

FORCED 3-ADIC ANOMALIES OF S_3 CUBIC FIELDS

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ABSTRACT. Let k be a totally real cubic field and p a prime unramified in k . The p -adic regulator of k is a 2×2 determinant of p -adic logarithms of units and is always divisible by p^2 ; call k *anomalous at p* if it is divisible by p^3 . Among all 592 421 totally real cubic fields of discriminant up to 10^7 , the proportion of anomalous fields matches a random-matrix prediction at every prime $5 \leq p \leq 31$, but at $p = 3$ it is 0.381 instead of the predicted 0.407. We explain this. For cubic fields of fundamental discriminant D prime to 3, we show that the 3-class group and the ray class group modulo 9 of the quadratic field $\mathbb{Q}(\sqrt{D})$ can force every cubic field of discriminant D to be anomalous at 3. When the 3-class group of $\mathbb{Q}(\sqrt{D})$ is cyclic, we decide exactly when k is anomalous, in terms of one invariant for which we give an explicit formula by cubic residue and Hilbert symbols. The phenomenon comes from the failure of semisimplicity of $\mathbb{F}_3[S_3]$ and has no analogue for $p \geq 5$. The results fix one or two 3-adic digits of the regulator and say nothing about Leopoldt's conjecture itself.

1. INTRODUCTION

1.1. Leopoldt's conjecture for cubic fields as a 2×2 determinant. Let k be a totally real cubic field with fundamental units u_1, u_2 and let p be a prime unramified in k . Fix the three embeddings $\sigma_1, \sigma_2, \sigma_3: k \rightarrow \overline{\mathbb{Q}_p}$. The p -adic regulator

$$R_p(k) = \det(\log_p \sigma_i(u_j))_{1 \leq i, j \leq 2}$$

is well defined up to sign, and Leopoldt's conjecture [13] for k at p is the statement $R_p(k) \neq 0$. Leopoldt's conjecture is known for abelian fields by Brumer's p -adic Baker theorem [1]; for non-Galois cubic fields it is open. Hofmann, Johnston and Nickel [10, Appendix, Thm. A.2] verified it at $p = 3$ for every totally real Galois S_3 extension $E/\mathbb{Q}$ with $\text{Disc}(O_E) < 10^{20}$ (for an S_3 -extension, Leopoldt's conjecture for E and for its cubic subfield are equivalent [3]), and Ferri and Johnston [3] construct infinite families of totally real S_3 -extensions satisfying it at any prescribed finite set of primes.

Since $\log_p$ of a unit is divisible by p in the unramified completions, $v_p(R_p(k)) \geq 2$. We call $\text{depth}_p(k) := v_p(R_p(k)) - 2$ the *depth* and say k is *anomalous at p* if the depth is positive. In Gras's language $R_p(k)/p^2$ is the normalised regulator, and (when $p \nmid h_k$) k is anomalous at p exactly when it is not p -rational [6, 7]; see also [16, 9]. If the 2×2 matrix of reduced logarithms were a Haar-random matrix over $\mathbb{Z}_p$, the anomaly probability would be

$$q(p) = 1 - (1 - p^{-1})(1 - p^{-2}) = \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

For cyclic cubic fields the Galois symmetry changes the model, and Hofmann and Zhang [8] have studied the valuations of R_p in that case; non-Galois cubic fields are not treated there.

1.2. The survey. We computed $R_p(k)$ modulo a high power of p for every totally real cubic field of discriminant $d_k \leq 10^7$ (there are 592 421 of them) and every prime $3 \leq p \leq 31$ unramified in k . The unit groups were computed with PARI/GP [18] assuming the generalised

Riemann hypothesis; all numerical statements in this paper inherit that caveat, and none of the theorems depend on it. Table 1 shows the result.

p	fields	anomaly rate	$q(p)$	z
3	330903	0.3814	0.4074	-30.4
5	426455	0.2309	0.2320	-1.6
7	477175	0.1597	0.1603	-1.1
11	525213	0.0988	0.0984	+0.9
13	538483	0.0820	0.0824	-0.9
17	556937	0.0630	0.0621	+2.9
19	561968	0.0551	0.0553	-0.5
23	567513	0.0449	0.0453	-1.2
29	572657	0.0355	0.0356	-0.6
31	574070	0.0330	0.0333	-1.0

TABLE 1. Anomaly rates over all totally real cubic fields with $d_k \leq 10^7$ and p unramified (z = deviation from $q(p)$ in standard errors).

At every $p \geq 5$ the Haar value fits within the expected fluctuation. At $p = 3$ the rate is 0.3814 against $11/27 = 0.4074$, a deviation of 30 standard errors. The deficit is not a small-discriminant effect: the rate stabilises near 0.385 (Table 2). The purpose of this paper is to explain it.

1.3. Results. Let $D > 0$ be a fundamental discriminant with $3 \nmid D$, let $F = \mathbb{Q}(\sqrt{D})$ and let k be a cubic field with $d_k = D$. Then the Galois closure K of k is a cyclic cubic extension of F unramified at every place, so $h := \text{rk}_3 \text{Cl}(F) \geq 1$, and there are $(3^h - 1)/2$ such fields k . Put $d := \text{rk}_3 \text{Cl}_9(F)$, the 3-rank of the ray class group of F modulo 9. Let $\mathcal{G} = \text{Gal}(F_S(3)/F)$ be the Galois group of the maximal pro-3 extension of F unramified outside $S = \{3, \infty\}$; then $d = \dim_{\mathbb{F}_3} H^1(\mathcal{G}, \mathbb{F}_3)$. Let $\chi_K, \chi_{\text{cyc}} \in H^1(\mathcal{G}, \mathbb{F}_3)$ cut out K and $F \cdot \mathbb{Q}(\zeta_9)^+$.

Theorem A (forced anomaly). *One has $d \in \{h+1, h+2\}$, and $d = h+2$ if and only if every $x \in F^\times$ such that (x) is the cube of an ideal is a cube in $F \otimes \mathbb{Q}_3$. If $d = h+2$, then every cubic field k with $d_k = D$ satisfies $v_3(R_3(k)) \geq 3$.*

For $h = 1$ the condition reads: the fundamental unit ε_F and an element α with $(\alpha) = \mathfrak{b}^3$, $\mathfrak{b}$ of order 3 in $\text{Cl}(F)$, are both cubes in $F \otimes \mathbb{Q}_3$; equivalently the torsion $T_3(F)$ of the Galois group of the maximal abelian pro-3 extension of F unramified outside 3 is not cyclic (Proposition 3.1).

For $h = 1$ we can say exactly what happens. Let $\bar{\lambda}(E_k) \subset O_k/3$ be the image of the units of k under the reduced logarithm $u \mapsto \log_3(u)/3 \bmod 3$; it lies in the trace-zero plane, which contains the line $\mathbb{F}_3 \cdot 1$.

Theorem B ($h = 1$). *Suppose $h = 1$ and put $c = 1$ if $\chi_K \cup \chi_{\text{cyc}} = 0$ in $H^2(\mathcal{G}, \mathbb{F}_3)$ and $c = 0$ otherwise. Then*

$$\text{rk}_3 \text{Cl}_9(k) = d - 1 + c, \quad \dim_{\mathbb{F}_3} \bar{\lambda}(E_k) = 4 - d - c, \quad 1 \in \bar{\lambda}(E_k) \iff c = 0.$$

Consequently:

d	c	$\bar{\lambda}(E_k)$	$\text{depth}_3(k)$
2	0	<i>the trace-zero plane</i>	0
2	1	<i>a line</i> $\neq \mathbb{F}_3 \cdot 1$	≥ 1
3	0	$\mathbb{F}_3 \cdot 1$	≥ 1
3	1	0	≥ 2

In words: in the forced stratum the reduced logarithms of all units of k are *rational* modulo 3, i.e. every unit of k is, at every place above 3, a rational 1-unit times a local cube.

