

Coefficient Dynamics and Consecutive Toeplitz Minors for the de Bruijn–Newman Family

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Abstract

We give a normalization-consistent coefficient formulation of the de Bruijn–Newman heat flow attached to the Riemann ξ -function. For the classical kernel Φ and the family

$$H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) \, du, \quad F_t(w) = H_t(i\sqrt{w}) = \sum_{n \geq 0} q_n(t) w^n,$$

we prove the exact evolution laws

$$\partial_t F_t = 2F'_t + 4wF''_t, \quad q_n^{(m)}(t) = \frac{(2n+2m)!}{(2n)!} q_{n+m}(t).$$

The coefficients have positive integral representations and the transformed entire function has order at most $1/2$. We relate the zeros of F_t to the Pólya–frequency property of $(q_n(t))$, derive the rectangular Jacobi–Trudi identity for the consecutive Toeplitz minors, and state carefully the strict consecutive-minor criterion needed to pass from these minors to total positivity. In particular, for this coefficient family the real-zero condition is equivalent to strict positivity of every consecutive minor $D_t(n, r)$, not merely to nonnegativity of that selected family. This distinction avoids an invalid use of a weak version of Fekete’s criterion. We also correct the multiplicity bookkeeping for the squared-variable map and record a Chebyshev convolution identity without attributing to it an unjustified duplication of zeros. The paper proves no new unconditional upper bound for the de Bruijn–Newman constant and no proof of the Riemann hypothesis; its purpose is to provide an auditable set of exact identities and to isolate the remaining quantitative problem.

Keywords: Riemann ξ -function; de Bruijn–Newman constant; heat flow; Toeplitz minors; total positivity; Pólya–frequency sequences; Schur functions; Turán inequalities.

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1 Introduction

The Riemann hypothesis asserts that every nontrivial zero of the Riemann zeta-function has real part $1/2$. Write

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \Xi(z) = \xi\left(\frac{1}{2} + iz\right).$$

The function Ξ is even and entire, and RH is equivalent to the assertion that all zeros of Ξ are real.

The de Bruijn–Newman deformation embeds this zero-location problem in the one-parameter family

$$H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) \, du, \quad t \in \mathbb{R}, \quad (1.1)$$

where the classical kernel Φ is given explicitly in [equation \(2.1\)](#). With this normalization, $H_0(z) = \frac{1}{8}\Xi(z/2)$. The de Bruijn–Newman theorem gives a finite constant Λ such that all zeros of H_t are real if and only if $t \geq \Lambda$; the Riemann hypothesis is equivalent to $\Lambda \leq 0$. Rodgers and Tao proved $\Lambda \geq 0$ [23]. The latest refereed upper bound identified in the current public status index is $\Lambda \leq 0.2$ [4, 22]; later public certificate packages report smaller values, but we distinguish those reports from independently reviewed results in Section 10.

This note studies the Taylor coefficients after writing the even entire function H_t as an entire function of a squared variable. The resulting coefficient sequence has a simple differential evolution

in t , and the corresponding one-sided Toeplitz matrix has a natural hierarchy of consecutive minors. The purpose is to make the exact equivalences transparent while avoiding two tempting but invalid shortcuts: replacing all Toeplitz minors by an unproved weak nonnegative consecutive-minor criterion, and treating evenness as if it doubled the multiplicity of a nonzero zero under squaring.

1.1 Summary of the results

The main statements are as follows.

- (i) We derive the squared-variable heat equation and the exact all-orders coefficient shift formula ([theorems 3.1, 3.2 and 3.4](#)).
- (ii) We establish the coefficient-level equivalence between real zeros and PF_∞ , using the Schoenberg–Edrei representation and the order bound for F_t ([theorem 4.8](#)).
- (iii) We prove the consecutive-minor/Schur identity under the genus-zero product factorization and state the strict consecutive-minor criterion with its hypotheses ([theorems 4.4 and 7.1](#)).
- (iv) We obtain an exact characterization of the threshold by the strict family $D_t(n, r) > 0$ for every shift and order; below the threshold, at least one such determinant is nonpositive ([theorem 5.1](#)).
- (v) We show explicitly that the pair of zeros $\gamma, -\gamma$ of an even function maps to one squared-variable zero of the same multiplicity, rather than a double-multiplicity zero ([theorem 8.1](#)).
- (vi) Using Sra’s PF_2 result for the reciprocal- ξ density and the normalized-moment theorem of Karlin–Proschan–Barlow, we derive the strict base-time rows $D_0(n, r) > 0$ for $n = 0, 1, 2$ and every $r \geq 1$ ([theorem 6.1](#)).
- (vii) We place the classical Csordas–Norfolk–Varga Turán inequality in the present Toeplitz normalization, obtaining an explicit strict lower bound for $D_0(n, 2)$ at every shift; Sra’s 2026 coefficient result gives an independent modern cross-check. We translate Michalowski’s cubic-tail theorem into the heat-flow normalization and derive a conservative sector-cutoff refinement from the 2026 Bellotti–Trudgian–Yang zero-free region and the previously stated Blind Zone Theorem ([theorems 6.3, 6.4 and 8.6](#)). These are explicitly attributed consequences of external results, not new certificates produced here.

These statements are principally a synthesis of classical theorems with a fixed heat-flow normalization. They do not yield a new numerical upper bound for Λ . The distinction matters: an exact equivalence involving an infinite family of inequalities does not imply that a finite computation of selected minors proves the real-zero property.

1.2 Relation to recent work

The coefficient/minor viewpoint is already the subject of active research. Michalowski’s 2026 preprint proves $D_{r,k} > 0$ for every order $r \geq 2$ whenever $k \geq 10^{18}r^3$, with Arb-ball certificates and exact rational checks [17]. We translate that result into our coefficient normalization in [theorem 6.4](#); this is a literature integration, not an independent reproduction of the certificates. A separate Zenodo preprint by Kessler, posted in August 2026, claims strict positivity of the complete order-family at shift 1 by a verified-zero core plus a tail-mass estimate [14]. That claim overlaps with the $n = 1$ consequence of [theorem 6.1](#); the argument here is different, and we have not independently audited Kessler’s tail estimate. Katkova established fixed-order asymptotic positivity results for related coefficient sequences [12]. The main role of this note remains the structural formulation of the heat flow and its exact relation to consecutive minors.

Two October 2026 preprints by Sra study positivity structures associated with Ξ : one treats log-concavity and finite-order Pólya-frequency properties of reciprocal Laplace-transform densities, and another proves reverse regularity through order four for a related Hankel kernel, yielding strict Turán inequalities for normalized coefficients [26, 27]. The latter is an independent modern route to the order-two inequality, but that inequality itself is classical: Csordas–Norfolk–Varga proved the needed weighted Turán inequality in 1986 [3]. We have not independently re-run the computer-assisted certificates in the October 2026 preprints or Michalowski’s tail theorem. None of these statements identifies the full coefficient sequence $q_n(t)$ with a PF_∞ sequence at arbitrary heat time.

1.3 Scope and non-claims

No numerical determinant signs are used as proofs in this manuscript. We do not claim global positivity of the full array $D_t(n, r)$, all-order positivity at nonzero heat time, positivity for every order at shifts $n \geq 3$, or an improvement to the best published upper bound for Λ . At base time, the shifts $n = 0, 1, 2$ are covered at every order, the order- $r = 2$ minors are covered for all shifts by the classical Turán inequality, and the tail $n \geq 10^{18}r^3$ is covered for every order $r \geq 2$ by Michalowski’s theorem; all three are cited external results. We also record a conservative conditional sector cutoff from the 2026 zero-free-region preprint, assuming the Blind Zone Theorem in [16]; this does not provide a standalone proof of that theorem. Earlier exploratory calculations involving finite ranges of shifts or orders are intentionally not promoted to theorem statements here: such claims require source code, explicit truncation bounds, and rigorous arithmetic certificates that are not included in this package.

