

PERFECT T-EMBEDDINGS: EXISTENCE

ABSTRACT. This note gives an affirmative answer to an open question posed in [CLR21]. Namely, it demonstrates the existence of perfect t-embeddings for each sufficiently non-degenerate planar bipartite graph G with positive edge weights. The non-degeneracy condition on G is formulated in terms of its perfect and almost perfect matchings, see (4) and (5), and does not concern the choice of edge weights.

The proof and the main part of the writing were produced by GPT-6 Astra in response to a series of prompts. We manually added Section 1.1 to put this result into a bigger context and slightly edited the introduction in order to improve the readability. We plan to provide a better human-digested version of this result later.

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1. INTRODUCTION

1.1. The description of the problem. T-embeddings are geometric embeddings associated with planar bipartite graphs equipped with the dimer model, that is, with positive edge weights determining the dimer partition function. Introduced in [CLR23] and [Ken+22], these embeddings

encode properties of the dimer model in their geometry. Every sufficiently non-degenerate dimer graph admits a continuous family of t-embeddings of its augmented dual [Gal24, BLR26] that are parametrized by suitably defined boundary conditions. A natural question is whether such boundary conditions can be chosen ‘canonically’.

A possible answer to this question was proposed in the preprint [CLR21] where ‘perfect’ boundary conditions were introduced. These boundary conditions ultimately allow one to link the behavior of the dimer model on large graphs with the Lorentzian geometry of the corresponding t-surfaces, the latter being natural lifts of t-embeddings to the Minkowski space $\mathbb{R}^{2,2}$. Variants of this link for s-embeddings of the planar Ising model and Tutte’s embeddings of resistor networks have been discussed in [Che24, Section 2.7] and [Bas+25], respectively.

The relevance of ‘perfect’ boundary conditions for several particular cases was demonstrated in a series of works [CR24, BNR24b, BNR24a, BNR25, KV25], mostly for graphs with the outer face of degree 4, for which the existence result is known. However, it remained unclear whether perfect t-embeddings exist for general dimer graphs with higher degree of the outer face, which put some doubt on the general framework advocated in [CLR21]. This manuscript resolves the existence question in the affirmative: it is proved that perfect t-embeddings exist, though are not unique, for every sufficiently non-degenerate planar bipartite graph equipped with arbitrary positive edge weights. The non-degeneracy assumptions are formulated in terms of almost perfect matchings and have the same spirit as the ones introduced in [KS04, Section 4] and the later works [Gal24] and [BLR26].

1.2. T-embeddings and perfectness. Let G be a finite planar bipartite graph with positive edge weights. Its white and black vertex sets are denoted by W and B . We distinguish a simple outer boundary cycle. The *augmented dual* G^* is obtained from the usual planar dual by replacing its outer dual vertex by a cycle: each incidence with the old outer vertex is retained at a separate vertex of this cycle, in its cyclic order. Thus the new boundary vertices of G^* have degree three. Equivalently, attach one new leaf of the opposite color to each vertex of the outer cycle, put these leaves on the boundary of a disk, and take the disk dual. We denote this leaf-augmented bipartite graph by Γ .

Following [CLR23, Ken+22] we define a *t-embedding* to be a planar realization $\mathcal{T}$ of G^* whose edges are straight segments, whose bounded faces are convex polygons, and whose black face angles at each interior dual vertex sum to π . The white face angles then also sum to π . The edge lengths must be gauge equivalent to the prescribed weights: there are positive functions $g^\circ, g^\bullet$ such that

$$|\mathcal{T}(v') - \mathcal{T}(v)| = g^\circ(w)\chi_{wb}g^\bullet(b)$$

for every dual edge $vv' = (wb)^*$. Let us emphasize that the faces of $\mathcal{T}$ are assumed to have non-empty interior unless they correspond to degree-two interior primal vertices; in the latter case these faces must correspond to nonzero segments.

A convenient way to approach the construction of t-embeddings goes via the notion of Coulomb gauges introduced in [Ken+22]. Let K be a real Kasteleyn matrix, with rows indexed by W and columns by B . Its edge entries are $K_e = \pm\chi_e$, with the signs around a bounded face of length $2m$ having product $(-1)^{m-1}$. The matrix entry $K(w, b)$ is the sum over edges joining w and b ; for a simple graph it is just K_{wb} . This signing makes all perfect-matching terms in $\det K$ have the same sign. A pair of complex functions $\mathcal{F}^\circ$ on W and $\mathcal{F}^\bullet$ on B is a *Coulomb gauge* if

$$\sum_b K(w, b)\mathcal{F}^\bullet(b) = 0 \quad (w \notin \partial G), \quad \sum_w K(w, b)\mathcal{F}^\circ(w) = 0 \quad (b \notin \partial G).$$

Orient each original dual edge $e^* = (wb)^*$ with its black primal endpoint on the right and define (to agree with the boundary increments in (14), we use the following common overall minus sign in the

edge forms):

$$(1) \quad d\mathcal{T}(e^*) = -\mathcal{F}^\bullet(b)K_e\mathcal{F}^\circ(w), \quad d\mathcal{O}(e^*) = -\mathcal{F}^\bullet(b)K_e\overline{\mathcal{F}^\circ(w)}.$$

The two forms are closed and therefore have primitives, defined up to translations. The convention in [CLR21, Equation (4.2)] omits both minus signs; the two conventions differ by a rotation by π in each complex plane. Note that arbitrary Coulomb gauges give only a *t-realization*: the faces of $\mathcal{T}(G^*)$ can be degenerate, have self-overlaps and wrong orientation.

If $\mathcal{T}$ is a genuine t-embedding, we call the primitive $\mathcal{O}$ is its *origami map*. On a white face it is an orientation-preserving Euclidean isometry, and on a black face it is an orientation-reversing Euclidean isometry. Indeed, where the corresponding gauge is nonzero,

$$d\mathcal{O} = \frac{\overline{\mathcal{F}^\circ(w)}}{\mathcal{F}^\circ(w)}d\mathcal{T} \quad \text{on a white face,} \quad d\mathcal{O} = \frac{\mathcal{F}^\bullet(b)}{\overline{\mathcal{F}^\bullet(b)}}d\overline{\mathcal{T}} \quad \text{on a black face.}$$

The map $(\mathcal{T}, \mathcal{O})$ can be view as a piecewise-linear surface in $\mathbb{R}^{2,2} \cong \mathbb{C}^{1,1} = \mathbb{C}^2$ called the *origami lift*. If we equip $\mathbb{C}^{1,1}$ with the real quadratic form

$$q(z, w) = |z|^2 - |w|^2,$$

then every face of this surface will be contained in a totally null affine plane. The group $O(2, 2)$ of Lorentz isometries act naturally on these lifted surfaces; their action on the Coulomb factors is real linear. In particular, such an isometry send t-embedding to another t-embedding as soon as it belongs to the connected component of the identity in $O(2, 2)$ and keeps the boundary of $\mathcal{T}$ to be non self-intersecting.

As originally defined in [CLR21, Definition 2.2], a *perfect t-embedding* is a t-embedding whose boundary polygon is tangential to the unit circle, and every nonboundary edge incident to a boundary vertex lies on the interior angle bisector there. The boundary polygon need not be convex. After the origami map has been rotated and translated appropriately, this condition has the more invariant Lorentz description

$$(2) \quad |\mathcal{T}(v)|^2 - |\mathcal{O}(v)|^2 = 1, \quad \text{Im } \mathcal{O}(v) = 0 \quad (v \in \partial G^*).$$

Writing $\mathcal{W} = \mathcal{O} + i$ gives exactly

$$(3) \quad |\mathcal{T}(v)|^2 - |\mathcal{W}(v)|^2 = 0, \quad \text{Im } \mathcal{W}(v) = 1.$$

Note that these algebraic conditions in fact guarantee the properness of the embedding of one additionally assumes that the boundary has the winding 1 around the origin, the bisector edges are pointing inwards and the polygonal faces of $\mathcal{T}$ have non-empty interior, see [CLR21, Theorem 4.1]. The first two conditions boil down to concrete inequalities discussed in (17) and below.

The existence problem has two parts: finding such proper boundary data, and controlling what their (discrete) holomorphic extensions do in the interior. The main theorem here settles the first part for every positive weighting under the boundary matching hypotheses (4). The second part gives nonoverlapping, possibly collapsed face images under the same hypotheses. It follows by a reduction to the connected matching-covered component containing the outer boundary. For connected matching-covered graphs, generic weightings give nonzero edges; positive area of all faces of degree at least three additionally requires the local matching condition stated below. The results of [Gal24] turned out to be particularly useful for the second part of the theorem.

1.3. The matching hypotheses and the main theorem. Let the outer boundary of G be the simple counterclockwise cycle $w_1, b_1, \dots, w_n, b_n$, with $n \geq 2$, and let $H = G - V(\partial G)$ be its induced interior graph. Our boundary nondegeneracy condition is

$$(4) \quad H \text{ has a perfect matching,} \quad G[V(H) \cup \{w_i, b_j\}] \text{ has a perfect matching for every } i, j.$$

The empty graph is understood to have a perfect matching. This condition involves only unweighted matching feasibility, and does not depend on which positive weights are prescribed.

We call G *matching-covered*, or a *dimer graph*, if it has a perfect matching and every edge occurs in some perfect matching. An *almost perfect matching* of the leaf-augmented graph Γ covers every original vertex of G once and each boundary leaf at most once. When vertices are deleted, the same convention applies to those that remain. For the strict interior conclusion we use the following additional condition:

- (5) For each $v \in V(H)$ with $\deg_G(v) \geq 3$, there are consecutive distinct neighbors u, u' of v such that $\Gamma - \{u, u'\}$ has an almost perfect matching.

Here consecutive refers to the cyclic order of edges at v in the planar drawing. The two neighbors have the same color. Equivalently, one can delete u, u' and a suitable set of original boundary vertices so that the remaining induced subgraph of G has a perfect matching: the vertices covered by leaf edges are precisely that additional boundary set. Thus (5), too, is a condition expressed entirely in matching combinatorics.

We use the following terminology for degeneracies. A realization is *nonstrictly proper* if its face images are convex, possibly collapsed, and have no overlapping two-dimensional interiors. For connected matching-covered graphs, the realizations below are limits of planar embeddings on the one-skeleton. In the full generality of (4), the conclusion concerns the dual edge realization and the polygons surrounding primal vertices; components away from the boundary may collapse to points. In particular, for a disconnected primal graph we do not require its full dual to be a cell decomposition of a disk. It is *strictly proper* if, in addition, every face of degree at least three has positive area and every face of degree two is a nonzero segment. Collinear successive sides of a positive-area convex polygon are allowed. A perfect boundary retains all the strict inward-angle conditions and winds once even when interior faces collapse.

Theorem 1.1. *Assume (4).*

- (1) *For every positive weighting of G , there exist $\alpha, \beta \in \mathbb{C}^n$ such that the Coulomb gauges*

$$\mathcal{F}^\circ(w) = \sum_{i=1}^n \beta_i K^{-1}(b_i, w), \quad \mathcal{F}^\bullet(b) = \sum_{i=1}^n \alpha_i K^{-1}(b, w_i)$$

have algebraically perfect proper boundary data: after normalization, the lifted boundary lies in (3), its inward directions satisfy the strict inequalities (17), and its projected boundary winds once counterclockwise. No genericity assumption on weights or additional interior nondegeneracy condition is required for this assertion.

- (2) *These gauges can be chosen so that the resulting realization of G^* is nonstrictly proper with a perfect boundary, for every positive weighting.*
 (3) *If G is connected and matching-covered and (5) holds, then for generic positive weights the realization is strictly proper. In particular, faces of degree at least three are positive-area convex polygons and faces of degree two are nonzero segments.*

Here generic means outside a proper real algebraic exceptional set in the positive edge-weight space.

Conclusions (1)–(2) hold under exactly (4), with no genericity requirement. Lemma 2.8 explains how the full gauges collapse all edge forms outside the boundary component of allowed edges. Connectedness and the matching-covered assumption are used only for the strict conclusion (3). The extra condition (5) removes faces forced to be flat by the graph: Proposition 2.7 shows its necessity for the stated generic positive-area conclusion.

1.4. Outline of the proof. Section 2 eliminates α and identifies the remaining equation with the critical-point equation of a real functional F_A on an open subset Ω_A of the oriented Grassmannian of two-planes. Positivity of two planes encodes all strict local boundary conditions and winding one. That section also explains the extension to the interior and the role of (5). Section 3 proves

that critical points cannot approach the boundary of Ω_A and that their total index is an integer I_n depending only on n . Section 4 computes

$$I_n \equiv (-1)^{n-1} \pmod{n},$$

using the uniform prism and its cyclic symmetry. This congruence makes the index nonzero and supplies the critical point. No uniqueness or extremum property of F_A is required.

For context, [Ken+22] introduced the Coulomb-gauge construction and treated the quadrilateral-boundary existence problem. Degeneracy qualifications for the general statement of that result are discussed in [Gal24, Remark 1.5] and [Gal26, Example 2.5, footnote 6]. Galashin proves existence of t-embeddings for all positive weights under a surplus condition, and gives a precise correspondence between positive boundary planes and t-immersions [Gal24, Theorem 1.3, Corollary 1.13]. For perfect t-embeddings, his Corollary 9.17 gives existence for *some* weights in specified matching strata; it does not assert existence at every prescribed weighting. The fixed-weight distinction is explicit in his Question 9.19 and Remark 9.24. The new boundary index argument in this paper addresses prescribed positive weights.

