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The continuum limit of
three-dimensional $SU(2)$
lattice Yang–Mills theory
on a torus V:
Short-distance and weak-coupling
asymptotics

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WARNING!

We have extensively audited this manuscript using large language models, including adversarial reviews by multiple models. However, the proof is too complex for Lean formalization at the present time, and it has not yet undergone verification by a human expert.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).

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<https://www.loc.gov/resource/g3200m.gct00126/?sp=106>

AUTHORS' NOTE

With the exception of this note, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited.

The manuscript began as an attempt to pursue the program of Magnen, Rivasseau, and Sénéor [132] using large language models. Our hope was that understanding the problem on the torus might offer some insight into the general case. The construction that resulted from this starting point is three-dimensional and lattice-based and draws extensively on Bałaban's renormalization group methods.

As we were verifying these results, we learned that Alpöge and Sussman announced that they appear to have a YM4 existence argument, following Bałaban's program (<https://github.com/ethanwsussman/Yang--Mills-LLM-experiment>). We are thrilled and look forward to learning more about these exciting developments. We hope that this manuscript nonetheless contributes to understanding constructive approaches to three-dimensional Yang–Mills theory. The first version of these results was posted to the first author's GitHub on October 17, 2026 (<https://github.com/pmaronow/3d-ym-torus>). Prior to posting the first version of this work, we did not inspect the ongoing work of Alpöge and Sussman. Further, to the best of our knowledge, the LLMs we used did not access their work during the mathematical development and drafting of this manuscript. This version contains additional clarifications relative to the first version; it has also been partitioned into five volumes.

We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We consider the Wilson lattice gauge measure for $SU(2)$ on a periodic three-dimensional torus of fixed size, with bare coupling $g^2 = g_3^2 a$ proportional to the lattice spacing a , as dictated by the superrenormalizability of three-dimensional Yang–Mills theory. Observables are Wilson loops of the lattice configuration evolved by the Wilson flow, the lattice gradient flow of the Wilson action, at positive flow time. We prove that, at every fixed value of the coupling, the joint laws of finitely many flowed Wilson loops converge as the lattice spacing tends to zero along the sequences of lattice sides nM^k , for fixed odd $M \geq 3$ and fixed positive n divisible by four. For each such refinement class, the limits are realized by a unique probability measure on the space of gauge orbits of Yang–Mills heat-flow trajectories at positive times, a version of the state space introduced by Cao and Chatterjee. At each fixed positive time, the energies of the selected trajectories $\pi_k(U)$, with U distributed under the lattice Wilson measures, are tight uniformly in the lattice spacing. The limiting measure is invariant under the lattice symmetries and the center symmetries. We also show that, at short distances, the rescaled deficits $(1 - W)/(g_3^2 \varepsilon)$ of the normalized fundamental Wilson loops W of loops $x + \varepsilon \ell$, flowed for time $\sigma \varepsilon^2$ with fixed $\sigma > 0$, converge to explicit quadratic functions of the heat-smoothed Gaussian free field. As the coupling tends to zero, the measure concentrates on flat connections, with Gaussian fluctuations of explicit covariance near regular commuting holonomies and fluctuations governed by the constant-mode quartic form near the central holonomies. The proof is a multiscale renormalization group analysis in the tradition of Balaban, carried out on a small-field event whose complement we show to have small probability, uniformly in the lattice spacing, using reflection positivity and chessboard estimates. It is combined with stochastic estimates for the lattice heat flow and a deterministic analysis of the Wilson flow for rough initial data.

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PART X

Gaussian short-distance limits

Reconstruction estimates and divergence-free currents

This section and Section 32 prove the short-distance theorem, Theorem 2.8. We study the heat-smoothed circulations defined in Definition 31.1: circulations, around small lattice loops of size ε , of the linear heat evolution of the reconstructed fine field, divided by $\sqrt{g_3^2 \varepsilon}$. They are linear in the link logarithms of one fixed gauge representative; flowed Wilson loops are compared with them in Lemma 32.11.

The main result of this section is an exact decomposition of the heat-smoothed circulation $I^{K,s}$ under the conditioned Wilson measure, proved in Lemma 31.8:

$$I^{K,s} = T_s + \sum_{j=1}^{K-s} \Gamma_j(\widehat{\theta}_{j-1})(w_j - \bar{w}_j) + N_{K,s}.$$

Here T_s is the contribution of the coarse field on the lattice of side n_s ; the j -th summand is linear in the centered normalized fluctuation variables of the j -th RG step, so that the sum is a martingale; and $N_{K,s}$ collects conditional means and nonlinear terms. For $a \leq \varepsilon \leq \varepsilon_0(s)$ and $\eta = g_3^2 \varepsilon$, we show that

$$|T_s| + \mathbb{E}|N_{K,s}| \leq \frac{Cr}{g_3 H_s} \sqrt{\varepsilon} + C g_3 D_s \sqrt{\varepsilon} + C \sqrt{\eta} [1 + \log(1/\eta)]^2,$$

uniformly in the number K of RG steps, where $H_s = L/n_s$ is the coarse mesh and D_s is bounded uniformly in s . Theorem 31.6 identifies the covariance of the martingale increments at the zero field:

$$\sum_{j=1}^{K-s} \Gamma_j(0) \Gamma_j(0)^\top = C^{a,\varepsilon,L} \otimes I_3 - C_{s,a,\varepsilon}^{\text{ret}}.$$

In the linearized Gaussian theory, $C^{a,\varepsilon,L} \otimes I_3$ is the covariance of the circulation of the lattice Maxwell Gaussian field, and $C_{s,a,\varepsilon}^{\text{ret}}$ is the covariance of its conditional expectation given the linearized coarse field. As $a \rightarrow 0$ and then $\varepsilon \rightarrow 0$, with s fixed, the right side converges to $\Sigma \otimes I_3$, where Σ is the covariance in (2.6). Section 32 uses these facts to show that the martingale is asymptotically Gaussian.

The normalization by $\sqrt{g_3^2 \varepsilon}$ comes from the Gaussian Maxwell field. If A has covariance $g_3^2(-\Delta)^{-1}$ in three dimensions, then $e^{\sigma \varepsilon^2 \Delta} A$ has typical size $g_3 \varepsilon^{-1/2}$, and its circulation around a loop of length of order ε has typical size $g_3 \sqrt{\varepsilon} = \sqrt{\eta}$.

Two ideas enter the proof of the decomposition. The first concerns the zero-fluctuation reconstruction, in which every fluctuation variable is set to zero. Let $\widehat{L}_i$ be the linearization at the zero field of the i -th zero-fluctuation reconstruction map; it takes link logarithms on the lattice of mesh a_{i-1} to link logarithms on the lattice of mesh a_i . Each $\widehat{L}_i$ is a constrained minimization of the Maxwell quadratic form. By Theorem 31.3, a product of them agrees, up to a periodic gradient, with a single constrained minimization over all the averaging steps at once:

$$a^{-1} \widehat{L}_k \cdots \widehat{L}_{j+1} v = a_j^{-1} E_j^a v + d_a f_j(v).$$

The circulation around a closed lattice loop annihilates the gradient exactly, and E_j^a obeys bounds that do not depend on $k - j$. The nonlinear Taylor remainders $\widehat{\mathcal{R}}_i$ of the reconstruction maps are inserted at their own scale and then propagated to the fine lattice by the linearizations:

$$\text{mesh } a_j \xrightarrow{\widehat{P}_{i-1} \circ \cdots \circ \widehat{P}_{j+1}} \text{mesh } a_{i-1} \xrightarrow{\widehat{\mathcal{R}}_i} \text{mesh } a_i \xrightarrow{\widehat{L}_k \cdots \widehat{L}_{i+1}} \text{mesh } a.$$

Converting link logarithms at mesh a_i into a potential at the scale a_j introduces the factor $a_j/a_i = B^{i-j}$. Since $\widehat{\mathcal{R}}_i$ vanishes to second order, every monomial contains at least two inputs, each carrying a contraction factor α_* for every preceding step. For $i - 1 - j$ preceding steps, the physical rescaling is therefore controlled by

$$\frac{a_j}{a_i} \alpha_*^{2(i-1-j)+1} = C_* (B \alpha_*^2)^{i-1-j}, \quad B \alpha_*^2 \leq \frac{1}{4}.$$

The propagated remainders are therefore summable. The result is an analytic field $\widehat{\mathcal{E}}_j^a$ on the fine lattice with the same circulations as the reconstruction, whose coefficients are bounded uniformly in the number of steps; we need no such bound for the composed gauge representative itself.