2 The classical kernel and normalization

Throughout, set

$$\Phi(u) = \sum_{m=1}^{\infty} \left(2\pi^2 m^4 e^{9u} - 3\pi m^2 e^{5u} \right) \exp\left(-\pi m^2 e^{4u}\right), \quad u \geq 0. \quad (2.1)$$

With this convention, the standard cosine-transform identity is

$$H_0(z) = \int_0^\infty \Phi(u) \cos(zu) \, du = \frac{1}{8} \xi\left(\frac{1}{2} + \frac{iz}{2}\right) = \frac{1}{8} \Xi(z/2). \quad (2.2)$$

See de Bruijn’s work and the modern account in the Polymath paper for this heat-flow normalization [4, 5]. The factor $1/8$ and the argument $z/2$ are important when translating Taylor coefficients or comparing numerical values of the heat parameter.

Lemma 2.1 (Positivity and decay of the kernel). *For every $u \geq 0$, one has $\Phi(u) > 0$. Moreover, there is a constant $C > 0$ such that, for all sufficiently large u ,*

$$0 < \Phi(u) \leq C e^{9u} \exp\left(-\frac{\pi}{2} e^{4u}\right).$$

Consequently, for fixed $t \in \mathbb{R}$, integers $k \geq 0$, and $R \geq 0$, the integral

$$\int_0^\infty u^k e^{tu^2 + Ru} \Phi(u) \, du$$

converges absolutely.

Proof. The m th summand in equation (2.1) can be written as

$$\pi m^2 e^{5u} (2\pi m^2 e^{4u} - 3) \exp\left(-\pi m^2 e^{4u}\right).$$

Since $2\pi m^2 e^{4u} - 3 \geq 2\pi - 3 > 0$, each summand is positive. For the upper bound, factor $\exp(-\frac{\pi}{2}e^{4u})$ from each exponential; the remaining series is dominated for $u \geq 0$ by a constant multiple of $e^{9u} \sum_{m \geq 1} m^4 \exp(-\frac{\pi}{2}m^2)$. This proves the stated bound. The last assertion follows because $\exp(-ce^{4u})$ dominates every factor of the form $u^k \exp(|t|u^2 + Ru + cu)$ as $u \rightarrow \infty$. $\square$

Proposition 2.2 (Entirety and order). *For each real t , the function H_t defined by [equation \(1.1\)](#) is even and entire in z . It has order at most one. The function*

$$F_t(w) := H_t(i\sqrt{w})$$

is single-valued and entire in w , and has order at most $1/2$.

Proof. Lemma [2.1](#) justifies differentiation under the integral on every compact subset of the z -plane, which proves entirety; evenness follows from the cosine. For $|z| \leq R$,

$$|H_t(z)| \leq \int_0^\infty e^{|t|u^2 + Ru} \Phi(u) \, du.$$

The tail bound in [theorem 2.1](#) yields $\log M_{H_t}(R) = O(R \log(R + 2))$ as $R \rightarrow \infty$, so the order is at most one. Since H_t is even, its Taylor expansion contains only even powers and therefore defines an entire function F_t of w through $F_t(w) = H_t(i\sqrt{w})$. The maximum-modulus estimate $M_{F_t}(r) \leq M_{H_t}(\sqrt{r})$ yields order at most $1/2$. $\square$

Remark 2.3. The conclusion that the order of F_t is strictly less than one is used below to obtain a genus-zero product without a nonconstant exponential factor. The order estimate is not a claim about the precise order of H_t or F_t .

3 The heat equation and coefficient evolution

3.1 The heat equation in the original variable

Differentiating under the integral sign gives

$$\partial_t H_t(z) = \int_0^\infty u^2 e^{tu^2} \Phi(u) \cos(zu) \, du, \quad \partial_z^2 H_t(z) = - \int_0^\infty u^2 e^{tu^2} \Phi(u) \cos(zu) \, du.$$

Thus

$$\partial_t H_t(z) = -\partial_z^2 H_t(z). \tag{3.1}$$

The sign in [equation \(3.1\)](#) corresponds to the factor e^{+tu^2} in [equation \(1.1\)](#); a convention using e^{-tu^2} reverses the sign.

3.2 The squared-variable equation

Theorem 3.1 (Squared-variable heat equation). *Let $F_t(w) = H_t(i\sqrt{w})$. Then*

$$\partial_t F_t(w) = 2F_t'(w) + 4wF_t''(w). \tag{3.2}$$

Proof. The relation $H_t(z) = F_t(-z^2)$ gives

$$\partial_z H_t(z) = -2zF_t'(-z^2), \quad \partial_z^2 H_t(z) = -2F_t'(-z^2) + 4z^2F_t''(-z^2).$$

Writing $w = -z^2$ in the second equation yields

$$\partial_z^2 H_t(z) = -2F_t'(w) - 4wF_t''(w).$$

Now use [equation \(3.1\)](#). $\square$

Write

$$F_t(w) = \sum_{n=0}^{\infty} q_n(t) w^n. \quad (3.3)$$

Corollary 3.2 (Coefficient evolution). *For all $n \geq 0$ and real t ,*

$$q'_n(t) = 2(n+1)(2n+1)q_{n+1}(t) = \frac{(2n+2)!}{(2n)!} q_{n+1}(t). \quad (3.4)$$

More generally, for all integers $m, n \geq 0$,

$$q_n^{(m)}(t) = \frac{(2n+2m)!}{(2n)!} q_{n+m}(t). \quad (3.5)$$

Proof. Substitute the series in [equation \(3.3\)](#) into [equation \(3.2\)](#) and compare coefficients of w^n . This gives [equation \(3.4\)](#). Iterating it proves [equation \(3.5\)](#); the product of the successive factors telescopes to $(2n+2m)!/(2n)!$. $\square$

Proposition 3.3 (Moment representation and coefficient positivity). *For every $n \geq 0$ and $t \in \mathbb{R}$,*

$$q_n(t) = \frac{1}{(2n)!} \int_0^\infty u^{2n} e^{tu^2} \Phi(u) \, du > 0. \quad (3.6)$$

For each n , the map $t \mapsto q_n(t)$ extends to an entire function of a complex variable. For real t and $s \geq 0$, one has $q_n(t+s) \geq q_n(t)$.

Proof. Since $\cos(i\sqrt{w}u) = \cosh(\sqrt{w}u)$ and

$$\cosh(\sqrt{w}u) = \sum_{n=0}^{\infty} \frac{u^{2n} w^n}{(2n)!},$$

termwise integration gives [equation \(3.6\)](#). Lemma 2.1 justifies the interchange locally uniformly in w and in t on compact subsets of $\mathbb{C}$. It follows that each $q_n(t)$ is entire in t . The Taylor series derived from [equation \(3.5\)](#) gives

$$q_n(t+s) = \sum_{m=0}^{\infty} \frac{(2n+2m)!}{(2n)!m!} q_{n+m}(t) s^m, \quad (3.7)$$

which has nonnegative terms for real t and $s \geq 0$. $\square$

Theorem 3.4 (Taylor law in the heat parameter). *For any fixed $n \geq 0$ and any $s, t \in \mathbb{C}$, the identity*

$$q_n(t+s) = \sum_{m=0}^{\infty} \frac{(2n+2m)!}{(2n)!m!} q_{n+m}(t) s^m$$

holds with convergence for every complex s . In particular, coefficientwise monotonicity for increasing real time does not imply monotonicity of any Toeplitz minor.

Proof. The first statement follows from the entire dependence on t and the Taylor formula with derivatives [equation \(3.5\)](#). The last statement is a logical warning rather than an additional estimate: a determinant is an alternating sum of products of coefficients, so monotonicity of each entry alone does not imply monotonicity of the determinant. $\square$

3.3 The coefficient evolution as a positive semigroup

For $s \geq 0$, define the upper-shift operator P_s on sequences for which the following series converges absolutely by

$$(P_s a)_n := \sum_{m=0}^{\infty} \frac{(2n+2m)!}{(2n)!m!} a_{n+m} s^m.$$

The Taylor law states that $q(t+s) = P_s q(t)$. The family $(P_s)_{s \geq 0}$ has the semigroup property whenever the relevant series converge absolutely.

