

COMPLETE PLANAR ARCS OVER NONSQUARE BINARY FIELDS

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ABSTRACT. Let $m'_2(q)$ be the largest size of a complete arc in $PG(2, q)$ other than a hyperoval. We prove

$$m'_2(q) \leq q - \lfloor q^{2/3} - q^{1/3} \rfloor \quad (q = 2^h \geq 512, h \text{ odd}).$$

The geometric statement underlying the proof is a degree obstruction for arbitrary geometrically irreducible Frobenius nonclassical plane curves; an origin from an arc is unnecessary. The argument uses the curve and a Frobenius twist of its dual, exact weighted point counts, and a local conductor inequality that allows arbitrary singularities. We give a sharper bound obtained by optimizing over powers of two, determine its leading constant according to $h \bmod 3$, and prove further refinements using a characteristic-two normal form. Exact finite computations give, for example, $m'_2(128) \leq 105$, $m'_2(512) \leq 452$, and $m'_2(2048) \leq 1875$.

1. INTRODUCTION

1.1. AI use. Apart from this subsection (and the first paragraph in the next one), this paper is AI generated. I had some notes making the modest improvement of my old bound $m'_2(q) \leq q - \sqrt{2q} + 2$ to $m'_2(q) \leq q - \sqrt{2q}$, (for q even and nonsquare) following the same strategy as the old result, which I uploaded to ChatGPT 5.6 to check the proof and suggest possible improvements. After some back and forth, ChatGPT generated the argument below using the dual curve, substantially improving the bound. I also had Claude Opus 5 run an audit of the manuscript and included corrections from the output of this audit.

1.2. Statements and conventions. Throughout $PG(2, q)$ denotes the projective plane over the finite field of q elements. An arc is a set of points in $PG(2, q)$ with no three collinear; it is complete if it cannot be enlarged. The quantity $m'_2(q)$ is defined to be the largest cardinality of a complete arc other than the $(q + 2)$ -arcs, or hyperovals. Segre's classical bound is $m'_2(q) \leq q - \sqrt{q} + 1$, q even which is tight when q is a square (see, e.g. [2] for the full historical development). I improved it to $m'_2(q) \leq q - \sqrt{2q} + 2$ in [9] for q nonsquare that had seen no improvement until the present work.

Write

$$(1.1) \quad k = q + 2 - t$$

for the size of an arc. Thus an upper bound on k is a lower bound on t .

All plane curves in the geometric assertions are geometrically irreducible. Their genus and their number N of rational points refer to the smooth projective normalization. We separately write M_{sm} for the number of smooth rational points on a plane model. A plane curve is q -Frobenius nonclassical if the tangent at a general point P contains $\text{Fr}_q(P)$; its nontrivial Frobenius order then equals its generic tangent contact order. In characteristic two this order is a power of two.

Theorem 1.1. *Suppose $q = 2^h \geq 512$ and h is odd. Then*

$$(1.2) \quad m'_2(q) \leq q - \lfloor q^{2/3} - q^{1/3} \rfloor.$$

More precisely, put

$$(1.3) \quad A_\nu(q) = \frac{q + 3\nu + 2}{\nu + 2},$$

$$(1.4) \quad B_\nu(q) = \frac{1 - \nu + \sqrt{(\nu - 1)^2 + 4\nu(q + 2)}}{2},$$

$$(1.5) \quad T(q) = \min_{\substack{\nu=2^e \\ 1 \leq e < h}} \max\{\lceil A_\nu(q) \rceil, \lceil B_\nu(q) \rceil\}.$$

Then

$$(1.6) \quad \boxed{m'_2(q) \leq q + 2 - T(q).}$$

Theorem 1.2. *Under the same assumptions on q , every q -Frobenius nonclassical nonlinear plane curve defined over $\mathbf{F}_q$ has degree*

$$(1.7) \quad d \geq T(q), \quad d > q^{2/3} - q^{1/3} + 1.$$

If $d \leq q/8$ and ν is its Frobenius order, then the more informative necessary conditions are

$$(1.8) \quad \boxed{(\nu + 2)d \geq q + 3\nu + 2, \quad d(d + \nu - 1) \geq \nu(q + 2).}$$

The curve theorem is useful because it isolates the main geometric work. The passage to arcs uses only the envelope construction and the Stöhr–Voloch bound. No assumption of ordinary singularities, rational tangent directions at singular points, or irreducibility of the entire envelope is made.

