

A COUNTEREXAMPLE TO HUPPERT'S CONJECTURE

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ABSTRACT. Let $S = \mathrm{PSL}_4(5)$ and let G be the unique subgroup of $\mathrm{PGL}_4(5)$ containing S with $[G : S] = 2$. We prove that G and S have the same set of irreducible complex character degrees. Meanwhile, G is centerless and is not isomorphic to $S \times A$ for any finite abelian group A . This provides a counterexample to Huppert's conjecture.

1. INTRODUCTION

For a finite group H , let $\mathrm{Irr}(H)$ denote the set of its irreducible complex characters and

$$\mathrm{cd}(H) = \{\chi(1) : \chi \in \mathrm{Irr}(H)\}$$

the set of its irreducible character degrees, considered without multiplicities. A natural problem in representation theory is to understand how much of the structure of H is encoded by the set $\mathrm{cd}(H)$. While $\mathrm{cd}(H)$ cannot distinguish between H and a direct product $H \times A$ with a finite abelian group A , Huppert [Hup00, p. 829] made the following conjecture:

Huppert's Conjecture. *Let S be a nonabelian finite simple group and H a finite group. If $\mathrm{cd}(H) = \mathrm{cd}(S)$, then $H \cong S \times A$ for some finite abelian group A .*

Huppert's conjecture has been verified for all sporadic simple groups [ADTW13, pp. 289–290], the alternating groups of degree at least 5 [BTZ17, Theorem 1.1], and numerous families of simple groups of Lie type of low rank (see, for instance, [Hup00, NTW12] and the references therein). For the four-dimensional projective special linear groups $\mathrm{PSL}_4(q)$, Nguyen, Tong-Viet, and Wakefield [NTW12, Theorem 1.1] established the conjecture for all prime powers $q \geq 13$. For $q \equiv 1 \pmod{4}$ in this range, they show that $\mathrm{cd}(H) \neq \mathrm{cd}(\mathrm{PSL}_4(q))$ for every $\mathrm{PSL}_4(q) < H \leq \mathrm{PGL}_4(q)$, since the degree

$$a(q) = \frac{1}{4}(q-1)^2(q^2+q+1)(q^2+1)$$

belongs to $\mathrm{cd}(\mathrm{PSL}_4(q))$ but not to $\mathrm{cd}(H)$. They explicitly note the exception at $q = 5$, where $a(5) = (5-1)(5^2+5+1)(5^2+1) = 3224$ is also a degree of $\mathrm{PGL}_4(5)$ [NTW12, proof of Proposition 6.1, case (i)].

In this paper, we show that at $q = 5$ the unique index-two diagonal extension of $\mathrm{PSL}_4(5)$ inside $\mathrm{PGL}_4(5)$ preserves the entire set of irreducible character degrees, thereby yielding a counterexample to Huppert's conjecture.

Theorem 1.1. *Let $\mathbb{F}_5$ denote the finite field with five elements, let $Z = \{aI_4 : a \in \mathbb{F}_5^\times\}$ be the scalar subgroup of $\mathrm{GL}_4(5)$, and define*

$$S = \mathrm{SL}_4(5)/Z = \mathrm{PSL}_4(5), \quad G = \{A \in \mathrm{GL}_4(5) : \det A \in \{1, 4\}\}/Z.$$

Then S is a nonabelian finite simple group, $S \trianglelefteq G$ with $[G : S] = 2$, $Z(G) = 1$, and

$$\begin{aligned} \mathrm{cd}(G) = \mathrm{cd}(S) = \{ & 1, 155, 403, 496, 650, 3224, 3875, 4030, 4836, 6448, \\ & 10075, 11904, 12400, 12896, 14976, 15625, 16120, 19344, 24180 \}. \end{aligned}$$

In particular, $G \not\cong S \times A$ for any finite abelian group A .

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The key is conjugation by $\bar{\delta} = \delta Z$, where $\delta = \text{diag}(4, 1, 1, 1)$. It moves only characters of degree 3224, while fixed characters realize both 3224 and $2 \cdot 3224 = 6448$. Clifford theory then gives $\text{cd}(G) = \text{cd}(S)$.

