

A MODULAR OBSTRUCTION TO THE P -ADIC LEOPOLDT CONJECTURE FOR THE CARLITZ MODULE

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ABSTRACT. Let L be a finite extension of $K = \mathbb{F}_q(\theta)$ and P a monic prime of $\mathbb{F}_q[\theta]$. Anglès asked whether the P -adic Leopoldt defect of Taelman’s module of units of the Carlitz module over L always vanishes, and Lucas conjectured this (together with the equivalence “ $L_P(E/\mathcal{O}_L) \neq 0$ iff $\exp_E$ is injective”) for Anderson modules. We show that both statements fail. For every odd prime p , the extension of $\mathbb{F}_p(\theta)$ generated by a root of $z^p - (-\theta)^{p-2}z - (-\theta)^{p-1}$ is real, has Galois group $\text{AGL}(1, p)$, and has positive P -adic Leopoldt defect and vanishing P -adic zeta value at $s = 1$ for *every* P . The obstruction comes from modular representation theory at the infinite place: we attach to every separable L/K an integer $\delta_{\text{gen}}(L)$, depending only on the Galois group of the Galois closure, its decomposition group at ∞ and the character by which this group acts on the Carlitz period, and prove that the P -adic defect is at least $\delta_{\text{gen}}(L)$ for all P . We prove that the defect vanishes, for every P , for real abelian extensions and for abelian extensions with cyclic Sylow p -subgroup, using Bosser’s P -adic Baker–Brumer theorem, and we show that a v -adic analogue of Papanikolas’s algebraic independence theorem for Carlitz logarithms would imply that the defect equals $\delta_{\text{gen}}(L)$ in general.

1. INTRODUCTION

Let $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q((1/\theta))$, and let C be the Carlitz module. For a finite extension L/K with integral closure $\mathcal{O}_L$ of A , Taelman [Tae10, Tae12] introduced the module of units $U(\mathcal{O}_L) \subset C(\mathcal{O}_L)$ and the class module $H(\mathcal{O}_L)$, and proved the class formula $\zeta_{\mathcal{O}_L}(1) = [H(\mathcal{O}_L)]_A \cdot R_{\infty, L}$, an analogue of the analytic class number formula.

Anglès [Ang25] proved a P -adic analogue for every monic prime P of A and every *real* L (no place of L above ∞ contains the Carlitz period $\tilde{\pi}$):

$$\zeta_{P, \mathcal{O}_L}(1) = [H(\mathcal{O}_L)]_A \frac{[C(\mathcal{O}_L/P\mathcal{O}_L)]_A}{[\mathcal{O}_L/P\mathcal{O}_L]_A} R_{P, L},$$

the analogue of Colmez’s p -adic class number formula. He defined the P -adic Leopoldt defect

$$\delta_P(U(\mathcal{O}_L)) = \text{rank}_A U(\mathcal{O}_L) - \text{rank}_{A_P} \langle U(P\mathcal{O}_L) \rangle_{A_P},$$

observed that for real L one has $\delta_P = 0 \iff R_{P, L} \neq 0 \iff \zeta_{P, \mathcal{O}_L}(1) \neq 0$, and asked whether $\delta_P(U(\mathcal{O}_L)) = 0$ always holds [Ang25, §6.3]. Lucas [Luc26] extended the class formula to Anderson t -modules E , stated the equality of ranks as a conjecture [Luc26, Conj. A], and conjectured that the P -adic L -series $L_P(E/\mathcal{O}_L)$ is non-zero if and only if $\exp_E$ is injective on $\text{Lie}_E(L_\infty)$ [Luc26, §4.7]. Besides $L = K$, the only known case is the P -th cyclotomic function field $L = K(\lambda_P)$ at the prime P itself, proved by Anglès and Taelman [AT15, Thm. 9.8] using Bosser’s P -adic Baker–Brumer theorem for Carlitz logarithms [Bos15]. (Theorem numbers of [Luc26] refer to arXiv:2504.03430v2.)

Main results. Our first result shows that the answer to Anglès’s question is negative.

Theorem 1.1. *Let p be an odd prime, $q = p$, and let $L_p = K(z)$ with*

$$f_p(z) = z^p - (-\theta)^{p-2}z - (-\theta)^{p-1} = 0.$$

Then L_p/K is a real separable extension of degree p , and for every monic prime P of A ,

$$\delta_P(U(\mathcal{O}_{L_p})) \geq 1, \quad R_{P,L_p} = 0, \quad \zeta_{P,\mathcal{O}_{L_p}}(1) = 0.$$

For $p = 3$ this is the real cubic field $K(z)$, $z^3 + \theta z - \theta^2 = 0$, whose Galois closure has group S_3 . In particular, for $E = C$ over $\mathcal{O}_{L_p}$, [Luc26, Conj. A] and the conjecture of [Luc26, §4.7] fail.

We give two independent proofs. The first (§4) uses only Taelman’s lattice theorem and the elementary theory of the P -adic Carlitz logarithm, and shows directly that $\delta_P \geq 1$. The second is analytic: in characteristic p the Goss zeta function of L_p factors as $\zeta_A \cdot \zeta_{F'}$, where $F' = K(\lambda_\theta) = K((- \theta)^{1/(p-1)})$ is the θ -th Carlitz cyclotomic field, and F' is not real, so $\zeta_{P,\mathcal{O}_{F'}}(1) = 0$ by [Ang25, Thm. 6.6].

The phenomenon behind Theorem 1.1 is general. Let L/K be separable with Galois closure M , put $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$, let $D \subset G$ be a decomposition group at ∞ , and, if $\tilde{\pi}$ lies in the completions of M at ∞ , let $\chi : D \rightarrow \mathbb{F}_q^\times$ be the character by which D acts on $\tilde{\pi}$. Put $\varepsilon_\chi = \sum_{d \in D} \chi(d)d^{-1} \in \mathbb{F}_q[G]$ ($\varepsilon_\chi = 0$ if $\tilde{\pi}$ is not in these completions), and let $r_L = \text{rank}_A U(\mathcal{O}_L)$.

Definition 1.2. $r_{\text{gen}}(L) = \max\{\text{rank}(F \mid \overline{\mathbb{F}}_p[G/H]) : F \in \overline{\mathbb{F}}_p[G], \varepsilon_\chi F = 0\}$, where F acts on the permutation module $\overline{\mathbb{F}}_p[G/H]$ by left multiplication, and $\delta_{\text{gen}}(L) = r_L - r_{\text{gen}}(L)$.