Corollary 1.3. *As h tends to infinity through odd integers,*

$$(1.9) \quad m'_2(2^h) \leq 2^h - c_h 2^{2h/3} + O(2^{h/3}),$$

where

$$c_h = \begin{cases} 1, & h \equiv 0 \pmod{3}, \\ 2^{1/3}, & h \equiv 1 \pmod{3}, \\ 2^{1/6}, & h \equiv 2 \pmod{3}. \end{cases}$$

Section 7 proves the additional exact bound

$$(1.10) \quad \boxed{m'_2(u^3) \leq u^3 - u^2 + \frac{u}{2}, \quad u = 2^{2r+1} \geq 8.}$$

Section 8 records finite bounds, including $q = 32, 128$, where the convenient restriction $t \leq q/8$ does not cover the critical range. The comparison with $q - \sqrt{2q}$ in that section refers to the bound that started this investigation. The classical nonsquare bound is $q - \sqrt{2q} + 2$ from [9], see subsection 1.1.

2. FROM A COMPLETE ARC TO A CURVE

Lemma 2.1. *Let q be even and let $\mathcal{A}$ be a complete k -arc other than a hyperoval, with $k = q + 2 - t$ and $k > t + 2$. Its Segre envelope has degree t . Every geometric irreducible component C is defined over $\mathbf{F}_q$, has degree $2 \leq d \leq t$, and contains at least kd distinct smooth $\mathbf{F}_q$ -points. In particular, $N \geq kd$ on its normalization.*

Proof. Use Segre's envelope theorem in the form of [2, Theorem 11]. For each $a \in \mathcal{A}$, let L_a be its line in the dual plane. The restriction of the degree- t envelope polynomial to L_a has exactly t distinct zeros, the points representing the unisecants through a . All these zeros are rational, and all intersections are simple.

Consequently the envelope is reduced, and each intersection with any L_a is smooth on the entire envelope. A geometric component of degree d meets each L_a in d such points. The resulting kd points are distinct: a unisecant cannot pass through two points of the arc. At any one of

these rational smooth points there is a unique geometric component, so Frobenius fixes that component. It is therefore defined over $\mathbf{F}_q$.

A line component would correspond to a rational point R in the original plane such that every line Ra , $a \in \mathcal{A}$, is a unisecant. The point R cannot belong to the arc, since an L_a is not an envelope component. Thus R could be added to the arc, contradicting completeness. Hence $d \geq 2$. $\square$

Let λ be the nontrivial Frobenius order of the line series on C , using $\lambda = 1$ in the Frobenius classical case. Stöhr–Voloch [8], together with $2g - 2 \leq d(d - 3)$, gives

$$(2.1) \quad N \leq \frac{\lambda(2g - 2) + (q + 2)d}{2} \leq \frac{d\{\lambda(d - 3) + q + 2\}}{2}.$$

The normalization convention in this formula is also stated explicitly in [3, Theorem 1.1]. For an envelope component, therefore,

$$(2.2) \quad \lambda(d - 3) \geq q + 2 - 2t.$$

If $3t < q + 5$, the value $\lambda = 1$ is impossible, because $d \leq t$. Thus every component is Frobenius nonclassical in this range. In particular, this applies when $t \leq q/8$.

3. A CURVE AND A FROBENIUS TWIST OF ITS DUAL

Let q be a power of two and let $C/\mathbf{F}_q$ be a nonlinear Frobenius nonclassical plane curve of degree d , and let ν be its generic tangent contact order. In this section assume $\nu < q$, and write

$$(3.1) \quad \mu = q/\nu.$$

In the applications $d \leq q/8$, so this assumption is automatic: a generic tangent contains the two distinct points P and $P^{(q)}$, and hence $\nu + 1 \leq d$.

We use two standard Gauss-map facts. The generic contact theorem identifies the inseparable Gauss degree with ν [4, Theorem 2.1]. For a Frobenius nonclassical plane curve the Gauss map is purely inseparable [5]; see also [1, Theorem 1.1].

Lemma 3.1. *There is a geometrically irreducible plane curve $C^\dagger/\mathbf{F}_q$ of degree D , where D is the degree of the ordinary dual of C , with the following properties:*

- (1) C and $C^\dagger$ have isomorphic function fields over $\mathbf{F}_q$, and thus have the same genus and the same normalization point count N ;
- (2) $C^\dagger$ is q -Frobenius nonclassical of order $\mu = q/\nu$;
- (3) the ordinary dual of $C^\dagger$ has degree d .