The invariant c can be computed explicitly (Theorem 5.2). Write a generator x of the relevant Kummer group of F as a norm $N_{K/F}(Y)$. Then c is the vanishing of a product of two symbols: a cubic residue symbol at an ideal recording how the 3-class group of F capitulates in K , and cubic Hilbert symbols at the primes above 3 of the local Hilbert-90 cofactors of Y . When $|\text{Cl}(F)[3^\infty]| = 3$ and ε_F is a local cube, the first symbol is trivial and c is a local condition at 3 on one unit of K . The proof of Theorem 5.2 is given in outline (one cochain identification is sketched); the formula agrees with c on all 2500 fields on which we tested it. We conjecture that $c = 1$ with probability $1/3$ for $d = 2$ and $1/9$ for $d = 3$ (Conjecture 6.1), and explain why this seems out of reach of current methods.

The reason is a parity phenomenon. By Poitou–Tate duality $H^2(\mathcal{G}, \mathbb{F}_3)$ is dual to a group of Kummer classes of F on which $\text{Gal}(F/\mathbb{Q})$ acts by -1 , so $H^2(\mathcal{G}, \mathbb{F}_3)$ has no invariants under $\text{Gal}(F/\mathbb{Q})$ (Proposition 3.1). Hence the cup product of two sign-type characters of $\mathcal{G}$ vanishes. When $d = h + 2$ there is a sign-type character χ_E independent of the class-group characters, and the vanishing of $\chi_K \cup \chi_E$ lets χ_E lift to the non-semisimple $\mathbb{F}_3[S_3]$ -module $[\text{sgn}; 1]$, the reduction modulo 3 of the reflection representation. Through Shapiro’s lemma the lift becomes a cyclic cubic extension of k unramified outside 3 and independent of $k \cdot \mathbb{Q}(\zeta_9)^+$, and two such extensions are exactly what anomaly means (Lemma 2.1). For $p \geq 5$ the module $\mathbb{F}_p[S_3/S_2]$ is semisimple and the mechanism disappears (Section 7), in accordance with Table 1.

1.4. The mixture law. Among the 259 403 fields of the survey with fundamental discriminant prime to 3, the strata have weights 0.879 ($h = 1, d = 2$), 0.092 ($h = 1, d = 3$), 0.028 ($h = 2, d = 3$) and 0.0004 ($h = 2, d = 4$). Theorem A accounts for the second and fourth stratum (all 24 009 fields are anomalous), and Theorem B reduces the first to the vanishing of one element of $\mathbb{F}_3$, which Theorem 5.2 computes explicitly. If that element is equidistributed (Conjecture 6.1), the anomaly rate over fundamental discriminants should be close to $0.879 \cdot \frac{1}{3} + 0.092 + 0.028 \cdot \frac{11}{27} + 0.0004 \approx 0.397$; the observed value is 0.390, the difference coming from a small deficit in the first stratum that decreases with D (Table 5). Fields of non-fundamental discriminant $d_k = f^2 D$, $f > 1$, are not covered by our results; their observed rate is 0.351.

1.5. Context and organisation. The ingredients — the ray class group modulo p^2 as a test of p -rationality [7], the Shafarevich–Koch formula, Poitou–Tate duality and Shapiro’s lemma [17, 11, 5] — are standard. We have not found Theorems A or B in the literature on p -rational fields [16, 9, 7, 14] or on p -adic regulators of cubic fields [8, 10, 3], but a statement of this kind may well be implicit in that literature and we make no strong priority claim. In a much deeper setting, p -adic regulators of units — determinants built from p -adic logarithms and valuations of units — give the leading terms of p -adic L -functions, as in the Gross–Stark conjecture proved by Dasgupta, Kakde and Ventullo [2]; here only the reduction modulo 3 of such a determinant matters.

Section 2 fixes notation and proves the depth criterion. Section 3 studies $\mathcal{G}$, and Section 4 proves Theorems A and B. Section 5 gives an explicit formula for c . Section 6 discusses the

generic stratum, Section 7 the primes $p \geq 5$, Section 8 the computations, and Section 9 the relation with Leopoldt's conjecture and with the function-field analogue.

2. SET-UP AND THE DEPTH CRITERION

Throughout, D, F, h, d, k, K are as in the introduction. Write σ for the generator of $\text{Gal}(F/\mathbb{Q})$ and $\text{Gal}(K/\mathbb{Q}) = S_3 = \langle \tau, s \rangle$ with $\tau^3 = s^2 = 1$, $k = K^s$, $F = K^\tau$. The prime 3 is unramified in K .

2.1. Reduced logarithms. For $u \in O_k$ prime to 3 put $\lambda(u) = \frac{1}{3} \log_3(u) \in O_k \otimes \mathbb{Z}_3$, where $\log_3(u) = \log_3(u^N)/N$ with $N = \text{lcm}(3^f - 1 : f \leq 3) = 104$, and let $\bar{\lambda}(u) \in O_k/3$ be its reduction. Since $\log(1 + 3y) \equiv 3y \pmod{9}$,

$$\bar{\lambda}(u) = N^{-1} \frac{u^N - 1}{3} \pmod{3}.$$

The same definitions apply to F and K .

Let $C_k = \ker(\text{Cl}_9(k)[3^\infty] \rightarrow \text{Cl}(k)[3^\infty])$.

Lemma 2.1. *Let k be a totally real cubic field in which 3 is unramified, with unit group E_k .*

- (1) $\bar{\lambda}(E_k)$ lies in the trace-zero plane of $O_k/3$, and $C_k \cong (O_k/3)/\bar{\lambda}(E_k)$.
- (2) $\text{depth}_3(k) \geq 1$ if and only if $\dim \bar{\lambda}(E_k) \leq 1$, if and only if $\dim C_k \geq 2$.
- (3) If $\bar{\lambda}(E_k) = 0$ then $\text{depth}_3(k) \geq 2$.

Proof. Let $M = (\sigma_i \lambda(u_j))$, a 3×2 matrix over the ring of integers of an unramified extension of $\mathbb{Q}_3$; then $R_3(k) = 9 \det(M_{i,j \leq 2})$. Each column of M sums to $\text{Tr } \lambda(u_j) = \lambda(N_{k/\mathbb{Q}} u_j) = \lambda(\pm 1) = 0$, so the three 2×2 minors of M agree up to sign, and $3 \mid \det M_{\leq 2}$ if and only if the reduction $\bar{M}$ has rank at most 1. As $k \otimes \mathbb{Q}_3$ is unramified, the embeddings induce an isomorphism $(O_k/3) \otimes \bar{\mathbb{F}}_3 \cong \bar{\mathbb{F}}_3^3$ carrying $\bar{\lambda}(u_j)$ to the j -th column of $\bar{M}$; linear dependence over $\bar{\mathbb{F}}_3$ and over $\mathbb{F}_3$ coincide. This proves the first equivalence in (2), the trace statement in (1), and (3) (if $M \equiv 0 \pmod{3}$ then $9 \mid \det M_{\leq 2}$).

Taking 3-primary parts in the exact sequence $E_k \rightarrow (O_k/9)^\times \rightarrow \text{Cl}_9(k) \rightarrow \text{Cl}(k) \rightarrow 1$, note that the 3-Sylow subgroup of $(O_k/9)^\times$ is $(1 + 3O_k)/(1 + 9O_k)$, isomorphic to $O_k/3$ via $1 + 3y \mapsto y$ because $(1 + 3y)(1 + 3y') \equiv 1 + 3(y + y') \pmod{9}$. The 3-component of the image of a unit u is u^{Nm} for any m with $Nm \equiv 1 \pmod{3}$, and corresponds to $\bar{\lambda}(u)$. Hence $C_k \cong (O_k/3)/\bar{\lambda}(E_k)$, and $\dim C_k = 3 - \dim \bar{\lambda}(E_k)$. $\square$

No hypothesis on Leopoldt's conjecture is used. Lemma 2.1 is the cubic case of Gras's ray-class test for p -rationality [7].