Theorem 1.3. *For every finite separable L/K and every monic prime P , $\delta_P(U(\mathcal{O}_L)) \geq \delta_{\text{gen}}(L) \geq 0$. Moreover $\delta_{\text{gen}}(L) = 0$ if M is real or if $p \nmid |D|$, and $\delta_{\text{gen}}(L_p) = 1$.*

Thus a positive defect requires an imaginary Galois closure in which ∞ is wildly ramified or has residue degree divisible by p . There is no number-field analogue. Over $\overline{\mathbb{Q}}$ the archimedean logarithm $\log |\cdot|$ is single-valued and gives the generic rank, and $\mathbb{Q}[G]$ is semisimple. Here the ∞ -adic Carlitz logarithm is defined only modulo $\tilde{\pi}A$, and the Galois module $\ker \exp_C \cong \text{Ind}_D^G \chi$ need not split off in characteristic p . Dually, the Goss zeta function of a real field sees only Brauer characters and can contain an “odd” constituent.

Bosser’s theorem settles the abelian case under a cyclicity hypothesis.

Theorem 1.4. *Let L/K be a finite abelian extension. If L is real, or if the Sylow p -subgroup of $\text{Gal}(L/K)$ is cyclic (for instance if $p \nmid [L : K]$), then $\delta_P(U(\mathcal{O}_L)) = 0$ for every monic prime P .*

This extends [AT15, Thm. 9.8] from $L = K(\lambda_P)$ to all such L and all P .

Conjecture 1.5. *For every finite separable L/K and every monic prime P , $\delta_P(U(\mathcal{O}_L)) = \delta_{\text{gen}}(L)$. In particular the defect does not depend on P .*

Theorem 1.6 (conditional). *Assume Hypothesis (VP_P) of §6: P -adic Carlitz logarithms of algebraic points that are linearly independent over K are algebraically independent over K . Then $\delta_P(U(\mathcal{O}_L)) = \delta_{\text{gen}}(L)$ for every finite separable L/K .*

Hypothesis (VP_P) is the P -adic analogue of Papanikolas’s theorem [Pap08]. It is open. The v -adic linear independence criterion of Chen [Che25] and its extension to all places by Tsui [Tsu26] give the linear part; we describe in Remark 6.1 what is missing. Theorem 1.6 is the function-field form of the classical observation that p -adic Schanuel-type statements imply Leopoldt’s conjecture (cf. [CM09, §3]), with the important difference that its conclusion is $\delta_P = \delta_{\text{gen}}$ and not $\delta_P = 0$. By Theorem 1.1, no transcendence input can give $\delta_P = 0$ in general.

Organisation. §2 collects facts on Taelman units, completions at ∞ and the P -adic logarithm, and quotes the two results of [Ang25] that we use. §3 proves Theorem 1.3 except for the computation $\delta_{\text{gen}}(L_P) = 1$. §4 treats the family L_P and proves Theorem 1.1. §5 proves Theorem 1.4, and §6 proves Theorem 1.6 and discusses (VP_P) .

2. PRELIMINARIES

Throughout, L/K is a finite separable extension of degree n . We fix an algebraic closure of K_∞ with completion $\mathbb{C}_\infty$, and, for a monic prime P of degree d , an algebraic closure of K_P with completion $\mathbb{C}_P$. We write v_P for the valuation of $\mathbb{C}_P$ with $v_P(P) = 1$.

2.1. The Carlitz module. $C_\theta = \theta + \tau$, where τ is the q -power Frobenius. The exponential $\exp_C(x) = \sum_{i \geq 0} x^{q^i} / D_i$, with $D_i = \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j})$, has coefficients in K , is entire on $\mathbb{C}_\infty$, satisfies $\exp_C(ax) = C_a(\exp_C x)$ for $a \in A$, and has kernel $\tilde{\pi}A$ [Gos96, Ch. 3]. Here

$$\tilde{\pi} = (-\theta)^{q/(q-1)} \prod_{i \geq 1} (1 - \theta^{1-q^i})^{-1}$$

for a suitable choice of $(-\theta)^{1/(q-1)}$ (see [ABP04, Cor. 5.1.4] and [Che25, Rem. 3.1.2]). The product lies in $1 + \theta^{-1} \mathbb{F}_q[[\theta^{-1}]]$ and $(-\theta)^{q/(q-1)} = (-\theta) \cdot (-\theta)^{1/(q-1)}$, so $\tilde{\pi}^{q-1} \in K_\infty$ and $K_\infty(\tilde{\pi}) = K_\infty((-\theta)^{1/(q-1)})$. Since $v_\infty(-\theta) = -1$ is prime to $q-1$ and $\mu_{q-1} = \mathbb{F}_q^\times \subset K_\infty$, the extension $K_\infty(\tilde{\pi})/K_\infty$ is Galois and totally ramified of degree $q-1$, and $\sigma \mapsto \sigma(\tilde{\pi})/\tilde{\pi}$ identifies its Galois group with $\mathbb{F}_q^\times$.

2.2. Taelman units. Put $L_\infty = L \otimes_K K_\infty \cong \prod_{v|\infty} L_v$. The map $\exp_C$ acts on L_∞ componentwise. A place $v \mid \infty$ of L is *complex* if some (equivalently, since $K_\infty(\tilde{\pi})/K_\infty$ is Galois, every) K_∞ -embedding $L_v \hookrightarrow \mathbb{C}_\infty$ has image containing $\tilde{\pi}$. It is *real* otherwise, and L is *real* if all its places above ∞ are real; this is the notion of [Ang25, §2]. Let c_L be the number of complex places.

Lemma 2.1. $N_L := \ker(\exp_C|_{L_\infty})$ is a free A -module of rank c_L , equal to $\bigoplus_{v \text{ complex}} A\varpi_v$, where $\varpi_v \in L_v$ is the element mapped to $\tilde{\pi}$ by a fixed embedding $L_v \hookrightarrow \mathbb{C}_\infty$.

Proof. Let $x \in L_v$ and $\iota : L_v \hookrightarrow \mathbb{C}_\infty$. Then $\exp_C(x) = 0$ iff $\iota(x) \in \tilde{\pi}A$. If v is real, this forces $x = 0$. If v is complex, the solutions are $A\varpi_v$. $\square$

Let $\Lambda_L = \{x \in L_\infty : \exp_C(x) \in \mathcal{O}_L\}$. By Taelman [Tae10, Thm. 1] (see also [Ang25, §1]), Λ_L is a discrete and co-compact A -submodule of L_∞ . Hence it is an A -lattice: it is free of rank n , and the natural map $\Lambda_L \otimes_A K_\infty \rightarrow L_\infty$ is an isomorphism. The module of units is $U(\mathcal{O}_L) = \exp_C(\Lambda_L) = \exp_C(L_\infty) \cap \mathcal{O}_L$. Since $N_L \subset \Lambda_L$,

$$(1) \quad U(\mathcal{O}_L) \cong \Lambda_L / N_L, \quad r_L := \text{rank}_A U(\mathcal{O}_L) = n - c_L.$$

Write $U_L := U(\mathcal{O}_L) \otimes_A K$, a K -vector space of dimension r_L . By exactness of $\otimes_A K$, $U_L = (\Lambda_L \otimes_A K) / (N_L \otimes_A K)$.