Proof. Choose affine coordinates with x separating and put $K = \mathbf{F}_q(C) = \mathbf{F}_q(x, y)$. The two Gauss-map facts imply that its Gauss field is K^ν . Thus its tangent coefficients can be written using $a, b \in K$ as

$$(3.2) \quad y = a^\nu + xb^\nu, \quad \mathbf{F}_q(a, b) = K,$$

So, $y^\mu = a^q + x^\mu b^q$. Frobenius nonclassicality gives $y^q = a^\nu + x^q b^\nu$, so

$$(3.3) \quad y^\mu = a + x^\mu b, \quad a^q + a = x^\mu(b^q + b).$$

Let $C^\dagger$ be the plane model parametrized by the coordinates (b, a) . Its function field is K over $\mathbf{F}_q$. The ordinary dual of C has coordinates (b^ν, a^ν) , so $C^\dagger$ is obtained by applying the inverse coefficient Frobenius to that dual equation; in particular its degree is D and it is defined over $\mathbf{F}_q$.

Differentiating the second identity in (3.3) and, using that $\nu < q$, so $\mu\nu > 1$ and $d(x^\mu) = 0$, gives $da = x^\mu db$. Here $db \neq 0$, since otherwise $da = db = 0$ would contradict $K = \mathbf{F}_q(a, b)$. In coordinates (B, A) on $C^\dagger$, the tangent line at (b, a) is

$$A + x^\mu B + y^\mu = 0.$$

Its Gauss field is $\mathbf{F}_q(x^\mu, y^\mu) = K^\mu$, and its image is a coefficient Frobenius twist of C . The tangent therefore has generic contact order μ , and its image has degree d . The second identity of (3.3) says precisely that (b^q, a^q) lies on this tangent, proving Frobenius nonclassicality. $\square$

It is the curve $C^\dagger$, with its $\mathbf{F}_q$ -isomorphic function field, that is used below. This avoids identifying the original curve with the normalization of its ordinary dual without accounting for the twist.

Lemma 3.2 (Weighted point identity). *With the preceding notation,*

$$(3.4) \quad \sum_{P \in C(\mathbf{F}_q)_{\text{plane}}} \text{mult}_P(C) = (q+1)d - \nu D.$$

Consequently

$$(3.5) \quad \boxed{N \leq (q+1)d - \nu D, \quad N \leq (q+1)D - \mu d.}$$

Proof. On the normalization, the sections $X_i X_j^q - X_j X_i^q$ have total degree $(q+1)d$. After removal of their common divisor they define the Gauss map: generically the line joining P and $P^{(q)}$ is the tangent at P . The resulting base-point-free system has degree νD .

Their common zeros are exactly the branches centered at rational plane points. At such a center take rational affine coordinates (u, v) vanishing there. The minimum valuation of the cross-product sections is the minimum of the valuations of $u^q - u$ and $v^q - v$, namely $m_\beta = \min\{\text{ord}_\beta u, \text{ord}_\beta v\}$. Summing over the branches at P gives $\text{mult}_P(C)$, proving (3.4). A rational point of the normalization contributes at least one to this weighted sum; nonrational branches at rational centers only increase the sum. Hence the first inequality in (3.5); apply the same argument to $C^\dagger$ for the second. $\square$

4. THE CONDUCTOR AND TWO SYMMETRIC DEGREE INEQUALITIES

Work over an algebraic closure for local computations. If $C : F = 0$ is Frobenius non-classical of degree d , let B be the common base divisor of its first partial derivatives on the normalization, and set

$$E = \deg B, \quad \Delta = p_a(C) - g, \quad p_a(C) = \frac{(d-1)(d-2)}{2}.$$

At each geometric branch β , choose an affine chart, a uniformizer z , and a generic separating linear coordinate x , and put

$$\rho_\beta = \text{ord}_z(dx/dz) = \min\{\text{ord}_z(dx/dz), \text{ord}_z(dy/dz)\}, \quad R = \sum_\beta \rho_\beta.$$

The plane conductor/different identity is

$$(4.1) \quad E = 2\Delta + R.$$

Locally, if c_β is the conductor exponent, it says that the polar base valuation is $c_\beta + \rho_\beta$, and $\sum c_\beta = 2\Delta$. For an individual branch, choose a separating coordinate x , and let $f(x, Y) \in k[[x]][Y]$ be the monic distinguished polynomial defining its completed local ring, obtained by formal Weierstrass preparation. Then $\text{ord}(f_y) = 2\delta_\beta + \text{ord}(dx/dz)$; intersections with the other branches supply the remaining conductor contributions. These are identities of differentials and conductors, valid also for wild projections. See [7, Theorem 1.3 and Section 3] for the arbitrary-characteristic conductor/Jacobian formulation.