2.2. Ray class groups as Galois cohomology. Let $S = \{3, \infty\}$ and let $F_S(3)$ be the maximal pro-3 extension of F unramified outside S . Since K/F is unramified, $K \subset F_S(3)$, and $F_S(3)/\mathbb{Q}$ is Galois. Put

$$\mathcal{G} = \text{Gal}(F_S(3)/F), \quad \tilde{\Gamma} = \text{Gal}(F_S(3)/\mathbb{Q}), \quad \Gamma_k = \text{Gal}(F_S(3)/k), \quad H = \text{Gal}(F_S(3)/K) \triangleleft \tilde{\Gamma},$$

so that $\tilde{\Gamma}/H \cong S_3$ and $[\tilde{\Gamma} : \mathcal{G}] = 2$. Let $F^{\text{ur}}(3)$ be the maximal unramified pro-3 extension of F and define $\tilde{\Gamma}', \Gamma'_k$ in the same way.

Lemma 2.2. *For $L \in \{F, k, K\}$, $\text{rk}_3 \text{Cl}_9(L) = \dim H^1(\text{Gal}(F_S(3)/L), \mathbb{F}_3)$; for $L \in \{F, k\}$, $\text{rk}_3 \text{Cl}(L) = \dim H^1(\text{Gal}(F^{\text{ur}}(3)/L), \mathbb{F}_3)$.*

Proof. Let $v \mid 3$ be a place of L . As $L_v/\mathbb{Q}_3$ is unramified, log maps $1 + 3O_v$ isomorphically onto $3O_v$, so $(1 + 3O_v)^3 = 1 + 9O_v$ and every character of order 3 of $L_v^\times$ is trivial on $1 + 9O_v$. Real places do not ramify in extensions of odd degree. Hence the cyclic cubic extensions of L unramified outside S correspond to the characters of order 3 of $\text{Cl}_9(L)$. Each such extension M/L lies in $F_S(3)$: the Galois closure of MK over F is a compositum of cyclic cubic extensions of K unramified outside S , hence a 3-extension of F unramified outside S . The unramified statement is proved in the same way. $\square$

2.3. Some $\mathbb{F}_3[S_3]$ -modules. Write $\mathbf{1}$ and sgn for the trivial and sign characters of S_3 over $\mathbb{F}_3$, viewed as $\tilde{\Gamma}$ -modules through $\tilde{\Gamma} \rightarrow S_3$. Let $P = \mathbb{F}_3[S_3/\langle s \rangle]$ be the permutation module on the three cosets, with basis e_1, e_2, e_3 , where e_1 is the coset fixed by s . Its submodules are $0 \subset \mathbb{F}_3(e_1 + e_2 + e_3) \subset N \subset P$, where $N = \{\sum a_i e_i : \sum a_i = 0\}$; the successive quotients are $\mathbf{1}$, sgn , $\mathbf{1}$ and P is uniserial. Thus there are exact sequences

$$(1) \quad 0 \rightarrow \mathbf{1} \rightarrow N \rightarrow \text{sgn} \rightarrow 0, \quad 0 \rightarrow N \rightarrow P \xrightarrow{\Sigma} \mathbf{1} \rightarrow 0, \quad 0 \rightarrow \text{sgn} \rightarrow P/\mathbf{1} \rightarrow \mathbf{1} \rightarrow 0,$$

none of which splits. By Shapiro's lemma, $H^i(\Gamma_k, \mathbb{F}_3) \cong H^i(\tilde{\Gamma}, P)$, and the map induced by the augmentation Σ is corestriction $H^i(\Gamma_k, \mathbb{F}_3) \rightarrow H^i(\tilde{\Gamma}, \mathbb{F}_3)$. Since $[\tilde{\Gamma} : \mathcal{G}] = 2$ is prime to 3, for every $\mathbb{F}_3[\tilde{\Gamma}]$ -module M we have $H^i(\tilde{\Gamma}, M) = H^i(\mathcal{G}, M)^\sigma$ (invariants under $\tilde{\Gamma}/\mathcal{G}$). We write $H^i(\mathcal{G}, \mathbb{F}_3)^\pm$ for the ± 1 -eigenspaces of σ ; then $H^i(\tilde{\Gamma}, \mathbf{1}) = H^i(\mathcal{G}, \mathbb{F}_3)^+$ and $H^i(\tilde{\Gamma}, \text{sgn}) = H^i(\mathcal{G}, \mathbb{F}_3)^-$.

3. THE RESOLVENT FIELD

Let $W = \{x \in F^\times : (x) = \mathfrak{b}^3 \text{ for an ideal } \mathfrak{b}\}/F^{\times 3}$; then $\dim W = 1 + h$, spanned by ε_F and elements $\alpha_1, \dots, \alpha_h$ attached to a basis of $\text{Cl}(F)[3]$. Let

$$V_S = \{x \in W : x \in F_v^{\times 3} \text{ for all } v \mid 3\}.$$

Proposition 3.1. (1) σ acts on W , and hence on V_S , by -1 .

(2) $H^2(\mathcal{G}, \mathbb{F}_3)^+ = 0$; that is, $H^2(\tilde{\Gamma}, \mathbf{1}) = 0$.

(3) $d = 1 + \dim V_S$, and V_S is the kernel of $\bar{\lambda}: W \rightarrow (O_F/3)^-$, a map to a line. Hence $d \in \{h+1, h+2\}$, and $d = h+2$ if and only if every element of W is a cube in $F \otimes \mathbb{Q}_3$.

(4) $\text{rk}_3 T_3(F) = d - 1$. For $h = 1$: ε_F is a cube in $F \otimes \mathbb{Q}_3$ if and only if the local part of $T_3(F)$ is nonzero, and $d = 3$ if and only if ε_F and α_1 are both cubes in $F \otimes \mathbb{Q}_3$, if and only if $T_3(F)$ is not cyclic.

Proof. (1) If $x \in W$ is σ -invariant modulo cubes, then $x^2 \equiv x\sigma(x) = N_{F/\mathbb{Q}}(x)$, so $x \equiv r := N_{F/\mathbb{Q}}(x)^2$ modulo cubes, with $r \in \mathbb{Q}^\times$ and $(r)O_F$ a cube. For a prime p unramified in F this gives $3 \mid v_p(r)$; for p ramified it gives $3 \mid 2v_p(r)$. So $r = \pm m^3$ with $m \in \mathbb{Q}$, a cube in F . As 2 is invertible, $W = W^-$.

(2) Inflation $H^2(\mathcal{G}, \mathbb{F}_3) \rightarrow H^2(G_S(F), \mathbb{F}_3)$ is injective and $\tilde{\Gamma}$ -equivariant: in the five-term exact sequence for $\text{Gal}(\bar{F}_S/F_S(3)) \triangleleft G_S(F)$ the obstruction $H^1(\text{Gal}(\bar{F}_S/F_S(3)), \mathbb{F}_3)^\mathcal{G}$ vanishes, because a nonzero invariant homomorphism would cut out a cyclic cubic extension of $F_S(3)$ that is Galois and pro-3 over F and unramified outside S . For $v \in S$ the local groups $H^2(F_v, \mathbb{F}_3)$ vanish: for $v \mid 3$ they are dual to $H^0(F_v, \mu_3) = 0$ since $F_v/\mathbb{Q}_3$ is unramified, and for v real they are killed by 2. Hence $H^2(G_S(F), \mathbb{F}_3) = \text{III}_S^2(\mathbb{F}_3)$, which by Poitou–Tate duality [15, Ch. I, Thm. 4.10], [17, Ch. VIII] is dual to $\text{III}_S^1(\mu_3)$. By Kummer theory $H^1(F, \mu_3) = F^\times/F^{\times 3}$, and the classes unramified outside S and locally trivial at S form exactly V_S . The duality pairing is invariant under conjugation by a lift of σ , since the local invariant maps satisfy $\text{inv}_{\sigma v} \circ \sigma = \text{inv}_v$. So $\text{III}_S^2(\mathbb{F}_3)^+$ is orthogonal to $V_S^- = V_S$, hence zero.

