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ONE OF $\zeta(7), \zeta(9), \ldots, \zeta(21)$ IS IRRATIONAL

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Abstract. We prove that at least one of the eight numbers $\zeta(7), \zeta(9), \ldots, \zeta(21)$ is irrational. Among published results, the best of this kind not involving $\zeta(5)$, a consequence of a theorem of Lai and Zhou, is that one of $\zeta(7), \ldots, \zeta(35)$ is irrational. The same statement was obtained earlier, independently and by the same construction, in an unrefereed Lean formalisation by CoolRmal, and J. Liu has since announced a Lean proof for $\zeta(7), \ldots, \zeta(19)$ by a determinant method. We use Zudilin's hypergeometric construction: sums of fourth derivatives of rational functions with quintuple zeros give linear forms in $1, \zeta(7), \ldots, \zeta(21)$ with no $\zeta(3)$ or $\zeta(5)$ term. The improvement comes from spreading the poles over eighteen blocks with nearby offsets, a sharper p-adic bound on the denominators, and a saddle-point analysis of a cotangent contour integral. The estimates rest on exact rational and interval-arithmetic certificates. The whole proof, including these certificates, is formalised in Lean 4 with Mathlib and checked by the Lean kernel using only the standard axioms.

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1. Introduction

Let $\zeta(s) = \sum_{m\geq 1} m^{-s}$ be the Riemann zeta function.

Theorem 1.1. *At least one of the eight numbers* $\zeta(7), \zeta(9), \zeta(11), \ldots, \zeta(21)$ *is irrational.*

1.1. **Context.** The even values $\zeta(2k)$ are nonzero rational multiples of π^{2k}, hence transcendental. Among the odd values, $\zeta(3)$ is irrational by Apéry's theorem [1]. Rivoal [13] and Ball and Rivoal [2] proved that infinitely many odd values are irrational, and Fischler, Sprang and Zudilin [7] sharpened the count. For short windows, Zudilin [18] proved that one

of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (see also [20, Thm. 3]); Rivoal [14] and Zudilin [19, Thm. 1] proved it for the nine numbers $\zeta(5), \ldots, \zeta(21)$; and Zudilin gave an elementary proof for $\zeta(5), \ldots, \zeta(25)$ [21].

A statement "one of $\zeta(5), \ldots, \zeta(N)$ is irrational" holds as soon as $\zeta(5)$ is irrational, and then it says nothing about $\zeta(7), \zeta(9), \ldots$. The known explicit results about windows that start at $\zeta(7)$ are the following.

- Zudilin [19, Thm. 1] proved that one of $\zeta(7), \zeta(9), \ldots, \zeta(37)$ is irrational, and more generally [19, Thm. 2] that for every odd $b \geq 1$ one of $\zeta(b+2), \zeta(b+4), \ldots, \zeta(8b-1)$ is irrational. (Theorem numbers and statements of [19] refer to the arXiv version, arXiv:math/0104249, where we checked them.)
- Lai and Zhou [11] proved that at least two of $\zeta(5), \zeta(7), \ldots, \zeta(35)$ are irrational. This implies, with no hypothesis on $\zeta(5)$, that one of $\zeta(7), \ldots, \zeta(35)$ is irrational: if $\zeta(7), \ldots, \zeta(35)$ were all rational, at most one number in the list $\zeta(5), \ldots, \zeta(35)$ could be irrational.
- Rivoal and Zudilin [15] proved that at least two of $\zeta(5), \ldots, \zeta(69)$ are irrational, which gives the same conclusion for $\zeta(7), \ldots, \zeta(69)$.

So among refereed results the window $\zeta(7), \ldots, \zeta(35)$ is the shortest of this kind that we know of. Theorem 1.1 shortens it to $\zeta(7), \ldots, \zeta(21)$, which is Rivoal's window of nine numbers with $\zeta(5)$ removed.

Unrefereed Lean formalisations. After this work was done we learned of two results that appeared first as Lean repositories, neither refereed.

- CoolRmal [3] proved Theorem 1.1 earlier, by the same derivative form of Zudilin's construction ($r = 5$) with a different pole profile; that formalisation was completed on 29 September 2026, before ours (22 October 2026). It also gives one of $\zeta(b+2), \ldots, \zeta(6b-1)$ for every odd $b \geq 3$, which improves Zudilin's $8b - 1$, and in particular one of $\zeta(9), \ldots, \zeta(33)$. Our proof was found independently; its pole profile, p-adic refinement and saddle-point treatment differ, and the method sections below are unchanged.
- J. Liu [12] announced, with a Lean formalisation (6 October 2026), that one of $\zeta(7), \ldots, \zeta(19)$ is irrational. The method is different: a growing determinant of higher-derivative linear forms, with decay $e^{-n^2/2}$ and nonvanishing proved by reduction modulo auxiliary primes. This is shorter than our window; see Section 10.3.

We have not checked either proof beyond reading the public descriptions.

A proof that $\zeta(5)$ is irrational has recently been announced in an unrefereed preprint by A. Fauzan, with Lean formalisations (for instance [5]). Theorem 1.1 does not use the arithmetic nature of $\zeta(5)$ or $\zeta(3)$ in any way: neither number occurs in our linear forms (Corollary 3.6). Theorem 1.1 neither implies nor is implied by the result of Lai and Zhou. Our first literature search covered arXiv and the web but not GitHub or Zenodo, where the two formalisations above appeared; it did not cover MathSciNet or zbMATH. We would be grateful to hear of other prior work.

1.2. **The method.** We follow the approach of Ball and Rivoal [2] and Zudilin [19, 20] through linear forms with rational coefficients, and the construction is not new: it is the derivative form of Zudilin's well-poised hypergeometric construction [20, §8, (8.2)–(8.4)], in which $\frac{1}{(r-1)!}\sum_{t\ge 0} R^{(r-1)}(t)$ is summed; here $r=5$, so the derivative has order 4. For n in an arithmetic progression we put

$$R_n(t)=\frac{(2t+n)\prod_{l=1}^{rn}(t-l)^5\prod_{l=n+1}^{(1+r)n}(t+l)^5}{\prod_j\prod_{k=\delta_j n}^{(1-\delta_j)n}(t+k)^{m_j}},\qquad L_n=\sum_{t\ge 1}R_n''''(t).$$

Differentiating four times raises every pole order by 4, so $\zeta(3)$ cannot occur; $\zeta(5)$ drops out because the residues of R_n add up to 0; and the symmetry $R_n(-n-t)=-R_n(t)$ removes the even zeta values (Section 3). Since all zeros and poles are integers there is no 2-adic or 3-adic cost of the kind that half-integer or third-integer shifts produce. As far as we can tell, Zudilin's theorem for $\zeta(7),\dots,\zeta(37)$ is of the same type, with all poles in a single block; our single-block instance Z1 of Section 8 reaches the same window. The analytic method, a saddle point of the Ball–Rivoal–Zudilin type, is also standard.

Compared with Zudilin's treatment, our proof uses the following ingredients. (CoolRmal's formalisation [3] treats the same construction with its own pole profile and estimates; we have not compared the two in detail.)

(1) *A near-continuous pole profile.* The denominator of R_n is a product of eighteen blocks, one at each offset $\delta_j=j/80$, $1\le j\le 18$, with $r=11/12$. Zudilin's parametrisation allows such profiles; the profile used here was found by a numerical search (Section 10.2). Even $S=18$ is useful: the top index is then $S+3=21$, so the window $\zeta(7),\dots,\zeta(21)$ gets one block more than an odd S would.

(2) *An arithmetic refinement.* If $k<p$, the harmonic sums $H_k^{(j)}$ are p-integral, and the Taylor coefficients of R_n at the pole $-k$ are p-integral unless another pole lies at a distance $\equiv 0 \pmod p$ (Corollary 4.4(b′)). So such poles cost nothing at p, and the others cost less than in the standard estimate (Definition 4.6). This lowers the arithmetic integral by an amount of order 1 per unit of n.

(3) *The saddle point for the* cot *kernel.* L_n is a contour integral of $R_n''''(z)\pi\cot\pi z$. Four integrations by parts move the derivatives onto the kernel, and $\sin^5 u\cdot\cot''''u = 2\cos 3u+22\cos u$ (Lemma 6.3) turns the integrand into $R_n(z)$ times an entire function whose leading exponential is $e^{-3\pi i z}$. The resulting phase $\Psi(x)-3\pi i x$ has a saddle x_0 with $\operatorname{Re}\Omega\le -10.1167485$, which gives both the size of L_n and its nonvanishing for infinitely many n (Section 6). The elementary bound on the real axis would not suffice here.

(4) *Certified numerics.* The arithmetic integral is a sum over 30,137 intervals, evaluated exactly in rational arithmetic (Section 5). The

saddle point, the steepest-descent contour and all bounds along it are certified by interval arithmetic.

(5) *Formalisation.* Every step, including all computations, has been formalised in Lean 4 [4] with Mathlib [16] (Section 9).

The exponent that must be negative is $Q = \operatorname{Re}\Omega + A_{\mathrm{cert}} < -0.1981$, see (7.1). Here $\operatorname{Re}\Omega$ controls the size of L_n (Proposition 6.16) and A_{cert} is the arithmetic integral (Proposition 5.5). There is no 2-adic term.

The same machine gives, with larger margins, the windows $\zeta(7), \ldots, \zeta(N)$ for $N = 23, 25, 29, 31, 33, 37$ (Section 8); those instances rest on certificates computed in Python and Arb and are not formalised. It does not reach $N = 19$ (Section 10.3).

1.3. **The formalisation.** The statement of Theorem 1.1 and its complete proof, including the prime number theorem, complex Stirling bounds, the contour-integral representation, the arithmetic certificate and the interval-arithmetic saddle-point certificate, are formalised in Lean 4 with Mathlib and checked by the Lean kernel. The proof depends only on Lean's three standard axioms (`propext`, `Classical.choice`, `Quot.sound`). It contains no `sorry` and no `native_decide`: every proof, including every computation, is checked by the kernel. The formal statement uses Mathlib's `riemannZeta` and is described in Section 9.

The prose proof below is self-contained, except that certain finite computations are only described: the exact value of a rational sum with 30,137 terms (Proposition 5.5) and the interval-arithmetic checks of the saddle-point data (Theorem 6.10). Both are decidable statements about rational data, proved by `decide +kernel`. The proof in this paper is the one that is formalised; Section 9.4 notes where the formal proof is organised differently.

1.4. **Organisation and notation.** Section 2 defines the class of rational functions and the instance used for Theorem 1.1. Section 3 shows that $L_n \in \mathbb{Q} + \mathbb{Q}\zeta(7) + \cdots + \mathbb{Q}\zeta(21)$. Section 4 constructs a common denominator D_n, and Section 5 bounds its size through the arithmetic integral. Section 6 bounds $|L_n|$ and proves that $L_n \neq 0$ for infinitely many n. Section 7 combines these results. Section 8 states the general theorem and the other instances, Section 9 describes the formalisation, and Section 10 contains remarks on the parameter search, the limits of the method and effectivity.

Throughout, p denotes a prime and v_p the p-adic valuation on $\mathbb{Q}$, with $v_p(0) = +\infty$; $\mathbb{Z}_{(p)}$ is the ring of rationals with $v_p \geq 0$. $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$ and $\{\cdot\}$ are the floor, ceiling and fractional part; $[\,P\,]$ is 1 if P holds and 0 otherwise; $\#$ denotes cardinality; intervals $[a, b]$ of integers are written like real intervals. $\theta(y) = \sum_{p \leq y} \log p$ is Chebyshev's function. For a power series g in ε, $[\varepsilon^M]g$ is the coefficient of ε^M. We write $(i)_4 := i(i+1)(i+2)(i+3)$ and $H_k^{(j)} := \sum_{m=1}^{k} m^{-j}$ (so $H_0^{(j)} = 0$).

2. The construction

2.1. The class of rational functions.

Definition 2.1. A *parameter set* $\mathcal{P} = (r, (\delta_j, m_j)_{j=1}^{J}, q)$ consists of $r \in \mathbb{Q}_{>0}$, offsets $\delta_j \in \mathbb{Q} \cap [0, \frac{1}{2})$ with multiplicities $m_j \in \mathbb{N}_{\geq 1}$, and $q \in \mathbb{N}_{\geq 1}$ with $rq \in \mathbb{Z}$ and $\delta_j q \in \mathbb{Z}$ for all j. We assume $S := \sum_j m_j \geq 2$. The integer n always runs through $q\mathbb{N} = \{q, 2q, 3q, \dots\}$. Put

$$a_j := \delta_j n, \quad b_j := n - a_j, \quad I_j := [a_j, b_j] \cap \mathbb{Z},$$
$$B_k := \sum_j m_j [\, k \in I_j \,] \ (0 \leq k \leq n),$$
$$G := \sum_j m_j (1 - 2\delta_j) - 10r, \qquad \mathcal{N} := (2 + r)n + 1,$$

and let $N_{\mathcal{P}}$ be the largest odd integer $\leq S + 4$ (so $N_{\mathcal{P}} = S + 4$ for odd S and $S + 3$ for even S).

Each I_j is a nonempty integer interval, symmetric under $k \mapsto n - k$, with $\#I_j = (1 - 2\delta_j)n + 1$; so $B_{n-k} = B_k$ and $0 \leq B_k \leq S$. The *zero set* is

$$\mathcal{Z}_n := [1, rn] \cup [-(1 + r)n, \, -n - 1] \subset \mathbb{Z},$$

two runs of rn consecutive integers on either side of the pole region $[-n, 0]$; each zero will have multiplicity 5. For $e \in \{0, 1\}$ define

(2.1)
$$R_n(t) := \frac{(2t + n)^e \prod_{l=1}^{rn} (t - l)^5 \prod_{l=n+1}^{(1+r)n} (t + l)^5}{\prod_j \prod_{k \in I_j} (t + k)^{m_j}} = \frac{(2t + n)^e \prod_{z \in \mathcal{Z}_n} (t - z)^5}{\prod_{k=0}^{n} (t + k)^{B_k}},$$

and

$$L_n := \sum_{t=1}^{\infty} R_n''''(t). \tag{2.2}$$

Lemma 2.2. *$\sum_{k=0}^{n} B_k = \sum_j m_j \#I_j = (G + 10r)n + S$ and $\deg R_n = e - Gn - S$. In particular $Gn \in \mathbb{Z}$. If $G > 0$, then $\deg R_n \leq -2$, so $R_n(t) = O(t^{-2})$, $R_n''''(t) = O(t^{-6})$ as $t \to \infty$, and the series (2.2) converges absolutely.*

Proof. $\sum_j m_j((1 - 2\delta_j)n + 1) = (G + 10r)n + S$, and the numerator has degree $e + 10rn$. Hence $Gn = e + 10rn - S - \deg R_n \in \mathbb{Z}$. If $G > 0$ then $Gn \geq 1$ and $\deg R_n \leq 1 - 1 - S \leq -2$. □

Lemma 2.3 (Symmetry)**.** *If $e \equiv 1 + S + Gn \pmod 2$, then $\deg R_n$ is odd and $R_n(-n - t) = -R_n(t)$.*

Proof. The first claim is Lemma 2.2. The map $z \mapsto -n - z$ sends $[1, rn]$ onto $[-n - rn, -n - 1]$ and back, and $-n - t - z = -(t - (-n - z))$; the e-factor becomes $2(-n - t) + n = -(2t + n)$; and $-n - t + k = -(t + (n - k))$ with $B_{n-k} = B_k$. So every linear factor changes sign and the multisets

of zeros and poles are preserved, whence $R_n(-n-t) = (-1)^{\deg R_n} R_n(t) = -R_n(t)$. □

If qG is even, then Gn is even for every $n \in q\mathbb{N}$, and the condition of Lemma 2.3 fixes $e \equiv 1 + S \pmod 2$ independently of n. This is hypothesis (H6) of Section 8. In the language of hypergeometric series, this symmetry together with the factor $2t + n$ makes R_n *very-well-poised*.