(Suggested correction from Claude: Theorem 1.3 of [7] is “ X is a local complete intersection iff $\widehat{\mathcal{J}}(\mathcal{O}_X) = \mathfrak{C}_X$ ”, an lci characterization, not a conductor/Jacobian formula. What this section needs is $\text{Jac } X \cdot \mathcal{O}X' = \mathfrak{C}_X \cdot \text{Jac}(X'/X)$, which appears there only as the intermediate equality (3.8.1). Citing the theorem as stated implies you need the hard direction; you need the trivial one (a plane curve is a hypersurface, hence lci), and [7] dispatch that direction in one line, by appeal to Remark 2.6(1) with the parenthetical “which is still true in our case” plus Lemma 3.11. An alternative reference is [6].)

The polar system and (4.1) give

$$(4.2) \quad \nu D = d(d-1) - E = 2g - 2 + 2d - R.$$

Use $E^\dagger, R^\dagger, \Delta^\dagger$ for $C^\dagger$; then

$$(4.3) \quad \mu d = D(D-1) - E^\dagger = 2g - 2 + 2D - R^\dagger.$$

In particular,

$$(4.4) \quad \nu D \leq d(d-1), \quad \mu d \leq D(D-1).$$

Lemma 4.1 (Local genus loss). *Every individual geometric branch satisfies*

$$2\delta_\beta \geq (\nu - 1)\rho_\beta.$$

Consequently

$$(4.5) \quad (\nu - 1)R \leq 2\Delta.$$

This includes $\nu = 2$.

Proof. The inseparability of the Gauss map gives, locally, $y = A(z)^\nu + xB(z)^\nu$. By the choice of x , $B^\nu = y'/x'$ is regular, and then A is regular as well. Center x, y at the branch point. For every nonidentity embedding σ of the separable extension $\overline{\mathbf{F}}_q((z))/\overline{\mathbf{F}}_q((x))$ into its normal closure,

$$y - \sigma y = (A - \sigma A)^\nu + x(B - \sigma B)^\nu.$$

Differences of regular power series are divisible by $z - \sigma z$ in the integral closure. Thus

$$v(y - \sigma y) \geq \nu v(z - \sigma z).$$

Sum over the embeddings, with the valuation normalized by $v(z) = 1$. The product of the conjugate differences of y is f_y , up to a unit, for a Weierstrass equation of the branch. The corresponding product for z generates the different of $\overline{\mathbf{F}}_q[[z]]$ over $\overline{\mathbf{F}}_q[[x]]$, of valuation ρ_β . These rings are monogenic for this purpose by Weierstrass preparation. Therefore

$$2\delta_\beta + \rho_\beta = v(f_y) \geq \nu \rho_\beta.$$

Finally $\sum_\beta \delta_\beta \leq \Delta$, since intersections of distinct branches make additional nonnegative contributions to Δ . $\square$

Proposition 4.2. *The degrees of C and $C^\dagger$ satisfy*

$$(4.6) \quad d(q + 2\nu + 1 - d) \leq \nu(\nu + 1)D,$$

$$(4.7) \quad D(q + 2\mu + 1 - D) \leq \mu(\mu + 1)d.$$

Proof. Substitute $R = 2g - 2 + 2d - \nu D$ into (4.5). Rearrangement gives

$$\nu(2g - 2) \leq d(d - 2\nu - 1) + \nu(\nu - 1)D.$$

By $R^\dagger \geq 0$ and (4.3),

$$\nu(2g - 2) \geq qd - 2\nu D.$$

Combining proves (4.6); interchange the two curves for (4.7). $\square$

Lemma 4.3 (Smooth rational points). *For C ,*

$$(4.8) \quad N \geq M_{\text{sm}} \geq d(q - d + 2).$$

The same statement holds for $C^\dagger$, with D in place of d .

Proof. The weighted point identity and the polar degree formula give the exact identity

$$\begin{aligned} M_{\text{sm}} &= (q + 1)d - \nu D - \sum_{P \in \text{Sing}(C)(\mathbf{F}_q)} \text{mult}_P(C) \\ &= d(q - d + 2) + 2\Delta + R - \sum_{P \in \text{Sing}(C)(\mathbf{F}_q)} \text{mult}_P(C). \end{aligned}$$

At every singular point of multiplicity $m \geq 2$, $2\delta_P \geq m(m-1) \geq m$. Thus the correction is nonnegative. Every smooth rational plane point lifts to a rational normalization point. $\square$

The lower bound is the one supplied by Borges–Homma [3, Theorem 1.3]; the proof above records it directly from the identities already needed here. The distinction between smooth plane points, weighted plane points, and normalization points is essential.

