

Rigidity of Newtonian Centered Co-Circular Central Configurations

Chih-Yao Wu*

October 23, 2026

Abstract

We prove that a Newtonian central configuration of finitely many positive masses on a circle whose center is the center of mass must be a regular polygon with equal masses. The proof selects an extremal mass-gap pair and lifts the circle to a periodic configuration on the line. A positive integral representation of the circular kernel converts the force equations into sharp weighted quadrature bounds. A scalar convexity estimate forces equality at the selected pair, and the equality conditions determine every mass and gap. The argument applies to every finite number of bodies and includes all analytic and rational estimates needed for the conclusion.

1 Introduction

A central configuration is an arrangement of point masses whose gravitational accelerations are proportional to their displacements from the center of mass. Equal masses at the vertices of a regular polygon give the simplest circular examples. The converse asks whether a circle centered at the center of mass can support any other Newtonian central configuration.

This is the Newtonian case of Problem 12 in the list of Albouy, Cabral and Santos [1]. Cors, Hall and Roberts [2] established uniqueness results for fixed positive masses and, in particular, regularity when the masses are equal. Hampton [3] proved local isolation of the equal-mass regular polygon among centered co-circular relative equilibria. Wang [4] obtained a criterion yielding the Newtonian classification for $n \leq 6$. Tang and Zhu [5] excluded further prescribed patterns of unequal masses. These results distinguish two issues: the geometry once the masses are equal, and the exclusion of unequal masses. Here both follow from one equality argument, for every finite number of bodies.

Theorem 1. *Let $n \geq 3$, let $m_1, \dots, m_n > 0$, and let distinct points $r_1, \dots, r_n$ lie on a circle of radius $R > 0$ and center O . Suppose that O is their center of mass and that, for some $G > 0$ and $\lambda \in \mathbb{R}$,*

$$G \sum_{j \neq i} m_j \frac{r_j - r_i}{|r_j - r_i|^3} = -\lambda(r_i - O), \quad i = 1, \dots, n. \quad (1)$$

Then the masses are equal and the points form a regular n -gon. Conversely, every equal-mass regular n -gon satisfies these conditions.

The argument takes place in the plane of the circle. It therefore applies to circles in any Euclidean ambient space. The two-body case consists of equal antipodal masses, as follows directly from centering.

*Correspondence: andersonwu2000@gmail.com. AI agents developed the argument and assisted with manuscript preparation under the author's coordination. This manuscript has not received an independent human expert review.

The proof begins by selecting a body with maximal mass relative to its nearest angular gap. After rescaling, that body has mass 1 and a neighbor at distance 1 with mass $b \leq 1$. The same selection gives a separation bound for every pair in the periodic lift of the circle. A convex quadrature inequality then bounds the contribution of any exterior distribution by a weighted lattice.

The bridge between this one-dimensional inequality and Newtonian gravity is a positive integral representation of the circular kernel. It turns the radial and tangential equations at the selected pair into two balance identities. Those identities first imply $b > 7/10$; a mixed-kernel estimate and one exact rational comparison then give $b = 1$. At that point the quadrature bounds are equalities. Every change of mass or gap produces a strictly positive loss, so equality forces the entire lift to be the unit lattice. Its projection is the regular polygon.

Sections 2–4 establish the normalization, quadrature inequality and kernel representation. Sections 5–7 force equality of the selected masses, and Sections 8–9 recover the entire configuration. Appendix A evaluates the scalar integral used in the periodization. Appendix B gives a complementary interpretation through a conserved interaction across cuts of the lifted configuration. All estimates needed for the theorem are proved in the text, including the finite rational calculation.

We use angular gaps in radians. A sum over physical bodies omits the target body; a sum over the periodic lift omits only the particular target point. This distinction accounts for the self-image term in Section 4.

2 An extremal pair and the periodic lift

Set $M = \sum_i m_i$, $p_i = m_i/M$, $u_i = (r_i - O)/R = (\cos \theta_i, \sin \theta_i)$, and $\lambda_* = \lambda R^3/(GM)$. Put $t_i = (-\sin \theta_i, \cos \theta_i)$. For $i \neq j$, define

$$K_{ij} = |u_j - u_i|^{-1}, \quad \Omega_{ij} = \frac{\sin(\theta_j - \theta_i)}{|u_j - u_i|^3},$$

with zero diagonal entries. Because

$$(u_j - u_i) \cdot u_i = -\frac{1}{2}|u_j - u_i|^2, \quad (u_j - u_i) \cdot t_i = \sin(\theta_j - \theta_i),$$

the normalized force is exactly

$$F_i := \sum_{j \neq i} p_j \frac{u_j - u_i}{|u_j - u_i|^3} = -\frac{1}{2}(Kp)_i u_i + (\Omega p)_i t_i. \quad (2)$$

Thus (1) is equivalent to

$$Kp = c\mathbf{1}, \quad \Omega p = 0, \quad c = 2\lambda_* > 0. \quad (3)$$

Positivity of c follows from the positive off-diagonal entries of K . Conversely, (3) gives (1), and pairwise cancellation gives $\sum_i p_i F_i = 0$, hence $\sum_i p_i u_i = 0$. We will use the given centering directly.