2. FROM BOUNDARY GAUGES TO A CRITICAL POINT ON A GRASSMANNIAN

We first express the boundary of a Coulomb-gauge construction using only an $n \times n$ block of the inverse Kasteleyn matrix. We then eliminate one of the two boundary coefficient vectors. The resulting conditions consist of one equation and strict geometric inequalities. In a canonical real vertex gauge, the inequalities are precisely positivity of two oriented two-planes. Finally, we identify the equation as the critical-point equation of a real function on their common domain. This section proves a reduction to *proper boundary data*; the final subsections control the extended gauges and explain both collapsed and strictly proper interior realizations.

2.1. Boundary sources and the inverse Kasteleyn matrix. Let W, B denote the white and black vertex sets. We use a real Kasteleyn matrix K with rows indexed by W , columns indexed by B , and signed edge entries K_e of absolute value equal to the prescribed positive edge weight. Thus the nonzero monomials in $\det K$ have the same sign. Write $Z(J)$ for the weighted perfect-matching partition function of a subgraph J , with $Z(\emptyset) = 1$. The interior graph H has a perfect matching by hypothesis; combining one with either alternating matching of the outer cycle shows that G has a perfect matching. Hence $|\det K| = Z(G) > 0$.

The boundary measurement matrix is

$$(6) \quad A_{ij} = K^{-1}(b_i, w_j), \quad 1 \leq i, j \leq n.$$

Its row index is a boundary black vertex and its column index a boundary white vertex. In particular, transposes in the following formulas are ordinary transposes, not adjoints.

Lemma 2.1 (Boundary minors and a canonical sign gauge). *The matrix A is invertible. After multiplying rows and columns of K by suitable real signs, its boundary block satisfies*

$$(7) \quad A_{ij} > 0 \quad (i \geq j), \quad A_{ij} < 0 \quad (i < j).$$

Every boundary-cycle edge belongs to a perfect matching and satisfies

$$(8) \quad K(w_j, b_i) A_{ij} > 0.$$

Proof. If $I, J \subset \{1, \dots, n\}$ have the same cardinality, Jacobi's complementary-minor identity gives

$$(9) \quad |\det A_{I,J}| = \frac{Z(G - \{b_i : i \in I\} - \{w_j : j \in J\})}{Z(G)}.$$

To justify the absence of cancellation in the numerator, compare two matchings of the deleted graph. Their symmetric difference is a union of alternating simple cycles. All deleted vertices lie on the original outer boundary and outside each surviving cycle, so the Kasteleyn sign comparison is the

same as in the original graph. All determinant monomials of the restricted matrix therefore have one sign. Taking $I = J = \{1, \dots, n\}$ gives $|\det A| = Z(H)/Z(G) > 0$.

Deleting any boundary white vertex and any boundary black vertex leaves two even paths of the outer cycle, allowing empty paths. Their unique matchings, together with a matching of H , show that every entry of A is nonzero. Similarly, each boundary edge extends to an alternating outer-cycle matching and then to a matching of G . Differentiation of the determinant with respect to a boundary edge gives

$$K(w_j, b_i)K^{-1}(b_i, w_j) = \mathbb{P}\{w_j b_i \text{ belongs to the random perfect matching}\} > 0,$$

which proves (8).

For clarity, the full sign pattern can be obtained without choosing weights specially. Normalize the signs of $K(w_i, b_i)$ to be positive and those of $K(w_{i+1}, b_i)$, $i < n$, to be negative. These choices fix a spanning path of the outer cycle. Comparing the two alternating cycle matchings, completed by the same matching of H , forces $K(w_1, b_n) > 0$ as well. The matrix of the bare weighted cycle consequently has positive diagonal, negative subdiagonal, and positive top-right entry. Its inverse is positive on and below the diagonal and negative above it: write that matrix as $P(I - L)$, where P is positive diagonal, L has positive subdiagonal and negative wrap entry, and $L^n = -aI$ with $a > 0$. Then

$$(I - L)^{-1} = \frac{I + L + \dots + L^{n-1}}{1 + a}$$

has exactly the asserted signs. For each cofactor of the original K , choose the matching consisting of a fixed matching of H and the two remaining outer paths. Comparing it with a full outer-cycle matching and the same matching of H gives exactly the bare-cycle cofactor sign. Sign coherence makes this the sign of the entire cofactor. Thus (7) holds. $\square$

An invertible real diagonal change $A \mapsto UAV$ does not change the boundary problem: replace β by $U^{-1}\beta$ and α by $V^{-1}\alpha$. The boundary increments below are unchanged, and the specified inward directions are multiplied by positive factors. We may therefore use (7) whenever positivity is needed.

For coefficient vectors $\alpha, \beta \in \mathbb{C}^n$, define functions on all white and black vertices by

$$(10) \quad \mathcal{F}^\circ(w) = \sum_{i=1}^n \beta_i K^{-1}(b_i, w), \quad \mathcal{F}^\bullet(b) = \sum_{i=1}^n \alpha_i K^{-1}(b, w_i).$$

Matrix multiplication gives their source equations explicitly:

$$(11) \quad \sum_b K(w, b) \mathcal{F}^\bullet(b) = \begin{cases} \alpha_i, & w = w_i, \\ 0, & w \notin \partial G, \end{cases}$$

$$(12) \quad \sum_w K(w, b) \mathcal{F}^\circ(w) = \begin{cases} \beta_i, & b = b_i, \\ 0, & b \notin \partial G. \end{cases}$$

Put

$$(13) \quad c = A^T \beta, \quad d = A \alpha.$$

Then $c_i = \mathcal{F}^\circ(w_i)$ and $d_i = \mathcal{F}^\bullet(b_i)$ are their boundary values. The source equations are precisely the closure equations for the Coulomb-gauge edge forms at the interior vertices.

2.2. The lifted boundary and its properness conditions. For $b, a \in \mathbb{C}$, introduce the null vector

$$Q(b, a) = (ba, \bar{b}a) \in \mathbb{C}^{1,1}, \quad |ba|^2 - |\bar{b}a|^2 = 0.$$

Let $\Theta_k = (z_k, w_k)$ denote the $2n$ boundary vertices of the origami lift, with cyclic indices. Summing the edge forms around a boundary white or black vertex, using (11)–(12), gives

$$(14) \quad \Theta_{2i} - \Theta_{2i-1} = Q(c_i, \alpha_i), \quad \Theta_{2i} - \Theta_{2i+1} = Q(\beta_i, d_i).$$

These increments close: their first coordinates have equal sums $\sum_i c_i \alpha_i = \beta^T A \alpha = \sum_i \beta_i d_i$, and their second coordinates have equal sums $\sum_i \bar{c}_i \alpha_i = \bar{\beta}^T A \alpha = \sum_i \bar{\beta}_i d_i$. Thus the boundary lift is defined up to translation.

The inward dual edge at the even boundary vertex has direction $-K(w_i, b_i)Q(c_i, d_i)$; the one at the following odd boundary vertex has direction $K(w_{i+1}, b_i)Q(c_{i+1}, d_i)$. Equation (8) permits us to discard positive lengths and use the equivalent directions

$$(15) \quad \mathcal{I}_{2i} = -\text{sgn}(A_{ii})Q(c_i, d_i), \quad \mathcal{I}_{2i+1} = \text{sgn}(A_{i,i+1})Q(c_{i+1}, d_i).$$

At the wrap, $i+1=1$ and the indices on A are cyclic. These are inward directions; using outward directions reverses the inequalities below.

The target surface for a normalized perfect boundary is

$$(16) \quad \mathcal{Q} = \{(z, x+i) : x \in \mathbb{R}, |z|^2 - x^2 = 1\}.$$

We call the boundary data *algebraically perfect and proper* if, after an affine Lorentz isometry applied also to directions by its linear part, every Θ_k belongs to $\mathcal{Q}$ and, writing ι_k for the first coordinate of $\mathcal{I}_k$,

$$(17) \quad \text{Im}((z_{k+1} - z_k)\bar{\iota}_k) < 0, \quad \text{Im}((z_{k-1} - z_k)\bar{\iota}_k) > 0.$$

We also require the projected closed polygon to avoid the origin and have winding number one counterclockwise. These conditions concern the boundary polygon and its incident directions. They do not, by themselves, assert injectivity or nondegeneracy of the interior construction.

2.3. Eliminating one coefficient vector. Throughout this subsection indices are cyclic modulo n . For complex numbers define the real alternating form

$$[u, v] = \text{Im}(\bar{u}v).$$

Given β , let $c = A^T \beta$ and define

$$(18) \quad \delta_i = [c_i, \beta_i], \quad \varepsilon_i = [c_{i+1}, \beta_i].$$

When these numbers are nonzero, set

$$(19) \quad D_i = \frac{[\beta_{i-1}, \beta_i]}{\delta_i \varepsilon_{i-1}}, \quad E_i = \frac{[c_i, c_{i+1}]}{\delta_i \varepsilon_i},$$

and

$$(20) \quad X_i = -\frac{\text{Re}(\bar{c}_i \beta_i)}{\delta_i}, \quad Y_i = -\frac{\text{Re}(\bar{c}_{i+1} \beta_i)}{\varepsilon_i}.$$

Proposition 2.2 (One-vector boundary criterion). *Algebraically perfect proper boundary gauges exist if and only if there is a vector β for which the quantities in (18)–(20) are defined and satisfy*

$$(21) \quad D_i > 0, \quad E_i > 0,$$

$$(22) \quad A \text{diag}(D_i) c = \text{diag}(E_i) \beta,$$

$$(23) \quad A_{ii} \delta_i < 0, \quad A_{i,i+1} \varepsilon_i > 0,$$

$$(24) \quad \sum_{i=1}^n (\arctan X_i - \arctan Y_i) = \pi.$$

Here $\arctan$ takes values in $(-\pi/2, \pi/2)$. Given such a vector, take $\alpha_i = D_i c_i$ and use the vertices

$$(25) \quad \Theta_{2i} = \left(-\frac{c_i \beta_i}{\delta_i}, X_i + i \right), \quad \Theta_{2i+1} = \left(-\frac{c_{i+1} \beta_i}{\varepsilon_i}, Y_i + i \right).$$

They already belong to $\mathcal{Q}$, so no additional isometry is needed.

Proof. We first explain why the normalization by an isometry can be absorbed into the gauges when proving necessity. Under the real-linear identification

$$(z, w) = (x + iy, u + iv) \mapsto M(z, w) = \begin{pmatrix} x + u & y + v \\ -y + v & x - u \end{pmatrix},$$

one has $\det M = |z|^2 - |w|^2$ and

$$M(Q(r + is, p + iq)) = 2 \begin{pmatrix} r \\ -s \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix}.$$

The two families of maximal linear subspaces of rank-one matrices fix, respectively, the left and the right factor line. A determinant-preserving linear map either preserves both families or interchanges them. In the first case it has the form $M \mapsto LMR$; in the second it has the form $M \mapsto LM^T R$, with $\det L \det R = 1$. Here is a direct justification of this familiar rank-one fact: the images of the four matrix units fix two left and two right factor lines; the images of the matrix units therefore have the form $\lambda_{ij} u_i v_j^T$, where u_1, u_2 and v_1, v_2 are independent. The image of $(e_1 + e_2)(e_1 + e_2)^T$ has rank one, so $\lambda_{11} \lambda_{22} = \lambda_{12} \lambda_{21}$. Thus the four nonzero scalars factor as $\lambda_{ij} = r_i s_j$. Absorbing them into L, R proves the claim. Transposition handles the family-interchanging case. Consequently a family-preserving Lorentz map acts by independent invertible real-linear transformations on the two arguments of Q . These transformations commute with A , which is real, and therefore act simultaneously on all increments and directions (14)–(15). The isometry $(z, w) \mapsto (z, -\bar{w})$ interchanges the two families, preserves $\mathcal{Q}$, and leaves all first coordinates unchanged; indeed it sends $Q(b, a)$ to $Q(ia, -ib)$. Postcomposing with it if necessary handles the other case. Translations are absorbed into the choice of vertices. Thus it suffices to analyze gauges whose boundary vertices already lie on $\mathcal{Q}$.

Write $d = A\alpha$. Since every second-coordinate increment is real, (14) implies

$$\alpha_i = D_i c_i, \quad d_i = E_i \beta_i$$

for real numbers D_i, E_i . At this stage these letters denote real multipliers; we will derive (19) shortly. Strictness in (17) excludes zero edges, so these multipliers and their factors are nonzero.

Consider a nonzero c_i -edge through $(z, x + i)$, whose direction is $(D_i c_i^2, D_i |c_i|^2)$. Subtract the equations $|z|^2 - x^2 = 1$ at its two endpoints. Since the direction is null, the quadratic terms cancel and give

$$\operatorname{Re} \left(z \frac{\bar{c}_i^2}{|c_i|^2} \right) = x.$$

The same null equation now gives

$$z \frac{\bar{c}_i^2}{|c_i|^2} = x + i\sigma, \quad \sigma \in \{1, -1\}.$$

Thus the first-coordinate line is tangent to the unit circle. At a vertex shared with a β_i -edge the two directions cannot be parallel: otherwise the inward direction would be parallel to both, contradicting (17). Hence $\delta_i \neq 0$. The imaginary sign for the β_i -line is $-\sigma$; if it were the same, the two displayed tangent-line equations would coincide. Solving the two equations yields

$$z = \sigma \frac{c_i \beta_i}{\delta_i}, \quad x = \sigma \frac{\operatorname{Re}(\bar{c}_i \beta_i)}{\delta_i}.$$

Applying the same argument at the odd vertices shows $\varepsilon_i \neq 0$. Propagating along the edges shows that the same σ occurs on every c -line, and $-\sigma$ on every β -line.