2.3. The Galois closure. Let M be the Galois closure of L , $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. Fix a K -embedding $\iota : M \hookrightarrow \mathbb{C}_\infty$ and extend it K_∞ -linearly to $\iota : M_\infty \rightarrow \mathbb{C}_\infty$. This map factors through the projection to one factor M_{w_0} ; let $D = \{g \in G : gw_0 = w_0\}$ be the decomposition group. For $g \in G$ let $\iota_g = \iota \circ g$; it factors through $M_{g^{-1}w_0}$. The group G acts on M_∞ and on Λ_M and commutes with $\exp_C$, so it acts on $U(\mathcal{O}_M)$, on U_M and on N_M .

Lemma 2.2. (a) *The map $M_\infty \otimes_{K_\infty} \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty^G$, $x \otimes c \mapsto (c\iota_g(x))_{g \in G}$, is an isomorphism. Similarly, for $a = gH \in G/H$ the map $\rho_a = \iota_g|_{L_\infty}$ is well defined, and $L_\infty \otimes_{K_\infty} \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty^{G/H}$, $x \otimes c \mapsto (c\rho_a(x))_a$, is an isomorphism.*

- (b) *Either every place of M above ∞ is real (we say M is real), or every such place is complex.*
- (c) *Suppose M is not real. Let $\varpi \in M_\infty$ be the element supported on M_{w_0} with $\iota(\varpi) = \tilde{\pi}$. There is a homomorphism $\chi : D \rightarrow \mathbb{F}_q^\times$ with $d(\varpi) = \chi(d)\varpi$ for $d \in D$. The elements $z_h = h^{-1}(\varpi)$, $h \in D \setminus G$, form an A -basis of N_M , and*

$$\iota_g(z_h) = \begin{cases} \chi(gh^{-1})\tilde{\pi} & gh^{-1} \in D, \\ 0 & \text{otherwise.} \end{cases}$$

- (d) *The places of L above ∞ correspond to the orbits of D on G/H . The place belonging to the orbit of gH is complex if and only if M is not real and χ is trivial on $D \cap gHg^{-1}$, which is the stabiliser of gH in D .*

Proof. (a) The ι_g are the $|G| = [M : K]$ distinct K -embeddings $M \rightarrow \mathbb{C}_\infty$, and M/K is separable. The second statement follows in the same way, since ρ_{gH} runs over the n embeddings of L .

(b) For any h , the element h induces an isomorphism $M_{h^{-1}w_0} \cong M_{w_0}$ over K_∞ , and whether a completion contains $\tilde{\pi}$ is invariant under K_∞ -isomorphism.

(c) Let e_{w_0} be the idempotent of M_{w_0} . Then $\varpi^{q-1} = \tilde{\pi}^{q-1}e_{w_0}$ with $\tilde{\pi}^{q-1} \in K_\infty$. An element $d \in D$ fixes e_{w_0} and is K_∞ -linear, so $(d\varpi)^{q-1} = \varpi^{q-1}$. Hence $d\varpi = \zeta\varpi$ with $\zeta \in \mu_{q-1}(M_{w_0}) = \mathbb{F}_q^\times$, and $\chi(d) := \zeta$ is multiplicative. Next, $\exp_C(\varpi)$ is supported on w_0 with $\iota(\exp_C \varpi) = \exp_C(\tilde{\pi}) = 0$, so $\exp_C(\varpi) = 0$ and $\exp_C(z_h) = h^{-1}\exp_C(\varpi) = 0$. The element z_h is supported on the factor $M_{h^{-1}w_0}$, and these factors are distinct for distinct cosets Dh . By Lemma 2.1 applied to M , the z_h form a basis. Finally, $\iota_g(z_h) = \iota(gh^{-1}\varpi)$. This is $\chi(gh^{-1})\tilde{\pi}$ if $gh^{-1} \in D$. Otherwise $gh^{-1}\varpi$ is supported away from w_0 , and the value is 0.

(d) The embeddings ρ_{gH} and $\rho_{g'H}$ define the same place of L iff they differ by a continuous K_∞ -automorphism of $\iota(M_{w_0})$, that is, iff $g'H \in DgH$. The completion of L at this place is identified by g with the closure of gL in M_{w_0} , whose Galois group over itself inside $\text{Gal}(M_{w_0}/K_\infty) = D$ is $D \cap \text{Gal}(M/gL) = D \cap gHg^{-1}$. It contains $\tilde{\pi}$, i.e. ϖ , iff this group fixes ϖ , i.e. iff χ is trivial on it. $\square$

If M is real we put $N_M = 0$ and $\varepsilon_\chi = 0$. Otherwise $\varepsilon_\chi = \sum_{d \in D} \chi(d)d^{-1} \in \mathbb{F}_q[G]$. A direct computation gives

$$(2) \quad d\varepsilon_\chi = \chi(d)\varepsilon_\chi, \quad \varepsilon_\chi d = \chi(d)\varepsilon_\chi \quad (d \in D).$$

2.4. The P -adic logarithm and the defect. Let $\mathcal{D}_P = \{x \in \mathbb{C}_P : v_P(x) > 1/(q^d - 1)\}$. On $\mathcal{D}_P$ the series $\exp_C$ and its inverse $\text{Log} = \log_C = \sum_i x^{q^i}/\ell_i$, with $\ell_i = \prod_{j=1}^i (\theta - \theta^{q^j})$, converge and are mutually inverse isometric bijections of $\mathcal{D}_P$, with $\text{Log}(C_a x) = a \text{Log}(x)$ for $a \in A$ [Bos15, §1]. Put $\mathcal{O}_{L,P} = \mathcal{O}_L \otimes_A A_P = \prod_{\mathfrak{p}|P} \mathcal{O}_{L_{\mathfrak{p}}}$. Since $P\mathcal{O}_{L_{\mathfrak{p}}} \subset \mathcal{D}_P$, the map Log is a bijection $C(P\mathcal{O}_{L,P}) \rightarrow P\mathcal{O}_{L,P}$. By continuity, the A -module structure of $C(P\mathcal{O}_{L,P})$ extends uniquely to an A_P -module structure for which Log is A_P -linear ([Ang25, §6.2]).

Let $U(P\mathcal{O}_L) = U(\mathcal{O}_L) \cap P\mathcal{O}_L$. It has finite index in $U(\mathcal{O}_L)$, since $U(\mathcal{O}_L)/U(P\mathcal{O}_L)$ embeds in the finite module $C(\mathcal{O}_L/P\mathcal{O}_L)$. Anglès [Ang25, §6.3] defines

$$\delta_P(U(\mathcal{O}_L)) = \text{rank}_A U(\mathcal{O}_L) - \text{rank}_{A_P} \langle U(P\mathcal{O}_L) \rangle_{A_P},$$

where $\langle U(P\mathcal{O}_L) \rangle_{A_P}$ is the A_P -submodule of $C(P\mathcal{O}_{L,P})$ generated by $U(P\mathcal{O}_L)$.