The second idea concerns conditional means. The conditional law of the fluctuation variables is interacting. Using its density, the local covariance estimates of Section 13, and global color-rotation symmetry, we show that the conditional means of the nonlinear terms are of order t_j^2 times the size of the coarse field. Their conditional variances are bounded independently of the number of fluctuation variables.

We keep four kinds of variables apart. The link logarithms $\widehat{\theta}_j$ on the lattice of mesh a_j are dimensionless; the potentials $\widehat{\theta}_j/a_j$ are in physical units; the normalized fluctuation variables w_j are distributed according to the conditional fluctuation kernel k_j of

Theorem 31.5, which is supported on a product of balls; and independent standard Gaussian coordinates occur only in the linearized calculation of Theorem 31.6. All circulations are computed for the same gauge representative and with the componentwise heat semigroup.

The inputs come from Sections 4, 8, 10 to 13, 15, 16, and 23. Section 31.1 fixes notation. Section 31.2 defines the circulations and computes their covariance under the lattice Maxwell Gaussian field. Section 31.3 compares linearized reconstructions and controls the nonlinear remainders. Section 31.4 estimates the coefficients and the conditional means. Section 31.5 proves the covariance identity, Section 31.6 proves the decomposition and sums over scales, and Section 31.7 records that all estimates hold at every fixed positive coupling. All results are used in Section 32.

31.1 Setting and conventions

Fix $L > 0$, an odd integer $M \geq 3$, and an upper bound $\Gamma > 0$ on the dimensionless coupling. Let $J, B = M^J$, the parameter tuple Θ , the reconstruction maps, and the minimal coarse side n_Γ be as in Theorem 23.5; they are chosen before the coupling. Fix $0 < \gamma \leq \Gamma$ and set $g_3^2 = \gamma/L$. Fix $n_* \in 4\mathbb{N}$ and the sequence of lattice sides $N_K = n_* B^K$. For a coarse index $s \geq 0$ with $n_s \geq n_\Gamma$ and for $K \geq s$, set

$$n_s = n_* B^s, \quad H_s = L/n_s, \quad k = K - s,$$

$$a_j = H_s B^{-j} \quad (0 \leq j \leq k), \quad a = a_k = L/N_K,$$

$$d_j = J(k - j), \quad g_j = g_3 \sqrt{a_j}, \quad \lambda_B = M^{(J-1)/2}, \quad t_j = \lambda_B g_j, \quad p_j = A_w \log(1/t_j).$$

Here a_j is the mesh of the lattice reached after j RG steps of the reconstruction, starting from the coarse lattice of side n_s . The integer d_j is the number of averaging steps between the fine lattice and the lattice of mesh a_j ; it determines the effective classical action used at that scale. In the notation of Section 26, d_j is the depth $\mathfrak{d}_j$, not the increment $x_j - x_{j-1}$ of the heat path. The number t_j is the largest coupling among the J intermediate scales of the j -th RG step, and p_j is the fluctuation cutoff radius, with the amplitude A_w fixed in Θ . The couplings $g_{j,\ell}$ of the intermediate scales satisfy $g_{j,\ell} \leq t_j$, and the ratios $g_{j,\ell}/t_j$ depend only on M and J .

Let $\nu_{K,s} = \nu_{N_K, n_s, \gamma} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, n_s, \gamma})$ be the conditioned Wilson measure. We use the joint realization of Remark 15.11, whose fine-configuration marginal is $\nu_{K,s}$. The reconstruction and the fine configuration have the same joint gauge-invariant observables there. The coarse field $\hat{\theta}_0$ is the array of link logarithms, on the lattice of side n_s , of the chosen representative of the coarse configuration; it satisfies $\|\hat{\theta}_0\|_\infty \leq r$. For $1 \leq j \leq k$,

$$\hat{\theta}_j = \hat{\Phi}_j(\hat{\theta}_{j-1}, g_j, w_j),$$

where $\widehat{\Phi}_j$ is the reconstruction map of Theorem 15.13 for the j -th RG step, at the classical recursion index d_j , the $w_j = (w_{j,\ell})_\ell$ are the normalized fluctuation variables of that step, and $g_{j,\cdot} w_j$ is the vector with components $g_{j,\ell} w_{j,\ell}$. By Definition 4.9, the heat-smoothed circulation is $I^{K,s} = \mathcal{I}_a(\widehat{\theta}_k)$, with $\mathcal{I}_a$ as in Definition 31.1 below. We write

$$\mathcal{F}_j = \sigma(\widehat{\theta}_0, w_1, \dots, w_j), \quad \bar{w}_j = \mathbb{E}[w_j \mid \mathcal{F}_{j-1}], \quad b_j = \|\widehat{\theta}_j\|_\infty.$$

The zero-fluctuation reconstruction maps and their linear and nonlinear parts are

$$\widehat{P}_j(v) = \widehat{\Phi}_j(v, 0), \quad \widehat{L}_j = D\widehat{P}_j(0), \quad \widehat{\mathcal{R}}_j(v) = \widehat{P}_j(v) - \widehat{L}_j v,$$

and for $i \geq j$ we set $\widehat{\Pi}_{i \leftarrow j} = \widehat{P}_i \circ \dots \circ \widehat{P}_{j+1}$, which is the identity if $i = j$. For $1 \leq j \leq k$, a field θ on the lattice of mesh a_{j-1} , and fluctuation variables w of the j -th step, define

$$G_j(\theta, w) = \mathcal{I}_a(\widehat{\Pi}_{k \leftarrow j} \widehat{\Phi}_j(\theta, g_{j,\cdot} w)), \quad \Gamma_j(\theta) = D_w G_j(\theta, 0), \quad \omega_j = \min\{a_j/\varepsilon, \sqrt{\varepsilon/a_j}\},$$

$$R_j^{\text{cur}} = G_j(\widehat{\theta}_{j-1}, w_j) - G_j(\widehat{\theta}_{j-1}, 0) - \Gamma_j(\widehat{\theta}_{j-1}) w_j.$$

The matrix $\Gamma_j(\widehat{\theta}_{j-1})$ is $\mathcal{F}_{j-1}$ -measurable, and the circulation Taylor remainder R_j^{cur} is the nonlinear part of the change in the circulation caused by the fluctuations of the j -th step.