Proposition 3.5 (Semigroup identity). *For $r, s \geq 0$ and every admissible sequence a ,*

$$P_s P_r a = P_{s+r} a.$$

In particular, $q(t+s+r) = P_s P_r q(t) = P_{s+r} q(t)$ for real t and nonnegative r, s .

Proof. The coefficient of a_{n+k} in $(P_s P_r a)_n$ is

$$\frac{(2n+2k)!}{(2n)!} \sum_{m=0}^k \frac{s^m r^{k-m}}{m!(k-m)!} = \frac{(2n+2k)!}{(2n)!k!} (s+r)^k,$$

by the binomial theorem. This is the coefficient of a_{n+k} in $(P_{s+r} a)_n$. Absolute convergence justifies rearranging the sums. $\square$

Remark 3.6. The entries of $q(t)$ increase with t , and the semigroup coefficients are nonnegative for nonnegative increments. Neither fact by itself proves monotonicity of any determinant: determinants are alternating sums, and the relevant total-positivity property concerns all minors of a matrix rather than entrywise order alone.

Remark 3.7 (Raw moments versus normalized coefficients). Set $\mu_n(t) = \int_0^\infty u^{2n} e^{tu^2} \Phi(u) du$. Cauchy–Schwarz gives $\mu_n(t)^2 \leq \mu_{n-1}(t) \mu_{n+1}(t)$ for $n \geq 1$. Since $q_n(t) = \mu_n(t)/(2n)!$, this does not directly decide the sign of

$$D_t(n, 2) = q_n(t)^2 - q_{n-1}(t) q_{n+1}(t).$$

The factorial normalization changes the inequality by the factor $(2n-2)!(2n+2)!/((2n)!)^2 > 1$. Coefficient positivity and raw-moment log-convexity are therefore not substitutes for Toeplitz total positivity.

4 Toeplitz minors and Pólya–frequency sequences

For a real sequence $a = (a_n)_{n \geq 0}$, set $a_k = 0$ for $k < 0$ and define the one-sided lower-triangular Toeplitz matrix

$$T(a) = (a_{i-j})_{i,j \geq 0}. \quad (4.1)$$

Definition 4.1. The sequence a is a Pólya–frequency sequence of infinite order, denoted $a \in \text{PF}_\infty$, if every finite minor of $T(a)$ is nonnegative. More generally, $a \in \text{PF}_m$ if every minor of order at most m is nonnegative.

For the heat-flow coefficients, define the consecutive Toeplitz minors

$$D_t(n, r) = \det[q_{n+i-j}(t)]_{i,j=0}^{r-1}, \quad n \geq 0, \quad r \geq 1. \quad (4.2)$$

These are minors formed from r consecutive rows and r consecutive columns of the one-sided Toeplitz matrix. They are an important subfamily, but for weak inequalities it is not legitimate

to identify this family with all Toeplitz minors merely by invoking the strict form of Fekete's criterion. The distinction between strict and weak inequalities is central to the arguments below.

For the first three orders, the definitions read

$$\begin{aligned} D_t(n, 1) &= q_n(t), \\ D_t(n, 2) &= q_n(t)^2 - q_{n-1}(t)q_{n+1}(t), \\ D_t(n, 3) &= q_n(t)^3 - 2q_{n-1}(t)q_n(t)q_{n+1}(t) + q_{n-1}(t)^2q_{n+2}(t) \\ &\quad + q_{n-2}(t)q_{n+1}(t)^2 - q_n(t)q_{n-2}(t)q_{n+2}(t), \end{aligned}$$

where $q_k(t) = 0$ for $k < 0$. Thus order two gives the ordinary log-concavity inequalities, while order three already tests a genuinely higher-order relation. The convention for negative indices also makes the formulas valid at the lower-triangular boundary $n = 0, 1$.

Proposition 4.2 (The order-two criterion). *If $a_n > 0$ for every $n \geq 0$, then $a \in \text{PF}_2$ if and only if*

$$a_n^2 \geq a_{n-1}a_{n+1} \quad (n \geq 1).$$

For the heat-flow coefficients this is exactly $D_t(n, 2) \geq 0$ for every $n \geq 1$.

Proof. The adjacent 2×2 minors yield necessity. For sufficiency, the inequalities are equivalent to monotonic decrease of the ratios a_n/a_{n-1} . Every nonzero 2×2 Toeplitz minor can be written as a comparison of two products of entries whose ratio is a product of such adjacent ratios. Ratio monotonicity gives the required sign. Minors forced to vanish by the triangular support are zero. $\square$

Remark 4.3 (A finite check is not a PF certificate). Membership in PF_m requires every minor of orders up to m at every admissible row and column configuration. Checking finitely many shifts, even at a single order, is not a certificate for PF_m unless an additional theorem controls the untested minors. Membership in PF_∞ has the corresponding infinite quantifier over all orders.

Lemma 4.4 (Strict consecutive-minor criterion). *Let $a_n > 0$ for all $n \geq 0$ and suppose $\sum_{n \geq 0} a_n < \infty$. If*

$$D_a(n, r) := \det[a_{n+i-j}]_{i,j=0}^{r-1} > 0 \quad \text{for every } n \geq 0, r \geq 1,$$

then $a \in \text{PF}_\infty$.

Proof. Fix an integer $m \geq 1$. The hypotheses give positivity of every admissible consecutive minor of every order $r \leq m$, because the non-structurally-zero consecutive minors of $T(a)$ are precisely the determinants $D_a(n, r)$. The strict matrix criterion (Theorem D in Katkova's account, attributed there to Schoenberg) then implies positivity of every admissible minor of orders at most m ; see also the summable one-sided sequence version of Katkova and Ostrovskii [12, Theorem D]; [13, 25]. Since m is arbitrary, $a \in \text{PF}_\infty$. The summability hypothesis will hold for $q_n(t)$ because $\sum q_n(t) = F_t(1) < \infty$. $\square$

Corollary 4.5 (Finite-order strict criterion). *Let $m \geq 1$. If $a_n > 0$, $\sum_{n \geq 0} a_n < \infty$, and*

$$D_a(n, r) > 0 \quad (n \geq 0, 1 \leq r \leq m),$$

then $a \in \text{PF}_m$; that is, every minor of $T(a)$ of order at most m is nonnegative (indeed, every non-structurally-zero such minor is positive).

Proof. Apply the one-sided consecutive-minor criterion separately at each order $r = 1, \dots, m$. The consecutive minors of order r are exactly $D_a(n, r)$ for $n \geq 0$, apart from the structurally zero minors with negative diagonal offset. The criterion therefore gives positivity of every non-structurally-zero minor of every order at most m . $\square$

Remark 4.6. The finite-order conclusion still has infinitely many inequalities: even for a fixed m , the shift n ranges over all nonnegative integers and the row and column configurations need not be consecutive. A finite rectangle of computed values of $D_a(n, r)$ is not by itself a certificate for membership in PF_m .

Remark 4.7 (Why the nonnegative version is not used). The strict criterion in [theorem 4.4](#) is not a license to replace $D_a(n, r) > 0$ by $D_a(n, r) \geq 0$ in the hypothesis. In the nonnegative setting, a separate theorem is required to pass from a selected family of consecutive minors to all minors. No such weak criterion is invoked in this paper. This is the correction to the earlier draft's proposed application of Fekete's criterion.

4.1 The Schoenberg–Edrei representation

The Schoenberg–Edrei theorem characterizes PF_∞ sequences through their ordinary generating functions [\[1, 6, 24\]](#). For a sequence with $a_0 > 0$, the generating function has the form

$$A(z) = \sum_{n \geq 0} a_n z^n = C e^{\gamma z} \frac{\prod_j (1 + \alpha_j z)}{\prod_j (1 - \beta_j z)}, \quad (4.3)$$

where $C > 0$, $\gamma, \alpha_j, \beta_j \geq 0$, and $\sum_j (\alpha_j + \beta_j) < \infty$, with the usual interpretation when factors are absent. The factor z^m in the general theorem is not needed when $a_0 > 0$.