Lemma 2.3. *Fix a K -embedding $\iota_P : M \hookrightarrow \mathbb{C}_P$. There is a unique K -linear map $\ell_P : U_M \rightarrow \mathbb{C}_P$ with $\ell_P(u) = \text{Log}(\iota_P u)$ for $u \in U(P\mathcal{O}_M)$. It is injective.*

Proof. The map $u \mapsto \text{Log}(\iota_P u)$ is A -linear on $U(P\mathcal{O}_M)$, and $U(P\mathcal{O}_M) \otimes_A K = U_M$. Hence it extends uniquely. For injectivity: if $\ell_P(\sum a_k u_k) = 0$ with $a_k \in A$ and $u_k \in U(P\mathcal{O}_M)$, then $x = \sum C_{a_k}(u_k) \in P\mathcal{O}_M$ satisfies $\text{Log}(\iota_P x) = 0$. Since $\iota_P x \in \mathcal{D}_P$ and Log is injective there, $x = 0$. $\square$

Lemma 2.4. *Let $u_1, \dots, u_{r_L} \in U(P\mathcal{O}_L)$ be a K -basis of U_L , and $g_1 H, \dots, g_n H$ the cosets of H . Then*

$$\delta_P(U(\mathcal{O}_L)) = r_L - \text{rank}_{\mathbb{C}_P}(\ell_P(g_j u_i))_{i \leq r_L, j \leq n}.$$

Proof. $\langle U(P\mathcal{O}_L) \rangle_{A_P}$ is a finitely generated submodule of the free A_P -module $C(P\mathcal{O}_{L,P}) \cong P\mathcal{O}_{L,P}$. Its rank is the K_P -dimension of the span of $\text{Log}(U(P\mathcal{O}_L))$ in $L \otimes_K K_P$, which is the span of the $\text{Log}(u_i)$. Now $L \otimes_K \mathbb{C}_P \cong \mathbb{C}_P^n$ via the embeddings $\iota_P \circ g_j|_L$. The logarithm has coefficients in K and commutes with these continuous embeddings, so $\text{Log}(u_i)$ maps to the row $(\text{Log}(\iota_P g_j u_i))_j = (\ell_P(g_j u_i))_j$, using $g_j u_i \in U(P\mathcal{O}_M)$. Finally $\text{rank}_A U(\mathcal{O}_L) = r_L$. $\square$

Remark 2.5. For real L , Anglès's regulator $R_{P,L}$ is the determinant of the K_P -linear map $L_P \rightarrow L_P$ sending an A -basis (b_i) of $\mathcal{O}_L$ to the Iwasawa logarithms of an A -basis of $U(\mathcal{O}_L)$ [Ang25, §6.2]. The Iwasawa logarithm is A -linear and agrees with Log on $C(P\mathcal{O}_{L,P})$, so these logarithms span the same K_P -space as $\text{Log}(U(P\mathcal{O}_L))$. Hence $R_{P,L} = 0$ if and only if $\delta_P(U(\mathcal{O}_L)) \geq 1$.

2.5. P -adic zeta values. For a non-zero ideal I of $\mathcal{O}_L$ let $n(I) \in A$ be the monic generator of the Fitting ideal of $\mathcal{O}_L/I$; it is multiplicative in I . Following [Ang25, §4], put

$$Z_{P,\mathcal{O}_L}(z) = \sum_{d \geq 0} U_{P,d} z^d, \quad U_{P,d} = \sum_{\substack{I \text{ prime to } P \\ \deg n(I)=d}} n(I)^{-1} \in K.$$

By [Ang25, Lemma 4.2], $v_P(U_{P,d}) \rightarrow \infty$, and $\zeta_{P,\mathcal{O}_L}(1) := \sum_d U_{P,d} \in K_P$ [Ang25, Def. 4.3]. Unique factorisation of ideals gives, in $K[[z]]$,

$$(3) \quad Z_{P,\mathcal{O}_L}(z) = \prod_{\mathfrak{p} \nmid P} (1 - n(\mathfrak{p})^{-1} z^{\deg n(\mathfrak{p})})^{-1}, \quad n(\mathfrak{p}) = a^{f(\mathfrak{p})} \text{ if } \mathfrak{p} \cap A = aA.$$

We use two results of Anglès.

Fact A ([Ang25, Thm. 6.6]). *If L/K is not real, then $\zeta_{P,\mathcal{O}_L}(1) = 0$. (See also [Luc26, Thm. 4.23].)*

Fact B ([Ang25, Thm. 6.7]). *If L/K is real, then $\zeta_{P,\mathcal{O}_L}(1) = c_{P,L} R_{P,L}$ with $c_{P,L} = [H(\mathcal{O}_L)]_A [C(\mathcal{O}_L/P\mathcal{O}_L)]_A / [\mathcal{O}_L/K^\times]$. (See also [Luc26, Thm. B].)*

Remark 2.6. The isomorphism $C(\mathcal{O}_L/P\mathcal{O}_L) \cong \prod_{\mathfrak{p} \mid P} A/(n(\mathfrak{p})-1)$ stated in [Ang25, §6.2] (and its z -twisted form) is incorrect for ramified P . The correct value of $[\tilde{C}(\mathcal{O}_L/\mathfrak{P}^e)]$ is $n^{e-1}(n - z^{\deg n})$ with $n = n(\mathfrak{P})$. This only changes a non-zero normalising factor, so Fact B holds as stated. Our proof of $\delta_P \geq 1$ in Theorem 1.1 uses neither Fact A nor Fact B.

3. THE FORCED DEFECT

Keep the notation of §2.3. Let $u_1, \dots, u_r$ ($r = r_L$) be a K -basis of U_L and $e_1, \dots, e_s$ a K -basis of U_M . Write $g_j u_i = \sum_k c_{ijk} e_k$ with $c_{ijk} \in K$, and let $\mathcal{N}(X) = (\sum_k c_{ijk} X_k)_{i,j} \in M_{r \times n}(K[X_1, \dots, X_s])$. For a field $\Omega \supset K$ and a K -linear map $\ell : U_M \rightarrow \Omega$, put $\mathcal{N}(\ell) = \mathcal{N}(\ell(e_1), \dots, \ell(e_s)) = (\ell(g_j u_i))_{i,j}$.

Let ρ be the rank of $\mathcal{N}(X)$ over $K(X)$. Then $\text{rank } \mathcal{N}(\ell) \leq \rho$ for every ℓ , since every minor of size $> \rho$ vanishes identically. Equality holds for some $\mathbb{C}_\infty$ -valued ℓ , since a non-zero ρ -minor does not vanish on all of $\mathbb{C}_\infty^s$. The number ρ does not depend on the choice of bases.