Each projected inward vector is now a nonzero real multiple of its vertex. Put $e_k = z_{k+1} - z_k$ and $f_k = \text{Im}(e_k \bar{z}_k)$. Because $\text{Im}(e_{k-1} \bar{z}_k) = f_{k-1}$, the two inequalities at vertex k say that f_k and f_{k-1} have the same nonzero sign. Along a tangent line the argument changes with this sign. Winding number one therefore forces every $f_k > 0$. On the two alternating kinds of oriented edges this reads

$$f_{2i-1} = -\sigma D_i |c_i|^2 > 0, \quad f_{2i} = -\sigma E_i |\beta_i|^2 > 0.$$

The affine isometry $(z, w) \mapsto (z, -w + 2i)$ preserves $\mathcal{Q}$ and all projected data. On gauges it is implemented by $\beta \mapsto i\beta$, $\alpha \mapsto -i\alpha$; it changes the signs of σ, D_i, E_i . We may thus arrange $\sigma = -1$ and $D_i, E_i > 0$. This gives exactly (25).

For the remaining computations, the two elementary determinant identities are

$$(26) \quad \frac{\beta_i}{\delta_i} - \frac{\beta_{i-1}}{\varepsilon_{i-1}} = -D_i c_i, \quad \frac{c_i}{\delta_i} - \frac{c_{i+1}}{\varepsilon_i} = -E_i \beta_i,$$

where D_i, E_i are now the explicit expressions in (19). For example, multiplying the first identity by $\delta_i \varepsilon_{i-1}$ reduces it to

$$\beta_i [c_i, \beta_{i-1}] - \beta_{i-1} [c_i, \beta_i] = -[\beta_{i-1}, \beta_i] c_i.$$

The second follows from the same three-vector determinant identity. Using (25) and (26) gives

$$(27) \quad z_{2i} - z_{2i-1} = D_i c_i^2, \quad X_i - Y_{i-1} = D_i |c_i|^2,$$

$$(28) \quad z_{2i} - z_{2i+1} = E_i \beta_i^2, \quad X_i - Y_i = E_i |\beta_i|^2.$$

Thus the multipliers previously introduced really are (19), and $d = A\alpha$ is exactly (22).

Substitution into (15) gives the projected inward vectors

$$\iota_{2i} = \text{sgn}(A_{ii}) E_i \delta_i z_{2i}, \quad \iota_{2i+1} = -\text{sgn}(A_{i,i+1}) E_i \varepsilon_i z_{2i+1}.$$

As $f_k > 0$, inequalities (17) hold precisely when both displayed real multipliers are negative. This is (23). Finally rotate a c_i -line to $z = x - i$. Along its oriented segment x increases from Y_{i-1} to X_i , contributing $\arctan X_i - \arctan Y_{i-1}$ to the argument. A following β_i -line rotates to $z = x + i$ and is traversed with x decreasing from X_i to Y_i , contributing $\arctan X_i - \arctan Y_i$. Summing gives a total argument change $2 \sum_i (\arctan X_i - \arctan Y_i)$, proving (24).

Conversely, suppose the listed conditions hold. The identity

$$|c_i \beta_i|^2 - \text{Re}(\bar{c}_i \beta_i)^2 = \delta_i^2$$

and its odd counterpart put all vertices (25) on $\mathcal{Q}$. The edge identities and (22) give precisely the prescribed increments with $\alpha_i = D_i c_i$. Positivity gives increasing argument along every edge, and (24) gives winding one. The radial formulas for the inward vectors and (23) give both strict properness inequalities. This proves sufficiency. $\square$

2.4. Proper boundary data and positive two-planes. A rank-two vector $\beta \in \mathbb{C}^n$ represents an oriented real two-plane: take the row span, with its ordered row orientation, of

$$B = \begin{pmatrix} \text{Re } \beta_1 & \cdots & \text{Re } \beta_n \\ \text{Im } \beta_1 & \cdots & \text{Im } \beta_n \end{pmatrix}.$$

Its ordered 2×2 minors are $[\beta_i, \beta_j]$, $i < j$. Two such matrices represent the same oriented plane exactly when they differ by left multiplication by a real 2×2 matrix with positive determinant. Let $\text{Gr}^{\text{or}}(2, n)$ denote this oriented Grassmannian, and let $\text{Gr}_{>0}(2, n)$ be the set where every ordered minor is positive. It has real dimension $2n - 4$. Since A is real, the columns of BA represent $c = A^T \beta$. Define

$$(29) \quad \Omega_A = \{L \in \text{Gr}_{>0}(2, n) : LA \in \text{Gr}_{>0}(2, n)\}.$$

Proposition 2.3 (The admissible domain). *In the canonical sign gauge (7), the strict conditions (21), (23), and winding one are equivalent to membership in Ω_A . Thus the geometry in the one-vector criterion is imposed exactly by positivity of B and BA ; the remaining condition is (22).*

Proof. For a positive plane B , expansion gives

$$\delta_i = \sum_j A_{ji} [\beta_j, \beta_i] < 0.$$

Indeed, for $j < i$ the first factor is negative and the bracket positive; for $j > i$ the first factor is positive and the bracket negative. The same calculation gives $\varepsilon_i < 0$ for $i < n$ and $\varepsilon_n = [c_1, \beta_n] > 0$. The local sign tests therefore hold. If BA is also positive, the numerators and denominators in (19) have matching signs, including at the wrap, so all D_i, E_i are positive.

To keep track of the winding and prove the converse, extend both sequences *antiperiodically*: $\beta_{i+n} = -\beta_i$, $c_{i+n} = -c_i$. The wrap brackets change sign, but D, E , the vertices, and the functional introduced below do not change. All local conditions become

$$(30) \quad [c_i, \beta_i] < 0, \quad [c_{i+1}, \beta_i] < 0, \quad [\beta_i, \beta_{i+1}] > 0, \quad [c_i, c_{i+1}] > 0$$

for every integer i . Choose argument lifts C_i of c_i so that $0 < C_{i+1} - C_i < \pi$. The two mixed inequalities uniquely specify a lift B_i of the argument of β_i with

$$C_{i+1} - \pi < B_i < C_i.$$

The intervals for consecutive indices give $-\pi < B_{i+1} - B_i < \pi$. Since the corresponding bracket is positive, actually $0 < B_{i+1} - B_i < \pi$. If $C_{i+n} = C_i + m\pi$, antiperiodicity makes m odd and positivity makes it positive. The defining intervals force $B_{i+n} = B_i + m\pi$ as well. The reconstructed vertex arguments have consistent lifts

$$\arg z_{2i} = C_i + B_i, \quad \arg z_{2i+1} = C_{i+1} + B_i.$$

Every consecutive increment lies in $(0, \pi)$, and all increments over one circuit sum to $2m\pi$. Since the segments are tangent to the unit circle, these increments equal their actual argument changes. The winding number is therefore m .

If B and BA are positive, their column directions each traverse one half-turn, so $m = 1$. Conversely, winding one forces $m = 1$; every ordered pair of columns is then separated by an angle strictly between zero and π . Thus all ordered minors of B and BA are positive. This proves both implications. $\square$

For later use, these boundary polygons are simple. Along each edge the polar argument increases strictly, and in the admissible case its total increase is 2π . The polygon is therefore a radial graph over the circle: each ray meets it once. This supplies the simplicity needed in the interior extension argument.

2.5. The variational equation. On the open set where the mixed brackets in (18) are nonzero, define

$$(31) \quad F_A(\beta) = \sum_{i=1}^n \log |\delta_i| - \sum_{i=1}^n \log |\varepsilon_i|.$$

It is a smooth real-valued function. A common real-linear map $L : \mathbb{C} \rightarrow \mathbb{C}$ multiplies every bracket by $\det L$. The n numerator and n denominator factors cancel, so F_A is invariant under every invertible such map. In particular it descends to a smooth function on $\Omega_A \subset \text{Gr}^{\text{or}}(2, n)$. Nonzero mixed brackets imply real rank two: if all β_i lay on one real line, all c_i would lie on it too and every mixed bracket would vanish.

Proposition 2.4 (The critical-point reduction). *For any variation $\eta \in \mathbb{C}^n$,*

$$(32) \quad dF_A(\beta)[\eta] = \sum_i [\eta_i, E_i \beta_i - (A \operatorname{diag}(D_i) c)_i].$$

Consequently the critical points of F_A on Ω_A are exactly the vectors satisfying (22), modulo the common orientation-preserving real-linear action. Every such point gives algebraically perfect proper boundary gauges by $\alpha_i = D_i c_i$ and (25).

Proof. The variation of c is $A^T \eta$. Differentiating each logarithm and then reindexing the terms involving c_{i+1} gives

$$\begin{aligned} dF_A[\eta] &= \sum_i \left[(A^T \eta)_i, \frac{\beta_i}{\delta_i} - \frac{\beta_{i-1}}{\varepsilon_{i-1}} \right] + \sum_i \left[\frac{c_i}{\delta_i} - \frac{c_{i+1}}{\varepsilon_i}, \eta_i \right] \\ &= - \sum_i [(A^T \eta)_i, D_i c_i] + \sum_i [\eta_i, E_i \beta_i] \\ &= \sum_i [\eta_i, E_i \beta_i - (A \operatorname{diag}(D_i) c)_i]. \end{aligned}$$

The second line uses (26); the third uses that A is real. Nondegeneracy of $[\cdot, \cdot]$ proves that the derivative vanishes for all η exactly when (22) holds. The quotient projection from rank-two matrices to the oriented Grassmannian is a submersion, and F_A is constant on its fibers. Therefore criticality upstairs is also equivalent to criticality of the descended function. The reconstruction now follows from Propositions 2.2 and 2.3. $\square$

For $n = 2$ this already proves existence. Indeed, the canonical matrix has the form $A = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$ with $a, b, c, d > 0$, so $\det A = ad + bc > 0$. The positive oriented Grassmannian $\operatorname{Gr}_{>0}(2, 2)$ is a single point, and A preserves its orientation. Hence Ω_A is that point, where every smooth function is critical. We may therefore assume $n \geq 3$ in the index argument.

At this point the boundary existence problem has become concrete: prove that F_A has a critical point in the open set Ω_A . The next section constructs an index protected from the boundary of this set and shows that it is independent of the graph and its weights. The uniform-prism calculation will show that this index is nonzero.

2.6. From the boundary solution to an interior realization. We now state precisely which interior conclusions follow from the boundary solution. The input is a critical point in Ω_A , so that the notation of Proposition 2.2 applies:

$$c = A^T \beta, \quad \alpha_i = D_i c_i, \quad d = A\alpha, \quad d_i = E_i \beta_i, \quad D_i, E_i > 0.$$

The extensions (10) solve the Coulomb equations. Their boundary values and sources are nonzero, and their boundary primitive has the null-slice, angle, and winding properties already proved. Interior nonvanishing is a separate question.

An *almost perfect matching* (APM) of the leaf-augmented graph Γ is a matching covering every interior vertex exactly once; boundary leaves may be uncovered. For a set X of vertices, an APM of $\Gamma \setminus X$ is interpreted in the same way, with surviving original vertices still interior. Define

$$(33) \quad k_{\min}(\Gamma) = \min \left\{ \min_{\emptyset \neq B_0 \subset B} (|N_\Gamma(B_0)| - |B_0|), \min_{\emptyset \neq W_0 \subset W} (|N_\Gamma(W_0)| - |W_0|) \right\}.$$

The sets in these minima contain original vertices, which are the interior vertices of Γ ; their neighborhoods may include leaves. This is the surplus of [Gal24, Equation (1.3)].

Lemma 2.5. *If G is connected and every edge belongs to a perfect matching, then $k_{\min}(\Gamma) \geq 1$.*

Proof. Hall's inequality gives $|N_G(B_0)| \geq |B_0|$ for every black set B_0 . If equality holds for a nonempty proper set, every perfect matching pairs B_0 with all of $N_G(B_0)$. Therefore no edge between $N_G(B_0)$ and $\mathbf{B} \setminus B_0$ is allowed. By the hypothesis on edges, no such edge exists; by definition no edge joins B_0 to $\mathbf{W} \setminus N_G(B_0)$. This contradicts connectedness. Thus proper nonempty sets have $|N_G(B_0)| \geq |B_0| + 1$. For $B_0 = \mathbf{B}$, the n white boundary leaves give $|N_\Gamma(B_0)| = |B_0| + n$. The proof for white sets is identical. $\square$

The boundary-plane dictionary. The orientation conventions in the correspondence theorem matter: Galashin labels disk boundary vertices *clockwise*, whereas our original cycle is counterclockwise. Number the leaves by

$$u_{2j-1} = \text{the white leaf attached to } b_i, \quad u_{2j} = \text{the black leaf attached to } w_i, \quad i = n+1-j.$$

Set $t_i = (-1)^{n-i}$ and give both leaf edges of the pair i weight one and Kasteleyn sign t_i . This extends our canonical Kasteleyn signs to the boundary convention in [Gal24, Definition 2.8]. Indeed the boundary face between the two leaves of a pair has sign product $t_i^2 \operatorname{sgn} K(w_i, b_i) = 1$, and the next boundary face has sign product $t_i t_{i-1} \operatorname{sgn} K(w_i, b_{i-1}) = 1$ for $i > 1$. The final face has the prescribed exceptional sign $(-1)^{n-1}$. Extend the gauges to leaves by

$$\mathcal{F}^\circ(u_{2j-1}) = -t_i \beta_i, \quad \mathcal{F}^\bullet(u_{2j}) = -t_i \alpha_i.$$

Now the holomorphicity equations hold at every original vertex.