2. ACTION OF THE DIAGONAL INVOLUTION ON CONJUGACY CLASSES

Throughout the remainder of the paper, we work over the finite field $\mathbb{F}_5$ and maintain the notation of Theorem 1.1. We write I_4 for the 4×4 identity matrix and E_{ij} ($1 \leq i, j \leq 4$) for the standard matrix units. Since $a^4 = 1$ for all $a \in \mathbb{F}_5^\times$, the center $Z = \{aI_4 : a \in \mathbb{F}_5^\times\}$ of $\text{GL}_4(5)$ has order 4 and is contained in $\text{SL}_4(5)$. For $A \in \text{GL}_4(5)$, we denote its image in $\text{PGL}_4(5) = \text{GL}_4(5)/Z$ by $\bar{A} = AZ$. For a finite group H and an element $h \in H$, we write $[h]_H = \{xhx^{-1} : x \in H\}$ for the conjugacy class of h in H , and $C_H(h)$ for its centralizer in H .

Let $\delta = \text{diag}(4, 1, 1, 1) \in \text{GL}_4(5)$. For $a \in \mathbb{F}_5$, we write

$$J_4(a) = aI_4 + E_{12} + E_{23} + E_{34} = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}$$

for the 4×4 Jordan block with eigenvalue a .

Lemma 2.1. *The group $S = \text{PSL}_4(5)$ is nonabelian simple, $G = S \rtimes \langle \bar{\delta} \rangle$ with $[G : S] = 2$, and $C_{\text{PGL}_4(5)}(S) = 1$. In particular, $Z(G) = 1$.*

Proof. It is a standard fact that $S = \text{PSL}_4(5)$ is a nonabelian finite simple group of order $|\text{SL}_4(5)|/|Z| = 7\,254\,000\,000$. Since $\delta^2 = I_4$ and $\det \delta = 4 \notin (\mathbb{F}_5^\times)^4 = \{1\}$, the element $\bar{\delta}$ has order 2 in $\text{PGL}_4(5)$, normalizes S , and does not belong to S . As $\{1, 4\}$ is the unique subgroup of order 2 in $\mathbb{F}_5^\times$, we have $G = S \rtimes \langle \bar{\delta} \rangle$ and $[G : S] = 2$.

To see that $C_{\text{PGL}_4(5)}(S) = 1$, suppose that $\bar{A} \in \text{PGL}_4(5)$ centralizes S , with representative $A \in \text{GL}_4(5)$. For every pair of distinct indices $1 \leq i \neq j \leq 4$, the transvection $I_4 + E_{ij}$ lies in $\text{SL}_4(5)$, so there exists $c_{ij} \in \mathbb{F}_5^\times$ such that

$$A(I_4 + E_{ij})A^{-1} = c_{ij}(I_4 + E_{ij}).$$

Comparing eigenvalues on both sides forces $c_{ij} = 1$. Hence A commutes with E_{ij} for all $i \neq j$, which implies that A is a scalar matrix; that is, $A \in Z$ and $\bar{A} = 1$. Thus $C_{\text{PGL}_4(5)}(S) = 1$, and since $S \leq G \leq \text{PGL}_4(5)$, it follows that $Z(G) \leq C_{\text{PGL}_4(5)}(S) = 1$. $\square$

We next show that conjugation by $\bar{\delta}$ fixes every conjugacy class of S that does not arise from a single Jordan block of size 4.

Lemma 2.2. *Let $A \in \text{SL}_4(5)$. If A is not $\text{GL}_4(5)$ -conjugate to $J_4(a)$ for any $a \in \mathbb{F}_5^\times$, then $4 \in \det C_{\text{GL}_4(5)}(A)$ and*

$$[\delta A \delta^{-1}]_{\text{SL}_4(5)} = [A]_{\text{SL}_4(5)}, \quad [\bar{\delta} \bar{A} \bar{\delta}^{-1}]_S = [\bar{A}]_S.$$

Proof. It suffices to find $X \in C_{\text{GL}_4(5)}(A)$ with $\det X = 4$: then $h = \delta X^{-1} \in \text{SL}_4(5)$ satisfies $hAh^{-1} = \delta A \delta^{-1}$.

Regard $V = \mathbb{F}_5^4$ as an $\mathbb{F}_5[t]$ -module with t acting as A , and write

$$V \cong \bigoplus_{\nu} \mathbb{F}_5[t]/(f_{\nu}^{j_{\nu}}),$$

where the f_{ν} are monic irreducible polynomials. We will construct X on one summand and let it act as the identity on all the others.

Fix a summand $W = \mathbb{F}_5[t]/(f^j)$, with $\deg f = r$. Since $A|_W$ acts by multiplication by t , multiplication by any polynomial $b(t)$ commutes with $A|_W$. We therefore seek such a multiplication map with determinant 4.