Coefficient norms. We use the norms of Definition 4.6. For a matrix $T = (T_{eu})$ from link variables on a lattice of mesh h to link variables on a lattice of mesh a' , the row norm is $\sup_e \sum_u |T_{eu}|$ and the volume-normalized column norm is $\sup_u (a'^3/h^3) \sum_e |T_{eu}|$; we write $\|T\|_{\text{row}+\text{col}}$ for the larger of the two. For a tensor of higher degree, a column sum fixes one input index, called the distinguished input, sums over all outputs and all other inputs, and carries the same volume factor. Every coefficient is weighted by $e^{a_{\text{sp}}^t}$, where t is the length of a shortest connected network, with internal vertices allowed, joining the locations of its indices. Write

$$F(v) = \sum_{m \geq 1} T_m[v^{\otimes m}], \quad T_m = \frac{D^m F(0)}{m!}, \quad \phi(z) = \sum_{m \geq 1} f_m z^m, \quad f_m \geq 0.$$

We write $F \prec \phi$ if, for every $m \geq 1$, the row norm of T_m and each volume-normalized column norm of T_m with any one input slot distinguished are at most f_m . Differentiating marks an input once: the degree- $(m-1)$ tensor of DF is $mT_m = D^m F(0)/(m-1)!$, and is bounded by the corresponding coefficient mf_m of ϕ' . These bounds on absolute values of coefficients are stronger than bounds on the values of F , and they allow compositions of arbitrary length below.

Parameters and constants. By Lemma 15.16 and Theorem 23.5, there exists a constant C_* bounding the first derivative of the zero-fluctuation reconstruction maps, fixed before J is chosen, such that

$$\alpha_* = C_*/B < 1, \quad q = B\alpha_*^2 = C_*^2/B \leq \frac{1}{4}.$$

Further, there exist a constant β_J and radii r_{eff} and r , chosen in this order, with

$$\frac{\beta_J r_{\text{eff}}}{1 - \alpha_*} < \frac{1}{8}, \quad r < \frac{r_{\text{eff}}}{4},$$

such that all reconstructed fields $\hat{\theta}_j$ and all their zero-fluctuation continuations lie in the ball $\{v : \|v\|_\infty \leq r_{\text{eff}}\}$, on which the bounds of Lemma 15.16 hold; we call it the reconstruction domain. The deterministic coefficient bounds below hold on this domain. The conditional fluctuation kernels, and all statements about their expectations, concern coarse fields θ with $e^\theta \in B_r$, the small-link domain of Definition 4.4, together with the product-domain and support conditions of Theorem 13.8. On the support of $\nu_{K,s}$ these conditions hold at every step: the coarse configurations of all RG steps lie in D_r , and the thresholds in Theorem 15.13 keep the images of the reconstruction maps inside B_r , so that $e^{\hat{\theta}_j} \in B_r$ for $0 \leq j \leq k$. We write E_q for the bound, fixed in Θ , on the weighted analytic norm of the input effective interaction, which is produced by the integrations at finer scales.

Unless stated otherwise, C and c denote positive constants that may depend on $L, \gamma, M, J, \Theta, n_*$, the analytic domains of Theorem 23.5, the loops, and σ ; this is the dependence described in item (5) of Definition 4.14. They do not depend on K, j, d_j , the point x , the lattice offset (defined below), or the coarse field. We say explicitly when a constant does not depend on s ; otherwise it may depend on s through H_s and n_s . All estimates are on the periodic torus. In the limit statements $\gamma > 0$ and s are fixed, the number of steps K tends to infinity first, $\varepsilon \downarrow 0$ second, and no rate uniform as $\gamma \downarrow 0$ is claimed.

31.2 Divergence-free currents and the Maxwell covariance

We define the circulations and the limiting covariance Σ , compute the covariance of the circulation of the lattice Maxwell Gaussian field, and show that it converges to Σ when the lattice spacing and then the loop size tend to zero.

Definition 31.1. Fix $\sigma > 0$ and finitely many smooth closed loops $\ell_i : [0, 1] \rightarrow \mathbb{R}^3$, $1 \leq i \leq m$. For $\varepsilon > 0$ put $\eta = g_3^2 \varepsilon$, and for $p \in \mathbb{R}^3$ define

$$\hat{J}_i(p) = \int_0^1 \dot{\ell}_i(v) e^{-ip \cdot \ell_i(v)} dv, \quad \Sigma_{ih} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{-2\sigma|p|^2}}{|p|^2} \overline{\hat{J}_i(p)} \cdot \hat{J}_h(p) dp.$$

Let $G = (G_i^c)_{1 \leq i \leq m, 1 \leq c \leq 3}$ be a centered Gaussian vector with covariance $\Sigma \otimes I_3$, write $G_i = (G_i^1, G_i^2, G_i^3)$, and set $Z_i = |G_i|^2/8$. In Theorem 2.8 and Section 32 this vector is denoted C , and $(Z_i)_i$ is denoted Z_G .

For a lattice mesh $a = L/N \leq \varepsilon$ and $x \in \mathbb{T}_L^3$, let $\ell_{i,\varepsilon,a,x}$ be a closed oriented lattice cycle approximating the loop $x + \varepsilon \ell_i$ in $\mathbb{T}_L^3$, such that its physical length and diameter are at

most $C_\ell \varepsilon$, it has zero winding when $\varepsilon \leq \varepsilon_{\text{loop}}$, and there is a based homotopy between it and $x + \varepsilon \ell_i$ whose swept area, counted with multiplicity, is at most $C_\ell \varepsilon a$. The constants C_ℓ and $\varepsilon_{\text{loop}}$ depend only on the loops; they are uniform in x and in the choice of rounding of the loop to the lattice, which we call the lattice offset. For a link potential A on the lattice of mesh a , in physical units, set

$$(\mathcal{C}_{a,\varepsilon,x}A)_i^c = \frac{a}{\sqrt{\eta}} \sum_{e \in \ell_{i,\varepsilon,a,x}} \text{sgn}(e) (P_{\sigma\varepsilon^2}^a A)_{\text{dir}(e)}^c(x_{\text{positive}}(e)), \quad \mathcal{I}_a(v) = \mathcal{C}_{a,\varepsilon,x}(v/a),$$

for $1 \leq i \leq m$ and $1 \leq c \leq 3$. Here $P_t^a = e^{t\Delta_a}$ is the lattice heat semigroup, acting componentwise; $\text{dir}(e)$ is the coordinate direction of e ; $x_{\text{positive}}(e)$ is the starting vertex of the positively oriented link underlying e ; and $\text{sgn}(e) = -1$ if e is traversed against its positive orientation, so that a reversed link contributes minus the value on its positive orientation. The coefficient array $e \mapsto a \text{sgn}(e)$ is a divergence-free lattice current, since the cycle is closed. We call $\mathcal{C}_{a,\varepsilon,x}A$ the heat-smoothed circulation of A , and $\mathcal{I}_a(v)$ that of the link logarithms v .

Finally, for $k \in (2\pi/L)\mathbb{Z}^3$ (a frequency) set

$$J_{i,\varepsilon,a,x}(k) = a \sum_{e \in \ell_{i,\varepsilon,a,x}} \text{sgn}(e) e_{\text{dir}(e)} e^{-ik \cdot x_{\text{positive}}(e)}, \quad \lambda_a(k) = 4a^{-2} \sum_{r=1}^3 \sin^2(ak_r/2),$$

$$C_{ih}^{a,\varepsilon,L} = \frac{1}{\varepsilon L^3} \sum_{k \neq 0} \frac{e^{-2\sigma\varepsilon^2\lambda_a(k)}}{\lambda_a(k)} \overline{J_{i,\varepsilon,a,x}(k)} \cdot J_{h,\varepsilon,a,x}(k),$$

where k ranges over representatives of $(2\pi/L)\mathbb{Z}^3$ modulo $(2\pi/a)\mathbb{Z}^3$ with $|ak_r| \leq \pi$, and

$$C_{ih}^{\varepsilon,L} = \left(\frac{\varepsilon}{L}\right)^3 \sum_{k \in (2\pi/L)\mathbb{Z}^3 \setminus \{0\}} \frac{e^{-2\sigma|\varepsilon k|^2}}{|\varepsilon k|^2} \overline{\widehat{J}_i(\varepsilon k)} \cdot \widehat{J}_h(\varepsilon k).$$

The matrix Σ is the covariance of heat-smoothed circulations of a massless transverse field on $\mathbb{R}^3$: before smoothing, the Fourier covariance of that field is $|p|^{-2}(I - pp^\top/|p|^2)$ in each of three independent colors, and the transverse projection drops out against $\widehat{J}_i$ because $p \cdot \widehat{J}_i(p) = 0$.

