

A note of the Gorenstein symmetry conjecture (I)

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Abstract

Over any field, we reduce the Gorenstein symmetry conjecture to an explicit family of endomorphism algebras over a fixed five-dimensional commutative local symmetric algebra.

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1 Introduction

Self-injective algebras and algebras of finite global dimension have finite injective dimension on both sides. The Gorenstein symmetry conjecture asks whether finiteness on one side forces finiteness on the other. It is recorded among the homological conjectures for Artin algebras by Auslander, Reiten and Smalø [1, p. 410, Conjecture (13)]. Reductions through derived decompositions relate this question to smaller or more restricted classes of algebras; Han and Qin study this approach using recollements [3, Section 5]. Endomorphism algebras of generators over symmetric algebras form the class of gendo-symmetric algebras introduced by Fang and Koenig [2].

Realization of algebras as endomorphism quotients by maps through a fixed object occurs in controlled wildness [4]. A companion Gorenstein symmetry reduction uses the input-dependent symmetric trivial extension [5]. Here both the symmetric base and a summand of the generator are fixed, and the construction preserves each one-sided finiteness condition. All algebras and modules are finite-dimensional over a field k , and algebras have an identity. For an algebra A , the symbols $\text{id}(A_A)$ and $\text{id}({}_A A)$ denote the injective dimensions of its right and left regular modules. The Gorenstein symmetry conjecture for A is the equivalence

$$\text{id}(A_A) < \infty \iff \text{id}({}_A A) < \infty.$$

Write $D = \text{Hom}_k(-, k)$ for vector-space duality. An algebra C is symmetric if $C \cong DC$ as bimodules. A C -module is a generator if a finite direct sum of copies of it contains C as a direct summand. We use ordinary composition for multiplication in endomorphism algebras. The opposite algebra R^{op} has multiplication in the reverse order to R .

Define the commutative algebra

$$C = k1 \oplus kx \oplus kp \oplus kq \oplus kw, \quad x^2 = pq = qp = w, \quad (1)$$

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where all other products of two elements of $\{x, p, q, w\}$ are zero. Put $J = \text{rad } C$, the Jacobson radical, $U = C/(w)$ and

$$H = C \oplus U \oplus J \oplus k,$$

where $k = C/J$ is the simple module. Thus $\dim_k H = 14$.

Theorem 1.1. *For every nonzero finite-dimensional k -algebra R , there is an explicit finite left C -module M_R such that, for*

$$E_R = \text{End}_C(H \oplus M_R),$$

one has

$$\begin{aligned} \text{id}((E_R)_{E_R}) < \infty &\iff \text{id}({}_R R) < \infty, \\ \text{id}({}_{E_R} E_R) < \infty &\iff \text{id}(R_R) < \infty. \end{aligned}$$

If $d = \dim_k R$ and $n = d(2d + 3)$, then

$$\dim_k M_R = 2n, \quad \dim_k E_R = 53 + 16n + n^2 + d.$$

In particular, the Gorenstein symmetry conjecture over k is equivalent to its restriction to this family of gendo-symmetric algebras.

The construction encodes left multiplication in R by two operators on R^{2d+3} . Factoring the resulting endomorphisms through H gives the quotient R^{op} . Two explicit approximation sequences give finite projective resolutions of that quotient on both sides. The fixed algebra $\text{End}_C(H)$ has finite global dimension, which permits the separate transfer of the two injective-dimension conditions.

2 An idempotent quotient criterion

Here modules are right modules. We write pd for projective dimension and gl. dim for global dimension, the supremum of the projective dimensions of modules. The category $D^b(\text{mod } E)$ is the bounded derived category of finite right E -modules. We use cohomological complexes: $d_X^i : X^i \rightarrow X^{i+1}$, $H^i(X) = \ker d_X^i / \text{im } d_X^{i-1}$, and $X[1]^i = X^{i+1}$ with differential $-d_X^{i+1}$. For an idempotent e , Te denotes the image of multiplication by e , taken termwise for complexes. The symbol $\text{thick}(T)$ denotes closure under finite sums, shifts, cones and direct summands. The category $K^b(\text{inj } E)$ consists of the objects represented by bounded complexes of finite injective modules. Derived tensor and Hom are denoted by $\otimes^{\mathbf{L}}$ and $\mathbf{R}\text{Hom}$; they are computed using projective or injective resolutions in the corresponding variables. For a module X , $\text{add } X$ consists of the summands of finite direct sums of copies of X .

