

Illumination of complex convex bodies

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Abstract

For every complex 1-unconditional convex body $K \subset \mathbb{C}^n$, we prove

$$\text{ill}(K) \leq (2^{m+1} - 1) \binom{n}{m} + \sum_{k=m+1}^n \binom{n}{k} - (n - m), \quad m = \lceil 2n/3 \rceil.$$

This bound is asymptotic to $3^{n+1}/\sqrt{\pi n}$. We further obtain $\text{ill}^*(K) \leq b_*^n$ and $\text{ill}(K) = O(n \log(n+1)b_*^n)$, where $b_* = 2.4805534697\dots$ is defined by an explicit scalar equation. For normalized bodies other than $\mathbb{D}^n$ that contain a full coordinate $(n-1)$ -polydisc, we prove the sharp bound $\text{ill}(K) \leq 2^{n+1} - 2$. We also prove the sharp bound 14 for every normalized complex 1-unconditional body in $\mathbb{C}^3$ other than $\mathbb{D}^3$, and the sharp bound 30 in $\mathbb{C}^4$ for every normalized non-polydisc. The four-dimensional proof includes one finite covering assertion verified by exact integer arithmetic; its complete verifier is included. Under full coordinate permutation symmetry, we improve the three-dimensional bound to 12 by an explicit twelve-vector construction and an exact finite covering certificate. For complex Hanner bodies, we compute both illumination numbers exactly and determine the equality cases in their dimension bounds. If every extreme point has all coordinates nonzero, we prove $\text{ill}(K) \leq 2^{n+1} - 1$ and a quantitative strict fractional bound $\text{ill}^*(K) < 2^n$ for normalized non-polydiscs. For real 1-symmetric non-cubes we obtain bounds 12 in dimension four, 22 in dimension five, and $5 \cdot 2^{n-3}$ for every $n \geq 6$. We also determine the leading asymptotic maximum in the real 1-symmetric non-cube class: it is $(1 + o(1))2^{n-1}$. An explicit binary construction proves this without an external coding theorem. A connection to connected domination in cube graphs improves the upper estimate to $2^{n-1}(1 + (1 + o(1))/n)$.

Draft status. The proofs included here have been checked internally; external review and literature priority remain outstanding. Results from the wider research corpus that have not yet been checked are listed separately in Section 10.

1 Notation and preliminary facts

Write $[n] = \{1, \dots, n\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\mathbb{R}_+ = [0, \infty)$. A convex body is a compact convex set with nonempty interior in the underlying real space. It is complex 1-unconditional if it is invariant under independent rotations of its coordinates. Its Minkowski norm is denoted by $\|\cdot\|_K$. For $z \in \mathbb{C}^n$, write $|z| = (|z_1|, \dots, |z_n|)$ and $\text{supp}(z) = \{j \in [n] : z_j \neq 0\}$. We also write

$$M_x = \{j \in [n] : |x_j| = \|x\|_\infty\}, \quad Z_x = \{j \in [n] : x_j = 0\}.$$

A nonzero vector d illuminates $x \in \partial K$ if $x + td \in \text{int } K$ for some $t > 0$. The least cardinality of a family illuminating every boundary point is $\text{ill}(K)$. A finite nonnegative Borel measure μ on $\mathbb{C}^n \setminus \{0\}$ fractionally illuminates K if

$$\mu\{d : d \text{ illuminates } x\} \geq 1 \quad (x \in \partial K).$$

The infimum of the total masses of such measures is $\text{ill}^*(K)$.

We call K normalized when $e_j \in \partial K$ for every $j \in [n]$. Positive diagonal rescaling gives this normalization, and illumination numbers are invariant under invertible linear maps. Coordinate contractions preserve membership, as proved below; consequently a normalized body satisfies $K \subseteq \mathbb{D}^n$ and contains each unit coordinate disc.

Lemma 1. *For a complex 1-unconditional convex body K , the following statements hold.*

1. *If $|w_j| \leq |z_j|$ for every j , then $\|w\|_K \leq \|z\|_K$.*
2. *For every real linear subspace $E \subseteq \mathbb{C}^n$,*

$$\text{relint}(K \cap E) = E \cap \text{int } K.$$

3. *Suppose $z \in K$, J is a nonempty subset of $\text{supp}(z)$, and the point q obtained from z by setting the coordinates in J to zero belongs to $\text{int } K$. If $w_j = z_j$ for $j \notin J$ and $|w_j| < |z_j|$ for $j \in J$, then $w \in \text{int } K$.*

Proof. Invariance under all coordinate rotations implies central symmetry. Because K has nonempty interior, its center 0 is interior: if an open Euclidean ball of radius r around a lies in K , the same is true around $-a$, and convexity puts the ball of radius r around 0 in K . Its Minkowski functional is therefore a norm, with $\text{int } K = \{z : \|z\|_K < 1\}$.

Every number $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ is a convex combination of two unit complex numbers. Explicitly, if $\lambda = se^{i\theta}$, $0 \leq s \leq 1$, then

$$\lambda = \frac{1+s}{2}e^{i\theta} + \frac{1-s}{2}(-e^{i\theta}).$$

Thus contracting a single coordinate by λ preserves membership in K , by convexity and rotation invariance. Applying these contractions one coordinate at a time proves coordinatewise monotonicity of membership. Apply the same fact to every dilation of K to obtain the norm inequality in part 1. Coordinates with $z_j = 0$ also have $w_j = 0$, so cause no exception.

The inclusion from right to left in part 2 follows by intersecting an ambient open ball with E . For the reverse inclusion, let $y \in \text{relint}(K \cap E)$. If $y = 0$ there is nothing to prove. Otherwise relative interior gives an $s > 0$ with $(1+s)y \in K \cap E$. Consequently $\|y\|_K \leq (1+s)^{-1} < 1$, as required.

For part 3 choose $\rho \in (0, 1)$ such that $|w_j| \leq \rho|z_j|$ for all $j \in J$; there are finitely many strict ratios, so one common ρ exists. Put $y = \rho z + (1-\rho)q$. Then

$$\|y\|_K \leq \rho\|z\|_K + (1-\rho)\|q\|_K < 1.$$

On J we have $y_j = \rho z_j$, and off J we have $y_j = w_j = z_j$. Part 1 therefore gives $\|w\|_K \leq \|y\|_K < 1$. $\square$

We use the real pairing

$$[u, z] = \text{Re} \sum_{j \in [n]} \bar{u}_j z_j, \quad h_K(u) = \max_{z \in K} [u, z].$$

A direction d illuminates $x \in \partial K$ if and only if

$$[u, d] < 0 \quad \text{for every nonzero } u \text{ with } [u, x] = h_K(u). \quad (1)$$

Here is a proof that also covers nonsmooth boundary points. If $x + td \in \text{int } K$ for some $t > 0$, then $[u, x + td] < h_K(u) = [u, x]$ for every nonzero supporting normal at x , proving necessity. For sufficiency, suppose no positive step enters $\text{int } K$. For each integer $j \geq 1$, separation at $x + j^{-1}d \notin \text{int } K$ supplies a Euclidean unit vector u^j with $[u^j, x + j^{-1}d] \geq h_K(u^j)$. Since

$h_K(u^j) \geq [u^j, x]$, we have $[u^j, d] \geq 0$. Pass to a convergent subsequence of these unit vectors, with limit u . The support function is continuous, so $[u, x] \geq h_K(u)$; the reverse inequality holds because $x \in K$. Thus u is a nonzero supporting normal at x with $[u, d] \geq 0$, contradicting (1).

For a normalized unconditional body, put $S = \text{supp}(x)$ and $a_j = |u_j|$ for a supporting normal u at x . Independent coordinate rotations give $u_j = |u_j| x_j / |x_j|$ whenever $j \in S$: the maximum of $[u, \cdot]$ on the coordinate-rotation orbit of x is $\sum_j |u_j| |x_j|$, and x already attains that maximum. Thus, writing $h = h_K(u) > 0$,

$$h = \sum_{j \in S} a_j |x_j|, \quad a_j \leq h \quad (j \in [n]), \quad h \leq \sum_{j \in S} a_j. \quad (2)$$

The middle inequality follows from the coordinate discs in K ; the last uses $K \subseteq \mathbb{D}^n$.

2 General upper bounds

2.1 Directions indexed by coordinate supports

Theorem 2. *For $n \geq 1$, put $m = \lceil 2n/3 \rceil$ and*

$$C_n = (2^{m+1} - 1) \binom{n}{m} + \sum_{k=m+1}^n \binom{n}{k} - (n - m). \quad (3)$$

Every complex 1-unconditional convex body $K \subset \mathbb{C}^n$ satisfies $\text{ill}(K) \leq C_n$. Moreover,

$$C_n \sim \frac{3^{n+1}}{\sqrt{\pi n}}.$$

We use the cyclic construction in the proof of Proposition 2.7 of [1]. For $b \geq 1$, put $q_b = 2^{b+1} - 1$. The vectors

$$\left\{ \left(\exp \frac{2\pi i 2^{j-1} t}{q_b} \right)_{j \in [b]} : t \in \mathbb{Z}/q_b \mathbb{Z} \right\} \quad (4)$$

meet every product of b open semicircles, and hence illuminate $\mathbb{D}^b$. We use this published result as an external input.

Lemma 3. *Append to (4) the column of frequency 2^b . Every projection of the resulting $(b+1)$ -column matrix onto b columns meets every product of open semicircles.*

Proof. Since $2^{b+1} \equiv 1 \pmod{q_b}$, multiplication of the row parameter t by 2 cyclically permutes the $b+1$ frequencies. It is a permutation of the rows because q_b is odd. Any omitted column can therefore be moved to the last position, reducing the assertion to (4). $\square$

Lemma 4. *Let a finite rooted tree have nonempty coordinate sets as its nodes. Its root is a set T of cardinality b . Each node has at most two children, and each child is obtained by deleting one coordinate from its parent. Then q_b directions suffice, simultaneously for every normalized complex 1-unconditional body, to illuminate all boundary points whose supports occur in the tree.*

Proof. Arrange the vectors of (4), with their additional coordinate from Lemma 3, as rows of a matrix. Label b columns by T , leaving one additional column. At a node with support S , let G be the set of rows satisfying fixed open-semicircle conditions on $b - |S|$ columns outside S . Exactly one column outside S remains unrestricted. At the root, G contains every row, and the additional column is unrestricted.

The active columns S and the constrained columns together comprise exactly b columns. Lemma 3 implies that G , projected onto S , meets every product of open semicircles. Indeed, prescribe the desired semicircles on S and retain all the fixed constraints.

Suppose the children are $S \setminus \{i\}$ and $S \setminus \{j\}$. Split G by imposing, in its unrestricted column, respectively the upper and lower open semicircles. These conditions are disjoint. The previously unrestricted column becomes constrained, and the deleted column i or j becomes the unrestricted column of the corresponding child. Thus the invariant persists. For a single child, use just one of the two semicircles. Rows satisfying neither chosen condition stop at the current node; rows at leaves also stop. Every row consequently visits one nested path

$$T = S_0 \supset S_1 \supset \cdots \supset S_r, \quad |S_{\ell-1} \setminus S_\ell| = 1.$$

The group of a node is exactly the set of rows visiting it. In particular, rows that stop early remain available at that node and all its ancestors.

For a node S , define on the torus of unit complex coordinates

$$g_S(\zeta) = \max_{p \in G} \min_{j \in S} [-\operatorname{Re}(\overline{\zeta_j} p_j)].$$

The covering property makes g_S positive everywhere. It is continuous, being a finite maximum of finite minima of continuous functions, so its minimum on the compact torus is positive. Take $\beta > 0$ no larger than these minima for all the finitely many nodes. Thus, for every node S and every $(\zeta_j)_{j \in S}$ with $|\zeta_j| = 1$, some row $p \in G$ satisfies

$$\operatorname{Re}(\overline{\zeta_j} p_j) \leq -\beta \quad (j \in S). \quad (5)$$

Choose $0 < \delta < \min\{1, \beta/n\}$.

For a row with path $S_0, \dots, S_r$, give amplitude 1 to the coordinates in S_r . The coordinate deleted at step ℓ receives amplitude $\delta^{r-\ell+1}$, and coordinates outside T receive zero. Multiply these positive amplitudes by the corresponding entries of the row to obtain d . At every visited node S_ℓ ,

$$|d_j| \geq \delta^{r-\ell} \quad (j \in S_\ell), \quad |d_j| \leq \delta^{r-\ell+1} \quad (j \notin S_\ell). \quad (6)$$

Now let $x \in \partial K$ have support S_ℓ . Select a visiting row satisfying (5) for $\zeta_j = x_j/|x_j|$, and put $a = \delta^{r-\ell}$. For every supporting normal, (2) and (6) give

$$\begin{aligned} [u, d] &\leq -\beta a \sum_{j \in S_\ell} |u_j| + \delta a \sum_{j \notin S_\ell} |u_j| \\ &\leq a [-\beta + (n - |S_\ell|)\delta] \sum_{j \in S_\ell} |u_j| < 0. \end{aligned}$$

The last sum is positive since $h_K(u) > 0$. The criterion (1) completes the proof. $\square$

Proof of Theorem 2. By diagonal rescaling we may assume that K is normalized. Write $a_k = \binom{n}{k}$ and $a_{n+1} = 0$. For $k \in [n-1]$, between the k -sets and the $(k+1)$ -sets, the Boolean incidence graph has degrees $n-k$ and $k+1$, respectively. We assign a k -set to a containing parent, allowing each parent at most two children. Write $N(\mathcal{A})$ for the set of neighbors of a vertex family $\mathcal{A}$.

If $a_k \leq 2a_{k+1}$, edge counting gives, for every family $\mathcal{A}$ of k -sets,

$$|N(\mathcal{A})| \geq \frac{n-k}{k+1} |\mathcal{A}| \geq \frac{|\mathcal{A}|}{2}.$$

Hall's theorem, after replacing each upper vertex by two slots, assigns every k -set to a parent. If $a_k \geq 2a_{k+1}$, reverse edge counting gives, for every family $\mathcal{B}$ of upper vertices,

$$|N(\mathcal{B})| \geq \frac{k+1}{n-k} |\mathcal{B}| \geq 2|\mathcal{B}|.$$

To apply Hall's theorem to the duplicated upper vertices, take any set of slots and let $\mathcal{B}$ be their underlying vertices. There are at most $2|\mathcal{B}|$ slots and at least that many lower neighbors by the displayed estimate. Thus Hall's theorem fills both slots of every upper vertex. The unused k -sets become roots. The top level is handled separately: $[n]$ is a root and has no parent. Consequently, including $k = n$ with $a_{n+1} = 0$, the numbers of roots are

$$r_k = \max\{a_k - 2a_{k+1}, 0\}.$$

The choices between successive pairs of layers are independent: a vertex's parent and its children are governed by different matchings. Together they form a binary forest containing every nonempty subset exactly once.

The positive root counts occur precisely at $k \geq m = \lceil 2n/3 \rceil$. Applying Lemma 4 separately to the trees gives a family of at most

$$\begin{aligned} & \sum_{k=m}^n (a_k - 2a_{k+1})(2^{k+1} - 1) \\ &= (2^{m+1} - 1)a_m + \sum_{k=m+1}^n a_k. \end{aligned}$$

directions. Here the last equality follows from

$$\sum_{k=m}^n (a_k - 2a_{k+1})2^{k+1} = 2^{m+1}a_m, \quad \sum_{k=m}^n (a_k - 2a_{k+1}) = a_m - \sum_{k=m+1}^n a_k.$$

For every $k \geq m$ the number r_k of roots at level k is positive. Prescribe the nested sets $S_k = [k]$, $m \leq k \leq n$. The set S_n is already the top root. For $m \leq k < n$, the matching between levels k and $k+1$ leaves $r_k > 0$ lower vertices unmatched. Apply a coordinate permutation to this matching alone so that one unmatched vertex becomes S_k . Containment and all capacities are preserved. Whether a k -set is a root depends only on this matching; its children are assigned by the separate matching between levels $k-1$ and k . The permutations therefore impose no compatibility condition across level pairs. We can make every S_k a root at once.

Use the upper and lower open semicircles at every split in the tree construction. In each root's matrix, the row indexed by $t = 0$ has entry 1 in every column. It lies on the boundary of both semicircles, so it stops at the root and is never needed in a descendant's group. Take one common positive margin β for all groups in the forest, and choose $0 < \delta < \min\{1, \beta/n\}$.

Remove the $n - m + 1$ rows indexed by $t = 0$ associated with the roots $S_m, S_{m+1}, \dots, S_n$, and replace them by the single direction

$$d_j = \begin{cases} 1, & j \leq m, \\ \delta^{j-m}, & j > m. \end{cases}$$

At support S_k , its smallest active amplitude is δ^{k-m} , and every inactive amplitude is at most δ^{k-m+1} . Whenever the deleted row supplies the margin β at that root, the replacement has positive real coordinates and the required amplitude gap. Writing $a = \delta^{k-m}$, the same normal estimate gives

$$[u, d] \leq a[-\beta + (n - k)\delta] \sum_{j \in S_k} |u_j| < 0$$

for every nonzero supporting normal at any point for which that row was selected. The sum is positive by (2). At all other nodes and for all other selected rows the construction is unchanged. This saves $n - m = \lfloor n/3 \rfloor$ directions and proves the asserted upper bound. To obtain its asymptotic form, Stirling's formula with $m = 2n/3 + O(1)$ gives

$$2^{m+1} \binom{n}{m} \sim \frac{3^{n+1}}{\sqrt{\pi n}}.$$

All remaining terms in C_n have absolute value at most $2^{n+1} + n$ and are of smaller order, proving the asymptotic statement. $\square$

Taking every nonempty support as a single-vertex tree in Lemma 4 gives the comparison bound

$$\text{ill}(K) \leq \sum_{k=1}^n \binom{n}{k} (2^{k+1} - 1) = 2 \cdot 3^n - 2^n - 1.$$

Remark 1. The directions just constructed work for every normalized body. Appendix A gives a lower bound for this stronger requirement. It does not give a lower bound for the largest illumination number when the directions may be chosen separately for each body.

2.2 An upper bound with a smaller exponential base

Unconditionality here means invariance under independent rotations of the complex coordinates. Let $K \subset \mathbb{C}^n$ be such a convex body, and put

$$L = K \cap \mathbb{R}_+^n.$$

Then L is compact and convex and is a coordinatewise downset: if $y \in L$ and $0 \leq x_j \leq y_j$ for every $j \in [n]$, then $x \in L$. Its n -dimensional volume is positive because $0 \in \text{int } K$, and $z \in K$ if and only if $(|z_1|, \dots, |z_n|) \in L$. In particular, coordinatewise domination of moduli implies domination of the Minkowski norm. No normalization of the coordinate radii is needed.

Let $c_* \in (0, 1/2)$ be the unique solution of

$$\cos(\pi c_*) \cosh(\pi(1 - c_*)) = 1. \quad (7)$$

The product on the left is strictly decreasing on $(0, 1/2)$, and its endpoint values lie on opposite sides of one. Numerically,

$$c_* = 0.4031358373055328\dots, \quad b_* = c_*^{-1} = 2.480553469728144\dots$$

Theorem 5. *For every complex 1-unconditional convex body $K \subset \mathbb{C}^n$,*

$$\text{ill}^*(K) \leq b_*^n, \quad (8)$$

and

$$\text{ill}(K) \leq \lceil eb_*^n [2n \log(1 + 32b_*^2(n+1)^2) + 1] \rceil. \quad (9)$$

In particular,

$$\text{ill}(K) = O(n \log(n+1)b_*^n).$$

Replacing b_* by $5/2$ gives the simpler explicit estimate

$$\text{ill}(K) \leq \lceil e(5/2)^n [2n \log(1 + 200(n+1)^2) + 1] \rceil. \quad (10)$$

The minimum of this bound and Theorem 2 is valid in every dimension. Numerical comparisons are given in Appendix C.