Lemma 31.2. *In the notation of Definition 31.1, the following statements hold.*

1. *We have $p \cdot \widehat{J}_i(p) = 0$ and $|\widehat{J}_i(p)| \leq C_\ell \min\{|p|, 1\}$. The integral defining Σ converges, and Σ is a real symmetric positive semidefinite matrix; it may be singular.*
2. *For $v_1, \dots, v_m \geq 0$, with $V = \text{diag}(v_i)$,*

$$\mathbb{E} e^{-\Sigma_i v_i Z_i} = \det\left(I + \frac{1}{4}\Sigma^{1/2}V\Sigma^{1/2}\right)^{-3/2},$$

$$\mathbb{E} Z_i = \frac{3}{8}\Sigma_{ii}, \quad \text{Cov}(Z_i, Z_h) = \frac{3}{32}\Sigma_{ih}^2.$$

3. The functional $\mathcal{C}_{a,\varepsilon,x}$ vanishes on every periodic lattice gradient. If $\varepsilon \leq \varepsilon_{\text{loop}}$, it also vanishes on every constant field. For the initial potential $A^{K,s} = a_{N_K}^{-1} \widehat{\theta}_{K-s}$ of Definition 4.9, we have $I^{K,s} = \mathcal{I}_{a_{N_K}}(\widehat{\theta}_{K-s})$.
4. Let $\varepsilon \leq \varepsilon_{\text{loop}}$, and let A be the physical Maxwell Gaussian field on the lattice of mesh a , in three independent colors: the centered Gaussian field on link potentials modulo gradients and constants whose density is proportional to $\exp(-\|d\theta\|_{\ell^2}^2 / (2g_3^2 a))$, where $\theta = aA$ and d is the dimensionless plaquette coboundary. Equivalently, its covariance operator for the physical scalar product

$$\langle u, v \rangle_a = a^3 \sum_e u_e \cdot v_e$$

is $g_3^2 (-\Delta_a)^+ P_{\text{tr},a}$, where $P_{\text{tr},a}$ is the projection onto lattice divergence-free fields and $(-\Delta_a)^+$ is the inverse of $-\Delta_a$ on fields with zero mean. Then $\mathbb{E}[(\mathcal{C}_{a,\varepsilon,x} A)_i^c (\mathcal{C}_{a,\varepsilon,x} A)_h^{c'}] = \delta_{cc'} C_{ih}^{a,\varepsilon,L}$. The longitudinal part of the covariance drops out by the exact closedness of the cycles, before any limit is taken.

5. For each fixed $\varepsilon \in (0, \varepsilon_{\text{loop}}]$, $C^{a,\varepsilon,L} \rightarrow C^{\varepsilon,L}$ as $a \downarrow 0$, uniformly in x and the lattice offset. Further, $C^{\varepsilon,L} \rightarrow \Sigma$ as $\varepsilon \downarrow 0$.

The constants in these statements depend only on the loops, L , σ , C_ℓ , and $\varepsilon_{\text{loop}}$. The covariances $C^{a,\varepsilon,L}$, $C^{\varepsilon,L}$, and Σ do not depend on g_3 .

Proof. We verify the statements in order.

Step 1: the continuum current. Let $l_i = \int_0^1 |\dot{\ell}_i(v)| dv$, and let y_i be a point on the i -th loop. Integrating the derivative of $v \mapsto e^{-ip \cdot \ell_i(v)}$ over the closed loop, we find $p \cdot \widehat{J}_i(p) = 0$. Since $\int_0^1 \dot{\ell}_i = 0$, we also have

$$\widehat{J}_i(p) = \int_0^1 \dot{\ell}_i(v) [e^{-ip \cdot \ell_i(v)} - e^{-ip \cdot y_i}] dv, \quad |\widehat{J}_i(p)| \leq l_i \min\{2, |p| \text{diam } \ell_i\}.$$

The integrand defining Σ is therefore bounded near zero and bounded by $C|p|^{-2}e^{-2\sigma|p|^2}$ for $|p| \geq 1$; both bounds are integrable in three dimensions. For real numbers u_i , the quadratic form of Σ is

$$(2\pi)^{-3} \int_{\mathbb{R}^3} \frac{e^{-2\sigma|p|^2}}{|p|^2} \left| \sum_i u_i \widehat{J}_i(p) \right|^2 dp \geq 0.$$

The substitution $p \mapsto -p$, together with $\widehat{J}_i(-p) = \overline{\widehat{J}_i(p)}$, shows that the entries are real, and interchanging i and h shows that Σ is symmetric.

Step 2: the Gaussian comparison vector. Let Z^1, Z^2, Z^3 be independent standard Gaussian vectors in $\mathbb{R}^m$ and set $G^c = \Sigma^{1/2} Z^c$; this constructs G also when Σ is singular. Then $\sum_i v_i Z_i = (1/8) \sum_c (Z^c)^\top \Sigma^{1/2} V \Sigma^{1/2} Z^c$. Diagonalizing the positive semidefinite matrix

$\Sigma^{1/2}V\Sigma^{1/2}$ by an orthogonal matrix reduces the Laplace transform to a product, over its eigenvalues λ and the three colors, of

$$(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-(1+\lambda/4)z^2/2} dz = (1 + \lambda/4)^{-1/2},$$

which gives the determinant formula. For a centered Gaussian pair (X, Y) we have $\mathbb{E}(X^2Y^2) = \mathbb{E}X^2 \mathbb{E}Y^2 + 2(\mathbb{E}XY)^2$; this is Isserlis' formula [112]. Applying it to the coordinates of G , we obtain $\mathbb{E}Z_i = (3/8)\Sigma_{ii}$ and $\text{Cov}(Z_i, Z_h) = (1/64)\Sigma_c 2\Sigma_{ih}^2 = (3/32)\Sigma_{ih}^2$.

Step 3: gradients and constants. For a vertex function f , let $d_a f$ be the field whose value on the positive link from y to $y + ae_r$ is $(f(y + ae_r) - f(y))/a$. The heat semigroup commutes with d_a , since the lattice Laplacian is a sum of translations, which commute. The oriented sum of $a(P_t^a d_a f)_{\text{dir}(e)}(x_{\text{positive}}(e))$ around the lattice cycle telescopes to zero. The heat semigroup fixes constant fields, and the sum of the oriented displacement vectors $a \text{sgn}(e)e_{\text{dir}(e)}$ vanishes for a cycle of zero winding; hence the current also annihilates constants. Both assertions use the convention that a reversed link is evaluated on its positive orientation, at the starting vertex of that positive link. The identity $I^{K,s} = \mathcal{I}_{a_{N_K}}(\widehat{\theta}_{K-s})$ is the substitution $A = a_{N_K}^{-1}\widehat{\theta}_{K-s}$ in Definition 4.9; the gauge representative is not changed.