Lemma 2.1. *Let $e \in E$ be an idempotent, and set $B = eEe$, $I = EeE$, $Q = E/I$. Suppose*

$$I_E \in \text{thick}(eE), \quad \text{gl. dim } B < \infty, \quad \text{pd}_E(Q) < \infty.$$

For every finite right E -module N ,

$$\text{id}(N_E) < \infty \iff \mathbf{R}\text{Hom}_E(Q, N) \in K^b(\text{inj } Q).$$

The right Q -action on the derived Hom is induced by precomposition with left multiplication on Q .

Proof. The condition $I_E \in \text{thick}(eE)$ gives $\text{pd}(I_E) < \infty$, and $0 \rightarrow I \rightarrow E \rightarrow Q \rightarrow 0$ gives $\text{pd}(Q_E) < \infty$. Thus $\mathbf{R}\text{Hom}_E(Q, N)$ has bounded cohomology.

For $T \in D^b(\text{mod } E)$, the natural evaluation map

$$(Te) \otimes_B^{\mathbf{L}} eE \longrightarrow T$$

is induced by $te \otimes eu \mapsto teu$ and is an isomorphism for $T = eE$. Its isomorphism class is preserved under shifts, cones and summands, so it is an isomorphism on $\text{thick}(eE)$. Since $Ie = Ee$, applying it to I gives the multiplication isomorphism

$$Ee \otimes_B^{\mathbf{L}} eE \simeq I.$$

To retain both E -actions, compute the derived tensor by the two-sided bar complex with degree $-r$ term

$$Ee \otimes_k B^{\otimes_k r} \otimes_k eE \quad (r \geq 0),$$

where $B^{\otimes_k 0} = k$. Its differential is the alternating sum of adjacent multiplications: for $r \geq 1$,

$$\begin{aligned} d(u \otimes b_1 \otimes \cdots \otimes b_r \otimes v) &= ub_1 \otimes b_2 \otimes \cdots \otimes b_r \otimes v \\ &\quad + \sum_{i=1}^{r-1} (-1)^i u \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_r \otimes v \\ &\quad + (-1)^r u \otimes b_1 \otimes \cdots \otimes b_{r-1} \otimes b_r v. \end{aligned}$$

Here $u \in Ee$, $v \in eE$ and $b_i \in B$. The usual bar resolution is a projective resolution over B because k is a field; its augmented complex contracts by insertion of 1_B . The outside factors carry the left and right E -actions. Multiplication to I , given in degree zero by $u \otimes v \mapsto uv$, is therefore a bimodule chain map. After forgetting the left action it is the evaluation quasi-isomorphism above. Its cone has zero cohomology as a bimodule as well. The short exact sequence defining Q gives a bimodule triangle

$$Ee \otimes_B^{\mathbf{L}} eE \longrightarrow E \longrightarrow Q \longrightarrow (Ee \otimes_B^{\mathbf{L}} eE)[1]. \quad (2)$$

Tensoring with Q gives $Q \otimes_E^{\mathbf{L}} Q \simeq Q$, since $eQ = 0$. Write Inf for restriction along $E \twoheadrightarrow Q$, called inflation. For a right Q -complex X , associativity gives

$$(\text{Inf } X) \otimes_E^{\mathbf{L}} Q \simeq X \otimes_Q^{\mathbf{L}} (Q \otimes_E^{\mathbf{L}} Q) \simeq X.$$

This is the counit of derived tensor adjoint to inflation, so inflation is fully faithful. The same bimodule isomorphism gives full faithfulness for left modules.

Apply $\mathbf{R}\text{Hom}_E(-, N)$ to (2). Tensor–Hom adjunction and projectivity of eE give

$$\text{Inf } \mathbf{R}\text{Hom}_E(Q, N) \longrightarrow N \longrightarrow \mathbf{R}\text{Hom}_B(Ee, Ne) \longrightarrow (\text{Inf } \mathbf{R}\text{Hom}_E(Q, N))[1]. \quad (3)$$

The third term belongs to $K^b(\text{inj } E)$. Indeed, Ne has a bounded injective resolution over B . For a right B -injective V , the right E -module $\text{Hom}_B(Ee, V)$ is injective, because

$$\text{Hom}_E(Y, \text{Hom}_B(Ee, V)) \cong \text{Hom}_B(Ye, V)$$

and $Y \mapsto Ye$ is exact.