Combining (2.1) with (4.8) gives

$$(4.9) \quad \nu(d-3) \geq q+2-2d, \quad \text{equivalently} \quad (\nu+2)d \geq q+3\nu+2.$$

Applying (4.8) to $C^\dagger$ and (3.5) to C also gives

$$(4.10) \quad H(D) \leq 0, \quad H(X) = X(q+\nu+2-X) - (q+1)d.$$

5. THE LARGE DUAL AND THE SECOND SCALAR INEQUALITY

Lemma 5.1 (Large dual). *Let $q = 2^h$ with h odd. If C is Frobenius nonclassical and $d \leq q/8$, then*

$$(5.1) \quad \boxed{D > q + \nu + 2 - 2d.}$$

Proof. We have

$$H(2d) = d(q+2\nu+3-4d) > 0.$$

The quadratic H is symmetric about $(q+\nu+2)/2$, so (4.10) implies

$$(5.2) \quad D < 2d \quad \text{or} \quad D > q + \nu + 2 - 2d.$$

Suppose $D < 2d$, and set $r = D/d$. From the lower bound for C and the upper bound for $C^\dagger$,

$$r \geq \frac{q-d+2+\mu}{q+1} > \frac{7}{8}.$$

Since $D < 2d \leq q/4$, (4.10) gives

$$r \leq \frac{q+1}{q+\nu+2-D} < \frac{4}{3}.$$

Because $\nu\mu = q = 2^h$ and h is odd, at least one of $\nu^2 \leq q/2$ and $\mu^2 \leq q/2$ holds. If $\nu^2 \leq q/2$, then (4.6) implies

$$q+1-d \leq r\nu^2 + (r-2)\nu < \frac{2q}{3},$$

whereas $q+1-d \geq 7q/8+1$. If $\mu^2 \leq q/2$, then (4.7) implies

$$r(q+1-D) \leq \mu^2 + (1-2r)\mu \leq \frac{q}{2}.$$

But its left side is greater than $(7/8)(3q/4) = 21q/32$. Both alternatives are impossible. This excludes the first case of (5.2). $\square$

This is precisely where the nonsquare hypothesis enters. For square q , the balanced case $\nu = \mu = \sqrt{q}$ is available, and the preceding dichotomy fails.

Proposition 5.2. *Under the hypotheses of Lemma 5.1,*

$$(5.3) \quad \boxed{d(d+\nu-1) \geq \nu(q+2).}$$

Proof. Set $U = d(d-1)/\nu$. By (4.4), $U \geq D$. Lemma 5.1 puts D strictly to the right of the vertex of H , since $d \leq q/8$. The quadratic is decreasing there, so $H(U) \leq H(D) \leq 0$. Direct factorization gives

$$(5.4) \quad H(U) = \frac{d(d-1-\nu)}{\nu^2} \{ \nu(q+2) - d(d+\nu-1) \}.$$

The first factor is positive. Indeed, a generic tangent gives $d \geq \nu + 1$; equality would imply $D \leq U = d$, contradicting Lemma 5.1. The expression in braces is therefore nonpositive, as required. $\square$

Equations (4.9) and (5.3) prove the scalar assertions in Theorem 1.2. Their dominant terms are $d \gtrsim q/\nu$ and $d \gtrsim \sqrt{q\nu}$. Balancing them gives $\nu \asymp q^{1/3}$ and $d \asymp q^{2/3}$.

6. UNIFORM BOUND, OPTIMIZATION, AND THE ARC CONCLUSION

6.1. Eliminating the Frobenius order. Assume $d \leq q/8$. The two scalar inequalities give

$$\frac{q + 2 - 2d}{d - 3} \leq \nu \leq \frac{d(d - 1)}{q + 2 - d}.$$

The denominators are positive: (4.9) forces $d > 3$. Multiplication and simplification yield

$$(6.1) \quad P_q(d) := d^3 - 6d^2 + 3(q + 3)d - (q + 2)^2 \geq 0.$$

For real x ,

$$P'_q(x) = 3((x - 2)^2 + q - 1) > 0.$$

Put $Q = q^{1/3}$. An exact substitution gives

$$(6.2) \quad P_q(Q^2 - Q + 1) = -Q^2(3Q^2 - 4Q + 3) < 0.$$

Thus

$$(6.3) \quad d > Q^2 - Q + 1.$$

For $q \geq 512$, we have $Q \geq 8$ and $q/8 \geq Q^2 > Q^2 - Q + 1$. Therefore (6.3) also holds automatically if $d > q/8$. This proves the uniform curve bound globally.