Step 4: the lattice covariance. We take the representatives k with $|ak_r| \leq \pi$ and define

$$d_r(k) = a^{-1}(e^{iak_r} - 1), \quad \text{so that} \quad \lambda_a(k) = \sum_r |d_r(k)|^2.$$

The Fourier transform of the boundary of the cycle is zero, which is the identity

$$\sum_r \overline{d_r(k)} J_{i,\varepsilon,a,x,r}(k) = 0 \tag{31.1}$$

at every frequency, with repeated links counted with multiplicity. Indeed, the left side equals $\sum_e \text{sgn}(e)(e^{-ik \cdot y_+(e)} - e^{-ik \cdot y_-(e)})$, where $y_{\pm}(e)$ are the endpoints of the positive link underlying e , and this sum telescopes around the closed cycle. At $k = 0$, zero winding implies $J_i(0) = 0$. On each nonzero frequency, the algebraic identity

$$\sum_{r < q} |d_r A_q - d_q A_r|^2 = \lambda_a |A|^2 - \left| \sum_r \overline{d_r} A_r \right|^2$$

identifies the inverse of the Maxwell quadratic form, restricted to transverse fields, as $\lambda_a^{-1} P_{\text{tr},a}$. This normalization agrees with the Wilson normalization of Theorem 8.3: writing $\theta = aA$, the quadratic exponent is

$$\frac{\|d\theta\|_{\ell^2}^2}{2g_3^2 a} = \frac{a^3}{2g_3^2} \sum_y \sum_{r < q} |(d_a A)_{rq}(y)|^2,$$

where d on the left is dimensionless and d_a on the right is in physical units. With $\hat{A}(k) = a^3 \sum_y e^{-ik \cdot y} A(y)$, the covariance of the Fourier coefficients is $g_3^2 L^3 \mathbf{1}_{k'=-k} \lambda_a(k)^{-1} P_{\text{tr},a}(k)$. Pairing twice with the cycle, applying the two heat factors $e^{-\sigma \varepsilon^2 \lambda_a(k)}$, and dividing by η , we obtain the covariance

$$\frac{g_3^2}{\eta L^3} \sum_{k \neq 0} \frac{e^{-2\sigma \varepsilon^2 \lambda_a(k)}}{\lambda_a(k)} \overline{J_i(k)} \cdot J_h(k),$$

in each color separately. Here (31.1) removed the projection $P_{\text{tr},a}$. Since $g_3^2/\eta = 1/\varepsilon$, this is the formula for $C^{a,\varepsilon,L}$.

Step 5: the limit $a \downarrow 0$. We lift the small cycle and the loop $x + \varepsilon \ell_i$ to a ball of diameter $C\varepsilon$ in $\mathbb{R}^3$. For a C^1 one-form f , replacing its integral over each oriented link by the value at the starting vertex of the positive link changes the integral by at most $a^2 \|Df\|_\infty / 2$ per link; summing over at most $C_\ell \varepsilon / a$ links gives an error at most $C\varepsilon a \|Df\|_\infty$. For a piecewise smooth based homotopy $H(u, v)$ between the polygon and the loop, differentiating $\int f(H) \partial_v H dv$ in u and integrating by parts in v gives

$$\oint_{\text{polygon}} f - \oint_{x+\varepsilon \ell_i} f = \int_0^1 \int_0^1 df(H) (\partial_u H, \partial_v H) du dv.$$

The endpoint terms cancel because the paths are closed and the homotopy is based; for a piecewise smooth homotopy the formula holds piece by piece, and the internal boundaries cancel. The swept area bounds this difference by $C\varepsilon a \|df\|_\infty$. Applying both estimates to $f(y) = e^{-ik \cdot y} dy_r$, and using $\oint_{x+\varepsilon \ell_i} e^{-ik \cdot y} dy = \varepsilon e^{-ik \cdot x} \hat{J}_i(\varepsilon k)$, we obtain for each fixed ε and k

$$|J_{i,\varepsilon,a,x}(k) - \varepsilon e^{-ik \cdot x} \hat{J}_i(\varepsilon k)| \leq C\varepsilon a |k|, \quad (31.2)$$

uniformly in x and the lattice offset; a rounding of the common base point is included in the same estimate. Subtracting from each exponential in the definition of J_i its value at one vertex of the cycle, which is allowed since $J_i(0) = 0$, and using the bounds on length and diameter, we find

$$|J_i(k)| \leq C\varepsilon \min\{1, \varepsilon |k|\}, \quad \frac{4}{\pi^2} |k|^2 \leq \lambda_a(k) \leq |k|^2. \quad (31.3)$$

The second assertion follows in each coordinate from $2|u|/\pi \leq |\sin u| \leq |u|$ for $|u| \leq \pi/2$. For fixed $\varepsilon > 0$, we extend the summand defining $C^{a,\varepsilon,L}$ by zero outside the chosen representatives. By (31.3) its absolute value is bounded by a summable Gaussian function of k on the fixed reciprocal lattice $(2\pi/L)\mathbb{Z}^3$, uniformly in a , x , and the offset. Given a tolerance, we first truncate this Gaussian tail. On the finitely many remaining frequencies, (31.2) and $\lambda_a(k) \rightarrow |k|^2$ give uniform convergence. Substituting $J_i(k) = \varepsilon e^{-ik \cdot x} \hat{J}_i(\varepsilon k)$, the two translation phases cancel and the prefactor becomes $\varepsilon^2 / (\varepsilon L^3 |k|^2) = (\varepsilon/L)^3 |\varepsilon k|^{-2}$. This proves $C^{a,\varepsilon,L} \rightarrow C^{\varepsilon,L}$.

Step 6: the limit $\varepsilon \downarrow 0$. Let $\delta = 2\pi\varepsilon/L$ and, for $p \neq 0$, define

$$f_{ih}(p) = \frac{e^{-2\sigma|p|^2}}{|p|^2} \widehat{f}_i(p) \cdot \widehat{f}_h(p).$$

By Step 1, f_{ih} is bounded near zero, smooth away from zero, and has a Gaussian tail. We have $C_{ih}^{\varepsilon,L} = (2\pi)^{-3} \delta^3 \sum_{v \in \mathbb{Z}^3 \setminus \{0\}} f_{ih}(\delta v)$. For $r > 0$, the contribution of the points with $|\delta v| < r$ is at most $C(r + \delta)^3$, as one sees by counting the disjoint cubes of side δ centered at these points inside the ball of radius $r + \sqrt{3}\delta$; the corresponding integral is at most Cr^3 . On a fixed annulus $r \leq |p| \leq R$, uniform continuity bounds the difference between the integral over each cube and the sampled value by the modulus of continuity at $\sqrt{3}\delta$, and the boundary cubes have total volume tending to zero. The tails are uniform for $0 < \delta \leq 1$: for $R \geq 2$, the absolute sum over the shell $2^q R \leq |p| < 2^{q+1}R$ is at most $C(2^{q+1}R + 1)^3 e^{-c_\sigma(2^q R)^2}$, and these bounds are summable in q with sum tending to zero as $R \rightarrow \infty$. Letting $\delta \downarrow 0$, then $r \downarrow 0$, and then $R \rightarrow \infty$, we obtain $C_{ih}^{\varepsilon,L} \rightarrow (2\pi)^{-3} \int f_{ih} = \Sigma_{ih}$. The constants used depend only on the stated data, and g_3 appeared only through $g_3^2/\eta = 1/\varepsilon$. $\square$

31.3 Linearized reconstruction and nonlinear error terms

This section constructs the field $\widehat{\mathcal{E}}_j^a$ described at the beginning of the section and proves the deterministic bounds that convert coefficient bounds into bounds for circulations. The linear comparison follows the constrained Gaussian calculations for Balaban's averaging operation [10, Section A, (1.1)–(1.8), pp. 18–19, and Section D, (1.47)–(1.49), p. 26, and (1.64)–(1.67), p. 29]; the averaging map, the effective actions, and the gauge representatives used here are those of Sections 8, 10, 11, and 15, and the comparison is proved below for them.