Let $t = \text{pd}_E(Q)$. If V is right Q -injective, derived adjunction yields

$$\text{Ext}_E^j(Y, \text{Inf } V) \cong \text{Hom}_Q(\text{Tor}_j^E(Y, Q), V) = 0 \quad (j > t).$$

Thus inflation sends Q -injectives to modules of E -injective dimension at most t , and sends bounded injective complexes into $K^b(\text{inj } E)$. If the derived Hom in the statement is a bounded

injective complex over Q , the first and third terms of (3) have finite injective dimension over E . This subcategory is thick, so N does also.

Conversely, take a bounded injective resolution $V^\bullet$ of N over E . For every right E -injective V , the module $\text{Hom}_E(Q, V)$ is right Q -injective, since

$$\text{Hom}_Q(Y, \text{Hom}_E(Q, V)) \cong \text{Hom}_E(\text{Inf } Y, V)$$

and inflation is exact. Consequently $\text{Hom}_E(Q, V^\bullet)$ is the required bounded injective representative. $\square$

3 The fixed algebra and its endomorphism algebra

The multiplication in (1) is associative: a product of three radical basis elements vanishes in either order. The coefficient of w defines a linear functional $\varepsilon : C \rightarrow k$. The pairing $(a, b) \mapsto \varepsilon(ab)$ pairs 1 with w and, on the ordered basis x, p, q , has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is nondegenerate in every characteristic. Hence $C \cong DC$ as bimodules. In particular C is injective on both sides. For a finite left C -module X , its radical is JX , its socle is $\text{soc } X = \{u \in X : Ju = 0\}$, and its top is X/JX . One has

$$J = kx \oplus kp \oplus kq \oplus kw, \quad J^2 = kw, \quad J^3 = 0, \quad \text{soc } C = kw.$$

Let $W = kx \oplus kp \oplus kq$, so $U = k \oplus W$ and $W^2 = 0$ in U .

Lemma 3.1. *For $H = C \oplus U \oplus J \oplus k$, the algebra $B = \text{End}_C(H)$ has finite global dimension and dimension 53.*

Proof. First put $H_0 = C \oplus U \oplus k$ and $B_0 = \text{End}_C(H_0)$. Every finite C -module X has a short exact sequence

$$0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow X \longrightarrow 0 \quad (4)$$

with $H_1, H_2 \in \text{add } H_0$, which remains exact under $\text{Hom}_C(H_0, -)$. To construct it, if $wu \neq 0$ for some $u \in X$, the map $C \rightarrow X$, $1 \mapsto u$, is injective. Indeed, every nonzero ideal of the local algebra C contains its simple socle kw , whereas the kernel omits w . Injectivity of C splits this summand. Repetition leaves a summand killed by w .

In that summand $JX \subseteq \text{soc } X$. A complement of JX in $\text{soc } X$ splits off as a sum of simple modules: choose a vector-space complement containing JX to this semisimple subspace; it is stable under C . For the remaining module, $wX = 0$ and $\text{soc } X = JX$. A projective cover $U^t \twoheadrightarrow X$ over U has kernel contained in $\text{rad}(U^t) = W^t$, hence semisimple. Its radical maps onto $JX = \text{soc } X$. Maps from C lift by projectivity; maps from U lift by U -projectivity; maps from k lift by this surjection on socles. This gives (4); split summands are handled by identity maps.

Every finite right B_0 -module has a projective presentation

$$\text{Hom}_C(H_0, X_1) \longrightarrow \text{Hom}_C(H_0, X_0) \longrightarrow Y \longrightarrow 0, \quad X_i \in \text{add } H_0.$$

The map comes from a map $X_1 \rightarrow X_0$, since $\text{Hom}_C(H_0, -)$ is fully faithful on $\text{add } H_0$. Its kernel is $\text{Hom}_C(H_0, \ker(X_1 \rightarrow X_0))$. Apply (4) to this kernel module. The resulting length-one projective resolution of the second syzygy gives $\text{pd } Y \leq 3$, so $\text{gl. dim } B_0 \leq 3$. Vector-space duality gives the same Ext-vanishing bound for left B_0 -modules.