To determine the possible determinants, put $K = \mathbb{F}_5[t]/(f) \cong \mathbb{F}_{5^r}$ and let β be the image of $b(t)$ in K . If $\beta \neq 0$, multiplication by $b(t)$ defines an invertible map T on W . The filtration

$$W \supset fW \supset \cdots \supset f^j W = 0$$

has j successive quotients isomorphic to K , on each of which T acts by multiplication by β . Thus

$$\det T = N_{K/\mathbb{F}_5}(\beta)^j.$$

Consequently, to obtain $\det T = 4$, it suffices to choose $c \in \mathbb{F}_5^\times$ with $c^j = 4$ and then choose $\beta \in K^\times$ with $N_{K/\mathbb{F}_5}(\beta) = c$, using the surjectivity of the norm map. For $j = 1, 2, 3$, we may take $c = 4, 2, 4$, respectively.

Finally, the hypothesis guarantees a summand with $j \leq 3$: otherwise, since $\dim V = 4$, the decomposition would consist of a single summand $\mathbb{F}_5[t]/((t-a)^4)$, making A conjugate to $J_4(a)$. On such a summand choose T as above and extend it by the identity on the remaining summands. The resulting map X commutes with A and has determinant 4, as required. $\square$

We now analyze the remaining conjugacy classes, which come from the Jordan blocks $J_4(a)$ with $a \in \mathbb{F}_5^\times$.

Proposition 2.3. *Let $N = E_{12} + E_{23} + E_{34}$, $u = I_4 + N = J_4(1)$, and $d = \text{diag}(2, 1, 1, 1) \in \text{GL}_4(5)$. For $k \in \mathbb{Z}/4\mathbb{Z}$, define $u_k = d^k u d^{-k} \in \text{SL}_4(5)$ and $\mathcal{C}_k = [\overline{u_k}]_S$. Then the following hold:*

- (i) *Every conjugacy class of S not fixed by conjugation by $\bar{\delta}$ belongs to $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$.*
- (ii) *The four conjugacy classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are pairwise distinct in S , and $|C_S(\overline{u_k})| = 125$ for each $k \in \mathbb{Z}/4\mathbb{Z}$.*
- (iii) *For each $k \in \mathbb{Z}/4\mathbb{Z}$,*

$$\bar{\delta} \mathcal{C}_k \bar{\delta}^{-1} = \mathcal{C}_{k+2} = \{s^2 : s \in \mathcal{C}_k\}.$$

Proof. (i) and (ii). For any $a \in \mathbb{F}_5^\times$, setting $D_a = \text{diag}(a^3, a^2, a, 1) \in \text{GL}_4(5)$ gives $D_a J_4(a) D_a^{-1} = a J_4(1) = au$. Since $aI_4 \in Z$, we have $\overline{au} = \overline{u}$ in S , and hence for every $P \in \text{GL}_4(5)$,

$$\overline{PJ_4(a)P^{-1}} = \overline{(PD_a^{-1})u(PD_a^{-1})^{-1}}.$$

By Lemma 2.2, any S -conjugacy class not fixed by conjugation by $\bar{\delta}$ has a representative of the form $\overline{PuP^{-1}}$ with $P \in \text{GL}_4(5)$.

Since u is a single Jordan block,

$$C_{\text{GL}_4(5)}(u) = \{a_0 I_4 + a_1 N + a_2 N^2 + a_3 N^3 : a_0 \in \mathbb{F}_5^\times, a_1, a_2, a_3 \in \mathbb{F}_5\}.$$

This centralizer has order $4 \cdot 5^3 = 500$ and lies in $\text{SL}_4(5)$, since its elements have determinant $a_0^4 = 1$.

For $P, Q \in \text{GL}_4(5)$,

$$[PuP^{-1}]_{\text{SL}_4(5)} = [QuQ^{-1}]_{\text{SL}_4(5)} \iff \det P = \det Q. \quad (2.1)$$

Indeed, a conjugating $h \in \text{SL}_4(5)$ gives $Q^{-1}hP \in C_{\text{GL}_4(5)}(u)$; conversely, take $h = QP^{-1}$. Since $\det(d^k) = 2^k$, the u_k represent exactly four $\text{SL}_4(5)$ -classes.