Order the angles cyclically. Write $g_i > 0$ for the successive angular gaps, including the seam, so $\sum_i g_i = 2\pi$. Every $g_i < \pi$. Indeed, a gap of size at least π places all points in a closed semicircle. A normal vector pointing toward its midpoint has nonnegative scalar product with all points. If the containing arc has length less than π , all products are positive. If its length is exactly π , at most its two endpoints have zero product; $n \geq 3$ supplies an interior point. Positive masses then contradict centering.

Define

$$d_i = \min(g_{i-1}, g_i), \quad \rho = \max_i \frac{p_i}{d_i}.$$

The maximum exists and is positive. Choose a maximizing atom, relabel it 0, set $a = p_0$ and $\delta = d_0 = a/\rho$, and choose the orientation so a nearest neighbor lies in the positive angular direction. Rescale arcs and masses by

$$x_i = \theta_i/\delta \pmod{L}, \quad q_i = p_i/a, \quad L = 2\pi/\delta. \quad (4)$$

After rotation, the anchor is at 0 with mass 1, its next point is at 1 with mass $b \in (0, 1]$, and $L > 2$. For every atom, maximality of ρ gives

$$q_i \leq \min(g_{i-1}/\delta, g_i/\delta). \quad (5)$$

In particular $b \leq 1$. The useful constraint is the local mass-gap bound (5); other normalized masses may exceed 1.

Let $X = \{x_i + kL : 0 \leq i < n, k \in \mathbb{Z}\}$, with the same mass at every image. Consecutive gaps are the rescaled cyclic gaps. For any distinct lifted atoms $x < y$, the first gap to the right of x and last gap to the left of y are at most $y - x$. Equation (5) therefore gives

$$|x - y| \geq \max(q_x, q_y). \quad (6)$$

The inequality applies throughout the periodic lift, including different images of the same body. The lift has a positive minimum gap, a finite maximum gap, and bounded positive masses, since the base configuration is finite.

For $0 < t < L$, put

$$h_L(t) = \frac{\pi}{L} \csc \frac{\pi t}{L} = \frac{\delta}{2 \sin(\delta t/2)}.$$

The exact finite radial and derivative rows are

$$\sum_{j \neq i} q_j h_L(x_j - x_i \pmod{L}) = c/\rho, \quad \sum_{j \neq i} q_j h'_L(x_j - x_i \pmod{L}) = 0. \quad (7)$$

To check the constants, write $h(\theta) = [2 \sin(\theta/2)]^{-1}$ for $0 < \theta < 2\pi$. Then $h_L(t) = \delta h(\delta t)$, so the first sum is $\delta c/a = c/\rho$. Moreover,

$$h'(\theta) = -\frac{\sin \theta}{[2 \sin(\theta/2)]^3}, \quad h'_L(t) = \delta^2 h'(\delta t).$$

Consequently the second sum is $-\delta^2(\Omega p)_i/a = 0$. Modulo differences always take their representative in $(0, L)$; self-images are absent from (7).

We shall prove that these balance equations saturate a sharp bound for separated weighted points. Its equality case will make the lift a unit-mass integer lattice.

3 A sharp bound for separated weighted points

The separation bound (6) limits the mass that can be placed near an anchor. We need to convert this geometric restriction into a bound for a weighted sum of kernel values. The comparison distribution will have equal masses at equally spaced points.

The two sums below describe that comparison: $V(t, s)$ includes the point at distance t , whereas $R(t, s)$ starts one step later. Inequality (8) allows one spacing to be replaced by the next spacing while paying for the mass at the intervening point. Iterating this one-step inequality will prove (9).

Lemma 1 (Weighted quadrature). *Let ϕ be continuous, nonnegative, decreasing and convex on $[0, \infty)$, tending to zero. At the arguments in question suppose the following sums are finite, and define*

$$V(t, s) = s \sum_{j \geq 0} \phi(t + js), \quad R(t, s) = s \sum_{j \geq 1} \phi(t + js), \quad s > 0.$$

Then $V(t, s)$ is nondecreasing and $R(t, s)$ nonincreasing in s . Moreover,

$$\min(s, h)\phi(t) + V(t + h, h) \leq V(t, s). \quad (8)$$

If a countable positive source $t_i \geq d > 0$ satisfies $q_i \leq t_i$ and $|t_i - t_j| \geq \max(q_i, q_j)$, then

$$\sum_i q_i \phi(t_i) \leq d \sum_{j \geq 1} \phi(jd). \quad (9)$$

Proof. The nonnegative distributional second derivative μ of a convex decreasing function tending to zero represents it as

$$\phi(u) = \int_{(0, \infty)} (z - u)_+ d\mu(z). \quad (10)$$

For completeness, its right derivative is increasing and tends to zero: a strictly negative limiting derivative would force ϕ below zero. Thus $-\phi'_+(u) = \mu((u, \infty))$ away from the harmless endpoint convention. Integrating from u to infinity and using Tonelli gives (10); continuity supplies the endpoint value. All applied extensions below are continuous, so no endpoint jump is present.