Identify a complex number with its column $(\operatorname{Re} z, \operatorname{Im} z)^T$. The two boundary matrices in the correspondence are

$$(34) \quad \begin{aligned} \lambda_{2j-1} &= t_i \beta_i, & \lambda_{2j} &= t_i c_i, \\ \tilde{\lambda}_{2j-1} &= -t_i \bar{d}_i, & \tilde{\lambda}_{2j} &= t_i \bar{\alpha}_i, \end{aligned} \quad i = n+1-j.$$

These formulas follow directly from the boundary-restriction convention in [Gal24, Equation (2.9)]: take the alternating sign of that restriction, and reverse the imaginary coordinate of the black holomorphic function as in its Equation (1.24). If $C = \operatorname{Meas}(\Gamma, \chi)$ is the n -plane whose Plücker coordinates are APM partition functions, holomorphic extension gives

$$\lambda \in C, \quad \tilde{\lambda} \in C^\perp.$$

One can check their orthogonality directly: since A is real,

$$\sum_i c_i \otimes \bar{\alpha}_i = \sum_i \beta_i \otimes \bar{d}_i,$$

and this is $\lambda \tilde{\lambda}^T = 0$ after substituting (34).

For completeness we verify the strict signs and the winding conditions. Extend c, β antiperiodically. The signs proved in (30) say

$$[c_i, \beta_i] < 0, \quad [c_i, \beta_{i-1}] < 0.$$

Consequently all consecutive brackets of both matrices in (34) are positive. Choose increasing arguments $\theta_i = \arg \beta_i$ and $\chi_i = \arg c_i$ with $\theta_{i+n} = \theta_i + \pi$ and $\chi_{i+n} = \chi_i + \pi$. The mixed signs permit the simultaneous choices

$$0 < \chi_i - \theta_i < \pi, \quad 0 < \chi_i - \theta_{i-1} < \pi.$$

The successive turning angles of λ in pair i and between pair i and pair $i-1$ are

$$\chi_i - \theta_i, \quad \pi + \theta_{i-1} - \chi_i.$$

Their sum over the n decreasing indices is $(n-1)\pi$. The positive factors D_i, E_i do not change the arguments of $\tilde{\lambda}$; its corresponding turning angles are

$$\pi + \theta_i - \chi_i, \quad \chi_i - \theta_{i-1},$$

which sum to $(n + 1)\pi$. Thus the pair has exactly the positivity and sign-flip conditions of [Gal24, Equation (1.10)], with parameters $(k, N) = (n, 2n)$.

Weak embeddings and generic nonzero edges. A *weak embedding*, in the sequential sense used here, means a realization whose restriction to the one-skeleton is a uniform limit of restrictions of piecewise-linear embeddings of the disk. The continuous-family definition in [Gal26, Definition 3.2] implies this property. In particular, its faces do not have overlapping two-dimensional interiors; flat faces and collapsed edges are allowed. A *weak t-embedding* additionally has the holomorphic, convexity, and angle properties of that paper.

Proposition 2.6 (Interior conclusions). *Assume that G is connected and every edge belongs to a perfect matching, in addition to the boundary matching hypotheses of the main theorem.*

- (1) *For every positive weighting, there is a choice of the proper boundary gauges supplied by the main theorem whose extension is a weak embedding with a perfect boundary. Faces and edges may collapse.*
- (2) *Outside a proper algebraic exceptional set of positive weights, every extension of admissible boundary data has nonzero values at every vertex. In particular, every original and augmented boundary dual edge is a nonzero segment, and the resulting perfect-boundary realization is a weak embedding.*
- (3) *If $k_{\min}(\Gamma) \geq 2$, the realization is a genuine t-embedding with positive-area faces for every positive weighting.*

Proof. Lemma 2.5 gives $k_{\min}(\Gamma) \geq 1$. The explicit dictionary above puts the boundary pair in the domain of the origami-amplituhedron correspondence. Therefore [Gal26, Theorem 2.27] makes its holomorphic extension a weak t-immersion for all weights. Its boundary is the simple tangential polygon already constructed, so it is a weak t-embedding in that paper’s terminology.

For generic positive weights, [Gal24, Corollary 4.26(4)] states that every such holomorphic extension is nonzero at every vertex. The exceptional set can be chosen independently of the boundary planes. More explicitly, the weights admitting admissible boundary data with a zero extended gauge form a semialgebraic set: the positivity conditions are finite determinant inequalities, and the source and critical-point equations become polynomial after clearing their nonzero denominators. Projecting this set to weight space preserves semialgebraicity. The cited uniform genericity says that it has empty interior, so it lies in a proper real algebraic subset. All dual edges are consequently nonzero, and [Gal26, Theorem 3.3(2)] says that the realization is a weak embedding in the topological sense defined above. The nonnegative exterior turns supplied by the weak immersion inequalities, together with the total-turn formula of [Gal26, Proposition 2.23], make each face image convex, possibly flat. Although we set the added leaf weights to one, this entails no restriction on genericity: an arbitrary positive leaf weighting can be normalized to one by vertex gauges at the leaves, without changing weights on G .

To obtain the asserted choice at an arbitrary positive weighting, approximate it by generic positive weightings and choose a boundary critical point for each. The uniform compactness and boundary exclusion proved in the next section give a subsequence of these planes converging to a point of Ω_A , and the critical-point equation persists in the limit. Normalize two independent gauge coordinates and use the explicit boundary vertices (25) to fix the constants of integration. Then the reconstructed gauges and primitives converge, because all their formulas are continuous on this compact set. Each approximating primitive is a weak embedding; choose a piecewise-linear embedding within distance $1/r$ of the r th primitive. Their diagonal sequence converges uniformly to the limiting primitive. Convexity is preserved in this limit. Two positive-area limiting faces cannot have intersecting interiors: a common point away from their boundaries would have winding number one with respect to each approximating face boundary, contradicting disjointness in a sufficiently close planar embedding. Hence it is a weak embedding, and its boundary retains all the strict properness properties. This proves (1).

Under the stronger surplus hypothesis, the correspondence [Gal24, Theorem 1.3] gives a genuine t-immersion, including positive-area convex faces and nonzero edges, directly for every weighting. Its simple boundary makes it a t-embedding by [Gal24, Lemma 6.38]. The null-slice and bisector conditions already established make it perfect. This proves (3). $\square$

The role of [CLR21, Theorem 4.1] can now be stated without hiding an assumption. It characterizes perfect Coulomb realizations by their normalized hyperboloid boundary, inward-angle signs, and winding, under its explicit assumption that *both full gauge functions are nonzero everywhere* and all boundary sources are nonzero. Our boundary calculation verifies the sources and all the boundary conditions. Generic nonvanishing follows from the preceding proposition; at special weights it need not hold. Positive-area interior faces are supplied by the surplus or corner condition, rather than inferred from boundary nonvanishing.

Proposition 2.7 (A local matching criterion for generic strictness). *For each interior vertex v of G of degree at least three, impose*

(35) *some consecutive pair of distinct neighbors u, u' of v satisfies: $\Gamma \setminus \{u, u'\}$ has an APM.*

For generic positive weights, the weak embedding in Proposition 2.6 then has positive area at every dual face of degree at least three; degree-two interior faces are nonzero segments. Thus it is proper in the convention allowing these degree-two segments. Conversely, if (35) fails at an interior vertex of degree at least three, that face is flat for every admissible boundary plane at generic positive weights.

Proof. The neighbors u, u' have the same color. For the appropriate holomorphic gauge, [Gal24, Corollary 4.26(3)] says that the signed determinant at consecutive neighbors is nonnegative, and is strictly positive exactly when the deleted graph has an APM. In geometric terms, multiply this determinant by the two incident Kasteleyn weights and by $|\mathcal{F}(v)|^2$. Up to the fixed choice of orientation, the result is the determinant of the two consecutive oriented sides of the dual face v . At generic weights all gauge values are nonzero. Therefore a pair in (35) gives two noncollinear sides of this weakly convex face, which consequently has positive area. At a boundary primal vertex the two augmented boundary vertices and the adjoining inward edge already give a strict angle, so no extra condition is necessary there. A degree-two interior face has two nonzero opposite sides and is a nonzero segment.

Conversely, if all consecutive deletions fail, all consecutive neighbor determinants vanish. At generic weights those neighbor gauge values are nonzero, so they all lie in a common real line in $\mathbb{C}$. Multiplication by $\mathcal{F}(v)$ and the real Kasteleyn weights puts all sides of the dual face on a common line. Thus the face is flat. $\square$

2.7. Removing components invisible to the boundary gauges. The boundary theorem does not require connectedness or that every edge belong to a perfect matching. The same generality is possible for the conclusion about nonoverlapping face images: the boundary gauges collapse everything outside one matching-covered component. We give the reduction explicitly, rather than assume that the full graph is already nondegenerate.

Call an edge of G *allowed* if it belongs to a perfect matching, and let G^{allow} consist of all allowed edges and all vertices. Under (4), every edge of the outer cycle is allowed: combine a matching of H with either alternating cycle matching. Consequently all boundary vertices lie in one connected component G_0 of G^{allow} . We call G_0 the *boundary core*. It inherits its embedding and weights from G and has the same simple outer boundary cycle. Let $H_0 = G_0 - V(\partial G)$.

Lemma 2.8 (Boundary-core reduction). *Assume (4). Then:*

- (1) *G_0 is connected and matching-covered, and H_0 and every $G_0[V(H_0) \cup \{w_i, b_j\}]$ have perfect matchings.*

(2) The restriction K_0 of K to G_0 is a Kasteleyn matrix for G_0 , and

$$K^{-1}(b_i, w_j) = K_0^{-1}(b_i, w_j) \quad (1 \leq i, j \leq n).$$

(3) For the full Coulomb extensions (10), the restrictions to G_0 are precisely the extensions computed using K_0 . On every edge wb outside G_0 ,

$$\mathcal{F}^\circ(w)\mathcal{F}^\bullet(b) = 0, \quad \overline{\mathcal{F}^\circ(w)}\mathcal{F}^\bullet(b) = 0.$$

Thus both lifted edge forms vanish off the boundary core.

Proof. Fix a perfect matching M of G and label its pairs by (w_s, b_s) , $1 \leq s \leq N$. Form a directed graph on these pairs, with an arrow $s \rightarrow t$ whenever $w_s b_t$ is an edge of G . An edge is allowed if and only if its endpoints belong to the same strongly connected component of this directed graph. For a nonmatching edge, this follows in one direction by taking its alternating cycle in the symmetric difference of M with a matching containing it. In the other direction, a directed cycle containing the edge is an M -alternating cycle, and reversing its matching status produces a perfect matching containing the edge. Matching edges themselves give the loops $s \rightarrow s$.

It follows that the connected components of G^{allow} are exactly these strongly connected components, with each pair contributing one white and one black vertex. In particular, every component is balanced, and every perfect matching restricts to a perfect matching on each component. Each allowed edge remains allowed within its component. This proves the matching-covered assertion for G_0 .

Take a perfect matching of H and complete it by an alternating outer-cycle matching. Every edge of the resulting full matching is allowed, so its restriction to G_0 shows that H_0 is matchable. Likewise, a matching of $G[V(H) \cup \{w_i, b_j\}]$ extends to a full matching by matching the two even paths left on the outer cycle after w_i, b_j are removed. Restricting this full matching to the boundary core proves the restored-pair assertion in (1).

All other allowed components are connected, balanced, disjoint from G_0 , and contained in individual faces of G_0 . Hence each face of G_0 encloses an even number of deleted vertices. The usual Kasteleyn cycle-sign rule therefore remains valid on restriction to G_0 . Equivalently, any alternating cycle of G_0 encloses a union of balanced deleted components, so its determinant sign comparison is unchanged. Thus K_0 is Kasteleyn.

Order the strongly connected components topologically. Since K has white rows and black columns, it is block upper triangular in this order. Its diagonal blocks are the Kasteleyn matrices of the allowed components and are invertible: a component has a perfect matching, and the determinant monomials have one sign. The inverse is block upper triangular as well, and its block on any component is the inverse of that component's diagonal block. In particular, its boundary-core block is K_0^{-1} , proving (2) and the restriction statement in (3).