Comparing eigenvalues for both sides of $hu_i h^{-1} = cu_j$ forces $c = 1$. Thus the classes $\mathcal{C}_k$ remain pairwise distinct in S , and $C_S(\overline{u_k}) = C_{\text{SL}_4(5)}(u_k)/Z$ has order $500/4 = 125$. This completes (i) and (ii).

(iii) Since $d^2 = \text{diag}(4, 1, 1, 1) = \delta$, we have $\delta u_k \delta^{-1} = d^{k+2} u d^{-(k+2)} = u_{k+2}$, and therefore $\bar{\delta} \mathcal{C}_k \bar{\delta}^{-1} = \mathcal{C}_{k+2}$. To compare this with the squaring map, define

$$B = \begin{pmatrix} 3 & 4 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}_4(5).$$

A direct matrix computation over $\mathbb{F}_5$ gives $BN = (2N + N^2)B$, so

$$BuB^{-1} = I_4 + 2N + N^2 = (I_4 + N)^2 = u^2.$$

Hence $u_k^2 = d^k u^2 d^{-k} = (d^k B) u (d^k B)^{-1}$. Since $\det B = 24 \equiv 4 = 2^2 \pmod{5}$, the conjugating matrix $d^k B$ has determinant $\det(d^k B) = 2^{k+2} = \det(d^{k+2})$. By the determinant criterion (2.1), u_k^2 is $\text{SL}_4(5)$ -conjugate to u_{k+2} . Passing to the quotient S , we conclude that $\{s^2 : s \in \mathcal{C}_k\} = \mathcal{C}_{k+2} = \bar{\delta} \mathcal{C}_k \bar{\delta}^{-1}$. $\square$

3. CHARACTER DEGREES AND PROOF OF THEOREM 1.1

We now determine how conjugation by $\bar{\delta}$ permutes $\text{Irr}(S)$ using the ordinary character table of $S = \text{PSL}_4(5)$, which is listed as L4(5) in the GAP Character Table Library, version 1.3.11 [Bre25], and given in Appendix A. Throughout this section, all row labels $\chi_1, \dots, \chi_{49}$ and column indices $1, \dots, 49$ refer to the tables in Appendix A. For $\chi \in \text{Irr}(S)$ and $s \in S$, write $\chi^{\bar{\delta}}(s) = \chi(\bar{\delta}^{-1} s \bar{\delta})$.

Proposition 3.1. *Conjugation by $\bar{\delta}$ interchanges χ_9 with χ_{10} and χ_{11} with χ_{12} , where*

$$\chi_9(1) = \chi_{10}(1) = \chi_{11}(1) = \chi_{12}(1) = 3224,$$

and fixes all other 45 irreducible characters of S .

Proof. The centralizer orders $c_j = |C_S(g_j)|$ listed in Tables A.1–A.3 in Appendix A show that columns 11, 12, 13, 14 (labeled 5e, 5f, 5g, 5h) are the unique conjugacy classes of S whose centralizers have order 125. By Proposition 2.3(ii), these four columns correspond to the classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Moreover, the row $p_2(j)$ in Table A.1 shows that squaring interchanges the classes indexed by 11 and 12, and those indexed by 13 and 14. By Proposition 2.3(i) and (iii), conjugation by $\bar{\delta}$ induces the same permutation (11 12)(13 14) on the columns of the character table and fixes all remaining 45 columns.

In Table A.1, the 4×4 submatrix formed by rows $\chi_9, \chi_{10}, \chi_{11}, \chi_{12}$ and columns 11, 12, 13, 14 is

	11 (5e)	12 (5f)	13 (5g)	14 (5h)
χ_9	−1	−1	A	A'
χ_{10}	−1	−1	A'	A
χ_{11}	A	A'	−1	−1
χ_{12}	A'	A	−1	−1

$$A = \frac{3 - 5\sqrt{5}}{2}, \quad A' = \frac{3 + 5\sqrt{5}}{2}.$$

Furthermore, χ_9 and χ_{10} agree on all columns outside $\{13, 14\}$, while χ_{11} and χ_{12} agree on all columns outside $\{11, 12\}$. For every other row χ_i ($i \notin \{9, 10, 11, 12\}$), the entries in columns 11 and 12 are equal, and the entries in columns 13 and 14 are equal. Since $A \neq A'$, conjugation by $\bar{\delta}$ interchanges χ_9 and χ_{10} , and χ_{11} and χ_{12} , while fixing all other irreducible characters of S . Reading column 1 of Table A.1 gives $\chi_9(1) = \chi_{10}(1) = \chi_{11}(1) = \chi_{12}(1) = 3224$. $\square$

Proof of Theorem 1.1. By Lemma 2.1, $S = \text{PSL}_4(5)$ is a nonabelian simple group, $G = S \rtimes \langle \bar{\delta} \rangle$ contains S as a normal subgroup of index 2, and $Z(G) = 1$.