2.2. **The instance Z7.** Theorem 1.1 uses the following parameter set, called Z7 (the last column of Table 2).

- $r = 11/12$ and $q = 240$: $n = 240N_0$ with $N_0 \in \mathbb{N}_{\geq 1}$, so $rn = 220N_0$ and $(1+r)n = 460N_0$.
- Eighteen blocks $(\delta_j, m_j) = (j/80, 1)$, $1 \leq j \leq 18$, so $a_j = 3jN_0$.
- $e = 1$.

quantity	value
r, q, e	11/12, 240, 1
blocks (δ_j, m_j)	$(j/80, 1)$ for $j = 1, \ldots, 18$
$S = \sum_j m_j$, $N_{\mathcal{P}}$	18, 21
$\sum_j m_j(1-2\delta_j)$	$18 - \frac{171}{40} = \frac{549}{40}$
$G = \sum_j m_j(1-2\delta_j) - 10r$	$\frac{549}{40} - \frac{55}{6} = \frac{547}{120}$; $qG = 1094$
zero set $\mathcal{Z}_n$	$[1, 220N_0] \cup [-460N_0, -240N_0 - 1]$, each of order 5
$\sum_k B_k$	$3294N_0 + 18$
$\deg R_n$	$-1094N_0 - 17$ (odd)
$\mathcal{N} = (2+r)n + 1$	$700N_0 + 1$

TABLE 1. The parameters of Z7.

All blocks have $\delta_j \geq \frac{1}{80}$, so $B_k = 0$ for $k < 3N_0$: there is no pole at 0, and every pole of R_n lies in $[-n+3N_0, -3N_0]$. The e-factor vanishes at $t = -n/2$, which is the central pole ($B_{n/2} = 18$); so that pole has order 17, and the Taylor data below absorb this automatically ($u_e = 0$ in Lemma 4.3). By Lemma 2.3 and $qG = 1094 \equiv 0$, $S = 18$, the choice $e = 1$ gives $R_n(-n-t) = -R_n(t)$ for every $n \in 240\mathbb{N}$.

3. The linear form lies in $\mathbb{Q} + \mathbb{Q}\zeta(7) + \cdots + \mathbb{Q}\zeta(21)$

In this section $\mathcal{P}$ is any parameter set with $G > 0$ and e as in Lemma 2.3. For $0 \leq k \leq n$ define the power series in $\mathbb{Q}[[\varepsilon]]$

$$f_k(\varepsilon) := (n - 2k + 2\varepsilon)^e \prod_{z \in \mathcal{Z}_n} (-k - z + \varepsilon)^5 \prod_{\substack{0 \leq k' \leq n \\ k' \neq k}} (k' - k + \varepsilon)^{-B_{k'}}, \tag{3.1}$$

with $(d+\varepsilon)^{-1} = \sum_{M \geq 0} (-1)^M d^{-M-1} \varepsilon^M$ for $d \neq 0$. Note $-k - z \neq 0$, because no zero lies in the pole region $[-n, 0]$. If $t_0 := -k$, then $f_k(\varepsilon)$ is the Taylor

expansion of $R_n(t_0+\varepsilon)(t_0+\varepsilon+k)^{B_k}$ at $\varepsilon=0$. We write $u_z := -k-z$, $u_e := n-2k$ and $d_{k'} := k'-k$ when k is fixed.

Lemma 3.1 (Partial fractions)**.** *Put $a_{i,k} := [\varepsilon^{B_k-i}]f_k$ for $0 \le k \le n$, $1 \le i \le B_k$, and $a_{i,k} := 0$ otherwise. Then $a_{i,k} \in \mathbb{Q}$ and, for every $t \in \mathbb{C} \setminus \{-k : B_k \ge 1\}$,*

$$R_n(t) = \sum_{k=0}^{n} \sum_{i=1}^{S} \frac{a_{i,k}}{(t+k)^i}.$$

Proof. By Lemma 2.2 the numerator of (2.1) has smaller degree than the denominator, so R_n is the sum of its principal parts at the distinct poles $-k$. The principal part at $t_0=-k$ is $\sum_{i=1}^{B_k} a_{i,k}(t-t_0)^{-i}$, where $a_{i,k}$ is the coefficient of $(t-t_0)^{B_k-i}$ in the Taylor expansion of $R_n(t)(t-t_0)^{B_k}$, which is f_k. Since $B_k \le S$, all $i \le S$ are covered. □

Lemma 3.2. *For integers $j \ge 2$ and $k \ge 0$, $\sum_{t\ge 1}(t+k)^{-j} = \zeta(j) - H_k^{(j)}$.*

Proof. Substitute $m = t+k$. □

Proposition 3.3 (Expansion)**.** *Define*

$$\tau_i := (i)_4 \sum_{k=0}^{n} a_{i,k}, \qquad \rho_n := -\sum_{k=0}^{n} \sum_{i=1}^{S} (i)_4\, a_{i,k}\, H_k^{(i+4)}.$$

Then $\rho_n, \tau_i \in \mathbb{Q}$ and $L_n = \rho_n + \sum_{i=1}^{S} \tau_i\, \zeta(i+4)$.

Proof. All poles are ≤ 0, so for real $t > 0$ Lemma 3.1 may be differentiated termwise: $R_n''''(t) = \sum_{k,i} a_{i,k}(i)_4 (t+k)^{-i-4}$. Sum over $t \ge 1$; this is a finite sum of absolutely convergent series, and Lemma 3.2 with $j = i+4 \ge 5$ gives the claim. □

Lemma 3.4 (Symmetry of the coefficients)**.** *$a_{i,n-k} = (-1)^{i+1} a_{i,k}$ for all i, k. Consequently $\tau_i = 0$ for every even i.*

Proof. By Lemmas 2.3 and 3.1,

$$R_n(t) = -R_n(-n-t) = -\sum_{k,i} \frac{a_{i,k}}{(-n-t+k)^i} = \sum_{k,i} \frac{(-1)^{i+1} a_{i,k}}{(t+(n-k))^i},$$

and uniqueness of the partial-fraction decomposition gives $a_{i,n-k} = (-1)^{i+1} a_{i,k}$. Summing over k gives $\tau_i = (-1)^{i+1}\tau_i$. □

Lemma 3.5 (The residues add up to zero)**.** *$\sum_k a_{1,k} = 0$; hence $\tau_1 = 0$.*

Proof. As $t \to +\infty$, $tR_n(t) \to 0$ because $\deg R_n \le -2$, while $t\sum_{k,i} a_{i,k}(t+k)^{-i} \to \sum_k a_{1,k}$. □

Corollary 3.6. *$L_n = \rho_n + \sum_i \tau_i\, \zeta(i+4)$, where i runs over the odd integers with $3 \le i \le S$. Hence $L_n \in \mathbb{Q} + \mathbb{Q}\zeta(7) + \mathbb{Q}\zeta(9) + \cdots + \mathbb{Q}\zeta(N_{\mathcal{P}})$; for Z7,*

$$L_n = \rho_n + \sum_{j=3}^{10} \tau_{2j-3}\, \zeta(2j+1) \in \mathbb{Q} + \mathbb{Q}\zeta(7) + \mathbb{Q}\zeta(9) + \cdots + \mathbb{Q}\zeta(21).$$

In particular $\zeta(3)$ and $\zeta(5)$ do not occur.

Proof. In Proposition 3.3 only $\zeta(i+4)$ with $1 \le i \le S$ occurs. Lemma 3.4 removes the even i and Lemma 3.5 removes $i = 1$. For Z7, $i = 2j - 3$ with $3 \le j \le 10$. □

4. Denominators

In this section $\mathcal{P}$ is any parameter set with $G > 0$, e is as in Lemma 2.3, and $n \in q\mathbb{N}$ with $n > r$. For $0 \le k \le n$ we have $0 < |u_z| < \mathcal{N}$ for $z \in \mathcal{Z}_n$ (indeed $|u_z| \le (1+r)n$), $|u_e| \le n < \mathcal{N}$, and $|d_{k'}| \le n < \mathcal{N}$.

4.1. **Valuations of the Taylor coefficients.**

Definition 4.1. Let p be a prime and $\nu, \lambda \in \mathbb{Z}$. A power series $g \in \mathbb{Q}[[\varepsilon]]$ is *(ν, λ)-bounded at p* if $v_p([\varepsilon^M]g) \ge \nu - \lambda M$ for all $M \ge 0$.

Lemma 4.2. *Let p be a prime and $\lambda \in \mathbb{Z}$.*

1. *If g_1 is (ν_1, λ)-bounded and g_2 is (ν_2, λ)-bounded at p, then $g_1 g_2$ is $(\nu_1 + \nu_2, \lambda)$-bounded at p.*
2. *If $u, c \in \mathbb{Q}$ with $v_p(u) \ge \nu$ and $v_p(c) \ge \nu - \lambda$, then $u + c\varepsilon$ is (ν, λ)-bounded at p.*
3. *If $d \in \mathbb{Q} \setminus \{0\}$ with $v_p(d) \le \lambda$ and $\nu \le -v_p(d)$, then $(d+\varepsilon)^{-1}$ is (ν, λ)-bounded at p.*

Proof. (1) $[\varepsilon^M](g_1 g_2) = \sum_{a+b=M} [\varepsilon^a]g_1\, [\varepsilon^b]g_2$, and each term has valuation $\ge \nu_1 - \lambda a + \nu_2 - \lambda b$. (2) is immediate (also when $u = 0$). (3) The coefficient of ε^M is $(-1)^M d^{-M-1}$, of valuation $-v_p(d) - M v_p(d) \ge \nu - \lambda M$. □

Lemma 4.3 (Lemma A)**.** *Let p be a prime, $\lambda \ge 0$ an integer and $0 \le k \le n$. Let $\nu_E \in \mathbb{Z}$, $\nu_z \in \mathbb{Z}$ $(z \in \mathcal{Z}_n)$ and $\nu_{k'} \in \mathbb{Z}$ $(k' \ne k,\ B_{k'} \ge 1)$ satisfy*

- *$v_p(u_e) \ge \nu_E$ and $v_p(2) \ge \nu_E - \lambda$ (only if $e = 1$; put $\nu_E := 0$ if $e = 0$);*
- *$v_p(u_z) \ge \nu_z$ and $\nu_z \le \lambda$ for every $z \in \mathcal{Z}_n$;*
- *$v_p(d_{k'}) \le \lambda$ and $\nu_{k'} \le -v_p(d_{k'})$ for every $k' \ne k$ with $B_{k'} \ge 1$.*

Put $\nu := \nu_E + 5\sum_z \nu_z + \sum_{k' \ne k} B_{k'} \nu_{k'}$. Then f_k is (ν, λ)-bounded at p, and $v_p(a_{i,k}) \ge \nu - \lambda(B_k - i)$ for $1 \le i \le B_k$.

Proof. Apply Lemma 4.2(2) to $u_e + 2\varepsilon$ and to each $u_z + \varepsilon$ (here $v_p(1) = 0 \ge \nu_z - \lambda$), Lemma 4.2(3) to each $(d_{k'} + \varepsilon)^{-1}$, and Lemma 4.2(1) to the product (3.1). □

For a prime p and $0 \le k \le n$ define

$$c_p(k) := 5\,\#\{z \in \mathcal{Z}_n : p \mid k + z\} + e\,[\,p \mid n - 2k\,],$$

$$P_{p,j}(k) := \#\{k' \in I_j \setminus \{k\} : p \mid k' - k\}, \qquad P_p(k) := \sum_j m_j P_{p,j}(k),$$

$$\nu_p(k) := 5 \sum_{z \in \mathcal{Z}_n} v_p(k+z) - \sum_j m_j \sum_{k' \in I_j \setminus \{k\}} v_p(k' - k),$$

and $\lambda_p := \lfloor \log_p \mathcal{N} \rfloor$. Note $\sum_{k' \neq k} B_{k'} g(k') = \sum_j m_j \sum_{k' \in I_j \setminus \{k\}} g(k')$ for any g.

Corollary 4.4. *Let $0 \leq k \leq n$ and $1 \leq i \leq B_k$.*

(b′) *If $P_p(k) = 0$, i.e. no pole $-k' \neq -k$ of R_n has $p \mid k' - k$, then $a_{i,k} \in \mathbb{Z}_{(p)}$.*
(a) *If $p > n$, then $a_{i,k} \in \mathbb{Z}_{(p)}$.*
(b) *If $p^2 > \mathcal{N}$, then $v_p(a_{i,k}) \geq c_p(k) - P_p(k) - (B_k - i)$.*
(c) *$v_p(a_{i,k}) \geq \nu_p(k) - \lambda_p(B_k - i)$.*

Proof. We apply Lemma 4.3; $v_p(u_z) = v_p(k + z)$.

(b′) Take $\lambda = 0$ and all exponents 0. The hypotheses hold because all u's are integers and $v_p(d_{k'}) = 0$ for every pole $-k' \neq -k$.
(a) For $k' \neq k$, $0 < |k' - k| \leq n < p$, so $P_p(k) = 0$; apply (b′).
(b) Take $\lambda = 1$, $\nu_E = [\, p \mid u_e \,]$, $\nu_z = [\, p \mid u_z \,]$ and $\nu_{k'} = -[\, p \mid d_{k'} \,]$. Since $0 < |d_{k'}| < \mathcal{N} < p^2$, $v_p(d_{k'}) = [\, p \mid d_{k'} \,] \leq 1$; and $v_p(u_e) \geq \nu_E$ also when $u_e = 0$. Here $\nu = c_p(k) - P_p(k)$.
(c) Take $\lambda = \lambda_p$, $\nu_E = 0$, $\nu_z = v_p(u_z)$ and $\nu_{k'} = -v_p(d_{k'})$. Since $0 < |u_z|, |d_{k'}| < \mathcal{N} < p^{\lambda_p + 1}$, both valuations are $\leq \lambda_p$. Here $\nu = \nu_p(k)$. □

Part (b′) is the new ingredient. It needs no condition on p, and it is used for $k < p$, where also the harmonic sums are p-integral (Lemma 4.5): the pole $-k$ then costs nothing at p unless another pole lies at a distance divisible by p. (The e-factor term $v_p(n - 2k)$ is dropped from $\nu_p(k)$ in (c); this only weakens the bound.)