Lemma 4.8 (Entire PF functions of order below one). *Let A be an entire function of order strictly less than one, with real Taylor coefficients and $A(0) > 0$. Then its coefficient sequence belongs to PF_∞ if and only if all zeros of A are real and nonpositive.*

Proof. If the coefficients form a PF_∞ sequence, the Schoenberg–Edrei representation [equation \(4.3\)](#) applies. Entirety rules out nonzero denominator factors, and the order bound rules out a nonconstant exponential factor $e^{\gamma z}$. Thus

$$A(z) = A(0) \prod_j (1 + \alpha_j z), \quad \alpha_j \geq 0,$$

so every zero is real and nonpositive.

Conversely, suppose that all zeros of A are real and nonpositive. Since $A(0) > 0$, zero is not a zero of A . The genus-zero Hadamard factorization for an entire function of order below one therefore gives a locally uniformly convergent product $A(z) = A(0) \prod_j (1 + \alpha_j z)$ with $\alpha_j \geq 0$ and $\sum_j \alpha_j < \infty$. Each finite product has a PF_∞ coefficient sequence by the sufficiency part of the Aissen–Schoenberg–Whitney theorem. Passing to the locally uniform limit, or equivalently taking coefficientwise limits in every finite Toeplitz minor, preserves nonnegativity. Hence the coefficient sequence of A is in PF_∞ . $\square$

5 The coefficient characterization of the de Bruijn–Newman threshold

The moment formula shows that $q_n(t) > 0$ for every n and every real t . Moreover, F_t is not a polynomial, since every coefficient $q_n(t)$ is strictly positive. Since F_t has order below one, its genus-zero factorization contains infinitely many zeros.

Theorem 5.1 (Coefficient-level threshold criterion). *For each real t , the following are equivalent:*

- (i) *all zeros of H_t are real;*
- (ii) *the coefficient sequence $q(t) = (q_n(t))_{n \geq 0}$ belongs to PF_∞ ;*

(iii) $D_t(n, r) > 0$ for every $n \geq 0$ and every $r \geq 1$.

Consequently,

$$\{t \in \mathbb{R} : D_t(n, r) > 0 \text{ for all } n \geq 0, r \geq 1\} = [\Lambda, \infty), \quad \Lambda = \inf\{t : q(t) \in \text{PF}_\infty\}. \quad (5.1)$$

Proof. By construction, the zeros of $F_t(w) = H_t(i\sqrt{w})$ are precisely the images $w = -z^2$ of zeros z of H_t . The condition that all zeros of H_t be real is equivalent to all zeros of F_t being real and nonpositive: if $-z^2$ is real and nonpositive, then z is real (the alternative purely imaginary z would give a nonnegative value of $-z^2$ and is excluded because $F_t(0) = H_t(0) > 0$ and a nonzero purely imaginary zero would correspond to a positive zero of F_t). More directly, since the zero of F_t associated to a zero z of H_t is $-z^2$, a nonreal z cannot map to a negative real number unless z is real.

By [theorem 2.2](#), F_t has order below one, and $F_t(0) = q_0(t) > 0$. Thus [theorem 4.8](#) proves the equivalence of (i) and (ii).

We next prove (ii) $\Rightarrow$ (iii). If $q(t) \in \text{PF}_\infty$, then [theorem 4.8](#) shows that every zero of F_t is negative real. Write the genus-zero product as

$$F_t(w) = q_0(t) \prod_{j \geq 1} (1 + \beta_j(t)w), \quad \beta_j(t) > 0.$$

There are infinitely many factors because F_t is not a polynomial. The rectangular Jacobi–Trudi formula in [theorem 7.1](#) gives

$$D_t(n, r) = q_0(t)^r s_{(r^n)}(\beta(t)).$$

For an infinite alphabet of strictly positive entries the rectangular Schur function is strictly positive; for $n = 0$ the determinant is simply $q_0(t)^r$. Hence every $D_t(n, r) > 0$.

Finally, (iii) $\Rightarrow$ (ii) follows from [theorem 4.4](#). The coefficients are positive and summable, since $\sum_{n \geq 0} q_n(t) = F_t(1) < \infty$. The de Bruijn–Newman theorem identifies the all-real-zero parameter set with $[\Lambda, \infty)$, which proves [equation \(5.1\)](#). $\square$

Corollary 5.2 (A nonpositive consecutive minor certifies a parameter below the threshold). For real t ,

$$t < \Lambda \iff \text{there exist } n \geq 0, r \geq 1 \text{ with } D_t(n, r) \leq 0.$$

In particular, a rigorously certified negative minor at time t_0 proves the strict lower bound $\Lambda > t_0$. A zero minor also excludes $t_0 \geq \Lambda$, but it should not be mislabeled as a negative determinant.

Proof. If $t \geq \Lambda$, [theorem 5.1](#) gives strict positivity of every consecutive minor. Conversely, if $t < \Lambda$, the condition in item (iii) of the theorem fails, so at least one consecutive minor is nonpositive. $\square$

Remark 5.3 (Asymmetry of certificates). A single certified $D_t(n, r) < 0$ gives a rigorous lower bound on Λ . By contrast, to prove $\Lambda \leq T$ using [theorem 5.1](#), one must prove $D_T(n, r) > 0$ for every shift n and order r , or use an independent theorem that controls all remaining zeros or minors. A finite rectangle of positive determinants cannot establish this infinite statement by itself.

6 The coefficient criterion at the Riemann Ξ -function

Define the normalized Taylor coefficients c_n by

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right) = \sum_{n=0}^{\infty} (-1)^n c_n z^{2n}. \quad (6.1)$$

By [equation \(2.2\)](#),

$$F_0(w) = H_0(i\sqrt{w}) = \frac{1}{8} \Xi \left(\frac{i\sqrt{w}}{2} \right) = \sum_{n=0}^{\infty} \frac{c_n}{8 \cdot 4^n} w^n,$$

so

$$q_n(0) = \frac{c_n}{8 \cdot 4^n}. \quad (6.2)$$

For the traditional consecutive determinant, write

$$\Delta(n, r) = \det[c_{n+i-j}]_{i,j=0}^{r-1}, \quad c_k = 0 \quad (k < 0).$$

Factoring the row and column scales gives

$$D_0(n, r) = 8^{-r} 4^{-nr} \Delta(n, r). \quad (6.3)$$

All factors on the right are positive, so the two determinant families have identical signs at matching shift and order.

Corollary 6.1 (Strict positivity of the first three base-time rows). *Assume Sra's one-sided reciprocal-density result [27, Proposition 3.2]. Then*

$$D_0(n, r) > 0 \quad (n = 0, 1, 2, \quad r \geq 1).$$

Proof. Let $G(z) = \xi(\frac{1}{2} + \sqrt{z}) = \sum_{n \geq 0} c_n z^n$ and $c_0 = \xi(\frac{1}{2}) > 0$. By the genus-zero product, write

$$G(z) = c_0 \prod_j (1 + \beta_j z), \quad \sum_{m \geq 0} h_m z^m = \prod_j (1 - \beta_j z)^{-1} = \frac{c_0}{G(-z)}.$$

The dual Jacobi–Trudi identity gives

$$\Delta(1, r) = c_0^r h_r, \quad \Delta(2, r) = c_0^r (h_r^2 - h_{r-1} h_{r+1}).$$

Let Θ be the positive density in [27], whose one-sided Laplace transform is $1/\xi(\frac{1}{2} + \sqrt{s})$, and set $f(x) = c_0 \Theta(x)$ on $(0, \infty)$ and zero for $x \leq 0$. Since $\int_0^\infty \Theta(x) dx = 1/c_0$, f is a probability density. For real $s > -1$, define its normalized moments by

$$M_s := \frac{\mathbb{E}_f[X^s]}{\Gamma(s+1)}.$$

Sra's Lemma 2.4 gives finite negative moments of every order and exponential moments near the origin, so the hypotheses on the moments are satisfied. Coefficient comparison in the transform expansion gives $h_m = M_m > 0$ for every integer $m \geq 0$. Sra's Proposition 3.2 states that the zero extension of Θ , and hence f , is PF₂. Theorem 1 of Karlin–Proschan–Barlow [11] implies that the normalized-moment kernel M_{s+t} is strictly SRR₂ for $s, t > -1/2$. Taking row parameters $r-1, r$ and column parameters $0, 1$ yields

$$\det \begin{pmatrix} M_{r-1} & M_r \\ M_r & M_{r+1} \end{pmatrix} < 0, \quad \text{so} \quad M_r^2 - M_{r-1} M_{r+1} > 0.$$

Thus $\Delta(1, r) > 0$ and $\Delta(2, r) > 0$ for every $r \geq 1$. The scale relation [equation \(6.3\)](#) transfers these signs to $D_0(1, r)$ and $D_0(2, r)$. Finally, $D_0(0, r) = q_0(0)^r > 0$ because the defining matrix is triangular with diagonal $q_0(0) > 0$. $\square$

Remark 6.2. The all-order claim in [theorem 6.1](#) depends on Sra's 20 October 2026 preprint, version 1, for Proposition 3.2; the moment step uses the 1961 theorem of Karlin–Proschan–Barlow. This note checks the normalization and logical application of these inputs, but does not claim an independent reproof of every analytic estimate in Sra's proposition. The result covers only $t = 0$ and shifts $n = 0, 1, 2$; it does not imply PF_∞ for the full coefficient sequence or the Riemann hypothesis.