2.2.1 A one-coordinate inequality

Define a locally finite measure on $\mathbb{C}$ by

$$d\nu_1(z) = \frac{d\rho d\theta}{2\pi}, \quad z = \rho e^{i\theta}, \quad \rho \geq 0, \quad 0 \leq \theta < 2\pi.$$

Equivalently its density with respect to plane area is $(2\pi|z|)^{-1}$. The density is locally integrable. In particular, $\nu_1(\{|z| \leq R\}) = R$ for $R \geq 0$, and ν_1 has no atom at zero. Write $\nu = \nu_1^{\otimes n}$. Then

$$\nu(K) = \text{vol}_n(L), \quad \nu(aK) = a^n \nu(K) \quad (a > 0). \quad (11)$$

The homogeneity degree is n , although the ambient real dimension is $2n$.

Let

$$H(s) = \nu_1\{z : |z| \leq 1, \Re z \leq s\}.$$

For $-1 < s < 1$, direct integration gives

$$H(s) = \frac{1}{2} + \frac{\arcsin s + s \operatorname{arccosh}(1/|s|)}{\pi}, \quad (12)$$

where the second term is interpreted continuously at $s = 0$. Also $H(s) = 0$ for $s \leq -1$, and $H(s) = 1$ for $s \geq 1$. For $0 < s < 1$, conditioning on the radius gives

$$H(s) = s + \int_s^1 \left(\frac{1}{2} + \frac{\arcsin(s/\rho)}{\pi} \right) d\rho.$$

Integration by parts yields

$$\int_s^1 \arcsin(s/\rho) d\rho = \arcsin s - \frac{\pi s}{2} + s \int_s^1 \frac{d\rho}{\sqrt{\rho^2 - s^2}},$$

which proves (12) for positive s . Symmetry and continuity give the remaining values. Differentiating the resulting expression gives

$$H'(s) = \frac{1}{\pi} \int_s^1 \frac{d\rho}{\sqrt{\rho^2 - s^2}} = \frac{\operatorname{arccosh}(1/s)}{\pi}.$$

Lemma 6. *For every $0 < c \leq c_*$,*

$$H(s) \geq c + (1 - c)s \quad (s \leq 1). \quad (13)$$

Proof. On $[0, 1]$, the function H is concave, so $H(s) \geq (1 + s)/2 \geq c + (1 - c)s$. On $[-1, 0]$, it is convex, and the minimum of $H(s) - (1 - c)s$ occurs at

$$s = -\operatorname{sech}(\pi(1 - c)).$$

By (12), that minimum equals

$$\frac{1}{2} - \frac{\arcsin(\operatorname{sech}(\pi(1 - c)))}{\pi}.$$

It is at least c precisely when $\cos(\pi c) \cosh(\pi(1 - c)) \geq 1$, which holds for $c \leq c_*$. For $s \leq -1$, the right side of (13) is nonpositive. $\square$

The simple choice $c = 2/5$ is certified without numerical approximations. The needed inequality is $\cosh(3\pi/5) \geq 1 + \sqrt{5}$. Using $\pi > 25/8$, the Taylor lower bound gives

$$\cosh(3\pi/5) > 1 + \frac{(15/8)^2}{2} + \frac{(15/8)^4}{24} = \frac{321729}{98304} > \frac{13}{4} > 1 + \sqrt{5}.$$

2.2.2 A comparison of product measures

We first record the measure-comparison fact used below.

Lemma 7. *Let $c > 0$, and let $m_1, \dots, m_n$ be locally finite Borel measures on $[0, \infty)$ such that $m_j([0, s]) \geq cs$ for every $j \in [n]$ and $s \geq 0$. For every bounded Borel coordinatewise downset $A \subset [0, \infty)^n$,*

$$(m_1 \otimes \dots \otimes m_n)(A) \geq c^n \text{vol}_n(A).$$

Proof. In one dimension the cumulative inequality implies $\int f dm_j \geq c \int f ds$ for every non-negative decreasing function f of bounded support. Indeed, apply the layer-cake formula; each superlevel set of f is an initial interval. The claim for open initial intervals follows by taking limits from below, and an initial interval consisting only of zero causes no problem. Apply this comparison successively to the coordinates of 1_A , using Tonelli's theorem. At each stage the function of every remaining coordinate is decreasing. The argument allows atoms at zero. $\square$

Fix $0 < c \leq c_*$ and a point $x \in K$. Independent rotations allow us to assume $x = r = (r_1, \dots, r_n) \in L$. Both K and ν are invariant under these rotations. Define $F_r : \mathbb{C}^n \rightarrow [0, \infty)^n$ by

$$(F_r(d))_j = \max \left\{ 0, \frac{|d_j| - (1-c)r_j}{c}, r_j + \frac{\Re d_j}{c} \right\}. \quad (14)$$

When $r_j = 0$, this is simply $|d_j|/c$. Putting $y = F_r(d)$, we have in every coordinate

$$|d_j| \leq (1-c)r_j + cy_j, \quad \Re d_j \leq c(y_j - r_j). \quad (15)$$

For $r_j > 0$, let m_{r_j} be the image measure of ν_1 under the corresponding one-coordinate map. For $s \geq 0$, set

$$R = (1-c)r_j + cs, \quad B = c(s - r_j).$$

Then

$$\{d_j : (F_r(d))_j \leq s\} = \{|d_j| \leq R, \Re d_j \leq B\}.$$

The coordinate map is continuous. This preimage is bounded and has finite ν_1 -measure, so m_{r_j} is a locally finite Borel measure. Since $R - B = r_j > 0$, Lemma 6 and the degree-one homogeneity of ν_1 imply

$$m_{r_j}([0, s]) = RH(B/R) \geq cR + (1-c)B = cs. \quad (16)$$

For $r_j = 0$, the pushforward is exactly c times Lebesgue measure on $[0, \infty)$. Because F_r acts separately in each coordinate, the image of $\nu_1^{\otimes n}$ under F_r is the product of these coordinate image measures. This equality holds first on rectangles and then on Borel sets by uniqueness of locally finite product measures. Hence Lemma 7, applied to $(1-\varepsilon)L$, gives

$$\nu(F_r^{-1}((1-\varepsilon)L)) \geq c^n(1-\varepsilon)^n \nu(K), \quad 0 < \varepsilon < 1. \quad (17)$$

Moreover, this preimage lies in K : if $y \in (1-\varepsilon)L$, then $|d| \leq (1-c)r + cy$, whose K -norm is at most $1 - c\varepsilon$. The closed scaled set $(1-\varepsilon)L$, including its coordinate faces, is intentional; the measures in (16) may have an atom at zero.

2.2.3 A uniform estimate for interior points

Lemma 8. *Let $0 < c \leq c_*$, and let D have probability distribution $\nu|_K/\nu(K)$. For $0 < \varepsilon < 1$, put*

$$\tau = \frac{c\varepsilon}{4}, \quad \gamma = \frac{c^2\varepsilon^2}{8}, \quad p = [c(1-\varepsilon)]^n.$$

For every $x \in K$,

$$\mathbb{P}\{x + \tau D \in (1-\gamma)K\} \geq p. \quad (18)$$

The same distribution, step τ , contraction γ , and lower bound p work for every point x .

Proof. Rotate x to $r \in L$ and consider the preimage in (17). We show that every d in this preimage satisfies the desired finite contraction. For $0 < \tau \leq 1$, the two scalar inequalities in (15) imply

$$|r_j + \tau d_j| \leq (1 - \tau c + \tau^2)r_j + (\tau c + \tau^2)y_j. \quad (19)$$

Set $A = r_j + \tau c(y_j - r_j)$, so $A \geq (1 - \tau c)r_j \geq r_j/2$. Expanding the square and using (15) gives

$$\begin{aligned} |r_j + \tau d_j|^2 &\leq A^2 + \tau^2[(1 - 2c)r_j^2 + 2cr_j y_j] \\ &\leq A^2 + \tau^2 r_j(r_j + y_j). \end{aligned}$$

On the other hand,

$$[A + \tau^2(r_j + y_j)]^2 - A^2 \geq 2A\tau^2(r_j + y_j) \geq \tau^2 r_j(r_j + y_j).$$

Taking square roots proves (19).

Monotonicity of the unconditional norm and its triangle inequality now yield

$$\begin{aligned} \|x + \tau d\|_K &\leq (1 - \tau c + \tau^2) \|r\|_K + (\tau c + \tau^2) \|y\|_K \\ &\leq 1 - \tau c\varepsilon + \tau^2(2 - \varepsilon). \end{aligned}$$

For $\tau = c\varepsilon/4$, the last expression is

$$1 - \frac{c^2\varepsilon^2(2 + \varepsilon)}{16} \leq 1 - \frac{c^2\varepsilon^2}{8} = 1 - \gamma.$$

The mass estimate (17), and containment of its preimage in K , prove (18). Rotation invariance returns the statement to the original x . $\square$

For $x \in \partial K$, the event in (18) consists of illuminating directions. Consequently $\mathbb{P}\{D \text{ illuminates } x\} \geq [c(1 - \varepsilon)]^n$ for every $\varepsilon > 0$. The event on the left is independent of ε and is Borel: it is the union, over positive rational q , of the open sets $\{d : x + qd \in \text{int } K\}$. Taking the supremum of the lower bounds as $\varepsilon \downarrow 0$, the measure $c^{-n}\nu|_K/\nu(K)$ fractionally illuminates K . Its mass at zero is zero, so it defines a measure on the allowed nonzero directions. Taking $c = c_*$ proves (8).

2.2.4 A finite set of illuminating directions

Let $\eta = \gamma/2$. Choose a maximal finite η -separated set $\mathcal{N} \subset K$ in the norm of K . Such a set can be formed by successively adding points at distance at least η ; the packing estimate below bounds the number of additions. Maximality makes it an η -net, meaning that every point of K has distance at most η from it. Its cardinality is at most

$$N_0 = \left(1 + \frac{2}{\eta}\right)^{2n} = \left(1 + \frac{4}{\gamma}\right)^{2n}. \quad (20)$$

Indeed, the translates $z + (\eta/2)\text{int } K$, for $z \in \mathcal{N}$, are pairwise disjoint and lie in $(1 + \eta/2)K$. Comparison of ordinary Lebesgue volumes, in real dimension $2n$, gives (20).

Choose M independent vectors distributed as D in Lemma 8. For each net point, the probability that all of them fail is at most $(1 - p)^M \leq e^{-pM}$. Thus

$$M = \left\lceil \frac{\log N_0 + 1}{p} \right\rceil$$

makes the probability that any net point is missed at most $e^{-1} < 1$. There is therefore a choice $d^1, \dots, d^M$ such that every $z \in \mathcal{N}$ satisfies $\|z + \tau d^j\|_K \leq 1 - \gamma$ for some $j \in [M]$.

For arbitrary $x \in K$, take $z \in \mathcal{N}$ with $\|x - z\|_K \leq \eta$, and use the vector serving z . Then

$$\|x + \tau d^j\|_K \leq \|x - z\|_K + \|z + \tau d^j\|_K \leq \eta + 1 - \gamma = 1 - \gamma/2 < 1.$$

These vectors illuminate all of ∂K . The zero vector has probability zero under ν , so all sampled vectors are nonzero with probability one. Intersecting this event with the positive-probability covering event gives nonzero illuminating directions. We have proved the more flexible bound

$$\text{ill}(K) \leq \left\lceil \frac{2n \log(1 + 32/(c^2 \varepsilon^2)) + 1}{[c(1 - \varepsilon)]^n} \right\rceil, \quad 0 < \varepsilon < 1. \quad (21)$$

Take $\varepsilon = 1/(n + 1)$ and use $(1 - \varepsilon)^n = (1 + 1/n)^{-n} \geq e^{-1}$. This proves (9) with $c = c_*$, and (10) with $c = 2/5$.

Remark 2. Choosing $\varepsilon = 1/(n \log n)$, for $n \geq 2$, in (21) gives the sharper asymptotic prefactor

$$\text{ill}(K) \leq (4 + o(1)) n \log n b_*^n.$$

The finite-dimensional parameter ε can also be optimized directly. Neither refinement changes the exponential base.

The constant c_* is optimal for the affine inequality (13): values $c > 1/2$ fail at $s = 0$, and for $c_* < c \leq 1/2$ the negative minimizer used in its proof violates the inequality. This does not assert optimality of the resulting illumination bound.

3 A full coordinate hyperplane section

We now prove a sharp bound for a class defined by an exact coordinate-section condition. Throughout this section, $n \geq 2$ and $K \subset \mathbb{C}^n$ is a normalized complex 1-unconditional convex body. Write $\mathbb{D} = \{x \in \mathbb{C} : |x| \leq 1\}$, $[n] = \{1, \dots, n\}$, and $\mathbf{1} = (1, \dots, 1)$. By a *full coordinate hyperplane section* we mean that, after permuting coordinates,

$$K \cap H = \mathbb{D}^{n-1} \times \{0\}, \quad H = \{x \in \mathbb{C}^n : x_n = 0\}.$$

Since normalization implies $K \subseteq \mathbb{D}^n$, this is equivalent to $\mathbb{D}^{n-1} \times \{0\} \subseteq K$.

Theorem 9. *Let $n \geq 2$ and let $K \subset \mathbb{C}^n$ be a normalized complex 1-unconditional convex body. If $K \neq \mathbb{D}^n$ and K contains a full coordinate $(n - 1)$ -polydisc, then*

$$\text{ill}(K) \leq 2^{n+1} - 2. \quad (22)$$

This bound is attained, for every $n \geq 2$, by

$$H_n = \mathbb{D}^{n-2} \times \{(x_{n-1}, x_n) \in \mathbb{C}^2 : |x_{n-1}| + |x_n| \leq 1\}.$$

For $n = 2$ the first factor is omitted.

We use Lemma 1 and the cyclic polydisc construction (4). The remaining estimates are proved below.

3.1 Supporting normals

Lemma 10. *Suppose $\mathbb{D}^{n-1} \times \{0\} \subseteq K$, and let $u \neq 0$ be a supporting normal at $x \in \partial K$ for the real pairing $[u, x] = \text{Re} \sum_j \bar{u}_j x_j$. Set $b_j = |u_j|$ and $r_j = |x_j|$. Then*

$$h_K(u) = \sum_{j \in [n]} b_j r_j, \quad \sum_{j < n} b_j (1 - r_j) \leq b_n r_n. \quad (23)$$

If $x_j \neq 0$, then $u_j = b_j x_j / |x_j|$.

Proof. The orbit of x under independent coordinate rotations lies in K . The maximum of $[u, \cdot]$ on that orbit is $\sum_j b_j r_j$. Since x itself maximizes $[u, \cdot]$ on all of K , equality holds in each of the scalar inequalities $\operatorname{Re}(\overline{u_j} x_j) \leq b_j r_j$. This proves both the first equality and the alignment when $x_j \neq 0$. No alignment assertion is needed when $x_j = 0$. Maximizing the same functional on the contained polydisc gives $h_K(u) \geq \sum_{j < n} b_j$. Subtracting this from the first equality gives the second inequality in (23). $\square$

3.2 Projections onto proper coordinate subsets

Lemma 11. *Put $M = 2^n - 1$ and*

$$p_j^k = \exp\left(\frac{2\pi i 2^{j-1} k}{M}\right), \quad 0 \leq k < M, \quad j \in [n]. \quad (24)$$

There exists $c_n \in (0, 1)$, depending only on n , such that for every nonempty proper subset $S \subset [n]$ and every family of unit complex numbers $(\zeta_j)_{j \in S}$ there is an index k satisfying

$$\operatorname{Re}(\overline{\zeta_j} p_j^k) > c_n \quad (j \in S). \quad (25)$$

Proof. First consider a projection onto $n - 1$ coordinates. The n column frequencies in (24) are 2^a , $a = 0, \dots, n - 1$, taken modulo M . Since $2^n \equiv 1 \pmod{M}$, multiplying all frequencies by a power of 2 cyclically permutes these n exponents. This multiplication also just permutes the rows, because M is odd. Choose the cyclic shift that sends the omitted exponent to $n - 1$. The retained frequencies are then $1, 2, \dots, 2^{n-2}$, up to coordinate order. They are exactly the cyclic construction (4) in dimension $n - 1$, whose denominator is $2^n - 1$.

Hence every such projection meets every product of $n - 1$ open semicircles. Every smaller nonempty projection has the same property: extend its coordinate set to one of size $n - 1$ and prescribe arbitrary open semicircles on the added coordinates. In particular, for any $(\zeta_j)_{j \in S}$ one row has strictly positive real pairing with all the ζ_j on S .

To obtain a margin independent of these unit complex numbers, define on the compact torus $\{(\zeta_j)_{j \in S} : |\zeta_j| = 1\}$ the continuous function

$$f_S(\zeta) = \max_{0 \leq k < M} \min_{j \in S} \operatorname{Re}(\overline{\zeta_j} p_j^k).$$

It is continuous because both extrema are over finite sets, and the preceding paragraph shows $f_S(\zeta) > 0$ at every point. Its attained minimum α_S is therefore positive. There are only finitely many nonempty proper subsets S . Thus

$$c_n = \frac{1}{2} \min_{\emptyset \neq S \subsetneq [n]} \alpha_S$$

is positive and at most $1/2$. A row attaining the maximum in f_S has every pairing at least $\alpha_S \geq 2c_n > c_n$, proving the claim. $\square$

3.3 The illuminating directions and the boundary cases

Proof of the upper bound in Theorem 9. Permute coordinates so that the full section is $\{x_n = 0\}$, and set

$$\gamma = \|\mathbf{1}\|_K^{-1}, \quad \delta = \frac{c_n(1 - \gamma)}{2}. \quad (26)$$

We have $0 < \gamma < 1$. Indeed, if $\mathbf{1} \in K$, coordinatewise monotonicity would imply $\mathbb{D}^n \subseteq K$; normalization would then imply $K = \mathbb{D}^n$, contrary to the hypothesis. Thus $\|\mathbf{1}\|_K > 1$, and in particular $\delta > 0$.

Every $x \in K$ has at least one coordinate with $|x_j| \leq \gamma$. To see this, suppose $r_j = |x_j| > \gamma$ for all j . Choose λ with $\max_j \gamma/r_j < \lambda < 1$. Rotation invariance and Lemma 1 give

$$1 = \|\gamma \mathbf{1}\|_K \leq \|\lambda(r_1, \dots, r_n)\|_K = \lambda \|x\|_K \leq \lambda < 1,$$

a contradiction.

Using the vectors from Lemma 11, define

$$\begin{aligned} d^{k,0} &= -(p_1^k, \dots, p_{n-1}^k, 0), \\ d^{k,1} &= -(\delta p_1^k, \dots, \delta p_{n-1}^k, p_n^k), \quad 0 \leq k < M. \end{aligned} \tag{27}$$

These are $2M$ nonzero vectors, fixed before choosing the boundary point. We show that their union illuminates every $x \in \partial K$.

