functor. Use F_R, G_R , and their adjunction unit η as in Proposition 4.1. These functors preserve $\mathcal{D}$: derived tensor can be computed on bounded complexes of finite projectives, and derived Hom on bounded complexes of finite injectives. The right R -action on $\mathrm{Hom}_R(M, I)$ is $(f \cdot r)(m) = f(rm)$, using the left action on M . Thus the usual tensor–Hom adjunction gives $F_R \dashv G_R$ on $\mathcal{D}$.

For the left adjoint, let $\mathbb{S}_R = - \otimes_R^{\mathbf{L}} DR$. On finite projectives this is the Nakayama equivalence to finite injectives: $P \otimes_R DR \cong D \operatorname{Hom}_R(P, R)$. Since bounded complexes of either projectives or injectives model $\mathcal{D}$, it induces an autoequivalence. Finite-projective evaluation, applied to the Hom complexes, gives natural isomorphisms

$$D \operatorname{Hom}_{\mathcal{D}}(X, Y) \cong \operatorname{Hom}_{\mathcal{D}}(Y, \mathbb{S}_R X) \quad (X, Y \in \mathcal{D}).$$

Define $L_R = \mathbb{S}_R^{-1} G_R \mathbb{S}_R$. Then

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}}(L_R X, Y) &\cong D \operatorname{Hom}_{\mathcal{D}}(Y, G_R \mathbb{S}_R X) \\ &\cong D \operatorname{Hom}_{\mathcal{D}}(F_R Y, \mathbb{S}_R X) \\ &\cong \operatorname{Hom}_{\mathcal{D}}(X, F_R Y). \end{aligned}$$

These isomorphisms establish $L_R \dashv F_R$.

Morphisms from the thick image. Assume $\operatorname{id}(A_A) < \infty$. By Proposition 4.1, choose an integer $a \geq 1$ such that $G_R^a(\eta_R)$ is invertible. The objects $Y \in \mathcal{D}$ for which

$$G_R^a(\eta_Y) : G_R^a Y \longrightarrow G_R^{a+1} F_R Y$$

is invertible form a thick subcategory. Closure under shifts and direct summands follows from naturality and additivity; closure under triangles follows by applying the two exact functors to a triangle and using its long exact Hom sequences. This subcategory contains R , so it is all of $\mathcal{D}$.

Iterated adjunction therefore shows, for every $Y \in \mathcal{D}$ and $r \in \mathbb{Z}$, that composition with $\eta_{Y[r]}$ induces an isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(F_R^a R, Y[r]) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(F_R^a R, G_R F_R Y[r]).$$

Under adjunction this is the map

$$\operatorname{Hom}_{\mathcal{D}}(F_R^a R, Y[r]) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F_R^{a+1} R, F_R Y[r])$$

induced by F_R . More explicitly, for a morphism $f : X \rightarrow Y[r]$, the adjoint of $F_R f$ is $G_R F_R(f) \eta_X = \eta_{Y[r]} f$.

For fixed Y , the objects X for which these maps are isomorphisms for all r form a thick subcategory: long exact Hom sequences give closure under triangles, and the maps respect shifts and summands. It contains $F_R^a R$. Since Y was arbitrary, we have proved

$$\operatorname{Hom}_{\mathcal{D}}(X, Y[r]) \xrightarrow{F_R} \operatorname{Hom}_{\mathcal{D}}(F_R X, F_R Y[r]) \quad \text{is an isomorphism} \quad (4.1)$$

for every $X \in \operatorname{thick}(F_R^a R)$, every $Y \in \mathcal{D}$, and every $r \in \mathbb{Z}$.

Stabilization and the two-sided conclusion. Choose a quasi-inverse Ψ^{-1} and transport the three functors by

$$F = \Psi F_R \Psi^{-1}, \quad L = \Psi L_R \Psi^{-1}, \quad G = \Psi G_R \Psi^{-1}.$$

They are k -linear exact functors on $\mathcal{K}$ with $L \dashv F \dashv G$. Both ΨR and H generate $\mathcal{K}$ as a thick subcategory. Exact functors preserve shifts, triangles, and retracts; applying F^d to the finite thick constructions expressing each generator in terms of the other gives

$$\operatorname{thick}_{\mathcal{K}}(F^d \Psi R) = \operatorname{thick}_{\mathcal{K}}(F^d H) \quad (d \geq 0).$$

Set $\mathcal{T}_d = \text{thick}_{\mathcal{K}}(F^d H)$. Equation (4.1) transports to the morphism condition of Theorem 3.4 on $\mathcal{T}_a$. That theorem gives admissibility of $\mathcal{T}_a$, an autoequivalence $F|_{\mathcal{T}_a}$, and

$$\mathcal{T}_a^\perp = \ker F^a, \quad F(\mathcal{T}_a^\perp) \subseteq \mathcal{T}_a^\perp.$$

In particular, F^a annihilates the whole right perpendicular.

Let $\mathcal{T}_R = \Psi^{-1}(\mathcal{T}_a) \subseteq \mathcal{D}$. Conjugating the adjoints of the inclusion by Ψ gives both adjoints of $\mathcal{T}_R \hookrightarrow \mathcal{D}$. Preservation of all Hom groups gives $\mathcal{T}_R^\perp = \Psi^{-1}(\mathcal{T}_a^\perp)$. Thus $F_R|_{\mathcal{T}_R}$ is an autoequivalence, F_R preserves $\mathcal{T}_R^\perp$, and $F_R^a(\mathcal{T}_R^\perp) = 0$. These are the hypotheses of Proposition 4.2 for the original R -bimodule M . It follows that $A = R \ltimes M$ is Iwanaga–Gorenstein, proving the implication from finite right injective dimension to finite left injective dimension.

The opposite-algebra implication. Write M^{op} for the R^{op} -bimodule with underlying vector space M and actions $r^{\text{op}} \cdot m \cdot s^{\text{op}} = smr$. Then

$$A^{\text{op}} \cong R^{\text{op}} \ltimes M^{\text{op}}.$$

Vector-space duality is an exact contravariant equivalence between the bounded derived categories of finite right modules over an algebra and its opposite. Hence the composite

$$\text{D}^b(\text{mod } R^{\text{op}}) \xrightarrow{D} \text{D}^b(\text{mod } R) \xrightarrow{\Psi} \text{D}^b(\text{mod } H) \xrightarrow{D} \text{D}^b(\text{mod } H^{\text{op}})$$

is a covariant k -linear triangle equivalence; its first and last arrows are contravariant. The algebra H^{op} is hereditary: for finite left H -modules U', V' , duality identifies their Ext^2 with $\text{Ext}_H^2(DV', DU') = 0$. Dimension shifting and projective covers, as above, give projective dimension at most one. If $\text{id}(A_A) < \infty$, then A^{op} has finite right injective dimension. The implication already proved applies to this opposite trivial extension and yields finite left injective dimension for A^{op} , equivalently $\text{id}(A_A) < \infty$. $\square$

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