We now add J . Its endomorphism algebra is U . More explicitly, an endomorphism of J has the form

$$w \mapsto aw, \quad v \mapsto av + \ell(v)w \quad (v \in W), \quad a \in k, \quad \ell \in DW.$$

Indeed, write an endomorphism on $J/J^2 = W$ as F and its action on w as multiplication by a . Commutation with multiplication by every $u \in W$ gives $uF(v) = a(uv)$ for all $v \in W$. Nondegeneracy forces $F(v) = av$. The remaining coefficient $\ell : W \rightarrow k$ is arbitrary. The product pairing identifies W with DW , and this identifies the displayed algebra with U . The maps factoring through H_0 are exactly those with $a = 0$. Every such map factors as $J \rightarrow U \rightarrow J$, with $v \mapsto \ell(v)x$, $w \mapsto 0$ on the first arrow and $1 \mapsto x$ on the second. Conversely, a map $J \rightarrow C$ has image in $J = \{c \in C : wc = 0\}$, and a map $C \rightarrow J$ is multiplication by an element of J , so their composite has image in $J^2 = kw$. Maps through U or k have image in kw , since $J \rightarrow U$ kills w and maps W into W , while $U \rightarrow J$ maps W into kw .

Let $e_0 \in B$ project to H_0 . It follows that $B/Be_0B \cong k$; below, k is a B -module through this quotient. Consider the surjection

$$U \otimes_k W \longrightarrow J, \quad \bar{c} \otimes v \mapsto cv.$$

It is an isomorphism on tops. On radicals it is the multiplication $W \otimes_k W \rightarrow kw$, of rank one, so its kernel is k^8 . Thus

$$0 \longrightarrow k^8 \longrightarrow U^3 \longrightarrow J \longrightarrow 0. \quad (5)$$

The last map is a right add H_0 -approximation: maps from C and U lift through this surjection, and maps from k lift because the radical multiplication is onto kw . Its image after $\text{Hom}_C(J, -)$ is precisely the radical of $\text{End}_C(J)$, because any top-to-socle map lifts into the radical of U^3 . Consequently (5) gives the right B -projective resolution

$$0 \longrightarrow \text{Hom}_C(H, k^8) \longrightarrow \text{Hom}_C(H, U^3) \longrightarrow (1 - e_0)B \longrightarrow k \longrightarrow 0.$$

The two highest terms belong to $\text{add}(e_0B)$. Thus $\text{pd}(k_B) \leq 2$; decomposing the kernel of $B \rightarrow k$ as e_0B plus the kernel of $(1 - e_0)B \rightarrow k$ puts Be_0B in $\text{thick}(e_0B)$.

Evaluation, as in Lemma 2.1, gives

$$Be_0 \otimes_{B_0}^{\mathbf{L}} e_0B \simeq Be_0B.$$

Take a left B_0 -projective resolution of e_0B of length at most three. Tensoring with Be_0 represents the displayed derived tensor, as a left B -complex, by summands of finite sums of Be_0 in degrees $-3, \dots, 0$. Its sole cohomology is Be_0B , so $\text{pd}_B(Be_0B) \leq 3$ and $\text{pd}_B(k) \leq 4$. For any finite right B -module Y , tensoring the corresponding bimodule triangle gives

$$(Ye_0) \otimes_{B_0}^{\mathbf{L}} e_0B \longrightarrow Y \longrightarrow Y \otimes_B^{\mathbf{L}} k \longrightarrow ((Ye_0) \otimes_{B_0}^{\mathbf{L}} e_0B)[1].$$

The first term is represented by right B -projectives in degrees $-3, \dots, 0$, using $\text{gl. dim } B_0 \leq 3$. The third has cohomology only in degrees $-4, \dots, 0$, since $\text{pd}_B(k) \leq 4$. It is a complex of k -spaces with right B -action through the quotient $B \rightarrow k$, so its splitting into cohomology over k is also B -linear. Resolving each copy of k_B in length two represents this term by right B -projectives in degrees $-6, \dots, 0$. If the first and third terms are represented by complexes $L^\bullet$ and $R^\bullet$, then Y is represented by the cone of $R^\bullet[-1] \rightarrow L^\bullet$, whose degree j term is $L^j \oplus R^j$. Thus every finite right B -module has projective dimension at most six, giving $\text{gl. dim } B \leq 6$.