Lemma 4.5 (Lemma B)**.** *Let p be a prime, $\lambda \geq 0$, $j \geq 1$ and $k \geq 0$, and suppose $v_p(m) \leq \lambda$ for all $1 \leq m \leq k$. Then $v_p(H_k^{(j)}) \geq -j\lambda$. In particular, for $0 \leq k \leq n$: $H_k^{(j)} \in \mathbb{Z}_{(p)}$ if $k < p$; $v_p(H_k^{(j)}) \geq -j$ if $p^2 > \mathcal{N}$; and $v_p(H_k^{(j)}) \geq -j\lambda_p$ always.*

Proof. Each term m^{-j} has valuation $\geq -j\lambda$. For the special cases use $\lambda = 0$ (if $k < p$), $\lambda = 1$ (as $k \leq n < \mathcal{N} < p^2$) and $\lambda = \lambda_p$ (as $k < \mathcal{N}$). □

4.2. **The common denominator.**

Definition 4.6. For $0 \leq k \leq n$ with $B_k \geq 1$ and a prime p put

$$e_p(k) := \begin{cases} B_k + 4 - c_p(k) + P_p(k), & k \geq p, \\ B_k - 1 - c_p(k) + P_p(k), & k < p \text{ and } P_p(k) \geq 1, \\ 0, & k < p \text{ and } P_p(k) = 0, \end{cases}$$

and define the integer E_p by

$$E_p := \begin{cases} 0, & p > n, \\ \max\bigl(0,\ \max_{k:\, B_k \geq 1} e_p(k)\bigr), & p \leq n,\ p^2 > \mathcal{N}, \\ \max_{0 \leq k \leq n}\bigl(-\nu_p(k)\bigr) + (S+4)\lambda_p, & p^2 \leq \mathcal{N} \end{cases}$$

(the cases are applied in this order). Put $D_n := \prod_{p\le n} p^{E_p} \in \mathbb{Q}_{>0}$.

Thus $v_p(D_n) = E_p$ for every prime p. We do not need $E_p \ge 0$ in the third range, so we only assert $D_n \in \mathbb{Q}_{>0}$.

Proposition 4.7 (Integrality)**.** *$D_n\rho_n \in \mathbb{Z}$ and $D_n\tau_i \in \mathbb{Z}$ for all i.*

Proof. It suffices to show $v_p(\tau_i) \ge -E_p$ and $v_p(\rho_n) \ge -E_p$ for every prime p. Since $(i)_4 \in \mathbb{Z}$, it suffices to bound $v_p(a_{i,k})$ and $v_p(a_{i,k}H_k^{(i+4)})$ from below by $-E_p$, for $0 \le k \le n$ and $1 \le i \le B_k$ (the other $a_{i,k}$ vanish).

- $p > n$: $a_{i,k} \in \mathbb{Z}_{(p)}$ by Corollary 4.4(a), and $H_k^{(i+4)} \in \mathbb{Z}_{(p)}$ by Lemma 4.5 since $k \le n < p$.
- $p \le n$, $p^2 > \mathcal{N}$, $k \ge p$: by Corollary 4.4(b) and Lemma 4.5,

$$v_p\big(a_{i,k}H_k^{(i+4)}\big) \ge c_p(k) - P_p(k) - (B_k - i) - (i+4) = -e_p(k) \ge -E_p,$$

  and $v_p(a_{i,k}) \ge c_p(k) - P_p(k) - B_k + 1 \ge -e_p(k)$.
- $p \le n$, $p^2 > \mathcal{N}$, $k < p$: $H_k^{(i+4)} \in \mathbb{Z}_{(p)}$ by Lemma 4.5. If $P_p(k) \ge 1$, Corollary 4.4(b) gives $v_p(a_{i,k}) \ge c_p(k) - P_p(k) - B_k + 1 = -e_p(k)$. If $P_p(k) = 0$, Corollary 4.4(b′) gives $v_p(a_{i,k}) \ge 0 = -e_p(k)$.
- $p^2 \le \mathcal{N}$: by Corollary 4.4(c) and Lemma 4.5, since $B_k \le S$,

$$v_p\big(a_{i,k}H_k^{(i+4)}\big) \ge \nu_p(k) - \lambda_p(B_k - i) - \lambda_p(i+4) \ge \nu_p(k) - (S+4)\lambda_p \ge -E_p,$$

  and the same bound holds without the factor $H_k^{(i+4)}$. □

Corollary 4.8. *$D_nL_n = D_n\rho_n + \sum_i (D_n\tau_i)\,\zeta(i+4)$ (i odd, $3 \le i \le S$) is a linear form in $1, \zeta(7), \dots, \zeta(N_{\mathcal{P}})$ with integer coefficients.*

4.3. **The small primes.** Let $s_p(N)$ denote the sum of the base-p digits of $N \ge 0$, and $\ell_p(N) := \lfloor \log_p N \rfloor$ for $N \ge 1$, $\ell_p(0) := 0$.

Lemma 4.9 (Legendre-type bounds)**.** *Let p be a prime.*

(1) *For $N \ge 0$: $(p-1)v_p(N!) = N - s_p(N)$ and $0 \le s_p(N) \le (p-1)(\ell_p(N)+1)$.*
(2) *For integers $1 \le A \le B+1$:*

$$(p-1)\sum_{d=A}^{B} v_p(d) \ge (B-A+1) - (p-1)(\ell_p(B)+1),$$

$$(p-1)\sum_{d=A}^{B} v_p(d) \le (B-A+1) + (p-1)(\ell_p(A-1)+1).$$

Proof. (1) is Legendre's formula, and each of the at most $\ell_p(N)+1$ digits is $\le p-1$. (2) The sum equals $v_p(B!) - v_p((A-1)!)$, so $(p-1)$ times it is $(B-A+1) - s_p(B) + s_p(A-1)$; use (1). □

Lemma 4.10. *Let p be a prime, $0 \le k \le n$ and $1 \le j \le J$. Then $(p-1)\sum_{k'\in I_j\setminus\{k\}} v_p(k'-k) \le \#I_j + 2(p-1)(\lambda_p+1)$.*

Proof. The numbers $|k'-k|$ with $k' \in I_j$, $k' > k$, form an interval $[A_1, B_1]$ of positive integers, and those with $k' < k$ form another interval $[A_2, B_2]$; their total length is at most $\#I_j$, and $A_i - 1 \le n < \mathcal{N}$. Apply the upper bound of Lemma 4.9(2) to each (an empty interval contributes 0). □

Proposition 4.11. *Let p be a prime with $p^2 \le \mathcal{N}$. Then $p \le n$ and*

$$E_p - G\, v_p(n!) \le (4S + 14 + G)(\lambda_p + 1). \tag{4.1}$$

Consequently

$$\sum_{p^2 \le \mathcal{N}} \big(E_p - G\, v_p(n!)\big) \log p \le 2(4S + 14 + G)\lfloor \sqrt{\mathcal{N}} \rfloor \log \mathcal{N}.$$

For Z7 the constants are $4S + 14 + G \le 91$ and $2(4S + 14 + G) \le 182$.

Proof. Since $n > r$, $\mathcal{N} = (2 + r)n + 1 < (n + 1)^2$, so $p \le n$. Fix k. As z runs through $\mathcal{Z}_n$, $k + z$ runs through the two intervals $[k + 1, k + rn]$ and $[k - (1 + r)n, k - n - 1]$, each consisting of rn consecutive nonzero integers of absolute value $< \mathcal{N}$. By the lower bound of Lemma 4.9(2), applied to the absolute values,

$$(p - 1) \cdot 5 \sum_{z \in \mathcal{Z}_n} v_p(k + z) \ge 10rn - 10(p - 1)(\lambda_p + 1).$$

By Lemma 4.10 and Lemma 2.2,

$$(p - 1) \sum_j m_j \sum_{k' \in I_j \setminus \{k\}} v_p(k' - k) \le (G + 10r)n + S + 2S(p - 1)(\lambda_p + 1).$$

Hence $(p-1)(-\nu_p(k)) \le Gn + S + (2S+10)(p-1)(\lambda_p+1)$. By Lemma 4.9(1) and $\ell_p(n) \le \lambda_p$, $(p - 1)v_p(n!) \ge n - (p - 1)(\lambda_p + 1)$. Therefore

$$\begin{aligned}(p - 1)\big(E_p - G v_p(n!)\big) \le{}& S + (2S + 10)(p - 1)(\lambda_p + 1) \\ &+ (S + 4)(p - 1)\lambda_p + G(p - 1)(\lambda_p + 1),\end{aligned}$$

and dividing by $p - 1 \ge 1$ gives (4.1). For the sum, each term is at most $(4S + 14 + G)(\lambda_p \log p + \log p) \le 2(4S + 14 + G) \log \mathcal{N}$, since $p^{\lambda_p} \le \mathcal{N}$ and $p \le \mathcal{N}$, and there are at most $\lfloor \sqrt{\mathcal{N}} \rfloor$ primes in the range. For Z7, $4 \cdot 18 + 14 + \frac{547}{120} < 91$. □

There is no exceptional prime here: $p = 2$ and $p = 3$ are covered by the same count, because all zeros and poles of R_n are integers. (For the β-analogue [10], with half-integer poles, the prime 2 costs a term proportional to n.)

Proposition 4.12. *For every $n \in q\mathbb{N}$ with $n > r$,*

$$\log D_n - G \log n! \le 2(4S + 14 + G)\lfloor \sqrt{\mathcal{N}} \rfloor \log \mathcal{N} + \sum_{\substack{p \le n \\ p^2 > \mathcal{N}}} \big(E_p - G\lfloor n/p \rfloor\big) \log p.$$

Proof. By definition and Legendre's formula, $\log D_n - G\log n! = \sum_{p\le n}(E_p - Gv_p(n!))\log p$. Split the sum into the primes with $p^2 \le \mathcal{N}$ (Proposition 4.11) and those with $p^2 > \mathcal{N}$, for which $p^2 > \mathcal{N} > n$ and so $v_p(n!) = \lfloor n/p \rfloor$. □

5. The arithmetic integral

In this section we bound the last sum in Proposition 4.12 by $n(A_{\mathrm{cert}} + o(1))$. The definitions are for a general parameter set; the certificate is for Z7, where

(5.1)
$$A_{\mathrm{cert}} := \frac{1175799466448418772261184074309615 3}{1185451381619680151243401356902400} = 9.9185802528\ldots$$

5.1. **The step function.**

Definition 5.1. For real $x \ge 1$ and $0 \le y \le x$ put

$$\mathcal{S}(x,y) := \sum_j m_j \,[\,\delta_j x \le y \le (1-\delta_j)x\,],$$

$$P(x,y) := \sum_j m_j \,\#\{m \in \mathbb{Z}\setminus\{0\} :\ \delta_j x \le y+m \le (1-\delta_j)x\},$$

$$c(x,y) := 5\big(\lfloor y + rx\rfloor - \lfloor y\rfloor\big) + 5\big(\lceil y - x\rceil - \lceil y-(1+r)x\rceil\big),$$

$$b(x,y) := \begin{cases} \mathcal{S}+4-c+P, & y \ge 1,\\ \mathcal{S}-1-c+P, & y<1 \text{ and } P\ge 1,\\ 0, & y<1 \text{ and } P=0,\end{cases}$$

$$g(x,y) := \begin{cases} b(x,y), & \mathcal{S}(x,y) > 0,\\ 0, & \mathcal{S}(x,y) = 0,\end{cases}$$

(arguments (x,y) omitted on the right), and

$$e^*(x) := \max\Big(0, \max_{0\le y\le x} g(x,y)\Big) - G\lfloor x\rfloor.$$

The maximum exists because $g(x,\cdot)$ takes finitely many values on $[0,x]$. For Z7 no block contains $y=0$, so $g(x,0)=0$ and the outer $\max(0,\cdot)$ is redundant; the formalisation defines e^* without it. The formalisation also writes the pole count of block j by cases: $\lfloor y-\delta_j x\rfloor + \lfloor (1-\delta_j)x - y\rfloor$ in the block, and $\lfloor (1-\delta_j)x - y\rfloor - \lfloor \delta_j x - y\rfloor + [\,\delta_j x - y \in \mathbb{Z}\,]$ (resp. the mirror expression) to the left (resp. right) of it; by $\lceil a\rceil = \lfloor a\rfloor + 1 - [\,a\in\mathbb{Z}\,]$ this is the count above.

Proposition 5.2 (Pointwise bound)**.** *Let $p \le n$ be a prime. Then $\max\big(0, \max_{k:B_k\ge 1} e_p(k)\big) - G\lfloor n/p\rfloor \le e^*(n/p)$. In particular $E_p - G\lfloor n/p\rfloor \le e^*(n/p)$ whenever $p^2 > \mathcal{N}$.*

Proof. Let $0 \le k \le n$ with $B_k \ge 1$, and put $x := n/p \ge 1$ and $y := k/p \in [0,x]$. Since $a_j/p = \delta_j x$ and $b_j/p = (1-\delta_j)x$, we have $B_k = \mathcal{S}(x,y)$, and, writing $k' = k + mp$, $P_{p,j}(k) = \#\{m \ne 0 : \delta_j x \le y+m \le (1-\delta_j)x\}$,

so $P_p(k) = P(x,y)$. The zeros $z \in [1, rn]$ with $p \mid k+z$ correspond to the multiples of p in $[k+1, k+rn]$, of which there are $\lfloor (k+rn)/p \rfloor - \lfloor k/p \rfloor = \lfloor y + rx \rfloor - \lfloor y \rfloor$. The zeros $z = -l$, $l \in [n+1, (1+r)n]$, with $p \mid l-k$ correspond to the multiples of p in $[n+1-k, (1+r)n-k]$, of which there are $\lfloor (1+r)x - y \rfloor - \lfloor x - y \rfloor = \lceil y - x \rceil - \lceil y - (1+r)x \rceil$. Hence $c_p(k) = c(x,y) + e[\, p \mid n - 2k\,] \geq c(x,y)$. Finally $k \geq p$ if and only if $y \geq 1$. Comparing the three cases of Definition 4.6 with those of b, we get $e_p(k) \leq b(x,y) = g(x,y)$. Take the maximum over k and subtract $G\lfloor x \rfloor$. □

Lemma 5.3 (Uniform bound)**.** *For $x \geq 1$, $e^*(x) \leq S + 14 + G\{x\}$.*

Proof. For each block, $[\, \delta_j x \leq y \leq (1-\delta_j)x\,] + \#\{m \neq 0 : \dots\} = \#\{m \in \mathbb{Z} : \delta_j x \leq y + m \leq (1-\delta_j)x\} \leq (1 - 2\delta_j)x + 1$. Each of the two zero counts in c is at least $rx - 1$, so $c \geq 10rx - 10$. Hence, if $y \geq 1$, $b = \sum_j m_j([\,\cdot\,] + \#\{\cdot\}) + 4 - c \leq S + (G + 10r)x + 4 - 10rx + 10 = S + 14 + Gx$, and the other two cases of b are smaller or 0. So $\max(0, \max_y g) \leq S + 14 + Gx$, and $e^*(x) \leq S + 14 + Gx - G\lfloor x \rfloor$. □

For Z7 this reads $e^*(x) \leq 32 + G\{x\}$.

