

A note of the Cartan determinant conjecture

Qiyue Tang

Abstract

We prove the Cartan determinant conjecture for every corner of finite global dimension in an Artin algebra of global dimension at most two. Consequently, the conjecture holds for torsionless-finite Artin algebras.

Scholarly credit

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1 Introduction

Finite projective resolutions express simple modules as integer combinations of indecomposable projective modules. In 1954, Eilenberg established the resulting unimodularity of the Cartan matrix for finite-dimensional algebras of finite global dimension [3]. The Cartan determinant conjecture asks whether its sign is always positive. Zacharia proved this for Artin algebras of global dimension at most two in 1983 [6].

An Artin algebra is an associative unital algebra that is finite as a module over a commutative Artin ring. Let A be a nonzero Artin algebra. Choose representatives $S_1, \dots, S_n$ of its simple right modules and their indecomposable projective covers $P_1, \dots, P_n$. Its integral Cartan matrix is

$$C_A = ([P_j : S_i])_{1 \leq i, j \leq n},$$

where $[P_j : S_i]$ is composition multiplicity. Write $\text{gl. dim } A$ for global dimension. If it is finite, choose a finite projective resolution of each S_j and write its degree- q term as $\bigoplus_i P_i^{a_{iqj}}$ for integers $q \geq 0$, with nonnegative multiplicities a_{iqj} that vanish beyond the resolution. Taking alternating composition multiplicities gives $C_A Q = I$, where $Q_{ij} = \sum_{q \geq 0} (-1)^q a_{iqj}$ and I is the $n \times n$ identity matrix. Hence $\det C_A = \pm 1$.

Two reduction methods compare Cartan determinants with those of an idempotent corner or an idempotent quotient. Burgess–Fuller develop matrix reduction using projective resolutions [1, Lemma 1.5]. Ingalls–Paquette compare an algebra with a corner under a finite-global-dimension hypothesis on the Yoneda algebra associated with the deleted idempotent [4, Proposition 6.1]. For representation-finite Artin algebras, Chen–Guo prove the conjecture [2, Theorem 1.1].

Theorem A. *Let R be an Artin algebra with $\text{gl. dim } R \leq 2$, and let $e \in R$ be a nonzero idempotent. If $A = eRe$ has finite global dimension, then $\det C_A = 1$.*

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The proof filters the category of finite projective right R -modules by the projective dimensions of the evaluated modules Pe . Each positive layer, and a modified initial layer, has an ordinary object quotient whose endomorphism algebra has global dimension at most two. Integer Cartan column operations give a factorization of $\det C_R$ into $\det C_A$ and the determinants of these auxiliary algebras. Zacharia's theorem determines their signs.

A finitely generated right A -module is torsionless if it embeds in a finitely generated projective module. The algebra A is torsionless-finite if there are only finitely many isomorphism classes of indecomposable torsionless right modules; see Ringel [5]. The category of torsionless modules is closed under kernels. Its additive generator therefore gives an endomorphism algebra of global dimension at most two, to which Theorem A applies.

Theorem B. *Let A be a torsionless-finite Artin algebra. If $\text{gl. dim } A < \infty$, then $\det C_A = 1$.*

2 Kernels and Cartan factorization

Fix a commutative Artin ring K . All algebras are finite K -algebras, and modules are finitely generated right modules unless specified otherwise. Write $\text{mod } B$ for their category, $\text{proj } B$ for its full subcategory of projective modules, $\text{pd}_B N$ for projective dimension, and $\text{gl. dim } B$ for global dimension on all right modules. For integers $q \geq 0$, $\text{Ext}_B^q(-, -)$ is the degree- q derived functor of $\text{Hom}_B(-, -)$. We put $\text{pd}_B 0 = -\infty$. The symbol $\ell_K V$ denotes the composition length of a finite K -module V , and $\text{rad } B$ denotes the Jacobson radical of a ring B . Endomorphism rings are multiplied by composition, with the rightmost map applied first; $\text{Hom}_{\mathcal{C}}(X, Y)$ denotes the morphisms from X to Y in a category $\mathcal{C}$.