Proposition 3.1. *For every K -linear $\ell : U_M \rightarrow \mathbb{C}_\infty$ there is $F \in \mathbb{C}_\infty[G]$ with $\varepsilon_\chi F = 0$ and $\text{rank } \mathcal{N}(\ell) = \text{rank}(F \mid \mathbb{C}_\infty[G/H])$, and every such F arises in this way. Consequently $\rho = r_{\text{gen}}(L)$.*

Proof. Step 1. Since $U_M = (\Lambda_M \otimes_A K)/(N_M \otimes_A K)$ and $\Lambda_M \otimes_A \mathbb{C}_\infty = (\Lambda_M \otimes_A K_\infty) \otimes_{K_\infty} \mathbb{C}_\infty \cong M_\infty \otimes_{K_\infty} \mathbb{C}_\infty$, we get

$$U_M \otimes_K \mathbb{C}_\infty \cong (M_\infty \otimes_{K_\infty} \mathbb{C}_\infty) / \langle z_h \rangle_{\mathbb{C}_\infty}.$$

Thus $\mathbb{C}_\infty$ -valued K -linear maps ℓ on U_M correspond bijectively to $\mathbb{C}_\infty$ -linear forms λ on $M_\infty \otimes \mathbb{C}_\infty$ vanishing on all z_h .

Step 2. By Lemma 2.2(a), every linear form is $\lambda = \sum_g f(g) \iota_g$ for a unique $f : G \rightarrow \mathbb{C}_\infty$. Put $F = \sum_g f(g)g$. By Lemma 2.2(c), $\lambda(z_h) = \tilde{\pi} \sum_{d \in D} \chi(d) f(dh)$. The sum is the coefficient of h in $\varepsilon_\chi F$. So λ kills every z_h iff $\varepsilon_\chi F = 0$. If M is real there is no condition.

Step 3. Choose $\lambda_i \in \Lambda_L \otimes_A K$ lifting u_i , and $\nu_1, \dots, \nu_{c_L}$ a K -basis of $N_L \otimes_A K$. Then $\beta = (\lambda_1, \dots, \lambda_r, \nu_1, \dots, \nu_{c_L})$ is a K -basis of $\Lambda_L \otimes_A K$, hence a K_∞ -basis of L_∞ , and the matrix $\hat{R} = (\rho_a(\beta_i))_{i,a}$ is invertible by Lemma 2.2(a). Since $g_j \lambda_i \in \Lambda_M \otimes K$ lifts $g_j u_i$,

$$\ell(g_j u_i) = \lambda(g_j \lambda_i) = \sum_g f(g) \iota(g g_j \lambda_i) = \sum_{a \in G/H} m_{aj} \rho_a(\lambda_i), \quad m_{aj} = \sum_{g: g g_j H = a} f(g).$$

The matrix $m = (m_{aj})$ is the matrix of left multiplication by F on $\mathbb{C}_\infty[G/H]$ in the basis $([g_j H])_j$, since $F[g_j H] = \sum_g f(g)[g g_j H]$. The rows of $\hat{R}m$ are the rows of $\mathcal{N}(\ell)$, followed by the rows $(\lambda(g_j \nu_k))_j$, which vanish because $g_j N_L \subset N_M$. Hence $\text{rank } \mathcal{N}(\ell) = \text{rank}(\hat{R}m) = \text{rank}(m) = \text{rank}(F \mid \mathbb{C}_\infty[G/H])$.

Step 4. The space $S = \{F : \varepsilon_\chi F = 0\}$ is defined over $\mathbb{F}_q$, and the condition $\text{rank}(F \mid \mathbb{C}_\infty[G/H]) \geq k$ is the non-vanishing of some k -minor, a polynomial with coefficients in $\mathbb{F}_q$ in the coordinates of S . Such a polynomial vanishes on $S(\mathbb{C}_\infty)$ iff it vanishes on $S(\overline{\mathbb{F}}_p)$, because $\overline{\mathbb{F}}_p$ is infinite. Hence $\rho = \max_{F \in S(\mathbb{C}_\infty)} \text{rank} = \max_{F \in S(\overline{\mathbb{F}}_p)} \text{rank} = r_{\text{gen}}(L)$. $\square$

Theorem 3.2. *For every monic prime P , $\delta_P(U(\mathcal{O}_L)) \geq \delta_{\text{gen}}(L) \geq 0$.*

Proof. Choose the u_i in $U(P\mathcal{O}_L)$. By Lemma 2.4, $\delta_P = r_L - \text{rank } \mathcal{N}(\ell_P)$ with ℓ_P as in Lemma 2.3. Hence $\delta_P \geq r_L - \rho = \delta_{\text{gen}}(L)$ by Proposition 3.1. Finally, $\rho \leq r_L$ because $\mathcal{N}$ has r_L rows. $\square$

Proposition 3.3. (a) *If M is real, then $\delta_{\text{gen}}(L) = 0$.*

(b) *If $p \nmid |D|$, then $\delta_{\text{gen}}(L) = 0$.*

Proof. (a) Here L is real, so $r_L = n$, and $F = 1$ has rank n .

(b) We may assume M is not real. Then $e_\chi = |D|^{-1} \varepsilon_\chi$ is an idempotent by (2), and $\varepsilon_\chi(1 - e_\chi) = 0$. The image of e_χ on a D -module V is the χ -eigenspace $\{v : dv = \chi(d)v\}$. For $V = \overline{\mathbb{F}}_p[G/H] = \bigoplus_{\text{orbits}} \overline{\mathbb{F}}_p[D/(D \cap gHg^{-1})]$, the χ -eigenspace of an orbit summand has dimension 1 if $\chi|_{D \cap gHg^{-1}} = 1$ and 0 otherwise. By Lemma 2.2(d) its total dimension is c_L . So $\text{rank}(1 - e_\chi) = n - c_L = r_L$. $\square$

Lemma 3.4. *Suppose M is not real and $D = G$. Then $\varepsilon_\chi F = 0$ iff $\chi(F) := \sum_g f(g)\chi(g) = 0$. If $\overline{\mathbb{F}}_p[G/H]$ has a filtration by G -submodules with a one-dimensional subquotient on which G acts through χ , then $\det(F \mid \overline{\mathbb{F}}_p[G/H]) = 0$ for every such F , and $r_{\text{gen}}(L) \leq n - 1$.*

Proof. By (2), $\varepsilon_\chi F = \chi(F)\varepsilon_\chi$, and $\varepsilon_\chi \neq 0$. The determinant of F is the product of its determinants on the subquotients of the filtration, and on the given one F acts by $\chi(F) = 0$. $\square$

4. THE FAMILY L_p

Let p be an odd prime and $q = p$. Let s be a root of $x^{p-1} + \theta$, put $F' = K(s)$, let y be a root of $y^p - y - s$, put $M = F'(y)$, and set $z = ys^{p-2}$.