5.2. **Step structure.** Writing $\#\{m \in \mathbb{Z} : \delta x \leq y + m \leq (1-\delta)x\} = 1 + \lfloor y - \delta x \rfloor - \lceil y - (1-\delta)x \rceil$, $[\, \delta x \leq y \leq (1-\delta)x\,] = [\, \lfloor y - \delta x \rfloor \geq 0\,] \cdot [\, \lceil y - (1-\delta)x \rceil \leq 0\,]$ and $[\, y \geq 1\,] = [\, \lfloor y \rfloor \geq 1\,]$, we see that $g(x,y)$ depends on y only through the numbers $\lfloor y - sx \rfloor$ and $\lceil y - sx \rceil$ for the slopes

$$s \in \mathcal{T} := \{0,\ -r,\ 1,\ 1+r\} \cup \{\delta_j,\ 1 - \delta_j : 1 \leq j \leq J\}.$$

(The slopes 0 and 1 also carry the constraint $0 \leq y \leq x$.) For Z7, $\mathcal{T} = \{\sigma/240\}$ with $\sigma \in \{0, -220, 240, 460\} \cup \{3j, 240 - 3j : 1 \leq j \leq 18\}$. Let $X := 400$. The *event set* $\mathcal{E} \subset [1, X]$ is the union of

- the points $1 + j/20$ $(0 \leq j < 40)$ and $3 + j/2$ $(0 \leq j \leq 2(X-3))$, which include every integer in $[1, X]$;
- all $x \in (1, X)$ of the form $x = m/(s_1 - s_2)$ with $s_1, s_2 \in \mathcal{T}$, $s_1 > s_2$, and $m \in \mathbb{Z}$.

This is a finite set of rationals, $1 = x_0 < x_1 < \cdots < x_L = X$. For Z7 it has 30,138 elements, so $L = 30{,}137$.

Lemma 5.4. *e^* is constant on each open interval $(x_{\ell-1}, x_\ell)$.*

Proof. Let $J = (x_{\ell-1}, x_\ell)$ and $x, x' \in J$. Consider the discrete set $C(x) := \{sx + m : s \in \mathcal{T},\ m \in \mathbb{Z}\}$. For two of its elements the difference $(s - s')x + (m - m')$ is an affine function of x; if it is not identically zero, its only zero is an event point or lies outside $(1, X)$, so it does not vanish on J. Hence $sx + m \mapsto sx' + m$ is a well-defined, strictly increasing bijection $C(x) \to C(x')$, which extends to an increasing bijection $\pi : \mathbb{R} \to \mathbb{R}$. For every $s \in \mathcal{T}$ and every y, $sx + m \leq y \iff sx' + m \leq \pi(y)$, so $\lfloor \pi(y) - sx' \rfloor = \lfloor y - sx \rfloor$, and likewise for ceilings. Moreover $\pi(0) = 0$ and $\pi(x) = x'$ (slopes 0 and 1), so π maps $[0, x]$ onto $[0, x']$, and $\lfloor x \rfloor = \lfloor x' \rfloor$ because $\mathcal{E}$ contains every integer of $[1, X]$. Therefore $g(x', \pi(y)) = g(x, y)$ for all y, and $e^*(x') = e^*(x)$. □

The proof also shows how e^* is computed exactly at a rational point ξ: $g(\xi, \cdot)$ is constant between consecutive points of $C(\xi)$, so the maximum over $[0, \xi]$ is attained at a point of $C(\xi)$ or as a right limit there. In the formalisation, with $\xi = P/Q$ and $M = 240Q$, all points of $C(\xi)$ lie in $\frac{1}{M}\mathbb{Z}$, and $e^*(\xi)$ is a maximum over the residues modulo M of the scaled offsets of one closed-form candidate each; that this maximum equals $\max_{0 \le y \le \xi} g(\xi, y)$, and is attained, is proved in general.

5.3. **The certificate.**

Proposition 5.5 (Certified computation, Z7)**.** *Let* $\xi_\ell := (x_{\ell-1} + x_\ell)/2$ *and* $A_{\mathrm{body}} := \sum_{\ell=1}^{L} e^*(\xi_\ell)\big(\frac{1}{x_{\ell-1}} - \frac{1}{x_\ell}\big)$*. Then*

$$\sum_{\ell=1}^{L} \Big\lceil 2^{96} e^*(\xi_\ell)\Big(\frac{1}{x_{\ell-1}} - \frac{1}{x_\ell}\Big)\Big\rceil = U := 778841997893779320302794447562.$$

Consequently $A_{\mathrm{body}} \le U/2^{96} = 9.8303680\ldots$ *and*

$$\frac{U}{2^{96}} + \frac{33}{400} + \frac{G}{2 \cdot 399} = A_{\mathrm{cert}},$$

with A_{cert} *as in* (5.1)*. (Note* $33/400 + G/798 = 10559/119700$*, and* $33 = S + 15$*.)*

Proof. This is a finite computation with explicit rationals: enumerate $\mathcal{E}$, evaluate $e^*(\xi_\ell)$ exactly, round each term up to a multiple of 2^{-96}, and add. The last identity is exact rational arithmetic. The computation has been carried out by an exact-rational Python implementation and by the Lean kernel, with the same result; see Section 9.5. □

5.4. **From primes to the integral.**

Lemma 5.6 (Tail)**.** *Let* $X \ge 2$ *be an integer,* $C, G' \ge 0$*, and* $h : [X, \infty) \to \mathbb{R}$ *with* $h(y) \le C + G'\{y\}$*. For every* $\varepsilon > 0$*, for all sufficiently large* n*,*

$$\sum_{p \le n/X} h(n/p) \log p \le n\Big(\frac{C}{X} + \frac{G'}{2(X-1)}\Big) + \varepsilon n.$$

Proof. Choose an integer $K_0 \ge 1$ with $G'/(2K_0(X-1)) \le \varepsilon/4$ and an integer $M > X$ with $(C + G') \log 4/M \le \varepsilon/4$. The primes $p \le n/M$ contribute at most $(C+G')\theta(n/M) \le (C+G')(\log 4)\, n/M \le \varepsilon n/4$, by Chebyshev's bound $\theta(y) \le y \log 4$. Cut $[X, M)$ into the $K_0(M - X)$ cells $[m + \frac{i-1}{K_0}, m + \frac{i}{K_0})$, $X \le m < M$, $1 \le i \le K_0$. On such a cell $h \le C + G'i/K_0$. The primes with n/p in the cell $[a, b)$ are those in $(n/b, n/a]$, and by the prime number theorem $\theta(n/a) - \theta(n/b) = n(1/a - 1/b) + o(n)$. Since there are finitely many cells, for large n the primes $n/M < p \le n/X$ contribute at most

$$n \sum_{m=X}^{M-1} \sum_{i=1}^{K_0} \Big(C + \frac{G'i}{K_0}\Big)\Big(\frac{1}{m + \frac{i-1}{K_0}} - \frac{1}{m + \frac{i}{K_0}}\Big) + \frac{\varepsilon n}{4}.$$

The C-part telescopes to $C(1/X-1/M) \le C/X$. In the G'-part each bracket is at most $1/(K_0 m^2)$, so the inner sum is at most $G'(K_0+1)/(2K_0 m^2)$; since $\sum_{m \ge X} m^{-2} \le 1/(X-1)$, the G'-part is at most $\frac{G'}{2(X-1)} + \frac{\varepsilon}{4}$. □

Proposition 5.7. *For Z7 there are $\varepsilon_n \to 0$ such that, for all large $n \in 240\mathbb{N}$,*

$$\sum_{\substack{p \le n \\ p^2 > \mathcal{N}}} \big(E_p - G\lfloor n/p \rfloor\big) \log p \le n(A_{\mathrm{cert}} + \varepsilon_n).$$

Proof. Fix $\varepsilon > 0$ and define h on $[1,\infty)$ by $h(x) := e^*(x)$ for $x < 400$ and $h(x) := \max(e^*(x), 0)$ for $x \ge 400$. Let n be large; in particular $n/\sqrt{\mathcal{N}} \ge 400$, which holds as soon as $N_0 \ge 400^2$.

(1) By Proposition 5.2, each term of the sum is at most $h(n/p)\log p$. Every prime $p \le n$ not in the sum has $p^2 \le \mathcal{N}$, hence $n/p \ge 400$ and $h(n/p) \ge 0$. So the sum is at most $\sum_{p \le n} h(n/p)\log p$.

(2) *Primes $n/400 < p \le n$*, i.e. $n/p \in [1, 400)$. Let $v_\ell := e^*(\xi_\ell)$. By Lemma 5.4, $h(n/p) = v_\ell$ when $n/p \in (x_{\ell-1}, x_\ell)$, and by the prime number theorem $\sum_{x_{\ell-1} < n/p < x_\ell} \log p = n(1/x_{\ell-1} - 1/x_\ell) + o(n)$ for each of the finitely many ℓ. At most $L+1$ primes have $n/p \in \mathcal{E}$, and each contributes $O(\log n)$, since $|e^*| \le S + 14 + 400G$ on $[1,400]$ by Lemma 5.3. Hence this part is at most $nA_{\mathrm{body}} + \varepsilon n/2$ for large n.

(3) *Primes $p \le n/400$.* By Lemma 5.3, $h(x) \le 32 + G\{x\} \le 33 + G\{x\}$ for $x \ge 400$ (the certificate uses $33 = S+15$). Lemma 5.6 with $X = 400$, $C = 33$, $G' = G$ bounds this part by $n(33/400 + G/798) + \varepsilon n/2$.

Adding up and using Proposition 5.5, the sum is at most $n(A_{\mathrm{cert}} + \varepsilon)$ for large n. □

The same argument works for any parameter set, with $C = S + 15$ and with A_{cert} the certified bound for that set.

Corollary 5.8. *For Z7 there are $\varepsilon_n' \to 0$ with $\log D_n \le G \log n! + n(A_{\mathrm{cert}} + \varepsilon_n')$ for all large $n \in 240\mathbb{N}$.*

Proof. Combine Propositions 4.12 and 5.7; the term $182\lfloor\sqrt{\mathcal{N}}\rfloor \log \mathcal{N}$ is $O(\sqrt{n}\log n) = o(n)$. □

6. Size and nonvanishing of L_n

We extend R_n to the rational function of $z \in \mathbb{C}$ given by (2.1). It has real coefficients, and its poles are the points $-k$ with $B_k \ge 1$, all ≤ 0 (for Z7, all $\le -3N_0$). Put $k(z) := \pi\cot(\pi z)$ and $k_j := k^{(j)}$.

6.1. **An integral representation.**

Lemma 6.1. *$R_n''''(t) = 0$ for every integer $t \in [1, rn]$.*

Proof. Near such t, $R_n(s) = (s-t)^5 g(s)$ with g holomorphic at t, so all derivatives of order ≤ 4 vanish at t. □

Lemma 6.2. *Let $c > \max\{-k : B_k \geq 1\}$ and $j \geq 0$. There is $C = C(n, c, j)$ with $|R_n^{(j)}(z)| \leq C(1 + |z|^2)^{-1}$ whenever $\operatorname{Re} z \geq c$. Moreover $R_n^{(j)}(\bar{z}) = \overline{R_n^{(j)}(z)}$.*

Proof. By Lemma 2.2, $|R_n(w)| \leq C_1|w|^{-2}$ for $|w| \geq R_1$. Cauchy's estimate on circles of radius $\frac{1}{2}$ gives $|R_n^{(j)}(z)| \leq C_1'|z|^{-2}$ for $|z| \geq 2R_1 + 2$, and on the compact set $\{\operatorname{Re} z \geq c,\ |z| \leq 2R_1 + 2\}$, which contains no pole, $R_n^{(j)}$ is bounded. The symmetry holds because R_n has real coefficients. □

Lemma 6.3 (The kernel)**.** *With $\mathsf{C} := \cot(\pi z)$ and $\mathsf{S} := 1/\sin(\pi z)$,*

$$k_1 = -\pi^2\mathsf{S}^2, \quad k_2 = 2\pi^3\mathsf{C}\mathsf{S}^2,$$
$$k_3 = -2\pi^4(\mathsf{S}^4 + 2\mathsf{C}^2\mathsf{S}^2), \quad k_4 = 8\pi^5(\mathsf{C}^3\mathsf{S}^2 + 2\mathsf{C}\mathsf{S}^4).$$

Consequently $\sin^5(\pi z)\, k_4(z) = \pi^5 T(z)$ with the entire function

$$T(z) := 8\cos(\pi z)\big(\cos^2(\pi z) + 2\big) = 2\cos 3\pi z + 22\cos \pi z$$
$$= e^{-3\pi i z}\big(1 + 11\mathbf{q} + 11\mathbf{q}^2 + \mathbf{q}^3\big), \qquad \mathbf{q} := e^{2\pi i z},$$

and $|T(z)| \leq 24e^{3\pi \operatorname{Im} z}$ for $\operatorname{Im} z \geq 0$. The kernel k has a simple pole with residue 1 at every integer; $k_0, \dots, k_4$ are bounded on every vertical line $\operatorname{Re} z = c \notin \mathbb{Z}$ and on $\{|\operatorname{Im} z| \geq 1\}$.

Proof. Use $\mathsf{C}' = -\pi\mathsf{S}^2$ and $\mathsf{S}' = -\pi\mathsf{C}\mathsf{S}$ and differentiate. Then $\sin^5(\pi z)k_4 = 8\pi^5\cos(\pi z)(\cos^2(\pi z) + 2)$, and $8\cos^3 u + 16\cos u = 2\cos 3u + 22\cos u$. Expanding the cosines in exponentials gives the $\mathbf{q}$-form, and $|\mathbf{q}| \leq 1$ for $\operatorname{Im} z \geq 0$. The last statements are standard. □

Proposition 6.4. *Put $G_n(z) := R_n(z)k_4(z)$. For every real $c \in (0, rn+1) \setminus \mathbb{Z}$,*

$$L_n = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R_n''''(z)\, k(z)\, dz = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_n(z)\, dz$$
$$= -\frac{1}{\pi}\operatorname{Re}\int_0^\infty G_n(c+iy)\, dy.$$

Proof. First equality. Let $N > c$ be an integer and $Y > 1$, and integrate $R_n''''k$ counterclockwise around the rectangle with vertices $c \pm iY$, $N + \frac{1}{2} \pm iY$. Inside, the integrand is meromorphic with simple poles at the integers $t \in (c, N]$, with residue $R_n''''(t)$. On the horizontal sides k is bounded and $|R_n''''(z)| \leq C(1+|z|^2)^{-1} \leq CY^{-2}$, so these sides contribute $O(N/Y^2)$, which tends to 0 as $Y \to \infty$. On $\operatorname{Re} z = N + \frac{1}{2}$, $|k| \leq \pi$ and $|R_n''''| \leq C(1 + |z|^2)^{-1}$, so that side tends to 0 as $N \to \infty$. On $\operatorname{Re} z = c$ the integral converges absolutely. Hence $-\int_{c-i\infty}^{c+i\infty} R_n''''k\, dz = 2\pi i \sum_{t>c} R_n''''(t) = 2\pi i\, L_n$, where the last equality uses Lemma 6.1: every integer $1 \leq t < c$ satisfies $t \leq rn$.

Second equality. On the line $\operatorname{Re} z = c$ put $W := R_n'''k_0 - R_n''k_1 + R_n'k_2 - R_nk_3$. Then $W' = R_n''''k_0 - R_nk_4$ (the other terms cancel in pairs). The k_j are bounded on the line and $R_n^{(j)}(c + iy) \to 0$ as $|y| \to \infty$ (Lemma 6.2), so

$W \to 0$ at both ends, and both $R_n''''k_0$ and $R_n k_4$ are integrable on the line. By the fundamental theorem of calculus, $\int_{\operatorname{Re} z=c}(R_n''''k_0 - R_n k_4)\,dz = 0$.

