

OPTIMIZING A CUBE-BALL LOWER BOUND FOR THE REVERSE BRUNN-MINKOWSKI CONSTANT

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ABSTRACT. We optimize a single term in the cube-Euclidean-ball construction of V. Patankar to obtain $C_{70} \geq \exp(1/(2e)) > 1.20194336847$. The strongest public proposal located and reproduced in our dated source audit is Patankar's 1.201927, whose specified witness evaluates near 1.2019271425725. A separate finite-dimensional witness certifies 1.20194336. Two separately written arithmetic implementations, using Arb and rational logarithm/Stirling bounds respectively, verify the numerical comparisons and reject corrupted or excessive claims. The geometric reduction and passage to the limit are proved here. This unrefereed preprint is not peer reviewed and not formally verified; exhaustive priority is not asserted.

1. STATEMENT AND COMPARISON

Let C_{70} be the infimum of C such that for every pair of centrally symmetric convex bodies $K, L \subset \mathbb{R}^n$, in every dimension, there is a linear map u with $|\det u| = 1$ and

$$|uK + L|^{1/n} \leq C(|K|^{1/n} + |L|^{1/n}).$$

This is Tao and contributors' entry 70a [1]. The existence of a finite constant is Milman's theorem; we address a lower bound, uniformly against every permitted relative position.

Theorem 1. *One has*

$$C_{70} \geq \exp\left(\frac{1}{2e}\right) > \frac{120194336847}{10^{11}} = 1.20194336847.$$

An explicit finite witness also gives $C_{70} > 3756073/3125000 = 1.20194336$.

The table's merged lower bound is merely 1. The comparison relevant to this note is the stronger open proposal of Patankar [2], 1.201927, from dimension 300000 and cube side/ball radius 13/1000. We reproduced his unchanged public interval script before searching, and an independently implemented Arb recurrence encloses its value near 1.20192714257250284496098903. Toledo also publicly replayed this witness [3]. The new printed lower bound exceeds that witness by more than 0.000016225897. It is a modest optimization of the same construction, not a new class of convex bodies.

2. THE POSITIONING REDUCTION

Write $\kappa_j = \pi^{j/2}/\Gamma(1+j/2)$, with $\kappa_0 = 1$. Take $K = [-a/2, a/2]^n$ and $L = B_2^n$.

Lemma 2 (Patankar's cube-ball criterion). *For every $a > 0$ and every $n \geq 1$,*

$$C_{70} \geq \frac{\left(\sum_{k=0}^n \binom{n}{k} a^{n-k} \kappa_k\right)^{1/n}}{a + \kappa_n^{1/n}}.$$

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Proof. The parallelotope $P = uK$ is generated by vectors $v_1, \dots, v_n$ with volume a^n . Let $\delta_i > 0$ be their squared Gram–Schmidt lengths. For any j -element subset S , the squared volume of its spanned parallelotope is at least $\prod_{i \in S} \delta_i$: projecting against only earlier selected vectors cannot reduce length more than projecting against all earlier vectors. Multiplying over S and using $\prod_i \delta_i = a^{2n}$ gives the principal-minor product inequality usually attributed to Szász. AM–GM now gives

$$V_j(P) = \sum_{|S|=j} \text{vol}_j(v_i : i \in S) \geq \binom{n}{j} a^j.$$

Here V_j are intrinsic volumes, with the usual zonotope normalization. Steiner’s formula therefore yields

$$|P + B_2^n| = \sum_{j=0}^n V_j(P) \kappa_{n-j} \geq \sum_{k=0}^n \binom{n}{k} a^{n-k} \kappa_k.$$

This holds for every determinant-one-in-absolute-value u , while the input root volumes are a and $\kappa_n^{1/n}$. Hence every admissible universal constant is at least the displayed ratio. Orthogonal equal-length generators attain equality in this positioning comparison. This is the existing construction and reduction of [2]. $\square$

3. ONE TERM AND ITS LIMIT

For even integers $0 < k < n$, put $t = k/n$ and set

$$a = \frac{1-t}{t} \kappa_n^{1/n}.$$

Retaining just term k in Lemma 2 proves $C_{70} \geq R_{n,k}$, where

$$\begin{aligned} (1) \quad \log R_{n,k} &= \frac{\log \binom{n}{k}}{n} + (1-t) \log(1-t) + t \log t + \frac{\log \kappa_k - t \log \kappa_n}{n} \\ &= \frac{\log(n!) - \log(k!) - \log((n-k)!) - \log((k/2)!) + t \log((n/2)!)}{n} \\ (2) \quad &+ (1-t) \log(1-t) + t \log t. \end{aligned}$$

All powers of π cancel in the second expression. The chosen scale maximizes this single normalized term: differentiation of $(n-k) \log s - n \log(1+s)$ gives $s = (n-k)/k$.