(3) Every Galois extension of F of degree 3 unramified outside S lies in $F_S(3)$, so $H^1(\mathcal{G}, \mathbb{F}_3) = H^1(G_S(F), \mathbb{F}_3)$. Tate's global Euler characteristic formula [15, Ch. I, Thm. 5.1], [17, Ch. VIII] gives $h^0 - h^1 + h^2 = \sum_{v \text{ real}} h^0(F_v, \mathbb{F}_3) - [F : \mathbb{Q}] = 0$ for $G_S(F)$ with coefficients $\mathbb{F}_3$, so $d = 1 + h^2(G_S(F), \mathbb{F}_3) = 1 + \dim V_S$; this is the Shafarevich–Koch formula in our case [11, 17]. Every class of W has a representative x prime to 3 (move $\mathfrak{b}$ in its class), and such an x is a cube in F_v exactly when $x^N \in 1 + 9O_v$, i.e. when $\bar{\lambda}(x)$ vanishes at v . Since $N_{F/\mathbb{Q}}(x) = \pm m^3$ with $3 \nmid m$, we get $\text{Tr } \lambda(x) = 3\lambda(m) \equiv 0 \pmod{3}$, so $\bar{\lambda}(x) \in (O_F/3)^{\text{Tr}=0} = (O_F/3)^-$, which is a line. Hence $h \leq \dim V_S \leq h + 1$.

(4) The Galois group X_F of the maximal abelian pro-3 extension of F unramified outside 3 is $\mathbb{Z}_3 \oplus T_3(F)$ (Leopoldt's conjecture holds for F), so $\text{rk}_3 T_3(F) = d - 1$. The local part of $T_3(F)$ is the torsion of $U_F^1/\bar{E}_F$; under λ it is $I_F/\mathbb{Z}_3\lambda(\varepsilon_F)$ with I_F the trace-zero line of $O_F \otimes \mathbb{Z}_3$, which is nonzero exactly when $\lambda(\varepsilon_F) \in 3I_F$. The remaining statements follow from (3) with $W = \langle \varepsilon_F, \alpha_1 \rangle$. $\square$

Remark 3.2. In the language of the introduction, the forced stratum is the case where “the unit part of $T_3(F)$ splits off its class-group part”; precisely, it means $\text{rk}_3 T_3(F) = h + 1$. When $9 \mid h_F$ this need not be a direct sum decomposition, and indeed 2428 of the 23 914 forced fields in our data have $9 \mid h_F$.

4. PROOF OF THE THEOREMS

Theorem 4.1 (rank formula). *For every cubic field k with $d_k = D$ one has $\text{rk}_3 \text{Cl}_9(k) = d - 1 + c$, where $c = 1$ if $\chi_K \cup \chi_{\text{cyc}} = 0$ in $H^2(\mathcal{G}, \mathbb{F}_3)$ and $c = 0$ otherwise.*

Proof. By Lemma 2.2 and Shapiro's lemma, $\text{rk}_3 \text{Cl}_9(k) = \dim H^1(\tilde{\Gamma}, P)$. We use:

- $H^0(\tilde{\Gamma}, \mathbf{1}) = \mathbb{F}_3$ and $H^0(\tilde{\Gamma}, \text{sgn}) = 0$;
- $H^1(\tilde{\Gamma}, \mathbf{1}) = \text{Hom}(\tilde{\Gamma}, \mathbb{F}_3) = \mathbb{F}_3\chi_{\text{cyc}}$, because $\mathbb{Q}(\zeta_9)^+$ is the only cyclic cubic field unramified outside 3;
- $H^1(\tilde{\Gamma}, \text{sgn}) = H^1(\mathcal{G}, \mathbb{F}_3)^-$, of dimension $d - 1$, containing χ_K (because σ acts on $\text{Gal}(K/F)$ by -1);
- $H^2(\tilde{\Gamma}, \mathbf{1}) = 0$ (Proposition 3.1(2)) and $H^2(\tilde{\Gamma}, \text{sgn}) = H^2(\mathcal{G}, \mathbb{F}_3)$.

The first sequence of (1) gives, since $H^0(\tilde{\Gamma}, \mathbf{1}) \rightarrow H^0(\tilde{\Gamma}, N) = N^{S_3} = \mathbb{F}_3(e_1 + e_2 + e_3)$ is an isomorphism,

$$0 \rightarrow H^1(\tilde{\Gamma}, \mathbf{1}) \rightarrow H^1(\tilde{\Gamma}, N) \rightarrow H^1(\tilde{\Gamma}, \text{sgn}) \rightarrow H^2(\tilde{\Gamma}, \mathbf{1}) = 0,$$

so $\dim H^1(\tilde{\Gamma}, N) = d$, and also that $H^2(\tilde{\Gamma}, N) \rightarrow H^2(\tilde{\Gamma}, \text{sgn})$ is injective. In the second sequence, $H^0(\tilde{\Gamma}, P) = \mathbb{F}_3(e_1 + e_2 + e_3)$ maps to $3 = 0$ in $H^0(\tilde{\Gamma}, \mathbf{1})$, so the connecting map $\mathbb{F}_3 = H^0(\tilde{\Gamma}, \mathbf{1}) \rightarrow H^1(\tilde{\Gamma}, N)$ is injective and

$$0 \rightarrow H^1(\tilde{\Gamma}, N)/\mathbb{F}_3 \rightarrow H^1(\tilde{\Gamma}, P) \xrightarrow{\text{cor}} H^1(\tilde{\Gamma}, \mathbf{1}) \xrightarrow{\delta} H^2(\tilde{\Gamma}, N).$$

Thus $\dim H^1(\tilde{\Gamma}, P) = d - 1 + \dim \ker \delta$. Composing δ with the injection $H^2(\tilde{\Gamma}, N) \rightarrow H^2(\tilde{\Gamma}, \text{sgn})$ gives, by naturality, the connecting map of the third sequence of (1), which is the cup product with its extension class in $H^1(\tilde{\Gamma}, \text{Hom}(\mathbf{1}, \text{sgn})) = H^1(\mathcal{G}, \mathbb{F}_3)^-$. With the section $1 \mapsto e_1$ that class is the cocycle $g \mapsto (g - 1)e_1$; it vanishes on a lift of s , and for $g \in \mathcal{G}$ acting as $\tau^{\chi_K(g)}$ it equals $\chi_K(g)(e_2 - e_1)$ modulo constants, because $e_3 - e_1 \equiv -(e_2 - e_1)$ there. So the class is χ_K , and $\delta(\chi_{\text{cyc}}) = 0$ if and only if $\chi_K \cup \chi_{\text{cyc}} = 0$. $\square$

Remark 4.2 (the witness). The proof also says where the extensions come from. Modulo the class of the extension P (which maps to χ_K), $H^1(\tilde{\Gamma}, N)$ is spanned by χ_{cyc} and by lifts to

$N = [\text{sgn}; 1]$ of the sign-type characters other than χ_K ; such lifts exist because their obstruction $\chi_K \cup \chi$ lies in $H^2(\tilde{\Gamma}, 1) = 0$. When $h = 1$ and $d = 3$ the extra sign-type character χ_E cuts out a cyclic cubic extension E/F , ramified at 3, with $E/\mathbb{Q}$ an S_3 -extension. Its lift gives a cyclic cubic extension L_E/k unramified outside 3 and different from $k \cdot \mathbb{Q}(\zeta_9)^+$, whose Galois closure over F is a non-abelian group of order 27 containing EK . This field is the explicit cause of the anomaly.