Third equality. $G_n(c-iy) = \overline{G_n(c+iy)}$, because R_n and k_4 commute with complex conjugation. Hence $\int_{c-i\infty}^{c+i\infty} G_n\,dz = i\int_{\mathbb{R}} G_n(c+iy)\,dy = 2i\operatorname{Re}\int_0^\infty G_n(c+iy)\,dy$. □

Remark 6.5. One can also work with $R_n''''k$ directly: on the upper half of the line split $\pi\cot\pi z = -\pi i + K_+(z)$ with $K_+(z) = -2\pi i\mathbf{q}/(1-\mathbf{q})$ and integrate the constant part exactly. This gives $L_n = -R_n'''(c) - \frac{1}{\pi}\operatorname{Im}\big(\int R_n'''' K_+\big)$, with a boundary term $-R_n'''(c)$ (an Euler–Maclaurin term) that has to be bounded separately. The phase is the same: $\sin^5(\pi z)\,e^{2\pi i z}$ has leading exponential $e^{-3\pi i z}$, which is the origin of the phase 3π in both routes. Moving the four derivatives onto the kernel, as above, removes the boundary term, the Faà di Bruno expression for R_n'''' and all asymptotics of the digamma function and its derivatives; only Stirling's formula for $\log\Gamma$ is needed.

6.2. **Gamma factorisation.** We use the following factors (c_i, s_i, u_i, β_i):

$$(5,+1,0,0),\quad (5,-1,r,1),\quad (5,+1,1+r,1),\quad (-5,+1,1,1),$$
$$\text{and for each block } j:\quad (m_j,+1,\delta_j,0),\quad (-m_j,+1,1-\delta_j,1).$$

Put $\alpha_i(x) := s_i x + u_i$. One checks directly that

(6.1)
$$\sum_i c_i s_i = 0, \qquad \sum_i c_i u_i = -G, \qquad \sum_i c_i\big(\beta_i - \tfrac12\big) = -S, \qquad \sum_i c_i = 10,$$

so that $\sum_i c_i\alpha_i(x) = -G$ for all x. For Z7 there are 40 factors, with $\sum_i |c_i| = 56$ and $\sum_i |c_i(\beta_i - \frac12)| = 28$.

Lemma 6.6. *For $z \in \mathbb{C}\setminus\mathbb{Z}$,*

$$R_n(z) = (-1)^{rn}(2z+n)^e\,E_n(z)\,\frac{\sin^5\pi z}{\pi^5}, \qquad E_n(z) := \prod_i \Gamma(s_i z + n u_i + \beta_i)^{c_i}.$$

Hence, for $\operatorname{Im} z > 0$, $G_n(z) = (-1)^{rn}(2z+n)^e E_n(z)\,T(z)$. For Z7, $(-1)^{rn} = 1$ and $e = 1$.

Proof. $\prod_{l=1}^{rn}(z-l) = (-1)^{rn}\Gamma(rn+1-z)/\Gamma(1-z) = (-1)^{rn}\Gamma(z)\Gamma(rn+1-z)\sin(\pi z)/\pi$ by the reflection formula, $\prod_{l=n+1}^{(1+r)n}(z+l) = \Gamma(z+(1+r)n+1)/\Gamma(z+n+1)$, and $\prod_{k=a_j}^{b_j}(z+k) = \Gamma(z+b_j+1)/\Gamma(z+a_j)$. Raise to the appropriate powers and use $(-1)^{5rn} = (-1)^{rn}$. The second statement follows from Lemma 6.3. □

For $\operatorname{Im} z > 0$ every argument $s_i z + n u_i + \beta_i$ has nonzero imaginary part, so E_n is holomorphic and zero-free there, and G_n involves no singular kernel.

6.3. **Stirling's formula in sectors.** Let $\log\Gamma$ denote the branch of the logarithm of Γ on $\mathbb{C}\setminus(-\infty,0]$ that is real on $(0,\infty)$.

Theorem 6.7 (cf. [6, §5.11(ii)])**.** *Let $0<\varphi\le\pi$ and $\varsigma:=\sin(\varphi/2)$. For $w\neq 0$ with $|\arg w|\le\pi-\varphi$,*

$$\Big|\log\Gamma(w)-\big((w-\tfrac12)\log w-w+\tfrac12\log 2\pi\big)\Big|\le\frac{1}{12\varsigma^2|w|}.$$

Proof. This is the case $n=1$ of the error bound in [6, §5.11(ii)]: the remainder of the Stirling series [6, (5.11.1)] after its first term is at most the first neglected term $1/(12|w|)$ times $\sec^2(\frac12\arg w)\le 1/\varsigma^2$. We recall the argument, which is the one formalised. With $Q_1(t):=\frac12\{t\}(1-\{t\})\in[0,\frac18]$ one has $\log\Gamma(w)=(w-\frac12)\log w-w+\frac12\log 2\pi+\int_0^\infty Q_1(t)(t+w)^{-2}\,dt$. If $|\arg w|\le\pi-\varphi$ and $t\ge 0$, then $|t+w|\ge\varsigma(t+|w|)$. For $\varrho>0$, the identity $\int_0^1\frac{u(1-u)/2}{(u+x)^2}du=(x+\frac12)\log(1+\frac1x)-1\le\frac{1}{12x}-\frac{1}{12(x+1)}$, summed over the unit intervals, gives $\int_0^\infty Q_1(t)(t+\varrho)^{-2}dt\le 1/(12\varrho)$. Take $\varrho=|w|$. □

Corollary 6.8. *Let $0<\varphi\le\pi$, $\varsigma=\sin(\varphi/2)$, $a>0$, and let $\alpha\in\mathbb{C}$ with $|\alpha|\ge a$ and $|\arg\alpha|\le\pi-\varphi$. Let $\beta\in[0,2]$ and $n\ge 4/a$. Then*

$$\Big|\log\Gamma(n\alpha+\beta)-\big[n\alpha\log n+n(\alpha\log\alpha-\alpha)+(\beta-\tfrac12)(\log n+\log\alpha)+\tfrac12\log 2\pi\big]\Big|\le\frac{9+1/(6\varsigma^2)}{an}.$$

Proof. Put $w=n\alpha+\beta$ and $u=\beta/(n\alpha)$, so $|u|\le 2/(an)\le\frac12$ and $|w|\ge an/2$. Adding $\beta\ge 0$ moves the argument towards 0, so $|\arg w|\le|\arg\alpha|$, and $\log w=\log n+\log\alpha+\log(1+u)$ with principal logarithms. Insert this into Theorem 6.7 and use $|\log(1+u)-u|\le|u|^2$ and $|\log(1+u)|\le\frac32|u|$ for $|u|\le\frac12$. □

6.4. **The phase and the integrand.** On the set of $x\in\mathbb{C}$ with $\alpha_i(x)\notin(-\infty,0]$ for all i (it contains the open upper half-plane and the real interval $(\max_j(-\delta_j),r)$) define, with principal logarithms,

$$\Psi(x):=\sum_i c_i\,\alpha_i(x)\log\alpha_i(x),\qquad f(x):=\Psi(x)-3\pi i x,$$

$$\psi_0(x):=\sum_i c_i(\beta_i-\tfrac12)\log\alpha_i(x),\qquad A(x):=(2x+1)^e\,e^{\psi_0(x)}.$$

By (6.1), $\Psi'(x)=\sum_i c_i s_i\log\alpha_i(x)$, $\Psi''(x)=\sum_i c_i/\alpha_i(x)$ and $\Psi'''(x)=-\sum_i c_i s_i/\alpha_i(x)^2$. We say that the *sector condition* (a,η) *holds at* x if $|\alpha_i(x)|\ge a$ and $|\arg\alpha_i(x)|\le\pi-\eta$ for every i. Put

$$\lambda_n:=-Gn\log n+Gn-S\log n+5\log 2\pi.$$

Proposition 6.9. *Let $a>0$ and $0<\eta\le\pi$. There are n_1 and a function $B(\varrho)$, depending only on $\mathcal{P}$, a and η, with the following property. Let $n\ge n_1$,*

and let x satisfy $\operatorname{Im} x > 0$, $\operatorname{Re} x \geq 0$ and the sector condition (a, η). Then

$$G_n(nx) = (-1)^{rn}\, n^e\, e^{\lambda_n + nf(x)} A_n(x)$$

for a number $A_n(x)$ with

(U) $|A_n(x)| \leq B(|x|)$;
(M) $|A_n(x) - A(x)| \leq B(|x|)\big(n^{-1} + e^{-2\pi n \operatorname{Im} x}\big)$.

Proof. Let $\varsigma = \sin(\eta/2)$ and apply Corollary 6.8 to each factor, with $\alpha = \alpha_i(x)$ and $\beta = \beta_i$; note $s_i(nx) + nu_i + \beta_i = n\alpha_i(x) + \beta_i$. Summing with weights c_i and using (6.1), the main terms add up to $-Gn\log n + n\Psi(x) + Gn - S\log n + \psi_0(x) + 5\log 2\pi = \lambda_n + n\Psi(x) + \psi_0(x)$. Hence $E_n(nx) = \exp(\lambda_n + n\Psi(x) + \psi_0(x) + e_n)$ with $|e_n| \leq \sum_i |c_i|\,(9 + 1/(6\varsigma^2))/(an)$. By Lemmas 6.6 and 6.3, $(2nx+n)^e = n^e(2x+1)^e$ and $T(nx) = e^{-3\pi i n x} Q(\mathbf{q})$ with $Q(\mathbf{q}) = 1 + 11\mathbf{q} + 11\mathbf{q}^2 + \mathbf{q}^3$ and $|\mathbf{q}| = e^{-2\pi n \operatorname{Im} x}$. So the formula holds with

$$A_n(x) := (2x+1)^e\, e^{\psi_0(x) + e_n}\, Q(\mathbf{q}).$$

Since $a \leq |\alpha_i(x)| \leq |x| + |u_i| + 2$, $|\psi_0(x)|$ and $|2x+1|$ are bounded in terms of a and $|x|$. With $|Q| \leq 24$, $|Q - 1| \leq 23|\mathbf{q}|$ and $|e^{e_n} - 1| \leq 2|e_n|$ for large n, (U) and (M) follow. □

(For Z7 the formalisation proves (U) and (M) in the explicit forms $|A_n| \leq 72B^2$ and $|A_n - A| \leq 100B^2\delta_n$, where $B \geq 1$ bounds $|2x+1|$ and $|e^{\psi_0(x)}|$, and $\delta_n \leq 1$ bounds $|e_n|$ and $|\mathbf{q}|$.)

6.5. **The certified saddle data.**

Theorem 6.10 (Certified saddle data, Z7)**.** *For Z7 put $a := 1/100$, $\eta := \pi - 2.76$, $\rho := 1/20$, $\delta_0 := 10^{-7}$, $\eta_{\mathrm{far}} := 0.0199414$ and*

$$c_0 := 0.89216430079417582861020 76 + 0.18291207184510905389248 22\, i.$$

There are $x_0 \in \mathbb{C}$, $d \in \mathbb{C}$ with $|d| = 1$, and a polyline $v_0, \ldots, v_{23} \in \mathbb{C}$ with

$$v_0 = x_c := 0.817411103201, \qquad v_5 = x_0 - \rho d, \qquad v_6 = x_0 + \rho d,$$
$$v_{23} = x_{\mathrm{top}} := 2.051154018351 + 12.001186644670\, i,$$

and $\operatorname{Re} v_k \geq 0$ for all k, such that the following hold, where $\Omega := f(x_0)$ and $K := |\Psi''(x_0)|$.

(S0) *$|x_0 - c_0| \leq 10^{-12}$, and x_0 is the unique zero of $\Psi' - 3\pi i$ in this disc. Moreover*

$$-10.1167486 \leq \operatorname{Re}\Omega \leq -10.1167485,$$
$$-7.704987202 \leq \operatorname{Im}\Omega \leq -7.704987200, \qquad \sin(240\ \operatorname{Im}\Omega) < -0.9.$$

(S1) *$K \geq 28$, $\Psi''(x_0)d^2 = -K$, and there is $M \leq 311$ with $|\Psi'''(x)| \leq M$ for $|x - x_0| \leq \rho$ and $M\rho \leq \frac{3}{2}K$.*

(S1′) *$\rho < \operatorname{Im} x_0$, so the closed disc $|x - x_0| \leq \rho$ lies in $\operatorname{Im} x > 0$; and $\operatorname{Im}(v_k + t(v_{k+1} - v_k)) > 0$ for $0 \leq k < 23$ and $0 < t \leq 1$.*

(S2) *For $1 \leq k < 23$, $k \neq 5$, and every $x \in [v_k, v_{k+1}]$: $\operatorname{Re} f(x) \leq \operatorname{Re}\Omega - \eta_{\mathrm{far}}$.*

(S3) *For every* $y \geq 0$*:* $\operatorname{Re} f(x_{\mathrm{top}} + iy) \leq \operatorname{Re}\Omega - \eta_{\mathrm{far}} - (12 - 3\pi)y$.
(S4) *The sector condition* (a, η) *holds on the whole contour: at* $x_0 + sd$ *for* $|s| \leq \rho$*, on the segments of (S2), at* $x_{\mathrm{top}} + iy$ *for* $y \geq 0$*, and on the segments of (S5).*
(S5) $\delta_0 \leq x_c < r = \frac{11}{12}$*, and for every real* ϵ *with* $|\epsilon| \leq \delta_0$ *and every* $x \neq v_0 + \epsilon$ *on the segment* $[v_0 + \epsilon, v_1]$*:* $\operatorname{Re} f(x) \leq \operatorname{Re}\Omega - \eta_{\mathrm{far}}$.

Method of proof. All statements are verified by exact rational interval arithmetic, with verified enclosures of π, log, arctan and arg (series with explicit tail bounds after argument reduction) and complex boxes. The verification is carried out by the Lean kernel; see Section 9.5.