Reading the distinct entries in column 1 of Table A.1 yields the 19-element set $\text{cd}(S)$ in Theorem 1.1.

Since G/S is cyclic of prime order 2, Clifford theory (see, e.g., [Isa76, Theorem 6.11 and Corollaries 6.17 and 11.22]) partitions $\text{Irr}(G)$ into two types of characters:

- (a) Each G -invariant character $\chi \in \text{Irr}(S)$ extends to two distinct irreducible characters of G , each of degree $\chi(1)$.
- (b) The two characters in each orbit $\{\chi, \chi^{\bar{\delta}}\} \subset \text{Irr}(S)$ of size 2 induce to the same irreducible character $\text{Ind}_S^G(\chi) \in \text{Irr}(G)$ of degree $2\chi(1)$.

By Proposition 3.1, the only characters of S not fixed by conjugation by $\bar{\delta}$ are $\chi_9, \chi_{10}, \chi_{11}, \chi_{12}$, all of degree 3224. Since $\chi_7, \chi_8 \in \text{Irr}(S)$ also have degree 3224 and are G -invariant, every degree in $\text{cd}(S)$ is realized by at least one G -invariant irreducible character of S , so $\text{cd}(S) \subseteq \text{cd}(G)$. On the other hand, $\bar{\delta}$ interchanges χ_9 with χ_{10} and χ_{11} with χ_{12} , so these characters induce to irreducible characters of G of degree $2 \cdot 3224 = 6448$, which already belongs to $\text{cd}(S)$ because $\chi_{16}(1) = \chi_{17}(1) = 6448$. Therefore, $\text{cd}(G) = \text{cd}(S) \cup \{2 \cdot 3224\} = \text{cd}(S)$.

Finally, if $G \cong S \times A$ for some finite abelian group A , comparing orders gives $|A| = [G : S] = 2$, which implies $|Z(G)| = |Z(S \times A)| = |A| = 2$, a contradiction to $Z(G) = 1$. $\square$

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APPENDIX A. THE ORDINARY CHARACTER TABLE OF $\text{PSL}_4(5)$

The following reproduces the ordinary character table $\text{L4}(5)$ from CTblLib 1.3.11 [Bre25], with its row and column order unchanged. For a class representative g_j , the headers give its class name, order o_j , centralizer order $c_j = |C_S(g_j)|$, and square-class number $p_2(j)$. Column 1, giving character degrees, is repeated in every block; $c_1 = |S| = 7\,254\,000\,000$. A dot denotes zero. Noninteger values are defined after the last block.