For the dimension, the entries of the following table are $\dim_k \text{Hom}_C(X, Y)$, with source in the row and target in the column:

	C	U	J	k
C	5	4	4	1
U	4	4	4	1
J	4	9	4	3
k	1	3	1	1

The first row is evaluation at 1. A map from U is determined by an element annihilated by w . Maps from k are socle elements. A map $J \rightarrow U$ kills w and maps W arbitrarily to the radical W , giving dimension nine. The remaining entries follow from the displayed form of $\text{End}_C(J)$ and the duality $\text{Hom}_C(J, C) \cong DJ$. Their sum is 53. $\square$

4 Encoding the input algebra

Let R be nonzero, put $d = \dim_k R$ and $A = R^{\text{op}}$, choose a basis $a_1, \dots, a_d$ over k , and put $m = 2d + 3$, $V = R^m$. Right multiplication makes V a left A -module; the same action will identify A with the endomorphism quotient. Let $L_a : R \rightarrow R$ denote left multiplication by a . Define operators $P, T \in \text{End}_k(V)$ by

$$P = \begin{pmatrix} 0 & 1_R & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1_R \\ 0 & & & 0 \end{pmatrix}, \quad T = \text{diag}(\underbrace{1_R, \dots, 1_R}_{d+1}, 0, L_{a_1}, \dots, L_{a_d}, 0). \quad (6)$$

An entry 1_R here means the identity map of the vector space R ; P has m block rows. Both operators commute with componentwise right multiplication by R .

Lemma 4.1. *The common centralizer of P and T in $\text{End}_k(V)$ consists precisely of componentwise right multiplications by elements of R and is isomorphic to $A = R^{\text{op}}$.*

Proof. Writing an endomorphism as an $m \times m$ block matrix, the equality $FP = PF$ says that entries below the diagonal vanish and the entries on each upper diagonal are equal. Indeed, its (i, j) equation is $F_{i,j-1} = F_{i+1,j}$, with zero boundary entries. Denote the block on the s th upper diagonal by F_s .

For $1 \leq s \leq d+1$, take the diagonal positions $d+2-s$ and $d+2$ of T , which contain 1_R and 0. For $d+2 \leq s \leq m-1$, use positions $m-s$ and m , again containing 1_R and 0. The corresponding block equation in $FT = TF$ forces $F_s = 0$. The sole remaining block F_0 commutes with all L_{a_i} , hence with every left multiplication. For $r \in R$, this gives $F_0(r) = rF_0(1)$, so it is right multiplication. Composition of right multiplications reverses multiplication in R , giving R^{op} . $\square$

Define $M = M_R = V \oplus V$ by

$$x(v, z) = (0, v), \quad p(v, z) = (0, Pv), \quad q(v, z) = (0, Tv), \quad w(v, z) = 0. \quad (7)$$

All products of two radical actions vanish, so (1) holds. Its radical and socle both equal $0 \oplus V$. The right R -action makes V and M left A -modules, with $r^{\text{op}} \cdot v = vr$; V is free of rank m over A . These actions commute with C .

Lemma 4.2. *The C -endomorphisms of M are precisely*

$$\begin{pmatrix} F & 0 \\ L & F \end{pmatrix}, \quad F \in A, \quad L \in \text{End}_k(V). \quad (8)$$

The remaining Hom spaces have the parameterizations

space	parameters	space	parameters
$\text{Hom}_C(C, M)$	V^2	$\text{Hom}_C(M, C)$	$(DV)^2$
$\text{Hom}_C(U, M)$	V^2	$\text{Hom}_C(M, U)$	$(DV)^3$
$\text{Hom}_C(J, M)$	V^3	$\text{Hom}_C(M, J)$	$(DV)^2$
$\text{Hom}_C(k, M)$	V	$\text{Hom}_C(M, k)$	DV

The left A -action in the left column acts on the output V ; the right A -action in the right column acts on DV by precomposition. The endomorphisms of M factoring through $\text{add } H$ are exactly the lower blocks L in (8).

Proof. Commutation with x first forces the two diagonal blocks to be equal and the upper-right block to vanish. Commutation with p, q requires that diagonal block to centralize P, T . Lemma 4.1 gives (8).

Maps from C or U to M are determined by an arbitrary image of 1, since $wM = 0$. Maps from k have image in $0 \oplus V$. For $f : J \rightarrow M$, write $f(w) = (0, z)$. The relation $x^2 = w$ forces the first component of $f(x)$ to be z , while $xp = xq = 0$ makes those of $f(p), f(q)$ zero. Then $pq = w$ gives $f(w) = pf(q) = 0$. Thus f kills w and maps W arbitrarily to $0 \oplus V$.