Theorem 31.3. *Assume the setting of Section 31.1 and the loop data of Definition 31.1. Fix a coarse index s with $n_s \geq n_\Gamma$ and $K \geq s$. Then the following statements hold.*

1. *On the larger analytic domain fixed in Theorem 23.5, the zero-fluctuation reconstruction maps satisfy*

$$\widehat{P}_j \prec \phi(z) = \frac{\alpha_* z}{1 - \beta_j z}, \quad (v \mapsto \widehat{P}_j(v) - D\widehat{P}_j(0)v) \prec \frac{\alpha_* \beta_j z^2}{1 - \beta_j z},$$

with output rows, every volume-normalized column with a distinguished input, and a fixed spatial weight. The constant β_j is chosen on that domain before r_{eff} and r .

2. *There exist deterministic matrices E_j^a , $0 \leq j \leq k$, with $E_k^a = I$, each a scalar multiple of the identity on the color factor, with rows and volume-normalized columns (with factor a^3/a_j^3)*

bounded by a constant, and with a fixed positive spatial decay rate measured in units of a_j , such that

$$a^{-1}\widehat{L}_k \cdots \widehat{L}_{j+1}v - a_j^{-1}E_j^a v = d_a f_j(v)$$

for a periodic vertex function $f_j(v)$ on the fine lattice depending linearly on v . The difference has no constant part, because the block averaging constraint fixes the componentwise means in physical units. These bounds do not depend on $k - j$.

3. Let T be a matrix from link variables on a lattice of mesh h to link variables on the fine lattice, with row norm and volume-normalized column norm at most A_0 , and with at most a fixed number of input coordinates per cell of side h . Then for $0 \leq t \leq \sigma \varepsilon_{\text{loop}}^2$,

$$\sup_y \|\text{row}_y(P_t^a T)\|_{\ell^2} \leq C A_0 \min\{1, h^{3/2}(t + a^2)^{-3/4}\}, \quad \|P_t^a T\|_{\ell^\infty \rightarrow \ell^\infty} \leq A_0.$$

Consequently, for $a \leq \varepsilon \leq \varepsilon_{\text{loop}}$,

$$\left\| \mathcal{C}_{a,\varepsilon,x} \frac{g_3 \sqrt{h}}{h} T \right\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq C A_0 \min\{h/\varepsilon, \sqrt{\varepsilon/h}\},$$

$$\left\| \mathcal{C}_{a,\varepsilon,x} \frac{g_3 \sqrt{h}}{h} T \right\|_{\ell^\infty \rightarrow \mathbb{R}^{3m}} \leq C A_0 \sqrt{\varepsilon/h},$$

where ℓ^2 and ℓ^∞ refer to the counting norms on the inputs.

4. For $i > j$ put $r' = i - 1 - j$ and $D = \beta_j / (1 - \alpha_*)$. Then

$$(v \mapsto \widehat{\mathcal{R}}_i(\widehat{\Pi}_{i-1 \leftarrow j}(v))) \prec C \alpha_*^{2r'+1} \frac{z^2}{(1 - Dz)^2},$$

for rows and for columns with the normalization a_i^3/a_j^3 . Derivatives with a distinguished input are bounded by derivatives of the same series, with the same spatial decay rate.

5. Define the field on the fine lattice

$$\widehat{\mathcal{E}}_j^a(v) = E_j^a v + \sum_{i=j+1}^k \frac{a_j}{a_i} E_i^a \widehat{\mathcal{R}}_i(\widehat{\Pi}_{i-1 \leftarrow j}(v)).$$

For $\|v\|_\infty \leq r_{\text{eff}}$, it is equivariant under global color rotations, vanishes at $v = 0$, and satisfies

$$\mathcal{I}_a(\widehat{\Pi}_{k \leftarrow j}(v)) = \mathcal{C}_{a,\varepsilon,x}(a_j^{-1} \widehat{\mathcal{E}}_j^a(v)).$$

Its rows and its columns with normalization a^3/a_j^3 are bounded by a power series independent of k , j , and d_j . In particular, on the reconstruction domain,

$$\|D \widehat{\mathcal{E}}_j^a(v)\|_{\text{row+col}} \leq C, \quad \|D \widehat{\mathcal{E}}_j^a(v) - D \widehat{\mathcal{E}}_j^a(0)\|_{\text{row+col}} \leq C \|v\|_\infty,$$

and every fixed higher derivative obeys the corresponding bound. The conversion to physical units is governed by the identities

$$\frac{a_j}{a_i} \alpha_*^{2r'+1} = B^{r'+1} \alpha_*^{2r'+1} = C_* q^{r'}, \quad \frac{a_i^3}{a_j^3} \cdot \frac{a_i^3}{a_j^3} = \frac{a_i^3}{a_j^3}.$$

The constants in the coefficient bounds depend only on M, J, Θ , the analytic domains and their ratios, the spatial decay rates, and the number of coordinates per cell; the constants in the heat and circulation bounds may in addition depend on L, σ, C_ℓ , and $\varepsilon_{\text{loop}}$. None depends on K, s, j, d_j , or the coarse field in the reconstruction domain.

Remark 31.4. The identity in part (5) is an identity between circulations. It does not say that the composed nonlinear reconstruction $\hat{\Pi}_{k \leftarrow j}$ coincides with a single reconstruction over all intermediate scales, and no inverse of the Maxwell operator on the coarse lattice is used.

Proof. The proof has six steps. Step 1 recalls the analytic input. Step 2 proves the linear comparison of part (2), which is the source of the exact cancellation of gradients. Step 3 proves the heat and circulation bounds of part (3). Steps 4 and 5 show that coefficient bounds survive compositions of arbitrary length and prove part (4). Step 6 assembles the field $\hat{\mathcal{E}}_j^a$.

Step 1: the analytic input. The coefficient bound on the larger domain and the order in which the constants are chosen are given by Lemma 15.16. The identification of C_* with the constant C_{CP} of that lemma and the choice of J are part of Theorem 23.5; in particular we use $q \leq 1/4$, and no constant of an all-order bound at fixed J is used to choose J . Restricting the joint coefficient series of Lemma 15.16 to zero fluctuation variables gives the two majorants of part (1), with the stated output and distinguished-input conventions. Its domain inclusions hold on the reconstruction domain.

Step 2: the linear comparison. For $j < k$ we apply Lemma 11.9 once, with $J(k-j)$ averaging steps, preceding classical recursion index zero, coarse mesh a_j , and relative fine spacing $u = a/a_j$. The logarithm of the representative given by that lemma is $u \bar{A}_u(v)$, and we define

$$E_j^a = D \bar{A}_u(0).$$

By the first-derivative clause of that lemma, we have the row bound and the column factor $u^3 = a^3/a_j^3$, with one positive spatial decay rate in units of a_j , uniformly in the number $J(k-j)$ of averaging steps. We use no bound on the dependence on fluctuation variables at this growing number of steps. By global color equivariance, each 3×3 color block of E_j^a commutes with every rotation. Commuting with the rotations by π about the coordinate axes makes the block diagonal, and commuting with the rotations that

permute the axes makes its three diagonal entries equal. Hence E_j^a is a scalar multiple of the identity on the color factor. We set $E_k^a = I$.

