

Robin's Inequality and Superabundant Counterexamples

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Abstract

Robin's criterion states that the Riemann Hypothesis holds if and only if $\sigma(n) < e^\gamma \cdot n \cdot \log \log n$ for every $n > 5040$. Let n' be the squarefree kernel of n , p_k its largest prime factor, and $d = e^\gamma / \zeta(2) \approx 1.08276$. We prove that Robin's inequality holds for every $n > 5040$ with $d \cdot \log \log n' \leq \log \log n$, and for every $n > 5040$ whose prime factors are all below $(\log n)^{1/d}$, and that both results remain valid with d replaced by 1.016. Our main results concern superabundant numbers, among which there are infinitely many counterexamples to Robin's inequality if the Riemann Hypothesis is false. Using the classical error term in the prime number theorem, we show that every superabundant counterexample satisfies $\log \log n - \log p_k < C \cdot \exp(-c \sqrt{\log p_k})$ for absolute constants $C, c > 0$. Since the least counterexample satisfies $\log n > p_k$, we deduce that the Riemann Hypothesis is true if there are constants $c > 0$ and $0 < \alpha < 2$ with $c \geq 0.0562/31.03^{2-\alpha}$, and a threshold $N \leq 10^{10^{13.11485}}$, such that $\log \log n - \log p_k \geq c / \log^\alpha(p_k)$ for every superabundant number $n > N$ with $\log n > p_k$. We also give explicit versions of these bounds: every counterexample satisfies $p_k^{1.000002} > \log n$, and every superabundant counterexample satisfies $-0.010101/(\log \log n)^3 < \log \log n - \log p_k < 0.056201/(\log \log n)^2$. An enumeration of the superabundant numbers up to 10^{40} shows that only 3 of the 244 exceeding 5040 satisfy $e^{0.0562} > 1 + (\log \log n - \log p_k) \cdot \log^2(p_k)$, namely those with $p_k > \log n$, the largest of which is 10685862914126400. Testing the hypothesis of the sufficient condition on the same range, we find that all 168 superabundant numbers between 10685862914126400 and 10^{40} satisfy $\log \log n - \log p_k > 0$, and that for each $\alpha \in \{1, 2, 3, 4, 5\}$ the minimum of $(\log \log n - \log p_k) \cdot \log^\alpha(p_k)$ over them is positive, ranging from 0.0223 for $\alpha = 1$ to 6.19 for $\alpha = 5$, and is attained at the same superabundant number, whose largest prime factor is 59.

Keywords: Riemann Hypothesis, Robin's inequality, Sum-of-divisors function, Superabundant numbers, Squarefree kernel, Prime numbers, Chebyshev function

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1. Introduction

The Riemann Hypothesis (RH) is the conjecture that the Riemann zeta function has its zeros only at the negative even integers and at complex numbers with real part $\frac{1}{2}$ [1]. It has been called the "Holy Grail of Mathematics" [2, 3], and numerous equivalent formulations exist [4]. One of particular interest here is Robin's criterion [5], which asserts that RH is equivalent to the inequality

$$\sigma(n) < e^\gamma \cdot n \cdot \log \log n \quad \text{for every integer } n > 5040. \quad (1)$$

Here $\gamma \approx 0.57721$ is the Euler–Mascheroni constant, $\sigma(n) = \sum_{d|n} d$ is the sum-of-divisors function, and $\log$ denotes the natural logarithm. Grönwall's theorem [6] states that $\limsup_{n \rightarrow \infty} \sigma(n)/(n \log \log n) = e^\gamma$, so (1) asserts that this limiting value is never attained beyond 5040. A closely related elementary reformulation in terms of harmonic numbers is due to Lagarias [7], and the extremal integers relevant to both criteria have been studied computationally by Briggs [8] and structurally by Caveney, Nicolas and Sondow [9].

It is known that (1) holds for many classes of integers: odd integers exceeding 9 and squarefree integers exceeding 30 [10], sums of two squares exceeding 720 [11], and t -free integers for $t = 5, 7, 11$ and 20 [10, 12, 13, 14]. In a

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complementary direction, Akbary and Friggstad [15] showed that the least counterexample to (1), should one exist, must be a superabundant number, and Nazardonyavi and Yakubovich [16] refined this to the smaller class of extremely abundant numbers. The author has also contributed to this line of results [17].

Let $\text{core}(n)$ denote the squarefree kernel of n , that is, the product of the distinct primes dividing n [10], and write $n' = \text{core}(n)$. In an earlier version of this work [18] the author showed that (1) holds for every $n > 5040$ with $\zeta(2) \cdot \log \log n' \leq \log \log n$, and for every $n > 5040$ all of whose prime factors q satisfy $q < \sqrt[\beta]{\log n}$, where $\beta = e^\gamma$. The purpose of the present paper is to replace both constants $\zeta(2) \approx 1.64493$ and $e^\gamma \approx 1.78107$ by the single smaller constant

$$d = \frac{e^\gamma}{\zeta(2)} = \frac{6 \cdot e^\gamma}{\pi^2} \approx 1.08276.$$

Our main results are the following.

Theorem 1.1. *Let $n > 5040$ and $n' = \text{core}(n)$. If $d \cdot \log \log n' \leq \log \log n$, then Robin's inequality holds for n .*

Theorem 1.2. *Let $n > 5040$. If every prime q that divides n satisfies $q < \sqrt[\beta]{\log n}$, then Robin's inequality holds for n .*

1.1. Summary of the Main Results

Let p_k denote the largest prime factor of n , and call n a *counterexample* if $n > 5040$ and (1) fails for n . Besides Theorems 1.1 and 1.2, the main results of this paper are the following.

- (1) Theorems 1.1 and 1.2 remain valid with d replaced by 1.016 (Theorems 3.4 and 3.5).
- (2) Every sufficiently large superabundant number n satisfies $p_k > \sqrt[\beta]{\log n}$ and $\theta(p_k) > \sqrt[\beta]{\log n}$ (Theorem 4.4), and the least counterexample satisfies $\sqrt[\beta]{\log n} \leq p_k < \log n$ (Corollary 4.2).
- (3) If the Riemann Hypothesis is false, there are infinitely many superabundant counterexamples (Proposition 5.3).
- (4) Every counterexample satisfies $p_k > e^{30.54}$ (Proposition 5.1) and $p_k^{1.000002} > \log n$ (Proposition 5.2).
- (5) Every superabundant counterexample satisfies

$$-\frac{0.0101}{\log^3(p_k)} < \log \log n - \log p_k < \frac{0.0562}{\log^2(p_k)},$$

$$e^{0.0562} > 1 + (\log \log n - \log p_k) \cdot \log^2(p_k)$$

(Theorem 5.6), and, in terms of $\log \log n$ alone,

$$-\frac{0.010101}{(\log \log n)^3} < \log \log n - \log p_k < \frac{0.056201}{(\log \log n)^2},$$

$$(\log n)^{\log \log n - \log p_k} < e^{\frac{0.056201}{\log \log n}}$$

(Corollary 5.7). Moreover, $\log \log n - \log p_k < O\left(\exp\left(-c_2 \sqrt{\log p_k}\right)\right)$ for an absolute constant $c_2 > 0$ (Corollary 5.10). If there are constants $c > 0$ and $0 < \alpha < 2$ with $c \geq 0.0562/31.03^{2-\alpha}$, and $N \leq 10^{10^{13.11485}}$, such that every superabundant number $n > N$ with $\log n > p_k$ satisfies $\log \log n - \log p_k \geq \frac{c}{\log^\alpha(p_k)}$, then the Riemann Hypothesis is true (Proposition 5.15).

- (6) The least counterexample satisfies $\log p_k < \log \log n$ (Theorem 5.11), $1 < (\log n)^{\log \log n - \log p_k} < e^{\frac{0.056201}{\log \log n}}$ (Corollary 5.7), and $\log n \cdot \log \log \frac{n}{p_k} > p_k \cdot \log p_k$ (Theorem 5.13).
- (7) Among the 244 superabundant numbers $5040 < n \leq 10^{40}$, exactly 3 satisfy $e^{0.0562} > 1 + (\log \log n - \log p_k) \cdot \log^2(p_k)$, namely the 3 with $p_k > \log n$ (Proposition 6.4).