Case 1: $x_n = 0$. By Lemma 1, x lies on the relative boundary of $K \cap \{x_n = 0\} = \mathbb{D}^{n-1} \times \{0\}$. Consequently $\|x\|_\infty = 1$, and $M_x = \{j \in [n-1] : |x_j| = 1\}$ is nonempty. Choose k with the margin (25) on M_x , taking $\zeta_j = x_j$. For $j \in M_x$ and $0 < t < 2c_n$,

$$|x_j - tp_j^k|^2 = 1 - 2t \operatorname{Re}(\overline{x_j} p_j^k) + t^2 < 1 - 2tc_n + t^2 < 1.$$

For every remaining $j < n$, we initially have $|x_j| < 1$, so $|x_j - tp_j^k| < 1$ for all sufficiently small positive t , by continuity. This includes coordinates that are zero. There are finitely many coordinates, so one $t > 0$ satisfies all these inequalities. The last coordinate remains zero. Hence $x + td^{k,0}$ is in the relative interior of the full section, and therefore in $\operatorname{int} K$ by Lemma 1.

Case 2: $x_n \neq 0$ and $r_j > \gamma$ for all $j < n$. The preceding observation forces $r_n \leq \gamma < 1$. Normalization gives $\|x_n e_n\|_K = r_n < 1$, so $q = x_n e_n$ is interior. All first $n-1$ coordinates of x are nonzero. Choose k with (25) on $[n-1]$, using $\zeta_j = x_j/r_j$. For each $j < n$ we have

$$|x_j - tp_j^k|^2 < r_j^2 - 2tc_n r_j + t^2 < r_j^2 \quad \text{whenever } 0 < t < 2c_n r_j.$$

Choose t smaller than all these finitely many positive upper bounds. The point $x + td^{k,0}$ strictly contracts every coordinate in $[n-1]$ and leaves the last one unchanged. Apply part 3 of Lemma 1 with $J = [n-1]$ and $q = x_n e_n$ to conclude that $x + td^{k,0} \in \operatorname{int} K$.

Case 3: $x_n \neq 0$ and $r_i \leq \gamma$ for some $i < n$. Let

$$E = Z_x \cup \{i\}, \quad S = [n] \setminus E.$$

Then $\emptyset \neq E \subseteq [n-1]$, $n \in S$, and every coordinate of x on S is nonzero. In particular S is a nonempty proper subset. Also $r_j \leq \gamma$ for every $j \in E$. Choose a single k with (25) on S , using $\zeta_j = x_j/r_j$. This choice uses x and is independent of its supporting normal.

Now let $u \neq 0$ be any supporting normal at x , and put $b_j = |u_j|$. Since $0 \leq r_j \leq 1$, all terms on the left below are nonnegative. Lemma 10 gives

$$(1 - \gamma) \sum_{j \in E} b_j \leq \sum_{j \in E} b_j (1 - r_j) \leq \sum_{j < n} b_j (1 - r_j) \leq b_n r_n \leq b_n. \tag{28}$$

On S the normal aligns with x and the chosen entry has positive pairing at least c_n . On E no alignment or sign estimate is required: the absolute value of each contribution is at most δb_j , including when $x_j = 0$. It follows that

$$\begin{aligned} [u, d^{k,1}] &\leq -c_n b_n - c_n \delta \sum_{j \in S \setminus \{n\}} b_j + \delta \sum_{j \in E} b_j \\ &\leq -\left(c_n - \frac{\delta}{1 - \gamma}\right) b_n - c_n \delta \sum_{j \in S \setminus \{n\}} b_j \\ &= -\frac{c_n}{2} b_n - c_n \delta \sum_{j \in S \setminus \{n\}} b_j < 0. \end{aligned}$$

For the final strict inequality, if the last expression were zero, positivity of c_n and δ would force $b_n = 0$ and $b_j = 0$ for every $j \in S \setminus \{n\}$. Equation (28) would then force all b_j on E to vanish as well, contradicting $u \neq 0$. Thus the same $d^{k,1}$ pairs negatively with every nonzero supporting normal at x . The illumination criterion (1) proves that $d^{k,1}$ illuminates x .

These three cases exhaust ∂K . The parameter c_n was chosen solely from n , and δ was then fixed from K ; neither depends on the boundary point or its supporting normal. We have constructed at most $2M = 2^{n+1} - 2$ illuminating directions. $\square$

3.4 Sharpness and immediate consequences

Proof of sharpness in Theorem 9. The body H_n in the theorem is normalized, complex 1-unconditional, and contains $\mathbb{D}^{n-1} \times \{0\}$. It is not $\mathbb{D}^n$, since $e_{n-1} + e_n \notin H_n$. The upper bound already proved therefore applies to H_n .

For the lower bound consider its two boundary tori

$$\begin{aligned} T_- &= \{x : |x_j| = 1 \ (j < n), \ x_n = 0\}, \\ T_+ &= \{x : |x_j| = 1 \ (j \leq n-2 \text{ or } j = n), \ x_{n-1} = 0\}. \end{aligned}$$

Let $\mathcal{V}$ be any finite illuminating family for H_n , and let $\mathcal{V}_-$ and $\mathcal{V}_+$ consist of the directions that illuminate at least one point of T_- and T_+ , respectively. If $d \in \mathcal{V}_-$ illuminates $x \in T_-$ at some time $t > 0$, then interior membership gives

$$|x_{n-1} + td_{n-1}| + t|d_n| < 1.$$

The reverse triangle inequality gives $|x_{n-1} + td_{n-1}| \geq 1 - t|d_{n-1}|$, so $|d_{n-1}| > |d_n|$. Interchanging the last two coordinates shows that $d \in \mathcal{V}_+$ requires $|d_n| > |d_{n-1}|$. Consequently $\mathcal{V}_- \cap \mathcal{V}_+ = \emptyset$.

Let π_- be projection onto coordinates $1, \dots, n-1$. If d illuminates $x \in T_-$ in H_n , then $\pi_-(x + td) \in \text{int } \mathbb{D}^{n-1}$ for the same t : the first $n-2$ moduli are strictly below one, and the displayed inequality also makes the $(n-1)$ st modulus strictly below one. Thus $\pi_-(\mathcal{V}_-)$ illuminates every point of $(\partial \mathbb{D})^{n-1}$.

For completeness, illuminating this torus suffices to illuminate the entire polydisc, in any dimension m . Every point $x \in \mathbb{D}^m$ is a finite convex combination $x = \sum_{j \in [N]} \lambda_j y^j$ of torus points: express each scalar coordinate as a convex combination of two unit complex numbers and multiply the coordinatewise weights. Discard terms of zero weight. If a direction d illuminates one y^j , say $y^j + td \in \text{int } \mathbb{D}^m$, then

$$x + \lambda_j td = \lambda_j(y^j + td) + \sum_{a \neq j} \lambda_a y^a \in \text{int } \mathbb{D}^m.$$

The last inclusion follows coordinatewise from a positive weight on a vector all of whose coordinate moduli are strictly below one. The exact polydisc illumination theorem [1] now implies $|\mathcal{V}_-| \geq 2^n - 1$. The projection omitting coordinate $n-1$ gives $|\mathcal{V}_+| \geq 2^n - 1$ in exactly the same way. Since these families are disjoint,

$$|\mathcal{V}| \geq |\mathcal{V}_-| + |\mathcal{V}_+| \geq 2(2^n - 1) = 2^{n+1} - 2.$$

Together with the upper bound, this proves equality. $\square$

Corollary 12. *Every normalized complex 1-unconditional body $K \subset \mathbb{C}^2$ with $K \neq \mathbb{D}^2$ satisfies $\text{ill}(K) \leq 6$, and this is sharp. In $\mathbb{C}^3$ a full coordinate bidisc gives the sharp bound 14; in $\mathbb{C}^4$ a full coordinate tridisc gives the sharp bound 30, in both cases assuming K is not the full polydisc.*

Proof. In dimension two every normalized body contains a unit coordinate disc, so the section hypothesis is automatic. Apply Theorem 9 with $n = 2, 3, 4$. Its examples H_2, H_3, H_4 establish the respective sharpness claims. $\square$

Remark 3. The directions (27) depend on K through γ and δ . The directions are chosen separately for each body. In dimensions three and four the corollary only covers the stated section classes. Closeness to $\mathbb{D}^n$ does not by itself imply the required exact section containment.

4 Three complex dimensions

Theorem 13. *Every normalized complex 1-unconditional convex body $K \subset \mathbb{C}^3$ with $K \neq \mathbb{D}^3$ satisfies $\text{ill}(K) \leq 14$. The bound is sharp.*

Theorem 9 already handles bodies containing a full coordinate bidisc. We construct fourteen directions whenever at least one coordinate bidisc is missing. The construction uses two copies of a seven-vector family, each with three directions tending to each coordinate axis. The partition of the seven rows comes from an additional coordinate, as in Lemma 4. It is also the two-dimensional case of the coloring in [2, Proposition 4.4], with the omitted row restored.

4.1 Seven directions in a coordinate plane

Set $\eta = e^{2\pi i/7}$. By Lemma 3, every two-coordinate projection of

$$\{(\eta^t, \eta^{2t}, \eta^{4t}) : t \in \{0, \dots, 6\}\}$$

meets every product of two open semicircles. Replace the last entry by $h_t = \eta^{3-4t}$. Conjugation and rotation preserve open semicircles, so the same projection property holds for the columns η^t, η^{2t}, h_t . Partition the rows by the sign of the imaginary part of the additional coordinate:

$$U = \{t : \text{Im } h_t > 0\} = \{0, 2, 4\}, \quad V = \{t : \text{Im } h_t < 0\} = \{1, 3, 5\}, \quad h_6 = 1.$$

The entries η^t , $t \in U$, meet every open semicircle: prescribe that semicircle in the first column and the upper open semicircle in the additional column. Similarly, the entries η^{2t} , $t \in V$, meet every open semicircle, by prescribing the lower open semicircle in the additional column. All seven rows still satisfy arbitrary open-semicircle conditions on the two visible coordinates simultaneously.

For $\varepsilon > 0$, define

$$p^t(\varepsilon) = -(a_t \eta^t, b_t \eta^{2t}), \quad (a_t, b_t) = \begin{cases} (1, \varepsilon), & t \in U, \\ (\varepsilon, 1), & t \in V, \\ (1, 1), & t = 6. \end{cases} \quad (29)$$

For any $x_1, x_2 \neq 0$, the two semicircle conditions give a row with

$$\text{Re}(\overline{x_j} p_j^t(\varepsilon)) < 0 \quad (j \in [2]).$$

Positive amplitudes do not change these signs. Since

$$|x_j + s d_j|^2 = |x_j|^2 + 2s \text{Re}(\overline{x_j} d_j) + s^2 |d_j|^2,$$

both moduli strictly decrease for a sufficiently small positive step s . As ε tends to zero, the U -rows tend to three directions on the first axis whose unit entries meet every open semicircle, and the V -rows do the same on the second axis. Thus the additional coordinate determines both the partition and the two axis limits.

4.2 Proof of the bound

Proof of Theorem 13. If K contains a full coordinate bidisc, apply Theorem 9. Otherwise permute the coordinates so that $e_2 + e_3 \notin K$. In fact, the following argument only requires this one missing bidisc. Embed (29) in the two planes by setting

$$\begin{aligned} A_\varepsilon &= \{(p_1^t(\varepsilon), p_2^t(\varepsilon), 0) : t \in \{0, \dots, 6\}\}, \\ B_\varepsilon &= \{(p_1^t(\varepsilon), 0, p_2^t(\varepsilon)) : t \in \{0, \dots, 6\}\}. \end{aligned}$$

We choose one $\varepsilon > 0$ for which these fourteen directions illuminate K .

We first record why any three unit entries meeting every open semicircle on each coordinate axis illuminate a normalized complex 1-unconditional body $L \subset \mathbb{C}^2$ different from $\mathbb{D}^2$. At a boundary point x with both coordinates nonzero, their moduli cannot both equal one, since coordinate contractions and rotations would then give $L = \mathbb{D}^2$. Choose a coordinate to decrease so that the unchanged coordinate has modulus less than one. Setting the chosen coordinate to zero leaves an interior point of L , since every axis point of modulus less than one is interior. One of the three directions on the chosen axis strictly decreases that modulus for a small positive step, and Lemma 1(3) puts the new point in $\text{int } L$. If only one coordinate is nonzero, decrease it in the same way; deleting it leaves the interior point zero. This proves the assertion for all boundary points of L .

Now take $L = K \cap \{x_1 = 0\}$, identified with a body in $\mathbb{C}^2$. It is normalized and is not a bidisc. The V -rows from the two families tend to three pure directions on axis 2 and three on axis 3. By the preceding argument, these limits illuminate ∂L in its plane. Lemma 1(2) identifies relative interior in the section with ambient interior intersected with that plane. Thus they also illuminate $\partial K \cap \{x_1 = 0\}$ in K . The U -rows of either family tend to three pure directions on axis 1 that illuminate its unit circle. The limiting directions therefore illuminate the compact set

$$E = (\partial K \cap \{x_1 = 0\}) \cup \{ze_1 : |z| = 1\}.$$

There exists $\varepsilon_0 > 0$ such that $A_\varepsilon \cup B_\varepsilon$ illuminates all of E whenever $0 < \varepsilon < \varepsilon_0$. Indeed, for each $x \in E$, choose one limiting direction $d(0)$ and a step $s > 0$ with $x + sd(0) \in \text{int } K$. Openness supplies neighborhoods of x and $d(0)$ such that the same step enters the interior for every pair in those neighborhoods. Finitely many of the point neighborhoods cover E . The corresponding finitely many directions $d(\varepsilon)$ converge to their limits, so the minimum of their positive admissible upper bounds on ε gives one ε_0 valid on all of E . Fix $\varepsilon \in (0, \varepsilon_0)$.

Let $x \in \partial K \setminus E$. Then $x_1 \neq 0$ and at least one of x_2, x_3 is nonzero. If $x_3 = 0$, a direction in A_ε strictly decreases the first two moduli, and Lemma 1(3), with the interior point zero, shows that it illuminates x . If $x_2 = 0$, use B_ε instead.

It remains to consider three nonzero coordinates. The moduli $|x_2|$ and $|x_3|$ cannot both equal one: deleting the first coordinate and rotating the others would put $e_2 + e_3$ in K . If $|x_3| < 1$, choose a direction in A_ε decreasing the first two moduli. It leaves the third coordinate unchanged, and $x_3 e_3 \in \text{int } K$, so Lemma 1(3) applies. Otherwise $|x_2| < 1$, and a direction in B_ε works by the same argument with the interior point $x_2 e_2$. These cases exhaust the boundary and prove $\text{ill}(K) \leq 14$.

Finally, the body $\mathbb{D} \times \{(z, w) : |z| + |w| \leq 1\}$ has illumination number 14 by the sharpness part of Theorem 9. Hence the bound is sharp. $\square$

5 Four complex dimensions

Theorem 14. *Every normalized complex 1-unconditional convex body $K \subset \mathbb{C}^4$ different from $\mathbb{D}^4$ satisfies*

$$\text{ill}(K) \leq 30.$$

The bound is sharp. Consequently every complex 1-unconditional convex body in $\mathbb{C}^4$ has illumination number at most 31, and equality holds precisely when its positive diagonal normalization is $\mathbb{D}^4$.

The proof separates bodies according to which coordinate bidiscs they contain. A full coordinate bidisc on $\{i, j\}$ is present exactly when $e_i + e_j \in K$. Three constructions below cover every body with a missing bidisc. The remaining case uses a finite covering assertion, verified by the exact integer program in Appendix B. Every geometric argument applies to all boundary points and all their supporting normals.

5.1 Two disjoint missing bidiscs

Proposition 15. *Let $K \subset \mathbb{C}^4$ be normalized and complex 1-unconditional. If $e_1 + e_2 \notin K$ and $e_3 + e_4 \notin K$, then $\text{ill}(K) \leq 30$.*

Proof. Put

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\} < 1.$$

For $j \in \{3, 4\}$, use Lemma 4 with root $\{1, 2, j\}$, children $\{1, j\}$ and $\{2, j\}$, singleton children $\{1\}, \{j\}$ below $\{1, j\}$, and singleton child $\{2\}$ below $\{2, j\}$. Write $D_j(\delta)$ for its fifteen directions, embedded with the remaining coordinate zero. Keep the angular entries and the row partitions fixed and let $\delta \in (0, 1)$ be the amplitude parameter of that construction. There is $c > 0$, independent of δ , such that for every node S and every choice of coordinate arguments on S , a row of its group is inward with angular margin c at every coordinate of S ; if its smallest amplitude on S is a , every amplitude outside S but inside its root is at most δa .

In each family the rows belonging to the singleton nodes $\{1\}, \{2\}$ converge, as $\delta \rightarrow 0$, to directions on their respective coordinate axes, whose unit entries meet every open semicircle. These limiting directions illuminate

$$E = \partial K \cap \{x_3 = x_4 = 0\}.$$

Indeed, the coordinate section on $\{1, 2\}$ is not a bidisc. At a boundary point with both coordinates nonzero, at least one modulus is less than one. Decrease the other coordinate, using an inward axis direction. Deleting that coordinate leaves an axis point of modulus less than one, which is interior to K ; strict contraction toward it therefore enters the interior. If only one coordinate is nonzero, decrease it instead, with the interior point zero.

The set E is compact. At each of its points choose one limiting direction and one positive step entering into K . Openness gives neighborhoods of both the point and that direction for which the same step enters the interior. Take a finite cover of E by these point neighborhoods. Convergence of the finitely many corresponding rows gives $\delta_0 > 0$ such that $D_3(\delta) \cup D_4(\delta)$ illuminates E whenever $0 < \delta < \delta_0$. Fix

$$0 < \delta < \min \left\{ \delta_0, 1, \frac{c(1-\gamma)}{4} \right\}.$$

Let $x \in \partial K \setminus E$. Choose $j \in \{3, 4\}$ and the other index k so that $x_j \neq 0$ and $|x_k| \leq \gamma$. Such a choice exists: if one of x_3, x_4 vanishes, omit that coordinate; otherwise omit a coordinate with the smaller modulus. Put $R = \{1, 2, j\}$ and $S = \text{supp}(x) \cap R$. Every subset of R containing j is a node of the chosen tree, so the row property applies to S .

For a nonzero supporting normal u at x , put $b_i = |u_i|$, $r_i = |x_i|$ and $h = [u, x]$. Normal alignment and normalization give $h = \sum_i b_i r_i > 0$ and $b_i \leq h$. Consequently

$$(1 - r_k)h \leq \sum_{i \in S} b_i r_i \leq \sum_{i \in S} b_i, \quad b_\ell \leq \frac{1}{1 - \gamma} \sum_{i \in S} b_i \quad (\ell \in R \setminus S).$$

The sum over S is positive. Choose a row $d \in D_j(\delta)$ having angular margin c on S . This choice depends only on x , not on u . Its omitted coordinate is zero, and there are at most two inactive coordinates inside R , whence

$$[u, d] \leq a \left(-c + \frac{2\delta}{1-\gamma} \right) \sum_{i \in S} b_i < 0.$$

The normal criterion proves that d illuminates x . Along with the compact-set argument for E , this proves the bound. $\square$

5.2 Two present bidiscs and a missing bidisc

Proposition 16. *Let $K \subset \mathbb{C}^4$ be a normalized complex 1-unconditional convex body. Suppose that*

$$e_1 + e_3 \in K, \quad e_1 + e_4 \in K, \quad e_3 + e_4 \notin K.$$

Then $\text{ill}(K) \leq 30$.