We now show that uE_j^a represents the composed linearization for circulations. First let v be real; all minimizations and comparisons of nonnegative quadratic forms in this paragraph are on real tangent spaces. By the quadratic part of the nested minimization identity of Theorem 10.10, each linearized reconstruction minimizes the inherited nonnegative Maxwell quadratic form over the free coordinates, on which this form is positive definite. If a quadratic form $Q(x, y, z)$ has positive blocks in the constrained variables y and z , then

$$\inf_z \inf_y Q(x, y, z) = \inf_{(y, z)} Q(x, y, z).$$

Indeed, every value of the problem on the right bounds the left side from above, and for each fixed z the inner minimizer realizes the infimum over y . Completing the squares in the positive blocks shows that both infima are attained at the classical tangents. We iterate this equality over the finitely many averaging steps. Since the indices $d_j = J(k - j)$ are compatible, the inherited forms are exactly the successive Schur complements. Thus the composed tangent $\hat{L}_k \cdots \hat{L}_{j+1}v$ and the tangent $uE_j^a v$ minimize the same quadratic form on the fine lattice subject to the same linearized averaging constraint v . The gauge transformations of Theorem 15.13 depend only on the coarse field and equal the identity at the tree roots; they preserve the constraint and the gauge orbit of the minimizer. Their derivatives at zero add periodic gradients, which do not affect this conclusion.

Let X and Y be these two minimizers, and let Q_0 be the Maxwell quadratic form. Since the constraint is linear, the midpoint satisfies it. By the parallelogram identity, we have

$$Q_0\left(\frac{X+Y}{2}\right) = \frac{1}{2}Q_0(X) + \frac{1}{2}Q_0(Y) - \frac{1}{4}Q_0(X-Y).$$

Minimality and nonnegativity imply $Q_0(X-Y) = 0$; in particular $d(X-Y) = 0$. We determine the kernel. Let $Z = X - Y$ and fix a nonzero lattice frequency, with $b_r = e^{iak_r} - 1$. Zero curl says $b_r \hat{Z}_q = b_q \hat{Z}_r$. Choosing an index r with $b_r \neq 0$, we have $\hat{Z}_q = b_q(\hat{Z}_r/b_r)$ for every q ; that is, every nonzero-frequency part of Z is a periodic gradient. The zero-frequency part is a constant field, and its componentwise means in physical units are its constant components.

This constant vanishes for $X - Y$. By Lemma 8.1, the derivative of the composed block averaging map is the box and segment average plus a periodic gradient on the coarse lattice, and its version in physical units preserves each componentwise mean. Explicitly, if $m' = a_j/a$ is the total blocking factor and $K_{m'}$ the box and segment average, the box and segment formula counts each fine link m' times before its factor m'^{-3} , so that

$$\sum_Y (K_{m'} Z)_r(Y) = m'^{-2} \sum_y Z_r(y), \quad (m'a)^3 \sum_Y \frac{(K_{m'} Z)_r(Y)}{m'a} = a^3 \sum_y \frac{Z_r(y)}{a}.$$

The coarse gradient has mean zero. Since $X - Y$ satisfies the homogeneous constraint, its mean in physical units is zero. The same argument applies to coarse fields with nonzero constant components, which are fixed by the constraint. We conclude that $a^{-1}(X - Y) = d_a f_j(v)$ for a periodic vertex function, chosen with mean zero, which depends linearly on v by the frequency formula above. This proves part (2), and by Lemma 31.2 the circulation annihilates the gradient. The maps in this identity are real linear; their complex-linear extensions give the same identity for complex v .

Step 3: heat and circulation bounds. Let $R(T)$ and $C(T)$ be the row norm and the volume-normalized column norm of T . For a vector z of inputs, by the Cauchy–Schwarz inequality with the weights $|T_{eu}|$, we have

$$\sum_e \left| \sum_u T_{eu} z_u \right|^2 \leq R(T) \sum_{e,u} |T_{eu}| |z_u|^2 \leq R(T) C(T) (h/a)^3 \sum_u |z_u|^2,$$

since $\sum_e |T_{eu}| \leq (h/a)^3 C(T)$. Thus $\|T\|_{\ell^2 \rightarrow \ell^2} \leq A_0 (h/a)^{3/2}$. We use Lemma 16.1 with the fixed horizon $T_{\text{heat}} = \sigma \varepsilon_{\text{loop}}^2$ and difference order zero. Its hypotheses hold on the periodic fine lattice of mesh $a = L/N_K$, with $N_K \geq 4$, and all times considered lie in $[0, T_{\text{heat}}]$. Since the heat kernel of that lemma is a density with respect to $a^3 \sum_y$, the row of P_t^a at y , as a vector in the counting norm, is $a^3 p_t^a(y, \cdot)$, and its ℓ^2 norm is at most $Ca^{3/2}(t + a^2)^{-3/4}$. Applying this row to T , we obtain

$$\|\text{row}_y(P_t^a T)\|_{\ell^2} \leq \|a^3 p_t^a(y, \cdot)\|_{\ell^2} \|T\|_{\ell^2 \rightarrow \ell^2} \leq CA_0 h^{3/2} (t + a^2)^{-3/4}.$$

Since the heat kernel is positive with unit mass, the absolute row sums of $P_t^a T$ are at most A_0 ; this bounds both the ℓ^2 norm of each row and the $\ell^\infty \rightarrow \ell^\infty$ norm, including at $t = 0$. The finitely many directions and input types are absorbed into C . Since the horizon is fixed, the constants depend on its ratio to L^2 , and the constant mode of the heat semigroup is included.

For $\varepsilon \leq \varepsilon_{\text{loop}}$ the time $t = \sigma \varepsilon^2$ lies in $[0, T_{\text{heat}}]$, and $a \leq \varepsilon$ implies $(t + a^2)^{-3/4} \leq C_\sigma \varepsilon^{-3/2}$. The circulation sums the rows over at most $C_\ell \varepsilon/a$ links with the weight $a/\sqrt{\eta}$, so the prefactor multiplying a single row is

$$\frac{\varepsilon}{\sqrt{\eta}} \cdot \frac{g_3 \sqrt{h}}{h} = \sqrt{\varepsilon/h}.$$

Multiplying it by $\min\{1, C(h/\varepsilon)^{3/2}\}$ proves the ℓ^2 circulation bound, and multiplying it by the absolute row bound proves the ℓ^∞ bound. Summing over the fixed number of loop and color components gives a fixed factor.

Step 4: compositions of coefficient bounds. In the composition of two tensors, fixing an output and summing over all inputs uses the row bound of the outer tensor and a row

bound of the inner tensor at every input slot of the outer tensor. If instead we fix one input of the inner tensor, we follow the chain of factors through which this input enters. Along this chain we use the column bound of the inner tensor and the column bound of the outer tensor with the corresponding distinguished input; all other factors use rows. For example, let B be a bilinear tensor from mesh a_{i-1} to mesh a_i and G a linear tensor from mesh a_j to mesh a_{i-1} , and consider $(B(Gv, Gv))_e = \sum_{y,y',u,u'} B_{e;yy'} G_{yu} G_{y'u'} v_u v_{u'}$. Fixing u in the first slot, we have

$$\sum_{e,u'} \left| \sum_{y,y'} B_{e;yy'} G_{yu} G_{y'u'} \right| \leq \sum_y |G_{yu}| \sum_{e,y'} |B_{e;yy'}| \sum_{u'} |G_{y'u'}| \leq \frac{a_j^3}{a_{i-1}^3} C(G) \cdot \frac{a_{i-1}^3}{a_i^3} C_1(B) \cdot R(G),$$

where $C_1(B)$ is the column norm of B with its first input distinguished. After multiplication by the volume factor a_i^3/a_j^3 , the volume factors cancel:

$$\frac{a_i^3}{a_{i-1}^3} \cdot \frac{a_{i-1}^3}{a_j^3} = \frac{a_i^3}{a_j^3}.$$

In general there is exactly one chain of factors carrying a fixed input slot. Its column bound uses the same homogeneous majorant coefficient as the row bound; summing over the possible positions of a differentiated input introduces exactly the degree factor represented by differentiating the scalar power series, once. The Taylor coefficients $D^n/n!$ distribute the input indices among the input slots of the outer tensor with exactly the multiplicities of scalar substitution: ordering the slots first and then dividing by their factorials counts each partition once. We conclude that composition of the scalar majorants majorizes these tensor contractions, also with a distinguished input, and that no constant is lost at successive levels.