For clarity, the stronger support statement follows from the directed acyclic graph of components. Write the boundary sources as vectors a on white vertices and b on black vertices, supported on G_0 . The full source equations are

$$K\mathcal{F}^\bullet = a, \quad K^T\mathcal{F}^\circ = b.$$

Block back-substitution shows that $\mathcal{F}^\bullet$ can be nonzero only on components from which G_0 is reachable, whereas $\mathcal{F}^\circ$ can be nonzero only on components reachable from G_0 . Suppose an edge joins a white vertex in a component C to a black vertex in a component D and both gauge values are nonzero. The directed component graph then contains

$$G_0 \longrightarrow C \longrightarrow D \longrightarrow G_0,$$

where some of the paths may have length zero. A directed acyclic graph has no such cycle unless $C = D = G_0$. Thus every noncore edge has a zero factor in its Coulomb product. The origami product has the same zero factor, which proves (3). $\square$

Corollary 2.9 (All-weight nonoverlap without an interior hypothesis). *Under (4), for every positive weighting the proper boundary gauges can be chosen so that the resulting full dual realization has convex, possibly collapsed, face images with no intersections of their two-dimensional interiors. No connectedness or edge-allowedness assumption is needed for this statement.*

Proof. By Lemma 2.8, the connected matching-covered graph G_0 satisfies the same boundary matching hypotheses and has exactly the same boundary matrix A . Apply Proposition 2.6(1) to G_0 , and use the same coefficient vectors α, β to extend the gauges to G . Their lifted forms agree on core edges and vanish on every other edge.

Here is the effect on the planar dual. Every face f of G lies in a unique face f_0 of G_0 . Send its dual vertex to the dual vertex of f_0 , and leave the augmented outer boundary fixed. Endpoints of a dual edge corresponding to a noncore primal edge have the same image; a dual edge crossing a core primal edge maps to that core edge's dual. This defines a collapse map from the full dual to the core dual. By the vanishing statement of Lemma 2.8, the full primitive is the core primitive composed with this map, up to an overall translation.

Consequently a polygon surrounding a core primal vertex has exactly the image of the corresponding core dual face; additional zero sides may repeat some of its vertices. Every polygon surrounding a noncore primal vertex is a point, since all its incident edge forms vanish. Thus the positive-area face images are precisely the positive-area core face images, which are convex and have pairwise disjoint interiors. The boundary has not changed. This gives the asserted nonstrict realization. For a disconnected full graph, this statement concerns these face polygons and the dual edge realization; it does not assert that the full combinatorial dual is a cell decomposition of a disk or that it has an injective realization before the collapse. $\square$

3. REDUCTION TO THE SYMMETRIC PRISM

The preceding section reduced the boundary problem to a critical point of F_A on the open set Ω_A of positive two-planes. We now show that its critical points remain in a compact subset of that open set, even as the graph and its weights change. This lets us attach a total integer index to all the critical points, including degenerate ones. A deformation of totally nonnegative boundary measurements then shows that this index depends only on n , the number of boundary vertices of either color. Consequently it suffices to compute the index for one particularly symmetric graph, the uniform prism.

We retain the assumptions that the induced interior graph H has a perfect matching and that every graph $G[V(H) \cup \{w_i, b_j\}]$ has a perfect matching. The arguments in this section concern boundary measurements; they do not require any nonvanishing assertion for gauges at interior vertices.

3.1. The boundary plane and its matching coordinates. Use the boundary matrix A from (6), in the canonical sign gauge (7), and put $S = A^{-1}$. Ordering the interior vertices before the boundary vertices, block Gaussian elimination gives

$$(36) \quad S = K_{\partial\partial} - K_{\partial H} K_{HH}^{-1} K_{H\partial}.$$

Here K_{HH} is invertible because H is matchable. Set $D = \text{diag}((-1)^1, \dots, (-1)^n)$, and define an $n \times 2n$ real matrix C by specifying its odd and even columns:

$$(37) \quad C_{2i-1} = e_i, \quad (C_2 \mid C_4 \mid \dots \mid C_{2n}) = DSD.$$

The e_i are the standard coordinate vectors of $\mathbb{R}^n$. For an n -element subset $I \subset \{1, \dots, 2n\}$, let $\Delta_I(C)$ be the determinant of the indicated columns, in increasing order. Define the odd and even index sets

$$O = \{1, 3, \dots, 2n-1\}, \quad E = \{2, 4, \dots, 2n\}.$$

Thus $\Delta_O(C) = 1$. This section uses the original counterclockwise order $w_1, b_1, \dots, w_n, b_n$. Accordingly, this matrix C is expressed in a different signed coordinate order from the clockwise boundary measurement plane in the dictionary (34).

Lemma 3.1 (Matching formula for all boundary coordinates). *For every n -element set I , let*

$$\mathcal{D}(I) = \{w_i : 2i - 1 \in I\} \cup \{b_j : 2j \notin I\}.$$

Then

$$(38) \quad \Delta_I(C) = \frac{Z(G - \mathcal{D}(I))}{Z(H)} \geq 0.$$

In particular, the row space of C is a totally nonnegative n -plane, meaning that all its ordered maximal minors are nonnegative. Moreover, $\Delta_E(C) > 0$, and every coordinate obtained from either O or E by a single exchange with the opposite parity is strictly positive.

Proof. We include the signs in the proof, since they will also give the strict signs of S . Normalize K as in Lemma 2.1: the outer diagonal edges have positive signs, the subdiagonal edges have negative signs, and the wrap edge has positive sign. An interior row sign, if needed, can arrange $\det K_{HH} = Z(H) > 0$ without changing these outer signs. In the case $H = \emptyset$, this determinant is 1. A matching of H completed by all outer diagonal edges then has positive determinant sign, so $\det K = Z(G) > 0$ as well.

Write

$$P = \{i : 2i - 1 \in I\}, \quad V = \{j : 2j \in I\}, \quad U = \{1, \dots, n\} \setminus P, \quad Q = \{1, \dots, n\} \setminus V.$$

The residual graph in (38) has boundary whites w_U and boundary blacks b_V . Put $p = |P| = |Q|$, so $|U| = |V| = n - p$. Restricting the block elimination in (36) to these rows and columns gives

$$(39) \quad \det K_{H \cup w_U, H \cup b_V} = Z(H) \det S_{U,V}.$$

Expanding $\Delta_I(C)$ in its identity columns gives

$$(40) \quad \Delta_I(C) = (-1)^{\sigma + \sum_{i \in U} i + \sum_{j \in V} j} \det S_{U,V}, \quad \sigma = \sum_{i \in P} i + \frac{p(p+1)}{2} + \#\{(j, i) \in V \times P : j < i\}.$$

Indeed, the position of an identity column $2i - 1$ among the selected columns is its rank among P , plus the number of selected even columns before it; this gives σ . The other two sums arise from the two factors D in (37).

We next identify this determinant sign with the positive matching sign. Pair the deleted vertices w_P, b_Q by nonintersecting arcs outside the outer cycle, each joining opposite colors. Such a pairing always exists: in a cyclic list with equally many vertices of each color, remove an adjacent opposite-colored pair, pair the remaining list inductively, and insert the removed arc next to the boundary. Denote the resulting bijection by $\pi : P \rightarrow Q$. Give the added edge $w_i b_{\pi(i)}$ sign $+1$ if $i \leq \pi(i)$ and -1 if $i > \pi(i)$. These signs extend the Kasteleyn signing. To check this, cut the boundary between b_n and w_1 and draw the nonintersecting arcs on its other side. The boundary interval from w_i to b_j , together with the new arc, bounds a region with no vertices away from that interval. If $i \leq j$, the interval has $j - i$ negative edges and length $2(j - i) + 1$, requiring sign $+1$ on its closing edge. If $i > j$, the interval has $i - j$ negative edges and length $2(i - j) - 1$, requiring sign -1 . The same check applies with nested arcs: multiplying the face sign conditions inside a closing arc gives this interval condition, since the signs of internal arcs occur twice. Thus the augmented graph has a Kasteleyn matrix extending K .

Here is the elementary parity calculation needed for these arcs. For a pairing π , let $\text{inv}(\pi)$ be the number of inversions of the sequence of black indices paired with the increasing white indices, and

let $\text{cr}(\pi)$ count crossing pairs of arcs. Then

$$(41) \quad \text{inv}(\pi) + \#\{i \in P : i > \pi(i)\} \equiv \frac{p(p-1)}{2} + \#\{(j, i) \in Q \times P : j < i\} + \text{cr}(\pi) \pmod{2}.$$

For completeness, this is a sum of identities for pairs of arcs. For white indices $i < i'$, compare the four endpoints $2i - 1, 2i' - 1, 2\pi(i), 2\pi(i')$ on the line. The parity of

$$1 + \mathbf{1}_{\pi(i) < i'} + \mathbf{1}_{\pi(i') < i} + \mathbf{1}_{\pi(i) > \pi(i')}$$

is 1 precisely when the arcs cross. Summing these identities accounts for every term in (41); the comparisons of an arc's own endpoints give $\#\{i > \pi(i)\}$. Since our arcs do not cross, the signed determinant of their matching matrix on rows P , columns Q has sign

$$(-1)^{p(p-1)/2 + \#\{(j, i) \in Q \times P : j < i\}} = (-1)^\sigma.$$

The last equality follows from

$$\#\{(j, i) \in V \times P : j < i\} = \sum_{i \in P} (i - 1) - \#\{(j, i) \in Q \times P : j < i\}.$$

Assign a separate positive variable to each added edge. The coefficient of the product of all these variables in the determinant of the augmented Kasteleyn matrix is a positive sum of the residual matching weights. It is exactly $Z(G - \mathcal{D}(I))$: a matching using all added edges is otherwise a matching of the residual graph. On the other hand, Laplace expansion makes that coefficient

$$(-1)^{\sum_{i \in U} i + \sum_{j \in V} j + \sigma} \det K_{H \cup w_U, H \cup b_V} = Z(H) \Delta_I(C),$$

where the interior row and column offsets cancel modulo two. This proves (38), including the case where the residual graph has no matching.

Finally, $I = E$ gives $\Delta_E = Z(G)/Z(H) > 0$. A single exchange from O restores exactly one white and one black vertex to H , so its partition function is positive by hypothesis. A single exchange from E removes one white and one black vertex from G . A matching of H together with the matchings of the two outer paths, each having an even number of vertices, supplies its matching. This proves all the strict assertions. $\square$

Corollary 3.2 (Two strict sign patterns). *In the canonical gauge,*

$$(42) \quad \text{sgn } A_{ij} = \begin{cases} +1 & i \geq j, \\ -1 & i < j, \end{cases} \quad \text{sgn } S_{ij} = \begin{cases} +1 & i \leq j, \\ -1 & i > j. \end{cases}$$

Conversely, for a plane represented as in (37) with positive odd and even pivots, these strict patterns are equivalent to positivity of all the single exchanges from O and E .

Proof. Expansion in the identity columns gives

$$(43) \quad \Delta_{O \setminus \{2i-1\} \cup \{2j\}}(C) = \begin{cases} S_{ij} & i \leq j, \\ -S_{ij} & i > j. \end{cases}$$

Likewise, using the even columns as a pivot and $S^{-1} = A$,

$$(44) \quad \frac{\Delta_{E \setminus \{2j\} \cup \{2i-1\}}(C)}{\Delta_E(C)} = \begin{cases} A_{ji} & j \geq i, \\ -A_{ji} & j < i. \end{cases}$$

These two formulas and Lemma 3.1 prove the claim. In particular, the matching assumption is used to make every entry of S nonzero; no positivity assumption on larger minors of S is being imposed. $\square$

3.2. A compact domain and nonvanishing mixed brackets. Let $\mathcal{G} = \text{Gr}^{\text{or}}(2, n)$ be the oriented real Grassmannian, represented by rank-two real $2 \times n$ matrices modulo left multiplication by matrices of positive determinant. Write

$$X = \text{Gr}_{\geq 0}(2, n), \quad X^\circ = \text{Gr}_{> 0}(2, n)$$

for its closed nonnegative and open positive parts. Thus, if the columns of B are represented by complex numbers $\beta_1, \dots, \beta_n$, $B \in X$ means $[\beta_i, \beta_j] \geq 0$ for $i < j$, and $B \in X^\circ$ means that all these inequalities are strict. Recall that $c = A^T \beta$ is represented by BA . In this notation the admissible set from (29) and its compact container are

$$(45) \quad \Omega_A = X^\circ \cap \{B : BA \in X^\circ\}, \quad K_A = X \cap \{B : BA \in X\}.$$

We do not need to assert that K_A is the closure of Ω_A .

Lemma 3.3 (A nonvanishing linear combination). *Let $v_1, \dots, v_n \in \mathbb{R}^2$ have rank two and satisfy $\det(v_i, v_j) \geq 0$ for $i < j$. If the nonzero real coefficients $s_1, \dots, s_n$ have at most one sign change, then $\sum_i s_i v_i \neq 0$. Some of the vectors v_i may themselves vanish.*

Proof. After multiplying all coefficients by -1 if needed, a proposed zero sum with a sign change can be written

$$\sum_{i \leq k} a_i v_i = \sum_{j > k} b_j v_j, \quad a_i, b_j > 0.$$

The determinant of these equal sums is $\sum_{i \leq k < j} a_i b_j \det(v_i, v_j) = 0$. Every term is nonnegative, so every cross determinant vanishes. If both groups contain a nonzero vector, all vectors in the two groups are proportional, contrary to rank two. If one group consists only of zero vectors, the other group has a positive linear combination equal to zero. Taking the determinant with its first nonzero vector forces every later vector to be proportional to it, again contradicting rank two. The same last argument treats coefficients all having the same sign. $\square$

From now on in this section we use antiperiodic notation

$$\beta_{i+n} = -\beta_i, \quad c_{i+n} = -c_i.$$

It is equivalent to the cyclic convention in (18)–(19): changing both the wrap numerator and its associated denominator leaves D_i, E_i and the vertices unchanged. With this notation all the mixed brackets have the same sign.