A map $f : M \rightarrow U$ has a top coefficient $\alpha \in DV$ on its first summand. Commutation with x gives $f(0, v) = \alpha(v)x$. Commutation with p gives

$$\alpha(Pv)x = \alpha(v)p.$$

The independence of x, p implies $\alpha = 0$. Hence f kills the second summand and maps the first arbitrarily to W .

Every map $M \rightarrow C$ has image in J , since $wM = 0$. All maps $M \rightarrow J$ are

$$(v, z) \mapsto \ell(v)x + \ell(Pv)q + \ell(Tv)p + (s(v) + \ell(z))w, \quad \ell, s \in DV. \quad (9)$$

To derive it, $f(0, z) = \ell(z)w$ for some $\ell \in DV$, because $0 \oplus V = xM$ and $xJ = kw$. Write $f(v, 0) = a(v)x + b(v)p + c(v)q + s(v)w$, with $a, b, c, s \in DV$. Commutation with x, p, q gives $a(v) = \ell(v)$, $c(v) = \ell(Pv)$ and $b(v) = \ell(Tv)$, respectively. Conversely, multiplying its right side by x, p, q gives respectively $\ell(v)w$, $\ell(Pv)w$, $\ell(Tv)w$, exactly the images required by (7). Maps to k are arbitrary functionals on the first summand.

These formulas also show that a composition $M \rightarrow X \rightarrow M$ for $X = C, U, J, k$ has only a lower block. For $X = C$, the first map has image in J , while the second maps J into $0 \oplus V$ and kills w . For $X = J$, the second map sends all of J into $0 \oplus V$ and kills w ; by (9), the first map sends $0 \oplus V$ into kw . Each composite therefore has image in $0 \oplus V$ and kills $0 \oplus V$. For $X = U, k$, the first map kills $0 \oplus V$, and its image under the second map lies in $0 \oplus V$, by the displayed parameterizations. Conversely, every rank-one lower block factors through $M \twoheadrightarrow k \rightarrow M$, and all linear maps $V \rightarrow V$ are sums of rank-one maps. This proves the factorization assertion. $\square$

Put $G = H \oplus M$, $E = \text{End}_C(G)$, and let $f, g \in E$ be projections onto H, M , so $f + g = 1$. Lemma 4.2 gives

$$fEf = B, \quad I = EfE, \quad E/I \cong A. \quad (10)$$

Indeed, all matrix blocks meeting H vanish in the quotient, and the kernel in the M -diagonal block is exactly the factorization ideal just computed. Moreover $\dim_k \text{Hom}_C(H, M) = \dim_k \text{Hom}_C(M, H) = 8n$ where $n = \dim_k V$, while (8) gives $\dim_k \text{End}_C(M) = n^2 + d$. Lemma 3.1 proves the dimension formula in Theorem 1.1.

5 Resolutions of the quotient

Tensor products in the next sequences are over k ; C acts on their first factor and A on the coefficient V . There are short exact sequences compatible with both actions:

$$0 \longrightarrow k \otimes V^2 \longrightarrow U \otimes V \xrightarrow{\pi} M \longrightarrow 0, \quad (11)$$

$$0 \longrightarrow M \xrightarrow{\iota} J \otimes V \longrightarrow k \otimes V^2 \longrightarrow 0. \quad (12)$$

Here $\pi(1 \otimes v) = (v, 0)$, so its radical map is $(1, P, T) : W \otimes V \rightarrow V$. Its kernel is parameterized by

$$(u, v) \mapsto p \otimes u - x \otimes Pu + q \otimes v - x \otimes Tv.$$

The second injection is

$$\iota(v, z) = x \otimes v + q \otimes Pv + p \otimes Tv + w \otimes z.$$

Multiplying by x, p, q gives $w \otimes v, w \otimes Pv, w \otimes Tv$, respectively. The x -coordinate and then the w -coordinate show injectivity. The last arrow in (12) is explicitly

$$x \otimes a + p \otimes b + q \otimes c + w \otimes z \mapsto (b - Ta, c - Pa) \in V^2, \quad a, b, c, z \in V.$$

It is onto by taking $a = z = 0$. Its kernel has $b = Ta$ and $c = Pa$, so is exactly the image of ι . Multiplication by J has image in $w \otimes V$, which this arrow kills; the target therefore has trivial radical action. All maps are A -linear because P, T commute with that action.