For a triangular kernel $(z - u)_+$, set $A_0 = (z - t)_+$ and $m = \lfloor A_0/s \rfloor$. Finite summation gives

$$R = smA_0 - \frac{1}{2}s^2m(m+1), \quad V = s(m+1)A_0 - \frac{1}{2}s^2m(m+1).$$

On each interval where m is fixed,

$$\partial_s R = m[A_0 - (m+1)s] \leq 0, \quad \partial_s V = (m+1)[A_0 - ms] \geq 0.$$

The formulas agree at breakpoints, including the zero terms at integer A_0/s . Integration against μ , justified by positivity, proves the two monotonicities. If $h \geq s$, the left side of (8) is $s\phi(t) + R(t, h) \leq s\phi(t) + R(t, s)$. If $h \leq s$, it is $V(t, h) \leq V(t, s)$.

For a finite selected subset $t_1 < \dots < t_m$, put $s_1 = t_1 \geq d$ and $s_i = t_i - t_{i-1}$ for $i \geq 2$. The assumptions give $q_i \leq \min(s_i, s_{i+1})$ for $i < m$, and $q_m \leq s_m$. The last point satisfies $q_m \phi(t_m) \leq V(t_m, s_m)$. Suppose that the points after t_i have already been bounded by

$V(t_{i+1}, s_{i+1})$. Since $t_{i+1} = t_i + s_{i+1}$, inequality (8), with $t = t_i$, $s = s_i$ and $h = s_{i+1}$, gives

$$\begin{aligned} \sum_{r=i}^m q_r \phi(t_r) &\leq q_i \phi(t_i) + V(t_{i+1}, s_{i+1}) \\ &\leq \min(s_i, s_{i+1}) \phi(t_i) + V(t_i + s_{i+1}, s_{i+1}) \\ &\leq V(t_i, s_i). \end{aligned}$$

Backward induction therefore yields

$$\sum_{i=1}^m q_i \phi(t_i) \leq V(t_1, s_1) = R(0, s_1) \leq R(0, d).$$

Exhausting the countable source by finite subsets proves (9) by positivity. If a differentiable kernel has the required shape only on $[d, \infty)$, extend it below d by its tangent at d . This extension is continuous, nonnegative, decreasing and convex; all sampled values in (9) remain unchanged. $\square$

For strictness we will use a more specialized elementary observation. If $\phi(t) = e^{-ut}$, $u > 0$, then for $t > 0$

$$V(t, s) = e^{-ut} \frac{s}{1 - e^{-us}}, \quad R(t, s) = e^{-ut} \frac{s}{e^{us} - 1}. \quad (11)$$

The first is strictly increasing and the second strictly decreasing in $s > 0$. Indeed, their derivative numerators are $1 - (1 + us)e^{-us} > 0$ and $e^{us} - 1 - use^{us} < 0$, respectively; these signs follow by differentiating from zero. The same strictness holds for integrals against any positive Laplace density nonzero on a set of positive measure. At $t = 0$ only R is needed and remains finite for the kernels used here.

4 The periodic kernel and the two balance identities

Fix $L \geq 2$. For $t > 0$, define

$$\begin{aligned} w(s) &= \tanh(Ls/2), \quad k(t) = \int_0^\infty w(s) e^{-st} ds, \quad U(t) = -k'(t), \\ C(t) &= k(t) - k(t+1), \quad A(t) = C(t) - U(t+1), \quad B(t) = U(t) - C(t). \end{aligned} \quad (12)$$

Here C is the change in potential over a unit step, while A and B compare this change with the force at the two endpoints. All differentiations at positive t are justified by exponential domination. Since $0 < w(s) \leq Ls/2$,

$$k(t) \leq L/(2t^2), \quad U(t) \leq L/t^3. \quad (13)$$

For $0 < t < L$, geometric summation and Tonelli give

$$\sum_{j \in \mathbb{Z}} k(|t + jL|) = \int_0^\infty \frac{e^{-st} + e^{-s(L-t)}}{1 + e^{-Ls}} ds = \frac{\pi}{L} \csc \frac{\pi t}{L} = h_L(t). \quad (14)$$

Substituting $v = e^{-Ls}$ and inverting the variable in the second term reduces the integral to

$$\frac{1}{L} \int_0^\infty \frac{v^{t/L-1}}{1+v} dv.$$

Appendix A evaluates this integral and completes the proof of (14).

The nonzero images of a body at its own target contribute

$$S := 2 \sum_{j \geq 1} k(jL) = 2 \int_0^\infty \frac{ds}{e^{Ls} + 1} = \frac{2 \log 2}{L}. \quad (15)$$

Consequently the full lifted radial row at atom i , excluding only that particular lifted atom, is

$$\sum_{y \in X \setminus \{x_i\}} q_y k(|y - x_i|) = c/\rho + Sq_i. \quad (16)$$

The extra term Sq_i records the images of the target body and must therefore be kept when two rows are subtracted. Opposite self-images cancel in the derivative row. Equation (13), finiteness per period, and bounded masses justify absolute summation and differentiation of the image series away from their singularities, so (7) gives exactly zero lifted derivative row as well.