Proposition 3.4 (Compactness and smooth extension). *Assume only that A and $S = A^{-1}$ satisfy (42). Then K_A is compact. At every point of K_A , all β_i, c_i are nonzero and*

$$(46) \quad \delta_i = [c_i, \beta_i] < 0, \quad \varepsilon_i = [c_{i+1}, \beta_i] < 0.$$

Consequently

$$F_A = \sum_i \log |\delta_i| - \sum_i \log |\varepsilon_i|$$

extends to a smooth function on a neighborhood of K_A in $\mathcal{G}$. These assertions hold uniformly when A varies in a compact family satisfying (42).

Proof. The oriented Grassmannian is compact. Its subset X , defined by closed minor inequalities, is compact as well, and $B \mapsto BA$ is a continuous automorphism. This proves compactness of K_A .

If $c_j = \sum_i A_{ij} \beta_i$ were zero, the j th column of A would contradict Lemma 3.3: its coefficients are strictly negative and then strictly positive. Similarly, $\beta_i = \sum_j S_{ji} c_j$ cannot vanish, since the i th column of S has strictly positive and then strictly negative entries.

For every i , expand

$$\delta_i = \sum_j A_{ji} [\beta_j, \beta_i].$$

Terms with $j < i$ are nonpositive because $A_{ji} < 0$; terms with $j > i$ are nonpositive because $A_{ji} > 0$ and $[\beta_j, \beta_i] \leq 0$. Since $\beta_i \neq 0$ and the columns have rank two, at least one term is negative. For $i < n$, the same calculation with $A_{j,i+1}$ gives $\varepsilon_i < 0$: terms with $j \leq i$ have a negative coefficient, while those with $j \geq i+1$ have a positive one. For the wrap,

$$\varepsilon_n = [-c_1, \beta_n] = - \sum_j A_{j1} [\beta_j, \beta_n] < 0,$$

since every $A_{j1} > 0$ and at least one bracket is positive.

A change of representative by $L \in \mathrm{GL}_2^+(\mathbb{R})$ multiplies each bracket by $\det L$. The n logarithms of this factor cancel between the two sums defining F_A . Thus the function is well-defined on $\mathcal{G}$. Local smooth representatives give a smooth extension wherever the mixed brackets do not vanish. For uniformity, choose representatives with orthonormal rows, a compact space of choices. The joint set of pairs (A, B) with $B \in K_A$ is compact for a compact family of A , and the just-proved strict nonvanishing bounds all bracket magnitudes away from zero there. $\square$

3.3. Excluding every boundary critical point. The critical-point equation from (22) is

$$(47) \quad d = A\alpha, \quad \alpha_i = D_i c_i, \quad d_i = E_i \beta_i,$$

where, with the antiperiodic convention,

$$D_i = \frac{[\beta_{i-1}, \beta_i]}{\delta_i \varepsilon_{i-1}}, \quad E_i = \frac{[c_i, c_{i+1}]}{\delta_i \varepsilon_i}.$$

The differential identity proving (47) only requires nonzero mixed brackets. By Proposition 3.4, it therefore remains valid at every point of K_A .

Theorem 3.5 (No critical points on the inadmissible boundary). *Under (42), every critical point of F_A in K_A belongs to Ω_A .*

Proof. We first describe the limiting boundary polygons, before imposing the critical-point equation. For any nonnegative rank-two sequence of nonzero columns β_i , one can choose arguments so that

$$\arg \beta_1 \leq \arg \beta_2 \leq \cdots \leq \arg \beta_n \leq \arg \beta_1 + \pi.$$

All antiperiodic successive increments, including the increment from β_n to $-\beta_1$, belong to $[0, \pi)$ and sum to π . An increment equal to π would force all columns to lie on one line, contrary to rank two. The same statements hold for the c_i . It follows from (46) that $D_i, E_i \geq 0$ on K_A .

Recall the projected vertices from (25):

$$(48) \quad z_{2i} = -\frac{c_i \beta_i}{\delta_i}, \quad z_{2i+1} = -\frac{c_{i+1} \beta_i}{\varepsilon_i}.$$

Both scalar prefactors are positive. Consecutive argument increments of these vertices are alternately the increments of c_i and of β_i just described. Each lies in $[0, \pi)$, and their sum is 2π . Each lifted vertex has the form $(z, x + i)$ with $|z|^2 - x^2 = 1$, and consecutive lifted vertices have null difference. Therefore every nonconstant projected segment is tangent to the unit circle. Explicitly, if the real lift coordinate changes by $h \neq 0$ and the projection by $h\nu$, nullity gives $|\nu| = 1$; subtracting the two quadric equations gives $\mathrm{Re}(z\bar{\nu}) = x$ at the initial endpoint, whence $|\mathrm{Im}(z\bar{\nu})| = 1$. If $h = 0$, nullity forces a zero segment. Thus all segments avoid the origin, and the polygon has winding number one even when some edges have collapsed.

Suppose a point in $K_A \setminus \Omega_A$ were critical. Because neither sequence has a zero column, failure of strict positivity implies a vanishing cyclically adjacent bracket in at least one sequence: in the argument description, any zero ordered minor makes either an interval of increments, or its complementary interval, have sum zero. Hence some D_i or E_i vanishes.

Use (47) to define α, d . Their real ranks are equal, since A is real and invertible. If this common rank were two, then multiplying the c and β columns by the nonnegative numbers D_i, E_i would make both α and d nonnegative two-planes. When $D_i = 0$, the identity

$$0 = \alpha_i = \sum_j S_{ij} d_j$$

contradicts Lemma 3.3, because a row of S has at most one strict sign change. When $E_i = 0$, apply the same lemma to $0 = d_i = \sum_j A_{ij} \alpha_j$. Thus the common rank is at most one. It is not zero: the argument increments of β sum to π , so at least one adjacent bracket, and hence at least one D_i , is positive.

All entries of α and d consequently lie on the same real line $\ell \subset \mathbb{C}$. The line is common because each sequence is a real linear combination of the other. The projected edge identities (28) are

$$z_{2i} - z_{2i-1} = D_i c_i^2, \quad z_{2i+1} - z_{2i} = -E_i \beta_i^2.$$

If the first edge is nonzero, then $c_i = \alpha_i / D_i \in \ell$; if the second is nonzero, then $\beta_i = d_i / E_i \in \ell$. All nonzero edges are therefore parallel to the line ℓ^2 , so the closed polygon is contained in an affine line. A closed curve contained in a line and avoiding the origin has winding zero. This contradicts the winding-one conclusion above and proves the theorem. $\square$

Corollary 3.6 (Uniform confinement of critical points). *For any compact continuous family $A(t)$ satisfying (42), the joint critical set*

$$\{(t, B) : B \in K_{A(t)}, dF_{A(t)}(B) = 0\}$$

is compact and contained in the open set $\{(t, B) : B \in \Omega_{A(t)}\}$.

Proof. Proposition 3.4 gives compactness of the joint container and a smooth extension in the plane variable, continuous with its derivatives in t . Its critical subset is closed, hence compact. Theorem 3.5 excludes every point of that subset outside the stated open set. $\square$

3.4. The total index and its continuation. Fix any smooth positive-definite metric on $\mathcal{G}$. By the preceding corollary the full critical set in Ω_A has a relatively compact open neighborhood U with $\bar{U} \subset \Omega_A$ and with ∇F_A nonzero on ∂U . Define

$$(49) \quad I(A) = \text{ind}(\nabla F_A, U).$$

Here the index is the usual local index of a vector field: perturb it inside U , keeping it fixed near the boundary, so that its zeros are isolated and nondegenerate, and sum the signs of the determinants of its derivatives at those zeros. Local degree makes this integer independent of the perturbation and of U . If F_A already has only nondegenerate critical points, this is simply

$$(50) \quad I(A) = \sum_{B \in \text{Crit}(F_A) \cap \Omega_A} (-1)^{\mu(B)},$$

where $\mu(B)$ is the number of negative eigenvalues of its Hessian. The sum is finite by compactness. The same definition applies to nonisolated or degenerate critical sets. An empty critical set has index zero. Changing the metric does not change the index: a convex interpolation of positive-definite metrics gives a homotopy of gradients with exactly the same zero set.

Proposition 3.7 (Index continuation). *Along a compact continuous path $A(t)$ satisfying (42), the integer $I(A(t))$ is constant.*

Proof. Fix t_0 and choose an isolating neighborhood U as above for the full critical set at t_0 . For t close to t_0 , the compact set $\bar{U}$ remains in $\Omega_{A(t)}$, and its boundary remains free of zeros. Furthermore every admissible critical point at such a t still lies in U . Otherwise a sequence of counterexamples with $t \rightarrow t_0$ would, by Corollary 3.6, converge to a critical point in $K_{A(t_0)}$ outside U . The boundary-exclusion theorem makes that limit admissible, contradicting the choice of U . Homotopy invariance

of the local index on this one neighborhood now gives local constancy in t . A locally constant integer on an interval is constant. No connectedness assumption on Ω_A is involved. $\square$

3.5. One common deformation for all boundary measurements. We now connect the different matrices supplied by graphs. The intermediate matrices will be legitimate real boundary measurements; they need not come from the same graph. Define $\mathcal{C}_n$ to be the set of totally nonnegative oriented n -planes in $\mathbb{R}^{2n}$ for which the odd pivot, the even pivot, and every single exchange from these pivots are positive. Lemma 3.1 places all our graph measurements in $\mathcal{C}_n$.

Lemma 3.8 (A common strictly positive limit). *Let $m = 2n$ and let J be the symmetric $m \times m$ matrix whose two adjacent diagonals are 1 and whose other entries are zero. For every $t > 0$, the matrix $P(t) = \exp(tJ)$ has all minors strictly positive. For every totally nonnegative n -plane L , the plane $LP(t)$ is strictly positive when $t > 0$, and it converges as $t \rightarrow \infty$ to the same strictly positive plane L_* , independent of L . The convergence is uniform over the compact nonnegative Grassmannian.*

Proof. For $1 \leq k \leq m$, consider the induced infinitesimal operator $J^{[k]}$ on $\bigwedge^k \mathbb{R}^m$. In the standard basis indexed by increasing k -subsets, its off-diagonal entries are 1 precisely when one index moves one step to an unoccupied neighboring location, and are zero otherwise. Such a move requires no reordering of the wedge, so no negative sign occurs. Any two k -subsets can be connected by these moves, for example by moving each one successively to $\{1, \dots, k\}$. Thus $J^{[k]}$ is nonnegative and its adjacency graph is connected. For any two basis indices, some power of $J^{[k]}$ has a positive entry between them. The exponential power series therefore shows that every entry of $\exp(tJ^{[k]})$ is positive for $t > 0$. The identity

$$\bigwedge^k \exp(tJ) = \exp(tJ^{[k]})$$

identifies these entries with the ordered $k \times k$ minors of $P(t)$. This proves strict total positivity. Cauchy–Binet then shows that $LP(t)$ has all maximal minors positive whenever L has nonnegative maximal minors, at least one of them nonzero.

For the limit, the eigenvalues of J are

$$\lambda_r = 2 \cos \frac{r\pi}{m+1}, \quad r = 1, \dots, m.$$

This follows directly by substituting the vector with coordinates $\sin(jr\pi/(m+1))$ in the eigenvalue equation. The eigenvalues are strictly decreasing. Hence $J^{[n]}$ has a unique largest eigenvalue $\Lambda = \lambda_1 + \dots + \lambda_n$, whose eigenvector v_* is the wedge of the corresponding eigenvectors of J . Since $J^{[n]}$ is symmetric, nonnegative, and connected, its largest eigenvector can be chosen with all coordinates strictly positive. One can see this last fact directly from the strictly positive matrix $\exp(J^{[n]})$: maximizing its Rayleigh quotient permits replacement of a maximizer by its absolute values, and its eigenvalue equation then makes every coordinate positive; strict positivity also forces simplicity.

Normalize nonnegative Plücker vectors to Euclidean length one. Their scalar product with the positive unit vector v_* is bounded below by the smallest coordinate of v_* . Spectral decomposition therefore shows, uniformly for such vectors v , that

$$e^{-t\Lambda} v \exp(tJ^{[n]}) \longrightarrow \langle v, v_* \rangle v_*.$$

After normalization this is convergence to the plane L_* spanned by the first n eigenvectors of J . Its Plücker vector v_* is strictly positive, as required. $\square$

Reparametrize this flow by $t = s/(1-s)$ for $0 \leq s < 1$ and append L_* at $s = 1$. Lemma 3.8 gives a continuous path from every $L \in \mathcal{C}_n$ to one common endpoint, staying in $\mathcal{C}_n$ throughout. To turn it into a path of boundary matrices, represent $L(s)$ by a matrix and multiply on the left by the inverse of its odd-column block. This is possible and continuous because the odd pivot

is positive. Its odd columns are then the identity, so write its even-column block uniquely as $DS(s)D$ and put $A(s) = S(s)^{-1}$. The even pivot is also positive, so $S(s)$ is always invertible. The single-exchange formulas (43) and (44) imply the strict signs (42) for $A(s)$ and $S(s)$ at every s , including both endpoints. Thus any two graph boundary matrices are joined by a compact path to which Proposition 3.7 applies.