Lemma 5.1. *The quotient in (10) has right and left E -projective resolutions*

$$0 \rightarrow \text{Hom}_C(G, k \otimes V^2) \rightarrow \text{Hom}_C(G, U \otimes V) \rightarrow gE \rightarrow A \rightarrow 0, \quad (13)$$

$$0 \rightarrow \text{Hom}_C(k \otimes V^2, G) \rightarrow \text{Hom}_C(J \otimes V, G) \rightarrow Eg \rightarrow A \rightarrow 0. \quad (14)$$

In each resolution the two highest terms come from $\text{add}(fE)$, respectively $\text{add}(Ef)$. Consequently $I_E \in \text{thick}(fE)$, ${}_E I \in \text{thick}(Ef)$ and A has projective dimension at most two on each side. The first resolution is a sequence of (A, E) -bimodules, and the second of (E, A) -bimodules: A acts on each coefficient V , and on M by its diagonal action in (8).

Proof. For (13), maps from C, U to M lift along π by their arbitrary cyclic image. Maps from k, J have image in $0 \oplus V$ and lift into the submodule $x \otimes V$ of $\text{rad}_C(U \otimes V) = W \otimes V$. Indeed, π restricts to a C -module isomorphism $x \otimes V \rightarrow 0 \oplus V$, $x \otimes v \mapsto (0, v)$. Thus every map $X \rightarrow M$ with $X \in \text{add } H$ factors through π ; this is the right $\text{add } H$ -approximation property. Applied to source M , Lemma 4.2 shows that its image consists of lower blocks; every lower block is realized by the map $(v, z) \mapsto x \otimes Lv$ followed by π . Thus $\text{Hom}_C(G, -)$ applied to (11) has cokernel A and gives (13).

For (14), a map $M \rightarrow C$ extends along ι by injectivity of C . A map to J given by (9) extends to $J \otimes V \rightarrow J$ by

$$x \otimes v \mapsto \ell(v)x + s(v)w, \quad p \otimes v \mapsto \ell(v)p, \quad q \otimes v \mapsto \ell(v)q, \quad w \otimes v \mapsto \ell(v)w.$$

These assignments respect the multiplication in C . Maps to U or k kill the second summand of M and extend by prescribing their value on $x \otimes V$ and making them zero on $p \otimes V, q \otimes V, w \otimes V$. Thus every map $M \rightarrow X$ with $X \in \text{add } H$ factors through ι , the left approximation property. On maps to M , the image consists of lower blocks, all realized by sending $x \otimes v$ to $(0, Lv)$ and the other three components to zero. Applying $\text{Hom}_C(-, G)$ gives (14).

The terms are projective because $U, J, k \in \text{add } G$: for $X \in \text{add } G$, $\text{Hom}_C(G, X)$ is right E -projective and $\text{Hom}_C(X, G)$ is left E -projective. As $I_E = fE \oplus \ker(gE \rightarrow A)$, the first resolution puts I_E in $\text{thick}(fE)$. The second gives the left assertion in the same way. In the first resolution, left A acts by postcomposition on the target; in the second, right A acts by precomposition on the source. These actions commute with the E -actions, and all arrows are compatible with them by (11) and (12). The maps $gE \rightarrow A$ and $Eg \rightarrow A$ take the M -diagonal block modulo lower blocks, so the quotient actions are the regular A -actions. $\square$

6 Transfer of the two injective dimensions

Lemmas 3.1 and 5.1 meet all the hypotheses of Lemma 2.1, on either side of E .

The bimodule structures in Lemma 5.1 retain the A -action when computing derived Hom. To see this, let $P^\bullet \rightarrow A$ denote (13) in nonpositive degrees and let $E \rightarrow V^\bullet$ be an injective resolution of the right regular E -module. There are right A -linear quasi-isomorphisms

$$\mathrm{Hom}_E(P^\bullet, E) \longrightarrow \mathrm{Hom}_E(P^\bullet, V^\bullet) \longleftarrow \mathrm{Hom}_E(A, V^\bullet).$$

The first uses the bounded, termwise E -projective complex $P^\bullet$; the second uses the bounded-below injective complex $V^\bullet$. Both respect precomposition by A . Thus the first complex computes $\mathbf{R}\mathrm{Hom}_E(A, E)$ in the derived category of right A -modules. Using (14) and a left E -injective resolution gives the corresponding statement for left A -modules.