Proposition 4.1. (i) $[F' : K] = p - 1$ and $[M : F'] = p$. M/K is Galois, with group $G = \{\tau_{c,b} : s \mapsto cs, y \mapsto cy + b \ (c \in \mathbb{F}_p^\times, b \in \mathbb{F}_p)\}$. Its normal subgroup $N = \{\tau_{1,b}\} \cong \mathbb{F}_p$ and the subgroup $H = \{\tau_{c,0}\} \cong \mathbb{F}_p^\times$ satisfy $G = N \rtimes H$, with $\tau_{c,0}\tau_{1,b}\tau_{c,0}^{-1} = \tau_{1,c^{-1}b}$. Thus $G \cong \text{AGL}(1, p)$.

(ii) z is a root of f_p , $L_p := K(z) = M^H$ has degree p , and M is the Galois closure of L_p .

(iii) The finite primes of A ramified in M are exactly θ . One has $\theta \mathcal{O}_{L_p} = \mathfrak{p}_1 \mathfrak{p}_2^{p-1}$ with $f(\mathfrak{p}_1) = f(\mathfrak{p}_2) = 1$, and θ is totally ramified in F' .

(iv) ∞ has a single place in M , which is totally ramified, so $D = G$. M is not real, and $\chi(\tau_{c,b}) = c$.

(v) L_p is real and F' is not real.

Proof. (i) $x^{p-1} + \theta$ is Eisenstein at θ . Suppose $s = \beta^p - \beta$ with $\beta \in F'$. At the unique place of F' above ∞ we have $v(s) = -1$. If $v(\beta) < 0$, then $v(\beta^p - \beta) = pv(\beta) \neq -1$; if $v(\beta) \geq 0$, then $v(\beta^p - \beta) \geq 0$. This is a contradiction, so the Artin-Schreier polynomial $y^p - y - s$ is irreducible over F' . Next, $(cy + b)^p - (cy + b) = c(y^p - y) = cs$, so each $\tau_{c,b}$ defines an automorphism of M . These give $p(p-1) = [M : K]$ automorphisms, so M/K is Galois with group G . The conjugation formula is a direct computation on y .

(ii) We have $z^p = (y + s)s^{p(p-2)} = z s^{(p-1)(p-2)} + s^{(p-1)^2} = (-\theta)^{p-2}z + (-\theta)^{p-1}$. The element $\tau_{c,b}$ sends z to $(cy + b)c^{p-2}s^{p-2} = z + bc^{-1}s^{p-2}$, using $c^{p-1} = 1$. So the stabiliser of z is H , $[K(z) : K] = p$, and f_p is the minimal polynomial of z . It is separable, since $f'_p = -(-\theta)^{p-2} \neq 0$. The core of H in G is trivial, so M is the Galois closure.

(iii) The discriminant of $x^{p-1} + \theta$ is a unit times a power of θ , so F'/K is unramified at finite $a \neq \theta$ and totally ramified at θ . Since s is integral and $\frac{d}{dy}(y^p - y - s) = -1$, M/F' is unramified at all finite places. Hence the inertia group I_θ at a prime above θ has order $p-1$ and meets N trivially, so it is a complement of N . In $\text{AGL}(1, p)$ every complement is the stabiliser of a point and is self-normalising. Thus the decomposition group D_θ , which normalises I_θ , equals I_θ . As a G -set, G/H is the affine line on which G acts. A point stabiliser has two orbits on it, of sizes 1 and $p-1$. These give the primes of L_p above θ , with residue degrees 1 (because $D_\theta = I_\theta$) and ramification indices 1 and $p-1$.

(iv) Let $v_\infty(\theta) = -1$. The Newton polygon of f_p at ∞ has vertices $(0, -(p-1))$ and $(p, 0)$. The middle point $(1, -(p-2))$ lies strictly above the segment, because $(p-1)^2/p > p-2$. So all roots of f_p have valuation $-(p-1)/p$, and ∞ is totally ramified in L_p , with $e = p$. It is also totally ramified in F' , with $e = p-1$. Hence $e(M/\infty) = p(p-1) = |G|$: there is one place and $D = G$. The completion $F'_\infty = K_\infty(s)$ contains $(-\theta)^{1/(p-1)} = \zeta s$ for some $\zeta \in \mathbb{F}_p^\times$, so it contains $\tilde{\pi}$ (§2.1). Hence M is not real. Finally $\tilde{\pi} = \zeta sw$ with $\zeta, w \in K_\infty$, so $\tau_{c,b}(\tilde{\pi})/\tilde{\pi} = \tau_{c,b}(s)/s = c$.

(v) L_p has one place above ∞ , and its completion has degree p over K_∞ . Since $(p-1) \nmid p$, it cannot contain the degree- $(p-1)$ field $K_\infty(\tilde{\pi})$. On the other hand $F'_\infty \ni \tilde{\pi}$. $\square$

Theorem 4.2. $\delta_{\text{gen}}(L_p) = 1$. Consequently, for every monic prime P , $\delta_P(U(\mathcal{O}_{L_p})) \geq 1$, $R_{P,L_p} = 0$ and $\zeta_{P,\mathcal{O}_{L_p}}(1) = 0$.

Proof. Since L_p is real, $r_{L_p} = p$. Since $D = G$, Lemma 3.4 applies with constraint $\chi(F) = 0$.

The G -set. Identify G/H with $N \cong \mathbb{F}_p$ via $\tau_{1,b}H \mapsto b$. By Proposition 4.1(i), $\tau_{1,b'}$ acts by translation $x \mapsto x + b'$ and $\tau_{c,0}$ acts by $x \mapsto c^{-1}x$.

A filtration. Let $V = \overline{\mathbb{F}}_p[\mathbb{F}_p]$ with basis $[x]$. Let t be translation by 1, put $\mathbf{N} = t - 1$, and $V_k = \mathbf{N}^k V$ for $0 \leq k \leq p$. Since $\mathbf{N}^p = t^p - 1 = 0$ and $V = \overline{\mathbb{F}}_p[t] \cdot [0]$, we have $\dim V_k = p - k$,

and V_k/V_{k+1} is spanned by $\mathbf{N}^k[0]$. Translations act trivially on V_k/V_{k+1} , because $t = 1 + \mathbf{N}$. For $u \in \mathbb{F}_p^\times$ let $\mu_u[x] = [ux]$. Then $\mu_u t \mu_u^{-1} = t^u$ and $\mu_u \mathbf{N} \mu_u^{-1} = (1 + \mathbf{N})^u - 1 = u\mathbf{N} + O(\mathbf{N}^2)$. As $\mu_u[0] = [0]$, we get $\mu_u \mathbf{N}^k[0] \equiv u^k \mathbf{N}^k[0] \pmod{V_{k+1}}$, and each V_k is G -stable. So $\tau_{c,b}$ acts on V_k/V_{k+1} by c^{-k} . For $k = p - 2$ this is $c = \chi(\tau_{c,b})$.

The rank. By Lemma 3.4, $r_{\text{gen}}(L_p) \leq p - 1$. Conversely, $F = 1 - t$ satisfies $\chi(F) = 1 - \chi(t) = 0$, and its kernel on V is the line of translation invariants, so $\text{rank } F = p - 1$. Hence $r_{\text{gen}}(L_p) = p - 1$ and $\delta_{\text{gen}}(L_p) = 1$.