Theorem 3.9 (A universal boundary index). *For fixed n , every graph and every positive weighting satisfying the stated matching assumptions has the same total index*

$$I(A) = I_n.$$

If $I_n \neq 0$, every one of these boundary matrices admits algebraically perfect proper boundary gauges.

Proof. The common deformation and index continuation prove equality of the indices. A nonzero local index implies that the vector field has a zero. By Theorem 3.5, every zero in K_A is in Ω_A . The construction of the preceding section then gives $\alpha_i = D_i c_i$ and the required proper boundary gauges. $\square$

3.6. The uniform prism belongs to this class. For $n \geq 3$, take two concentric cycles with $2n$ vertices each and join corresponding vertices by radial spokes. Give the endpoints of each spoke opposite colors. Denote the outer vertices by w_i, b_i as before, and the inner vertices adjacent to them by B_i, W_i , respectively. All edge weights are 1. Define the antiperiodic shift by

$$(51) \quad T_{i+1,i} = 1 \quad (1 \leq i < n), \quad T_{1,n} = -1, \quad T^n = -I,$$

and put $O_0 = I - T$. With white rows ordered as $(w_1, \dots, w_n, W_1, \dots, W_n)$ and black columns as $(b_1, \dots, b_n, B_1, \dots, B_n)$, a Kasteleyn matrix and its boundary inverse block are

$$(52) \quad K_{\text{pr}} = \begin{pmatrix} O_0 & I \\ I & -O_0^T \end{pmatrix}, \quad A_{\text{pr}} = (I + O_0 O_0^T)^{-1} O_0^T.$$

To check the signing, each annular square has edge-sign product -1 , as required for a quadrilateral, and the inner $2n$ -gon has product $(-1)^{n-1}$, as required for its length. The matrix O_0 is invertible because $T^n = -I$ precludes eigenvalue 1. Taking the Schur complement of the lower-right block gives $S_{\text{pr}} = O_0 + O_0^{-T}$; its inverse is the matrix displayed in (52), since O_0 is normal.

The interior graph is the inner cycle, so it has a perfect matching. To match the graph with any one outer white w_i and outer black b_j restored, use their two spokes. The remaining inner cycle has lost one vertex of each color; each of its two remaining paths has an even number of vertices and can be matched. Thus the prism satisfies exactly the matching hypotheses used throughout this section. In particular its total index is I_n .

The fundamental Fourier vector

$$(53) \quad \beta_j = \exp\left(\frac{i(j-1)\pi}{n}\right)$$

represents a positive two-plane. Since A_{pr} commutes with T , it preserves that plane and acts there by a nonzero complex scalar, so the same plane lies in $\Omega_{A_{\text{pr}}}$. The next section uses the prism's cyclic symmetry to prove $I_n \neq 0$. The case $n = 2$ needs no such calculation: its positive Grassmannian consists of one oriented plane, which is admissible because $\det A > 0$, and its zero-dimensional index is $I_2 = 1$.

4. THE UNIFORM PRISM AND THE INDEX CALCULATION

The continuation argument reduces existence to a single symmetric boundary matrix. We first identify its central Fourier critical point and compute its Hessian. We then study the restriction of the functional to every nontrivial cyclic subgroup's fixed locus. A map defined by normalized mixed areas makes each of these restricted problems explicit. Finally, a small invariant perturbation separates the central point from free cyclic orbits and computes the total index modulo n .

4.1. The reference matrix and the Fourier point. Throughout this section $n \geq 3$; the case $n = 2$ is treated separately after Proposition 2.4. Recall the antiperiodic shift T and the prism matrix A from (51)–(52). In particular,

$$(54) \quad T^n = -I, \quad H = 3I - T - T^T, \quad A = H^{-1}(I - T)^T.$$

The preceding section verified both the Kasteleyn signing and the matching hypotheses for this graph.

Recall that $c = A^T \beta$ and

$$F_A(\beta) = \sum_i \log |[c_i, \beta_i]| - \sum_i \log |[c_{i+1}, \beta_i]|, \quad [z, w] = \text{Im}(\bar{z}w).$$

We use the antiperiodic extensions $\beta_{i+n} = -\beta_i$ and $c_{i+n} = -c_i$. They leave the functional and reconstructed vertices unchanged and make both lists of mixed brackets negative on the admissible domain Ω_A . The action of T has order n on oriented two-planes: $T^n = -I$ acts there trivially. Since T commutes with A , the induced cyclic action preserves both F_A and Ω_A .

Put $\theta = \pi/n$ and $\omega = e^{2\pi i/n}$. In formulas using Fourier series we index the coordinates by $j = 0, \dots, n-1$. The first Fourier plane is represented by

$$\beta_j = e^{i\theta j}.$$

All its ordered two-minors are positive. Define

$$\kappa = \frac{1 - e^{-i\theta}}{3 - 2\cos\theta}, \quad s = \text{Im } \kappa = \frac{\sin\theta}{3 - 2\cos\theta} > 0.$$

Then $c = \kappa\beta$. Hence c also represents a positive plane, and this point lies in Ω_A . Both families of mixed brackets equal $-s$. The coefficients in the critical equation are

$$D_j = \frac{\sin\theta}{s^2}, \quad E_j = |\kappa|^2 D_j.$$

Since $AA^T\beta = |\kappa|^2\beta$, the equation $A \text{diag}(D)c = \text{diag}(E)\beta$ holds. We denote this critical point by x_* .

Lemma 4.1 (The quotient Hessian). *At x_* the Hessian has $n-3$ negative eigenvalues, $n-3$ positive eigenvalues, and a two-dimensional kernel. A local quotient chart is*

$$(55) \quad \beta_j = e^{i\theta j} \left(1 + \sum_{k=1}^{n-2} h_k \omega^{kj} \right).$$

In this chart the kernel is the complex line $h_1 = z$, $h_k = 0$ for $k \neq 1$. A generator of the cyclic action acts on it as $z \mapsto \omega z$.

Proof. An arbitrary infinitesimal variation has the form

$$\eta_j = e^{i\theta j} \sum_{k \in \mathbb{Z}/n\mathbb{Z}} h_k \omega^{kj}.$$

Write

$$\kappa_k = \frac{1 - e^{-i(\theta + 2\pi k/n)}}{3 - 2\cos(\theta + 2\pi k/n)}, \quad L_k = \bar{\kappa} - \kappa_k.$$

For a real parameter t , set $\beta(t) = \beta + t\eta$ and $c(t) = A^T \beta(t)$. Differentiating the brackets at $t = 0$ gives

$$\begin{aligned} \dot{\delta}_j &= [(A^T \eta)_j, \beta_j] + [c_j, \eta_j] = \text{Im} \sum_k L_k h_k \omega^{kj}, \\ \dot{\varepsilon}_j &= [(A^T \eta)_{j+1}, \beta_j] + [c_{j+1}, \eta_j] = \text{Im} \sum_k (-\bar{L}_k) h_k \omega^{kj}. \end{aligned}$$

For the second line one can check each Fourier coefficient using $\bar{\kappa}e^{-i\theta} = -\kappa$ and $\kappa_k e^{i(\theta+2\pi k/n)} = -\bar{\kappa}_k$.

The sums of second derivatives of the two bracket families agree. To verify this directly, the identity $A^T = -TA$ and orthogonality of the Fourier modes imply

$$\sum_j [(A^T \eta)_j, \eta_j] = \sum_j [(A^T \eta)_{j+1}, \eta_j].$$

Equivalently, the k th mode contributes respectively $-n \operatorname{Im} \kappa_k |h_k|^2$ and $-n \operatorname{Im}(\kappa_k e^{i(\theta+2\pi k/n)}) |h_k|^2$; these are equal because $\kappa_k e^{i(\theta+2\pi k/n)} = -\bar{\kappa}_k$. It follows from $\delta_j = \varepsilon_j = -s$ and $(\log |a(t)|)'' = a''/a - (a'/a)^2$ that only the squares of the first variations contribute to the Hessian. For complex numbers a_k ,

$$\sum_{j=0}^{n-1} \left(\operatorname{Im} \sum_k a_k \omega^{kj} \right)^2 = \frac{n}{2} \sum_k |a_k|^2 - \frac{n}{2} \operatorname{Re} \sum_k a_k a_{-k}.$$

Applying this identity to the two first variations yields

$$(56) \quad \operatorname{Hess}_{x_*} F_A(\eta, \eta) = -\frac{n}{s^2} \sum_k \operatorname{Im}(L_k L_{-k}) \operatorname{Im}(h_k h_{-k}).$$

Here is the coefficient calculation determining the signs. Set $a = \cos \theta$, $b = \sin \theta$, $\phi = 2\pi k/n$, $C = \cos \phi$, and $d_{\pm} = 3 - 2 \cos(\theta \pm \phi)$. Then

$$L_{\pm k} = \frac{\cos(\theta \pm \phi) - a - i\{bd_{\pm} + (3 - 2a) \sin(\theta \pm \phi)\}}{(3 - 2a)d_{\pm}}.$$

Multiplication, using $\cos(\theta \pm \phi) = aC \mp b \sin \phi$ and $\sin(\theta \pm \phi) = bC \pm a \sin \phi$, gives

$$\operatorname{Im}(L_k L_{-k}) = \frac{4b(C-1)(C-\cos 2\theta)}{(3-2a)^2 d_+ d_-}, \quad \frac{\operatorname{Im}(L_k L_{-k})}{s^2} = \frac{4(C-1)(C-\cos 2\theta)}{bd_+ d_-}.$$

The coefficient vanishes exactly for $k = 0, 1, -1$. For every other mode, $C < \cos 2\theta < 1$, so it is positive.

The real-linear coordinate changes $\eta = p\beta + q\bar{\beta}$ are exactly modes $0, -1$. Their tangent space is complementary to the chart (55), so that chart is a local slice for the quotient. After removing these directions, mode 1 is the entire kernel. Each distinct remaining pair $k, -k$ contributes a nonzero multiple of $\operatorname{Im}(h_k h_{-k})$, a real quadratic form of signature $(2, 2)$. When n is even, the self-paired mode $k = n/2$ contributes a nonzero multiple of $\operatorname{Im}(h_k^2)$, of signature $(1, 1)$. This proves the inertia. Shifting the index j and renormalizing the coefficient of $e^{i\theta j}$ multiplies h_k by ω^k , up to the choice of generator, proving the last assertion. $\square$

4.2. The normalized area map on fixed loci.

Theorem 4.2 (Rigidity on every nontrivial cyclic fixed locus). *Let $n = qm$, where $q \geq 2$ is any divisor of n . For the uniform prism, the restriction of F_A to the entire C_q -fixed part of Ω_A has exactly one critical point: the first Fourier plane. Its restricted Hessian is nondegenerate, with $m-1$ negative and $m-1$ positive eigenvalues.*

Here C_q is generated by the m th power of the cyclic shift. The theorem does not impose a prime-power condition on q . We prove it by constructing coordinates from mixed areas; neither global coordinates on Ω_A nor uniqueness of its unrestricted critical points are assumed.

Put $T_{i+1,i} = 1$, $T_{1,n} = -1$, and

$$H = 3I - T - T^T, \quad \gamma = H^{-1}\beta.$$

Then

$$c_i = \gamma_i - \gamma_{i-1}, \quad \beta_i = 3\gamma_i - \gamma_{i-1} - \gamma_{i+1}.$$

All indices are antiperiodic modulo n .

Lemma 4.3. *The map H^{-1} sends every strictly positive oriented two-plane to a strictly positive oriented two-plane.*

Proof. Set $r = (3 - \sqrt{5})/2$ and $Q = (I - rT)^{-1}$. Then $H^{-1} = rQQ^T$, and

$$Q_{ij} = \frac{1}{1 + r^n} \begin{cases} r^{i-j}, & i \geq j, \\ -r^{n+i-j}, & i < j. \end{cases}$$

After positive row and column scalings, this is the matrix with entries 1 below and on the diagonal and $-R$ above it, where $R = r^n > 0$. For two ordered rows, the column vectors occur in three successive blocks: $(1, 1)^T$, $(-R, 1)^T$, and $(-R, -R)^T$. Every ordered two-column determinant is nonnegative. Thus all ordered 2×2 minors of Q , and by Cauchy–Binet of H^{-1} , are nonnegative. Since H^{-1} is invertible, its second compound matrix is invertible, so each row of that nonnegative matrix has a positive entry. Applying it to a strictly positive Plücker vector proves the claim. $\square$

Let $\mathcal{U}_{q,m}$ denote the C_q -fixed locus inside Ω_A . Choose the complex coordinates on a fixed positive plane so that

$$\beta_{i+m} = \rho \beta_i, \quad \rho = e^{i\pi/q}.$$

This choice is unique up to multiplication by a nonzero complex scalar. First conjugate the finite-order induced real map to a rotation by an orientation-preserving real linear change of coordinates. Positivity then makes the argument increment over m steps lie between zero and π , and the q increments sum to π ; thus the induced rotation is precisely π/q . The corresponding complex eigenspace of the shift has dimension m . Hence the entire fixed positive locus, and in particular $\mathcal{U}_{q,m}$, is an open subset of $\mathbb{CP}^{m-1}$. The same twisted period holds for γ, c .