Define left and right exterior sums on this one actual source by

$$\mathcal{L}_K = \sum_{x < 0} q_x K(-x), \quad \mathcal{R}_K = \sum_{x > 1} q_x K(x - 1).$$

The left distances satisfy $t \geq 1$ and $q \leq t$; the right distances satisfy $t \geq b$ and $q \leq t$, by (6). Both inherit pairwise exclusion. Let $D = k(1) + S$, distinct from the earlier density ρ . Subtracting (16) at 0 and 1 gives

$$(b - 1)k(1) + \mathcal{L}_C - \mathcal{R}_C = S(1 - b), \quad \mathcal{L}_C - \mathcal{R}_C = (1 - b)D. \quad (17)$$

At 0 the direct right force is $bU(1)$, and at 1 the direct left force is $U(1)$. The two torque balances therefore give

$$\mathcal{L}_U - \mathcal{R}_{U(\cdot+1)} = bU(1), \quad \mathcal{R}_U - \mathcal{L}_{U(\cdot+1)} = U(1). \quad (18)$$

The kernels A and B were chosen to combine these radial and tangential balances. Indeed, their definitions give

$$\begin{aligned} \mathcal{L}_A + \mathcal{R}_B &= (\mathcal{R}_U - \mathcal{L}_{U(\cdot+1)}) + (\mathcal{L}_C - \mathcal{R}_C), \\ \mathcal{L}_B + \mathcal{R}_A &= (\mathcal{L}_U - \mathcal{R}_{U(\cdot+1)}) - (\mathcal{L}_C - \mathcal{R}_C). \end{aligned}$$

The first parentheses in each line are supplied by (18), and the remaining difference by (17). We obtain

$$\mathcal{L}_A + \mathcal{R}_B = U(1) + (1 - b)D, \quad \mathcal{L}_B + \mathcal{R}_A = bU(1) - (1 - b)D. \quad (19)$$

These identities couple the two exterior distributions. The positive kernels A and B will allow their contributions to be bounded by weighted lattices.

5 A strict lower bound for the neighboring mass

Before using the sharper mixed estimate, we first keep the neighboring mass away from zero. The two positive balance identities already provide such a bound. Write

$$p_A(s) = 1 - (1 + s)e^{-s}, \quad p_B(s) = s - 1 + e^{-s}.$$

Then A and B have Laplace densities wp_A and wp_B . Both densities are positive for $s > 0$, since $p'_A(s) = se^{-s} > 0$, $p'_B(s) = 1 - e^{-s} > 0$, and both vanish at zero. Further,

$$p_B - p_A = s - 2 + (s + 2)e^{-s}, \quad (p_B - p_A)' = p_A > 0,$$

so $B > A > 0$. The ratio $r = p_A/p_B$ strictly decreases because

$$p'_A p_B - p_A p'_B = e^{-s}[s^2 - 4 \sinh^2(s/2)] < 0. \quad (20)$$

The last inequality follows from $2 \sinh(s/2) > s$, by differentiating. Also $v(s) = w(s)/s$ strictly decreases: for $z > 0$, $\tanh z - z \operatorname{sech}^2 z > 0$, since its derivative is $2z \operatorname{sech}^2 z \tanh z > 0$ and its limit at zero is zero.

If r, v are decreasing and ν is a probability measure, then

$$\int r v d\nu - \int r d\nu \int v d\nu = \frac{1}{2} \iint (r(s) - r(t))(v(s) - v(t)) d\nu(s) d\nu(t) \geq 0. \quad (21)$$

Apply this to the normalized finite measure with density $sp_B(s)e^{-ts} ds$. Dividing by its positive integrals gives

$$\frac{A(t)}{B(t)} \geq \frac{\int_0^\infty sp_A(s)e^{-ts} ds}{\int_0^\infty sp_B(s)e^{-ts} ds} = \frac{t(3t+1)}{(t+1)(3t+2)} =: c(t). \quad (22)$$

The numerator integral is $t^{-2} - (t+1)^{-2} - 2(t+1)^{-3}$, and the denominator is $2t^{-3} - t^{-2} + (t+1)^{-2}$, which verify the displayed ratio directly. Moreover

$$c'(t) = \frac{2(6t^2 + 6t + 1)}{(t+1)^2(3t+2)^2} > 0, \quad 0 < c(t) < 1.$$

On the right, $A \geq c(b)B$. On the nonempty left, $B > A > c(b)A$. The left is nonempty by periodicity, or directly by the first balance in (18). Therefore (19) gives the strict inequality

$$bU(1) - (1-b)D > c(b)[U(1) + (1-b)D]. \quad (23)$$

Since $w = 1 - 2/(e^{Ls} + 1)$,

$$D = 1 + 2 \int_0^\infty \frac{1 - e^{-s}}{e^{Ls} + 1} ds > 1, \quad 0 < U(1) < 1. \quad (24)$$

Dividing (23) by $U(1)$, and weakening $D/U(1) > 1$ to 1 with its nonnegative coefficient $1 - b$, gives

$$2b - 1 > c(b)(2 - b). \quad (25)$$

For $b \leq 1/2$ this is impossible. Its numerator after multiplying by the positive denominator $(b+1)(3b+2)$ is

$$P(b) = 9b^3 + 2b^2 - 3b - 2.$$

On $[1/2, 7/10]$, $P'(b) = 27b^2 + 4b - 3 > 0$, but $P(7/10) = -33/1000 < 0$. Hence

$$b > 7/10. \quad (26)$$

6 A mixed-kernel bound

The preliminary estimate leaves $7/10 < b \leq 1$. To exclude its open part, we subtract one quarter of the second balance in (19) from the first. Before making that subtraction, we must check that the resulting kernels satisfy Lemma 1. The coefficient $1/4$ is chosen for two purposes: the left kernel has the required shape on $t \geq 1$, and the right kernel has a positive Laplace density. The endpoint $7/10$ and the eight-term truncation in Section 7 will then give a strict scalar bound. Only these properties of the constants are used.