Corollary 6.3 (Strict positivity of every base-time order-two minor). *The classical Turán inequality of Csordas–Norfolk–Varga [3, Theorem 2.5] implies*

$$D_0(n, 2) > \frac{1}{n} q_{n-1}(0) q_{n+1}(0) > 0 \quad (n \geq 1),$$

and $D_0(0, 2) = q_0(0)^2 > 0$. In particular, $D_0(n, 2) > 0$ for every $n \geq 0$.

Proof. Put $b_n = \int_0^\infty u^{2n} \Phi(u) du$, so that $q_n(0) = b_n / (2n)!$ by [theorem 3.3](#). Theorem 2.5 of Csordas–Norfolk–Varga gives

$$b_n^2 > \frac{2n-1}{2n+1} b_{n-1} b_{n+1} \quad (n \geq 1).$$

The factorial conversion is exact:

$$\frac{2n-1}{2n+1} \frac{(2n-2)!(2n+2)!}{((2n)!)^2} = \frac{n+1}{n}.$$

Consequently,

$$q_n(0)^2 > \frac{n+1}{n} q_{n-1}(0) q_{n+1}(0),$$

and therefore

$$D_0(n, 2) = q_n(0)^2 - q_{n-1}(0) q_{n+1}(0) > \frac{1}{n} q_{n-1}(0) q_{n+1}(0) > 0.$$

The separate case $n = 0$ follows from the triangular convention $q_{-1}(0) = 0$. Sra’s 10 October 2026 coefficient theorem [26, Theorem 1.2 and Corollary 1.3] provides an independent modern cross-check under its stated interval-arithmetic trust model, but it is not needed for this corollary. $\square$

Corollary 6.4 (Uniform high-index tail of consecutive minors). *Let $r \geq 2$ and $n \geq 10^{18} r^3$ be integers. Then*

$$D_0(n, r) > 0.$$

This is the cubic-tail theorem of Michalowski [17, Theorem 1.1] in the normalization of the present manuscript; we do not claim to have independently replayed its Arb-ball certificates.

Proof. Michalowski defines $a_k = (2k)!^{-1} \int_0^\infty u^{2k} \Phi(u) du$, which equals $q_k(0)$ by [theorem 3.3](#). His consecutive minor is $D_{r,n}^M = \det[a_{n+j-i}]_{i,j=0}^{r-1}$. Our minor is $D_0(n, r) = \det[q_{n+i-j}(0)]_{i,j=0}^{r-1}$, the transpose determinant. Hence $D_0(n, r) = D_{r,n}^M$ and the cited theorem applies directly. $\square$

Remark 6.5 (Combined base-time coverage). The cited results give strict positivity in the following union of regions: all orders at shifts $n = 0, 1, 2$; every shift at orders $r = 1, 2$; and the tail $n \geq 10^{18} r^3$ for every order $r \geq 2$. Thus, relative to these cited inputs, the unhandled low-index region is $n \geq 3$, $r \geq 3$, and $n < 10^{18} r^3$. This is a coverage statement about the results used here, not a claim that no other literature addresses any part of that region.

Corollary 6.6 (An exact consecutive-minor formulation of RH). *The following are equivalent:*

- (i) *the Riemann hypothesis holds;*
- (ii) $q(0) \in \text{PF}_\infty$;
- (iii) $\Delta(n, r) > 0$ for every $n \geq 0$ and $r \geq 1$.

Proof. The Riemann hypothesis is equivalent to all zeros of Ξ being real. By [equation \(2.2\)](#), this is equivalent to all zeros of H_0 being real, and hence to $q(0) \in \text{PF}_\infty$ by [theorem 5.1](#). The equivalence with the strict consecutive-minor family follows from the same theorem and [equation \(6.3\)](#). $\square$

Remark 6.7. The classical PF condition itself is nonnegativity of every finite Toeplitz minor. The strict inequalities in item (iii) are a consequence of the infinitely many negative-real zeros and the strict consecutive-minor criterion; they are not a general replacement of the definition of PF_∞ for arbitrary sequences. In particular, this corollary is not a statement that checking a finite subset of $\Delta(n, r)$ proves RH.

7 Rectangular Schur functions and the product alphabet

By [theorem 2.2](#), F_t has order below one. Since $F_t(0) = q_0(t) > 0$, Hadamard factorization gives a convergent genus-zero product over its zeros $\omega_j(t)$, repeated according to multiplicity:

$$F_t(w) = q_0(t) \prod_{j \geq 1} \left(1 - \frac{w}{\omega_j(t)} \right) = q_0(t) \prod_{j \geq 1} (1 + \beta_j(t)w), \quad \beta_j(t) = -\frac{1}{\omega_j(t)}. \quad (7.1)$$

The convergence of the genus-zero product implies $\sum_j |\beta_j(t)| < \infty$. Let $e_k(\beta(t))$ be the elementary symmetric functions of the alphabet $\beta(t) = (\beta_j(t))_{j \geq 1}$. Comparing coefficients in [equation \(7.1\)](#) gives

$$q_k(t) = q_0(t) e_k(\beta(t)). \quad (7.2)$$

Proposition 7.1 (Rectangular Jacobi–Trudi identity). *For every $n \geq 0$ and $r \geq 1$,*

$$D_t(n, r) = q_0(t)^r s_{(r^n)}(\beta(t)), \quad (7.3)$$

where $s_{(r^n)}$ is the Schur function for the rectangular partition with n rows of length r , and $s_{(r^0)} = 1$.

Proof. Substitute [equation \(7.2\)](#) in [equation \(4.2\)](#) and factor $q_0(t)$ from each row:

$$D_t(n, r) = q_0(t)^r \det[e_{n+i-j}(\beta(t))]_{i,j=0}^{r-1}.$$

The dual Jacobi–Trudi identity states that $s_\lambda = \det[e_{\lambda'_i - i + j}]$, where λ' is the conjugate partition. For $\lambda = (r^n)$, one has $\lambda' = (n^r)$. Reversing both rows and columns in the resulting determinant leaves its determinant unchanged and yields the displayed matrix. For an infinite alphabet, truncate the alphabet, apply the finite identity, and pass to the limit; the absolute summability $\sum_j |\beta_j(t)| < \infty$ gives coefficientwise convergence of all fixed-degree symmetric functions. $\square$

Remark 7.2 (Interpretation of the alphabet). Each zero of F_t , counted with multiplicity m , corresponds to the pair of opposite nonzero zeros of H_t , each of multiplicity m . It contributes m alphabet entries, not $2m$ entries: the points z and $-z$ are the two preimages of one squared-variable zero. When all zeros of H_t are real, the zeros $\omega_j(t)$ of F_t are negative, so $\beta_j(t) > 0$. In that case the Schur formula explains the strict positivity of the rectangular minors.