Define the positive mixed areas and consecutive areas

$$d_i = -[c_i, \beta_i], \quad e_i = -[c_{i+1}, \beta_i], \quad u_i = [\gamma_i, \gamma_{i+1}], \quad v_i = [\gamma_{i-1}, \gamma_{i+1}].$$

Here $d_i, e_i, u_i > 0$, the last inequality by Lemma 4.3. Direct expansion gives

$$(57) \quad d_i = 2u_{i-1} + u_i - v_i, \quad e_i = u_{i-1} + 2u_i - v_i.$$

These real quantities are periodic modulo m , and $\sum_{i=1}^m d_i = \sum_{i=1}^m e_i =: h > 0$. The normalized area map is

$$\Psi : \mathcal{U}_{q,m} \longrightarrow \Delta_{m-1}^\circ \times \Delta_{m-1}^\circ, \quad \Psi(\gamma) = ((d_i/h)_i, (e_i/h)_i),$$

where Δ_{m-1}° is the simplex of strictly positive coordinates summing to one. This is well defined on complex lines, and both source and target have real dimension $2m - 2$.

Lemma 4.4 (Infinitesimal rigidity of the area map). *The map Ψ is a local diffeomorphism at every point of $\mathcal{U}_{q,m}$.*

Proof. Take a tangent vector in the kernel and represent it by a variation $\dot{\gamma}$ with the same twisted period. Vanishing of the normalized area derivatives says $\dot{d}_i = \lambda d_i$, $\dot{e}_i = \lambda e_i$ for one real number λ . Replacing $\dot{\gamma}$ by $\dot{\gamma} - (\lambda/2)\gamma$ removes this scaling and makes all $\dot{d}_i, \dot{e}_i$ zero. Equation (57) then gives

$$\dot{u}_i = U \quad \text{for every } i, \quad \dot{v}_i = 3U$$

for a real constant U .

The elementary vector recurrence

$$u_{i-1}\gamma_{i+1} - v_i\gamma_i + u_i\gamma_{i-1} = 0$$

therefore differentiates to

$$u_{i-1}\dot{\gamma}_{i+1} - v_i\dot{\gamma}_i + u_i\dot{\gamma}_{i-1} = U\beta_i.$$

Divide by $u_{i-1}u_i$ and define the real symmetric antiperiodic Jacobi operator

$$(Jz)_i = \frac{z_{i+1}}{u_i} + \frac{z_{i-1}}{u_{i-1}} - \frac{v_i z_i}{u_{i-1}u_i}, \quad W = \text{diag}((u_{i-1}u_i)^{-1}).$$

Let Γ, B be the $n \times 2$ real matrices whose rows represent γ_i, β_i . Then

$$J\Gamma = 0, \quad J\dot{\Gamma} = UWB, \quad 0 = U\Gamma^T WB.$$

The matrix $\Gamma^T WB$ is invertible. Indeed, Cauchy–Binet gives

$$\det(\Gamma^T WB) = \sum_{i < j} W_i W_j [\gamma_i, \gamma_j] [\beta_i, \beta_j] > 0,$$

since both planes are strictly positive. Consequently $U = 0$.

Since $u_i > 0$, two initial values determine every real recurrence solution. The real and imaginary components of γ are independent solutions, so they span the solution space. The recurrence therefore says that $\dot{\gamma}$ is a common real linear infinitesimal change of coordinates. Its twisted-period condition forces that real linear map to commute with rotation by π/q . Since this rotation is nonreal, its commutant consists exactly of complex scalars. Such tangent vectors vanish in $\mathbb{CP}^{m-1}$. The derivative of Ψ is therefore injective, and equality of dimensions proves the lemma. $\square$

Completion of the proof of Theorem 4.2. On this fixed locus the functional factors as

$$F_A = q \left(\sum_i \log(d_i/h) - \sum_i \log(e_i/h) \right).$$

By the local diffeomorphism lemma, its critical points correspond to critical points of this function on the product of the two open simplexes. That function has a unique critical point:

$$d_i/h = e_i/h = 1/m \quad \text{for every } i.$$

Indeed, the Lagrange-multiplier equations on each simplex make all reciprocal coordinates equal. At the displayed point, on tangent vectors ξ, η with $\sum_i \xi_i = \sum_i \eta_i = 0$, the Hessian is

$$qm^2 \left(-\sum_i \xi_i^2 + \sum_i \eta_i^2 \right).$$

It has $m-1$ negative and $m-1$ positive eigenvalues.

It remains to show that the uniform area values have only one preimage. The equalities $d_i = e_i$ first imply that every u_i equals one positive constant u . Equality of all the d_i then implies that every v_i equals one constant v . The recurrence reduces to

$$\gamma_{i+1} - (v/u)\gamma_i + \gamma_{i-1} = 0.$$

The operator J now has constant coefficients and commutes with the antiperiodic shift. Its real kernel has dimension exactly two, by the nonzero second-order recurrence, and is the positive plane spanned by the real and imaginary parts of γ . This plane is therefore shift invariant. The antiperiodic shift has simple complex spectrum; its invariant real two-planes are conjugate Fourier pairs. Of these, only the first pair has all ordered two-minors positive. To see this, orient a pair by a frequency $\phi \in (0, \pi)$, so its minors are positive multiples of $\sin(\ell\phi)$ for $1 \leq \ell < n$. The frequencies are odd multiples of π/n . If $\phi > \pi/n$, choose $\ell = \lceil \pi/\phi \rceil < n$; then $\sin(\ell\phi) \leq 0$, a contradiction. The plane is therefore exactly the first Fourier plane. That plane is admissible and fixed by the subgroup. It is consequently the unique restricted critical point, and its index is $(-1)^{m-1}$. For $m=1$ the same conclusion is immediate from the one-point fixed locus, with index one. $\square$

4.3. The index modulo the boundary size.

Theorem 4.5 (The total index modulo n). *For every $n \geq 2$,*

$$I_n \equiv (-1)^{n-1} \pmod{n}.$$

In particular I_n is nonzero for every n , and the stated matching hypothesis guarantees perfect proper boundary gauges for every positive weighting and every boundary size.

Proof. The case $n = 2$ follows from $I_2 = 1$. Assume $n \geq 3$ and use the uniform prism reference with a C_n -invariant metric. By Theorem 4.2, the restriction of its functional to the fixed locus of every nontrivial subgroup of C_n has precisely one critical point, the fundamental Fourier plane x_* , and that restricted critical point is nondegenerate. At a fixed point of a subgroup, the gradient of an invariant function belongs to the fixed tangent space. Thus criticality of the restriction is equivalent to ambient criticality there.

For an entirely local way to regularize x_* , use the Fourier chart

$$\beta_j = e^{i\theta j} + \sum_{k=1}^{n-2} h_k e^{i(2k+1)\theta j}, \quad \theta = \pi/n.$$

The cyclic action multiplies h_k by $e^{2\pi i k/n}$ (or its inverse, according to the chosen generator). By Lemma 4.1, its Hessian kernel is the complex coordinate $z = h_1$, while the transverse Hessian has $n - 3$ negative and $n - 3$ positive eigenvalues. Add $\epsilon \chi|z|^2$, where $\epsilon > 0$ is sufficiently small and χ is an invariant cutoff, equal to one near x_* and compactly supported in the chart inside Ω_A . The center becomes a nondegenerate critical point of index $(-1)^{n-3}$. Since the weight-one representation is faithful, $z = 0$ on the fixed locus of every nontrivial subgroup. The perturbation therefore leaves every restricted function unchanged, and their only critical point remains x_* .

All remaining critical points consequently lie in the free-action locus. The original critical set is compact by Corollary 3.6. The perturbation has compact support inside Ω_A , so every new critical point lies in that support or in the original critical set. After removing a small isolating neighborhood of x_* , the remaining critical set is therefore a compact subset of the free-action locus. Ordinary Morse approximation on its smooth quotient by C_n , lifted to the covering, gives an arbitrarily small invariant perturbation making all these critical points nondegenerate. This can be supported in the free locus and away from the isolating boundary; no new fixed critical points or boundary critical points are introduced. It preserves the total index. Every noncentral critical orbit has exactly n points, all with the same local index. Hence their total contribution is divisible by n , and

$$I_n \equiv (-1)^{n-3} = (-1)^{n-1} \pmod{n}.$$

□

4.4. Conclusion of the existence proof.

Proof of Theorem 1.1. For $n = 2$, the observation after Proposition 2.4 gives an admissible critical point directly. For $n \geq 3$, Theorem 3.9 identifies the total index for every boundary matrix under (4) with that of the uniform prism. Theorem 4.5 makes this integer nonzero. An empty critical set would have index zero, so there is a critical point in Ω_A . Proposition 2.4 and the reconstruction in Proposition 2.2 give β and $\alpha_i = D_i c_i$, with every required boundary inequality strict. This proves (1) for every positive weighting.

Corollary 2.9, using Proposition 2.6 on the boundary core, chooses such solutions so that the full realization is nonstrictly proper, proving (2) under the same hypotheses. If G is connected and matching-covered and satisfies (5), Proposition 2.7 gives (3). Thus no additional interior hypothesis or genericity is required for (1)–(2). □

Remark 4.6. The congruence suffices for existence. The stronger possible formula $I_n = (-1)^{n-2}(n-1)$ is not needed and is not proved here. The argument also permits several critical points, degenerate critical points, and full cyclic orbits away from the Fourier point of the uniform prism.

REFERENCES

- [Bas+25] Mikhail Basok, Dmitry Chelkak, Benoît Laslier, and Marianna Russkikh, *Harmonic functions on Tutte embeddings and linearized Monge-Ampère equation*, arXiv:2511.06587 [math-ph], 2025. <https://doi.org/10.48550/arXiv.2511.06587>.
- [BLR26] Mikhail Basok, Benoît Laslier, and Marianna Russkikh, *T-embeddings as stationary measures on T-graphs*, arXiv:2610.24180 [math-ph], 2026. <https://doi.org/10.48550/arXiv.2610.24180>.
- [BNR24a] Tomas Berggren, Matthew Nicoletti, and Marianna Russkikh, *Perfect t-embeddings and lozenge tilings*, arXiv:2408.05441 [math.PR], 2024. <https://doi.org/10.48550/arXiv.2408.05441>.
- [BNR24b] Tomas Berggren, Matthew Nicoletti, and Marianna Russkikh, *Perfect t-embeddings of uniformly weighted Aztec diamonds and tower graphs*, International Mathematics Research Notices **2024** (2024), no. 7, 5963–6007. <https://doi.org/10.1093/imrn/rnad299>.
- [BNR25] Tomas Berggren, Matthew Nicoletti, and Marianna Russkikh, *Perfect t-embeddings of doubly periodic Aztec diamonds*, arXiv:2508.04938 [math.PR], 2025. <https://doi.org/10.48550/arXiv.2508.04938>.
- [Che24] Dmitry Chelkak, *Ising model and s-embeddings of planar graphs*, Annales Scientifiques de l'École Normale Supérieure **57** (2024), no. 5, 1271–1346. <https://doi.org/10.24033/asens.2591>.
- [CLR21] Dmitry Chelkak, Benoît Laslier, and Marianna Russkikh, *Bipartite dimer model: perfect t-embeddings and Lorentz-minimal surfaces*, arXiv:2109.06272 [math.PR], 2021. <https://doi.org/10.48550/arXiv.2109.06272>.
- [CLR23] Dmitry Chelkak, Benoît Laslier, and Marianna Russkikh, *Dimer model and holomorphic functions on t-embeddings of planar graphs*, Proceedings of the London Mathematical Society **126** (2023), no. 5, 1656–1739. <https://doi.org/10.1112/plms.12516>.
- [CR24] Dmitry Chelkak and Sanjay Ramassamy, *Fluctuations in the Aztec diamonds via a space-like maximal surface in Minkowski 3-space*, Confluentes Mathematici **16** (2024), 1–17. <https://doi.org/10.5802/cml.94>.
- [Gal24] Pavel Galashin, *Amplituhedra and origami, I: tree level*, arXiv:2410.09574 [hep-th], 2024. <https://doi.org/10.48550/arXiv.2410.09574>.
- [Gal26] Pavel Galashin, *Amplituhedra and origami, II: loop level*, arXiv:2606.03439 [hep-th], 2026. <https://doi.org/10.48550/arXiv.2606.03439>.
- [Ken+22] Richard Kenyon, Wai Yeung Lam, Sanjay Ramassamy, and Marianna Russkikh, *Dimers and circle patterns*, Annales Scientifiques de l'École Normale Supérieure **55** (2022), no. 3, 865–903. <https://doi.org/10.24033/asens.2507>.
- [KS04] Richard W. Kenyon and Scott Sheffield, *Dimers, tilings and trees*, Journal of Combinatorial Theory, Series B **92** (2004), no. 2, 295–317. <https://doi.org/10.1016/j.jctb.2004.07.001>.
- [KV25] David Keating and Hieu Trung Vu, *Perfect t-Embeddings of Uniformly Weighted Generalized Tower Graphs*, arXiv:2509.18791 [math-ph], 2025. <https://doi.org/10.48550/arXiv.2509.18791>.