Proof. Put

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\}, \quad c = 1 - \gamma.$$

The maximum exists by compactness. Normalization gives $\gamma \leq 1$. Equality would give a point of K with $|x_3| = |x_4| = 1$; coordinate contractions and rotations would then imply $e_3 + e_4 \in K$. Thus $c > 0$. Every $x \in K$ satisfies

$$|x_3| \leq \gamma \quad \text{or} \quad |x_4| \leq \gamma.$$

Set $\omega = e^{2\pi i/15}$ and

$$v_j^t = \omega^{2^{j-1}t} \quad (0 \leq t \leq 14, j \in [4]).$$

By Lemma 3, any three columns meet every product of three open semicircles. Define the following disjoint sets of rows:

$$\begin{aligned} G &= \{t : \text{Im } v_4^t > 0\}, & H &= \{t : \text{Im } v_4^t < 0\}, \\ H_1 &= \{t \in H : \text{Im } v_2^t > 0\}, & H_3 &= \{t \in H : \text{Im } v_2^t < 0\}. \end{aligned}$$

Each of G, H has seven elements. The only row with zero imaginary part in any column is row zero, so $H = H_1 \sqcup H_3$ and $\{0, \dots, 14\} = \{0\} \sqcup G \sqcup H_1 \sqcup H_3$.

The rows G meet every product of two open semicircles in columns $\{1, 2\}$, and also in columns $\{2, 3\}$. To see this, append to the two prescribed conditions the upper-open-semicircle condition on column 4 and use the three-column projection property. Similarly, H meets every product of two open semicircles in columns $\{1, 3\}$. The entries in column 1 of H_1 meet every open semicircle: prescribe that semicircle in column 1, the upper semicircle in column 2, and the lower semicircle in column 4. Prescribing the desired semicircle in column 3 and the lower semicircles in columns 2 and 4 proves the analogous assertion for column 3 of H_3 . The entries in column 2 of G also meet every open semicircle. Finally, all fifteen rows meet every product of three open semicircles in columns $\{1, 2, 3\}$.

These finitely many covering statements have a common positive margin. More precisely, choose $\beta > 0$ such that, for each pair

$$\begin{aligned} (R, S) \in \{ & (\{0, \dots, 14\}, \{1, 2, 3\}), (G, \{1, 2\}), (G, \{2, 3\}), \\ & (H, \{1, 3\}), (H_1, \{1\}), (G, \{2\}), (H_3, \{3\}) \}. \end{aligned}$$

and every family $(\zeta_j)_{j \in S}$ of unit complex numbers, some $t \in R$ satisfies

$$\text{Re}(\overline{\zeta_j} v_j^t) \geq \beta \quad (j \in S).$$

Indeed, for each fixed (R, S) the continuous function $\max_{t \in R} \min_{j \in S} \operatorname{Re}(\overline{\zeta_j} v_j^t)$ is positive on the compact torus. The minimum of its positive minima over the displayed finite list gives such a β .

Fix

$$0 < \delta < \min\{1, \beta c/4\}.$$

For each row define the three amplitudes by

$$(a_1^t, a_2^t, a_3^t) = \begin{cases} (\delta^2, 1, \delta), & t \in G, \\ (1, \delta^2, \delta), & t \in H_1, \\ (\delta, \delta^2, 1), & t \in H_3, \\ (1, 1, 1), & t = 0, \end{cases}$$

and set

$$d^t = -(a_1^t v_1^t, a_2^t v_2^t, a_3^t v_3^t, 0), \quad A = \{d^t : 0 \leq t \leq 14\}.$$

We prove that A illuminates every $x \in \partial K$ with $|x_4| \leq \gamma$.

Fix such an x , put $r_j = |x_j|$ and $S = \{j \in \{1, 2, 3\} : r_j > 0\}$, and let $u \neq 0$ be any supporting normal at x . Write $b_j = |u_j|$ and $h = h_K(u) > 0$. The normalized-normal estimates give

$$b_j \leq h, \quad h = \sum_{j \in [4]} b_j r_j \leq \sum_{j \in S} b_j + \gamma h, \quad h \leq \frac{1}{c} \sum_{j \in S} b_j.$$

In particular $S \neq \emptyset$ and $\sum_{j \in S} b_j > 0$ for every supporting normal. At every nonzero coordinate $u_j = b_j x_j / r_j$. We select the row below using only x , and show its strict inward inequality for every supporting normal u .

If $S = \{1, 2, 3\}$, take a row having the common margin on all three coordinates. All three amplitudes are positive, so

$$[u, d^t] \leq -\beta \sum_{j \in [3]} a_j^t b_j < 0.$$

If $S = \{2, 3\}$, select $t \in G$ with the margin on both active coordinates. If $S = \{1, 3\}$, select $t \in H$ with the margin on those coordinates. In either case the sole inactive coordinate among the first three has amplitude δ^2 , and both active amplitudes are at least δ . Consequently

$$[u, d^t] \leq -\beta \delta \sum_{j \in S} b_j + \delta^2 h \leq \left(-\beta \delta + \frac{\delta^2}{c}\right) \sum_{j \in S} b_j < 0.$$

Suppose $S = \{1, 2\}$. The two full bidiscs in the hypothesis give $b_1 + b_4 \leq h$ and $b_1 + b_3 \leq h$. Since $r_3 = 0$,

$$(1 - r_4)b_4 \leq b_2 r_2 - b_1(1 - r_1) \leq b_2, \quad b_4 \leq \frac{b_2}{c},$$

and hence

$$b_3 \leq b_2 r_2 + b_4 r_4 - b_1(1 - r_1) \leq b_2 + \gamma b_4 \leq \frac{b_2}{c}.$$

Choose $t \in G$ with the margin on coordinates 1 and 2. Then

$$[u, d^t] \leq -\beta \delta^2 b_1 - \beta b_2 + \delta b_3 \leq -\beta \delta^2 b_1 + \left(-\beta + \frac{\delta}{c}\right) b_2 < 0,$$

because $b_1 + b_2 > 0$.

Finally, suppose $S = \{i\}$ is a singleton. Choose a row from H_1 , G , or H_3 according as $i = 1$, $i = 2$, or $i = 3$, with the margin on coordinate i . Its amplitude there is one, and its other two amplitudes are at most δ . Thus

$$[u, d^t] \leq -\beta b_i + 2\delta h \leq \left(-\beta + \frac{2\delta}{c}\right) b_i < 0.$$

All nonempty subsets S have now been covered, with a single row chosen before the supporting normal. The normal criterion proves the asserted illumination by A .

Let P interchange coordinates 3 and 4, and put $B = PA$. The body PK satisfies the same hypotheses and has the same value of γ . Applying the preceding argument to PK and then applying P back shows that B illuminates every boundary point of K with $|x_3| \leq \gamma$. Every boundary point satisfies one of the two coordinate inequalities. Therefore $A \cup B$ illuminates K , and $|A \cup B| \leq 30$. $\square$

5.3 A coordinate triple containing all its bidiscs

Proposition 17. *Let $K \subset \mathbb{C}^4$ be normalized and complex 1-unconditional. If*

$$e_3 + e_4 \notin K, \quad e_1 + e_2, e_1 + e_4, e_2 + e_4 \in K,$$

then $\text{ill}(K) \leq 30$.

Proof. Set

$$\gamma = \max_{x \in K} \min\{|x_3|, |x_4|\} < 1.$$

The strict inequality follows from compactness, coordinate contractions and $e_3 + e_4 \notin K$. For a nonzero supporting normal u at x , write $r_j = |x_j|$, $b_j = |u_j|$ and $h = [u, x]$. The normal alignment and normalization give

$$h = \sum_{j \in [4]} b_j r_j > 0, \quad b_j \leq h \quad (j \in [4]).$$

Whenever $e_i + e_j \in K$, we also have $b_i + b_j \leq h$.

Put $\zeta = e^{2\pi i/15}$ and $v_j^t = \zeta^{2^{j-1}t}$ for $0 \leq t \leq 14$ and $j \in [4]$. Every three-coordinate projection of these fifteen rows meets every product of three open semicircles. Define

$$G = \{t : \text{Im } v_4^t > 0\}, \quad H = \{t : \text{Im } v_4^t < 0\}.$$

Both sets have seven elements and their complement is $\{0\}$. Prescribing the upper semicircle in column 4 shows that the G -rows meet every product of two open semicircles in columns $\{1, 3\}$ and in columns $\{2, 3\}$. Prescribing the lower semicircle gives the same property for the H -rows in columns $\{1, 2\}$. These statements also imply their restrictions to single columns.

For a positive parameter δ , define

$$d^{t,A} = \begin{cases} -(\delta v_1^t, \delta v_2^t, v_3^t, 0), & t \in G, \\ -(v_1^t, v_2^t, \delta v_3^t, 0), & t \in H, \\ -(1, 1, 1, 0), & t = 0. \end{cases}$$

Positive amplitudes preserve the three-coordinate covering property of all fifteen rows.

Use Lemma 4 for the tree with root $\{1, 2, 4\}$, children $\{1, 4\}$ and $\{2, 4\}$, singleton children $\{1\}$, $\{4\}$ below $\{1, 4\}$, and singleton child $\{2\}$ below $\{2, 4\}$. Denote its fifteen directions by $d^{t,B}$, embedded in coordinates 1, 2, 4, with coordinate 3 zero. The construction supplies the following property for every nonempty $S \subseteq \{1, 2, 4\}$ containing 4: for any coordinate arguments on S , one row has a positive angular margin on S , and its smallest active amplitude a is at least $1/\delta$ times every inactive amplitude within $\{1, 2, 4\}$. Take a common positive margin $c \in (0, 1)$ for these finitely many covering properties and the G, H properties above, and fix

$$0 < \delta < \frac{c(1 - \gamma)}{4}.$$

Here the rows are chosen from their coordinate arguments, independently of the supporting normal.

First suppose $r_3 \leq \gamma$ and $x_4 \neq 0$, and put $S = \text{supp}(x) \cap \{1, 2, 4\}$. Since $b_3 \leq h$,

$$(1 - r_3)h \leq \sum_{j \in S} b_j r_j \leq \sum_{j \in S} b_j.$$

The latter sum is positive. Every inactive normal coefficient is therefore at most $(1 - \gamma)^{-1} \sum_{j \in S} b_j$. Choose the row $d^{t,B}$ just described. There are at most two inactive coordinates within $\{1, 2, 4\}$, so

$$[u, d^{t,B}] \leq a \left(-c + \frac{2\delta}{1 - \gamma} \right) \sum_{j \in S} b_j < 0.$$

This row illuminates x .

Next suppose $x_3 \neq 0$ and $r_4 \leq \gamma$. If $x_1, x_2 \neq 0$, choose a direction $d^{t,A}$ inward in all three of its nonzero coordinates. Their normal coefficients have positive sum, since $b_4 \leq h$ and $r_4 < 1$. Hence $[u, d^{t,A}] < 0$.

If $x_1 = 0$ and $x_2 \neq 0$, the full bidiscs on $\{2, 4\}$ and $\{1, 2\}$ yield

$$(1 - r_4)b_4 \leq b_3 r_3 - b_2(1 - r_2) \leq b_3,$$

$$b_1 \leq b_3 r_3 + b_4 r_4 - b_2(1 - r_2) \leq \frac{b_3}{1 - r_4}.$$

Choose a G -row inward on $\{2, 3\}$. Then

$$[u, d^{t,A}] \leq \left(-c + \frac{\delta}{1 - \gamma} \right) b_3 - c\delta b_2 < 0.$$

For strictness, if $b_3 = 0$ the displayed bounds imply $b_1 = b_4 = 0$, and consequently $b_2 > 0$. If $x_2 = 0$ and $x_1 \neq 0$, interchange 1 and 2, using the full bidiscs on $\{1, 4\}$ and $\{1, 2\}$.

If $x_1 = x_2 = 0$, the full bidiscs on $\{1, 4\}$ and $\{2, 4\}$ give

$$b_1, b_2 \leq b_3 r_3 - (1 - r_4)b_4 \leq b_3.$$

Here $b_3 > 0$, since $r_4 < 1$. A G -row inward on coordinate 3 satisfies $[u, d^{t,A}] \leq (-c + 2\delta)b_3 < 0$.

Finally suppose $x_3 = x_4 = 0$. Choose an H -row inward on all nonzero coordinates among 1, 2. A zero coordinate among these two has zero normal coefficient. Indeed, if $x_2 = 0$, then $h = b_1 r_1 \geq b_1 + b_2$ and $r_1 \leq 1$ force $b_2 = 0$; the other case is identical. Moreover $b_3 \leq h \leq b_1 + b_2$, so

$$[u, d^{t,A}] \leq (-c + \delta)(b_1 + b_2) < 0.$$

These cases exhaust ∂K . If $x_4 \neq 0$ and $r_3 \leq \gamma$, the first case applies. If $x_4 \neq 0$ and $r_3 > \gamma$, the definition of γ gives $r_4 \leq \gamma$ and $x_3 \neq 0$, so the second case applies. When $x_4 = 0$, either $x_3 \neq 0$ and the second case applies, or the last case applies. Thus the fifteen vectors $d^{t,A}$ and fifteen vectors $d^{t,B}$ illuminate K . $\square$

5.4 All coordinate bidiscs

We first record a finite construction on four coordinates. Put

$$v_j^k = \exp \left(\frac{2\pi i 2^{j-1} k}{31} \right), \quad k \in [30], \quad j \in [4], \quad c = \sin(\pi/124).$$

Define two distinct indices $m(k), q(k) \in [4]$ by the following table.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m(k)$	1	2	1	4	3	2	3	4	1	4	1	4	3	2	4
$q(k)$	4	4	2	3	4	3	2	2	4	2	3	2	1	4	1

k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$m(k)$	2	1	2	4	2	3	3	1	3	2	3	4	1	2	1
$q(k)$	1	4	1	3	1	2	4	2	1	3	4	3	3	4	4

Lemma 18. *For every nonempty proper $S \subset [4]$ and every family of unit complex numbers $(a_j)_{j \in S}$, there is a row $k \in [30]$ such that*

$$m(k) \in S, \quad \operatorname{Re}(\overline{a_j} v_j^k) > c \quad (j \in S). \quad (30)$$

For every family $(a_j)_{j \in [4]}$ of unit complex numbers and every two-element set $L \subset [4]$, there is a row $k \in [30]$ satisfying either

$$\operatorname{Re}(\overline{a_j} v_j^k) > c \quad (j \in [4]), \quad (31)$$

or

$$q(k) \in L, \quad \operatorname{Re}(\overline{a_j} v_j^k) > c \quad (j \neq q(k)). \quad (32)$$

Proof. For a unit complex number a_j , the condition $\operatorname{Re}(\overline{a_j} z) > c$ defines an open arc on the unit circle of length $61/124$ turns. Since $61/124 > 15/31$, this arc contains fifteen consecutive points of the uniform grid with denominator 31. Indeed, the first grid point strictly after the left endpoint lies at most $1/31$ turns after it, so the fifteenth lies at most $15/31$ turns after it, strictly before the right endpoint. If the first residue is $b_j \in \{0, \dots, 30\}$, every retained row in

$$W(j, b_j) = \{k \in [30] : (2^{j-1}k - b_j) \bmod 31 < 15\}$$

satisfies the required strict inequality in coordinate j .

The integer verifier in Appendix B checks

$$\left(\bigcap_{j \in S} W(j, b_j) \right) \cap \{k : m(k) \in S\} \neq \emptyset \quad (33)$$

for every nonempty proper S and every choice of its block starts. These are exactly $4 \cdot 31 + 6 \cdot 31^2 + 4 \cdot 31^3 = 125054$ checks, and they prove (30).

For all $31^4 = 923521$ choices of four block starts, the verifier checks that either the full intersection is nonempty, or

$$\# \left\{ j \in [4] : \left(\bigcap_{i \neq j} W(i, b_i) \right) \cap \{k : q(k) = j\} \neq \emptyset \right\} \geq 3. \quad (34)$$

The first alternative proves (31). In the second alternative, the three or more eligible indices meet every pair L , which proves (32). There are 9341 empty full intersections; the cardinality in (34) is three in 1320 cases and four in 8021 cases. The displayed verifier exhausts these finite sets, uses integer arithmetic only, and returns the stated counts. The open-arc reduction proves the assertions for all continuous arguments, including endpoint configurations. $\square$

Lemma 19. *Suppose $K \subset \mathbb{C}^4$ is a normalized complex 1-unconditional convex body containing every coordinate disc. If $K \neq \mathbb{D}^4$, then $\operatorname{ill}(K) \leq 30$.*

Proof. If K contains a full coordinate tridisc, apply Theorem 9. Otherwise define

$$\gamma = \max_{i \in [4]} \|\mathbf{1} - e_i\|_K^{-1} < 1, \quad \delta = \frac{c(1-\gamma)}{4}, \quad \tau = \frac{c\delta(1-\gamma)}{2}. \quad (35)$$

Every coordinate triple of every $x \in K$ contains an index j with $|x_j| \leq \gamma$. To see this, if T is a coordinate triple and $r = \min_{j \in T} |x_j|$, monotonicity gives $r \|\mathbf{1}_T\|_K \leq \|x\|_K \leq 1$, hence $r \leq \gamma$. In particular, a point of full support has at least two coordinates of modulus at most γ .

Fix the thirty directions

$$d_j^k = -a_j^k v_j^k, \quad a_j^k = \begin{cases} 1, & j = m(k), \\ \tau, & j = q(k), \\ \delta, & j \notin \{m(k), q(k)\}. \end{cases} \quad (36)$$

Here $0 < \tau < \delta < 1$. We show that they illuminate K .

Fix $x \in \partial K$, put $r_j = |x_j|$, and let $u \neq 0$ be a supporting normal at x . With $b_j = |u_j|$ and $h = h_K(u)$, normal alignment and containment of the coordinate bidiscs give

$$h = \sum_{j \in [4]} b_j r_j > 0, \quad h \geq b_i + b_j \quad (i \neq j), \quad h \geq b_j. \quad (37)$$

At a nonzero coordinate, $u_j = b_j x_j / r_j$. Thus a row satisfying $\operatorname{Re}(\overline{x_j / r_j} v_j^k) > c$ contributes at most $-ca_j^k b_j$ to $[u, d^k]$. Any unchecked coordinate contributes at most $a_j^k b_j$. In each case below the row is chosen from x before the supporting normal is considered.

Suppose first that x has full support. Choose a pair L on which $r_j \leq \gamma$ and apply the second assertion of Lemma 18. A row satisfying (31) has negative pairing with every nonzero supporting normal. Otherwise put $q = q(k) \in L$. From $h \geq b_q$ we obtain

$$(1-\gamma)b_q \leq (1-r_q)b_q \leq \sum_{j \neq q} b_j r_j \leq \sum_{j \neq q} b_j.$$

All amplitudes outside q are at least δ , so

$$[u, d^k] \leq -c\delta \sum_{j \neq q} b_j + \tau b_q \leq -\frac{c\delta}{2} \sum_{j \neq q} b_j < 0.$$

The final sum cannot vanish: otherwise $u \neq 0$ would imply $h = b_q r_q < b_q$, contradicting (37).