Proof of Theorem 1.1. Apply $\mathrm{Hom}_E(-, E)$ to (13). Under evaluation on add G , the resulting cochain complex in degrees 0, 1, 2 is

$$\mathrm{Hom}_C(M, G) \longrightarrow \mathrm{Hom}_C(U \otimes V, G) \longrightarrow \mathrm{Hom}_C(k \otimes V^2, G). \quad (15)$$

The zero and first cohomologies vanish by left exactness of $\mathrm{Hom}_C(-, G)$ on (11). Write Z for its second cohomology. The right A -action is precomposition with the left A -action on (11). Lemma 4.2 therefore gives an isomorphism of right A -modules

$$\mathrm{Hom}_C(M, G) \cong A_A \oplus (DV)^{n+8}.$$

Here the n copies of DV in $\mathrm{End}_k(V)$ come from its input, and the other eight copies come from $\mathrm{Hom}_C(M, H)$. The later terms, with the same right A -action, are

$$\mathrm{Hom}_C(U \otimes V, G) \cong (DV)^{2n+13}, \quad \mathrm{Hom}_C(k \otimes V^2, G) \cong (DV)^{2n+12}.$$

Indeed, the Hom table gives $\dim_k \mathrm{Hom}_C(U, H) = 13$ and $\dim_k \mathrm{Hom}_C(k, H) = 6$, while the Hom spaces to M have dimensions $2n$ and n . Since V is free as a left A -module, DV is right A -injective. Consequently there is an exact sequence

$$0 \longrightarrow A_A \oplus I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow Z \longrightarrow 0$$

with all I_j injective. Dimension shifting in the second variable gives

$$\mathrm{Ext}_A^t(Y, Z) \cong \mathrm{Ext}_A^{t+2}(Y, A) \quad (t > 0)$$

for every finite right A -module Y . Thus Z has finite injective dimension exactly when A_A does. The complex (15) represents $Z[-2]$ in the derived category of right A -modules. Lemma 2.1 now gives

$$\mathrm{id}(E_E) < \infty \iff \mathrm{id}(A_A) < \infty.$$

For the left side, apply $\mathrm{Hom}_{E^{\mathrm{op}}}(-, E)$ to (14). It gives

$$\mathrm{Hom}_C(G, M) \longrightarrow \mathrm{Hom}_C(G, J \otimes V) \longrightarrow \mathrm{Hom}_C(G, k \otimes V^2), \quad (16)$$

again with cohomology only in degree two, by (12). Call that cohomology Z' . The left A -action acts on the output V , so

$$\mathrm{Hom}_C(G, M) \cong {}_A A \oplus V^{n+8}.$$

The remaining terms, with A acting on the coefficient V , are

$$\mathrm{Hom}_C(G, J \otimes V) \cong V^{2n+13}, \quad \mathrm{Hom}_C(G, k \otimes V^2) \cong V^{2n+12}.$$

Here $\dim_k \operatorname{Hom}_C(H, J) = 13$ and $\dim_k \operatorname{Hom}_C(H, k) = 6$, while the Hom spaces from M have dimensions $2n$ and n . These terms are left A -projective. Thus

$$0 \longrightarrow {}_A A \oplus P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow Z' \longrightarrow 0$$

is exact with all P_j projective. If ${}_A A$ has finite injective dimension, so do all these projectives and hence Z' . The opposite version of Lemma 2.1, applied to (16), gives $\operatorname{id}({}_E E) < \infty$.

Conversely, suppose $\operatorname{id}({}_E E) < \infty$. The finite left projective resolution of A from Lemma 5.1 makes ${}_E A$ have finite injective dimension: each finite E -projective has finite injective dimension, and this property is closed under finite exact sequences. The multiplication triangle (2) gives fully faithful derived inflation for $A = E/I$. For every finite left A -module Y , it follows that

$$\operatorname{Ext}_A^t(Y, A) \cong \operatorname{Ext}_E^t(Y, A).$$

The right side vanishes above a bound independent of Y . Hence $\operatorname{id}({}_A A) < \infty$, proving the second equivalence.

The algebra E is gendo-symmetric because C is symmetric and G contains C . The dimension formulas were proved after (10). Finally, as R ranges over all finite-dimensional k -algebras, so does R^{op} . Since $A = R^{\operatorname{op}}$, A_A is the left regular R -module and ${}_A A$ is the right regular R -module. This gives the two displayed formulas in the statement and transfers the conjecture in both directions. $\square$

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