The proofs combine explicit estimates of Rosser and Schoenfeld [19], of Axler [20] and of Aoudjit, Berkane and Dusart [21] with the numerical verification of (1) for all $5040 < n \leq 10^{10^{13.11485}}$ due to Morrill and Platt [14]. Section 4 treats superabundant numbers, Section 5 the largest prime factor of a counterexample, and Section 6 the computer verification of every numerical claim, including a rigorous enumeration of the 263 superabundant numbers up to 10^{40} .

2. Background and Ancillary Results

We define the *abundancy index* $I: \mathbb{N} \rightarrow \mathbb{Q}$ by $I(n) = \frac{\sigma(n)}{n}$, and we say that *Robin's inequality holds for n* if

$$I(n) < e^\gamma \cdot \log \log n.$$

Proposition 2.1. *Robin's inequality holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [5] [Theorem 1 pp. 188].*

As usual, φ denotes Euler's totient function. For every $n > 1$ we have $\frac{n'}{\varphi(n')} = \prod_{q|n} \frac{q}{q-1}$, where the product runs over the distinct primes dividing n .

Proposition 2.2. *For $n > 1$ [10] [(2.7) pp. 362]:*

$$I(n) < \prod_{q|n} \frac{q}{q-1} = \frac{n'}{\varphi(n')}.$$

The next two propositions are explicit estimates of Rosser and Schoenfeld.

Proposition 2.3. *For $x \geq 285$ [19] [Theorem 7], see also [20] [(6.1)]:*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log x} \cdot \left(1 - \frac{1}{2 \log^2(x)}\right).$$

Equivalently, for $x \geq 285$,

$$\prod_{p \leq x} \frac{p}{p-1} < e^\gamma \cdot \log x \cdot \left(1 - \frac{1}{2 \log^2(x)}\right)^{-1}.$$

Proposition 2.4. *For every integer $m \geq 3$ [19] [Theorem 15]:*

$$\frac{m}{\varphi(m)} < e^\gamma \cdot \log \log m + \frac{2.51}{\log \log m}.$$

Rosser and Schoenfeld in fact prove Proposition 2.4 with 2.50637 in place of 2.51 for every $m \geq 3$ except $m = 223092870$; the version with 2.51 holds without exception and is sufficient for our purposes.

Proposition 2.5. *Robin's inequality holds for every integer n with $5040 < n \leq 10^{10^{13.11485}}$ [14] [Theorem 5].*

The Chebyshev function $\theta(x)$ is defined by $\theta(x) = \sum_{p \leq x} \log p$, where the sum runs over all primes $p \leq x$. For a prime p , we write $p\# = \prod_{q \leq p} q$ for the primorial of p , so that $\log(p\#) = \theta(p)$. Throughout, $p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots$ denotes the sequence of all primes.

Ramanujan's unpublished notes [22] introduced generalized highly composite numbers, encompassing both superabundant and colossally abundant numbers; these were also studied by Alaoglu and Erdős [23]. Given the first k consecutive primes $p_1, \dots, p_k$, an integer of the form $\prod_{i=1}^k p_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_k \geq 1$ is called a *Hardy–Ramanujan integer* [10] [pp. 367]. A natural number n is called *superabundant* if $I(m) < I(n)$ for every natural number $m < n$.

Proposition 2.6. *Every superabundant number is a Hardy–Ramanujan integer [23] [Theorem 1 pp. 450].*

Proposition 2.7. *If $n > 5040$ is the smallest integer such that Robin's inequality does not hold for n , then n is superabundant [15] [Theorem 3, p. 273].*

Proposition 2.8. *If $n > 5040$ is the smallest integer such that Robin's inequality does not hold for n , and p denotes the largest prime factor of n , then $p < \log n$ [10] [Lemma 6.1, p. 369].*

The following explicit estimates of Axler and of Aoudjit, Berkane and Dusart, and the author's lower bound for the largest prime factor of the least counterexample are used in Section 5.

Proposition 2.9. For $x \geq 46,909,038$ [20] [Proposition 6.1]:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log x} \cdot \left(1 - \frac{1}{20 \log^3(x)} - \frac{3}{16 \log^4(x)}\right).$$

Equivalently, for $x \geq 46,909,038$,

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \cdot \log x \cdot \left(1 - \frac{1}{20 \log^3(x)} - \frac{3}{16 \log^4(x)}\right)^{-1}.$$

Proposition 2.10. For $x \geq 7,232,121,212$ [21]:

$$\theta(x) \geq \left(1 - \frac{0.01}{\log^3(x)}\right) \cdot x.$$

Proposition 2.11. If $n > 5040$ is the smallest integer for which Robin's inequality fails, then the largest prime factor p of n satisfies $p > e^{31.018189471}$ [17] [Theorem 4.2 pp. 748].

2.1. Numerical Constants

We use $\zeta(2) = \frac{\pi^2}{6} = 1.6449340668 \dots$ [24], $\gamma = 0.5772156649 \dots$ and $e^{\gamma} = 1.7810724179 \dots$. The following numerical facts are elementary; they were checked with high-precision arithmetic, and the computed values are listed in Table 1 of Section 6.

Lemma 2.12. Let $d = \frac{e^{\gamma}}{\zeta(2)}$. Then:

- (i) $d = 1.0827621 \dots$, so that $1.0827 < d < 1.0828$;
- (ii) $e^{\gamma} \cdot (d - 1) = \frac{e^{2\gamma}}{\zeta(2)} - e^{\gamma} = 0.1474054 \dots > 0.1474$;
- (iii) $e^{\gamma} \cdot (d - 1) \cdot \log^2(64) > 0.1474 \cdot 17.296 > 2.549 > 2.51$;
- (iv) $\theta(73) = \log(73\#) = 65.876 \dots > 64$;
- (v) $\left(1 - \frac{1}{2 \log^2(285)}\right)^{-1} = 1.01589 \dots < 1.016 < d$;
- (vi) $e^{\gamma} \cdot \log 285 \cdot \left(1 - \frac{1}{2 \log^2(285)}\right)^{-1} = 10.2275 \dots < 10.23$;
- (vii) $e^{\gamma} \cdot \log 313 = 10.2344 \dots > 10.23$ and $e^{313} < 10^{136}$.

2.2. Two Auxiliary Lemmas

Lemma 2.13. Let $x \geq 2$. Then:

- (a) if $x \leq 285$, then $\prod_{p \leq x} \frac{p}{p-1} < 10.23$;
- (b) if $x \geq 285$, then $\prod_{p \leq x} \frac{p}{p-1} < 1.016 \cdot e^{\gamma} \cdot \log x$.

Proof. Every factor $\frac{p}{p-1}$ exceeds 1, so the product is nondecreasing in x . For (a), Proposition 2.3 at $x = 285$ and Lemma 2.12 (vi) give

$$\prod_{p \leq x} \frac{p}{p-1} \leq \prod_{p \leq 285} \frac{p}{p-1} < e^{\gamma} \cdot \log 285 \cdot \left(1 - \frac{1}{2 \log^2(285)}\right)^{-1} < 10.23.$$

For (b), the function $x \mapsto \left(1 - \frac{1}{2 \log^2(x)}\right)^{-1}$ is decreasing on $[285, \infty)$, so by Proposition 2.3 and Lemma 2.12 (v),

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \cdot \log x \cdot \left(1 - \frac{1}{2 \log^2(285)}\right)^{-1} < 1.016 \cdot e^{\gamma} \cdot \log x.$$

□

Lemma 2.14. *Let $n > 5040$ satisfy $I(n) < 10.23$. Then Robin's inequality holds for n .*

Proof. If $\log n \geq 313$, then by Lemma 2.12 (vii),

$$e^\gamma \cdot \log \log n \geq e^\gamma \cdot \log 313 > 10.23 > I(n).$$

If $\log n < 313$, then $n < e^{313} < 10^{136} < 10^{10^{13.11485}}$ by Lemma 2.12 (vii), and Robin's inequality holds for n by Proposition 2.5. $\square$

3. Main Results

3.1. A Condition on the Squarefree Kernel

Proof of Theorem 1.1. Let $n > 5040$ satisfy $d \cdot \log \log n' \leq \log \log n$, where $n' = \text{core}(n)$. By Proposition 2.2,

$$I(n) < \frac{n'}{\varphi(n')}. \quad (2)$$

There are two cases.