Consequences. Theorem 3.2 gives $\delta_P \geq 1$. Remark 2.5 gives $R_{P,L_p} = 0$, and Fact B then gives $\zeta_{P,\mathcal{O}_{L_p}}(1) = 0$. $\square$

The vanishing of $\zeta_{P,\mathcal{O}_{L_p}}(1)$ also has a direct analytic proof, independent of Taelman units.

Proposition 4.3. *In $K[[z]]$, $Z_{P,\mathcal{O}_{L_p}}(z) = Z_{P,A}(z) Z_{P,\mathcal{O}_{F'}}(z)$ for every monic prime P .*

Proof. We compare the Euler factors (3) prime by prime. Write $X = z^{\deg a}/a$.

For $a \neq \theta$ the prime a is unramified in M , by Proposition 4.1(iii). Let $\phi \in G$ be a Frobenius at a prime above a . The primes of L_p (resp. F') above a correspond to the $\langle \phi \rangle$ -orbits on $G/H \cong \mathbb{F}_p$ (resp. $G/N \cong \mathbb{F}_p^\times$), with residue degree equal to the orbit length. Write $\phi = \tau_{c,b}$.

- If $c = 1$ and $b = 0$: L_p gives $(1 - X)^{-p}$, and F' gives $(1 - X)^{-(p-1)}$.
- If $c = 1$ and $b \neq 0$: ϕ is a p -cycle on $\mathbb{F}_p$, so L_p gives $(1 - X^p)^{-1} = (1 - X)^{-p}$ in characteristic p ; F' gives $(1 - X)^{-(p-1)}$.
- If c has order $m > 1$: ϕ has one fixed point and $(p-1)/m$ cycles of length m , so L_p gives $(1 - X)^{-1}(1 - X^m)^{-(p-1)/m}$; F' gives $(1 - X^m)^{-(p-1)/m}$.

In each case the factor of L_p equals $(1 - X)^{-1}$ times that of F' , and $(1 - X)^{-1}$ is the factor of A .

For $a = \theta$, Proposition 4.1(iii) gives $(1 - X)^{-2}$ for L_p and $(1 - X)^{-1}$ for F' .

Since the factors agree prime by prime, removing the factor at P on both sides preserves the identity. $\square$

Corollary 4.4. $\zeta_{P,\mathcal{O}_{L_p}}(1) = \zeta_{P,A}(1) \cdot \zeta_{P,\mathcal{O}_{F'}}(1) = 0$ for every P .

Proof. Let $\sum a_i z^i$ and $\sum b_j z^j$ be the two factors. By [Ang25, Lemma 4.2], $a_i \rightarrow 0$ and $b_j \rightarrow 0$ in K_P . Then $|a_i b_j|_P \rightarrow 0$ as $i + j \rightarrow \infty$, so the double series $\sum_{i,j} a_i b_j$ converges in the non-archimedean field K_P and may be summed in any order. Summing along $i + j = m$ shows that the coefficients of the product sum to $(\sum a_i)(\sum b_j)$. By Proposition 4.3 the product is $Z_{P,\mathcal{O}_{L_p}}$. Finally, F' is not real, so $\zeta_{P,\mathcal{O}_{F'}}(1) = 0$ by Fact A. $\square$

Remark 4.5. These identities have been checked by exact computation in $\mathbb{F}_p[[P]]$, using the factorisation of f_p and of $x^{p-1} + \theta$ modulo every prime of degree ≤ 6 ($p = 3, 5$) or ≤ 4 ($p = 7$). For $p = 3$ and $P = \theta \pm 1$, the partial sums of $L_P(\chi, 1) = \sum_{a \text{ monic}, P \nmid a} a(0)/a$, where $\zeta_{P,\mathcal{O}_{F'}} = \zeta_{P,A} L_P(\chi, \cdot)$, have P -adic valuation exactly $3^m - (m + 1)$ through degree $m \leq 6$. In particular they tend to 0, while $\zeta_{P,A}(1)$ is a unit.

5. ABELIAN EXTENSIONS

Fact C (Bosser [Bos15, Thm. 1]). *Let $\overline{K} \subset \mathbb{C}_P$ be the algebraic closure of K , and $\alpha_1, \dots, \alpha_m \in \mathcal{D}_P \cap \overline{K}$. If $\text{Log } \alpha_1, \dots, \text{Log } \alpha_m$ are linearly independent over K , then they are linearly independent over $\overline{K}$.*

Proof of Theorem 1.4. Let $G = \text{Gal}(L/K)$, so $M = L$, $H = 1$ and $r = r_L$, and write $\mathbf{C} = \mathbb{C}_P$. Choose a K -basis of U_L in $U(P\mathcal{O}_L)$. By Lemma 2.3, the logarithms ℓ_P of these basis elements

are K -linearly independent. They are logarithms of points of $\mathcal{D}_P \cap \overline{K}$, so by Fact C the $\overline{K}$ -linear extension of ℓ_P to $U_L \otimes_K \overline{K}$ is injective. Let $\tilde{\ell}$ be the $\mathbf{C}$ -linear extension to $U_L \otimes_K \mathbf{C}$. Make $\mathbf{C}[G]$ act on linear forms by $(x \cdot \phi)(u) = \phi(xu)$; since G is abelian this is a module structure. The columns of the matrix of Lemma 2.4 are the forms $g \cdot \tilde{\ell}$, so

$$\text{rank}(\ell_P(gu_i)) = \dim \mathbf{C}[G] \cdot \tilde{\ell} = |G| - \dim \text{Ann}(\tilde{\ell}),$$

where $\text{Ann}(\tilde{\ell}) = \{x : \tilde{\ell}(xU_L) = 0\}$ is an ideal of $\mathbf{C}[G]$.

Cyclicity. By the normal basis theorem $L_\infty \cong K_\infty[G]$. Hence $\Lambda_L \otimes_A K$ and $K[G]$ are $K[G]$ -modules that become isomorphic over K_∞ , and by the Noether–Deuring theorem [CR62, (29.7)] they are isomorphic. So U_L is a quotient of $K[G]$, i.e. a cyclic $K[G]$ -module.