6.2. The exact optimization over powers of two. For a fixed ν , the two scalar inequalities are exactly $d \geq A_\nu(q)$ and $d \geq B_\nu(q)$. Taking ceilings and minimizing gives $d \geq T(q)$ whenever $d \leq q/8$. To extend this conclusion to every d , observe that

$$(6.4) \quad T(q) \leq q/8 \quad (q \geq 512).$$

Indeed, choose $\nu = 8$ and $d = q/8$. The first scalar inequality is $q/4 \geq 26$. The second, after multiplication by 64, is $q^2 - 456q - 1024 \geq 0$, which holds for $q \geq 512$. Here $q/8$ is an integer. The conclusion is then immediate also for $d > q/8$. This completes Theorem 1.2.

6.3. Deduction for complete arcs. Let $\mathcal{A}$ be a complete arc other than a hyperoval, and put $t = q + 2 - |\mathcal{A}|$. If $t > q/8$, then both conclusions of Theorem 1.1 follow from (6.4) and $q/8 > Q^2 - Q + 1$. If $t \leq q/8$, Lemma 2.1 applies, and (2.2) excludes Frobenius classical components. Choose any component. It is Frobenius nonclassical and $d \leq t$, so Theorem 1.2 gives

$$t \geq T(q), \quad t > Q^2 - Q + 1.$$

Since t is an integer, the second inequality is $t \geq \lfloor Q^2 - Q \rfloor + 2$. Substitution into $k = q + 2 - t$ proves (1.2) and (1.6).

6.4. Leading constants. An adjacent power of two gives $T(q) = O(Q^2)$, where $Q = q^{1/3}$. The two scalar inequalities then force every minimizing ν to be bounded above and below by fixed positive multiples of Q ; this follows from the two displayed bounds on ν at the beginning of subsection 6.1. Uniformly for such ν ,

$$\max\{A_\nu(q), B_\nu(q)\} = \max\{q/\nu, \sqrt{q\nu}\} + O(Q).$$

The expression on the right decreases up to $\nu = Q$ and increases afterward. Thus its minimum on the powers of two is attained at a power adjacent to Q . Powers outside these adjacent choices cannot lower the leading term. Write $h = 3a + j$.

j	relevant values of ν	$\min \max\{q/\nu, \sqrt{q\nu}\}/q^{2/3}$
0	2^a	1
1	$2^a, 2^{a+1}$	$2^{1/3}$
2	2^{a+1}	$2^{1/6}$

This proves Corollary 1.3. The two values in the middle row tie at leading order; retaining the exact formulas resolves the rounding and lower-order terms.

7. NORMAL FORMS AND SHARPER EXACT FAMILIES

The following criterion is independent of finite-field Frobenius nonclassicality. It sharpens the sufficient condition $E < d/2$ in the characteristic-two argument of Hefez [4, Theorem 5.11, equation (5.14), and Corollary 5.16].

Lemma 7.1. *Let $C : F = 0$ be a geometrically irreducible plane curve in characteristic two with generic tangent contact $\nu = 2^e \geq 4$. If $E < d$, then*

$$(7.1) \quad F = \sum_{i=0}^2 X_i P_i^\nu, \quad d \equiv 1 \pmod{\nu}.$$

Over a perfect field the P_i have coefficients in the field of definition of F .

Proof. Put $F_i = \partial_i F$, $F_{ij} = \partial_i \partial_j F$, and $H_i = D_i^{(2)} F$, using Hasse derivatives. The quadratic coefficient on a generic tangent vanishes. The tangent direction with coordinates F_j, F_i in positions i, j therefore gives, on C ,

$$(7.2) \quad F_j^2 H_i + F_i^2 H_j = F_i F_j F_{ij}.$$

At a branch choose j with $\text{ord}(F_j) = b_\beta$, the polar base valuation, and put $a = \text{ord}(F_i) \geq b_\beta$. In local line-bundle trivializations all polynomial sections are regular, so

$$2b_\beta + \text{ord } H_i \geq \min\{2a + \text{ord } H_j, a + b_\beta + \text{ord } F_{ij}\} \geq a + b_\beta.$$

Hence

$$\text{ord}_\beta H_i \geq \text{ord}_\beta F_i - b_\beta.$$

If $H_i \neq 0$, then $F_i \neq 0$: otherwise (7.2) and $\deg H_i < d$ force $H_i = 0$. Summing the valuation inequality and using Bézout gives $d(d-2) \geq d(d-1) - E$, contrary to $E < d$. Thus all $H_i = 0$. Equation (7.2) also forces all mixed $F_{ij} = 0$; cases with $F_i = 0$ are automatic. Differentiated Euler then gives $(d-1)F_i = 0$, so d is odd. The F_i are squares, and Euler gives $F = \sum X_i Q_i^2$.