For an object W in an additive category, $\text{add } W$ is the full subcategory of direct summands of finite direct sums of W . For a collection of objects, add has the same closure meaning. All additive subcategories below are closed under isomorphisms and direct summands. We use K -linear, idempotent-complete Krull–Schmidt categories: every idempotent splits, and every object is a finite direct sum of indecomposable objects with local endomorphism rings, uniquely up to isomorphism and permutation. A local ring has a unique maximal right ideal, its Jacobson radical. The Hom spaces have finite K -length, and there are finitely many indecomposable isomorphism classes. A basic additive generator is the direct sum of one representative of each such class. The zero category is allowed as an auxiliary category; its Cartan determinant is the empty determinant 1.

If $\mathcal{C} = \text{add } W$, the functor $\text{Hom}_{\mathcal{C}}(W, -)$ sends Y to $\text{Hom}_{\mathcal{C}}(W, Y)$ and a map $f : Y \rightarrow Y'$ to $t \mapsto ft$. Its values are right $\text{End}_{\mathcal{C}}(W)$ -modules by $t \cdot b = tb$. It identifies $\mathcal{C}$ with $\text{proj } \text{End}_{\mathcal{C}}(W)$. Indeed, it is fully faithful on finite sums of W , hence on their direct summands, and every finite projective module is such a summand of a finite free module. A categorical kernel of $f : X \rightarrow Y$ is a morphism $u : T \rightarrow X$ with $fu = 0$ such that every $v : V \rightarrow X$ with $fv = 0$ factors uniquely through u .

Lemma 2.1. *If $\mathcal{C} = \text{add } W$ has kernels of all its morphisms, then $\text{gl. dim } \text{End}_{\mathcal{C}}(W) \leq 2$.*

Proof. Put $B = \text{End}_{\mathcal{C}}(W)$. A finite right B -module N has a projective presentation induced by a morphism $f : X_1 \rightarrow X_0$ in $\mathcal{C}$. The kernel property gives an exact sequence

$$0 \longrightarrow \text{Hom}(W, \ker f) \longrightarrow \text{Hom}(W, X_1) \longrightarrow \text{Hom}(W, X_0) \longrightarrow N \longrightarrow 0.$$

Its first three modules are projective, so $\text{pd}_B N \leq 2$. In particular, every simple right B -module has projective dimension at most two. This also bounds arbitrary modules: the radical of B is nilpotent, every module has a finite filtration by its radical powers with semisimple factors, and direct sums of projective resolutions of simple modules resolve those factors in length at most two. Successive extensions preserve this bound. $\square$

Cartan entries can be computed using lengths over K . Let $X_1, \dots, X_n$ represent the indecomposable isomorphism classes of $\mathcal{C}$, take $W = \bigoplus_i X_i$, and let L_i be the simple top of $\text{Hom}(W, X_i)$ over $B = \text{End}(W)$. Define the residual division ring

$$\Delta_i = \text{End}(X_i) / \text{rad } \text{End}(X_i).$$

Let $\varepsilon_i \in B$ be the projection of W onto X_i followed by inclusion. The right projective $\text{Hom}(W, X_j)$ is $\varepsilon_j B$; its simple top is $\varepsilon_j B / \varepsilon_j \text{rad } B$. The corner $\varepsilon_i B \varepsilon_i$ is $\text{End}(X_i)$, and the quotient $B / \text{rad } B$ is the product of the division rings Δ_i . Thus $N \mapsto N \varepsilon_i$ is an exact component functor, with value Δ_i on L_i and zero on the other simples. Applying it to a composition series, and using $\text{Hom}(W, X_j) \varepsilon_i \cong \text{Hom}(X_i, X_j)$, gives

$$[\text{Hom}(W, X_j) : L_i] = \frac{\ell_K \text{Hom}(X_i, X_j)}{\ell_K \Delta_i}. \quad (1)$$

Thus these ratios, with sources indexing rows and targets indexing columns, give C_B . The same formula on a subset of the objects gives the Cartan matrix of their endomorphism algebra. Replacing an additive generator by its basic generator is a Morita equivalence and preserves the Cartan matrix up to simultaneous relabelling.