(i) L real. Then $N_L = 0$ and $U_L = K[G]e$ is free, with $e \in U(P\mathcal{O}_L)$. Put $y_g = \ell_P(ge)$. The matrix of Lemma 2.4 for the basis $(he)_h$ is $(y_{gh})_{h,g}$. After replacing h by h^{-1} and transposing, it is the matrix of multiplication by $Y = \sum_g y_g g$ on $\mathbf{C}[G]$ in the basis G , so the two determinants agree up to sign. Write $G = G_p \times G_{p'}$. Then $\mathbf{C}[G] = \bigoplus_\psi e_\psi \mathbf{C}[G]$ over the characters $\psi : G_{p'} \rightarrow \overline{\mathbb{F}}_p^\times$, and $e_\psi \mathbf{C}[G] \cong \mathbf{C}[G_p]$ is a local ring with residue map $\sum x_k k \mapsto \sum x_k \psi(k_{p'})$. Multiplication by an element w of a local ring R with residue field $\mathbf{C}$ has determinant $\bar{w}^{\dim R}$, since $w - \bar{w}$ is nilpotent. Hence

$$\det(y_{gh}) = \pm \prod_\psi \left(\sum_{g \in G} \psi(g_{p'}) y_g \right)^{|G_p|}.$$

Each factor equals $\tilde{\ell}(\sum_g \psi(g_{p'}) ge)$. This is the value of the $\overline{K}$ -linear extension of ℓ_P at a non-zero element of $U_L \otimes \overline{K}$, hence non-zero. So the rank is $|G| = r_L$, and $\delta_P = 0$ by Lemma 2.4.

(ii) $G_p = \langle \gamma \rangle$ cyclic of order p^m . Each block $e_\psi \mathbf{C}[G] \cong \mathbf{C}[x]/(x^{p^m})$ with $x = \gamma - 1$ has only the ideals (x^j) . So every ideal of $\mathbf{C}[G]$, in particular $\text{Ann}(\tilde{\ell})$, is generated by elements $e_\psi x^j$ of $\overline{\mathbb{F}}_p[G]$. For $y \in \overline{\mathbb{F}}_p[G]$ and $u \in U_L$ we have $yu \in U_L \otimes \overline{K}$, and $\tilde{\ell}(yu) = 0$ forces $yu = 0$ by Fact C. Hence $\text{Ann}(\tilde{\ell}) = \text{Ann}(U_L \otimes \mathbf{C})$. Since $U_L \otimes \mathbf{C}$ is cyclic, it is isomorphic to $\mathbf{C}[G]/\text{Ann}(U_L \otimes \mathbf{C})$, and so

$$\text{rank} = \dim \mathbf{C}[G]/\text{Ann}(U_L \otimes \mathbf{C}) = \dim U_L = r_L.$$

Lemma 2.4 gives $\delta_P = 0$. □

Remark 5.1. Only the homogeneous form of Bosser's theorem is used. When G_p is not cyclic, the ideals of $\mathbf{C}[G_p]$ form continuous families, and linear independence over $\overline{K}$ no longer controls $\text{Ann}(\tilde{\ell})$. Indeed, for imaginary abelian L with non-cyclic Sylow p -subgroup contained in D , $\delta_{\text{gen}}(L)$ can be positive: numerically one finds $\delta_{\text{gen}} = 2$ for $G = D = (\mathbb{Z}/3)^2 \times \mathbb{Z}/2$ ($q = 3$) and $\delta_{\text{gen}} = 1$ for $G = D = (\mathbb{Z}/2)^2$ ($q = 2$). We have not constructed explicit fields with these data.

6. A CORRECTED CONJECTURE AND A CONDITIONAL THEOREM

Hypothesis (VP_P). Let $\alpha_1, \dots, \alpha_m \in \mathcal{D}_P \cap \overline{K}$ (with $\overline{K} \subset \mathbb{C}_P$). If $\text{Log } \alpha_1, \dots, \text{Log } \alpha_m$ are linearly independent over K , then they are algebraically independent over K .

Proof of Theorem 1.6. Choose the basis e_k of U_M in $U(P\mathcal{O}_M)$. The values $\ell_P(e_k) = \text{Log}(\iota_P e_k)$ are logarithms of points of $\mathcal{D}_P \cap \overline{K}$, and they are K -linearly independent by Lemma 2.3. By (VP_P) they are algebraically independent over K . Every minor of $\mathcal{N}(X)$ is a polynomial over K , so it vanishes at $X = \ell_P(e)$ only if it vanishes identically. Hence $\text{rank } \mathcal{N}(\ell_P) = \rho = r_{\text{gen}}(L)$, and Lemma 2.4 gives $\delta_P = \delta_{\text{gen}}(L)$. □

By Proposition 3.3, (VP_P) implies $\delta_P(U(\mathcal{O}_L)) = 0$ whenever the Galois closure of L is real or ∞ has decomposition group of order prime to p in it, for abelian and non-abelian L alike. By Theorem 4.2 it cannot imply more.

Remark 6.1 (status of (VP_P)). At ∞ the analogue is Papanikolas’s theorem [Pap08, Thm. 1.2.6]. Its proof combines the ABP criterion [ABP04] with a Tannakian (difference Galois) computation.

At a finite place $v = P$, three inputs are available.

- Chen [Che25] proved a v -adic ABP criterion for systems $\psi^{(-1)} = \Phi\psi$, with $\Phi \in M(\overline{K}[t])$ and ψ v -adically entire with algebraic coefficients, specialised at $\xi \in \overline{K}$. For $\deg v = 1$ he constructed such systems whose specialisations are v -adic Carlitz multiple polylogarithms. His Theorem 1.2.2 shows that the algebra they span is graded by weight and that $\overline{K}$ -linear relations come from K -linear ones.
- Tsui [Tsu26] extended this to all v .
- Mishiba [Mis11] built a v -adic Tannakian theory (*op. cit.*, Thm. 1.1) for *formal* periods in $K^{\text{sep}}((t-\zeta))$. Their specialisation requires his v -adic ABP conjecture (*op. cit.*, Conj. 6.3), which is proved only in rank one, and his formal polylogarithms are not identified with the convergent v -adic ones.

These results give the linear case of (VP_P) , which in the homogeneous form needed here is already Fact C. They do not give (VP_P) itself: the grading reduces $\overline{K}$ -algebraic relations among logarithms to K -linear relations among monomials of one weight, but does not exclude such relations.

What seems to be missing is a transfer of Papanikolas’s §§4–6 to Chen’s systems. For $\deg P = 1$:

- (i) the difference Galois group of the system attached to $(\Omega_\alpha, \log_C u_1, \dots, \log_C u_m)$ over $\overline{K}(t)$, inside the fraction field of the v -adic Tate algebra, whose σ -invariants are $\mathbb{F}_q(t)$;
- (ii) the equality of the transcendence degrees of values and functions, via Chen’s criterion applied to tensor powers;
- (iii) the computation of this group as $\mathbb{G}_m \rtimes \mathbb{G}_a^m$.

The last step needs the $\overline{K}$ -linear independence of 1, Chen’s v -adic period $\tilde{\Omega}$ and the $\tilde{\Omega} \log_C u_i$, which does not seem to follow formally from the grading.

Remark 6.2. Lucas [Luc26, Conj. B], after Caruso–Gazda [CG25], conjectures that $\text{ord}_{z=1} L_P$ is independent of P . Conjecture 1.5 predicts the same for the defect. We have not computed $\text{ord}_{z=1} L_P$ for L_p , and do not claim that it equals $\delta_{\text{gen}}(L_p)$.

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