Case I: $\log n' \geq 64$. Put $L' = \log \log n'$, so that $L' \geq \log 64 > 0$, and in particular $n' \geq 3$. By Lemma 2.12 (iii),

$$e^\gamma \cdot (d-1) \cdot L'^2 \geq e^\gamma \cdot (d-1) \cdot \log^2(64) > 2.51,$$

that is, $\frac{2.51}{L'} < e^\gamma \cdot (d-1) \cdot L'$. By Proposition 2.4 applied to $m = n'$, and by the hypothesis $d \cdot L' \leq \log \log n$,

$$\frac{n'}{\varphi(n')} < e^\gamma \cdot L' + \frac{2.51}{L'} < e^\gamma \cdot L' + e^\gamma \cdot (d-1) \cdot L' = e^\gamma \cdot d \cdot L' \leq e^\gamma \cdot \log \log n.$$

Together with (2) this gives $I(n) < e^\gamma \cdot \log \log n$.

Case II: $\log n' < 64$. Let $q_1 < \dots < q_r$ be the distinct primes dividing n , so that $r \geq 1$ and $n' = q_1 \cdots q_r$. Since $q_i \geq p_i$ for every i , we have $n' \geq p_1 \cdots p_r = p_r\#$, hence $\theta(p_r) \leq \log n' < 64 < \theta(73)$ by Lemma 2.12 (iv), and so $p_r \leq 71$. The function $t \mapsto \frac{t}{t-1}$ is decreasing on $(1, \infty)$, so $\frac{q_i}{q_i-1} \leq \frac{p_i}{p_i-1}$ for every i . By Lemma 2.13 (a),

$$\frac{n'}{\varphi(n')} = \prod_{i=1}^r \frac{q_i}{q_i-1} \leq \prod_{i=1}^r \frac{p_i}{p_i-1} \leq \prod_{p \leq 71} \frac{p}{p-1} < 10.23.$$

By (2), $I(n) < 10.23$, and Robin's inequality holds for n by Lemma 2.14. $\square$

Remark 3.1. Case II of the proof does not use the hypothesis of Theorem 1.1: Robin's inequality holds for every $n > 5040$ whose squarefree kernel satisfies $\text{core}(n) < e^{64}$. The hypothesis $d \cdot \log \log n' \leq \log \log n$ is needed only in Case I.

3.2. A Condition on the Prime Factors

Proof of Theorem 1.2. Let $n > 5040$, and suppose that every prime q dividing n satisfies $q < \sqrt[d]{\log n}$. Let P be the largest prime factor of n . Then $P^d < \log n$, that is,

$$d \cdot \log P < \log \log n. \quad (3)$$

Since every prime dividing n is at most P and every factor $\frac{p}{p-1}$ exceeds 1, Proposition 2.2 gives

$$I(n) < \prod_{q|n} \frac{q}{q-1} \leq \prod_{p \leq P} \frac{p}{p-1}.$$

There are two cases.

Case I: $P \geq 285$. By Lemma 2.13 (b), Lemma 2.12 (v) and (3),

$$I(n) < 1.016 \cdot e^\gamma \cdot \log P < e^\gamma \cdot d \cdot \log P < e^\gamma \cdot \log \log n.$$

Case II: $P < 285$. By Lemma 2.13 (a), $I(n) < 10.23$, and Robin's inequality holds for n by Lemma 2.14. $\square$

3.3. Comparison with the Earlier Constants

Since $d < \zeta(2)$, the hypothesis $\zeta(2) \cdot \log \log n' \leq \log \log n$ implies $d \cdot \log \log n' \leq \log \log n$ whenever $\log \log n' \geq 0$; and when $\log \log n' < 0$ we have $d \cdot \log \log n' < 0 < \log \log n$ for $n > 5040$, so the hypothesis of Theorem 1.1 holds in any case. Likewise, since $d < e^\gamma$ and $\log n > 1$, the condition $q < \sqrt[\gamma]{\log n}$ implies $q < \sqrt[d]{\log n}$. Hence Theorems 1.1 and 1.2 contain the two results of the earlier version [18].

Corollary 3.2. *Let $n > 5040$ and $n' = \text{core}(n)$. If $\zeta(2) \cdot \log \log n' \leq \log \log n$, or if every prime q dividing n satisfies $q < \sqrt[\gamma]{\log n}$, then Robin's inequality holds for n .*

3.4. The Exponent 1.016

The proofs of Theorems 1.1 and 1.2 extend to the smaller exponent 1.016.

Lemma 3.3. *Let $n > 5040$ satisfy $I(n) < 17.3$. Then Robin's inequality holds for n .*

Proof. If $\log n \geq 16600$, then $e^\gamma \cdot \log \log n \geq e^\gamma \cdot \log 16600 > 17.3 > I(n)$. If $\log n < 16600$, then $n < e^{16600} < 10^{7210} < 10^{10^{13.11485}}$, and Robin's inequality holds for n by Proposition 2.5. $\square$

Theorem 3.4. *Let $n > 5040$ and $n' = \text{core}(n)$. If $1.016 \cdot \log \log n' \leq \log \log n$, then Robin's inequality holds for n .*

Proof. By Proposition 2.2, $I(n) < \frac{n'}{\varphi(n')}$.

Case I: $\log n' \geq 13000$. Put $L' = \log \log n' \geq \log 13000 > 0$, so that $n' \geq 3$. Since $e^\gamma \cdot 0.016 \cdot \log^2(13000) > 2.51$, we have $\frac{2.51}{L'} < 0.016 \cdot e^\gamma \cdot L'$, and Proposition 2.4 gives

$$\frac{n'}{\varphi(n')} < e^\gamma \cdot L' + \frac{2.51}{L'} < 1.016 \cdot e^\gamma \cdot L' \leq e^\gamma \cdot \log \log n.$$

Case II: $\log n' < 13000$. Let $q_1 < \dots < q_r$ be the distinct primes dividing n . As in the proof of Theorem 1.1, $n' \geq p_r\#$, so $\theta(p_r) \leq \log n' < 13000 < \theta(14000)$, hence $p_r < 14000$, and by Lemma 2.13 (b),

$$\frac{n'}{\varphi(n')} \leq \prod_{i=1}^r \frac{p_i}{p_i - 1} \leq \prod_{p \leq 14000} \frac{p}{p - 1} < 1.016 \cdot e^\gamma \cdot \log 14000 < 17.3.$$

Thus $I(n) < 17.3$, and Robin's inequality holds for n by Lemma 3.3. $\square$

Theorem 3.5. *Let $n > 5040$. If every prime q that divides n satisfies $q < \sqrt[1.016]{\log n}$, then Robin's inequality holds for n .*

Proof. Let P be the largest prime factor of n , so that $1.016 \cdot \log P < \log \log n$, and $I(n) < \prod_{p \leq P} \frac{p}{p-1}$ by Proposition 2.2. If $P \geq 285$, Lemma 2.13 (b) gives $I(n) < 1.016 \cdot e^\gamma \cdot \log P < e^\gamma \cdot \log \log n$. If $P < 285$, Lemma 2.13 (a) gives $I(n) < 10.23$, and Robin's inequality holds for n by Lemma 2.14. $\square$

Corollary 3.6. *Theorems 3.4 and 3.5 imply Theorems 1.1 and 1.2.*

Proof. Since $d > 1.016$ by Lemma 2.12 (v), the hypothesis $d \cdot \log \log n' \leq \log \log n$ implies $1.016 \cdot \log \log n' \leq \log \log n$ when $\log \log n' \geq 0$, and when $\log \log n' < 0$ we have $1.016 \cdot \log \log n' < 0 < \log \log n$ for $n > 5040$. Likewise, $q < \sqrt[d]{\log n}$ implies $q < \sqrt[1.016]{\log n}$, because $\log n > 1$. $\square$

4. Superabundant Numbers

Let n be a superabundant number with largest prime factor p_k . By Proposition 2.6, $n = \prod_{i=1}^k p_i^{a_i}$ with $a_1 \geq \dots \geq a_k \geq 1$, so the primes dividing n are exactly the primes $p \leq p_k$. Therefore

$$\text{core}(n) = p_k\# \quad \text{and} \quad \log \text{core}(n) = \theta(p_k).$$

Theorems 1.1 and 1.2 thus take the following form.