Let $\mathcal{L} \subseteq \mathcal{C}$ be an additive subcategory. A right $\mathcal{L}$ -approximation of Y is a morphism $p_Y : U_Y \rightarrow Y$, with $U_Y \in \mathcal{L}$, through which every morphism from an object of $\mathcal{L}$ to Y factors. Such approximations exist here: take generators over K of the finitely many spaces $\text{Hom}(V, Y)$, where V runs over the indecomposable objects of $\mathcal{L}$, and combine their maps into a morphism from a finite direct sum of these objects.

The object quotient $\mathcal{Q} = \mathcal{C} / [\mathcal{L}]$ has the same objects as $\mathcal{C}$, with

$$\mathcal{Q}(X, Y) = \mathcal{C}(X, Y) / [\mathcal{L}](X, Y),$$

where $[\mathcal{L}](X, Y)$ consists of maps factoring through objects of $\mathcal{L}$. Composition is induced from $\mathcal{C}$. The class of a map f will be denoted by $\bar{f}$. For $f : X \rightarrow Y$ and $p : U \rightarrow Y$, the map $(f, p) : X \oplus U \rightarrow Y$ has restrictions f and p on the two summands.

Lemma 2.2. *Suppose each $Y \in \mathcal{C}$ has a right $\mathcal{L}$ -approximation $p_Y : U_Y \rightarrow Y$ whose kernel exists and belongs to $\mathcal{L}$. Suppose also that $\ker(f, p_Y)$ exists in $\mathcal{C}$ for every $f : X \rightarrow Y$. Then $\mathcal{Q}$ has kernels and is equivalent to $\text{proj } D$ for an Artin algebra D with $\text{gl. dim } D \leq 2$. If Γ and $\Gamma_{\mathcal{L}}$ are endomorphism algebras of basic generators of $\mathcal{C}$ and $\mathcal{L}$, respectively, then*

$$\det C_{\Gamma} = \det C_{\Gamma_{\mathcal{L}}} \det C_D. \quad (2)$$

Proof. Fix $f : X \rightarrow Y$, and write $T = \ker(f, p_Y)$. Let $h : T \rightarrow X$ and $v : T \rightarrow U_Y$ be the components of its kernel morphism. Then $fh = -p_Y v$, so $\bar{f}\bar{h} = 0$.

For a map $g : V \rightarrow X$ with $\bar{f}\bar{g} = 0$, write $fg = ab$ with $b : V \rightarrow V'$ and $a : V' \rightarrow Y$, where $V' \in \mathcal{L}$. By approximation, $a = p_Y s$ for some $s : V' \rightarrow U_Y$. Thus $(g, -sb) : V \rightarrow X \oplus U_Y$ factors through T , giving a factorization of $\bar{g}$ through $\bar{h}$.

For uniqueness, suppose $k : V \rightarrow T$ and $hk = cb$ factor through $V' \in \mathcal{L}$, with $c : V' \rightarrow X$. Lift fc through p_Y , writing $fc = p_Y t$. The morphism $(c, -t)$ induces $j : V' \rightarrow T$, and $h(k - jb) = 0$. Let $\kappa : \ker p_Y \rightarrow U_Y$ be the kernel map. The pair $(0, \kappa)$ induces $\iota : \ker p_Y \rightarrow T$. If $w : V \rightarrow T$ satisfies $hw = 0$, then $p_Y v w = -f h w = 0$. There is a unique $s : V \rightarrow \ker p_Y$ with $\kappa s = v w$. The pairs $(h, v)w$ and $(h, v)\iota s$ are equal; since the kernel morphism (h, v) is monic, $w = \iota s$. Thus ι is the kernel of h . In particular, $k - jb$ factors through $\ker p_Y \in \mathcal{L}$. Both jb and $k - jb$ vanish in $\mathcal{Q}$, so $\bar{k} = 0$. Hence $\bar{h}$ is a categorical kernel of $\bar{f}$.