Suppose x has exactly three nonzero coordinates. Let i be the missing coordinate, choose $t \in \operatorname{supp}(x)$ with $r_t \leq \gamma$, and put $R = \operatorname{supp}(x) \setminus \{t\}$. Apply (30) to R . Write $m = m(k) \in R$ and let l be the other index of R . The bidisc inequalities for $\{t, l\}$ and $\{i, l\}$ give

$$b_t(1-r_t) + b_l(1-r_l) \leq b_m r_m, \quad b_i \leq b_m r_m + b_t r_t - b_l(1-r_l).$$

Consequently

$$b_t \leq \frac{b_m}{1-\gamma}, \quad b_i \leq b_m + \frac{\gamma b_m}{1-\gamma} = \frac{b_m}{1-\gamma}.$$

The unchecked coordinates i, t have amplitude at most δ . It follows that

$$[u, d^k] \leq -cb_m - ca_l^k b_l + \delta(b_i + b_t) \leq -\frac{c}{2}b_m - ca_l^k b_l < 0.$$

For strictness, $b_m = b_l = 0$ would force $b_t = b_i = 0$ by the preceding estimates. The amplitude a_l^k may equal τ ; its positivity suffices.

Suppose x has exactly two nonzero coordinates. Apply (30) to their pair, and denote them by $m = m(k)$ and l . For either missing coordinate i , the bidisc inequality for $\{i, l\}$ implies

$$b_i \leq b_m r_m - b_l(1 - r_l) \leq b_m.$$

Therefore

$$[u, d^k] \leq -(c - 2\delta)b_m - ca_l^k b_l < 0.$$

Indeed, $c - 2\delta \geq c/2 > 0$, and $h > 0$ ensures that at least one of the active coefficients b_m, b_l is positive.

Finally, if x has only one nonzero coordinate j , then $r_j = 1$: the normalized axis section is the unit disc and its relative interior lies in $\text{int } K$. Now $h = b_j$, and the bidisc inequality for $\{i, j\}$ forces $b_i = 0$ for every $i \neq j$. The row supplied by (30) for the singleton $\{j\}$ has negative pairing with every nonzero supporting normal. There is no empty-support boundary point because $0 \in \text{int } K$.

In every case one of the fixed directions has negative pairing with all nonzero supporting normals at x . The strict normal criterion proves the claim. $\square$

5.5 Completion of the proof

Proof of Theorem 14. Suppose first that $K \neq \mathbb{D}^4$. If every coordinate bidisc is present, Lemma 19 applies. Otherwise let

$$\mathcal{E} = \{\{i, j\} \subset [4] : i \neq j, e_i + e_j \notin K\}.$$

If two members of $\mathcal{E}$ are disjoint, a coordinate permutation puts us in Proposition 15.

It remains that $\mathcal{E}$ is nonempty and its members pairwise intersect. Either all of them contain a common index, or they are exactly the three edges of a triangle. Indeed, if there are at least two edges, write two of them as $\{a, b\}$ and $\{a, c\}$. An edge avoiding a must be $\{b, c\}$. Once all three edges are present, every further edge would be disjoint from one of them. A family with only one edge has a common index as well.

In the common-index case, permute coordinates so that this index is 3 and one missing edge is $\{3, 4\}$. The bidiscs on $\{1, 2\}$, $\{1, 4\}$ and $\{2, 4\}$ are present, so Proposition 17 applies. In the triangle case, label its vertices 2, 3, 4. The bidiscs on $\{1, 3\}$ and $\{1, 4\}$ are present and the one on $\{3, 4\}$ is missing; Proposition 16 applies. These cases exhaust all possibilities and prove $\text{ill}(K) \leq 30$.

The normalized non-polydisc

$$H_4 = \mathbb{D}^2 \times \{(z_3, z_4) \in \mathbb{C}^2 : |z_3| + |z_4| \leq 1\}$$

has illumination number 30 by the sharpness argument in Theorem 9. Thus the bound is sharp. The exact polydisc theorem gives $\text{ill}(\mathbb{D}^4) = 31$. Finally, positive diagonal rescaling preserves illumination numbers and normalizes every complex 1-unconditional body, which proves the last assertion. $\square$

6 1-symmetric bodies in dimension three

A complex 1-unconditional body is *1-symmetric* if it is also invariant under every coordinate permutation. The bound $\text{ill}(K) \leq 14$ for non-polydiscs in this class follows from [2, Theorem 1.5]. The following construction improves it. It uses the reduction underlying [2, Proposition 2.8]; we include a complete geometric proof using supporting normals.

Theorem 20. *Every complex 1-symmetric convex body $K \subset \mathbb{C}^3$ which is not a complex affine image of $\mathbb{D}^3$ satisfies*

$$\text{ill}(K) \leq 12.$$

Proof. Put $\zeta = \exp(2\pi i/13)$ and define

$$v^r = (\zeta^r, \zeta^{3r}, \zeta^{4r}) \quad (r \in [12]), \quad (f(1), \dots, f(12)) = (2, 1, 1, 3, 1, 2, 2, 1, 3, 1, 3, 2). \quad (38)$$

We first verify that, with $\beta = \sin(\pi/26) > 0$, for every nonempty proper $S \subset [3]$ and every $(z_j)_{j \in S}$ with $|z_j| = 1$, there is $r \in [12]$ such that

$$f(r) \in S, \quad \operatorname{Re}(\overline{z_j} v_j^r) \leq -\beta \quad (j \in S). \quad (39)$$

All residues in the following verification are taken modulo 13. Set $(a_1, a_2, a_3) = (1, 3, 4)$ and $B_s = \{s, s+1, \dots, s+5\}$. A nonempty residue set meets every B_s if and only if its largest cyclic gap is at most six: a gap of length g leaves exactly $g-1$ consecutive missing residues. For singleton supports, the sets $\{a_j r : f(r) = j\}$ and their largest gaps are

j	$\{a_j r : f(r) = j\}$	largest gap
1	$\{2, 3, 5, 8, 10\}$	5
2	$\{3, 5, 8, 10\}$	6
3	$\{3, 5, 10\}$	6

For $i < j$, define

$$R_{ij}(s) = \{a_j r : a_i r \in B_s, f(r) \in \{i, j\}, r \in [12]\}.$$

Their largest cyclic gaps, calculated directly from (38), are

$(i, j) \setminus s$	0	1	2	3	4	5	6	7	8	9	10	11	12
(1, 2)	6	6	6	4	4	4	6	6	6	6	6	6	6
(1, 3)	4	4	4	5	6	5	4	4	4	5	6	6	5
(2, 3)	6	6	6	6	6	4	6	6	6	6	6	6	6

Every entry is at most six. Thus for any prescribed blocks B_{s_j} , $j \in S$, a row satisfies $f(r) \in S$ and $a_j r \in B_{s_j}$ for every $j \in S$. This finite assertion comprises $3 \cdot 13 + 3 \cdot 13^2 = 546$ cases; the accompanying verifier checks all of them with integer arithmetic.

For each unit z , the closed arc $\{w : |w| = 1, \operatorname{Re}(\overline{z}w) \leq -\beta\}$ has angular length $12\pi/13$, exactly six spacings between successive thirteenth roots of unity. It therefore contains six consecutive roots. Applying the finite assertion to one such block in each coordinate proves (39), including all endpoint cases.

Scale K so that $e_j \in \partial K$ for $j \in [3]$. This uses one common factor, since K is 1-symmetric. Then $K \subset \mathbb{D}^3$ and there is a unique $\rho > 0$ with $\rho(1, 1, 1) \in \partial K$. Coordinate contractions give $\rho\mathbb{D}^3 \subset K$, so $0 < \rho < 1$: equality would give $K = \mathbb{D}^3$. Fix

$$\Delta = \frac{\beta}{6}, \quad \delta = \min \left\{ \frac{\Delta}{2}, \frac{\beta\Delta(1-\rho)}{2\rho} \right\}. \quad (40)$$

For every $r \in [12]$ define a direction by

$$d_j^r = \begin{cases} v_j^r, & j = f(r), \\ \delta v_j^r, & j = 3 \neq f(r), \\ \Delta v_j^r, & j \neq f(r), j \neq 3. \end{cases} \quad (41)$$

These twelve nonzero directions are fixed before choosing a boundary point.

Let $x \in \partial K$. For any nonzero supporting normal u at x , write $b_j = |u_j|$ and $t_j = |x_j|$. Maximizing independently over coordinate rotations shows that

$$u_j = b_j \frac{x_j}{|x_j|} \quad (x_j \neq 0), \quad h_K(u) = \operatorname{Re} \sum_{j=1}^3 \overline{u_j} x_j = \sum_{j=1}^3 b_j t_j > 0.$$

There is no restriction on the argument of u_j when $x_j = 0$. Swapping any two coordinate moduli of x and choosing their arguments to maximize the same functional gives

$$(b_i - b_j)(t_i - t_j) \geq 0. \quad (42)$$

Indeed, the swapped point belongs to K , and subtracting its functional value from $h_K(u)$ gives the displayed inequality.

Suppose first that $M_x \neq [3]$. Apply (39) with $S = M_x$ and $z_j = x_j/|x_j|$, and put $k = f(r) \in M_x$ for the chosen row. Write $\alpha_j = |d_j^r|$. Then $\alpha_k = 1$ and $0 < \alpha_j \leq \Delta$ for $j \neq k$. For $j \notin M_x$, (42) gives $b_j \leq b_k$, including when $x_j = 0$. Consequently

$$\begin{aligned} \operatorname{Re} \sum_{j=1}^3 \overline{u_j} d_j^r &\leq -\beta b_k - \beta \sum_{j \in M_x \setminus \{k\}} \alpha_j b_j + \Delta \sum_{j \notin M_x} b_j \\ &\leq -(\beta - 3\Delta) b_k - \beta \sum_{j \in M_x \setminus \{k\}} \alpha_j b_j < 0. \end{aligned}$$

For the strict inequality, $\beta - 3\Delta = \beta/2 > 0$. If $b_k = 0$, all coefficients outside M_x vanish, and some coefficient on $M_x \setminus \{k\}$ is positive because $u \neq 0$. The row was chosen using x alone, so this holds for every supporting normal at x .

If $M_x = [3]$, all coordinate moduli equal ρ . Apply (39) with $S = \{1, 2\}$ and $z_j = x_j/|x_j|$ for $j \in \{1, 2\}$. Then $k = f(r) \in \{1, 2\}$, $\alpha_1, \alpha_2 \geq \Delta$, and $\alpha_3 = \delta$. The third unit coordinate disc is contained in K , hence

$$b_3 \leq h_K(u) = \rho(b_1 + b_2 + b_3), \quad b_3 \leq \frac{\rho}{1 - \rho}(b_1 + b_2).$$

In particular $b_1 + b_2 > 0$. By (40),

$$\operatorname{Re} \sum_{j=1}^3 \overline{u_j} d_j^r \leq -\beta \Delta(b_1 + b_2) + \delta b_3 \leq -\frac{\beta \Delta}{2}(b_1 + b_2) < 0.$$

Again the chosen row works for every supporting normal. The normal criterion (1) now shows that the directions (41) illuminate K . $\square$

Thus the bound $2^{n+1} - 2$ is not sharp in the 1-symmetric class when $n = 3$. It is sharp when $n = 2$, since the complex ℓ_1 unit ball has illumination number six. In dimension three that ball has illumination number nine by Theorem 21, so the largest illumination number among non-polydisc 1-symmetric bodies lies between nine and twelve. The construction does not establish that twelve is sharp, or settle sharpness in higher dimensions.

7 Complex Hanner bodies

A *complex Hanner body* is obtained recursively from the unit disc $\mathbb{D} \subset \mathbb{C}$ by taking Cartesian products and ℓ_1 sums on disjoint coordinate blocks. For bodies A and B with Minkowski norms $\|\cdot\|_A$ and $\|\cdot\|_B$, these operations are

$$\begin{aligned} A \times B &= \{(x, y) : \max\{\|x\|_A, \|y\|_B\} \leq 1\}, \\ A \oplus_1 B &= \{(x, y) : \|x\|_A + \|y\|_B \leq 1\}. \end{aligned}$$

Every resulting body $H \subset \mathbb{C}^n$ is normalized and complex 1-unconditional. Write $\operatorname{ext} H$ for its extreme points and define

$$\mathcal{S}(H) = \{\operatorname{supp}(x) : x \in \operatorname{ext} H\}, \quad F_H(t) = \sum_{S \in \mathcal{S}(H)} t^{|S|}.$$

These definitions depend on H , not on a choice of recursive description. For a nonempty $S \subseteq [n]$, put

$$E_S = \{z \in \mathbb{C}^n : z_j = 0 \text{ for } j \notin S\},$$

$$\mathbb{D}^S = \{z \in E_S : |z_j| \leq 1 \text{ for } j \in S\},$$

and let

$$T_S = \{z \in E_S : |z_j| = 1 \text{ for } j \in S\}$$

be its coordinate torus.

Theorem 21. *For every complex Hanner body H ,*

$$\text{ill}(H) = 2F_H(2) - F_H(1), \quad \text{ill}^*(H) = F_H(2). \quad (43)$$

The polynomial is computed recursively by

$$F_{\mathbb{D}}(t) = t, \quad F_{A \times B}(t) = F_A(t)F_B(t), \quad F_{A \oplus_1 B}(t) = F_A(t) + F_B(t). \quad (44)$$

We first describe the extreme points and explain why directions illuminating different tori must be counted separately.

Lemma 22. *For a complex Hanner body H ,*

$$\text{ext } H = \bigsqcup_{S \in \mathcal{S}(H)} T_S, \quad H = \text{conv}(\text{ext } H).$$

Every point of H is a finite convex combination of these extreme points. Moreover, $H \cap E_S = \mathbb{D}^S$ for every $S \in \mathcal{S}(H)$, and (44) holds.

Proof. The extreme points of a disc are its unit circle, and each point of the disc is a convex combination of two circle points. We prove the remaining assertions inductively, using

$$\text{ext}(A \times B) = \text{ext } A \times \text{ext } B, \quad (45)$$

$$\text{ext}(A \oplus_1 B) = (\text{ext } A \times \{0\}) \cup (\{0\} \times \text{ext } B). \quad (46)$$

For (45), if one coordinate block is not extreme, a nontrivial convex decomposition in that block gives one in the product. Conversely, a convex decomposition of a point whose two blocks are extreme must be constant in each block.

For (46), an extreme point lies on the boundary, where $\|x\|_A + \|y\|_B = 1$. If both blocks are nonzero, the identity

$$(x, y) = \|x\|_A \left(\frac{x}{\|x\|_A}, 0 \right) + \|y\|_B \left(0, \frac{y}{\|y\|_B} \right)$$

is a nontrivial convex decomposition. Thus an extreme point has one block zero, and its other block must be extreme in the corresponding factor. Conversely, suppose $x \in \text{ext } A$ and $(x, 0)$ is a convex combination, with both weights positive, of $(x^1, y^1), (x^2, y^2) \in A \oplus_1 B$. Since $x^1, x^2 \in A$, extremality gives $x^1 = x^2 = x$. Now $\|x\|_A = 1$, so the sum constraint forces $y^1 = y^2 = 0$. This proves extremality of $(x, 0)$; the other branch is identical.

Finite convex representations also persist under both operations. For a product, multiply the coefficients in finite convex representations of its two blocks. For a sum, use

$$A \oplus_1 B = \text{conv}((A \times \{0\}) \cup (\{0\} \times B))$$

and then the representations in the factors. The displayed identity follows directly from the norm inequality: for a point with $\alpha = \|x\|_A$, $\beta = \|y\|_B$ and $\alpha + \beta \leq 1$, use weights $\alpha, \beta, 1 - \alpha - \beta$

on the two normalized blocks and zero, omitting zero weights. The reverse inclusion is convexity. The induction therefore proves the finite convex-hull assertion as well as the extreme-point identities.

At a product, the supports are precisely $S \cup R$ with $S \in \mathcal{S}(A)$ and $R \in \mathcal{S}(B)$; their intersections with the two coordinate blocks recover S and R uniquely. At a sum, the support families are united without repetitions, since their nonempty sets lie in disjoint blocks. The two identities above now give the asserted disjoint union of tori and the polynomial recursion. The supports themselves may intersect; the tori for distinct supports are disjoint.

Finally, $T_S \subset H$ implies $\mathbb{D}^S = \text{conv} T_S \subset H \cap E_S$. The convex-hull identity follows by taking products of finite circle representations in the coordinates of S . The reverse inclusion follows from $H \subseteq \mathbb{D}^n$. $\square$

Lemma 23. *If $S, R \in \mathcal{S}(H)$ are distinct, no direction illuminates both a point of T_S and a point of T_R .*

Proof. We induct on the recursive construction. The assertion is vacuous for a disc. For $H = A \times B$, a direction illuminating (x, y) has projections illuminating x in A and y in B , since interior membership in the product requires interior membership in both factors. Distinct supports differ in at least one factor, where the induction hypothesis rules out a common direction.

For $H = A \oplus_1 B$, a direction $d = (v, w)$ illuminating $(x, 0)$, with $\|x\|_A = 1$, satisfies for some $s > 0$

$$1 > \|x + sv\|_A + s\|w\|_B \geq 1 - s\|v\|_A + s\|w\|_B.$$

Thus $\|v\|_A > \|w\|_B$. Illuminating a point $(0, y)$ in the other branch requires $\|w\|_B > \|v\|_A$, which is impossible for the same direction. If both tori lie in the same branch, projection of an illuminating direction illuminates the corresponding points of that factor. The induction hypothesis again applies. $\square$

Proof of Theorem 21. The recursion follows from Lemma 22. We first justify the reduction to extreme points for both illumination numbers. For $x \in \partial H$, let

$$\mathcal{I}_H(x) = \{d \in \mathbb{C}^n \setminus \{0\} : d \text{ illuminates } x\}.$$

By Lemma 22, write $x = \sum_{k \in [M]} \lambda_k y^k$, where $y^k \in \text{ext } H$, $\lambda_k > 0$ and $\sum_k \lambda_k = 1$. Fix one index k before choosing a direction. If $y^k + sd \in \text{int } H$, then

$$x + \lambda_k sd = \lambda_k (y^k + sd) + \sum_{h \neq k} \lambda_h y^h \in \text{int } H.$$

Indeed, its H -norm is at most $\lambda_k \|y^k + sd\|_H + 1 - \lambda_k < 1$. Consequently

$$\mathcal{I}_H(y^k) \subseteq \mathcal{I}_H(x). \tag{47}$$

A set illuminating every extreme point therefore illuminates the boundary. Likewise, a measure assigning mass at least one to $\mathcal{I}_H(y)$ for each extreme point y assigns mass at least one to $\mathcal{I}_H(x)$ for every boundary point x . No choice of the extreme point depending on the direction is needed.