Corollary 4.1. *Let $n > 5040$ be a superabundant number, and let p_k be its largest prime factor. Then Robin's inequality holds for n in each of the following cases:*

$$(a) \quad d \cdot \log \theta(p_k) \leq \log \log n;$$

$$(b) \quad p_k < \sqrt[d]{\log n}.$$

Proof. Part (a) is Theorem 1.1 with $\log \log n' = \log \theta(p_k)$. Part (b) is Theorem 1.2, since every prime dividing n is at most p_k . $\square$

Applying Corollary 4.1 to a hypothetical least counterexample gives the following.

Corollary 4.2. *Suppose that the Riemann Hypothesis is false. Let $n > 5040$ be the least integer for which Robin's inequality does not hold, and let p_k be its largest prime factor. Then*

$$\sqrt[d]{\log n} \leq p_k < \log n \quad \text{and} \quad \theta(p_k) > \sqrt[d]{\log n}.$$

Proof. By Propositions 2.1 and 2.7, such an n exists and is superabundant. Since Robin's inequality fails for n , both conditions of Corollary 4.1 fail. The failure of (b) gives $p_k \geq \sqrt[d]{\log n}$, and the failure of (a) gives $d \cdot \log \theta(p_k) > \log \log n$, that is, $\theta(p_k) > \sqrt[d]{\log n}$. Finally, $p_k < \log n$ by Proposition 2.8. $\square$

Proposition 4.3. *Let n run through the superabundant numbers, and let p_k denote the largest prime factor of n . Then $p_k \sim \log n$ [23]. Moreover, $\theta(x) \sim x$ as $x \rightarrow \infty$ [24].*

Theorem 4.4. *Every sufficiently large superabundant number n , with largest prime factor p_k , satisfies*

$$p_k > \sqrt[d]{\log n} \quad \text{and} \quad \theta(p_k) > \sqrt[d]{\log n}.$$

Proof. There are infinitely many superabundant numbers [23], so $\log n \rightarrow \infty$ and, by Proposition 4.3, $p_k \rightarrow \infty$ and $\theta(p_k) \sim p_k \sim \log n$. Write $p_k = \log n \cdot (1 + \delta_n)$ and $\theta(p_k) = \log n \cdot (1 + \varepsilon_n)$ with $\delta_n \rightarrow 0$ and $\varepsilon_n \rightarrow 0$. Since $d > 1$,

$$\frac{p_k^d}{\log n} = (1 + \delta_n)^d \cdot (\log n)^{d-1} \rightarrow \infty \quad \text{and} \quad \frac{\theta(p_k)^d}{\log n} = (1 + \varepsilon_n)^d \cdot (\log n)^{d-1} \rightarrow \infty,$$

so $p_k^d > \log n$ and $\theta(p_k)^d > \log n$ for every sufficiently large superabundant number n . $\square$

5. The Largest Prime Factor of a Counterexample

In this section we say that n is a *counterexample* if $n > 5040$ and Robin's inequality does not hold for n , that is, $I(n) \geq e^\gamma \cdot \log \log n$. For an integer n with largest prime factor p_k we write

$$L = \log \log n, \quad P = \log p_k, \quad x = L - P = \log \log n - \log p_k.$$

5.1. Every Counterexample Has a Large Prime Factor

Proposition 5.1. *Let n be a counterexample with largest prime factor p_k . Then*

$$p_k^{1.016} > \log n \quad \text{and} \quad \log p_k > 30.54.$$

In particular $p_k > 7,232,121,212 > 46,909,038$.

Proof. By Proposition 2.5, $n > 10^{10^{13.11485}}$, so $L > \log \log (10^{10^{13.11485}}) = 13.11485 \cdot \log 10 + \log \log 10 > 31.032$. If $p_k < 285$, then $I(n) < 10.23$ by Proposition 2.2 and Lemma 2.13 (a), and Robin's inequality would hold for n by Lemma 2.14, a contradiction. Hence $p_k \geq 285$, and by Proposition 2.2 and Lemma 2.13 (b),

$$e^\gamma \cdot L \leq I(n) < \prod_{p \leq p_k} \frac{p}{p-1} < 1.016 \cdot e^\gamma \cdot P.$$

Thus $1.016 \cdot P > L$, which is the first claim, and $P > \frac{31.032}{1.016} > 30.54$. Finally $e^{30.54} > 7,232,121,212$. $\square$

Proposition 5.1 applies to every counterexample, whereas Proposition 2.11 gives the slightly stronger bound $\log p_k > 31.018$ for the least one.

For $P > 0$ we write

$$\varepsilon(P) = \frac{1}{20P^3} + \frac{3}{16P^4},$$

which is decreasing in P .

Proposition 5.2. *Let n be a counterexample with largest prime factor p_k . Then*

$$p_k^{1.000002} > \log n \quad \text{and} \quad \log p_k > 31.0319.$$

Proof. Put $P = \log p_k$ and $L = \log \log n$. By Proposition 5.1, $P > 30.54$, so $p_k > 46,909,038$ and $(1 - \varepsilon(P))^{-1} \leq (1 - \varepsilon(30.54))^{-1} < 1.000002$. By Propositions 2.2 and 2.9, and since Robin's inequality fails for n ,

$$e^\gamma \cdot L \leq I(n) < \prod_{p \leq p_k} \frac{p}{p-1} < e^\gamma \cdot P \cdot (1 - \varepsilon(P))^{-1} < 1.000002 \cdot e^\gamma \cdot P.$$

Hence $L < 1.000002 \cdot P$, which is $p_k^{1.000002} > \log n$. Finally, $L > 31.032$ by Proposition 2.5, so $P > \frac{31.032}{1.000002} > 31.0319$. $\square$

5.2. Infinitely Many Superabundant Counterexamples

A natural number n is called *colossally abundant* if there exists $\varepsilon > 0$ such that $\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(m)}{m^{1+\varepsilon}}$ for every natural number $m > 1$ [23].

Proposition 5.3. *If the Riemann Hypothesis is false, then Robin's inequality fails for infinitely many colossally abundant numbers [5] [Section 4, Proposition 1]. Since every colossally abundant number is superabundant [23], there are infinitely many superabundant counterexamples.*

5.3. The Window for Superabundant Counterexamples

Theorem 5.4. *Let n be a superabundant counterexample with largest prime factor p_k . Then*

$$\left(1 - \frac{1}{20 \log^3(p_k)} - \frac{3}{16 \log^4(p_k)}\right)^{-1} > \frac{\log \log n}{\log p_k}.$$

Proof. By Proposition 5.1, $p_k > 46,909,038$. By Proposition 2.6, the prime factors of n are exactly the primes $p \leq p_k$. By Propositions 2.2 and 2.9, and since Robin's inequality fails for n ,

$$e^\gamma \cdot (\log p_k) \cdot \left(1 - \frac{1}{20 \log^3(p_k)} - \frac{3}{16 \log^4(p_k)}\right)^{-1} > \prod_{p \leq p_k} \frac{p}{p-1} > I(n) \geq e^\gamma \cdot \log \log n.$$

Dividing by $e^\gamma \cdot \log p_k > 0$ gives the claim. $\square$

The next result holds for every superabundant number with a sufficiently large prime factor, whether or not Robin's inequality fails.