Table A.1. Character table of S : columns 1–17.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
class	1a	2a	3a	3b	4a	4b	5a	5b	5c	5d	5e	5f	5g	5h	6a	6b	8a
o_j	1	2	3	3	4	4	5	5	5	5	5	5	5	5	6	6	8
c_j	$ S $	28800	23400	720	480	144	375000	37500	37500	625	125	125	125	125	720	72	24
$p_2(j)$	1	1	3	4	2	2	7	8	9	10	12	11	14	13	4	3	5
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	155	11	−1	5	7	−1	30	5	5	5	5	5	5	5	5	−1	1
χ_3	403	19	13	1	−5	−1	28	28	3	3	3	3	−2	−2	1	1	−1
χ_4	403	19	13	1	−5	−1	28	3	28	3	−2	−2	3	3	1	1	−1
χ_5	496	16	25	4	·	4	−4	21	21	−4	1	1	1	1	4	1	·
χ_6	650	26	26	−4	6	2	25	25	25	·	·	·	·	·	−4	2	·
χ_7	3224	8	26	5	4	−4	99	−26	−26	−1	−1	−1	−1	−1	5	2	−2
χ_8	3224	8	26	5	4	4	99	−26	−26	−1	−1	−1	−1	−1	5	2	2
χ_9	3224	−8	26	−4	·	·	−26	−26	49	−1	−1	−1	A	A'	4	−2	·
χ_{10}	3224	−8	26	−4	·	·	−26	−26	49	−1	−1	−1	A'	A	4	−2	·
χ_{11}	3224	−8	26	−4	·	·	−26	49	−26	−1	A	A'	−1	−1	4	−2	·
χ_{12}	3224	−8	26	−4	·	·	−26	49	−26	−1	A'	A	−1	−1	4	−2	·
χ_{13}	3875	35	−25	−1	11	−1	125	·	·	·	·	·	·	·	−1	−1	−1
χ_{14}	4030	46	−26	4	−6	2	155	5	5	5	·	·	·	·	4	−2	·
χ_{15}	4836	−60	·	6	4	·	211	36	36	11	1	1	1	1	6	·	·
χ_{16}	6448	16	13	4	·	−4	−52	73	−27	−2	−2	−2	3	3	4	1	·
χ_{17}	6448	16	13	4	·	−4	−52	−27	73	−2	3	3	−2	−2	4	1	·
χ_{18}	10075	43	13	−5	−1	−1	75	−50	75	·	·	·	·	·	−5	1	1
χ_{19}	10075	43	13	−5	−1	−1	75	75	−50	·	·	·	·	·	−5	1	1
χ_{20}	11904	·	48	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{21}	11904	·	48	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{22}	11904	·	48	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{23}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{24}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{25}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{26}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{27}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{28}	11904	·	−24	·	·	·	−96	4	4	4	−1	−1	−1	−1	·	·	·
χ_{29}	12400	16	1	4	·	4	−100	25	25	·	·	·	·	·	4	1	·
χ_{30}	12896	−32	26	8	·	·	−104	46	46	−4	1	1	1	1	−8	−2	·
χ_{31}	12896	−32	−52	−4	·	·	−104	46	46	−4	1	1	1	1	4	4	·
χ_{32}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{33}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{34}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{35}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{36}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{37}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{38}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{39}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{40}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{41}	14976	·	·	·	·	·	−24	−24	−24	1	1	1	1	1	·	·	·
χ_{42}	15625	25	25	−5	5	1	·	·	·	·	·	·	·	·	−5	1	−1
χ_{43}	16120	40	−26	1	4	4	−5	20	20	−5	·	·	·	·	1	−2	−2
χ_{44}	16120	40	−26	1	4	−4	−5	20	20	−5	·	·	·	·	1	−2	2
χ_{45}	19344	48	·	6	−8	·	94	−6	−6	−6	−1	−1	−1	−1	6	·	·
χ_{46}	19344	−48	·	−12	·	·	94	−6	−6	−6	−1	−1	−1	−1	12	·	·
χ_{47}	19344	−48	·	6	·	·	94	−6	−6	−6	−1	−1	−1	−1	−6	·	·
χ_{48}	19344	−48	·	6	·	·	94	−6	−6	−6	−1	−1	−1	−1	−6	·	·
χ_{49}	24180	−12	·	−6	−4	·	305	30	30	5	·	·	·	·	−6	·	·

Table A.2. Character table of S : columns 18–33 (with column 1 repeated).

