

A DEGREE-THREE LOCAL-WEIGHT CAP AND AN IMPROVED RELEVANT-VARIABLE BOUND

RICARDO PINHO

ABSTRACT. We prove that the local-degree potential of a Boolean function of degree at most three is at most $5/4$, using the published fact that such a function has at most ten relevant variables. Inserting this cap into Wellens's degree-block-sensitivity recurrence, with the existing exact certificates released by Épi, gives $C_{44} \leq 4.3576549226$. The reproduced preceding recurrence gives approximately 4.365467422574458; our exact degree-three cap lowers it by $1/128$. Two separately written exact-arithmetic programs check the finite certificates, recurrence, infinite tail and finite cases of the added lemma. The universal argument is written below; the published degree-three classification is cited, not re-enumerated. This preprint is not peer reviewed and not verified in a proof assistant; no optimality is claimed.

1. THE CONSTANT AND THE PRECEDING BOUND

For a Boolean function f , let $r(f)$ be its number of relevant variables and $d = \deg f$ the degree of its multilinear polynomial. We use the sign cube and sign-valued output when convenient; invertible affine changes of input and output preserve degree and local degrees. The constant is

$$C_{44} = \sup_{d \geq 1} \sup_{\deg f \leq d} \frac{r(f)}{2^d}.$$

Write $d_i(f)$ for the largest degree of a monomial involving a relevant coordinate i , and

$$\mathcal{D}(f) = \sum_{i \text{ relevant}} 2^{-d_i(f)}.$$

Since $d_i(f) \leq d$, $r(f)/2^d \leq \mathcal{D}(f)$.

Theorem 1. *For every Boolean function f of positive degree d ,*

$$r(f) \leq \frac{21788274613}{5000000000} 2^d = 4.3576549226 2^d.$$

Consequently $C_{44} \leq 4.3576549226$.

The merged table records 4.394 from Wellens [3, 1]. The stronger public bound located in the present audit is Épi's 4.3654674227 [4, 5]. Its degree-48 certificates and horizon-96 recurrence reproduce exactly at

$$\frac{172933981203227910490173314851}{39614081257132168796771975168} = 4.365467422574458 \dots$$

That extra decimal precision is not a new improvement. We keep these released polynomial certificates and change one analytic cap in the recurrence. The difference between the two exact horizon-96 bounds is $1/128$, a gain in the actual preceding method rather than a rounding gain.

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2. A SHARP DEGREE-THREE POTENTIAL CAP

We recall two ingredients. First, the influence argument gives

$$(1) \quad \mathcal{D}(f) \leq \frac{1}{2} I(f) \leq \frac{1}{2} \deg f.$$

Indeed, a nonzero derivative of local degree $d_i - 1$ is nonzero on at least a $2^{-(d_i-1)}$ fraction of the cube; summing gives the first inequality. The Fourier expression for total influence gives the second.

Second, for every set H of coordinates, the restriction inequality is

$$(2) \quad \mathcal{D}(f) \leq \sum_{i \in H \text{ relevant}} 2^{-d_i(f)} + \mathbb{E}_{\alpha \in \{-1,1\}^H} \mathcal{D}(f_\alpha).$$

For one restricted coordinate, any other coordinate surviving in both branches has local degree no greater than its original local degree. If it survives in only one branch, its original local degree is exactly one higher than in that branch. Thus its original weight is at most the average restricted weight in either case. Sum over the unrestricted coordinates and iterate to get (2). These are the usual local-weight facts of Chiarelli–Hatami–Saks and Wellens [2, 3].

We use the established theorem $R_3 = 10$: a Boolean function of degree at most three depends on at most ten variables. This comes from Tarannikov–Kirienko’s resilient-function result and its degree translation; Krotov–Valyuzhenich explicitly state this consequence in Section 3 and step (i) of their classification [6, 7]. Their account also cites Zverev’s computer-free argument. We rely on this published theorem; our programs do not independently re-enumerate all cubic Boolean functions.