- (S0): with $Y \approx 1/\Psi''(c_0)$, box evaluations show that $z \mapsto z - Y(\Psi'(z) - 3\pi i)$ is a contraction of the disc $|z - c_0| \leq 10^{-12}$ into itself, and the Banach fixed-point theorem gives existence and uniqueness. Ω is enclosed by a box evaluation of $f(c_0)$ (precision 2^{-80}, 40 series terms) widened by the mean-value bound $|f(x_0) - f(c_0)| \leq (10^{-16} + 30{\cdot}10^{-12}){\cdot}10^{-12}$. For the sine, one checks $240 \operatorname{Im}\Omega = (u - \frac{\pi}{2}) - 294{\cdot}2\pi$ with $|u| \leq 0.38$, so $\sin(240 \operatorname{Im}\Omega) = -\cos u \leq -(1 - u^2/2) < -0.9$.
- (S1), (S1′): $\Psi''(x_0)$ is enclosed in a box; $d := v/|v|$ with $v := 1 - \overline{\Psi''(x_0)}/K$ satisfies $\Psi''(x_0)d^2 = -K$; and M bounds $\sum_i |c_i|/|\alpha_i|^2$ on the disc, via lower bounds for $|\alpha_i|$ there. Numerically $K \approx 28.31$ and $M \approx 310.2$, so $M\rho \approx 15.5 \leq 42$; and $\operatorname{Im} x_0 \approx 0.1829$.
- (S1′), (S2), (S4), (S5): the vertices $v_1, \ldots, v_4$ and $v_7, \ldots, v_{23}$ are rational points with 12 decimals, obtained from a Runge–Kutta trace of the steepest-descent curves of $\operatorname{Re} f$ from $x_0 \pm \rho d$ and a greedy simplification (4 segments on the lower curve down to x_c, 16 on the upper curve up to x_{top}). On each segment, Taylor bounds on boxes show $\operatorname{Re} f \leq T_0 := -10.1367$; since $\operatorname{Re}\Omega \geq -10.1167486$, this gives $\operatorname{Re} f \leq \operatorname{Re}\Omega - 0.0199514$. The connectors $[v_4, v_5]$ and $[v_6, v_7]$ lie in discs of radius 10^{-9} around v_4 and v_7, on which $\operatorname{Re} f \leq T_0$ is checked directly. Moving the start point by $|\epsilon| \leq \delta_0$ changes f by at most $\delta_0 \sup|f'| \leq 10^{-5}$ on a box containing the widened first segment, which costs 10^{-5} of the margin; hence $\eta_{\mathrm{far}} = 0.0199414$. The sector condition is checked on boxes covering all segments.
- (S3): $\partial_y \operatorname{Re} f(x_{\mathrm{top}} + iy) = 3\pi - \operatorname{Im}\Psi'(x_{\mathrm{top}} + iy)$. Box evaluations give $\operatorname{Im}\Psi' \geq 12$ for $12.0012 \leq \operatorname{Im} \leq 40$, and for $Y := \operatorname{Im} x_{\mathrm{top}} + y \geq 40$,

$$\operatorname{Im}\Psi'(X + iY) \geq \frac{\pi}{2}\sum_i c_i - \sum_i \frac{|c_i||s_i X + u_i|}{Y} = 5\pi - \frac{142.86\ldots}{Y} \geq 12,$$

with $X := \operatorname{Re} x_{\mathrm{top}}$. Since $\operatorname{Re} f(x_{\mathrm{top}}) \leq T_0$, (S3) follows. On the ray $|\alpha_i| \geq 12$ and $|\arg \alpha_i| \leq \frac{\pi}{2} + \frac{1}{3} \leq 2.76$. □

These labels are those of the Lean structure `Z7.Analytic.SaddleData` (`ZetaIrrational/Z7/Analytic/SaddleData.lean`): (S0) is `S0`, `sin_ne`, `ReΩ_le`; (S1) is `S1_dir`, `S1_M`, `S1_ρ`; (S1′) is `S1'_disc`, `path_im`; (S2) and

(S3) are the fields `S2` and `S3`; (S4) is `S4_disc` together with the sector parts of `S2`, `S3` and `S2_first`; and (S5) is `v0_im`, δ_0`_le`, `v0_lt` and `S2_first`. The path fields `v_disc_lo`, `v_disc_hi`, `re_nonneg` are the conditions on the polyline stated before (S0).

At the saddle, $A(x_0) = (2x_0+1)e^{\psi_0(x_0)} \neq 0$, since $\operatorname{Re} x_0 > 0$. The numerical values are $x_0 \approx 0.8921643 + 0.1829121i$, $\operatorname{Re}\Omega \approx -10.11674855$ and $\sin(240 \operatorname{Im}\Omega) \approx -0.932453$.

6.6. **Deformation of the contour.**

Lemma 6.11. *Let $c \in (0, rn+1) \setminus \mathbb{Z}$, and let $w_0 = c, w_1, \dots, w_{K'} \in \mathbb{C}$ be such that every point of every segment $[w_k, w_{k+1}]$ other than w_0 has positive imaginary part, and $\operatorname{Re} w_k \geq 0$ for all k. Then*

$$i\int_0^\infty G_n(c+iy)\,dy = \sum_{k=0}^{K'-1} \int_{[w_k, w_{k+1}]} G_n(z)\,dz + i\int_0^\infty G_n(w_{K'}+iy)\,dy.$$

Proof. Let $\delta := \min_{m\in\mathbb{Z}} |c-m| \in (0, \frac{1}{2}]$; note $\delta \leq c$. On $U := \{\operatorname{Im} z > 0\} \cup \{|z-c| < \delta\}$, G_n is holomorphic: U contains no integer and no pole of R_n. U is star-shaped with respect to $c + i\delta/2$, so G_n has a primitive on U. For $Y > \operatorname{Im} w_{K'}$ consider the closed polygon $c \to c+iY \to \operatorname{Re} w_{K'} + iY \to w_{K'} \to w_{K'-1} \to \cdots \to w_0 = c$, which lies in U; the integral of G_n around it is 0. On the horizontal side, which lies in $\operatorname{Re} z \geq 0$, k_4 is bounded and $|R_n| \leq C(1+|z|^2)^{-1}$ by Lemma 6.2, so its contribution tends to 0 as $Y \to \infty$. The vertical integrals converge absolutely. □

6.7. **Laplace's method.**

Lemma 6.12. *Let $K_1, \rho_1 > 0$, $M_1 \geq 0$ with $M_1\rho_1 \leq \frac{3}{2}K_1$. Let $\phi, g : [-\rho_1, \rho_1] \to \mathbb{C}$ be continuous, with $|\phi(s) + K_1 s^2/2| \leq M_1|s|^3/6$, $|g| \leq A_0$ and $|g(s) - g(0)| \leq A_1|s|$. Then for $n > 0$*

$$\Big|\int_{-\rho_1}^{\rho_1} e^{n\phi(s)} g(s)\,ds - g(0)\sqrt{\frac{2\pi}{nK_1}}\Big| \leq \frac{1}{n}\Big(\frac{8A_0M_1}{3K_1^2} + \frac{2A_1}{K_1}\Big) + A_0 e^{-nK_1\rho_1^2/4}\sqrt{\frac{4\pi}{nK_1}}.$$

Moreover $\operatorname{Re}\phi(s) \leq 0$.

(The formalisation proves the same estimate with the larger constants $(16\sqrt{2\pi}/3)\,A_0M_1/K_1^2$ and $2\sqrt{2\pi}\,A_1/K_1$; only the order $O(1/n)$ is used.)

Proof. Write $\phi(s) = -K_1s^2/2 + \tau(s)$. Then $|\tau(s)| \leq M_1|s|^3/6 \leq M_1\rho_1 s^2/6 \leq K_1 s^2/4$, so $\operatorname{Re}\phi \leq -K_1s^2/4 \leq 0$ and $|e^{n\phi} - e^{-nK_1s^2/2}| \leq e^{-nK_1s^2/2} n|\tau| e^{n|\tau|} \leq \frac{nM_1|s|^3}{6} e^{-nK_1s^2/4}$. Split $\int e^{n\phi}g = g(0)\int e^{-nK_1s^2/2} + \int e^{-nK_1s^2/2}(g(s)-g(0)) + \int (e^{n\phi} - e^{-nK_1s^2/2})g$. The second integral is at most $A_1\int_{\mathbb{R}} |s| e^{-nK_1s^2/2} ds = 2A_1/(nK_1)$, and the third at most $\frac{nM_1A_0}{6}\int_{\mathbb{R}} |s|^3 e^{-nK_1s^2/4} ds = \frac{8A_0M_1}{3nK_1^2}$. Finally $\int_{|s|>\rho_1} e^{-nK_1s^2/2} ds \leq e^{-nK_1\rho_1^2/4}\int_{\mathbb{R}} e^{-nK_1s^2/4} ds$. □

6.8. **The asymptotic formula, nonvanishing and size.**

Proposition 6.13. *For Z7 let $\theta := \operatorname{Im}\Omega$ and $C_0 := d\,A(x_0)\sqrt{2\pi/K}$, so $C_0 \neq 0$; write $C_0 = |C_0|e^{i\phi_0}$. Then, as $n \to \infty$ in $240\mathbb{N}$,*

$$L_n = -\frac{1}{\pi}\, n^{3/2} e^{\lambda_n + n\operatorname{Re}\Omega}\big(|C_0|\sin(n\theta+\phi_0) + o(1)\big).$$

Proof. Let $c_n := \lfloor nx_c\rfloor + \frac12$. Since $0 < x_c < r$, we have $0 < c_n < rn+1$ and $c_n \notin \mathbb{Z}$, and $\chi_n := c_n/n - x_c$ satisfies $|\chi_n| \le \frac{1}{2n} \le \delta_0$ for large n. Let $J_n := i\int_0^\infty G_n(c_n+iy)\,dy$. By Proposition 6.4, $L_n = -\frac{1}{\pi}\operatorname{Im} J_n$. Apply Lemma 6.11 with $w_0 = c_n$ and $w_k = nv_k$ ($1 \le k \le 23$); the hypotheses hold by (S1′) and $\operatorname{Re} v_k \ge 0$, since $\operatorname{Im}(c_n + t(nv_1 - c_n)) = tn\operatorname{Im} v_1 > 0$ for $t > 0$. Thus J_n is the sum of the 23 segment integrals and the integral over the vertical ray from nx_{top}.

The Taylor segment. Parametrising $[nv_5, nv_6]$ by $z = n(x_0 + sd)$, $|s| \le \rho$, and using Proposition 6.9 (applicable by (S1′), (S4) and $\operatorname{Re}(x_0+sd) \ge 0$, which holds since $\operatorname{Re} v_5, \operatorname{Re} v_6 \ge 0$),

$$\int_{[nv_5,nv_6]} G_n(z)\,dz = n^2 d\, e^{\lambda_n + n\Omega}\mathcal{D}_n, \qquad \mathcal{D}_n := \int_{-\rho}^{\rho} e^{n\phi(s)} A_n(x_0+sd)\,ds,$$

where $\phi(s) := f(x_0+sd) - \Omega$. Since $f'(x_0) = \Psi'(x_0) - 3\pi i = 0$, $f''(x_0)d^2 = \Psi''(x_0)d^2 = -K$ and $|f'''| = |\Psi'''| \le M$ on the disc, Taylor's formula gives $|\phi(s) + Ks^2/2| \le M|s|^3/6$; by Lemma 6.12, $\operatorname{Re}\phi \le 0$. By (M) with $\operatorname{Im}(x_0 + sd) \ge \operatorname{Im} x_0 - \rho > 0$, $|\mathcal{D}_n - \int_{-\rho}^{\rho} e^{n\phi(s)}A(x_0+sd)\,ds| = O(n^{-1})$. The function A is holomorphic on $\operatorname{Im} x > 0$, hence bounded and Lipschitz on the disc, and Lemma 6.12 gives $\mathcal{D}_n = A(x_0)\sqrt{2\pi/(nK)} + O(n^{-1})$.

The far part. Let $\mathcal{E}_n$ be the sum of the other 22 segment integrals and the ray integral. On each of these segments, (S2), (S4) and (U) give $|G_n(z)| \le Cn\, e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})}$, and the segments have length $O(n)$. (For the first segment, $z = n((x_c+\chi_n) + t(v_1 - x_c - \chi_n))$ with $|\chi_n| \le \delta_0$, and (S5) applies.) On the ray, $z = n(x_{\text{top}} + iu)$ with $u \ge 0$, and (S3), (S4), (U) give $|G_n(z)| \le n\, e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})} e^{-n(12-3\pi)u} B(|x_{\text{top}}| + u)$; since $12 - 3\pi > 2.5$, its integral over $dz = in\,du$ is $O(n^2 e^{\lambda_n + n(\operatorname{Re}\Omega - \eta_{\text{far}})})$ for large n. Hence $|\mathcal{E}_n| = O(n^2 e^{\lambda_n + n\operatorname{Re}\Omega - n\eta_{\text{far}}})$.

Conclusion. Combining,

$$\begin{aligned} n^{-3/2}e^{-\lambda_n} J_n &= e^{n\Omega}\big(C_0 + O(n^{-1/2})\big) + O\big(n^{1/2}e^{-n\eta_{\text{far}}}\big)e^{n\operatorname{Re}\Omega} \\ &= C_0 e^{n\Omega} + o\big(e^{n\operatorname{Re}\Omega}\big), \end{aligned}$$

and $\operatorname{Im}(C_0 e^{n\Omega}) = |C_0| e^{n\operatorname{Re}\Omega}\sin(n\theta+\phi_0)$. □

Lemma 6.14. *For real u, v: $|\sin u| + |\sin(u+v)| \ge |\sin v|$.*

Proof. $\sin v = \sin(u+v)\cos u - \cos(u+v)\sin u$. □

Corollary 6.15. *For Z7 and all sufficiently large $n \in 240\mathbb{N}$, $L_n \neq 0$ or $L_{n+240} \neq 0$. In particular $L_n \neq 0$ for infinitely many $n \in 240\mathbb{N}$.*

Proof. By Lemma 6.14 with $v = 240\theta$ and (S0), $\max(|\sin(n\theta+\phi_0)|, |\sin((n+240)\theta+\phi_0)|) \geq \frac{1}{2}|\sin(240\theta)| > 0.45$ for every n. By Proposition 6.13, once the $o(1)$ terms are below $0.45|C_0|$, the corresponding L is nonzero. □

Proposition 6.16 (Size)**.** *For Z7 put* $V_{\mathrm{cert}} := -10.1167$*. There are* $\varepsilon_n'' \to 0$ *such that, for all large* $n \in 240\mathbb{N}$ *with* $L_n \neq 0$*,*

$$\log|L_n| \leq -G\log n! + n\big(V_{\mathrm{cert}} + \varepsilon_n''\big).$$

Proof. By Proposition 6.4 and the proof of Proposition 6.13, $|L_n| \leq |J_n|/\pi \leq Cn^{3/2}e^{\lambda_n + n\operatorname{Re}\Omega}$ for large n. By Stirling's upper bound $\log n! \leq n\log n - n + \frac{1}{2}\log n + 1$, $\lambda_n \leq -G\log n! + G(\frac{1}{2}\log n + 1) - 18\log n + 5\log 2\pi$. Finally $\operatorname{Re}\Omega \leq V_{\mathrm{cert}}$ by (S0). □

Remark 6.17. In the β-analogue [10] the size of L_n was bounded by an elementary estimate on the real axis. Here that route is too weak: on the real axis the integrand of L_n only gives the exponent $V_{\mathrm{up}} \approx -8.92$ for Z7 [numerical], against $A_{\mathrm{cert}} \approx 9.92$, so that $V_{\mathrm{up}} + A_{\mathrm{cert}} \approx +1.0$. The saddle point gains about 1.19 per unit of n, through the cancellation in the sum $\sum_t R_n''''(t)$, and this gain is essential for Z7. (For Z6 the real-axis exponent would leave a margin of only about 0.07; see Table 3.)