Suppose inductively $F = \sum X_i Q_i^r$ for a power of two $2 \leq r < \nu$. The tangent coefficient of degree r vanishes, giving

$$F_j^r D_i^{(r)} F = F_i^r D_j^{(r)} F \quad \text{on } C.$$

If $D_i^{(r)} F \neq 0$, valuation and Bézout yield

$$d(d-r) \geq r\{d(d-1) - E\}, \quad \text{hence} \quad E \geq (1 - 1/r)d^2 \geq d,$$

a contradiction. Thus all $D_i^{(r)} F = 0$. In the existing normal form this says $\sum X_j (\partial_i Q_j)^r = 0$; the three monomial residue classes modulo r are distinct, so every $\partial_i Q_j = 0$. Each Q_j is a square and the exponent upgrades to $2r$. Induction reaches ν . Finally $P_i^\nu = F_i$, so perfectness gives the assertion about coefficients. $\square$

Remark 7.2. In addition, if $C/\mathbf{F}_q$ is Frobenius nonclassical, $\nu < q$, $d \leq q$, and the lemma applies, then

$$(7.3) \quad (\nu - 1)d \leq q - 1.$$

Indeed, putting $a = (d - 1)/\nu$ and $\mu = q/\nu$, Frobenius nonclassicality implies $F \mid \sum X_i^\mu P_i$. The latter polynomial is nonzero because $a < \mu$, so its three groups of monomials have disjoint supports. Comparing degrees gives $d \leq \mu + a$, which is (7.3).

7.1. Cubic field sizes.

Theorem 7.3. *Let $q = u^3$, where $u = 2^{2r+1} \geq 8$. Every nonlinear q -Frobenius nonclassical plane curve defined over $\mathbf{F}_q$ has*

$$d \geq u^2 - u/2 + 2.$$

Consequently $m'_2(q) \leq q - u^2 + u/2$.

Proof. Suppose $d \leq d_1 = u^2 - u/2 + 1$. Then $d \leq q/8$ and Lemma 5.1 gives $D > q - 2d$. If $\nu \leq u/2$, then

$$(\nu + 2)d < (u/2 + 2)u^2 \leq 3u^3/4 < q,$$

contradicting (4.9). If $\nu \geq 2u$, then

$$\nu D > 2u(q - 2d) \geq \frac{3}{2}u^4 > d^2,$$

contradicting (4.4). Hence $\nu = u$.

The second scalar inequality excludes $d \leq u^2 - u/2$, since at $d_0 = u^2 - u/2$ its left side is

$$d_0(d_0 + u - 1) = u^4 - \frac{5}{4}u^2 + \frac{1}{2}u < u(q + 2).$$

Thus $d = d_1$. Put $L = q - d + 1$. It is to the right of the vertex of H , and

$$H(L) = u^3 - \frac{11}{4}u^2 + \frac{3}{2}u - 1 > 0.$$

Since D is also on the decreasing side and $H(D) \leq 0$, it follows that $D > L$, hence $D \geq q - d + 2$. Consequently

$$E = d(d - 1) - uD \leq \frac{3}{4}u^2 - \frac{3}{2}u < d.$$

Lemma 7.1 forces $d \equiv 1 \pmod{u}$, but $d = u^2 - u/2 + 1 \not\equiv 1 \pmod{u}$. This contradiction proves the degree bound. The arc deduction follows as in Section 6, since $d_1 \leq q/8$. $\square$

7.2. Twice a cube.

Theorem 7.4. *Let $q = 2u^3$, where $u = 4^m \geq 16$, and put*

$$T_u = 2u^2 - 4u + 11 - \left\lfloor \frac{20}{u + 2} \right\rfloor.$$

Every nonlinear q -Frobenius nonclassical plane curve defined over $\mathbf{F}_q$ has $d \geq T_u + 1$. Consequently

$$(7.4) \quad m'_2(q) \leq q + 1 - T_u.$$

For $u \geq 64$ this is $m'_2(q) \leq q - 2u^2 + 4u - 10$.

Proof. Suppose $d \leq T_u$. Polynomial division shows that

$$T_u = \left\lfloor \frac{2u^3 + 3u + 2}{u + 2} \right\rfloor \leq 2u^2 - 2u \leq q/8.$$

If $\nu \leq u/2$, then $(\nu + 2)d \leq (u/2 + 2)(2u^2 - 2u) < 2u^3$, contradicting the first scalar inequality. If $\nu \geq 2u$, the inequalities $D > q - 2d$ and $\nu D \leq d(d - 1) < d^2$ would imply

$$d^2 + 4ud > 4u^4.$$

For $d \leq 2u^2 - 2u$ the left side is at most $4u^4 - 4u^2$. Therefore $\nu = u$, and the first scalar inequality forces $d = T_u$.