The resulting kernels are

$$f_L = A - \frac{1}{4}B, \quad g_L = B - \frac{1}{4}A. \quad (27)$$

The density of g_L is $w(p_B - p_A/4) > 0$; consequently g_L is positive, strictly decreasing, and strictly convex for all $t > 0$.

To handle f_L , introduce only the explicit comparison kernels obtained by replacing w by 1:

$$\begin{aligned} A_\infty(t) &= \frac{1}{t(t+1)^2}, & B_\infty(t) &= \frac{1}{t^2(t+1)}, \\ f(t) &= \frac{3t-1}{4t^2(t+1)^2}, & g(t) &= \frac{3t+4}{4t^2(t+1)^2}. \end{aligned} \quad (28)$$

These follow by integrating $p_A e^{-st}$, $p_B e^{-st}$. This comparison replaces only the kernel density; the configuration remains fixed. Direct differentiation yields

$$\begin{aligned} f'(t) &= -\frac{9t^2 - t - 2}{4t^3(t+1)^3}, & f''(t) &= \frac{18t^3 + 2t^2 - 7t - 3}{2t^4(t+1)^4}, \\ -f'''(t) &= \frac{3(30t^4 + 10t^3 - 15t^2 - 15t - 4)}{2t^5(t+1)^5}. \end{aligned} \quad (29)$$

At $t = 1 + u$, $u \geq 0$, the three sign-controlling numerators are respectively

$$6 + 17u + 9u^2, \quad 10 + 51u + 56u^2 + 18u^3, \quad 6 + 105u + 195u^2 + 130u^3 + 30u^4.$$

Thus $-f', f'', -f''' > 0$ for $t \geq 1$. The ratio p_A/p_B decreases from 1 to 0, by (20) and its elementary endpoint limits. Hence $p_f = p_A - p_B/4$ changes sign exactly once, from positive to negative, at some $s_* > 0$. For $j = 0, 1, 2$,

$$\int_0^\infty s^{j+1} p_f(s) e^{-ts} ds = (-1)^{j+1} f^{(j+1)}(t) > 0 \quad (t \geq 1).$$

To pass from these comparison integrals to the actual kernel, we use the following elementary observation.

Lemma 2 (One sign change). *Suppose ψ is nonnegative on $(0, s_*)$ and nonpositive on (s_*, ∞) , and v is positive and nonincreasing. If ψ and $v\psi$ are absolutely integrable and $\int_0^\infty \psi(s) ds > 0$, then*

$$\int_0^\infty v(s)\psi(s) ds \geq v(s_*) \int_0^\infty \psi(s) ds > 0.$$

Proof. On each side of s_* , the two factors in $(v(s) - v(s_*))\psi(s)$ have the same sign. Integrating this nonnegative product proves the inequality. $\square$

Apply the lemma with $\psi(s) = s^{j+1}p_f(s)e^{-ts}$ and $v(s) = w(s)/s$, whose decrease was proved in Section 5. The exponential factor gives absolute integrability at infinity; near zero, $p_f(s) = O(s^2)$ and $w(s)/s$ is bounded. The preceding integral calculation supplies $\int \psi > 0$. Hence, for $j = 0, 1, 2$,

$$(-1)^j f_L^{(j)}(t) = \int_0^\infty s^j w(s) p_f(s) e^{-ts} ds > 0 \quad (t \geq 1).$$

These three signs say exactly that f_L is positive, strictly decreasing and strictly convex on the domain of the left exterior distances.

The kernels have summable tails. For example $p_A(s), p_B(s) = O(s^2)$ at zero, while $w(s) = O(s)$; splitting the integral at $s = 1$ shows $A, B, f_L, g_L = O(t^{-4})$ at infinity. Alternatively, for their positive combinations A, B , the bound $w \leq 1$ and (28) already gives $O(t^{-3})$. Either bound suffices.

Apply Lemma 1, with tangent extension below 1 for f_L and below b for g_L , to the actual left and right sources. This gives

$$\mathcal{L}_{f_L} \leq \sum_{j \geq 1} f_L(j), \quad \mathcal{R}_{g_L} \leq b \sum_{j \geq 1} g_L(jb). \quad (30)$$

Subtract one quarter of the second equality in (19) from the first. Together with (30), it implies

$$F_L(b) := \sum_{j \geq 1} f_L(j) + b \sum_{j \geq 1} g_L(jb) - (1 - b/4)U(1) - \frac{5}{4}(1 - b)D \geq 0. \quad (31)$$

Because $f_L + g_L = \frac{3}{4}[U(t) - U(t + 1)]$, telescoping gives

$$F_L(1) = 0 \quad (32)$$

for every real $L > 0$. The identity holds before any restriction on the period has been deduced.