Lemma 2. *If $\deg f \leq 3$, then $\mathcal{D}(f) \leq 5/4$. The constant $5/4$ is attained.*

Proof. Degrees at most two satisfy $\mathcal{D}(f) \leq 1$ by (1), so assume $\deg f = 3$. A relevant coordinate of local degree one would give $f = x_i a + g$, with a a nonzero constant. Both $g + a$ and $g - a$ must take values in $\{-1, 1\}$, forcing $a = \pm 1$ and $g = 0$. Such a function is a dictator, contrary to degree three. Hence all relevant local degrees are at least two.

If they are all three, the published ten-variable theorem immediately gives $\mathcal{D}(f) = r(f)/8 \leq 10/8 = 5/4$. Otherwise fix a coordinate i of local degree two. Write

$$f(x) = x_i A(y) + B(y), \quad A(y) = \frac{f(1, y) - f(-1, y)}{2}.$$

The nonzero polynomial A is affine and takes values only in $\{-1, 0, 1\}$. Every nonzero linear coefficient b_j has $|b_j| \geq 1/2$, since changing that input changes A by $2b_j$, a nonzero integer. Its total range has width $2 \sum_j |b_j| \leq 2$. Thus A depends on a set H of at most two coordinates. Let $s = |H|$ and $p = \Pr(A \neq 0)$ over those controls.

Whenever $A = \pm 1$, Booleanity forces $B = 0$, so the corresponding restricted function is $\pm x_i$, with potential $1/2$. Whenever $A = 0$, the restricted function is independent of i and has degree at most two. To see the degree claim, a cubic monomial disjoint from H would have the same coefficient in every restriction to H : a term containing that monomial and an additional control would have degree at least four. But one of the nonempty active branches is a dictator, which forces each such cubic coefficient to vanish. All restrictions therefore have degree at most two. Their potentials are at most one by (1).

Each coordinate in H has original local degree at least two. Applying (2) gives

$$\mathcal{D}(f) \leq \frac{s}{4} + \frac{p}{2} + (1 - p).$$

The possible nonzero affine ternary forms are, up to signs, coordinate permutations and omission of unused controls: 1 , x_j , $(1 + x_j)/2$, and $(x_j + x_k)/2$. For $(s, p) = (0, 1), (1, 1), (1, 1/2), (2, 1/2)$, respectively, the last bound is $1/2, 3/4, 1, 5/4$. This proves the inequality.

For sharpness define

$$Q(s, t, u, v) = \frac{1}{2}((s+t)u + (s-t)v),$$

and, on disjoint blocks x, y of four variables each, set

$$F(s, t, x, y) = \frac{1}{2}((s+t)Q(x) + (s-t)Q(y)).$$

For sign inputs exactly one summand is active in each expression, so F is Boolean. Its sixteen nonzero monomials are cubic, and each of its ten variables appears in one. Hence $\mathcal{D}(F) = 10/8 = 5/4$. Both checkers independently verify this small witness. $\square$

3. THE EXACT FINITE RECURRENCE AND TAIL

Let $\mathcal{B}_d$ be an upper bound for the block sensitivity of degree- d Boolean functions. Wellens's restriction of a degree- d monomial costs $d2^{-d}$ in potential and lowers block sensitivity by at least one. With $U(0, d) = 0$, a valid recursion is

$$(3) \quad U(b, d) = \min \left\{ c_d, \quad d2^{-d} + \max_{1 \leq k \leq d} U(b-1, k) \right\} \quad (1 \leq b \leq \mathcal{B}_d),$$

where $U(b, d) = U(\mathcal{B}_d, d)$ for $b > \mathcal{B}_d$ and

$$c_d = d/2 \quad (d \neq 3), \quad c_3 = 5/4.$$

Induction on b proves that $U(b, d)$ bounds the potential of every function with degree exactly d and block sensitivity at most b . Constants contribute zero. This is Wellens's recurrence with Lemma 2 replacing the degree-three influence cap $3/2$.