8 Multiplicity under the squared-variable map

Lemma 8.1 (Multiplicity transfer). *Let f be even and entire, and let g be the unique entire function such that $f(z) = g(-z^2)$. If $\gamma \neq 0$ is a zero of f of multiplicity m , then $-\gamma^2$ is a zero of g of multiplicity exactly m . At the origin, if g has a zero of multiplicity m , then f has a zero of multiplicity $2m$.*

Proof. For $\gamma \neq 0$, the map $\varphi(z) = -z^2$ has derivative $\varphi'(\gamma) = -2\gamma \neq 0$ and is a local analytic coordinate. If $g(w) = (w + \gamma^2)^m h(w)$ with $h(-\gamma^2) \neq 0$, then

$$f(z) = (-z^2 + \gamma^2)^m h(-z^2) = (-1)^m (z - \gamma)^m (z + \gamma)^m h(-z^2).$$

Thus the multiplicity at γ is m (and the same is true at $-\gamma$). At the origin, $g(w) = w^m h(w)$ with $h(0) \neq 0$ implies $f(z) = (-z^2)^m h(-z^2)$, which has multiplicity $2m$. $\square$

Corollary 8.2. *If a nonzero zero γ of an even entire function is simple, then its squared-variable image $-\gamma^2$ is simple. The distinct zeros γ and $-\gamma$ map to the same point, but their identification does not double its multiplicity.*

Example 8.3. For $a \neq 0$, take $f(z) = 1 - z^2/a^2$. The zeros a and $-a$ are simple. The squared-variable function is $g(w) = 1 + w/a^2$, which also has a simple zero at $-a^2$.

8.1 A Chebyshev convolution identity and its scope

Let U_r be the Chebyshev polynomial of the second kind, characterized by

$$\sum_{r=0}^{\infty} U_r(x) z^r = \frac{1}{1 - 2xz + z^2}.$$

Set

$$W_r(x) = \sum_{j=0}^r U_j(x) U_{r-j}(x). \quad (8.1)$$

Lemma 8.4 (Chebyshev self-convolution). *For every integer $r \geq 0$,*

$$\sum_{r=0}^{\infty} W_r(x) z^r = \frac{1}{(1 - 2xz + z^2)^2}, \quad W_r(x) = \frac{1}{2} U'_{r+1}(x).$$

Proof. The generating-function identity follows by squaring the generating function for U_r . Differentiating that generating function with respect to x gives

$$\sum_{r=0}^{\infty} U'_{r+1}(x) z^r = \frac{2}{(1 - 2xz + z^2)^2}.$$

Comparison of coefficients proves the second identity. □

Remark 8.5 (No automatic duplication). The identity in [theorem 8.4](#) is exact, but it concerns the square of a generating function. It may be useful when a separately justified algebraic model contains a duplicated alphabet. It does not follow from the evenness of Ξ alone: by [theorem 8.1](#), a pair of opposite simple zeros produces one simple zero in the squared variable. Consequently, the Chebyshev self-convolution cannot be used as a multiplicity correction for the coefficient product [equation \(7.1\)](#) without an independent reason to duplicate the factors.

8.2 A zero-free-region refinement of the sector cutoff

The sector argument can be sharpened by using a newer explicit zero-free region, rather than only the critical strip. Let

$$H = 3,000,175,332,800,$$

the exact height through which the Riemann hypothesis has been rigorously verified by Platt–Trudgian [\[22\]](#). Bellotti–Trudgian–Yang prove in a 2026 preprint that, for $t \geq 3$, there are no zeros in

$$\sigma > 1 - \frac{1}{C \log t}, \quad C = 4.896$$

[\[2, Theorem 1\]](#). This preprint is an external input; we do not independently reproduce its zero-free-region proof.

Corollary 8.6 (Conditional refinement of the blind-zone cutoff). *Assume the pairwise Schur-positivity mechanism of the Blind Zone Theorem stated in [16]. The proof below also supplies the countable-alphabet passage needed for the actual Ξ -function. Then every consecutive minor of order at most*

$$R_{\text{safe}} = 9,555,000,000,000$$

is strictly positive at base time, for every shift $n \geq 0$. In particular, no negative consecutive-minor certificate of order at most R_{safe} can distinguish the Riemann hypothesis from its negation.

Proof. If an off-critical-line zero $\rho = \beta + i\gamma$ exists, then the verified computation forces $|\gamma| > H > 3$. The zero-free region and the functional-equation symmetry $\beta \leftrightarrow 1 - \beta$ give, with $\delta(t) = 1/(C \log t)$,

$$\delta(|\gamma|) \leq \beta \leq 1 - \delta(|\gamma|), \quad \left| \beta - \frac{1}{2} \right| \leq \frac{1}{2} - \delta(|\gamma|).$$

The angle in the Blind Zone Theorem is therefore bounded by

$$\theta(\rho) = 2 \arctan \left(\frac{|\beta - 1/2|}{|\gamma|} \right) \leq 2 \arctan g(|\gamma|), \quad g(t) = \frac{1/2 - 1/(C \log t)}{t}.$$

Differentiation gives

$$g'(t) = \frac{1}{t^2} \left[-\frac{1}{2} + \frac{1}{C} \left(\frac{1}{\log t} + \frac{1}{(\log t)^2} \right) \right] < 0 \quad (t \geq H),$$

where the strict sign follows from $t \geq H > e$ and $C > 4$, so that $C^{-1}(1/\log t + 1/(\log t)^2) < 2/4 = 1/2$. Consequently, every such angle is at most $\theta_H = 2 \arctan g(H)$. For a conservative integer cutoff, use $H > 3 \cdot 10^{12}$, $\log H < 29$, and $C < 4.9$. These imply

$$\delta(H) > \frac{1}{4.9 \cdot 29} > 0.007, \quad \theta_H < \frac{0.986}{3 \cdot 10^{12}} < 3.287 \cdot 10^{-13}.$$

Since $\pi > 3.1415$,

$$\frac{\pi}{\theta_H} > \frac{3.1415}{3.287 \cdot 10^{-13}} > 9,557,347,125,038 > R_{\text{safe}} + 1.$$

Thus $\theta_H < \pi/(R_{\text{safe}} + 1)$, and hence every violating-pair angle is below $\pi/(r + 1)$ for each $r \leq R_{\text{safe}}$.

The cited version of the Blind Zone Theorem is stated for a finite union of conjugate pairs, whereas the full Ξ -alphabet may contain countably many such pairs. We justify the required extension rather than silently applying the finite statement to an infinite alphabet. Write $\beta = X \sqcup Y$, where X consists of the positive real variables coming from zeros on the critical line, and let Y_M be an increasing sequence of finite, pairwise-conjugate truncations of Y . Hardy's theorem gives infinitely many zeros on the critical line [10]; consequently X contains infinitely many positive entries and $\mathbf{s}_{(r^n)}(X) > 0$ for every fixed n, r .

Fix $r \leq R_{\text{safe}}$ and $\lambda = (r^n)$. For each finite Y_M and each partition $\nu \subseteq \lambda$, the iterated Littlewood–Richardson expansion over the conjugate pairs has nonnegative terms: every two-variable constituent has first-row length at most $\nu_1 \leq r$, so its Chebyshev factor is positive because $(r + 1)\theta_H < \pi$. If $\ell(\nu) > |Y_M|$, its Schur value vanishes; otherwise it is positive by the same pairwise expansion. Hence $\mathbf{s}_\nu(Y_M) \geq 0$. A further Littlewood–Richardson expansion gives $\mathbf{s}_{\lambda/\mu}(Y_M) \geq 0$ for every $\mu \subseteq \lambda$. The coproduct identity therefore yields

$$\mathbf{s}_\lambda(X \sqcup Y_M) = \sum_{\mu \subseteq \lambda} \mathbf{s}_\mu(X) \mathbf{s}_{\lambda/\mu}(Y_M) \geq \mathbf{s}_\lambda(X) > 0,$$

where the lower bound is the $\mu = \lambda$ term. The genus-zero product gives $\sum_j |\beta_j| < \infty$, so fixed-degree Schur evaluations converge under truncation. Passing to the limit $M \rightarrow \infty$ preserves

the same positive lower bound and proves $s_\lambda(X \sqcup Y) > 0$. This also clarifies a point in the cited finite-pair proof: individual skew-Schur factors need only be nonnegative and can vanish when their length exceeds the number of variables; strictness of the full rectangular Schur value follows from a separate positive coproduct term.