Remark 4.3 (c as a transfer). The image of cor is $\ker \delta$, so $c = 1$ if and only if corestriction $H^1(\Gamma_k, \mathbb{F}_3) \rightarrow H^1(\tilde{\Gamma}, \mathbb{F}_3)$ is nonzero, i.e. if and only if the transfer $\text{Gal}(\mathbb{Q}(\zeta_9)^+/\mathbb{Q}) \rightarrow \Gamma_k^{\text{ab}}/3$ is nonzero. By class field theory the transfer is the extension of ideals $\text{Cl}_9(\mathbb{Q})[3] = \langle [2] \rangle \rightarrow \text{Cl}_9(k)/3$, so $c = 0$ if and only if the class of $2O_k$ is trivial in $\text{Cl}_9(k)/3$.

Lemma 4.4. $\text{rk}_3 \text{Cl}(k) \leq h - 1$. In particular $\text{Cl}(k)[3] = 0$ when $h = 1$.

This bound is presumably classical: Gerth computes 3-ranks of class groups of non-Galois cubic fields in terms of the quadratic resolvent [4]. We have not consulted that paper, so we include a proof.

Proof. Run the argument of Theorem 4.1 with $\tilde{\Gamma}' = \text{Gal}(F^{\text{ur}}(3)/\mathbb{Q})$ in place of $\tilde{\Gamma}$ (Lemma 2.2). Now $H^1(\tilde{\Gamma}', 1) = 0$ (there is no unramified cyclic cubic extension of $\mathbb{Q}$) and $H^1(\tilde{\Gamma}', \text{sgn}) = \text{Hom}(\text{Cl}(F), \mathbb{F}_3)$ has dimension h . The second sequence gives $\text{rk}_3 \text{Cl}(k) = \dim H^1(\tilde{\Gamma}', N) - 1$, and the first gives $H^1(\tilde{\Gamma}', N) \hookrightarrow H^1(\tilde{\Gamma}', \text{sgn})$. $\square$

Proof of Theorem A. The first two statements are Proposition 3.1(3). Since C_k is the kernel of $\text{Cl}_9(k)[3^\infty] \rightarrow \text{Cl}(k)[3^\infty]$, Theorem 4.1 and Lemma 4.4 give

$$\dim C_k \geq \text{rk}_3 \text{Cl}_9(k) - \text{rk}_3 \text{Cl}(k) \geq (d - 1 + c) - (h - 1) = d - h + c.$$

If $d = h + 2$ this is at least 2, and Lemma 2.1 gives $\text{depth}_3(k) \geq 1$. $\square$

The same inequality shows that k is anomalous whenever $d = h + 1$ and $c = 1$.

Proof of Theorem B. For $h = 1$, Lemma 4.4 gives $\text{Cl}(k)[3] = 0$, so $C_k = \text{Cl}_9(k)[3^\infty]$ is elementary abelian of dimension $d - 1 + c$, and $\dim \bar{\lambda}(E_k) = 3 - (d - 1 + c)$ by Lemma 2.1. The class of the principal ideal $2O_k$ in $\text{Cl}_9(k)$ is the image of $2 \in (O_k/9)^\times$, whose 3-component is that of $-2 = 7 = 1 + 3 \cdot 2$, i.e. the class of $2 \cdot 1$ in $(O_k/3)/\bar{\lambda}(E_k)$. By Remark 4.3, $c = 0$ if and only if $1 \in \bar{\lambda}(E_k)$. The table follows: for $d = 3$, $c = 0$ the space $\bar{\lambda}(E_k)$ is a line containing 1; for $d = 3$, $c = 1$ it is 0 and Lemma 2.1(3) applies; for $d = 2$ it is the plane ($c = 0$) or a line not containing 1 ($c = 1$). $\square$

Remark 4.5 (units of K). The classical route through the unit group of K gives only part of this. For $h = 1$ and $|\text{Cl}(F)[3^\infty]| = 3$, the Herbrand quotient of E_K for the unramified extension K/F is $1/3$, so $H^1(\text{Gal}(K/F), E_K) \cong \mathbb{Z}/3$ and $\mathfrak{b}O_K = (\beta)$ is principal. In the forced stratum ε_F and $\eta = \alpha/\beta^3$ are units of K that are local cubes at 3 and independent modulo cubes. Both are of sign type, however, and by themselves impose no dependence on the reduced logarithms of E_k ; the additional input is the vanishing $H^2(\tilde{\Gamma}, 1) = 0$.

5. AN EXPLICIT FORMULA FOR c

Theorem B reduces the first free 3-adic digit to the invariant c , defined by a cup product. Here we evaluate that cup product. One might expect a formula in terms of the cubic Hilbert symbols $(\kappa_K, \zeta_3)_{\mathfrak{p}}$, $\mathfrak{p} \mid 3$, of the Kummer generators of K and of $\mathbb{Q}(\zeta_9)^+$ over $L = F(\zeta_3)$. Those symbols are the local components of $\chi_K \cup \chi_{\text{cyc}}$, and they vanish identically, because

$H^2(\mathcal{G}, \mathbb{F}_3)$ consists of III^2 only (Proposition 3.1). All the information is in the Poitou–Tate pairing with V_S , which mixes a local term at 3 with a class-group term.

Let $\lambda_{\mathbb{Q}}: \mathbb{Z}_3^\times \rightarrow \mathbb{F}_3$, $u \mapsto \frac{1}{3} \log_3(u) \bmod 3$, extended to rational numbers prime to 3.

Definition 5.1. Let $x \in V_S$, and let τ generate $\text{Gal}(K/F)$.

- (1) By the Hasse norm theorem $x = N_{K/F}(Y)$ for some $Y \in K^\times$: x is a local norm everywhere, since K/F is unramified and x is a local cube at 3. Replacing Y by $YW/\tau(W)$ for a suitable W , we may assume Y is a unit at every prime above 3.
- (2) Write $(x) = \mathfrak{b}^3$. Since $(Y)\mathfrak{b}^{-1}O_K$ has norm 1 and the ideal group of K is a permutation module, $(Y) = \mathfrak{b}O_K \cdot \tau(\mathfrak{A})/\mathfrak{A}$ for an ideal $\mathfrak{A}$ of K prime to 3.
- (3) For $v \mid 3$ let $\xi_v \in F_v$ be the cube root of x , and write $Y = \xi_v \tau(Z_v)/Z_v$ with $Z_v \in (K \otimes_F F_v)^\times$ a 1-unit (Hilbert 90).

Put

$$I(x) = \lambda_{\mathbb{Q}}(N_{K/\mathbb{Q}}\mathfrak{A}) - \lambda_{\mathbb{Q}}\left(\prod_{v \mid 3} N_{K \otimes F_v/\mathbb{Q}_3}(Z_v)\right) \in \mathbb{F}_3.$$

Over L , the first term is the cubic residue symbol $(\zeta_3/N_{K/F}\mathfrak{A})_3$, which records how $\mathfrak{b}$ capitulates in K . The second is the product of the cubic Hilbert symbols $(NZ_v, \zeta_3)_{\mathfrak{p}}$ at the primes above 3.

Theorem 5.2. $I(x)$ does not depend on the choices made, and $\langle \chi_K \cup \chi_{\text{cyc}}, x \rangle = \pm I(x)$ under $\text{Gal}(\mathbb{Q}(\zeta_9)^+/\mathbb{Q}) \cong \mathbb{F}_3$. Hence for $h = 1$:

- if $d = 2$, then $c = 1 \iff I(x) = 0$, where x generates V_S ;
- if $d = 3$, then $c = 1 \iff I(\varepsilon_F) = I(\alpha_1) = 0$.

Proof (outline). Independence of the choices.