Decompose each object using its indecomposable summands in $\mathcal{C}$. The summands in $\mathcal{L}$ become zero in $\mathcal{Q}$. For a summand X outside $\mathcal{L}$, the ideal $[\mathcal{L}](X, X)$ contains no invertible map, since an invertible factorization would make X a summand of an object of $\mathcal{L}$. It is therefore contained

in the radical of the local ring $\text{End}_{\mathcal{C}}(X)$. The quotient endomorphism ring remains local, with the same residual division ring. Distinct surviving summands remain nonisomorphic: mutually inverse quotient maps lift to maps whose composites differ from identities by radical elements, and an invertible composite makes one indecomposable object a summand of the other.

Let V be the sum of the surviving representatives and set $D = \text{End}_{\mathcal{Q}}(V)$. The projections onto these summands give a complete set of pairwise nonisomorphic primitive idempotents in the Artin algebra D . Thus all finite projective D -modules are represented by direct sums of those objects. Full faithfulness of $\text{Hom}_{\mathcal{Q}}(V, -)$ on these sums gives $\mathcal{Q} \simeq \text{proj } D$, including idempotent completeness. The kernel assertion and Lemma 2.1 imply $\text{gl. dim } D \leq 2$.

For the determinant, let W be the basic generator of $\mathcal{C}$ and put $P_Y = \text{Hom}_{\mathcal{C}}(W, Y)$. Let $I_Y = [\mathcal{L}](W, Y)$, the image of $\text{Hom}(W, p_Y)$. The actual kernel gives

$$0 \longrightarrow P_{\ker p_Y} \longrightarrow P_{U_Y} \longrightarrow I_Y \longrightarrow 0.$$

In the group of finite Γ -modules modulo short exact sequences, write $[N]$ for the class of a module N . The sequence yields

$$[P_Y/I_Y] = [P_Y] - [P_{U_Y}] + [P_{\ker p_Y}].$$

Order the indecomposable objects with those of $\mathcal{L}$ first. Keep the corresponding Cartan columns and replace each remaining column by the displayed combination. Only columns indexed by $\mathcal{L}$ are added to it, so the transformation has determinant one. The new high columns vanish on the low rows, since every map from a low object factors through $\mathcal{L}$. The low diagonal block is $C_{\Gamma_{\mathcal{L}}}$ by (1). The high diagonal block is C_D : the surviving residual division rings are unchanged, and its component spaces are exactly $\mathcal{Q}(X_i, X_j)$. The transformed matrix is therefore

$$\begin{pmatrix} C_{\Gamma_{\mathcal{L}}} & 0 \\ * & C_D \end{pmatrix},$$

which proves (2). □

3 The filtration of a corner

Proof of Theorem A. Put $A = eRe$ and $a = \text{gl. dim } A < \infty$. For $\mathcal{E} = \text{proj } R$, evaluation at e is the exact functor

$$F : \text{mod } R \longrightarrow \text{mod } A, \quad F(N) = Ne.$$

It preserves kernels by restriction to the e -components. If a map is surjective and $y = ye$ is in its target, choose a preimage x ; then xe maps to y . Hence F also preserves surjections and is exact.

The category $\mathcal{E}$ is closed under module kernels. Indeed, for a map $g : P_1 \rightarrow P_0$ between projectives, put $T = \ker g$. For any right R -module V , the short exact sequences with middle terms P_1 and P_0 give

$$\text{Ext}_R^1(T, V) \cong \text{Ext}_R^2(\text{im } g, V) \cong \text{Ext}_R^3(\text{coker } g, V) = 0.$$

Thus T is projective; it is finitely generated because R is Artinian.

For integers $0 \leq m \leq a$, define

$$\mathcal{E}_m = \{P \in \text{proj } R : \text{pd}_A(Pe) \leq m\}, \quad \mathcal{P} = \text{add}(eR).$$

These are finite additive subcategories, $\mathcal{P} \subseteq \mathcal{E}_0$, and $\mathcal{E}_a = \mathcal{E}$. Let Γ_m be the endomorphism algebra of a basic generator of $\mathcal{E}_m$.