For the ordinary lower bound, take any finite illuminating set D and, for each $S \in \mathcal{S}(H)$, let D_S consist of its directions illuminating at least one point of T_S . These groups are pairwise disjoint by Lemma 23. If d illuminates $x \in T_S$, then $H \subseteq \mathbb{D}^n$ implies $|x_j + sd_j| < 1$ for $j \in S$ at some positive step. Expanding the square gives

$$\text{Re}(\overline{x_j} d_j) < 0 \quad (j \in S). \tag{48}$$

Thus the projections of D_S illuminate every point of T_S in the polydisc $\mathbb{D}^S$. They also illuminate its entire boundary by the finite convex-combination argument above, applied to that polydisc. Every such projection is nonzero; even if some projections coincide, their number is at most $|D_S|$. The exact polydisc theorem [1, Theorem 1.4] therefore gives

$$|D| \geq \sum_{S \in \mathcal{S}(H)} |D_S| \geq \sum_{S \in \mathcal{S}(H)} (2^{|S|+1} - 1).$$

For the matching upper bound, embed an optimal illuminating set of $\mathbb{D}^S$ in E_S for each $S \in \mathcal{S}(H)$, with zero coordinates outside S . It has $2^{|S|+1} - 1$ directions. Lemma 22 gives $H \cap E_S = \mathbb{D}^S$, and Lemma 1(2) identifies the relative interior of this section with $E_S \cap \text{int } H$. The embedded directions therefore illuminate T_S in H . Their union illuminates all extreme points, and hence all of ∂H by (47). This proves

$$\text{ill}(H) = \sum_{S \in \mathcal{S}(H)} (2^{|S|+1} - 1) = 2F_H(2) - F_H(1).$$

For the fractional lower bound, define the direction sets

$$R_S = \bigcup_{x \in T_S} \mathcal{I}_H(x) \quad (S \in \mathcal{S}(H)).$$

They are open and pairwise disjoint. Openness follows because, for each fixed positive step, the condition $x + sd \in \text{int } H$ is open jointly in (x, d) ; taking the union over steps gives an open illumination relation. Its sections $\mathcal{I}_H(x)$ and their unions R_S are open. Disjointness is Lemma 23.

Let σ_S be Haar probability measure on T_S , that is, the product of the uniform probability measures on its coordinate circles. For any fixed direction d , the necessary conditions (48) hold on a set of σ_S -measure $2^{-|S|}$ if all d_j , $j \in S$, are nonzero, and on an empty set otherwise. Each nonzero coordinate imposes one open semicircle, of measure $1/2$. It follows that

$$\sigma_S\{x \in T_S : d \in \mathcal{I}_H(x)\} \leq 2^{-|S|} \mathbf{1}_{R_S}(d).$$

Here $\mathbf{1}_{R_S}$ is the indicator function of R_S . For a fractional illuminating measure μ , the open relation is Borel measurable, so Tonelli's theorem yields

$$\begin{aligned} 1 &\leq \int_{T_S} \mu(\mathcal{I}_H(x)) d\sigma_S(x) \\ &= \int_{\mathbb{C}^n \setminus \{0\}} \sigma_S\{x \in T_S : d \in \mathcal{I}_H(x)\} d\mu(d) \\ &\leq 2^{-|S|} \mu(R_S). \end{aligned}$$

Summing over the disjoint R_S gives $\mu(\mathbb{C}^n \setminus \{0\}) \geq \sum_S 2^{|S|} = F_H(2)$.

For the upper bound, view each σ_S instead as a probability measure on directions in E_S , still with unit moduli on S and zeros outside S , and put

$$\mu = \sum_{S \in \mathcal{S}(H)} 2^{|S|} \sigma_S.$$

All these directions are nonzero. At any $x \in T_S$, independent uniform coordinates satisfy (48) with probability exactly $2^{-|S|}$. For directions supported on S , these inequalities are sufficient: a sufficiently small positive step enters the relative interior of $\mathbb{D}^S$, hence the interior of H . Therefore $2^{|S|} \sigma_S(\mathcal{I}_H(x)) = 1$. The measure μ illuminates every extreme point and, by (47), every boundary point. Its total mass is $F_H(2)$, proving the fractional formula. $\square$

Corollary 24. *Let $H \subset \mathbb{C}^n$ be a complex Hanner body, and put*

$$B = \mathbb{D} \oplus_1 \mathbb{D} = \{(z, w) \in \mathbb{C}^2 : |z| + |w| \leq 1\}.$$

Then $\text{ill}^*(H) \leq 2^n$, with equality precisely when, after a coordinate permutation,

$$H = \mathbb{D}^p \times B^r, \quad p, r \in \mathbb{Z}_{\geq 0}, \quad p + 2r = n.$$

Factors with exponent zero are omitted. For these bodies,

$$F_H(t) = 2^r t^{p+r}, \quad \text{ill}(H) = 2^{n+1} - 2^r. \quad (49)$$

Moreover, $\text{ill}(H) \leq 2^{n+1} - 1$, with equality precisely for $H = \mathbb{D}^n$. For $n \geq 2$ and $H \neq \mathbb{D}^n$,

$$\text{ill}(H) \leq 2^{n+1} - 2,$$

with equality precisely for coordinate permutations of $\mathbb{D}^{n-2} \times B$.

Proof. We prove $F_H(2) \leq 2^n$ by induction, retaining the equality conditions. It holds for a disc. A product of factors A_1, A_2 of dimensions $a, b \geq 1$ satisfies $F_{A_1 \times A_2}(2) = F_{A_1}(2)F_{A_2}(2) \leq 2^{a+b}$, with equality if and only if both factors attain equality. At a sum,

$$F_{A_1 \oplus_1 A_2}(2) = F_{A_1}(2) + F_{A_2}(2) \leq 2^a + 2^b \leq 2^{a+b}.$$

The last inequality is equivalent to $2^{-a} + 2^{-b} \leq 1$. Both terms are at most $1/2$, so equality occurs exactly when $a = b = 1$. In that case the factors are discs and their sum is B . Inductively, equality is possible exactly for products of discs and copies of B , up to coordinate permutation. Conversely, every such product attains equality. The recursion gives $F_B(t) = 2t$ and then (49).

Let $c_H = F_H(1) = |\mathcal{S}(H)|$. Its recursion multiplies at a product and adds at a sum, starting from one for a disc. Thus $c_H = 1$ if and only if $H = \mathbb{D}^n$: a sum has at least two supports, while a product has one only when both factors do. The exact formula therefore gives

$$\text{ill}(H) = 2F_H(2) - c_H \leq 2^{n+1} - c_H.$$

This proves the ordinary bound and its polydisc equality case. If $H \neq \mathbb{D}^n$, then $c_H \geq 2$, proving the stricter bound. Equality in that bound implies

$$2(2^n - F_H(2)) + c_H = 2.$$

The first term is nonnegative and $c_H \geq 2$, so $F_H(2) = 2^n$ and $c_H = 2$. The classification already proved gives $H = \mathbb{D}^p \times B^r$, after a coordinate permutation, with $c_H = 2^r$. Consequently $r = 1$ and $p = n - 2$. Conversely, this body attains the stated value by (49). $\square$

For example, the complex ℓ_1 unit ball, obtained as the ℓ_1 sum of n discs, has $F_H(t) = nt$ and hence $\text{ill}(H) = 3n$ and $\text{ill}^*(H) = 2n$. In dimension four, $B \times B$ has illumination numbers 28 and 16, whereas $\mathbb{D}^2 \times B$ has 30 and 16. All numerical conclusions are preserved by invertible complex linear maps: directions are mapped bijectively and illuminating measures are pushed forward with unchanged total mass. The coordinate descriptions in Corollary 24 refer to the recursively defined class above; for its linear images they become statements of complex linear equivalence.

8 Extreme points with no zero coordinates

The condition considered here is the complex counterpart of the condition in Sun and Vritsiou's Theorem 5 [3]. Extreme points are understood in the underlying real vector space.

Theorem 25. *Let $K \subset \mathbb{C}^n$ be a normalized complex 1-unconditional convex body such that*

$$x \in \text{ext } K \implies \text{supp}(x) = [n]. \quad (50)$$

Then

$$\text{ill}(K) \leq 2^{n+1} - 1, \quad \text{ill}^*(K) \leq 2^n.$$

If $K \neq \mathbb{D}^n$, put

$$\gamma = \max\{s \geq 0 : s(1, \dots, 1) \in K\} = \|(1, \dots, 1)\|_K^{-1}.$$

Then $0 < \gamma < 1$ and

$$\text{ill}^*(K) \leq \left(2^{-n} + \frac{\arcsin((1-\gamma)/2)}{\pi 3^{n-1}}\right)^{-1} < 2^n. \quad (51)$$

Consequently, within the normalized class defined by (50), equality $\text{ill}^*(K) = 2^n$ holds exactly for $K = \mathbb{D}^n$.

Proof. We first record the reduction to extreme points for both assertions. Every boundary point y is a finite convex combination of extreme points. Choose one summand x with positive coefficient λ . If $\lambda < 1$, write $y = \lambda x + (1-\lambda)z$ with $z \in K$. Whenever $x + td \in \text{int } K$, convexity gives

$$y + \lambda td = \lambda(x + td) + (1-\lambda)z \in \text{int } K.$$

The case $\lambda = 1$ is immediate. Thus every direction illuminating x illuminates y . In particular, a family or measure illuminating every extreme point illuminates the whole boundary.

Set $N = 2^{n+1} - 1$ and $\zeta = \exp(2\pi i/N)$. The cyclic construction for the polydisc [1, Proposition 2.7] gives the directions

$$d^k = -(\zeta^k, \zeta^{2k}, \dots, \zeta^{2^{n-1}k}), \quad 0 \leq k < N,$$

with the following property: for every $\xi \in \mathbb{T}^n$, where $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$, some k satisfies

$$\text{Re}(\overline{\xi_j} d_j^k) < 0 \quad (j \in [n]).$$

Fix an extreme point x and write $x_j = r_j \xi_j$, where $r_j > 0$ and $\xi_j \in \mathbb{T}$. Every nonzero supporting normal at x has the form $u_j = b_j \xi_j$, with $b_j \geq 0$. The chosen direction therefore satisfies

$$[u, d^k] = \sum_{j \in [n]} b_j \text{Re}(\overline{\xi_j} d_j^k) < 0$$

for every such normal. The normal criterion proves the ordinary bound, followed by the reduction above.

Let σ be the product of normalized arc-length measures on $\mathbb{T}^n$. For the same extreme point x , the set

$$A_x = \{d \in \mathbb{T}^n : \text{Re}(\overline{\xi_j} d_j) < 0 \text{ for every } j \in [n]\}$$

consists of illuminating directions and has measure 2^{-n} . Thus $2^n \sigma$ fractionally illuminates K .

Assume now that $K \neq \mathbb{D}^n$. Since $0 \in \text{int } K$ and $K \subseteq \mathbb{D}^n$, we have $0 < \gamma \leq 1$. If $\gamma = 1$, coordinate rotations and contractions of $(1, \dots, 1) \in K$ give $\mathbb{D}^n \subseteq K$, a contradiction. For every $x \in K$, coordinatewise monotonicity also gives $\min_j |x_j| \leq \gamma$.

For each extreme point x , choose j_0 with $r_{j_0} \leq \gamma$. This choice depends only on x , not on a supporting normal. Since the unit coordinate discs lie in K , every nonzero supporting normal at x satisfies

$$b_{j_0} \leq \sum_{j \in [n]} r_j b_j, \quad (1 - \gamma)b_{j_0} \leq \sum_{j \neq j_0} b_j =: S.$$

Here $S > 0$: if $S = 0$, then $\gamma < 1$ forces $b_{j_0} = 0$, contrary to $u \neq 0$. Define

$$B_x = \left\{ d \in \mathbb{T}^n : \begin{array}{l} 0 < \operatorname{Re}(\overline{\xi_{j_0}} d_{j_0}) < (1 - \gamma)/2, \\ \operatorname{Re}(\overline{\xi_j} d_j) < -1/2 \quad (j \neq j_0) \end{array} \right\}.$$

Every direction in B_x satisfies, for every nonzero supporting normal at x ,

$$[u, d] < \frac{(1 - \gamma)b_{j_0} - S}{2} \leq 0.$$

Strictness follows from $S > 0$, including when $b_{j_0} = 0$. Thus B_x also consists of illuminating directions. It is disjoint from A_x , and independence of the coordinates gives

$$\sigma(B_x) = \frac{\arcsin((1 - \gamma)/2)}{\pi} \left(\frac{1}{3} \right)^{n-1}.$$

Indeed, $\cos \theta < -1/2$ occupies one third of the circle, and $0 < \cos \theta < a$ occupies the proportion $\arcsin(a)/\pi$. This lower bound for the measure of illuminating directions is uniform over all extreme points and therefore over the entire boundary, by the reduction at the start of the proof. The illumination relation is open in (x, d) , so all its direction sections are Borel. Scaling σ by the reciprocal of $\sigma(A_x) + \sigma(B_x)$ proves (51). Finally, $\operatorname{ill}^*(\mathbb{D}^n) = 2^n$ follows from Theorem 21, completing the equality assertion. $\square$

The ordinary bound is also the full-support specialization of the earlier construction by coordinate supports. The fractional estimate does not give the strict ordinary inequality $\operatorname{ill}(K) \leq 2^{n+1} - 2$; that assertion remains unresolved here for the general class (50).

9 An improvement for real 1-symmetric bodies

In this section, a real 1-symmetric convex body is invariant under all coordinate sign changes and permutations. The proper-support condition in [2, Proposition 2.9] admits smaller families than the $3 \cdot 2^{n-2} + 1$ pairs constructed there.

Theorem 26. *Let $n \geq 3$, and let $K \subset \mathbb{R}^n$ be a real 1-symmetric convex body that is not an affine image of $[-1, 1]^n$. Then*

$$\operatorname{ill}(K) \leq \begin{cases} 6, & n = 3, \\ 12, & n = 4, \\ 22, & n = 5, \\ 5 \cdot 2^{n-3}, & n \geq 6. \end{cases}$$

In particular, $\operatorname{ill}(K) \leq 3 \cdot 2^{n-2}$ for every $n \geq 3$, so the bound $3 \cdot 2^{n-2} + 1$ is not sharp in any dimension $n \geq 4$.

We first give the geometric implication, including a common choice of the magnitudes of all directions.

Lemma 27. Let $n \geq 2$ and let $\mathcal{B} \subset \{-1, 1\}^n \times [n]$ be finite. Suppose that for every nonempty proper $S \subset [n]$ and every $a \in \{-1, 1\}^S$ there is $(b, k) \in \mathcal{B}$ with

$$k \in S, \quad b_j = -a_j \quad (j \in S). \quad (52)$$

If $K \subset \mathbb{R}^n$ is 1-symmetric and is not an affine cube, then $\text{ill}(K) \leq |\mathcal{B}|$.

Proof. Scale K by one common factor so that $e_j \in \partial K$ for every $j \in [n]$. Coordinate contractions follow from convexity and sign invariance. Thus $K \subset [-1, 1]^n$, and there is a unique $\rho > 0$ with $\rho(1, \dots, 1) \in \partial K$. Moreover, $\rho[-1, 1]^n \subset K$, whence $0 < \rho < 1$; equality would give $K = [-1, 1]^n$. Set

$$\Delta = \frac{1}{2n}, \quad \delta = \min \left\{ \frac{\Delta}{2}, \frac{\Delta(1-\rho)}{2\rho} \right\} > 0. \quad (53)$$

For every $(b, k) \in \mathcal{B}$ define $d^{b,k} \in \mathbb{R}^n$ by

$$d_j^{b,k} = \begin{cases} b_j, & j = k, \\ \delta b_j, & j = n \neq k, \\ \Delta b_j, & j \neq k, j \neq n. \end{cases} \quad (54)$$

These directions are fixed before choosing a boundary point.

Fix $x \in \partial K$. For a nonzero supporting normal u at x , put $c_j = |u_j|$ and $t_j = |x_j|$. Maximizing over the coordinate sign changes gives

$$u_j = c_j \text{sign}(x_j) \quad (x_j \neq 0), \quad h_K(u) = \sum_{j=1}^n c_j t_j > 0.$$

At a zero coordinate of x , the sign of u_j is unrestricted. Exchanging the moduli of coordinates i, j of x and then choosing signs that maximize the same functional gives

$$(c_i - c_j)(t_i - t_j) \geq 0. \quad (55)$$

Indeed, the exchanged point belongs to K , and subtracting its functional value from $h_K(u)$ gives this inequality.

Suppose $M_x \neq [n]$. Choose (b, k) using (52) with $S = M_x$ and $a_j = \text{sign}(x_j)$. Write $\alpha_j = |d_j^{b,k}|$. Then $\alpha_k = 1$, $0 < \alpha_j \leq \Delta$ for $j \neq k$, and (55) gives $c_j \leq c_k$ for $j \notin M_x$, including the zero coordinates of x . Consequently

$$\begin{aligned} \langle u, d^{b,k} \rangle &\leq -c_k - \sum_{j \in M_x \setminus \{k\}} \alpha_j c_j + \Delta \sum_{j \notin M_x} c_j \\ &\leq -(1 - n\Delta)c_k - \sum_{j \in M_x \setminus \{k\}} \alpha_j c_j < 0. \end{aligned}$$

For strictness, $1 - n\Delta = 1/2$. If $c_k = 0$, every coefficient outside M_x vanishes, and some coefficient on $M_x \setminus \{k\}$ is positive because $u \neq 0$.

If $M_x = [n]$, then every $t_j = \rho$. Choose (b, k) using (52) with $S = [n-1]$ and $a_j = \text{sign}(x_j)$. Thus $k \neq n$, $\alpha_j \geq \Delta$ for $j < n$, and $\alpha_n = \delta$. Since $[-e_n, e_n] \subset K$,

$$c_n \leq h_K(u) = \rho \sum_{j=1}^n c_j, \quad c_n \leq \frac{\rho}{1-\rho} \sum_{j=1}^{n-1} c_j.$$

The sum on the right is positive. Hence

$$\langle u, d^{b,k} \rangle \leq -\Delta \sum_{j=1}^{n-1} c_j + \delta c_n \leq -\frac{\Delta}{2} \sum_{j=1}^{n-1} c_j < 0.$$

In both cases the pair (b, k) depends on x alone, and the inequality holds for every nonzero supporting normal at x . The supporting-normal criterion for illumination therefore shows that the directions (54) illuminate K . $\square$

Let Q_n denote the graph with vertex set $\{0, 1\}^n$, in which two vertices are adjacent when exactly one coordinate differs. We write $v \oplus e_j$ for the vertex obtained from v by changing coordinate j . A vertex cover is a set meeting every edge; equivalently, its complement is independent.

Lemma 28. *Let $A \subset \{0, 1\}^n$ be a vertex cover, and let $T : A \rightarrow A$ satisfy*

$$T(v) = v \oplus e_{k(v)} \quad (v \in A)$$

for some $k(v) \in [n]$. Suppose every directed cycle of T uses every coordinate direction. Then the family

$$\mathcal{B}(A, T) = \{((2v_j - 1)_{j \in [n]}, k(v)) : v \in A\}$$

satisfies (52) and has cardinality $|A|$.

Proof. Fix S and a as in (52). The vertices satisfying $2v_j - 1 = -a_j$ for every $j \in S$ form a proper face F of positive dimension. It contains an edge, so $A \cap F \neq \emptyset$. If $k(v) \notin S$ for every $v \in A \cap F$, then $T(A \cap F) \subset A \cap F$. Iterating T from any vertex of this finite nonempty set would give a directed cycle contained in F . Such a cycle cannot use any coordinate in S , a contradiction. Thus some $v \in A \cap F$ has $k(v) \in S$, which proves the assertion. $\square$

Lemma 29. *Suppose $A \subset \{0, 1\}^n$ is a vertex cover and $T : A \rightarrow A$ sends every vertex to an adjacent vertex. Suppose T has exactly one directed cycle, and that cycle uses all n coordinates. Then there is a vertex cover $A' \subset \{0, 1\}^{n+1}$ of cardinality $2|A|$ and an adjacent-vertex map $T' : A' \rightarrow A'$ with exactly one directed cycle, using all $n + 1$ coordinates.*

Proof. Choose an edge $a \rightarrow b$ of the directed cycle with $b = a \oplus e_1$. Define

$$A' = (A \times \{0\}) \cup ((A \oplus e_1) \times \{1\}), \quad A \oplus e_1 = \{v \oplus e_1 : v \in A\}. \quad (56)$$

Writing $I = \{0, 1\}^n \setminus A$, its complement is $(I \times \{0\}) \cup ((I \oplus e_1) \times \{1\})$. Both layers are independent. A vertical edge in the complement would require $v \in I \cap (I \oplus e_1)$, contradicting the independence of I . Thus A' is a vertex cover, and $|A'| = 2|A|$.