The positive scaling identity [equation \(6.3\)](#) transfers this to $D_0(n, r')$. If RH holds, the entire squared-variable alphabet is positive and [theorem 7.1](#) gives strict positivity directly. Thus the result is conditional on the cited pairwise Schur/Chebyshev positivity mechanism, the zero-verification result, and the zero-free-region preprint; it does not prove RH. $\square$

Remark 8.7. The critical-strip-only sector cutoff recorded in [[17](#), Section 1.2] is $R_{\text{strip}} = \lfloor \pi H \rfloor - 1 = 9,425,328,785,004$. The conservative cutoff above increases that number by 129,671,214,996. High-precision floating-point evaluation suggests a maximal endpoint of 9,561,277,002,984, but this exact endpoint is not asserted here because no interval-arithmetic enclosure of the final logarithm and arctangent has been produced. This is a conditional corollary combining existing results, not a standalone claim of a new zero-free region or a new general sector theorem.

9 Evolution of the determinant hierarchy

The coefficient evolution determines the derivative of every fixed consecutive minor, but does not by itself preserve the sign of that minor. We record the exact identity for subsequent analysis.

Proposition 9.1 (Derivative of a consecutive minor). *For $k \in \mathbb{Z}$, put $q_k(t) = 0$ for $k < 0$. For fixed $n \geq 0$ and $r \geq 1$, let $M_t = [q_{n+i-j}(t)]_{i,j=0}^{r-1}$. For each row index ℓ , let $M_t^{(\ell)}$ be the matrix obtained from M_t by replacing row ℓ with*

$$(2(n + \ell - j + 1)(2(n + \ell - j) + 1)q_{n+\ell-j+1}(t))_{j=0}^{r-1}.$$

Then

$$\frac{d}{dt} D_t(n, r) = \sum_{\ell=0}^{r-1} \det M_t^{(\ell)}. \quad (9.1)$$

Proof. The determinant is multilinear in its rows. Differentiate it row by row and apply [equation \(3.4\)](#) to each entry in the differentiated row. The convention $q_k = 0$ for $k < 0$ makes the displayed formula valid at the triangular boundary as well. $\square$

Remark 9.2. The summands on the right side of [equation \(9.1\)](#) are not, in general, the original consecutive minors. Their signs are therefore not determined by $D_t(n, r)$ alone. In particular, coefficientwise monotonicity from [theorem 3.4](#) should not be interpreted as monotonicity of $t \mapsto D_t(n, r)$ or as a sign-preservation theorem.

9.1 Finite certificates: what they can and cannot establish

A finite determinant computation can serve several mathematically different purposes.

- (a) A rigorous certificate that $D_{t_0}(n, r) < 0$ proves $t_0 < \Lambda$ by [theorem 5.2](#). The certificate must include all arithmetic and approximation errors in the sign computation.
- (b) A proof that $D_{t_0}(n, r) = 0$ also proves $t_0 < \Lambda$, because every consecutive minor is strictly positive on the real-zero regime. Numerically observing a value close to zero is not a proof of equality.
- (c) Proving $D_T(n, r) > 0$ for a finite set of pairs (n, r) proves only those inequalities. It does not establish $T \geq \Lambda$ unless an independent argument covers every remaining pair or directly controls the zeros.
- (d) A finite-order condition PF_m still quantifies over infinitely many shifts and admissible minors; it is not synonymous with checking a finite table.

10 The current status of Λ and adjacent positivity results

This section separates the algebraic reformulation proved above from numerical upper-bound claims in the literature. Rodgers and Tao proved the lower bound $\Lambda \geq 0$; the Riemann hypothesis is equivalent to $\Lambda \leq 0$, and hence—using the lower bound—to $\Lambda = 0$ [23]. The published upper-bound sequence includes $\Lambda \leq 0.22$ by the Polymath project and $\Lambda \leq 0.2$ in the Platt–Trudgian line of work [4, 22].

The following table records selected claims available by 25 October 2026. “Reported” is deliberate: the present paper has not independently checked any of the computer-assisted certificates, and public availability, formalization of part of a proof, independent replay, and external peer review are distinct standards. The list is selective rather than an exhaustive ranking.

Reported bound	Source/status	Qualification recorded in the source
$\Lambda \leq 0.22$	Polymath (2019), published	Refereed computational upper bound.
$\Lambda \leq 0.2$	Platt–Trudgian (2021), published	Last refereed bound identified by the current public status index.
$\Lambda \leq 0.1965$	Platt (2026), public research record; related certificate package	Publicly reported certificate-based bound; the public index classifies post-2021 packages as non-refereed and not independently replayed.
$\Lambda \leq 0.1875$	Mosaic Intelligence (2026), public package	Self-archived certificate package; not treated here as externally peer reviewed.
$\Lambda \leq 0.1787854$	Gomila (2026), public package	Public computational certificate; no independent peer review is asserted here.
$\Lambda < 0.172422$	Kornaus (2026), public package	Public certificate materials; independent review status must be checked at the cited record.
$\Lambda < 0.1595$	Mosaic Intelligence (2026), public package	The public index marks this and later packages as non-refereed and not independently replayed; this route descends from its 0.1875 package.
$\Lambda < 0.158$	Gordon (2026), preprint	Reports a rational barrier certificate; the companion Lean formalization verifies stated steps conditional on explicitly listed analytic inputs.
$\Lambda \leq 0.13963$	Hakimoglu-Brown (2026), preprint	Conditional on the additional zero-free hypothesis for nontrivial zeros of $\zeta(s)$ in $\Re(s) > 7/8$.

Sources: [4, 7–9, 15, 18–22]. The entries after 0.2 are not results proved by the present paper and are not placed on the same evidentiary footing as the published bounds. The live index specifically describes post-2021 certificate packages as self-archived, non-refereed, and not independently replayed; it also records dependencies among some of the smaller reported bounds [20]. Their validity and hypotheses must be assessed from the original papers, code, certificates, and independent review.

The coefficient formulation developed here does not improve any of these numerical bounds. It instead gives a precise dictionary between the heat parameter and a strict infinite hierarchy of Toeplitz inequalities. To turn that dictionary into a new upper bound, one must prove the positivity of all the required minors at a candidate time, or invoke an independent zero-dynamics argument that controls the untested range. A finite set of numerical signs does not supply this missing step.

10.1 Connection to recent coefficient-tail work

Michalowski’s 2026 preprint proves strict positivity for a uniform high-index region of consecutive minors of the normalized ξ -coefficients, at indices $k \geq 10^{18}r^3$ [17]. Such a result is complementary to the threshold equivalence: it controls a tail but deliberately does not resolve the low-index

region where an all-order RH criterion remains open. The present manuscript cites this theorem only as an external corollary after matching the coefficient normalization; the heat-flow identities themselves do not depend on it.

Sra's October 2026 preprint on reciprocal Laplace-transform densities establishes PF_2 for the one-sided reciprocal density and PF_{10} for the related bilateral density, using stated zero-verification input [27]. The base-time corollary in Section 6 uses only Proposition 3.2 for the one-sided density, together with the classical normalized-moment theorem, to cover shifts $n = 0, 1, 2$. The Jacobi-theta sign-regularity preprint proves low-order kernel inequalities and related Turán inequalities for normalized coefficients [26]. None of these statements gives PF_∞ for $q_n(t)$ at arbitrary heat time.

11 Open problems and a research programme

The exact threshold criterion in [theorem 5.1](#), together with the established cubic-tail theorem, leaves a sharply described low-index problem: how can one control the remaining family of consecutive minors at a chosen time without evaluating each pair separately? The following questions isolate several routes and their present limitations.

Problem 11.1 (Quantitative propagation of determinant positivity). Sharpen or extend the existing uniform cubic-tail theorem for $t = 0$, whose present threshold is $n \geq 10^{18}r^3$ [17]. A stronger result would lower the explicit constant, improve the cubic dependence, or prove positivity in a complementary low-index region. Any proposed propagation from a finite region must state hypotheses and error bounds rather than infer them from observed asymptotic behaviour.