Fix $0 < \theta < 1$, let even $n \rightarrow \infty$, and choose even k with $k/n \rightarrow \theta$. Stirling’s formula applied to (2) gives

$$\frac{\log \binom{n}{k}}{n} = -t \log t - (1-t) \log(1-t) + O(\log n/n),$$

and

$$\frac{-\log((k/2)!) + t \log((n/2)!)}{n} = -\frac{t}{2} \log t + O(\log n/n).$$

The remainders are uniform when t stays in a compact subinterval of $(0,1)$. Thus $R_{n,k} \rightarrow \exp(-\theta \log \theta/2)$. Taking $\theta = e^{-1}$ gives the first inequality in Theorem 1. Every finite witness bounds the same universal C_{70} from below, so passing to the limit requires no monotonicity assertion and no numerical extrapolation. The derivative of $-\theta \log \theta/2$ vanishes at e^{-1} , its second derivative is negative, and its limiting values at the endpoints are zero. This optimizes this limiting single-term criterion; no optimality over all convex-body pairs is claimed.

4. CERTIFICATE AND ARITHMETIC VERIFICATION

The small certificate `certificates/c70_candidate.json` specifies

$$n = 10^{12}, \quad k = 367879441172, \quad a/\kappa_n^{1/n} = 632120558828/367879441172.$$

The finite expression (2) gives

$$R_{n,k} = 1.2019433684431505669948027797355 \dots > 1.20194336.$$

The first checker evaluates factorial logarithms and ball volumes with 256-bit Arb arithmetic. The second uses only rational arithmetic: range-reduced

$$\log x = 2 \sum_{j=0}^{N-1} \frac{z^{2j+1}}{2j+1} + E, \quad z = \frac{x-1}{x+1}, \quad 0 \leq E \leq \frac{2z^{2N+1}}{(2N+1)(1-z^2)}$$

for $1 \leq x \leq 2$, with $N = 44$, and Robbins's factorial inequalities

$$(m + \tfrac{1}{2}) \log m - m + \tfrac{1}{2} \log(2\pi) + \frac{1}{12m+1} < \log(m!) < (m + \tfrac{1}{2}) \log m - m + \tfrac{1}{2} \log(2\pi) + \frac{1}{12m}.$$

Even the coarse bound $3 < \pi < 7/2$ suffices for its finite comparison. The limit's decimal lower endpoint is separately checked using Arb and, in the second implementation, a rational Taylor enclosure of e through term 60 followed by a rational logarithm comparison. This verifies the number, while the preceding section supplies the limiting theorem.

Run, from the accompanying source directory,

```
python verify_c70_arb.py certificates/c70_candidate.json
python independent/verify_c70_rational.py \
    certificates/c70_candidate.json
python rejection_controls.py
```

The controls include an excessive finite bound, an excessive limiting bound, incorrect scale, odd dimensions, invalid schema and a valid but insufficient half-dimensional term. The two programs share no search or arithmetic code and use different libraries and formulas. Both were authored by the same AI agent; this is algorithmic cross-checking, not independent human authorship or a proof-assistant development.

PROVENANCE AND LIMITATIONS

The dated, bounded public-source audit is in `PRIOR_ART.md` and `FINAL_LIVE_CHECK.md`. Patankar's construction and positioning argument are credited above; the present contribution is its optimized single-term limit and a small rigorous certificate. No stronger C70 bound was located in the sources checked; exhaustive priority is not asserted. This AI-prepared preprint is not peer reviewed and not formally verified.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. This does not claim line-by-line human verification.

REFERENCES

- [1] T. Tao and contributors, *Optimization Constants in Mathematics*, entry 70a, <https://teorth.github.io/optimizationproblems/constants/70a.html>.
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