7. Proof of Theorem 1.1

Put

$$\begin{aligned} Q := V_{\mathrm{cert}} + A_{\mathrm{cert}} &= -10.1167 + 9.9185802528\ldots \\ &= -0.1981197\ldots < 0. \end{aligned} \tag{7.1}$$

(With the certified upper end -10.1167485 of $\operatorname{Re}\Omega$ in place of V_{cert} one gets $Q \leq -0.1981682$.) Let $\ell_n := D_nL_n$, $n \in 240\mathbb{N}$. By Corollary 4.8, $\ell_n = a_n + \sum_{j=3}^{10} c_{n,j}\zeta(2j+1)$ with $a_n, c_{n,j} \in \mathbb{Z}$. If $L_n \neq 0$, Corollary 5.8 and Proposition 6.16 give, for large n,

$$\log|\ell_n| = \log D_n + \log|L_n| \leq n\big(Q + \varepsilon_n' + \varepsilon_n''\big),$$

the terms $\pm G\log n!$ cancelling. Since $Q < 0$ and $\varepsilon_n', \varepsilon_n'' \to 0$, it follows that $|\ell_n| \leq e^{nQ/2}$ for all large n (trivially when $L_n = 0$), so $\ell_n \to 0$. On the other hand $D_n > 0$, so by Corollary 6.15, $\ell_n \neq 0$ for infinitely many n.

Suppose now that $\zeta(7), \zeta(9), \ldots, \zeta(21)$ are all rational, and let $\Delta \geq 1$ be a common denominator. Then $\Delta\ell_n \in \mathbb{Z}$ for every n. For the infinitely many n with $\ell_n \neq 0$ this gives $|\Delta\ell_n| \geq 1$, which contradicts $\ell_n \to 0$. Hence at least one of $\zeta(7), \ldots, \zeta(21)$ is irrational. □

8. The general theorem and other instances

The proofs of Sections 2–7 use the specific parameters of Z7 only through the following hypotheses.

Theorem 8.1. *Let* $\mathcal{P}$ *be a parameter set (Definition 2.1), and suppose:*

(H1) $G > 0$*;*

(H2) A_{cert} *is a rational number with* $\sum_\ell e^*(\xi_\ell)(\frac{1}{x_{\ell-1}}-\frac{1}{x_\ell})+\frac{S+15}{X}+\frac{G}{2(X-1)} \le A_{\mathrm{cert}}$*, for the event set of Section 5 with* $X=400$*;*
(H3) *there are saddle data* $(x_0,d,K,M,\rho,v_0,\dots,v_{K_{\mathrm{top}}})$ *and constants* $a,\eta,\eta_{\mathrm{far}},\delta_0>0$ *satisfying the analogues of (S0)–(S5) of Theorem 6.10 (without the numerical values), with* $\operatorname{Re} v_k \ge 0$*, a disc segment* $[v_{K_d},v_{K_d+1}]=[x_0-\rho d,x_0+\rho d]$*, and some positive decay rate on the top ray in (S3);*
(H4) $\sin(q\operatorname{Im}\Omega)\neq 0$*;*
(H5) $Q:=\operatorname{Re}\Omega+A_{\mathrm{cert}}<0$*;*
(H6) $qG\in 2\mathbb{Z}$*, and* $e\equiv 1+S \pmod 2$*.*

Then at least one of $\zeta(7),\zeta(9),\dots,\zeta(N_{\mathcal{P}})$ *is irrational.*

Proof. Sections 3–5 are written for a general parameter set; (H1) and (H6) are what they use (Lemma 2.3 and the remark after it), and (H2) replaces Proposition 5.5, the tail constant being $S+15$ by Lemma 5.3. Section 6 uses only (H3), with λ_n and the amplitude $A(x)=(2x+1)^e e^{\psi_0(x)}$ as defined there, and the factor $(-1)^{rn}$ of Lemma 6.6 does not affect the argument, which uses only $|\sin(n\theta+\phi_0)|$; (H4) replaces the sine bound of (S0). The asymptotic formula becomes $L_n=\mp\frac{1}{\pi}n^{e+1/2}e^{\lambda_n+n\operatorname{Re}\Omega}(|C_0|\sin(n\theta+\phi_0)+o(1))$. Section 7 then needs (H5). □

Theorem 1.1 is Theorem 8.1 for Z7, and this is the case that is formalised. Seven instances have been certified; Tables 2 and 3 list them, with all bounds rounded in the safe direction. Z1 has a single block and reaches Zudilin's window $\zeta(7),\dots,\zeta(37)$; Z4–Z7 have near-continuous profiles.

	r	q	blocks $\delta:m$	S	$N_{\mathcal{P}}$	G; qG	e
Z1	5/4	4	$0:33$	33	37	41/2; 82	0
Z2	1	8	$0:21,\ \frac{1}{8}:4,\ \frac{1}{4}:4$	29	33	16; 128	0
Z3	1	8	$0:19,\ \frac{1}{8}:4,\ \frac{1}{4}:4$	27	31	14; 112	0
Z4	1	80	$0:2,\ \frac{1}{80}:1,\ \frac{1}{40}:2,\ \frac{3}{80}:1,\ \frac{1}{20}:1,\ \frac{1}{16}:1,$ $\frac{3}{40}:1,\ \frac{7}{80}:1,\ \frac{1}{10}:1,\ \frac{1}{8}:2,\ \frac{11}{80}:1,\ \frac{13}{80}:2,$ $\frac{7}{40}:1,\ \frac{3}{16}:1,\ \frac{1}{5}:1,\ \frac{17}{80}:1,\ \frac{9}{40}:1,\ \frac{1}{4}:2,\ \frac{23}{80}:2$	25	29	339/40; 678	0
Z5	1	80	$0:2;\ \frac{k}{80}:1,\ k=1,\dots,19$	21	25	25/4; 500	0
Z6	19/24	240	$\frac{k}{80}:1,\ k=1,\dots,19$	19	23	19/3; 1520	0
Z7	11/12	240	$\frac{k}{80}:1,\ k=1,\dots,18$	18	21	547/120; 1094	1

TABLE 2. The certified instances: parameters. Each gives "one of $\zeta(7),\zeta(9),\dots,\zeta(N_{\mathcal{P}})$".

	$A_{\mathrm{cert}} \le$	$\operatorname{Re}\Omega \le$	$Q \le$	$\sin(q \operatorname{Im}\Omega)$	x_0	V_{up}
Z1	35.7955767	−36.7604812	−0.9649	−0.709656	$1.2579692 + 0.0404800i$	−36.597
Z2	25.3852394	−26.4996942	−1.1144	0.997966	$1.0073349 + 0.0420146i$	−26.327
Z3	23.3827332	−23.7154311	−0.3326	0.825398	$1.0077298 + 0.0558821i$	−23.476
Z4	11.4875113	−16.0979480	−4.6104	−0.968332	$0.9994996 + 0.1230432i$	−15.461
Z5	10.8538930	−12.9889571	−2.1350	0.283577	$0.9869778 + 0.1636877i$	−12.027
Z6	10.7587916	−11.4775814	−0.7187	−0.272087	$0.7869951 + 0.1190009i$	−10.826
Z7	9.9185803	−10.1167485	−0.1981	−0.932453	$0.8921643 + 0.1829121i$	−8.923

	(S1) κ, M, ρ	(S1′)	(S2) η_{far}	(S3) $\operatorname{Im}\Psi'$	(S4) a, η	η_{cr}; x_c
Z1	118.54, 5527, 0.01031	0.0302	$5.65 \cdot 10^{-3}$	15.47	$1.92 \cdot 10^{-2}$, 0.672	0.2211; 1.2290
Z2	114.57, 5128, 0.01066	0.0314	$5.82 \cdot 10^{-3}$	15.53	$1.99 \cdot 10^{-2}$, 0.718	0.2256; 0.9783
Z3	86.68, 2904, 0.01410	0.0418	$7.68 \cdot 10^{-3}$	15.55	$2.66 \cdot 10^{-2}$, 0.788	0.2923; 0.9712
Z4	40.68, 622.6, 0.03076	0.0923	$1.69 \cdot 10^{-2}$	15.59	$6.06 \cdot 10^{-2}$, 1.029	0.5898; 0.9357
Z5	31.19, 364.8, 0.04105	0.1226	$2.30 \cdot 10^{-2}$	15.62	$8.29 \cdot 10^{-2}$, 1.138	0.7565; 0.9128
Z6	42.43, 670.7, 0.02977	0.0892	$1.65 \cdot 10^{-2}$	15.63	$5.93 \cdot 10^{-2}$, 1.077	0.5602; 0.7290
Z7	28.31, 301.4, 0.04614	0.1368	$2.63 \cdot 10^{-2}$	15.64	$9.48 \cdot 10^{-2}$, 1.210	0.8265; 0.8174

TABLE 3. The certified instances: certificates of the Python and Arb computations. A_{cert} is the exact rational of (H2), rounded up; $\operatorname{Re}\Omega$ is the upper end of the Arb enclosure (radius $\le 2 \cdot 10^{-7}$), rounded up; $Q = \operatorname{Re}\Omega + A_{\mathrm{cert}}$, rounded up. Second part: the Taylor-disc data $\kappa = -\operatorname{Re}(\Psi''(x_0)d^2)$, M, ρ; the margin $\operatorname{Im} x_0 - \rho$ of (S1′); the far margin; the tail value of $\operatorname{Im}\Psi'$ on the top ray (it must exceed 3π); the sector constants; and the crossing gap $\eta_{\mathrm{cr}} = \operatorname{Re}\Omega - \Psi(x_c)$ with the start point x_c (a cross-check, not needed by the proof). V_{up} is the real-axis exponent of the numerical model [numerical].

The exact values of A_{cert} are

$$\begin{aligned}
\text{Z1}&: \tfrac{28289277235925350790655713667586667}{790300921079786767495600904601600},\\
\text{Z2}&: \tfrac{20061978064991101891984267756512101}{79030092107978676749560090460160 0},\\
\text{Z3}&: \tfrac{131995682276919083918237338201033 7}{56450065791413340535400064614400},\\
\text{Z4}&: \tfrac{75654922793088526827628553510069}{65858410089982230624633408716 80},\\
\text{Z5}&: \tfrac{42889207933911821929176852419078 47}{39515046053989338374780045230080 0},\\
\text{Z6}&: \tfrac{67126443531685095605521656546399 1}{62392177979983165854915860889600},
\end{aligned}$$

and, for Z7, (5.1).

Status of the instances. For Z7 everything is kernel-checked (Section 9); the Lean saddle certificate of Theorem 6.10 uses its own contour (with $\rho = 1/20$ and 23 segments) and agrees with the Arb value of $\operatorname{Re}\Omega$ in Table 3, and the Lean value of A_{cert} is the same exact rational. For Z1–Z6 the certificates come from two programs that are not part of the repository: an exact-rational Python implementation (python-flint) of the sum in (H2), and a

ball-arithmetic computation with Arb of the saddle data for (H3), (H4). The latter checks a variant of (S0)–(S5): the Taylor radius is chosen with $\rho \le \kappa/M$, which is stronger than $M\rho \le \frac{3}{2}\kappa$, and (S2) is checked on the whole polyline starting at x_c, together with the crossing gap $\Psi(x_c) \le \operatorname{Re}\Omega - \eta_{\mathrm{cr}}$; the perturbation of the start point in (S5) is not part of that certificate. That perturbation changes $\operatorname{Re} f$ on the first segment by $O(1/n)$, because the sector condition holds on a neighbourhood of that segment, so it costs nothing in the limit [proof sketch]. So Z1–Z6 rest on these certificates and are not formalised. The corresponding windows $\zeta(7), \ldots, \zeta(N)$ for $N = 23, \ldots, 37$ are weaker than Theorem 1.1 in any case; their only interest is the larger margins.

As an independent check of the arithmetic part, for every profile type of Table 2 and for Z7 itself at $n = 240$, the exact content of the vector (ρ_n, τ_i) was computed with python-flint and compared with the exponents E_p for $\sqrt{\mathcal{N}} < p \le n$: the bound was never violated, it was attained at most primes (at 30 of the 39 primes $41 \le p \le 233$ for Z7 at $n = 240$), and $E_p = 0$ for $p > n$ [numerical]. At the same time the window was confirmed to be exactly $\zeta(7), \ldots, \zeta(N_{\mathcal{P}})$, and L_n was compared with the direct sum $\sum_t R_n''''(t)$ to 30 digits.

9. The formalisation

9.1. **What is formalised.** The repository `https://github.com/telaviv/zeta-irrationality` [9], at tag `v1.0`, contains a proof of Theorem 1.1 in Lean 4 [4] (version v4.35.0) with Mathlib [16] (version v4.35.0 [17]). All declaration names and file paths in this section refer to that tag; paths are relative to the repository root. The formalisation covers every step of the proof of Theorem 1.1, including the parts that a written proof would cite: the prime number theorem, Stirling's formula with explicit remainder in sectors, the contour-integral representation, the deformation of contours, and all numerical certificates. Much of the instance-independent infrastructure (prime number theorem, Stirling, interval arithmetic, contour and Laplace lemmas, partial fractions, valuations) is adapted from the formalisation of the β-analogue [10].

9.2. **The statement.** The definition that the reader has to check is, in `ZetaIrrational/Statement.lean` (namespace `ZetaIrr`):

```
def OneOfZeta7To21 :  Prop :=
    ∃ j ∈ ({7, 9, 11, 13, 15, 17, 19, 21} : Finset ℕ),
        Irrational (riemannZeta j).re
theorem ZetaIrr.Z7.oneOfZeta7To21 :  OneOfZeta7To21
```

The theorem is in `ZetaIrrational/Z7/MainUnconditional.lean`. Here `riemannZeta : ℂ → ℂ` is Mathlib's Riemann zeta function, the natural number `j` is cast to $\mathbb{C}$, and `Irrational x` is Mathlib's predicate $x \notin \mathbb{Q}$. Nothing defined in the project occurs in the statement. Mathlib's `zeta_nat_eq_tsum_of_gt_one` identifies `riemannZeta j` with $\sum_{m \ge 1} m^{-j}$ for $j \ge 2$; the

file `ZetaIrrational/Defs.lean` uses it to pass to the real series used internally (`riemannZeta_re_eq_zetaR`). The statement assumes nothing about $\zeta(3)$ or $\zeta(5)$.

The command `lake env lean Axioms.lean` prints

```
'ZetaIrr.Z7.oneOfZeta7To21' depends on axioms:
    [propext, Classical.choice, Quot.sound]
```

and the same axiom list for each of the main ingredients listed in `Axioms.lean`. These are Lean's three standard axioms, which Mathlib also uses. The development contains no `sorry`, no added axiom, no `native_decide` and no `implemented_by`, so no step relies on compiled code.

9.3. **Correspondence with the paper.** Table 4 lists the main formal counterparts. The theorem is `oneOfZeta7To21_of_certUnits` applied to the kernel-checked sum `CertA.bodyUnits_400`. It goes through `oneOfZeta7To21_of_parts`, whose inputs are the certificate inequality, Proposition 6.16 and Corollary 6.15, and through the criterion `exists_irrational_of_small_forms`, which formalises Section 7.

9.4. **Notes on the formal proof.** The formal proof is the proof of this paper. The following points are worth noting when reading the two side by side.