7 The neighboring masses are equal

Section 6 gives $F_L(b) \geq 0$ for a configuration satisfying the force equations. We will prove the opposite strict inequality whenever $7/10 \leq b < 1$. There are two estimates to establish:

$$F_L(b) \leq F(b) + \frac{1 - b}{18}, \quad F(b) < -\frac{1 - b}{18}.$$

The first controls the difference between the periodic kernel and an explicit comparison kernel. For the second, convexity will reduce the entire interval to a calculation at $b = 7/10$, because both functions vanish at $b = 1$.

The convex comparison. Define

$$F(b) = \sum_{j \geq 1} f(j) + b \sum_{j \geq 1} g(jb) - \frac{9}{4} + \frac{3}{2}b. \quad (33)$$

Since $A_\infty + B_\infty = t^{-2} - (t + 1)^{-2}$, $F(1) = 0$. Each summand $h_j(b) = bg(jb)$ is convex, since

$$h_j''(b) = \frac{9b^3j^3 + 24b^2j^2 + 16bj + 4}{2b^3j^2(bj + 1)^4} > 0. \quad (34)$$

The series is locally uniformly convergent on $b > 0$, with $O(j^{-3})$ tails on every compact positive b interval. Summing convexity inequalities proves convexity of F ; the corresponding derivative series also converge locally uniformly, directly from the rational formulas.

The finite-period correction. We next compare the two functions, keeping the self-image term. Let

$$\delta_U = 1 - U(1) = 2 \int_0^\infty \frac{se^{-s}}{e^{Ls} + 1} ds, \quad \delta_D = D - 1 = 2 \int_0^\infty \frac{1 - e^{-s}}{e^{Ls} + 1} ds. \quad (35)$$

Both are positive, and $\delta_U \leq 2/(L+1)^2$. Set $p_g = p_B - p_A/4 > 0$. The difference between the right comparison sum and the right finite-period sum is

$$H(b) = b \sum_{j \geq 1} [g(jb) - g_L(jb)] = 2 \int_0^\infty \frac{p_g(s)}{e^{Ls} + 1} \frac{b}{e^{bs} - 1} ds. \quad (36)$$

Tonelli justifies the identity. For each $s > 0$, the factor $b/(e^{bs} - 1)$ strictly decreases in b : its derivative numerator is $e^{bs} - 1 - bse^{bs} < 0$. Thus $H'(b) \leq 0$. Differentiation is locally dominated: $p_g = O(s^2)$ near zero, the factor is $O(s^{-1})$ and its b -derivative is $O(1)$ there, uniformly on compact positive b intervals, and exponential decay controls infinity.

Define the absolutely convergent, b -independent sum

$$J_L = \sum_{j \geq 1} [f_L(j) - f(j)].$$

Subtracting (33) from (31), with (35)–(36), gives the exact identity

$$F_L(b) - F(b) = J_L - H(b) + (1 - b/4)\delta_U - \frac{5}{4}(1 - b)\delta_D.$$

The established differentiability of F and H therefore also gives that of F_L , and

$$F'_L(b) - F'(b) = -H'(b) - \frac{1}{4}\delta_U + \frac{5}{4}\delta_D \geq -\frac{1}{4}\delta_U.$$

Set $D_L(b) = F_L(b) - F(b)$. Since $D_L(1) = 0$, the derivative bound gives

$$D_L(b) = - \int_b^1 D'_L(u) du \leq \frac{1 - b}{4} \delta_U.$$

Thus the lower bound on the derivative gives the required upper bound on the function. Using $L \geq 2$, we obtain

$$F_L(b) \leq F(b) + \frac{1}{4}(1 - b)\delta_U \leq F(b) + \frac{1 - b}{2(L+1)^2} \leq F(b) + \frac{1 - b}{18} \quad (0 < b \leq 1, L \geq 2). \quad (37)$$

One endpoint controls the interval. It remains to prove the strict bound for F . Set $\beta = 7/10$. For $t \geq 1$, $f(t) \leq A_\infty(t) \leq t^{-3}$; for every $t > 0$, $g(t) \leq B_\infty(t) \leq t^{-3}$. Since

$$\sum_{j > 8} j^{-3} \leq \int_8^\infty x^{-3} dx = 1/128,$$

we have

$$F(\beta) \leq E := \sum_{j=1}^8 [f(j) + \beta g(j\beta)] + \frac{1 + \beta^{-2}}{128} - \frac{9}{4} + \frac{3}{2}\beta = -\frac{1794985965991645696553}{84600393758389463834880}. \quad (38)$$

In particular,

$$E + \frac{1 - \beta}{18} = -\frac{76995880670364259861}{16920078751677892766976} < 0. \quad (39)$$

Both values follow by substituting the rational functions (28) into the eight-term sum in (38). Thus (39) is an exact rational inequality.