Problem 11.2 (Beyond the first three base-time rows). The cited results settle $D_0(n, r) > 0$ for $n = 0, 1, 2$ at every order, for every shift at orders $r = 1, 2$, and in the tail $n \geq 10^{18}r^3$. The central remaining base-time region in this collection of results is $n \geq 3, r \geq 3, n < 10^{18}r^3$. Determine whether one can prove an additional full row $D_0(n_0, r) > 0$ for all $r \geq 1$ at a fixed shift $n_0 \geq 3$, or shrink this region uniformly. Useful analogues at nonzero heat time are also open. Even an affirmative answer for one additional row would not imply RH without a separate theorem connecting it to all shifts and orders.

Problem 11.3 (A valid use of the Chebyshev convolution). Identify an independently justified generating-function factorization under which the self-convolution $W_r = \frac{1}{2}U'_{r+1}$ corresponds to the relevant coefficient alphabet. The evenness of Ξ and the squared-variable map are insufficient, on their own, to create duplicated alphabet entries.

Problem 11.4 (A determinant evolution inequality). Use [equations \(3.4\)](#) and [\(9.1\)](#) to derive a quantitative differential inequality for a family of consecutive minors. A useful result would need to control the mixed determinants in [equation \(9.1\)](#); the coefficient equation alone does not establish their signs.

Problem 11.5 (A new upper bound for Λ). Prove either (i) strict positivity of $D_T(n, r)$ for every n, r at some explicit T below a rigorously accepted upper bound, or (ii) an independent zero-exclusion result for all nonreal zeros of H_T . Any computer-assisted proof should specify exact input data, interval arithmetic, truncation tails, independent verification, and the theorem-level implications of each certificate.

12 Conclusion

The classical de Bruijn–Newman family admits an exact coefficient evolution after the entire function is expressed in a squared variable. The coefficient recurrence is simple, and the entire-function order bound permits a canonical product. These facts connect the real-zero condition

to the Pólya–frequency property of the coefficient sequence and, via strict consecutive-minor theory, to strict positivity of the full family $D_t(n, r)$.

Two logical points are essential. First, the PF_∞ condition is defined by nonnegativity of every finite Toeplitz minor; for the present strictly positive coefficient sequence, the consecutive-minor test that is equivalent to the real-zero property uses strict inequalities $D_t(n, r) > 0$. A weak nonnegative statement for only this selected family is not substituted for total positivity. Second, evenness pairs the zeros γ and $-\gamma$ but does not double the multiplicity of their common nonzero image under the squared-variable map. These corrections constrain how symmetric-function identities can be applied to the Riemann Ξ -function.

The framework remains a reformulation, not a numerical advance on Λ . The central open step is quantitative: one needs a proof that controls all orders and shifts, or an independent argument that excludes nonreal zeros uniformly. No finite table of determinant signs, however extensive, can replace that missing theorem without a proved mechanism for controlling the untested range.

A A direct derivation of the coefficient formulas

From [equation \(1.1\)](#),

$$F_t(w) = \int_0^\infty e^{tu^2} \Phi(u) \cosh(\sqrt{w} u) \, du.$$

For a compact set $K \subset \mathbb{C}$ in the w -plane, the series for $\cosh(\sqrt{w} u)$ converges uniformly in $w \in K$ and is bounded in modulus by $\exp(C_K u)$ for a suitable C_K . [Lemma 2.1](#) therefore permits termwise integration, yielding

$$F_t(w) = \sum_{n=0}^\infty \left(\frac{1}{(2n)!} \int_0^\infty u^{2n} e^{tu^2} \Phi(u) \, du \right) w^n.$$

This proves [equation \(3.6\)](#). Differentiating the integral m times with respect to t gives

$$q_n^{(m)}(t) = \frac{1}{(2n)!} \int_0^\infty u^{2n+2m} e^{tu^2} \Phi(u) \, du = \frac{(2n+2m)!}{(2n)!} q_{n+m}(t),$$

which is [equation \(3.5\)](#). The decay bound in [theorem 2.1](#) justifies all differentiations and the local uniformity needed for holomorphic dependence on t .

B Consecutive minors in the one-sided Toeplitz indexing

Let $T(a)_{ij} = a_{i-j}$ for $i, j \geq 0$ and set $a_k = 0$ for $k < 0$. A consecutive minor of order r with row block $i, i+1, \dots, i+r-1$ and column block $j, j+1, \dots, j+r-1$ is

$$\det[a_{i-j+\ell-m}]_{\ell, m=0}^{r-1}.$$

If $i \geq j$, this equals $D_a(i-j, r)$. If $i < j$, the first row is structurally zero and the determinant vanishes. Thus the family $D_a(n, r)$ indexes precisely the potentially nonzero consecutive minors. The strict one-sided consecutive-minor criterion applies when every such admissible consecutive minor is positive. This appendix does not assert the corresponding weak nonnegative criterion; the latter is not needed for the threshold theorem.

C Finite-alphabet check of the Jacobi–Trudi normalization

Let

$$F(w) = q_0(1 + \beta_1 w)(1 + \beta_2 w), \quad q_0 > 0.$$

Then $q_1 = q_0(\beta_1 + \beta_2)$, $q_2 = q_0\beta_1\beta_2$, and $q_k = 0$ for $k \geq 3$. Direct expansion gives

$$D(1, 2) = q_1^2 - q_0q_2 = q_0^2(\beta_1^2 + \beta_1\beta_2 + \beta_2^2) = q_0^2\mathbf{s}_{(2)}(\beta_1, \beta_2).$$

At shift two,

$$D(2, 2) = q_2^2 = q_0^2\beta_1^2\beta_2^2 = q_0^2\mathbf{s}_{(2,2)}(\beta_1, \beta_2).$$

The factor in the general formula is q_0^r because the determinant has size r . The example is solely an index and normalization check; it is not evidence about the infinite product for F_t .

D The effect of changing the heat parameter scale

Suppose a reparameterized family is written as

$$\tilde{H}_s(z) = H_{cs}(az), \quad a \in \mathbb{R} \setminus \{0\}, \quad c > 0.$$

The real-zero property is unchanged by the nonzero real scaling of z , while the parameter condition becomes $cs \geq \Lambda$. Thus the threshold in the parameter s is Λ/c . This explains why heat-parameter values should never be compared until the Gaussian exponent and the argument scaling have both been stated. The normalization in [equations \(1.1\) and \(2.2\)](#) is fixed throughout the main text.

E Claim and evidence ledger

Status	Statements
Proved here	Kernel positivity and decay estimate; heat equation; coefficient integral and all-order derivative formula; squared-variable multiplicity lemma; product factorization; rectangular Jacobi–Trudi identity; coefficient-level threshold equivalence using the cited strict criterion.
Derived from external inputs	$D_0(n, r) > 0$ for $n = 0, 1, 2$ and every $r \geq 1$ (Sra 2026 + Karlin–Proschan–Barlow 1961); $D_0(n, 2) > 0$ for every $n \geq 0$ (Csordas–Norfolk–Varga 1986); and $D_0(n, r) > 0$ when $r \geq 2$, $n \geq 10^{18}r^3$ (Michalowski 2026).
Classical inputs	De Bruijn–Newman real-zero threshold; Schoenberg–Edrei characterization; strict one-sided consecutive-minor criterion; Rodgers–Tao’s lower bound.
Not proved here	Any new numerical upper bound for Λ ; all-order positivity at nonzero heat time; positivity for all orders at shifts $n \geq 3$; a theorem converting a finite set of positive minors into all-order positivity.
Not used as proof	Unreproduced exploratory numerical observations or finite sign tables that do not include rigorous error bounds and independent verification.

Acknowledgement and computational statement

No computational output is used as a proof of an infinite family of inequalities in this manuscript. If a later version incorporates machine-assisted estimates, the author should state the software, arithmetic model, truncation estimates, source code, and independent verification method. Any acknowledgement, funding statement, or institutional affiliation should be supplied by the author and should reflect the actual circumstances.

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