- *The analytic part* uses the four integrations by parts of Proposition 6.4, not the kernel split of Remark 6.5. There is no boundary term, and only Stirling's formula for $\log\Gamma$ is needed, for shifts $\beta \in [0, 2]$ (the factor list contains $\Gamma(z)$ and $\Gamma(z + \delta_j n)$, with $\beta = 0$). The start of the contour near the real axis is handled by (S5); no separate condition at the crossing point x_c is needed, because the integrand G_n has no singular kernel.
- *The size of L_n* comes from the saddle point (Proposition 6.16), with $V_{\mathrm{cert}} = -10.1167$; see Remark 6.17.
- *Corollary 4.4(b′)* is proved without any condition on p, and only its integrality form is used.
- *The pointwise bound* (Proposition 5.2) holds for every prime $p \le n$.
- *Small primes.* Definition 4.6 uses $(S+4)\lambda_p$ (for Z7, $22\lambda_p$) and drops the e-factor from $\nu_p(k)$; Proposition 4.11 has the explicit constant 182.
- *The tail.* Lemma 5.3 gives $e^*(x) \le S + 14 + G\{x\}$; the certificate uses the slightly larger constant $S + 15 = 33$, which costs $1/400$ in A_{cert}.
- *The event set* for Z7 has 30,138 points, so the sum of Proposition 5.5 has 30,137 terms.
- *The saddle certificate* uses its own steepest-descent contour (Theorem 6.10: $\rho = 1/20$, 23 segments, $\eta_{\mathrm{far}} = 0.0199414$, $a = 1/100$, top ray from $x_{\mathrm{top}} \approx 2.0512 + 12.0012i$), generated independently of the Arb computation of Table 3; the two enclosures of $\mathrm{Re}\,\Omega$ agree.

paper	Lean declaration (namespace `ZetaIrr`)
Lemma 3.1, Prop. 3.3	`Z7.R_eq_pfR`, `PF.pf_eval`, `PFData.tsum_R4`
Lemmas 3.4, 3.5, Cor. 3.6	`Z7.pf_symmetric`, `PFData.tau_eq_zero_of_even`, `PF.pf_top_coeff`, `Z7.pf_tau_one`, `PFData.tsum_R4_odd`, `Z7.expansion_Z7`
Lemmas 4.2, 4.3, Cor. 4.4	`Arith.psge_mul`, `Arith.psge_inv`, `Z7.Arith.vge_a`, `Z7.Arith.a_int_of_nodvd`, `Z7.Arith.a_int`, `Z7.Arith.a_bound_sharp`, `Z7.Arith.a_bound_small`
Lemma 4.5	`Arith.vge_Hq`, `Arith.vge_Hq_of_lt`
Def. 4.6, Prop. 4.7	`Z7.Arith.Ep`, `Z7.Arith.D`, `Z7.Arith.int_rho`, `Z7.Arith.int_tau`
Props. 4.11, 4.12	`Z7.Arith.Ep_sub_le_small`, `Z7.Arith.small_primes_sum_le`, `Z7.Arith.log_D_sub_le`
Prop. 5.2, Lemma 5.3	`Z7.CertA.pointwise_le_eStar_prime`, `Z7.Esharp_sub_le_eStar`, `Z7.CertA.eStar_le_tail`
Lemma 5.4	`Z7.CertA.eStar_eq_mid`, `Z7.CertA.eStar_isGreatest`
Prop. 5.5	`Z7.CertA.bodyUnits_400`, `Z7.CertA.certFact_of_units`
Lemma 5.6, Prop. 5.7	`PNT.eventually_primeSum_lemma5_le`, `Z7.sharp_of_pointwise`, `Z7.arith_Z7`
prime number theorem	`PNT.theta_div_tendsto`
Lemma 6.3	`Analytic.hasDerivAt_kk`, `Analytic.sin_pow_five_mul_kk4`, `Analytic.T_eq_exp`
Prop. 6.4	`Analytic.hasSum_cot_eq_line_integral`, `Analytic.integral_line_by_parts4`, `Z7.Analytic.L_eq_half_line`
Lemma 6.6, Prop. 6.9	`Z7.Analytic.RC_eq_En`, `Z7.Analytic.Gfn_eq_PT`, `Z7.Analytic.Pfun_asymp`, `Z7.Analytic.Gfn_path`, `Z7.Analytic.norm_Gfn_le`, `Z7.Analytic.Gfn_disc_approx`
Thm. 6.7, Cor. 6.8	`Stirling.norm_J_le`, `Stirling.stirling_scaled`
Thm. 6.10	`Z7.Interval.Z7_saddleData` (a term of the structure `Z7.Analytic.SaddleData`)
Lemmas 6.11, 6.12	`Z7.Analytic.contour_eq_polyline_Gfn`, `Analytic.laplace_bound`
Prop. 6.13, Cor. 6.15	`Z7.Analytic.nonvan_of_saddleData`, `Z7.nonvan_Z7`
Prop. 6.16	`Z7.Analytic.size_of_saddleData`, `Z7.size_Z7`
Section 7	`exists_irrational_of_small_forms`, `oneOfZeta7To21_of_forms`, `Z7.oneOfZeta7To21_of_hyp`, `Z7.oneOfZeta7To21_of_parts`, `Z7.oneOfZeta7To21_of_certUnits`, `Z7.oneOfZeta7To21`

TABLE 4. Formal counterparts of the main steps.

Further, as for the β-analogue, a few standard tools are organised differently because Mathlib (at the version used) lacks them. Mathlib has no residue

theorem, so the first equality of Proposition 6.4 is proved by subtracting model principal parts at the integers $c < t \leq N$, removing the resulting removable singularities, applying Cauchy's theorem for rectangles and evaluating the model integrals explicitly. The deformation (Lemma 6.11) uses primitives of holomorphic functions on unions of discs. Stirling's formula is proved from Euler's limit formula for Γ, an exact Euler–Maclaurin identity and Mathlib's Stirling formula for $n!$. The prime number theorem in the form $\theta(y)/y \to 1$ is ported from the PrimeNumberTheoremAnd project [8], through the port in the `zeta5` repository of César Del Solar [5]; the ported files keep their Apache License 2.0. All error terms are explicit, except those coming from the prime number theorem and from the compactness arguments in Section 6 (the decay constants of Lemma 6.2 and the amplitude bounds A_0, A_1 of Lemma 6.12).

9.5. **How the certificates are checked.** The arithmetic certificate and the saddle certificate are decidable statements about rational data, proved by `decide +kernel`, i.e. by evaluation in the Lean kernel. The data (event lists, polyline vertices, rational enclosures) were produced by scripts that are not trusted: a wrong value would make the build fail, but could not make a false statement provable. The generators are in the repository (`scripts/certA/`, `scripts/saddle/gen_saddle_Z7.py`).

The arithmetic certificate (Proposition 5.5). The specification of e^* and of the event list (built with `List.mergeSort`) is what the soundness proofs use, but the kernel cannot evaluate it efficiently. The proof therefore passes through equivalent forms, each with a general proof of equivalence: a closed form of e^* that uses only natural-number arithmetic per block (the kernel has fast `Nat` primitives but none for `Int`); a decomposition of the event list into 133 windows $[1 + 3j, 4 + 3j)$, each sorted in the kernel and compared with literal data, together with a check that their concatenation is strictly increasing; and a decomposition of the rounded sum into 1,029 chunks, each checked by a separate `decide +kernel`. The resulting value U agrees exactly with an independent exact-rational Python implementation, and with the Python recomputation `scripts/certA/z7_model.py` included in the repository, which mirrors the Lean definitions.

The saddle certificate (Theorem 6.10). This uses a verified interval arithmetic with rational endpoints (rounded outwards to dyadic grids), complex boxes, verified enclosures of log, arctan, π and arg, the contraction lemma for (S0), and segment, disc and ray checks as described in the proof of Theorem 6.10. The phase multiplier (2π for the β-analogue, 3π here) is a parameter of the interval code.

9.6. **Cost.** With Mathlib cached, the arithmetic certificate dominates the build: its 1,029 chunk lemmas live in 30 files, which take about 3.3 CPU-hours in total and about 9.5 GB of memory each while the kernel evaluates them. The wall time is therefore set by how many of these files fit in memory at once (`LEAN_NUM_THREADS`): 2 on a machine with 32 GB of RAM, giving

a build of a few hours, and 5 with 64 GB. Everything else takes about 15 minutes at 6 threads. Running out of memory makes the build fail, but it is not a proof error, and rerunning the build continues where it stopped.

9.7. **What has to be trusted.** Beyond the Lean kernel and the three axioms, the reader has to trust that the definition displayed above expresses Theorem 1.1, and the Mathlib definitions it uses (`riemannZeta`, real and complex numbers, `Complex.re`, `Irrational`, `Finset`). Nothing else, in particular no numerical data and no generating script, needs to be trusted. Besides `lake build`, the proof was replayed module by module with Lean's `leanchecker` (134 of the 141 modules, including the main theorem and both certificates) and, in passes, with the independent kernel `nanoda` (60,465 of the 61,819 theorems in the closure of the main theorem, including the main theorem and every obligation of both certificates), with no error; these replays used the build of an earlier commit that differs from the cited one only in comments and two lemma names (see `TRUST.md` in the repository). Apart from the check by the Lean kernel, all verification of the formalisation so far was carried out by AI systems; no human expert has yet reviewed the Lean code.

10. Remarks

10.1. **The margin.** The proof needs $Q = \operatorname{Re}\Omega + A_{\mathrm{cert}} < 0$, and $Q \approx -0.198$. The two terms are of size about 10, so the margin is about 2%. Both numbers are certified: A_{cert} by exact rational arithmetic and $\operatorname{Re}\Omega$ by interval arithmetic. There is no third term: the 2-adic and 3-adic costs that twisted constructions pay are absent.

10.2. **How the parameters were found.** The parameters were found by a numerical search [numerical, not part of the proof]. A fast model of Q (the arithmetic integral on a coarse grid, and the saddle point or the real-axis exponent) was optimised over the block profile, with offsets on the grid $\frac{1}{80}\mathbb{Z} \cap [0, \frac{3}{8}]$, and over $r \in \frac{1}{24}\mathbb{Z}$ (later $\frac{1}{48}\mathbb{Z}$), and the best candidates were then certified. In every run the optimum was close to a uniform spread: one block at each of S consecutive grid points $k/80$, with r between about 0.8 and 1. Compared with a single block at $\delta = 0$, the spread profile is worth about 6 per unit of n at $S = 27$, and the refinement Corollary 4.4(b′) together with the exact counts of Section 5 about 1–2 per unit of n [numerical].

For $S = 17$ (which would also give $\zeta(7), \ldots, \zeta(21)$) the best profile found has certified $\operatorname{Re}\Omega + A_{\mathrm{cert}} \approx +0.553$, so it does not close; with $S = 18$ blocks the uniform profile with $r = 11/12$ gives Z7.

10.3. **Why not $\zeta(19)$.** To reach $\zeta(7), \ldots, \zeta(19)$ this class needs $S \le 16$. With uniform profiles the model gives $Q \approx +1.14$ for $S = 16$ ($r = 5/6$) and $Q \approx +1.82$ for $S = 15$, including the saddle gain; a finer offset grid $\frac{1}{160}\mathbb{Z}$ at $S = 17$ improved the model by less than 0.1 [numerical]. So $N = 19$ appears out of reach for this class [heuristic]; the window $\zeta(7), \zeta(9), \zeta(11), \zeta(13)$ is

about 4.5 per unit of n away. Liu's determinant method [12] does reach $\zeta(7), \dots, \zeta(19)$; the obstruction described here concerns single linear forms of this class only.

One idea seemed to close the gap and fails. Replace the outer part of each zero run by zeros of lower multiplicity m_2 (a *tapered* run): multiplicity 5 on $[1, r_1 n]$ and m_2 on $(r_1 n, r_2 n]$, mirrored. Since only the integers up to the crossing point $n x_c$ need quintuple zeros, the model suggested $Q \approx -0.8$ at $S = 16$ when the sum $\sum_t R_n''''(t)$ over the taper region was assumed to telescope. For $m_2 \geq 2$ it does not: at the integers of the taper region R_n'''' equals the real-axis envelope times terms of type $\sin^2(\pi t) = (1 - \cos 2\pi t)/2$, which alias to frequency 0 under integer sampling, so that sum has the size of the real-axis maximum, not of the saddle. The extended certificate confirmed this: the required bound on the taper region fails by 6.32 and 4.37 per unit of n for the two best candidates. For $m_2 = 1$ the constraints fail as well ($Q \approx +0.95$ at best) [numerical]. Ideas not tried include other zero multiplicities on the whole run, blocks of multiplicity 2 at small δ, and genuinely different families.

10.4. **Comparison with Lai–Zhou.** No N makes "one of $\zeta(7), \dots, \zeta(N)$" imply "two of $\zeta(5), \dots, \zeta(35)$": the irrationality of $\zeta(7)$ alone would satisfy the former. Conversely, Lai and Zhou's theorem gives only $N = 35$ for windows starting at $\zeta(7)$. If the irrationality of $\zeta(5)$ is accepted, Theorem 1.1 is the stronger of the two statements about $\zeta(7), \zeta(9), \dots$. A "two of" statement for windows starting at $\zeta(7)$ would need an additional elimination, as in [11, 15], which this family does not do. For the construction of Lai and Zhou with simple zeros and twisted sums, the same search gave only $Q \approx -0.68$ at their own window ($S = 33$) and $Q \approx +0.37$ at $S = 31$ in our model [numerical], so the derivative family is the better one here.

10.5. **Effectivity.** The theorem is proved for $n \geq n_0$ with an unspecified n_0, because the prime number theorem is used without an error term and some constants come from compactness. This does not affect the irrationality statement. The explicit error terms are large: for instance the small-prime term $182\sqrt{\mathcal{N}} \log \mathcal{N}$ of Proposition 4.12 falls below $0.198\, n$ only for n of order $2 \cdot 10^9$ [numerical estimate]. An effective version would need an explicit form of the prime number theorem and of the compactness constants.

10.6. **What remains outside Lean.** Nothing that the proof of Theorem 1.1 uses. The Lean statement is exactly the one displayed in Section 9, about Mathlib's `riemannZeta`, and it uses no hypothesis on $\zeta(3)$ or $\zeta(5)$. Outside Lean are only: the numerical search that found the parameters, the generators of the certificate data (both untrusted and not needed for correctness), the exact checks at small n quoted in Section 8, and the certificates of the instances Z1–Z6, which are not used for Theorem 1.1.

10.7. **Review.** Before the formalisation, a prose version of the proof of Theorem 8.1 (with the kernel split of Remark 6.5) was reviewed adversarially by four independent AI reviewers (algebra, arithmetic, analysis, certificates and logic), who tested its claims with their own code. They found no fatal or major mathematical issue; one presentation error (the values of Q had been rounded to nearest instead of up) and about twenty minor points were corrected. Among their checks: Lemma A and Lemma B in 186 small instances at every prime in range; the integrality of $p^{E_p}(\rho_n, \tau_i)$ at every prime for 16 instances with $n \leq 32$; an independent evaluation of e^* that reproduces the certified body sums; the identity behind Remark 6.5 at Z7 for $n = 240$ and 480, to 10^{-16}; and a re-run of both certificates for Z1, Z6 and Z7 [numerical]. The formalisation then found no gap; the points where the formal proof is organised differently are those of Section 9.4.

Acknowledgements and disclosure

The construction, the proofs, the parameter search, the certificate programs, the Lean formalisation and the text of this paper were produced largely by AI models (Claude, Anthropic), working under the direction of the author. Apart from the check by the Lean kernel, all verification so far, including the reviews mentioned above, was carried out by AI systems. No human expert has yet reviewed the mathematics or the Lean code; we would be grateful for corrections. The prime number theorem in the formalisation is ported from the PrimeNumberTheoremAnd project of A. Kontorovich, T. Tao and collaborators, through the `zeta5` repository of César Del Solar.

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