The complete successor assignments are

$$\begin{aligned} T'(v, 0) &= (T(v), 0) & (v \in A \setminus \{a\}), \\ T'(v \oplus e_1, 1) &= (T(v) \oplus e_1, 1) & (v \in A \setminus \{a\}), \\ T'(a, 0) &= (a, 1), & T'(b, 1) = (b, 0). \end{aligned} \quad (57)$$

The two exceptional targets belong to A' because $a, b \in A$ and $a = b \oplus e_1$. All assigned successors are adjacent. Before the two exceptions, the graph consists of two copies of the original directed cycle with their incoming trees. The exceptions remove the edges $(a, 0) \rightarrow (b, 0)$ and $(b, 1) \rightarrow (a, 1)$ and join the resulting two directed paths into one cycle. All other vertices still reach this cycle. The new cycle contains both copies of every original cycle vertex and uses a vertical edge. It takes both values in each coordinate, and therefore uses every coordinate direction. This also supplies an edge in coordinate 1 for the next application. $\square$

Proof of Theorem 26. For the initial sets below, identify $v \in \{0, 1\}^m$ with the integer $\sum_{j=1}^m 2^{j-1} v_j$. Thus integers differing by exclusive-or with 2^{j-1} are adjacent in coordinate j . The weight of a vertex is the sum of its binary coordinates.

In dimension three, take $A_3 = \{1, 2, 3, 4, 5, 6\}$ and let T_3 follow the cyclic list

$$(1, 3, 2, 6, 4, 5).$$

The two missing vertices are 0 and 7, so A_3 is a vertex cover. Consecutive vertices, including the last and first, are adjacent, and the cycle uses all three coordinates. Repeatedly applying Lemma 29 gives $3 \cdot 2^{n-2}$ vertices for every $n \geq 3$.

For dimension five, let O_5 be the set of all odd-weight vertices, and define

$$\begin{aligned} E_5 &= \{6, 9, 12, 23, 24, 27\}, & A_5 &= O_5 \cup E_5, \\ C_5 &= (4, 6, 7, 23, 19, 27, 26, 24, 25, 9, 8, 12). \end{aligned} \tag{58}$$

The complement of A_5 consists of even-weight vertices and is independent. The list C_5 is a simple cycle using all five coordinates and containing every vertex of E_5 . Every odd-weight vertex has a neighbor in E_5 , as seen from the following complete neighbor lists:

$v \in E_5$	$\{v \oplus e_j : j \in [5]\}$
6	7, 4, 2, 14, 22
9	8, 11, 13, 1, 25
12	13, 14, 8, 4, 28
23	22, 21, 19, 31, 7
24	25, 26, 28, 16, 8
27	26, 25, 31, 19, 11

Here and in the next table the entries use the integer encoding just defined. Their union is exactly O_5 . Define $T_5(v)$ to be the next vertex of C_5 when v occurs on that cycle, and otherwise to be the smallest integer in E_5 adjacent to v . This defines all assignments. Every vertex outside C_5 reaches it in one step. Hence Lemmas 28 and 27 give $\text{ill}(K) \leq 16 + 6 = 22$ when $n = 5$.

For dimension six, let O_6 be the set of all odd-weight vertices and define

$$\begin{aligned} E_6 &= \{0, 5, 23, 24, 43, 46, 51, 60\}, & A_6 &= O_6 \cup E_6, \\ C_6 &= (0, 16, 24, 56, 60, 44, 46, 42, 43, 59, 51, 55, 23, 7, 5, 1). \end{aligned} \tag{59}$$

Again A_6 is a vertex cover because its complement has even weight. Consecutive vertices in C_6 , including the last and first, are adjacent. This simple cycle uses all six coordinates and contains all of E_6 . The neighbor lists are

$v \in E_6$	$\{v \oplus e_j : j \in [6]\}$
0	1, 2, 4, 8, 16, 32
5	4, 7, 1, 13, 21, 37
23	22, 21, 19, 31, 7, 55
24	25, 26, 28, 16, 8, 56
43	42, 41, 47, 35, 59, 11
46	47, 44, 42, 38, 62, 14
51	50, 49, 55, 59, 35, 19
60	61, 62, 56, 52, 44, 28

Their union is exactly O_6 . Define T_6 by following C_6 on the cycle and, at every other vertex, choosing the smallest adjacent integer in E_6 . Thus $|A_6| = 32 + 8 = 40$, and every vertex reaches the unique cycle C_6 .

Starting from (A_6, T_6) , the assignments (56)–(57) give $40 \cdot 2^{n-6} = 5 \cdot 2^{n-3}$ vertices in each dimension $n \geq 6$. Apply Lemma 28 and then Lemma 27. Together with the dimension-three construction and its first extension, this proves all the asserted bounds. $\square$

These constructions establish upper bounds for each individual body. They do not prove that 12 , 22 , or $5 \cdot 2^{n-3}$ is attained in the real 1-symmetric class.

Proposition 30. *For $n \geq 3$, the real 1-symmetric bodies*

$$B_1^n = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n |x_j| \leq 1 \right\}, \quad P_n = \left\{ x \in [-1, 1]^n : \sum_{j=1}^n |x_j| \leq n-1 \right\}$$

are not affine cubes, and

$$\text{ill}(B_1^n) = 2n, \quad \text{ill}(P_n) = 2^{n-1}.$$

Proof. At a vertex σe_j of B_1^n , where $\sigma \in \{-1, 1\}$, a direction d illuminates the vertex if and only if

$$\sigma d_j + \sum_{i \neq j} |d_i| < 0.$$

This follows by evaluating the norm along a sufficiently short positive step. Necessity for any illuminating step follows from convexity, which permits replacing it by a smaller step. The inequality forces $|d_j| > \sum_{i \neq j} |d_i|$ and fixes the sign of d_j . One direction therefore illuminates at most one vertex. The $2n$ coordinate directions illuminate all $2n$ vertices, proving the first equality. We use here the elementary fact that illuminating every vertex of a polytope illuminates its boundary: in a convex representation of a boundary point, move any vertex of positive weight into the interior and keep the other vertices fixed.

The vertices of P_n are exactly the signed permutations of $(1, \dots, 1, 0)$. To see this, a point with $\sum_j |x_j| < n-1$ has a coordinate of modulus less than one that can be perturbed in both directions while preserving both constraints. If equality holds and two coordinate moduli lie strictly between zero and one, transfer a small amount from one modulus to the other, retaining the signs; again the point is not extreme. There cannot be exactly one nonintegral modulus, since the sum of the moduli is the integer $n-1$. The remaining possibility is precisely $n-1$ coordinates of modulus one and one zero. Conversely, in a convex decomposition of such a point, the cube constraints fix all its unit coordinates in every term, and the sum constraint then forces the remaining coordinate to be zero. Thus each displayed point is a vertex.

Fix a direction d and the index j of the zero coordinate of one of these vertices. Illumination requires d_i to have sign opposite to the vertex coordinate for every $i \neq j$, by the active cube constraints. These requirements determine at most one vertex with zero coordinate j ; if a required d_i is zero there is no such vertex. Consequently d illuminates at most n vertices. There are $n2^{n-1}$ vertices, so $\text{ill}(P_n) \geq 2^{n-1}$.

For the reverse inequality take

$$D = \left\{ d \in \{-1, 1\}^n : \prod_{j=1}^n d_j = 1 \right\}.$$

At a vertex x with $x_j = 0$, prescribe $d_i = -x_i$ for $i \neq j$ and choose the unique remaining sign making $d \in D$. For $0 < t < 1$,

$$\|x + td\|_\infty < 1, \quad \sum_{i=1}^n |x_i + td_i| = n-1 - (n-2)t < n-1.$$

Thus d illuminates x , and the 2^{n-1} members of D illuminate P_n . Both bodies are 1-symmetric directly from their definitions. Their vertex counts, respectively $2n$ and $n2^{n-1}$, differ from 2^n for $n \geq 3$, excluding affine cubes. $\square$

The bound six in dimension three is therefore sharp. In dimensions four and five the largest illumination number in the non-cube 1-symmetric class lies respectively in $[8, 12]$ and $[16, 22]$. For $n \geq 6$ the corresponding interval is $[2^{n-1}, 5 \cdot 2^{n-3}]$. No equality classification at the new upper bounds is asserted.

9.1 Asymptotic sharpness

For $n \geq 3$, let M_n be the largest illumination number among real 1-symmetric convex bodies in $\mathbb{R}^n$ that are not affine cubes. The maximum exists: this class is nonempty, and its illumination numbers are positive integers bounded by Theorem 26.

Theorem 31. *For every $n \geq 3$ and every power of two $m = 2^r \leq n$, $r \geq 1$,*

$$M_n \leq 2^{n-1} + \frac{2^{n-1} + 2^{m-1}}{m} + n - 2. \quad (60)$$

Consequently,

$$M_n \sim 2^{n-1} \quad (n \rightarrow \infty). \quad (61)$$

Proof. Index the first m coordinates by $a \in \mathbb{F}_2^r$ in increasing binary order; the first coordinate has label 0. For $v \in \mathbb{F}_2^m$, define

$$p(v) = \sum_a v_a \in \mathbb{F}_2, \quad s(v) = \sum_a v_a a \in \mathbb{F}_2^r, \quad C = \{v : p(v) = 0, s(v) = 0\}.$$

The columns $(1, a)$ span $\mathbb{F}_2^{r+1}$: they include $(1, 0)$, and adding this column to $(1, e_i)$ gives $(0, e_i)$. Thus

$$c := |C| = 2^{m-r-1} = \frac{2^{m-1}}{m}.$$

Every odd-weight vertex u of Q_m has precisely one neighbor in C , namely

$$q(u) = u \oplus e_{s(u)}.$$

Flipping that coordinate changes its parity and makes its syndrome zero. Conversely, the syndrome specifies the only coordinate whose flip can do so. Distinct elements of C have Hamming distance at least four: the distance is even, and a difference in exactly two distinct coordinates would have nonzero syndrome.

The graph on the odd vertices of Q_m joining vertices at Hamming distance two is connected. Indeed, partition the coordinates in which two odd vertices differ into pairs and flip one pair at a time. For adjacent vertices u, u' in this graph,

$$d_H(q(u), q(u')) \leq 1 + 2 + 1 = 4.$$

Every element of C is the image of an odd neighbor. It follows that the graph on C joining pairs at distance four is connected: images of the preceding paths have consecutive vertices either equal or at distance four. Choose its breadth-first spanning tree rooted at 0, resolving choices by increasing binary order. For each tree edge from a parent v to a child w , let a, b be the first two coordinate labels in which they differ, and define

$$w_{vw} = v \oplus e_a \oplus e_b.$$

This vertex is even and has distance two from both endpoints. Hence at most $c - 1$ additional even vertices connect the centers in C by paths with odd intervening vertices. When $c = 1$ the tree has no edges and no such vertices are needed.

Put $\ell = n - m$. For $z \in \mathbb{F}_2^\ell$, write $p(z)$ for its parity, taking $p(z) = 0$ if $\ell = 0$. Define

$$E = \{(v \oplus p(z)e_0, z) : v \in C, z \in \mathbb{F}_2^\ell\}, \quad W = \{(w_{vw}, 0) : vw \text{ is a tree edge}\}.$$

Both sets consist of even vertices, and

$$|E| = 2^\ell c = \frac{2^{n-1}}{m}, \quad |W| \leq c - 1.$$

Let O_n denote all odd vertices of Q_n . Every $(u, z) \in O_n$ is adjacent to E : flip the first-block coordinate labeled $s(u)$. The resulting first block has syndrome zero and parity $p(z)$, so adding $p(z)e_0$ gives an element of C .

For each fixed $v \in C$, changing one coordinate of z changes exactly two coordinates of $(v \oplus p(z)e_0, z)$. The corresponding even vertices are therefore joined by a path of length two with an odd intermediate vertex. The vertices in W connect these sets through their copies with $z = 0$. Every odd vertex has a neighbor in E , so the induced graph on

$$A_0 = O_n \cup E \cup W$$

is connected.

In the ordinary coordinate order $1, \dots, n$, define

$$v^i = \sum_{h=1}^i e_h \quad (0 \leq i \leq n), \quad v^{n+i} = \sum_{h=i+1}^n e_h \quad (1 \leq i \leq n-1). \quad (62)$$

The list $(v^0, \dots, v^{2n-1})$, closed by $v^{2n} = v^0$, is a simple cycle: successive edges flip coordinates $1, \dots, n, 1, \dots, n$. It has n even vertices, including $v^0 = 0 \in E$. Put

$$A = A_0 \cup \{v^0, \dots, v^{2n-1}\}.$$

All odd cycle vertices already belong to O_n , and hence

$$|A| \leq 2^{n-1} + 2^\ell c + (c-1) + (n-1) = 2^{n-1} + \frac{2^{n-1} + 2^{m-1}}{m} + n - 2.$$

The complement of A consists of even vertices and is independent, so A is a vertex cover. Its induced graph is connected, since every added even cycle vertex has an odd neighbor in O_n .

Let $h(u)$ be graph distance in $Q_n[A]$ from u to the cycle vertex set $V = \{v^0, \dots, v^{2n-1}\}$. Define the successor map by

$$\begin{aligned} T(v^i) &= v^{i+1} & (i \bmod 2n), \\ T(u) &= \min\{w \in A : w \sim u, h(w) = h(u) - 1\} & (u \in A \setminus V), \end{aligned} \quad (63)$$

where the minimum uses binary order. Connectedness ensures that the set is nonempty. Distances decrease until the displayed cycle is reached, so it is the only directed cycle, and it uses all coordinates. Lemmas 28 and 27 prove (60).

Choose the largest power of two $m \leq n$. Since $m > n/2$,

$$1 \leq \frac{M_n}{2^{n-1}} \leq 1 + \frac{2}{m} + \frac{n-2}{2^{n-1}} < 1 + \frac{4}{n} + \frac{n-2}{2^{n-1}}.$$

The lower bound is supplied by P_n in Proposition 30. The two bounds give (61). $\square$

The next argument improves the error term using a published graph theorem. A set D dominates a graph if every vertex outside D has a neighbor in D . Write $\gamma_c(Q_d)$ for the minimum size of such a set whose induced graph is connected.

Proposition 32. *For every $n \geq 3$,*

$$M_n \leq 2^{n-1} + \gamma_c(Q_{n-1}) + n - 1. \quad (64)$$

Consequently,

$$M_n \leq 2^{n-1} \left(1 + \frac{1 + o(1)}{n} \right) \quad (n \rightarrow \infty). \quad (65)$$

Proof. Choose a minimum connected dominating set $D \subset Q_{n-1}$. Translate it so that $0 \in D$, and put

$$E_D = \{(y, p(y)) : y \in D\} \subset Q_n.$$

Every odd vertex has the form $(z, 1 - p(z))$. If $z \in D$, its last-coordinate neighbor belongs to E_D . Otherwise take a neighbor $y \in D$ of z . Then $p(y) = 1 - p(z)$, so the corresponding $(y, p(y)) \in E_D$ is again adjacent to the odd vertex. Adjacent elements of D lift to even vertices at distance two and are joined by a path with an odd intermediate vertex. Thus $O_n \cup E_D$ is connected.

Add the cycle (62). At most $n - 1$ further even vertices are needed, since $0 \in E_D$. Its union with $O_n \cup E_D$ is a connected vertex cover, so the successor map (63) and the same two lemmas prove (64). Griggs proves

$$\gamma_c(Q_d) = (1 + o(1)) \frac{2^d}{d} \quad (d \rightarrow \infty)$$

in [4, Theorems 2 and 4]. Substitute $d = n - 1$ and use $n - 1 = o(2^{n-1}/n)$ to obtain (65). $\square$

The sufficient proper-support condition has a stricter lower bound than the known lower examples for individual bodies.

Proposition 33. *Let t_n be the least size of a family $\mathcal{B} \subset \{-1, 1\}^n \times [n]$ satisfying (52). Then, for $n \geq 3$,*

$$\left\lceil \frac{n2^{n-1}}{n-1} \right\rceil \leq t_n \leq 2^{n-1} + \gamma_c(Q_{n-1}) + n - 1.$$

In particular,

$$t_n = 2^{n-1} \left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right) \right).$$

Proof. Consider only supports $S = [n] \setminus \{j\}$. There are $n2^{n-1}$ requirements (S, a) . A fixed pair (b, k) meets exactly $n - 1$ of them: the omitted coordinate can be any $j \neq k$, and then $a = -b|_S$ is prescribed. Counting incidences gives $(n - 1)|\mathcal{B}| \geq n2^{n-1}$. This remains valid when several labels are used at the same sign vector, since they are distinct pairs.

The preceding construction supplies a family satisfying (52) of size at most the stated upper bound. Griggs's asymptotic formula then gives

$$\frac{2^{n-1}}{n-1} \leq t_n - 2^{n-1} \leq (1 + o(1)) \frac{2^{n-1}}{n-1} + n - 1,$$

which proves the assertion. $\square$

The asymptotic equality for t_n does not imply the same second-order term for M_n : the sign-family condition is sufficient for illumination, and need not describe an optimal illuminating family for a particular body. In particular, (61) settles the leading asymptotic value, while the exact maxima M_n for fixed $n \geq 4$ remain undetermined here.

10 Research status

The following table records the results and remaining claims in the research corpus.

Claim	Present status	Treatment here
Sharp bounds 6 in $\mathbb{C}^2$ and 14 in $\mathbb{C}^3$ for normalized non-polydiscs	internally verified, including sharpness	Corollary 12 and Theorem 13
A non-polydisc containing a full coordinate $(n-1)$ -polydisc has $\text{ill}(K) \leq 2^{n+1} - 2$	internally verified, including sharpness	full proof in Section 3
Every normalized non-polydisc in $\mathbb{C}^4$ has $\text{ill}(K) \leq 30$	internally verified; one covering assertion computationally certified	full proof in Section 5; exact verifier in Appendix B
Every non-polydisc complex 1-symmetric body in $\mathbb{C}^3$ has $\text{ill}(K) \leq 12$	internally verified; finite covering computationally certified	full proof and certificate in Section 6
Real 1-symmetric non-cubes: bounds 12, 22, and $5 \cdot 2^{n-3}$	internally verified; explicit finite starting families checked exactly	full proof in Section 9
Real 1-symmetric non-cubes: $M_n \sim 2^{n-1}$	internally verified; self-contained proof of the leading term	Theorem 31; sharper error in Proposition 32
Exact ordinary and fractional formulas for complex Hanner bodies	internally verified, including equality cases	full proof in Section 7
Failure of one cyclic construction in dimension five	reported failure of this construction	verification pending

The ordinary complex 1-unconditional problem is settled here through dimension four. Its sharp dimension-only bound remains open in this corpus in dimensions at least five. The general estimates proved above are weaker than the conjectured sharp bound. For bodies whose extreme points have no zero coordinates, Theorem 25 gives the conjectured ordinary upper bound and a strict fractional bound for non-polydiscs. The strict ordinary inequality $\text{ill}(K) \leq 2^{n+1} - 2$ remains unresolved here for that class beyond the cases already covered above. The source inventory, theorem dependency ledger, and dated research log accompanying this file record the provenance and audit state of every item.