j	1	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
class	1a	10a	10b	10c	12a	12b	12c	13a	13b	13c	15a	15b	15c	20a	24a	24b	30a
o_j	1	10	10	10	12	12	12	13	13	13	15	15	15	20	24	24	30
c_j	$ S $	600	100	100	36	36	24	39	39	39	75	75	30	20	24	24	30
$p_2(j)$	1	7	8	9	15	16	15	25	26	24	27	28	29	18	23	23	29
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	155	6	1	1	-1	-1	1	-1	-1	-1	-1	-1	·	2	1	1	·
χ_3	403	4	4	-1	-1	-1	1	·	·	·	-2	3	1	·	-1	-1	1
χ_4	403	4	-1	4	-1	-1	1	·	·	·	3	-2	1	·	-1	-1	1
χ_5	496	-4	1	1	-2	1	·	2	2	2	·	·	-1	·	·	·	-1
χ_6	650	1	1	1	2	2	·	·	·	·	1	1	1	1	·	·	1
χ_7	3224	3	-2	-2	-1	2	1	·	·	·	1	1	·	-1	1	1	·
χ_8	3224	3	-2	-2	1	-2	1	·	·	·	1	1	·	-1	-1	-1	·
χ_9	3224	2	2	-3	·	·	·	·	·	·	1	1	1	·	·	·	-1
χ_{10}	3224	2	2	-3	·	·	·	·	·	·	1	1	1	·	·	·	-1
χ_{11}	3224	2	-3	2	·	·	·	·	·	·	1	1	1	·	·	·	-1
χ_{12}	3224	2	-3	2	·	·	·	·	·	·	1	1	1	·	·	·	-1
χ_{13}	3875	5	·	·	-1	-1	-1	1	1	1	·	·	-1	1	-1	-1	-1
χ_{14}	4030	11	1	1	2	2	·	·	·	·	-1	-1	-1	-1	·	·	-1
χ_{15}	4836	-5	·	·	·	·	-2	·	·	·	·	·	·	1	-1	·	1
χ_{16}	6448	-4	1	1	2	-1	·	·	·	·	-2	3	-1	·	·	·	-1
χ_{17}	6448	-4	1	1	2	-1	·	·	·	·	3	-2	-1	·	·	·	-1
χ_{18}	10075	3	-2	3	-1	-1	-1	·	·	·	-2	3	·	-1	1	1	·
χ_{19}	10075	3	3	-2	-1	-1	-1	·	·	·	3	-2	·	-1	1	1	·
χ_{20}	11904	·	·	·	·	·	·	B	D	C	-2	-2	·	·	·	·	·
χ_{21}	11904	·	·	·	·	·	·	C	B	D	-2	-2	·	·	·	·	·
χ_{22}	11904	·	·	·	·	·	·	D	C	B	-2	-2	·	·	·	·	·
χ_{23}	11904	·	·	·	·	·	·	D	C	B	1	1	·	·	·	·	·
χ_{24}	11904	·	·	·	·	·	·	D	C	B	1	1	·	·	·	·	·
χ_{25}	11904	·	·	·	·	·	·	C	B	D	1	1	·	·	·	·	·
χ_{26}	11904	·	·	·	·	·	·	C	B	D	1	1	·	·	·	·	·
χ_{27}	11904	·	·	·	·	·	·	B	D	C	1	1	·	·	·	·	·
χ_{28}	11904	·	·	·	·	·	·	B	D	C	1	1	·	·	·	·	·
χ_{29}	12400	-4	1	1	-2	1	·	-2	-2	-2	1	1	-1	·	·	·	-1
χ_{30}	12896	8	-2	-2	·	·	·	·	·	·	1	1	-2	·	·	·	2
χ_{31}	12896	8	-2	-2	·	·	·	·	·	·	-2	-2	1	·	·	·	-1
χ_{32}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{33}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{34}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{35}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{36}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{37}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{38}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{39}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{40}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{41}	14976	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
χ_{42}	15625	·	·	·	1	1	-1	-1	-1	-1	·	·	·	·	-1	-1	·
χ_{43}	16120	-5	·	·	1	-2	1	·	·	·	-1	-1	1	-1	1	1	1
χ_{44}	16120	-5	·	·	-1	2	1	·	·	·	-1	-1	1	-1	-1	-1	1
χ_{45}	19344	-2	-2	-2	·	·	-2	·	·	·	·	·	·	1	2	·	1
χ_{46}	19344	2	2	2	·	·	·	·	·	·	·	·	-2	·	·	·	2
χ_{47}	19344	2	2	2	·	·	·	·	·	·	·	·	1	·	E	$-E$	-1
χ_{48}	19344	2	2	2	·	·	·	·	·	·	·	·	1	·	$-E$	E	-1
χ_{49}	24180	-7	-2	-2	·	·	2	·	·	·	·	·	-1	1	·	·	-1

Table A.3. Character table of S : columns 34–49 (with column 1 repeated).