Theorem 5.5. *Let n be a superabundant number, and let p_k be the largest prime factor of n . Assume $p_k \geq 7,232,121,212$. Then*

$$\log \log n > \log p_k - \frac{0.0101}{\log^3(p_k)}.$$

Proof. By Proposition 2.6, every prime $p \leq p_k$ divides n , so $n \geq p_k\#$ and $\log n \geq \theta(p_k)$. By Proposition 2.10,

$$\log \log n \geq \log \theta(p_k) \geq \log p_k + \log \left(1 - \frac{0.01}{\log^3(p_k)}\right).$$

Put $y = \frac{0.01}{\log^3(p_k)}$. Since $\log p_k \geq \log 7,232,121,212 > 22.7$, we have $0 < y < 10^{-6}$. For such y ,

$$-\log(1 - y) \leq \frac{y}{1 - y} < 1.01 \cdot y,$$

where the first inequality is $\log(1 + u) \geq \frac{u}{1+u}$ applied with $u = -y$. Therefore

$$\log \log n > \log p_k - 1.01 \cdot y = \log p_k - \frac{0.0101}{\log^3(p_k)},$$

as claimed. $\square$

Theorem 5.6. *Let n be a superabundant counterexample with largest prime factor p_k , and put $x = \log \log n - \log p_k$. Then*

$$-\frac{0.0101}{\log^3(p_k)} < x < \frac{0.0562}{\log^2(p_k)} < 0.0000603,$$

and consequently

$$e^{0.0562} > 1 + x \cdot \log^2(p_k).$$

Proof. By Proposition 5.1, $P = \log p_k > 30.54$ and $p_k > 7,232,121,212$. Write $\varepsilon = \varepsilon(P)$. Multiplying the inequality of Theorem 5.4 by $P > 0$ gives $L < \frac{P}{1-\varepsilon}$, that is,

$$x < \frac{P \cdot \varepsilon}{1 - \varepsilon} = \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16P}}{1 - \varepsilon(P)} \leq \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16 \cdot 30.54}}{1 - \varepsilon(30.54)} < \frac{0.0562}{P^2} < \frac{0.0562}{30.54^2} < 0.0000603.$$

The lower bound is Theorem 5.5. Finally, $x \cdot P^2 < 0.0562$, and since $e^y > 1 + y$ for every $y > 0$, we have $1 + x \cdot P^2 < 1 + 0.0562 < e^{0.0562}$. $\square$

Since every counterexample satisfies $p_k^{1.000002} > \log n$, the window of Theorem 5.6 can be expressed in terms of $\log \log n$ alone.

Corollary 5.7. *Let n be a superabundant counterexample with largest prime factor p_k , and put $x = \log \log n - \log p_k$. Then*

$$-\frac{0.010101}{(\log \log n)^3} < x < \frac{0.056201}{(\log \log n)^2},$$

and consequently

$$e^{0.056201} > 1 + x \cdot (\log \log n)^2 \quad \text{and} \quad (\log n)^x < e^{\frac{0.056201}{\log \log n}}.$$

If n is the least counterexample, then moreover

$$1 < (\log n)^{\log \log n - \log p_k} < e^{\frac{0.056201}{\log \log n}}.$$

Proof. Put $L = \log \log n$. By Proposition 5.2, $\log p_k > \frac{L}{1.000002} > 0$, hence $\frac{1}{\log^2(p_k)} < \frac{1.000002^2}{L^2}$ and $\frac{1}{\log^3(p_k)} < \frac{1.000002^3}{L^3}$. Since $0.0562 \cdot 1.000002^2 < 0.056201$ and $0.0101 \cdot 1.000002^3 < 0.010101$, Theorem 5.6 gives

$$-\frac{0.010101}{L^3} < -\frac{0.0101}{\log^3(p_k)} < x < \frac{0.0562}{\log^2(p_k)} < \frac{0.056201}{L^2}.$$

Multiplying the upper bound by $L^2 > 0$ gives $x \cdot L^2 < 0.056201$, and since $e^y > 1 + y$ for every $y > 0$, we obtain $1 + x \cdot L^2 < 1 + 0.056201 < e^{0.056201}$. Since $\log n > 1$, the function $t \mapsto (\log n)^t$ is increasing, so $(\log n)^x < (\log n)^{\frac{0.056201}{L^2}}$; and since $(\log n)^{\frac{1}{L}} = e$, we have $(\log n)^{\frac{0.056201}{L^2}} = e^{\frac{0.056201}{L}}$. If n is the least counterexample, then $x > 0$ by Theorem 5.11, so $(\log n)^x > 1$. $\square$

5.4. The Window under the Error Term of de la Vallée Poussin

The explicit results above use Proposition 2.9. Without explicit constants, the error term in the prime number theorem due to de la Vallée Poussin gives a window that shrinks faster than any power of $\frac{1}{\log p_k}$.

Proposition 5.8. *There is an absolute constant $c_0 > 0$ such that [25]*

$$\theta(x) = x + O\left(x \cdot \exp\left(-c_0 \sqrt{\log x}\right)\right).$$

Lemma 5.9. *There is an absolute constant $c_1 > 0$ such that*

$$\prod_{p \leq x} \frac{p}{p-1} = e^\gamma \cdot \log x \cdot \left(1 + O\left(\exp\left(-c_1 \sqrt{\log x}\right)\right)\right).$$

Proof. Write $\theta(t) = t + E(t)$, so that $|E(t)| \leq C \cdot t \cdot \exp(-c_0 \sqrt{\log t})$ for $t \geq 2$ and some constant C , by Proposition 5.8. By partial summation,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{\log p}{p} \cdot \frac{1}{\log p} = \frac{\theta(x)}{x \log x} + \int_2^x \theta(t) \cdot \frac{1 + \log t}{t^2 \log^2 t} dt.$$

Since $\frac{d}{dt} \left(\log \log t - \frac{1}{\log t} \right) = \frac{1 + \log t}{t \log^2 t}$, the contribution of the main term t of $\theta(t)$ equals $\frac{1}{\log x} + \log \log x - \frac{1}{\log x} - \log \log 2 + \frac{1}{\log 2}$. The integral of the error term converges absolutely on $[2, \infty)$, and its tail satisfies

$$\begin{aligned} \int_x^\infty |E(t)| \cdot \frac{1 + \log t}{t^2 \log^2 t} dt &\leq 2C \int_x^\infty \frac{\exp(-c_0 \sqrt{\log t})}{t \log t} dt = 2C \int_{\log x}^\infty \frac{e^{-c_0 \sqrt{u}}}{u} du \\ &\leq \frac{2C}{\log x} \cdot \frac{2}{c_0^2} (1 + c_0 \sqrt{\log x}) e^{-c_0 \sqrt{\log x}}, \end{aligned}$$

where we used $1 + \log t \leq 2 \log t$ for $t \geq e$ and $\int_v^\infty e^{-c_0 \sqrt{u}} du = \frac{2}{c_0} (1 + c_0 \sqrt{v}) e^{-c_0 \sqrt{v}}$. Together with $\frac{|E(x)|}{x \log x} \leq C \exp(-c_0 \sqrt{\log x})$, this gives a constant B with

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O\left(\exp\left(-c_1 \sqrt{\log x}\right)\right)$$

for every $0 < c_1 < c_0$. Next, put $h(p) = -\log\left(1 - \frac{1}{p}\right) - \frac{1}{p}$, so that $0 < h(p) < \frac{1}{p(p-1)}$. Then $\sum_p h(p) = H$ converges and $\sum_{p > x} h(p) < \sum_{m > x} \frac{1}{m(m-1)} \leq \frac{1}{x-1}$. Hence

$$\log \prod_{p \leq x} \frac{p}{p-1} = \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} h(p) = \log \log x + B + H + O\left(\exp\left(-c_1 \sqrt{\log x}\right)\right).$$

By Mertens' theorem [26], $\prod_{p \leq x} \frac{p}{p-1} \sim e^\gamma \cdot \log x$, so $B + H = \gamma$. Exponentiating, and using $e^y = 1 + O(y)$ for bounded y , gives the claim. $\square$

Corollary 5.10. *There are absolute constants $C_2 > 0$ and $c_2 > 0$ such that every superabundant counterexample n , with largest prime factor p_k , satisfies*

$$-\frac{0.0101}{\log^3(p_k)} < \log \log n - \log p_k < C_2 \cdot \exp\left(-c_2 \sqrt{\log p_k}\right).$$

In particular, if the Riemann Hypothesis is false, then infinitely many superabundant numbers n satisfy $\log \log n - \log p_k < O\left(\exp\left(-c_2 \sqrt{\log p_k}\right)\right)$.