- Replacing Y by $YW/\tau(W)$ replaces $\mathfrak{A}$ by $\mathfrak{A}(W)$ and Z_v by $Z_v W$. This changes I by $\lambda_{\mathbb{Q}}$ of the principal idèle $N_{K/\mathbb{Q}}(W)$, which is 0 by Artin reciprocity for $\mathbb{Q}(\zeta_9)^+/\mathbb{Q}$.
- Changing $\mathfrak{A}$ by an ideal of F , or Z_v by an element of $F_v^\times$, changes the norms by cubes.

The pairing. Use the explicit cochain description of the pairing $\text{III}^1 \times \text{III}^2 \rightarrow \mathbb{Q}/\mathbb{Z}$ in [15, Ch. I, §4, after Remark 4.19], with $\alpha = \chi_{\text{cyc}} \cup \chi_K$ and $\beta = x$. There one trivialises β and α locally at each $v \in S$, writes $\beta \cup \alpha = d\epsilon$ for a global 2-cochain ϵ with values in the multiplicative group (the S -units), and sums the local Brauer invariants of the resulting local 2-cocycles. At $v \mid 3$ trivialise x locally by its cube root ξ_v . Globally take $\epsilon = -\chi_{\text{cyc}} \cup \theta$, where θ is a 1-cochain with values in $\overline{F}^\times$ and $d\theta = \chi_K \cup x$; it exists because the cyclic algebra $(K/F, x)$ is split by Y .

- The local term at every finite place takes the form $\text{inv}_v(\chi_{\text{cyc}} \cup y_v)$, where $y_v \in F_v^\times/F_v^{\times 3}$ is the difference of the global splitting (from Y) and the local one (from ξ_v for $v \mid 3$, from a local generator of $\mathfrak{b}$ for $v \nmid 3$). Apart from this term, the local contributions at $v \notin S$ are unramified classes of degree 2, which vanish.
- Comparing two splittings of a cyclic algebra gives $y_v = N(Z_v)$, where Y is the local splitting element times $\tau(Z_v)/Z_v$. At $v \nmid 3$ this is a generator of $(N_{K/F}\mathfrak{A})_v$.
- Hence $\langle \alpha, \beta \rangle = \chi_{\text{cyc}}(\text{rec}_F(y))$ for the idèle $y = (y_v)$. Artin’s map for $\mathbb{Q}(\zeta_9)^+/\mathbb{Q}$ sends Frob_ℓ to $\zeta \mapsto \zeta^\ell$ and $u \in \mathbb{Z}_3^\times$ to $\zeta \mapsto \zeta^{u^{-1}}$, which gives $\pm I(x)$.

We have not written out the cochain identification $y_v = N(Z_v)$ in full. If it failed, $I(x)$ would depend on the choice of Y ; it is invariant in every randomised test below. For $d = 3$, use that $H^2(\mathcal{G}, \mathbb{F}_3) \cong V_S^\vee$ and $V_S = \langle \varepsilon_F, \alpha_1 \rangle$. $\square$

Corollary 5.3. Suppose $h = 1$, $d = 2$, $|\text{Cl}(F)[3^\infty]| = 3$ and ε_F is a cube in $F \otimes \mathbb{Q}_3$. Then $\varepsilon_F = N_{K/F}(\eta)$ for a unit η of K , up to cubes of units of F , and $c = 1$ if and only if

$\lambda_{\mathbb{Q}}(\prod_{v|3} N_{K \otimes F_v / \mathbb{Q}_3}(Z_v)) = 0$, where $\eta = \xi_v \tau(Z_v)/Z_v$. So in this substratum c is a purely local condition at 3 on one unit of K .

Proof. Here the capitulation kernel of K/F has order $3 = |H^1(\text{Gal}(K/F), E_K)| = 3[E_F : N_{K/F} E_K]_3$, so $\varepsilon_F \in N(E_K)E_F^3$. Take $Y = \eta e$ with $e \in E_F$. Then $(Y) = (1)$, so $\mathfrak{A} = 1$. $\square$

Numerical check. We evaluated I with PARI/GP on random fields of the scan of Section 8 and compared with $c = \text{rk}_3 \text{Cl}_9(k) - d + 1$:

- Theorem 5.2 held in every case: 1000 fields with $h = 1$, $d = 2$ and ε_F not a local cube; 1000 glued fields; 500 forced fields with $d = 3$.
- $I(x)$ was unchanged under 600 random replacements $Y \mapsto YW/\tau(W)$, while its two terms changed.
- The alternative sign $\lambda_{\mathbb{Q}}(N\mathfrak{A}) + \lambda_{\mathbb{Q}}(\prod N Z_v)$ disagrees with c on 43 of these fields.
- In the substratum of Corollary 5.3 the class-group term vanished in all 898 tested fields. Elsewhere it was nonzero in 3–25% of them, so both terms are needed in general.

6. THE GENERIC STRATUM

Suppose $h = 1$ and $d = 2$, which is the case for 88% of our fields. By Theorem B, k is anomalous if and only if $\chi_K \cup \chi_{\text{cyc}} = 0$ in the line $H^2(\mathcal{G}, \mathbb{F}_3) \cong V_S^\vee$, i.e. if and only if one element of $\mathbb{F}_3$ vanishes. The line V_S is spanned by $\alpha_1 \varepsilon_F^j$ for the unique j making it a local cube when ε_F is not a local cube, and by ε_F when ε_F is a local cube but α_1 is not (the “glued” case, $T_3(F)$ cyclic). In both cases $\mathcal{G}$ has two generators and one relation, and nothing distinguishes the two cases at the level of this invariant.

Conjecture 6.1. Order the real quadratic fields F with $3 \nmid D$ by discriminant. Among those with $h = 1$ and $d = 2$, the proportion with $c = 1$ tends to $1/3$, both when ε_F is a local cube and when it is not. Among those with $h = 1$ and $d = 3$, it tends to $1/9$.

By Theorem B, Theorem 5.2 and Corollary 5.3, the following are equivalent forms of the first statement:

- a cubic field of the stratum fails to be 3-rational ($T_3(k) \neq 0$) with probability $1/3$;
- the symbol $I(x)$ of Definition 5.1 vanishes with probability $1/3$;
- in the substratum of Corollary 5.3, the local symbol at 3 of the norm unit η vanishes with probability $1/3$.

We do not know how to prove any of them. Three obstacles stand in the way:

- The density of the stratum itself (3-rank of $\text{Cl}(F)$ exactly one) is a Cohen–Lenstra prediction, not a theorem; Davenport–Heilbronn controls only the mean of $|\text{Cl}(F)[3]|$.
- $c = 1$ is equivalent to the existence of a second cyclic cubic extension of k unramified outside 3. Counting such extensions means counting nonic fields, for which no parametrisation is known.
- The symbol $I(x)$ is built from a norm from K , so it is not a Frobenius condition in a field that stays fixed as F varies, and Chebotarev’s theorem does not apply.

A naive count misled us at first. Writing the reduced log matrix of k with one difference-type linear relation coming from the unit index of K and an inhomogeneous term e equal to the reduced logarithm of ε_F , a uniform model gives anomaly probability $1/3$ for $e \neq 0$ and $5/9$ for $e = 0$, predicting $5/9$ in the glued case. The data give 0.324. The correct inhomogeneous term is the image $\bar{\lambda}(W) \subset (O_F/3)^-$ of the whole group W , which by Proposition 3.1 is nonzero exactly when $d = h + 1$: in the glued case it is carried by α_1 , through the capitulation relation $\alpha_1 = \eta\beta^3$. It vanishes exactly in the forced case, where the answer is 1 rather than $5/9$.