For $d = 1, \dots, 48$ we retain Épi's released nodes $1 = x_0 < x_1 < \dots < x_d$ and rational weights w_i . The exact conditions checked are

$$\sum_{i=0}^d w_i x_i^j = 0 \quad (j = 1, \dots, d), \quad w_0 > 0, \quad w_0 + \sum_{i \geq 1: w_i < 0} w_i > 0.$$

If a degree- d Boolean function had block sensitivity at least x_d , Wellens's symmetrization criterion would give a polynomial p with $p(0) = 0$, $p(1) = 1$ and $0 \leq p(k) \leq 1$ for $1 \leq k \leq x_d$. The displayed identities would imply both $\sum_i w_i p(x_i) = 0$ and $\sum_i w_i p(x_i) > 0$, a contradiction. Thus $\mathcal{B}_d = x_d - 1$ is valid. We intersect these bounds with Wellens's universal inequality

$$3b(b-1) \leq 2(d^4 - d^2),$$

using the largest integer satisfying it at each higher degree as well. No floating linear-program status enters this certificate.

Wellens's Corollary 3.10, equations (3.14)–(3.15), supplies the infinite extension: for every horizon H ,

$$(4) \quad \mathcal{D}(f) \leq U(\mathcal{B}_H, H) + \frac{(H+1)^3}{2^{H+1}} + \sum_{k=H+2}^{\infty} \frac{2k^2 - k}{2^k}.$$

The tail in (4) is exactly

$$T_H = \frac{(H+1)^3 + 2H^2 + 11H + 19}{2^{H+1}}.$$

Our row-wise dyadic-integer computation at $H = 96$ gives

$$U(\mathcal{B}_{96}, 96) + T_{96} = \frac{172624496193406565421448533795}{39614081257132168796771975168} < \frac{21788274613}{5000000000}.$$

Together with $r(f)2^{-d} \leq \mathcal{D}(f)$, this proves Theorem 1.

4. REPRODUCIBLE VERIFICATION AND SCOPE

The original Épi artifact `sources/wellens-certificate.json` has SHA-256

`ce66f1097a7284e9845113fa1dc39267932cc934fd23d9019ff13d3dcb39cf49`.

The new small input is `certificates/c44_candidate.json`. The first checker tests all power-moment identities and uses a row-wise integer recurrence at horizon 96. The second verifies the different Lagrange-residue identities $w_i x_i \prod_{j \neq i} (x_i - x_j) = \text{constant}$, then computes by columns using rational fractions and the independent horizon 72. It sums the tail from separate geometric-series moment formulas. Both prove the same printed upper bound. Their finite derivative-case tests enumerate coefficient patterns and affine ternary truth tables, respectively; the ten-variable sharpness witness is checked by an integer Walsh transform and by direct evaluation of its cubic polynomial.

Run

```
python verify_c44_exact.py certificates/c44_candidate.json
python independent/verify_c44_fraction.py \
    certificates/c44_candidate.json
python rejection_controls.py
```

Controls reject a smaller unsupported cap, an excessive upper-bound claim, an altered external theorem, changed source digest, and corrupted weights or nodes even when the attacker also refreshes the digest. These programs were written separately, with no imports from search or from one another, by the same AI agent. They are algorithmic cross-checks; there has been no independent human review. They do not formalize the published ten-variable theorem, Wellens’s universal recurrence argument, or the full prose proof of Lemma 2.

Endpoint-refined LP searches through degree 40 and integer slice-count relaxations through degree 10 did not yield a stronger block-sensitivity cap. Those are bounded negative experiments, not proofs of optimality. Their failure prompted the local-weight lemma; the existing polynomial certificates themselves were not improved.

PROVENANCE AND LIMITATIONS

The bounded live-source audit and final refresh are in `PRIOR_ART.md` and `FINAL_LIVE_CHECK.md`. We build on the local potential of Chiarelli–Hatami–Saks, the recurrence, symmetrization and tail of Wellens, Épi’s exact degree certificates, and the published ten-variable theorem. The contribution is the sharpened weighted degree-three cap and its effect on that recurrence. No stronger C44 bound was located in the sources checked; exhaustive priority is not asserted. This AI-prepared preprint is not peer reviewed and not verified in a proof assistant. No optimality is claimed.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. AI agents also prepared the certificate, checkers and publication materials. This does not claim a line-by-line human verification.

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