Proof. By Lemma 5.9, there are constants $A > 0$ and $c_1 > 0$ with $\prod_{p \leq t} \frac{p}{p-1} \leq e^\gamma \cdot \log t \cdot (1 + A \exp(-c_1 \sqrt{\log t}))$ for every $t \geq 2$; for t in a bounded range this holds after enlarging A , because the ratio $\prod_{p \leq t} \frac{p}{p-1} / (e^\gamma \log t)$ is bounded there. Put $P = \log p_k$ and $L = \log \log n$. By Proposition 2.6 and Proposition 2.2, and since Robin's inequality fails for n ,

$$e^\gamma \cdot L \leq I(n) < \prod_{p \leq p_k} \frac{p}{p-1} \leq e^\gamma \cdot P \cdot (1 + A \exp(-c_1 \sqrt{P})),$$

so $L - P < A \cdot P \cdot \exp(-c_1 \sqrt{P})$. Put $c_2 = \frac{c_1}{2}$. The function $P \mapsto P \exp(-c_2 \sqrt{P})$ is bounded on $(0, \infty)$, so $A \cdot P \cdot \exp(-c_1 \sqrt{P}) \leq C_2 \cdot \exp(-c_2 \sqrt{P})$ for a suitable constant C_2 . The lower bound is Theorem 5.5, which applies because $p_k > 7,232,121,212$ by Proposition 5.1. The last statement follows from Proposition 5.3. $\square$

For the least counterexample the window is one-sided.

Theorem 5.11. *Suppose that the Riemann Hypothesis is false. Let $n > 5040$ be the least integer such that Robin's inequality fails, let p_k be the largest prime factor of n , and put*

$$x = \log \log n - \log p_k.$$

Then $x > 0$ and

$$e^{0.0562} > 1 + x \cdot \log^2(p_k).$$

Proof. By Proposition 2.7, n is a superabundant counterexample. By Proposition 2.8, $p_k < \log n$, hence $L > P$, that is, $x > 0$. The second inequality is part of Theorem 5.6. $\square$

Remark 5.12. An earlier version of Theorem 5.11, based on the estimate $\prod_{p \leq x} \frac{p}{p-1} \leq e^\gamma \cdot \log x \cdot \left(1 + \frac{0.2}{\log^3(x)}\right)$ of [21] [Lemma 2.7 (5) pp. 19], gave $e^{0.2} > 1 + x \cdot \log^2(p_k)$. Proposition 2.9 replaces 0.2 by 0.0562.

The proof of [10] [Lemma 6.1, p. 369] also yields the following inequality.

Theorem 5.13. *Let n be a superabundant counterexample with largest prime factor p_k , put $r = \frac{n}{p_k}$, and suppose that Robin's inequality holds for r ; by the proof of [10] [Lemma 6.1, p. 369], this is the case when n is the least counterexample. Then*

$$\log n \cdot \log \log r > p_k \cdot \log p_k.$$

Proof. Since σ is submultiplicative, $I(n) \leq I(p_k) \cdot I(r) = \left(1 + \frac{1}{p_k}\right) \cdot I(r)$. Since Robin's inequality fails for n and holds for r ,

$$e^\gamma \cdot \log \log n \leq I(n) \leq \left(1 + \frac{1}{p_k}\right) \cdot I(r) < \left(1 + \frac{1}{p_k}\right) \cdot e^\gamma \cdot \log \log r,$$

so $\log \log n < \left(1 + \frac{1}{p_k}\right) \cdot \log \log r$, that is,

$$\log \log n - \log \log r < \frac{\log \log r}{p_k}.$$

On the other hand, $\log r = \log n - \log p_k$, so

$$\log \log n - \log \log r = -\log\left(1 - \frac{\log p_k}{\log n}\right) > \frac{\log p_k}{\log n},$$

because $0 < \frac{\log p_k}{\log n} < 1$ and $-\log(1 - u) > u$ for $0 < u < 1$. Combining the two inequalities gives $\frac{\log \log r}{p_k} > \frac{\log p_k}{\log n}$, that is, $\log n \cdot \log \log r > p_k \cdot \log p_k$. $\square$

Corollary 5.14. *Suppose that the Riemann Hypothesis is false. Then there are infinitely many superabundant counterexamples n , and each of them satisfies*

$$\log p_k - \frac{0.0101}{\log^3(p_k)} < \log \log n < \log p_k + \frac{0.0562}{\log^2(p_k)}$$

and

$$e^{0.0562} > 1 + (\log \log n - \log p_k) \cdot \log^2(p_k),$$

where p_k is the largest prime factor of n . For the least counterexample, moreover, $\log p_k < \log \log n$.

Proof. Combine Proposition 5.3, Theorem 5.6 and Theorem 5.11. □

Proposition 5.15. *Suppose that there are constants $c > 0$, $0 < \alpha < 2$ and $N \leq 10^{10^{13.11485}}$ such that every superabundant number $n > N$ with $\log n > p_k$, where p_k is the largest prime factor of n , satisfies*

$$\log \log n - \log p_k \geq \frac{c}{\log^\alpha(p_k)}.$$

If the least counterexample exists, then its largest prime factor p_k satisfies $\log p_k < \left(\frac{0.0562}{c}\right)^{\frac{1}{2-\alpha}}$. In particular, if $c \geq \frac{0.0562}{31.03^{2-\alpha}}$, then the Riemann Hypothesis is true.

Proof. Suppose that the Riemann Hypothesis is false, and let n be the least counterexample, with largest prime factor p_k and $P = \log p_k$. By Proposition 2.5, $n > 10^{10^{13.11485}} \geq N$; by Theorem 5.11, $\log n > p_k$; by Proposition 2.7, n is superabundant. Hence the hypothesis applies to n , and Theorem 5.6 gives

$$\frac{c}{P^\alpha} \leq \log \log n - \log p_k < \frac{0.0562}{P^2},$$

that is, $P^{2-\alpha} < \frac{0.0562}{c}$, which is the first claim. By Proposition 5.2, $P > 31.0319 > 31.03$, so $c < \frac{0.0562}{P^{2-\alpha}} < \frac{0.0562}{31.03^{2-\alpha}}$. Therefore, if $c \geq \frac{0.0562}{31.03^{2-\alpha}}$, the least counterexample cannot exist, and the Riemann Hypothesis is true by Proposition 2.1. □

The precondition $\log n > p_k$ in Proposition 5.15 excludes the superabundant numbers with $p_k > \log n$, for which $\log \log n - \log p_k < 0$; examples are $n = 27720$, $n = 55440$ and $n = 10685862914126400$ (Proposition 6.4).

6. Computational Verification

Every numerical claim of this paper was checked by a self-contained Python 3.12 script, `verify_robin_d.py`, written with the help of Claude Opus 5.5 (Anthropic) and run by the author, which uses the arbitrary-precision library `mpmath` with 60 significant decimal digits and exact integer arithmetic wherever a comparison of rational numbers is involved. The script has five parts: it checks the constants of Lemma 2.12, it compares the bounds of Lemma 2.13 with the true values of the Mertens product, it checks the constants of the theorems with the exponent 1.016 and of Section 5, it enumerates all superabundant numbers up to 10^{40} and tests on each of them the conditions of Corollary 4.1 and the inequality of Theorem 5.11, and it tests on the same numbers the hypothesis of Proposition 5.15. All 33 checks performed by the script passed. We now describe each part and its output in detail.