For $\beta \leq b < 1$, set $\eta = (1 - b)/(1 - \beta) > 0$. Then $b = \eta\beta + (1 - \eta)1$. Convexity and $F(1) = 0$ give

$$F(b) \leq \eta F(\beta) < -\eta \frac{1 - \beta}{18} = -\frac{1 - b}{18}. \quad (40)$$

Equations (37), (40) imply $F_L(b) < 0$, contradicting (31). In view of (26), the only possibility is

$$b = 1. \quad (41)$$

8 Equality and regularity

At $b = 1$, the first equality of (19) is

$$\mathcal{L}_A + \mathcal{R}_B = U(1). \quad (42)$$

Both sources now have initial distance at least 1. Lemma 1 applied to the positive Laplace kernels A, B , with tangent extensions if needed, gives

$$\mathcal{L}_A \leq \sum_{j \geq 1} A(j), \quad \mathcal{R}_B \leq \sum_{j \geq 1} B(j).$$

Their bounds sum to $U(1)$, since $A + B = U(t) - U(t + 1)$. Therefore both inequalities are equalities.

To identify the equality case, we write the loss in quadrature as a sum of nonnegative local deficits. On either half-line, list its distances increasingly as $t_1 < t_2 < \dots$, and let K be its kernel, A or B . The list exists because the base source is finite and periodic. Define $s_1 = t_1 \geq 1$, $s_i = t_i - t_{i-1}$ for $i \geq 2$, and let V_K, R_K be the sums of Lemma 1 formed from the original Laplace kernel K . All sampled arguments in the deficit calculation below are at least 1, so the tangent extension used to apply the lemma changes none of these values. All s_i are bounded above and bounded below away from zero. Exclusion gives

$$q_i \leq \min(s_i, s_{i+1}).$$

For each $i \geq 1$, define nonnegative transition and mass deficits

$$\begin{aligned} T_i &= V_K(t_i, s_i) - \min(s_i, s_{i+1})K(t_i) - V_K(t_{i+1}, s_{i+1}), \\ M_i &= [\min(s_i, s_{i+1}) - q_i]K(t_i). \end{aligned}$$

Equation (8) gives $T_i \geq 0$, and $M_i \geq 0$. The boundary deficit is

$$B_0 = R_K(0, 1) - R_K(0, s_1) \geq 0.$$

For every $m \geq 1$, telescoping gives the exact identity

$$R_K(0, 1) - \sum_{i=1}^m q_i K(t_i) = B_0 + \sum_{i=1}^m (T_i + M_i) + V_K(t_{m+1}, s_{m+1}). \quad (43)$$

The terminal term tends to zero. For example $K(t) = O(t^{-3})$ and $s_i \in [\sigma, H]$ imply

$$V_K(t_i, s_i) \leq HC \sum_{j \geq 0} (t_i + j\sigma)^{-3} \longrightarrow 0.$$

Equality in quadrature makes the left side of (43) tend to zero. Every term on its right is nonnegative. In particular, B_0 is bounded above by that left side for every m , so $B_0 = 0$. For any fixed i , take $m \geq i$; then T_i and M_i are each bounded above by the same quantity. Letting $m \rightarrow \infty$ gives $T_i = M_i = 0$. This applies to each fixed index, and therefore to every gap and mass in the ray.

The positive Laplace densities wp_A, wp_B and (11) imply strict decrease of $R_K(0, s)$. Hence $B_0 = 0$ forces $s_1 = 1$. In the proof of (8), if $s_{i+1} > s_i$, its deficit is $R_K(t_i, s_i) - R_K(t_i, s_{i+1}) > 0$; if $s_{i+1} < s_i$, it is $V_K(t_i, s_i) - V_K(t_i, s_{i+1}) > 0$. Thus $T_i = 0$ forces $s_{i+1} = s_i$. Inductively every gap is 1. Finally $K(t_i) > 0$, so $M_i = 0$ forces $q_i = 1$.

Applied to both sides of (42), the exterior atoms are exactly the negative integers and $2, 3, \dots$, each of mass 1. Together with the anchors 0 and 1, the full lift is exactly the unit-mass integer lattice.

9 Completion of the proof

The anchor image at L is an occupied lattice point, so L is an integer. The half-open interval $[0, L)$ contains exactly L lattice atoms, but it also contains exactly the n distinct physical atoms. Hence $L = n$. The count uses the given period L , whether or not it is primitive.

All $q_i = 1$, so all $p_i = a$, and $\sum_i p_i = 1$ gives $a = 1/n$. Every angular gap equals $\delta = 2\pi/L = 2\pi/n$. Restoring M, R, O proves the theorem's forward implication.

Conversely, at the vertices of a regular n -gon with equal mass $m > 0$, tangential forces cancel in offset pairs $j, n-j$, and the radial row is constant by cyclic symmetry. The center of mass is the circle center, because the sum of the n -th roots of unity is zero. The inward coefficient is

$$\lambda = \frac{Gm}{4R^3} \sum_{j=1}^{n-1} \csc \frac{\pi j}{n} > 0.$$

This proves the converse directly. For $n = 2$, centering says $m_1 u_1 = -m_2 u_2$, whose norms give $m_1 = m_2$ and then $u_2 = -u_1$. The antipodal pair has $\lambda = Gm/(4R^3)$. For $n = 1$, its sole point is its center of mass and cannot lie at positive distance from it.