For $Q \in \text{proj } A$, the induced module $Q \otimes_A eR$ belongs to $\mathcal{P}$: a splitting of Q from a finite free A -module tensors to a splitting from a finite sum of eR . For $Y \in \text{proj } R$ we have

$$\text{Hom}_R(Q \otimes_A eR, Y) \cong \text{Hom}_A(Q, Ye), \quad (Q \otimes_A eR)e \cong Q. \quad (3)$$

The first bijection sends u to $q \mapsto u(q \otimes e)$; its inverse sends v to $q \otimes er \mapsto v(q)r$. In the second isomorphism, $q \otimes ere$ maps to $qere$. Thus F restricts to an equivalence $\mathcal{P} \simeq \text{proj } A$.

Positive layers. Fix $1 \leq m \leq a$ and $Y \in \mathcal{E}_m$. Choose a right $\mathcal{E}_{m-1}$ -approximation $p_Y : U_Y \rightarrow Y$. Since $\mathcal{P} \subseteq \mathcal{E}_{m-1}$, (3) lifts a projective cover of FY to a map from an object of $\mathcal{P}$, which then factors through p_Y . Hence Fp_Y is surjective.

The module kernel $K_Y = \ker_R p_Y$ belongs to $\text{proj } R$, and exactness of F gives

$$0 \longrightarrow FK_Y \longrightarrow FU_Y \longrightarrow FY \longrightarrow 0.$$

The projective-dimension bound for this sequence is

$$\text{pd}_A FK_Y \leq \max\{\text{pd}_A FU_Y, \text{pd}_A FY - 1\} \leq m - 1. \quad (4)$$

For every right A -module V and integer $q \geq 1$, the long exact Ext sequence has a segment

$$\text{Ext}_A^q(FU_Y, V) \longrightarrow \text{Ext}_A^q(FK_Y, V) \longrightarrow \text{Ext}_A^{q+1}(FY, V).$$

The outer terms vanish for $q > m - 1$, so $K_Y \in \mathcal{E}_{m-1}$.

For $f : X \rightarrow Y$ in $\mathcal{E}_m$, the kernel $T = \ker_R(f, p_Y)$ is again projective over R . Its evaluation fits into

$$0 \longrightarrow FT \longrightarrow FX \oplus FU_Y \xrightarrow{F(f, p_Y)} FY \longrightarrow 0.$$

The last map is surjective, and the same bound gives $\text{pd}_A FT \leq m$. Thus $T \in \mathcal{E}_m$. Lemma 2.2 now applies to $\mathcal{E}_{m-1} \subseteq \mathcal{E}_m$. Write D_m for the endomorphism algebra of the surviving basic generator of $\mathcal{E}_m/[\mathcal{E}_{m-1}]$. We obtain

$$\text{gl. dim } D_m \leq 2, \quad \det C_{\Gamma_m} = \det C_{\Gamma_{m-1}} \det C_{D_m}. \quad (5)$$

The initial layer. Let

$$\mathcal{Z} = \{P \in \text{proj } R : Pe = 0\}.$$

The kernel of any map between its objects is projective over R and evaluates to zero. Lemma 2.1 implies that the endomorphism algebra Z of a basic generator of $\mathcal{Z}$ has global dimension at most two.

Set $\mathcal{L}_0 = \text{add}(\mathcal{P} \cup \mathcal{Z})$ inside $\mathcal{E}_0$. The two subcategories have no common nonzero indecomposable object, since F is an equivalence on $\mathcal{P}$ and vanishes on $\mathcal{Z}$. Moreover,

$$\text{Hom}_R(eR, Z') \cong Z'e = 0 \quad (Z' \in \mathcal{Z}).$$

Therefore the Cartan matrix of the endomorphism algebra of a basic generator of $\mathcal{L}_0$ is block triangular, with diagonal blocks C_A and C_Z . Here the first block uses $\text{End}_R(eR) \cong eRe$, where $x \in eRe$ acts by left multiplication, and Morita invariance removes repeated summands of eR . Its determinant is $\det C_A \det C_Z$.