6.1. The Constants of Lemma 2.12

The script computes γ , e^γ , $\zeta(2) = \frac{\pi^2}{6}$ (and confirms that this agrees with $\zeta(2)$ evaluated directly to 50 digits) and $d = \frac{e^\gamma}{\zeta(2)}$, and then verifies each item of Lemma 2.12. The computed values are collected in Table 1. Every inequality of Lemma 2.12 holds with a visible margin. The two tightest margins are those of item (iii), where $e^\gamma \cdot (d-1) \cdot \log^2(64) = 2.5495 \dots$ exceeds the Rosser–Schoenfeld constant 2.51 by about 0.04, and of item (vii), where $e^\gamma \cdot \log 313 = 10.2344 \dots$ exceeds 10.23 by about 0.004. The script also confirms that $\theta(71) = 61.586 \dots < 64$, so that 71 is indeed the largest prime that can divide n' in Case II of the proof of Theorem 1.1.

Table 1: Values computed for Lemma 2.12. All digits shown are correct.

Item	Quantity	Computed value
	γ	0.577215664901533...
	e^γ	1.7810724179902...
	$\zeta(2)$	1.64493406684823...
(i)	$d = e^\gamma / \zeta(2)$	1.08276219326092...
(ii)	$e^\gamma \cdot (d - 1)$	0.147405459669...
(iii)	$\log^2(64)$	17.2963085011...
(iii)	$e^\gamma \cdot (d - 1) \cdot \log^2(64)$	2.54957030518...
(iv)	$\theta(73)$	65.8767545906...
(iv)	$\theta(71)$	61.5862951494...
(v)	$\left(1 - \frac{1}{2 \log^2(285)}\right)^{-1}$	1.01589793074...
(vi)	$e^\gamma \log 285 \left(1 - \frac{1}{2 \log^2(285)}\right)^{-1}$	10.2275448716...
(vii)	$e^\gamma \cdot \log 313$	10.2344040108...
(vii)	$313 / \log 10$	135.9341728...

6.2. The Mertens Product

The script computes the exact partial products $\prod_{p \leq x} \frac{p}{p-1}$ for every prime $x \leq 2000$ and compares them with the bounds of Lemma 2.13. The results are as follows.

- (1) The true value is $\prod_{p \leq 285} \frac{p}{p-1} = 10.2257527701\dots$, which lies below the analytic bound $10.2275\dots$ obtained from Proposition 2.3, and below 10.23, as Lemma 2.13 (a) asserts. The analytic bound is therefore sharp to within 0.002 at $x = 285$.
- (2) The true value is $\prod_{p \leq 71} \frac{p}{p-1} = 7.82486815925\dots$, the quantity actually used in Case II of the proof of Theorem 1.1. The bound 10.23 used there is thus generous.
- (3) For every prime x with $285 \leq x \leq 2000$ the inequality $\prod_{p \leq x} \frac{p}{p-1} < 1.016 \cdot e^\gamma \cdot \log x$ of Lemma 2.13 (b) holds. The largest value of the ratio $\prod_{p \leq x} \frac{p}{p-1} / (e^\gamma \log x)$ on this range is $1.014231153\dots$, attained at $x = 293$, compared with the analytic constant $1.01589\dots$ of Lemma 2.12 (v).
- (4) Among the primes $x \leq 2000$, the inequality $\prod_{p \leq x} \frac{p}{p-1} \geq d \cdot e^\gamma \cdot \log x$ holds only for $x \leq 23$. Thus the direct argument of Case I in the proof of Theorem 1.2 already works numerically for every prime P with $29 \leq P \leq 2000$; the threshold 285 was chosen only because it is the point from which Proposition 2.3 applies.

6.3. Enumeration of Superabundant Numbers

To enumerate the superabundant numbers rigorously we use the following reduction to Hardy–Ramanujan integers.

Lemma 6.1. *For every natural number m there is a Hardy–Ramanujan integer $m^* \leq m$ with $I(m^*) \geq I(m)$.*

Proof. For $m = 1$ take $m^* = 1$. Otherwise let $q_1 < \dots < q_r$ be the primes dividing m and $m = \prod_{i=1}^r q_i^{b_i}$. For a prime p and an integer $b \geq 0$ put $g(p, b) = \frac{\sigma(p^b)}{p^b} = \sum_{j=0}^b p^{-j}$, so that I is the product of the factors $g(q_i, b_i)$. We proceed in two steps.

Step 1. Put $m_1 = \prod_{i=1}^r p_i^{b_i}$. Since $q_i \geq p_i$ for every i , we have $m_1 \leq m$, and since $g(p, b)$ is nonincreasing in p for fixed b , we have $I(m_1) \geq I(m)$.

Step 2. Suppose that $p < q$ are primes carrying exponents $a < b$ respectively, and exchange the two exponents. The integer does not increase, because $p^b q^a \leq p^a q^b$ is equivalent to $p^{b-a} \leq q^{b-a}$. The abundancy index does not decrease, because $g(p, b) \cdot g(q, a) \geq g(p, a) \cdot g(q, b)$ is equivalent to $\frac{g(p, b)}{g(p, a)} \geq \frac{g(q, b)}{g(q, a)}$, and, with $t = \frac{1}{p}$,

$$\frac{g(p, b)}{g(p, a)} = 1 + \frac{t^{a+1} + \dots + t^b}{1 + t + \dots + t^a} = 1 + \frac{1 + t + \dots + t^{b-a-1}}{t^{-a-1} + t^{-a} + \dots + t^{-1}},$$

which is increasing in t , that is, decreasing in p . Applying finitely many such exchanges to m_1 sorts its exponents into nonincreasing order and produces a Hardy–Ramanujan integer $m^* \leq m_1 \leq m$ with $I(m^*) \geq I(m_1) \geq I(m)$. $\square$

Proposition 6.2. *Let $h_1 = 1 < h_2 < h_3 < \dots$ be the Hardy–Ramanujan integers in increasing order. Then n is superabundant if and only if $n = h_j$ for some j with $I(h_j) > I(h_i)$ for all $i < j$.*

Proof. If n is superabundant, then n is a Hardy–Ramanujan integer by Proposition 2.6, and $I(n) > I(h_i)$ for every $h_i < n$ by definition. Conversely, let $n = h_j$ be such a record, and let $m < n$. By Lemma 6.1 there is a Hardy–Ramanujan integer $m^* \leq m < n$ with $I(m) \leq I(m^*)$, and $I(m^*) < I(n)$ because n is a record. Hence $I(m) < I(n)$, and n is superabundant. $\square$

The script generates every Hardy–Ramanujan integer $h \leq 10^{40}$ together with $\sigma(h)$, by recursion over the primes with nonincreasing exponents. There are exactly 5,754,126 of them. It then sorts them and extracts the records of I as in Proposition 6.2. A comparison $I(h) > I(h')$ is decided exactly through the integer inequality $\sigma(h) \cdot h' > \sigma(h') \cdot h$, so no rounding enters this step. This yields exactly 263 superabundant numbers not exceeding 10^{40} . For every superabundant n with $5040 < n \leq 10^{40}$ the script evaluates, with 60-digit precision, Robin’s inequality and the two conditions of Corollary 4.1. The outcome is summarized in the next proposition.

Proposition 6.3 (Computer-assisted). *Among the 263 superabundant numbers $n \leq 10^{40}$:*

- (1) *Robin’s inequality holds for every superabundant n with $5040 < n \leq 10^{40}$;*
- (2) *condition (a) of Corollary 4.1, namely $d \cdot \log \theta(p_k) \leq \log \log n$, holds for exactly 86 superabundant numbers n with $5040 < n \leq 10^{40}$, and the largest of them is*

$$n = 92005279690628304000,$$

whose largest prime factor is $p_k = 41$;

- (3) *condition (b) of Corollary 4.1, namely $p_k < \sqrt[4]{\log n}$, holds for exactly 7 superabundant numbers n with $5040 < n \leq 10^{40}$, and the largest of them is $n = 48886437600$, whose largest prime factor is $p_k = 19$.*

Part (1) is already contained in Proposition 2.5; it serves as an independent consistency check of the enumeration. Condition (a) is equivalent to $\theta(p_k) \leq \sqrt[4]{\log n}$, whereas condition (b) is $p_k < \sqrt[4]{\log n}$.