Let $\sigma = (u + 2)d - (q + 3u + 2)$. It equals 2 when $u = 16$ and 20 when $u \geq 64$, so $\sigma < u$. By (4.6),

$$E = d(d - 1) - uD \leq \frac{d\{(u + 2)d - q - 3u - 2\}}{u + 1} = \frac{d\sigma}{u + 1} < d.$$

But $d \equiv 10 \pmod{u}$ for $u = 16$, and $d \equiv 11 \pmod{u}$ for $u \geq 64$, contradicting Lemma 7.1. The arc conclusion again follows from the envelope argument. $\square$

8. EXACT FINITE BOUNDS AND REPRODUCIBILITY

The table compares the earlier estimate with the consequences of the degree and point inequalities, first without and then with the normal-form criterion. All entries are upper bounds; they are not assertions that an arc or a curve with the limiting parameters exists.

q	Earlier $q - \sqrt{2q}$	Duality inequalities	With normal form
32	24	24	24
128	112	106	105
512	480	453	452
2048	1984	1876	1875
8192	8064	7736	7735
32768	32512	31761	31760
131072	130560	128208	128208
524288	523264	516343	516342

For $q \geq 512$ the middle column is exactly $q + 2 - T(q)$. The small fields 32, 128 are handled by the full inequalities below, without assuming $t \leq q/8$.

8.1. The finite necessary conditions. Fix q and t , with $3t < q + 5$, $t < q/2$, and $k = q + 2 - t$. A complete arc with this t must supply integers d, D and a power of two ν , with $\mu = q/\nu$, satisfying

$$(8.1) \quad 2 \leq \nu < q, \quad \nu + 1 \leq d \leq t, \quad D \geq \mu + 1,$$

$$(8.2) \quad \nu(d - 3) \geq q + 2 - 2t,$$

$$(8.3) \quad \nu D \leq d(d - 1), \quad \mu d \leq D(D - 1),$$

$$(8.4) \quad d(q + 2\nu + 1 - d) \leq \nu(\nu + 1)D,$$

$$(8.5) \quad D(q + 2\mu + 1 - D) \leq \mu(\mu + 1)d,$$

$$(8.6) \quad \begin{aligned} & \max\{kd, d(q - d + 2), D(q - D + 2)\} \\ & \leq \min\{(q + 1)d - \nu D, (q + 1)D - \mu d\}. \end{aligned}$$

These conditions require no chosen genus or assumed singularity type. To obtain the last column, also exclude a tuple if $\nu \geq 4$ and

$$E = d(d - 1) - \nu D < d$$

but either the congruence in Lemma 7.1 fails or (7.3) fails when $d \leq q$. Apply the analogous test to $C^\dagger$ using $\mu, D, E^\dagger$.

At the first threshold allowed by the unrefined conditions, the following are the *full* ranges of possible dual degrees in the cases that gain one point. There is just one pair (ν, d) in each row.

q	t	ν	d	Possible D	Largest E
128	24	4	24	136–138	8
512	61	8	61	453–457	36
2048	174	16	174	1876–1881	86
8192	458	16	458	13079–13081	42
32768	1009	32	1009	31761–31783	720
524288	7947	64	7947	986632–986669	2414

In every row, $E < d$ and $d \not\equiv 1 \pmod{\nu}$, so the entire threshold is excluded. For $q = 131072$ the first allowed degree is $d = 2866$, with $\nu = 64$, and the maximum possible E is 5778; the normal-form argument does not exclude that threshold.

8.2. What the computation establishes. The accompanying script `verify_arc_bounds.py` uses exact integer arithmetic. For each displayed q it finds the first t allowed by (8.1)–(8.6), with and without the normal-form test. It verifies failure at $t - 1$ and reports one tuple at t . For fixed q every condition relaxes as t increases: the upper limit on d increases, k decreases, and (8.2) weakens. The normal-form tests are independent of t . Thus failure at $t - 1$ certifies failure at every smaller t .

The script also checks the exact scalar optimization for $q \geq 512$ and the complete boundary ranges in the preceding table. It uses $\lfloor \sqrt{n} \rfloor$ only, never floating-point square roots. The retained tuple is a certificate that these necessary inequalities do not force a still stronger bound; it is not an equation of a curve.

The main $q^{2/3}$ bound, its optimized version, and the exact families in Section 7 have proofs independent of this computation. The finite search is needed only for the remaining numerical refinements and the two small fields.

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