For $Y \in \mathcal{E}_0$, the module FY is projective over A . Define

$$p_Y : (FY) \otimes_A eR \longrightarrow Y, \quad y \otimes er \longmapsto yr.$$

Every map $Q \otimes_A eR \rightarrow Y$ corresponds under (3) to $Q \rightarrow FY$ and therefore factors through p_Y by tensoring that map with eR . Also Fp_Y is an isomorphism. Choose a right $\mathcal{Z}$ -approximation $z_Y : Z_Y \rightarrow Y$ and form

$$q_Y = (p_Y, z_Y) : (FY) \otimes_A eR \oplus Z_Y \longrightarrow Y.$$

This is a right $\mathcal{L}_0$ -approximation. Its kernel is projective over R and evaluates to zero because Fq_Y is an isomorphism. Hence $\ker q_Y \in \mathcal{Z} \subseteq \mathcal{L}_0$.

For $f : X \rightarrow Y$ in $\mathcal{E}_0$, the kernel of (f, q_Y) is projective over R . After evaluation, projection onto the first component identifies the kernel of (Ff, Fq_Y) with FX ; its inverse is $x \mapsto (x, -(Fq_Y)^{-1}Ff(x))$. It is therefore projective over A , so $\ker(f, q_Y) \in \mathcal{E}_0$. Applying Lemma 2.2 to $\mathcal{L}_0 \subseteq \mathcal{E}_0$ gives an Artin algebra D_0 with

$$\text{gl. dim } D_0 \leq 2, \quad \det C_{\Gamma_0} = \det C_A \det C_Z \det C_{D_0}. \quad (6)$$

Since $\mathcal{E}_a = \text{proj } R$, its basic generator has endomorphism algebra Morita equivalent to R . Multiplying (5) and (6) yields

$$\det C_R = \det C_A \det C_Z \prod_{m=0}^a \det C_{D_m}. \quad (7)$$

Every nonzero algebra among $R, Z, D_0, \dots, D_a$ has global dimension at most two. Zacharia's theorem [6] gives determinant one for each of them. Zero categories contribute the empty determinant one. Thus (7) gives $\det C_A = 1$. $\square$

4 Finite submodule categories

For a finite right module M over an Artin algebra B , let $\text{Sub}(M)$ be the full subcategory of finite modules isomorphic to submodules of finite direct sums of M . It is closed under direct sums, direct summands, and module kernels: a kernel is a submodule of its source, which already embeds in a finite sum of M . We call $\text{Sub}(M)$ finite if it has finitely many indecomposable isomorphism classes.

Corollary 4.1. *Let B be an Artin algebra and M a nonzero finite right B -module. Suppose $\text{Sub}(M)$ is finite. If $\text{gl. dim } \text{End}_B(M) < \infty$, then $\det C_{\text{End}_B(M)} = 1$.*

Proof. Choose an additive generator W of $\text{Sub}(M)$ containing M as a direct summand; adjoining M to a basic generator achieves this. The category $\text{add } W = \text{Sub}(M)$ has its actual module kernels, so Lemma 2.1 gives $\text{gl. dim } \text{End}_B(W) \leq 2$. Let e be the projection of W onto M followed by inclusion. Extension by zero on the complementary summand identifies

$$e \text{End}_B(W) e \cong \text{End}_B(M).$$

Apply Theorem A. $\square$

Proof of Theorem B. The torsionless right A -modules are precisely $\text{Sub}(A_A)$: every finite projective module is a summand of a finite sum of copies of A , and every such sum is projective. Torsionless-finiteness makes this category finite. Corollary 4.1, applied to $M = A_A$, gives the result because left multiplication identifies $\text{End}_A(A_A)$ with A . $\square$

AI contributions

The mathematical arguments were developed with AI assistance in ChatGPT. Codex assisted in checking the arguments and cited inputs and in preparing and revising this manuscript.

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ZHILI COLLEGE, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
Email address: tangqy24@mails.tsinghua.edu.cn