6.4. Constants of Section 5 and the Quantity $x \cdot \log^2(p_k)$

For the theorems with the exponent 1.016, the script verifies that $\theta(14000) = 13867.29 \dots > 13000$, that $e^\gamma \cdot 0.016 \cdot \log^2(13000) = 2.5571 \dots > 2.51$, that $1.016 \cdot e^\gamma \cdot \log 14000 = 17.2756 \dots < 17.3$, that the true value $\prod_{p \leq 14000} \frac{p}{p-1} = 17.0192 \dots$ is below 17.3, that $e^\gamma \cdot \log 16600 = 17.3069 \dots > 17.3$, and that $e^{16600} < 10^{7210}$. The script also verifies the numerical facts used in Section 5: that $\log \log(10^{10^{13.11485}}) = 31.03209 \dots > 31.032$, that $\frac{31.032}{1.016} > 30.54$, that $e^{30.54} > 7,232,121,212 > 46,909,038$, that $(1 - \varepsilon(30.54))^{-1} = 1.0000019708 \dots < 1.000002$, that $\frac{\frac{1}{20} + \frac{3}{16 \cdot 30.54}}{1 - \varepsilon(30.54)} = 0.0561395 \dots < 0.0562$, that $\frac{0.0562}{30.54^2} = 0.0000602557 \dots < 0.0000603$, that $\frac{31.032}{1.000002} > 31.0319$, that $0.0562 \cdot 1.000002^2 < 0.056201$, that $0.0101 \cdot 1.000002^3 < 0.010101$, and, for Theorem 5.5, that $\log 7,232,121,212 = 22.7017 \dots > 22.7$ and $\frac{0.01}{22.7^3} = 8.549 \dots \cdot 10^{-7} < 10^{-6}$. For every superabundant number $5040 < n \leq 10^{40}$ it computes $x = \log \log n - \log p_k$ and $x \cdot \log^2(p_k)$, counts the superabundant numbers with $x < 0$, that is, with $p_k > \log n$, and counts those for which the inequality $e^{0.0562} > 1 + x \cdot \log^2(p_k)$ of Theorem 5.11 fails or holds. The output of this part of the script is summarized in the next proposition and in Table 2.

Proposition 6.4 (Computer-assisted). *Among the 244 superabundant numbers n with $5040 < n \leq 10^{40}$, and with p_k the largest prime factor of n and $x = \log \log n - \log p_k$:*

- (1) *exactly 3 satisfy $x < 0$, that is, $p_k > \log n$;*
- (2) *exactly 3 satisfy $e^{0.0562} > 1 + x \cdot \log^2(p_k)$, namely the 3 numbers of part (1), which are $n = 27720$ and $n = 55440$ with $p_k = 11$ and $n = 10685862914126400$ with $p_k = 37$, with $x \cdot \log^2(p_k) = -0.4173 \dots, -0.0403 \dots$ and $-0.0325 \dots$, respectively;*

(3) the remaining 241 satisfy $e^{0.0562} \leq 1 + x \cdot \log^2(p_k)$, and the largest of them is

$$n = 5330180429932843151696086400707299936000,$$

with $p_k = 83$, $x = 0.097216398 \dots$ and $x \cdot \log^2(p_k) = 1.8982622 \dots$;

(4) the maximum of $x \cdot \log^2(p_k)$ is $2.855801894 \dots$, attained at

$$n = 10567690344536672406900888090467424000,$$

with $p_k = 73$, and the minimum is $-0.4173252461 \dots$, attained at $n = 27720$ with $p_k = 11$.

Table 2: The quantity $x \cdot \log^2(p_k)$, with $x = \log \log n - \log p_k$, for the ten largest superabundant numbers $n \leq 10^{40}$. The window of Theorem 5.6 requires $x \cdot \log^2(p_k) < 0.0562$.

	n	p_k	$x \cdot \log^2(p_k)$
278282512406132373381723386382308832000	79		2.172675585
355345361995522876779739093380486662400	83		1.311470417
417423768609198560072585079573463248000	79		2.259925324
484561857266622104699644218246118176000	83		1.379577194
834847537218397120145170159146926496000	79		2.408162468
888363404988807191949347733451216656000	83		1.511996393
1776726809977614383898695466902433312000	83		1.662331969
2665090214966421575848043200353649968000	83		1.749739048
3553453619955228767797390933804866624000	83		1.811518921
5330180429932843151696086400707299936000	83		1.898262198

By Proposition 6.4, for the ten largest superabundant numbers below 10^{40} , $\log \log n$ exceeds $\log p_k$ by between $\frac{1.31}{\log^2(p_k)}$ and $\frac{2.41}{\log^2(p_k)}$, and the maximum over the whole range is $\frac{2.86}{\log^2(p_k)}$. Table 2 also shows that the largest prime factor of consecutive superabundant numbers need not be monotone: it alternates between 79 and 83 before settling at 83. The value of $x \cdot \log^2(p_k)$ is smaller whenever $p_k = 83$, and along the last five entries, where $p_k = 83$ is fixed, it increases from 1.51 to 1.90 as the exponents of the smaller primes grow.

6.5. The Hypothesis of Proposition 5.15

For every superabundant number $5040 < n \leq 10^{40}$, the script computes $x = \log \log n - \log p_k$ and the quantities $x \cdot \log^\alpha(p_k)$ for $\alpha = 1, 2, 3, 4, 5$. The output is summarized in the next proposition and in Table 3.

Proposition 6.5 (Computer-assisted). *Among the 244 superabundant numbers $5040 < n \leq 10^{40}$, with $x = \log \log n - \log p_k$:*

- (1) exactly 3 satisfy $x \leq 0$, namely $n = 27720$ with $x = -0.0725795 \dots$, $n = 55440$ with $x = -0.0070194 \dots$, and $n = 10685862914126400$ with $x = -0.0024977 \dots$; these 3 numbers have $p_k > \log n$ and are therefore excluded by the precondition $\log n > p_k$ of Proposition 5.15;
- (2) there are exactly 168 superabundant numbers n with $10685862914126400 < n \leq 10^{40}$, and all of them satisfy $x > 0$;

(3) for every $\alpha \in \{1, 2, 3, 4, 5\}$, the minimum of $x \cdot \log^\alpha(p_k)$ over these 168 numbers is attained at

$$n = 58144272988663390205476800,$$

with $p_k = 59$, and its value is given in Table 3.

Table 3: Minimum of $x \cdot \log^\alpha(p_k)$, with $x = \log \log n - \log p_k$, over the 168 superabundant numbers $10685862914126400 < n \leq 10^{40}$, and over the 50 largest superabundant numbers $n \leq 10^{40}$, where it is attained at $p_k = 79$.

α	minimum over the 168 numbers	minimum over the last 50
1	0.02239728649	0.2087437012
2	0.0913257743	0.9120947171
3	0.3723842643	3.985350303
4	1.518410781	17.41378032
5	6.191376815	76.08860503

For the ten largest superabundant numbers below 10^{40} , whose largest prime factors are 79 and 83, the values of x lie between $0.0671 \dots$ and $0.1261 \dots$, and those of $x \cdot \log^3(p_k)$ lie between $5.795 \dots$ and $10.522 \dots$.

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Declaration of interests

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Declaration of generative AI in scientific writing

The author used Claude Opus 5.5 to write the verification script `verify_robin_d.py` of Section 6 and to generate parts of the text of this manuscript. The experiments were run by the author. After using this tool, the author reviewed and edited the code and the content as needed and takes full responsibility for the content of the publication.

Data availability

The Python script `verify_robin_d.py` used in Section 6 is available in the preprint..

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