Observed: the anomaly rate is 0.3257 when ε_F is not a local cube (171 138 fields) and 0.3241 in the glued case (56 912 fields). Overall $P(c = 1 \mid h = 1, d = 2) = 0.3253 \pm 0.0010$, with a deficit relative to $1/3$ that is largest for small D (Table 5); above $4 \cdot 10^6$ the pooled rate is 0.328 (4.5 standard errors below $1/3$), so we cannot yet decide whether the limit is exactly $1/3$. For $d = 3$ we observe $P(c = 1) = 0.101 \pm 0.002$ against $1/9$. In the forced stratum the probability of depth ≥ 2 given $c = 0$ is 0.338, so the model “ $c = 1$ with probability $1/9$, otherwise one further uniform digit” predicts $P(\text{depth} \geq 2) \approx \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{3} = \frac{11}{27}$, against 0.405 observed.

7. WHY NOTHING IS FORCED FOR $p \geq 5$

Let $p \geq 5$ be unramified in K . Lemma 2.1 holds with 3 replaced by p and Cl_9 by Cl_{p^2} , with the same proof. The forcing mechanism of Section 4 used two features specific to $p = 3$: K lies in a pro-3 extension of F , so the arithmetic of F (the characters χ_K, χ_E) and that of k live in the cohomology of one group; and the permutation module $\mathbb{F}_3[S_3/\langle s \rangle]$ glues the sign character to the trivial one. For $p \geq 5$ neither holds.

Proposition 7.1. *Let $p \geq 5$ be unramified in K . As an $\mathbb{F}_p[S_3]$ -module, $\text{Cl}_{p^2}(K)/p \cong \mathbf{1}^a \oplus \text{sgn}^b \oplus \bar{\rho}^m$, with $\bar{\rho}$ the two-dimensional irreducible representation. Here $a = \text{rk}_p \text{Cl}_{p^2}(\mathbb{Q}) = 1$, $b = \text{rk}_p \text{Cl}_{p^2}(F) - 1$, and $\text{rk}_p \text{Cl}_{p^2}(k) = 1 + m$. In particular $\text{rk}_p \text{Cl}_{p^2}(k)$ does not depend on the sign-type part, which is the only part through which invariants of F enter.*

Proof. $\mathbb{F}_p[S_3]$ is semisimple, so the decomposition exists. Since $[K : k]$, $[K : F]$ and $[K : \mathbb{Q}]$ are prime to p , taking invariants under $\langle s \rangle$, $\langle \tau \rangle$ and S_3 is exact and identifies $\text{Cl}_{p^2}(k)/p$, $\text{Cl}_{p^2}(F)/p$ and $\text{Cl}_{p^2}(\mathbb{Q})/p$ with the corresponding invariants. Now $\mathbf{1}^{\langle s \rangle} = \mathbf{1}$, $\text{sgn}^{\langle s \rangle} = 0$ and $\dim \bar{\rho}^{\langle s \rangle} = 1$, while $\mathbf{1}^\tau = \mathbf{1}$, $\text{sgn}^\tau = \text{sgn}$ and $\bar{\rho}^\tau = 0$. Finally $\text{Cl}_{p^2}(\mathbb{Q})[p] \cong \mathbb{Z}/p$ corresponds to the degree- p subfield of $\mathbb{Q}(\zeta_p)$. $\square$

So for $p \geq 5$ the anomaly of k is a statement about the $\bar{\rho}$ -multiplicity alone, and no condition on the resolvent can force it. This matches Table 1, where the Haar law holds at every $p \geq 5$ (we do not claim that the Haar law is a theorem).

8. COMPUTATIONS

All computations use PARI/GP 2.17.3 [18] with `bnfinit` assuming GRH; the class and unit groups are not certified. Scripts and data are in the directory of this paper.

8.1. The survey. `fe14_survey.py` enumerates the totally real cubic fields with $d_k \leq 10^7$ (592 421 fields), computes the units, and computes the reduced logarithm matrix modulo p^8 for $3 \leq p \leq 31$, exactly in $\mathbb{Z}_p[x]/(f)$. Table 2 shows the rate at $p = 3$ by discriminant range.

d_k range (10^6)	fields	rate	z vs $11/27$
(0, 1]	31435	0.3666	-14.7
(1, 2]	32531	0.3791	-10.4
(2, 4]	66076	0.3798	-14.5
(4, 6]	66600	0.3854	-11.5
(6, 8]	67043	0.3834	-12.6
(8, 10]	67218	0.3852	-11.7

TABLE 2. Anomaly rate at $p = 3$ by discriminant.

8.2. The strata. `fe15_strata.py` and `fe15b_cube.py` compute d , h and whether ε_F is a local cube for the 254 176 resolvents of the 259 403 fields with fundamental d_k prime to 3 (Table 3). The cell ($h = 1$, $d = 3$, ε_F not a local cube) is empty, as Proposition 3.1(4) requires.

h	d	ε_F local cube	fields	anomaly rate	depth 0 / 1 / ≥ 2
1	2	no	171138	0.3257	0.674 / 0.217 / 0.109
1	2	yes	56912	0.3241	0.676 / 0.218 / 0.106
1	3	yes	23914	1.0000	0.000 / 0.595 / 0.405
2	3	no	5146	0.3984	0.602 / 0.255 / 0.143
2	3	yes	2198	0.3922	0.608 / 0.258 / 0.134
2	4	yes	95	1.0000	0.000 / 0.568 / 0.432

TABLE 3. Fields with fundamental discriminant prime to 3, by stratum.

8.3. The full check. `fe16b_scan.py` recomputes, for each of the 259 403 fields, $\text{rk}_3 \text{Cl}(k)$, $\text{rk}_3 \text{Cl}_9(k)$, $\bar{\lambda}(E_k)$, whether $1 \in \bar{\lambda}(E_k)$, whether $\bar{\lambda}(E_k) \subset \mathbb{F}_3 \cdot 1$, and the class of $2O_k$ in $\text{Cl}_9(k)/3$; for each resolvent it recomputes $\dim V_S$ from $\bar{\lambda}(\varepsilon_F)$ and $\bar{\lambda}(\alpha_i)$. `fe16c_analyze.py` checks every statement of Proposition 3.1(3), Theorem 4.1, Lemma 4.4, Theorems A and B, Remark 4.3 and Lemma 2.1 (against the independently computed depth of the survey). There were no exceptions and no computational errors (run time 1901 s on 24 processes). Table 4 gives the distribution of c .

h	d	c	fields	anomaly	depth ≥ 2	$1 \in \bar{\lambda}(E_k)$	$\bar{\lambda}(E_k) \subset \mathbb{F}_3 \cdot 1$
1	2	0	153859	0.0000	0.0000	1.0000	0.0000
1	2	1	74191	1.0000	0.3321	0.0000	0.0000
1	3	0	21493	1.0000	0.3376	1.0000	1.0000
1	3	1	2421	1.0000	1.0000	0.0000	1.0000
2	3	0	6589	0.3274	0.1131	0.7534	0.0807
2	3	1	755	1.0000	0.3788	0.0000	0.0596
2	4	0	90	1.0000	0.4000	0.9556	1.0000
2	4	1	5	1.0000	1.0000	0.0000	1.0000

TABLE 4. The invariant c of Theorem 4.1 on all 259 403 fields (proportions per row).

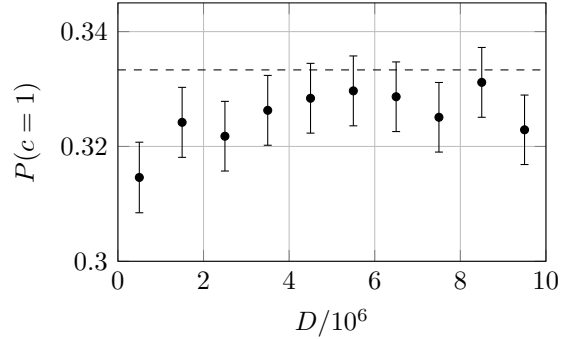
For $h = 2$ Lemma 4.4 is only an inequality and Theorem B does not apply; the table shows that $c = 1$ still forces the anomaly there (as Theorem A predicts via $\dim C_k \geq d - h + c$), while $d = 3$, $c = 0$ gives a rate of about $1/3$.