On a fixed circle, for a fixed total mass and oriented cyclic label order, the configuration is therefore unique modulo rotations of that circle.

A The periodization integral

The only special-function evaluation in (14) is

$$I(\beta) = \int_0^\infty \frac{x^{\beta-1}}{1+x} dx = \frac{\pi}{\sin \pi \beta}, \quad 0 < \beta < 1.$$

We evaluate it by a keyhole contour. Take the branch $z^{\beta-1} = \exp((\beta-1)(\log |z| + i \arg z))$, $0 < \arg z < 2\pi$, with a slit on the positive real axis. Integrate $z^{\beta-1}/(1+z)$ on the positively oriented keyhole contour of radii $\varepsilon < 1 < R$. The upper ray contributes the truncated real integral, and the lower ray, traversed backwards, contributes $-e^{2\pi i \beta}$ times it. The small circle is $O(\varepsilon^\beta)$, the large circle $O(R^{\beta-1})$, both tending to zero. The sole pole is at -1 , with residue $e^{i\pi(\beta-1)}$. Therefore

$$(1 - e^{2\pi i\beta})I(\beta) = 2\pi i e^{i\pi(\beta-1)}.$$

Using $1 - e^{2\pi i\beta} = -2ie^{i\pi\beta} \sin \pi\beta$ gives the stated value. This scalar evaluation completes the derivation of (14).

B A conserved interaction across cuts

The two balance identities in Section 4 arise from adjacent bodies. A conserved quantity also expresses the compatibility of the entire periodic configuration. We record it here to clarify the relation between local balance and the exterior distributions used in the proof.

Enumerate the lifted points increasingly as $(x_j)_{j \in \mathbb{Z}}$, with masses q_j . With $U = -k'$ as in (12), define the interaction across the cut after x_ℓ by

$$\Pi_\ell = \sum_{j \leq \ell < k} q_j q_k U(x_k - x_j).$$

The lift has a minimum gap $\sigma > 0$ and a maximum mass $Q < \infty$. For each index difference $r \geq 1$, exactly r pairs cross the cut, and their distances are at least $r\sigma$. The bound $U(t) \leq L/t^3$ therefore gives

$$0 < \Pi_\ell \leq Q^2 L \sigma^{-3} \sum_{r \geq 1} r^{-2} < \infty.$$

Moving the cut past $x_{\ell+1}$ removes its interactions with points on the left and adds its interactions with points on the right. Absolute convergence justifies this subtraction, yielding

$$\begin{aligned} \Pi_{\ell+1} - \Pi_\ell = q_{\ell+1} & \left(\sum_{k > \ell+1} q_k U(x_k - x_{\ell+1}) \right. \\ & \left. - \sum_{j < \ell+1} q_j U(x_{\ell+1} - x_j) \right) = 0. \end{aligned}$$

The last equality is the lifted tangential balance. Thus Π_ℓ is independent of the cut.

For the selected anchors 0 and 1, define the exterior exponential sums

$$X(s) = \sum_{x < 0} q_x e^{sx}, \quad Y(s) = \sum_{x > 1} q_x e^{-s(x-1)}.$$

Tonelli's theorem factors the interaction across the gap as

$$\Pi = \int_0^\infty s \tanh(Ls/2) e^{-s} (1 + X(s))(b + Y(s)) ds.$$

The separated-point inequality applied to the exponential kernel gives $X(s) \leq (e^s - 1)^{-1}$ and $Y(s) \leq b(e^{bs} - 1)^{-1}$. These bounds and the conserved interaction concern the same two exterior distributions. In particular, balance couples the two sides even when each is bounded separately. The main proof uses the linear identities (19) to obtain the stronger conclusion that both quadrature bounds are saturated.

References

- [1] A. Albouy, H. E. Cabral and A. A. Santos, Some problems on the classical n -body problem, *Celestial Mechanics and Dynamical Astronomy* **113** (2012), 369–375. doi:[10.1007/s10569-012-9431-1](https://doi.org/10.1007/s10569-012-9431-1).
- [2] J. M. Cors, G. R. Hall and G. E. Roberts, Uniqueness results for co-circular central configurations for power-law potentials, *Physica D* **280–281** (2014), 44–47. doi:[10.1016/j.physd.2014.05.003](https://doi.org/10.1016/j.physd.2014.05.003).
- [3] M. Hampton, Splendid isolation: local uniqueness of the centered co-circular relative equilibria in the N -body problem, *Celestial Mechanics and Dynamical Astronomy* **124** (2016), 145–153. doi:[10.1007/s10569-015-9656-x](https://doi.org/10.1007/s10569-015-9656-x).
- [4] Z. Wang, On the centered co-circular central configurations for the n -body problem, *Journal of Differential Equations* **361** (2023), 376–390. doi:[10.1016/j.jde.2023.02.066](https://doi.org/10.1016/j.jde.2023.02.066).
- [5] Z. Tang and S. Zhu, Perturbing masses: A study of centered co-circular central configurations in power-law n -body problems, *Physica D* **461** (2024), 134115. doi:[10.1016/j.physd.2024.134115](https://doi.org/10.1016/j.physd.2024.134115).