A One family for all normalized bodies

Let U_n be the least cardinality of a finite family of directions that illuminates every normalized complex 1-unconditional convex body in $\mathbb{C}^n$. The family must be chosen independently of the body. This is a stronger requirement than bounding $\text{ill}(K)$ separately for each K .

Proposition 34. Put $m = \lceil 2n/3 \rceil$ and

$$L_n = (2^{m+1} - 1) \binom{n}{m}, \quad B_n = L_n + \sum_{k=m+1}^n \binom{n}{k}.$$

With C_n defined in (3),

$$L_n \leq U_n \leq C_n \leq B_n, \tag{66}$$

and

$$U_n \sim \frac{3^{n+1}}{\sqrt{\pi n}}. \tag{67}$$

Proof. The directions in the proof of Theorem 2 depend only on n , the finite matchings and cyclic vectors, and their common positive margin. The normal estimates use only normalization of K . Hence the same family works for every normalized K , proving $U_n \leq C_n$.

For every m -set $S \subset [n]$, when $m < n$, consider the normalized body

$$K_S = \{z \in \mathbb{C}^n : \|z_S\|_\infty + \|z_{S^c}\|_\infty \leq 1\}.$$

At a point x of its coordinate torus

$$|x_j| = 1 \ (j \in S), \quad x_j = 0 \ (j \notin S),$$

the directional derivative of its norm is

$$\max_{j \in S} \operatorname{Re}(\overline{x_j} d_j) + \|d_{S^c}\|_\infty.$$

Illumination forces this derivative to be negative: for a convex function, an interior point along the ray forces a negative one-sided derivative at its initial point. Consequently

$$\operatorname{Re}(\overline{x_j} d_j) < -\|d_{S^c}\|_\infty \quad (j \in S), \quad \min_{j \in S} |d_j| > \max_{j \notin S} |d_j|.$$

Thus the m largest coordinate magnitudes of d determine S uniquely. The groups of directions serving these tori for different m -sets are disjoint. Within the group for S , projection must illuminate the m -polydisc torus, and hence the polydisc, requiring at least $2^{m+1} - 1$ directions by the exact polydisc illumination theorem [1]. Summation gives $U_n \geq L_n$. If $m = n$, the polydisc itself gives this conclusion.

Finally, $B_n - L_n \leq 2^n$. Stirling's formula, with $m = 2n/3 + O(1)$, yields

$$2^{m+1} \binom{n}{m} \sim \frac{3}{\sqrt{\pi n}} 3^n.$$

The same asymptotic holds for $C_n = B_n - (n - m)$; squeezing between the lower and upper bounds proves the claim. □

Corollary 35.

$$U_1 = 3, \quad U_2 = 7, \quad U_3 = 21, \quad U_4 = 60, \quad U_5 = 155.$$

Proof. For $n = 1, 2$, $B_n - L_n = 0$ and $n - m = 0$. For $3 \leq n \leq 5$, $B_n - L_n = 1 = n - m$. Thus $C_n = L_n$ in all five dimensions, and (66) gives the stated values. □

n	$2 \cdot 3^n - 2^n - 1$	C_n	L_n
2	13	7	7
3	45	21	21
4	145	60	60
5	453	155	155
6	1393	470	465
8	12865	3563	3556
10	117073	30653	30600
15	28665045	6149077	6147141
20	6972520225	1270070614	1270048920

These are general estimates, not assertions of the best possible bound for each individual body or each dimension. In particular, exact values of U_n concern the simultaneous normalized problem. The same asymptotic $C_n \sim 3^{n+1}/\sqrt{\pi n}$ holds, and the final additive refinement does not affect it.

B The four-coordinate finite verification

The following Python program verifies the finite assertions (33) and (34). The finite reduction to these assertions is proved in Lemma 18. All sets of rows are represented by integer bit masks; every loop ranges over the entire finite set indicated. The program requires only Python's standard library. Running it with Python 3.12.14 gives 125054 923521 9341 [0, 0, 0, 1320, 8021].

```
from itertools import combinations, product

m = [1,2,1,4,3,2,3,4,1,4,1,4,3,2,4,
      2,1,2,4,2,3,3,1,3,2,3,4,1,2,1]
q = [4,4,2,3,4,3,2,2,4,2,3,2,1,4,1,
      1,4,1,3,1,2,4,2,1,3,4,3,3,4,4]
assert len(m) == len(q) == 30
assert all(1 <= a <= 4 and 1 <= b <= 4 and a != b
           for a, b in zip(m, q))
w = [[sum(1 << (k-1) for k in range(1,31)
         if ((1 << j)*k-a) % 31 < 15)
      for a in range(31)] for j in range(4)]
main = [sum(1 << k for k in range(30) if m[k] == j+1)
        for j in range(4)]
tiny = [sum(1 << k for k in range(30) if q[k] == j+1)
        for j in range(4)]

tested = 0
for size in (1,2,3):
    for S in combinations(range(4), size):
        allowed = 0
        for j in S:
            allowed |= main[j]
        for starts in product(range(31), repeat=size):
            candidates = (1 << 30)-1
            for j, a in zip(S, starts):
                candidates &= w[j][a]
            assert candidates & allowed
            tested += 1
assert tested == 125054

holes = 0
histogram = [0]*5
for a,b,c,d in product(*w):
    if a & b & c & d:
        continue
    holes += 1
    candidates = [b&c&d, a&c&d, a&b&d, a&b&c]
    count = sum(bool(candidates[j] & tiny[j])
                for j in range(4))
    assert count >= 3
    histogram[count] += 1
assert holes == 9341
assert histogram == [0,0,0,1320,8021]
```

```
print(tested, 31**4, holes, histogram)
```

C Numerical comparison through dimension 100

The table compares the following estimates. Here $c < c_*$ is the positive rational approximation recorded in the accompanying data file, and $b = 1/c$:

$$\begin{aligned} A_n &= 2 \cdot 3^n - 2^n - 1, \\ C_n &= (2^{m+1} - 1) \binom{n}{m} + \sum_{k=m+1}^n \binom{n}{k} - (n - m), \quad m = \lceil 2n/3 \rceil, \\ E_n &= \lceil eb^n [2n \log(1 + 32b^2(n+1)^2) + 1] \rceil, \\ T_n &= \left\lceil \frac{2n \log(1 + 32/(c^2 \varepsilon_n^2)) + 1}{[c(1 - \varepsilon_n)]^n} \right\rceil. \end{aligned}$$

Here A_n is the sum of the bounds for individual supports, C_n is the bound in Theorem 2, E_n evaluates the explicit formula with $b = 1/c$, and T_n is (21) with a numerically chosen value of ε_n . The chosen parameters approximately minimize the real-valued expression before taking its ceiling; validity needs only $0 < \varepsilon_n < 1$.

The first dimension in this table with $E_n < C_n$ is $n = 51$; the first with $T_n < C_n$ is $n = 47$. Each remains smaller through $n = 100$, the end of this table. The smaller of C_n and T_n is bold. The last column is T_n/C_n : values below one mean $T_n < C_n$. These compare the displayed estimates, not necessarily the best known bounds in each individual dimension.

Numbers below 10^7 are printed exactly. Larger counts are rounded *upward* to six significant digits for display, so the displayed counts remain upper bounds; ratios are ordinary rounded approximations. Admissibility of the rational c was checked by exact rational lower bounds on $\cos(\pi c) \cosh(\pi(1 - c))$. The formulas remain valid for this smaller value of c . Integer ceilings agree when evaluated at 120 and 170 decimal digits. The integer evaluations are numerical and are not used in the proofs. Full computed counts and the chosen ε_n are recorded in the accompanying data file.

n	A_n	C_n	E_n	T_n	T_n/C_n
1	3	3	97	56	18.67
2	13	7	518	308	44
3	45	21	2047	1221	58.14
4	145	60	7103	4225	70.42
5	453	155	22891	13555	87.45
6	1393	470	70353	41470	88.23
7	4245	1329	209211	122774	92.38
8	12865	3563	607225	354861	99.6
9	38853	10711	1729986	1007086	94.02
10	117073	30653	4856875	2817201	91.91
11	352245	84379	1.34743×10^7	7789603	92.32
12	1058785	253240	3.70155×10^7	2.13328×10^7	84.24
13	3180453	731819	1.00851×10^8	5.79540×10^7	79.19
14	9549553	2049513	2.72846×10^8	1.56369×10^8	76.3
15	2.86651×10^7	6149077	7.33709×10^8	4.19428×10^8	68.21
16	8.60280×10^7	1.78895×10^7	1.96264×10^9	1.11929×10^9	62.57
17	2.58150×10^8	5.06892×10^7	5.22569×10^9	2.97352×10^9	58.66
18	7.74579×10^8	1.52071×10^8	1.38569×10^{10}	7.86814×10^9	51.74
19	2.32400×10^9	4.44521×10^8	3.66100×10^{10}	2.07461×10^{10}	46.67
20	6.97253×10^9	1.27008×10^9	9.64068×10^{10}	5.45279×10^{10}	42.93
21	2.09187×10^{10}	3.81023×10^9	2.53123×10^{11}	1.42909×10^{11}	37.51
22	6.27580×10^{10}	1.11768×10^{10}	6.62813×10^{11}	3.73568×10^{11}	33.42
23	1.88278×10^{11}	3.21332×10^{10}	1.73137×10^{12}	9.74214×10^{11}	30.32

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n	A_n	C_n	E_n	T_n	T_n/C_n
24	5.64843×10^{11}	9.63995×10^{10}	4.51251×10^{12}	2.53514×10^{12}	26.3
25	1.69455×10^{12}	2.83529×10^{11}	1.17370×10^{13}	6.58396×10^{12}	23.22
26	5.08367×10^{12}	8.19082×10^{11}	3.04702×10^{13}	1.70679×10^{13}	20.84
27	1.52511×10^{13}	2.45725×10^{12}	7.89653×10^{13}	4.41713×10^{13}	17.98
28	4.57534×10^{13}	7.24241×10^{12}	2.04313×10^{14}	1.14136×10^{14}	15.76
29	1.37261×10^{14}	2.10030×10^{13}	5.27839×10^{14}	2.94491×10^{14}	14.02
30	4.11782×10^{14}	6.30090×10^{13}	1.36176×10^{15}	7.58813×10^{14}	12.04
31	1.23535×10^{15}	1.86027×10^{14}	3.50857×10^{15}	1.95276×10^{15}	10.5
32	3.70604×10^{15}	5.41168×10^{14}	9.02878×10^{15}	5.01937×10^{15}	9.275
33	1.11182×10^{16}	1.62351×10^{15}	2.32076×10^{16}	1.28875×10^{16}	7.938
34	3.33544×10^{16}	4.79993×10^{15}	5.95884×10^{16}	3.30547×10^{16}	6.886
35	1.00064×10^{17}	1.39998×10^{16}	1.52846×10^{17}	8.46976×10^{16}	6.05
36	3.00190×10^{17}	4.19994×10^{16}	3.91676×10^{17}	2.16825×10^{17}	5.163
37	9.00568×10^{17}	1.24319×10^{17}	1.00279×10^{18}	5.54581×10^{17}	4.461
38	2.70171×10^{18}	3.63392×10^{17}	2.56518×10^{18}	1.41730×10^{18}	3.9
39	8.10511×10^{18}	1.09018×10^{18}	6.55652×10^{18}	3.61925×10^{18}	3.32
40	2.43154×10^{19}	3.23015×10^{18}	1.67454×10^{19}	9.23531×10^{18}	2.859
41	7.29460×10^{19}	9.45971×10^{18}	4.27366×10^{19}	2.35493×10^{19}	2.489
42	2.18838×10^{20}	2.83792×10^{19}	1.08994×10^{20}	6.00083×10^{19}	2.115
43	6.56514×10^{20}	8.41588×10^{19}	2.77789×10^{20}	1.52815×10^{20}	1.816
44	1.96955×10^{21}	2.46866×10^{20}	7.07545×10^{20}	3.88916×10^{20}	1.575
45	5.90863×10^{21}	7.40598×10^{20}	1.80107×10^{21}	9.89217×10^{20}	1.336
46	1.77259×10^{22}	2.19791×10^{21}	4.58201×10^{21}	2.51470×10^{21}	1.144
47	5.31777×10^{22}	6.45634×10^{21}	1.16505×10^{22}	6.38921×10^{21}	0.9896
48	1.59533×10^{23}	1.93691×10^{22}	2.96073×10^{22}	1.62251×10^{22}	0.8377
49	4.78599×10^{23}	5.75201×10^{22}	7.52029×10^{22}	4.11826×10^{22}	0.716
50	1.43580×10^{24}	1.69177×10^{23}	1.90924×10^{23}	1.04481×10^{23}	0.6176
51	4.30739×10^{24}	5.07531×10^{23}	4.84487×10^{23}	2.64950×10^{23}	0.522
52	1.29222×10^{25}	1.50809×10^{24}	1.22889×10^{24}	6.71587×10^{23}	0.4453
53	3.87665×10^{25}	4.44049×10^{24}	3.11567×10^{24}	1.70161×10^{24}	0.3832
54	1.16300×10^{26}	1.33215×10^{25}	7.89606×10^{24}	4.30966×10^{24}	0.3235
55	3.48899×10^{26}	3.96044×10^{25}	2.00031×10^{25}	1.09109×10^{25}	0.2755
56	1.04670×10^{27}	1.16729×10^{26}	5.06543×10^{25}	2.76130×10^{25}	0.2366
57	3.14009×10^{27}	3.50186×10^{26}	1.28226×10^{26}	6.98571×10^{25}	0.1995
58	9.42026×10^{27}	1.04158×10^{27}	3.24472×10^{26}	1.76668×10^{26}	0.1696
59	2.82608×10^{28}	3.07266×10^{27}	8.20785×10^{26}	4.46643×10^{26}	0.1454
60	8.47824×10^{28}	9.21797×10^{27}	2.07557×10^{27}	1.12882×10^{27}	0.1225
61	2.54347×10^{29}	2.74291×10^{28}	5.24691×10^{27}	2.85199×10^{27}	0.104
62	7.63041×10^{29}	8.09811×10^{28}	1.32598×10^{28}	7.20349×10^{27}	0.08895
63	2.28913×10^{30}	2.42944×10^{29}	3.34992×10^{28}	1.81891×10^{28}	0.07487
64	6.86737×10^{30}	7.23180×10^{29}	8.46070×10^{28}	4.59151×10^{28}	0.06349
65	2.06022×10^{31}	2.13667×10^{30}	2.13627×10^{29}	1.15873×10^{29}	0.05423
66	6.18064×10^{31}	6.41001×10^{30}	5.39245×10^{29}	2.92342×10^{29}	0.04561
67	1.85419×10^{32}	1.90876×10^{31}	1.36082×10^{30}	7.37376×10^{29}	0.03863
68	5.56257×10^{32}	5.64328×10^{31}	3.43322×10^{30}	1.85943×10^{30}	0.03295
69	1.66878×10^{33}	1.69299×10^{32}	8.65953×10^{30}	4.68773×10^{30}	0.02769
70	5.00632×10^{33}	5.04293×10^{32}	2.18365×10^{31}	1.18153×10^{31}	0.02343
71	1.50190×10^{34}	1.49187×10^{33}	5.50511×10^{31}	2.97733×10^{31}	0.01996
72	4.50568×10^{34}	4.47560×10^{33}	1.38756×10^{32}	7.50087×10^{31}	0.01676
73	1.35171×10^{35}	1.33355×10^{34}	3.49653×10^{32}	1.88931×10^{32}	0.01417
74	4.05512×10^{35}	3.94730×10^{34}	8.80907×10^{32}	4.75777×10^{32}	0.01205
75	1.21654×10^{36}	1.18419×10^{35}	2.21887×10^{33}	1.19789×10^{33}	0.01012
76	3.64961×10^{36}	3.52935×10^{35}	5.58782×10^{33}	3.01536×10^{33}	0.008544
77	1.09489×10^{37}	1.04523×10^{36}	1.40691×10^{34}	7.58889×10^{33}	0.00726
78	3.28465×10^{37}	3.13569×10^{36}	3.54165×10^{34}	1.90958×10^{34}	0.00609

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n	A_n	C_n	E_n	T_n	T_n/C_n
79	9.85393×10^{37}	9.34791×10^{36}	8.91379×10^{34}	4.80413×10^{34}	0.005139
80	2.95618×10^{38}	2.76975×10^{37}	2.24305×10^{35}	1.20842×10^{35}	0.004363
81	8.86853×10^{38}	8.30925×10^{37}	5.64334×10^{35}	3.03906×10^{35}	0.003657
82	2.66056×10^{39}	2.47767×10^{38}	1.41958×10^{36}	7.64168×10^{35}	0.003084
83	7.98168×10^{39}	7.34452×10^{38}	3.57029×10^{36}	1.92118×10^{36}	0.002616
84	2.39451×10^{40}	2.20336×10^{39}	8.97794×10^{36}	4.82918×10^{36}	0.002192
85	7.18351×10^{40}	6.57141×10^{39}	2.25725×10^{37}	1.21370×10^{37}	0.001847
86	2.15506×10^{41}	1.94877×10^{40}	5.67430×10^{37}	3.04988×10^{37}	0.001565
87	6.46516×10^{41}	5.84629×10^{40}	1.42619×10^{38}	7.66279×10^{37}	0.001311
88	1.93955×10^{42}	1.74398×10^{41}	3.58406×10^{38}	1.92499×10^{38}	0.001104
89	5.81865×10^{42}	5.17380×10^{41}	9.00552×10^{38}	4.83511×10^{38}	0.0009345
90	1.74560×10^{43}	1.55214×10^{42}	2.26246×10^{39}	1.21430×10^{39}	0.0007823
91	5.23678×10^{43}	4.63098×10^{42}	5.68314×10^{39}	3.04916×10^{39}	0.0006584
92	1.57104×10^{44}	1.37436×10^{43}	1.42737×10^{40}	7.65558×10^{39}	0.000557
93	4.71311×10^{44}	4.12306×10^{43}	3.58447×10^{40}	1.92185×10^{40}	0.0004661
94	1.41394×10^{45}	1.23038×10^{44}	9.00026×10^{40}	4.82396×10^{40}	0.0003921
95	4.24180×10^{45}	3.65268×10^{44}	2.25959×10^{41}	1.21069×10^{41}	0.0003315
96	1.27254×10^{46}	1.09581×10^{45}	5.67215×10^{41}	3.03815×10^{41}	0.0002773
97	3.81762×10^{46}	3.27055×10^{45}	1.42368×10^{42}	7.62311×10^{41}	0.0002331
98	1.14529×10^{47}	9.71253×10^{45}	3.57292×10^{42}	1.91251×10^{42}	0.0001969
99	3.43586×10^{47}	2.91376×10^{46}	8.96564×10^{42}	4.79761×10^{42}	0.0001647
100	1.03076×10^{48}	8.69779×10^{46}	2.24952×10^{43}	1.20337×10^{43}	0.0001384

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