8.4. Reproducibility and formal verification. Every table is regenerated from the stored JSON outputs by `make_tables.py`. The intermediate identities were in addition checked by hand on ten fields from each stratum (`fe16_check.py`), including three forced fields with $9 \mid h_F$.

We do not offer a formal (Lean) certificate. The weight of the proof is carried by Poitou–Tate duality, Tate’s Euler characteristic formula and class field theory (Lemma 2.2, Proposition 3.1), none of which is available in a proof assistant library at present. What remains — the structure of the permutation module P and the bookkeeping of the long exact sequences in the proof of Theorem 4.1 — is elementary, and formalising it alone would certify the least doubtful part of the argument while leaving the load-bearing inputs untouched. The PARI/GP checks instead

D (10^6)	fields	$P(c = 1)$	z vs $\frac{1}{3}$
[0, 1)	21939	0.3146	-5.89
[1, 2)	22659	0.3242	-2.92
[2, 3)	22773	0.3218	-3.70
[3, 4)	22860	0.3263	-2.26
[4, 5)	22942	0.3284	-1.59
[5, 6)	22962	0.3297	-1.18
[6, 7)	23027	0.3287	-1.51
[7, 8)	22942	0.3251	-2.65
[8, 9)	23010	0.3312	-0.70
[9, 10)	22936	0.3229	-3.35

TABLE 5. The generic stratum $h = 1$, $d = 2$: $P(c = 1)$ (the anomaly rate) by discriminant, with 95% intervals; dashed line $1/3$.



test every link of the chain on every field of the survey, including the inputs we cite (for instance $d = 1 + \dim V_S$ and $\text{rk}_3 \text{Cl}(k) \leq h - 1$), with the caveat that they are GRH-conditional.

9. RELATION TO LEOPOLDT'S CONJECTURE AND TO FUNCTION FIELDS

Theorems A and B force the first 3-adic digit of $R_3(k)/9$ (two digits when $d = 3$ and $c = 1$), and say nothing about $R_3(k) \neq 0$. Nothing in the argument controls the further digits, and the data are consistent with the next digit being random once the forced one is accounted for (depth exactly 1 in 59.5% of the forced fields). Writing $O_K \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3[S_3]$, the regulator of k is the ρ -factor of the group determinant of the reduced logarithms of one unit of K , an irreducible quadratic form in 3-adic logarithms, and the non-vanishing of such forms is the open (p -adic four exponentials) part of Leopoldt's conjecture for S_3 cubic fields. The non-maximality of $\mathbb{Z}_3[S_3]$ changes the distribution of the reduction of this form, but not its transcendence content.

There is a parallel in positive characteristic. For the Carlitz module over $\mathbb{F}_q(\theta)$ the analogue of Leopoldt's conjecture can fail, and the failure is forced by the same Brauer reduction: in characteristic 3 the reflection representation of S_3 reduces to the non-split extension $[\text{sgn}; \mathbf{1}]$, and for a real cubic extension of $\mathbb{F}_3(\theta)$ whose Galois closure is imaginary and wildly ramified at ∞ the Goss zeta function factors through the non-real quadratic resolvent, which forces a positive Leopoldt defect at every prime [12]. Over number fields the group algebra $\mathbb{Q}[S_3]$ is semisimple and no rank defect can be forced; what survives of the mechanism is the forced digit of Theorem A.

10. OPEN QUESTIONS

- (1) *Equidistribution of c .* Theorem 5.2 makes c an explicit symbol, and in the substratum of Corollary 5.3 a purely local condition on one unit of K . Prove Conjecture 6.1, even conditionally on a Cohen–Lenstra–Martinet-type hypothesis for T_3 of S_3 cubic fields that accounts for the Galois data of Theorems A and B, and explain the small-discriminant deficit of Table 5.
- (2) *The cochain identification.* Complete the outline proof of Theorem 5.2, for instance by identifying $\langle \chi_K \cup \chi_{\text{cyc}}, x \rangle$ with a Rédei-type triple symbol.
- (3) *The strata with $h \geq 2$.* For $h = 2$, $d = 3$ and $c = 0$ the anomaly rate is about $1/3$; a description analogous to Theorem B requires controlling $\text{Cl}(k)[3]$ and its interaction with C_k .

- (4) *Non-fundamental discriminants.* When $d_k = f^2 D$ with $f > 1$, K/F is ramified and K need not lie in a pro-3 extension of F unramified outside 3; the observed rate 0.351 calls for a separate analysis.
- (5) *Limits.* Is the anomaly rate in the generic stratum exactly $1/3$ in the limit, and is $P(c = 1 \mid h = 1, d = 3)$ exactly $1/9$?

REFERENCES

- [1] Armand Brumer, *On the units of algebraic number fields*, Mathematika **14** (1967), 121–124.
- [2] Samit Dasgupta, Mahesh Kakde, and Kevin Ventullo, *On the Gross–Stark conjecture*, Ann. of Math. (2) **188** (2018), no. 3, 833–870.
- [3] Fabio Ferri and Henri Johnston, *Applications of representation theory and of explicit units to Leopoldt’s conjecture*, Res. Number Theory **12** (2026), no. 2, Paper No. 32.
- [4] Frank Gerth, III, *Ranks of 3-class groups of non-Galois cubic fields*, Acta Arith. **30** (1976), no. 4, 307–322.
- [5] Georges Gras, *Class field theory: From theory to practice*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
- [6] ———, *The p -adic Kummer–Leopoldt constant: normalized p -adic regulator*, Int. J. Number Theory **14** (2018), no. 2, 329–337.
- [7] ———, *On p -rationality of number fields. Applications – PARI/GP programs*, Publ. Math. Besançon Algèbre Théorie Nr. (2019), no. 2, 29–51.
- [8] Tommy Hofmann and Yinan Zhang, *Valuations of p -adic regulators of cyclic cubic fields*, J. Number Theory **169** (2016), 86–102.
- [9] Jean-François Jaulent and Thong Nguyen Quang Do, *Corps p -rationnels, corps p -réguliers, et ramification restreinte*, J. Théor. Nombres Bordeaux **5** (1993), no. 2, 343–363.
- [10] Henri Johnston and Andreas Nickel, *On the p -adic Stark conjecture at $s = 1$ and applications*, J. Lond. Math. Soc. (2) **101** (2020), no. 3, 1320–1354, With an appendix by T. Hofmann, H. Johnston and A. Nickel.
- [11] Helmut Koch, *Galois theory of p -extensions*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.
- [12] Shawn Krisman, *A modular obstruction to the P -adic Leopoldt conjecture for the Carlitz module*, Hexagon preprint hexagon:2610.00525v1, 2026, <https://hexagonmath.org/2610.00525v1>.
- [13] Heinrich-Wolfgang Leopoldt, *Zur Arithmetik in abelschen Zahlkörpern*, J. Reine Angew. Math. **209** (1962), 54–71.
- [14] Hang Li and Derong Qiu, *On p -rationality of cubic and quartic number fields*, 2023, arXiv:2304.10157.
- [15] J. S. Milne, *Arithmetic duality theorems*, 2 ed., BookSurge, LLC, Charleston, SC, 2006.
- [16] Abbas Movahhedi and Thong Nguyen Quang Do, *Sur l’arithmétique des corps de nombres p -rationnels*, Séminaire de Théorie des Nombres, Paris 1987–88, Progr. Math., vol. 81, Birkhäuser Boston, 1990.
- [17] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, 2 ed., Grundlehren der mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2008.
- [18] The PARI Group, *PARI/GP version 2.17.3*, 2025, <http://pari.math.u-bordeaux.fr/>.

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