j	1	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
class	1a	31a	31b	31c	31d	31e	31f	31g	31h	31i	31j	39a	39b	39c	39d	39e	39f
o_j	1	31	31	31	31	31	31	31	31	31	31	39	39	39	39	39	39
c_j	$ S $	31	31	31	31	31	31	31	31	31	31	39	39	39	39	39	39
$p_2(j)$	1	42	43	35	34	37	36	39	38	41	40	48	49	45	44	47	46
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	155	.	.	.	.	.	.	.	.	.	.	-1	-1	-1	-1	-1	-1
χ_3	403	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_4	403	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_5	496	.	.	.	.	.	.	.	.	.	.	-1	-1	-1	-1	-1	-1
χ_6	650	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	.	.	.	.	.	.
χ_7	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_8	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_9	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{10}	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{11}	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{12}	3224	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{13}	3875	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1
χ_{14}	4030	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{15}	4836	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{16}	6448	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{17}	6448	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{18}	10075	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{19}	10075	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{20}	11904	.	.	.	.	.	.	.	.	.	.	C	C	D	D	B	B
χ_{21}	11904	.	.	.	.	.	.	.	.	.	.	D	D	B	B	C	C
χ_{22}	11904	.	.	.	.	.	.	.	.	.	.	B	B	C	C	D	D
χ_{23}	11904	.	.	.	.	.	.	.	.	.	.	K	$\bar{K}$	$\bar{M}$	M	$\bar{L}$	L
χ_{24}	11904	.	.	.	.	.	.	.	.	.	.	$\bar{K}$	K	M	$\bar{M}$	L	$\bar{L}$
χ_{25}	11904	.	.	.	.	.	.	.	.	.	.	L	$\bar{L}$	K	$\bar{K}$	$\bar{M}$	M
χ_{26}	11904	.	.	.	.	.	.	.	.	.	.	$\bar{L}$	L	$\bar{K}$	K	M	$\bar{M}$
χ_{27}	11904	.	.	.	.	.	.	.	.	.	.	M	$\bar{M}$	L	$\bar{L}$	K	$\bar{K}$
χ_{28}	11904	.	.	.	.	.	.	.	.	.	.	$\bar{M}$	M	$\bar{L}$	L	$\bar{K}$	K
χ_{29}	12400	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1
χ_{30}	12896	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{31}	12896	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{32}	14976	F	$\bar{F}$	$\bar{J}$	J	$\bar{I}$	I	$\bar{H}$	H	$\bar{G}$	G	.	.	.	.	.	.
χ_{33}	14976	$\bar{F}$	F	J	$\bar{J}$	I	$\bar{I}$	H	$\bar{H}$	G	$\bar{G}$	.	.	.	.	.	.
χ_{34}	14976	G	$\bar{G}$	F	$\bar{F}$	$\bar{J}$	J	$\bar{I}$	I	$\bar{H}$	H	.	.	.	.	.	.
χ_{35}	14976	$\bar{G}$	G	$\bar{F}$	F	J	$\bar{J}$	I	$\bar{I}$	H	$\bar{H}$	.	.	.	.	.	.
χ_{36}	14976	H	$\bar{H}$	G	$\bar{G}$	F	$\bar{F}$	$\bar{J}$	J	$\bar{I}$	I	.	.	.	.	.	.
χ_{37}	14976	$\bar{H}$	H	$\bar{G}$	G	$\bar{F}$	F	J	$\bar{J}$	I	$\bar{I}$	.	.	.	.	.	.
χ_{38}	14976	I	$\bar{I}$	H	$\bar{H}$	G	$\bar{G}$	F	$\bar{F}$	$\bar{J}$	J	.	.	.	.	.	.
χ_{39}	14976	$\bar{I}$	I	$\bar{H}$	H	$\bar{G}$	G	$\bar{F}$	F	J	$\bar{J}$	.	.	.	.	.	.
χ_{40}	14976	J	$\bar{J}$	I	$\bar{I}$	H	$\bar{H}$	G	$\bar{G}$	F	$\bar{F}$	.	.	.	.	.	.
χ_{41}	14976	$\bar{J}$	J	$\bar{I}$	I	$\bar{H}$	H	G	$\bar{G}$	F	$\bar{F}$	.	.	.	.	.	.
χ_{42}	15625	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_{43}	16120	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{44}	16120	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{45}	19344	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{46}	19344	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{47}	19344	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{48}	19344	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{49}	24180	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Bars denote complex conjugation. The noninteger values use the following notation:

$$\zeta_n = e^{2\pi i/n}, \quad \eta_{n,r} = \sum_{k=0}^{d_n-1} \zeta_n^{r5^k}, \quad d_{13} = d_{39} = 4, \quad d_{31} = 3,$$

$$A = \frac{3-5\sqrt{5}}{2}, \quad A' = \frac{3+5\sqrt{5}}{2}, \quad E = -i\sqrt{6},$$

$$(B, C, D) = (-\eta_{13,1}, -\eta_{13,4}, -\eta_{13,2}),$$

$$(F, G, H, I, J) = (\eta_{31,17}, \eta_{31,16}, \eta_{31,6}, \eta_{31,2}, \eta_{31,11}),$$

$$(K, L, M) = (-\eta_{39,23}, -\eta_{39,4}, -\eta_{39,14}).$$

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