The spatial weights are treated in the same contraction. We take the union of the connecting networks of the participating coefficients, joined at the contracted indices. It is a connected network joining all the external indices, so its length is at least their minimal network length. Every later input mesh is at most a_j , so the weight of each part of the network at its own scale dominates its weight in units of a_j . Multiplying the weights bounds the weight of the union with the initially fixed decay rate. This holds for compositions of arbitrary length, without a reduction of the decay rate at each step.

Step 5: iterated majorants and the remainder bound. Let $S_r = \sum_{v=0}^{r-1} \alpha_*^v$, with $S_0 = 0$. By induction, using the formula for ϕ , we have the identity

$$\phi^{\circ r}(z) = \frac{\alpha_*^r z}{1 - \beta_J S_r z}. \quad (31.4)$$

For the inductive step, we substitute the right side into $\alpha_* y / (1 - \beta_J y)$ and use $S_{r+1} = S_r + \alpha_*^r$. By Step 4, $\widehat{\Pi}_{i-1 \leftarrow j} \prec \phi^{\circ r'}$ with $r' = i - 1 - j$, and the remainder satisfies, coefficientwise,

$$\begin{aligned} \widehat{\mathcal{R}}_i \circ \widehat{\Pi}_{i-1 \leftarrow j} &\prec \frac{\alpha_* \beta_J (\phi^{\circ r'}(z))^2}{1 - \beta_J \phi^{\circ r'}(z)} = \frac{\beta_J \alpha_*^{2r'+1} z^2}{(1 - \beta_J S_{r'} z)(1 - \beta_J S_{r'+1} z)} \\ &\preceq \frac{\beta_J \alpha_*^{2r'+1} z^2}{(1 - Dz)^2}, \quad D = \frac{\beta_J}{1 - \alpha_*}. \end{aligned} \quad (31.5)$$

Here $\preceq$ denotes coefficientwise inequality of power series. In the equality we used $1 - \beta_J \phi^{\circ r'}(z) = (1 - \beta_J S_{r'+1} z) / (1 - \beta_J S_{r'} z)$, and the coefficientwise inequality holds because $\beta_J S_r \leq D$ for every r . This proves the row, column, and distinguished-derivative assertions of part (4), with $C = \beta_J$, enlarged once for the fixed convention on components. The bound $Dr_{\text{eff}} < 1/8$ makes all these substitutions convergent on the reconstruction domain. The two inputs that every monomial of the remainder contains each contribute a factor $\alpha_*^{r'}$; this is the origin of the exponent $2r' + 1$.

Step 6: the field $\widehat{\mathcal{E}}_j^a$. For a finite composition we have the identity

$$\widehat{\Pi}_{k \leftarrow j}(v) = \widehat{L}_k \cdots \widehat{L}_{j+1} v + \sum_{i=j+1}^k \widehat{L}_k \cdots \widehat{L}_{i+1} \widehat{\mathcal{R}}_i(\widehat{\Pi}_{i-1 \leftarrow j}(v)), \quad (31.6)$$

which follows by induction on $k - j$: to pass from $k - 1$ to k , apply $\widehat{L}_k$ to the identity at $k - 1$ and add $\widehat{\mathcal{R}}_k(\widehat{\Pi}_{k-1 \leftarrow j}(v))$. We apply the linear functional $\mathcal{I}_a$ to (31.6) and use the linear comparison of Step 2 on each summand, with the remainder $\widehat{\mathcal{R}}_i(\widehat{\Pi}_{i-1 \leftarrow j}(v))$ as input at scale i . Since the circulation annihilates the gradients, we obtain

$$\mathcal{I}_a(\widehat{\Pi}_{k \leftarrow j}(v)) = \mathcal{C}_{a,\varepsilon,x} \left(a_j^{-1} E_j^a v + \sum_{i=j+1}^k a_i^{-1} E_i^a \widehat{\mathcal{R}}_i(\widehat{\Pi}_{i-1 \leftarrow j}(v)) \right),$$

which is the identity of part (5). Each remainder is equivariant and vanishes at zero, and each E_i^a is a scalar multiple of the identity on the color factor; hence $\widehat{\mathcal{E}}_j^a$ has both properties.

Let C_E be a common bound for the rows and volume-normalized columns of all E_i^a . Composing E_i^a with a remainder multiplies its column normalization a_i^3/a_j^3 by a^3/a_i^3 , which gives a^3/a_j^3 . The factor a_j/a_i in the definition of $\widehat{\mathcal{E}}_j^a$ combines with (31.5) to give

$$\frac{a_j}{a_i} \alpha_*^{2r'+1} = B^{r'+1} \alpha_*^{2r'+1} = C_* q^{r'}.$$

Summing over i , a common majorant for $\widehat{\mathcal{E}}_j^a$ is

$$\Psi(z) = C_E z + \frac{C_E \beta_J C_*}{1 - q} \frac{z^2}{(1 - Dz)^2}. \quad (31.7)$$

By the fixed-input calculation of Step 4, the homogeneous column norms are bounded by the coefficients of Ψ ; the differentiated tensors, with an input marked only once, are bounded by its derivatives. It converges on a disk of radius larger than r_{eff} . Its first derivative bounds $D\hat{\mathcal{E}}_j^a$; its second derivative, integrated along the segment from 0 to v , gives $\|D\hat{\mathcal{E}}_j^a(v) - D\hat{\mathcal{E}}_j^a(0)\|_{\text{row+col}} \leq C\|v\|_\infty$; and each fixed further derivative is bounded by the corresponding derivative of Ψ , since $Dr_{\text{eff}} < 1/8$. All constants are functions of the coefficient data listed in the theorem; only Step 3 introduced the loops, the heat horizon, and the torus. This completes the proof. $\square$

31.4 Conditional current coefficients and means

We now bring in the conditioned Wilson measure. The next theorem collects the properties of the conditional fluctuation kernels that we need, bounds the coefficient matrices Γ_j and their dependence on the coarse field, and bounds the conditional mean and the conditional variance of the circulation Taylor remainders R_j^{cur} . The conditional kernels satisfy a uniform convexity bound on a product of balls. The resulting dimension-free Poincaré, logarithmic Sobolev, and covariance estimates are in the tradition of Brascamp and Lieb [33], Bakry and Émery [8], and Helffer and Sjöstrand [110]; their versions on products of balls are proved in Sections 5 and 13.

Theorem 31.5. *Assume the setting of Section 31.1 and the loop data of Definition 31.1, with one family of reconstruction maps on the whole sequence of lattices. There exists $s_0 < \infty$ such that for every integer $s \geq s_0$, every $K \geq s$, every $1 \leq j \leq K - s$, and every ε with $a = a_{K-s} \leq \varepsilon \leq \varepsilon_0(s)$, the following statements hold. One may take $\varepsilon_0(s) \leq \min\{cH_s, \varepsilon_{\text{loop}}, c/g_3^2\}$ for a fixed small $c > 0$.*

1. Under $\nu_{K,s}$, the conditional law of w_j given $\mathcal{F}_{j-1}$ is $k_j(\hat{\theta}_{j-1}, dw)$, where

$$k_j(\theta, dw) = \frac{\mathbf{1}_{\Omega_j}(w) e^{-V_j(\theta, w)} dw}{\int_{\Omega_j} e^{-V_j(\theta, w)} dw},$$

$$\Omega_j = \prod_{\ell=1}^{m_j} \bar{B}_{\mathbb{R}^3}(0, p_{j,\ell}), \quad p_{j,\ell} = p_j = A_w \log(1/t_j).$$

The potential V_j contains the part of the classical action beyond its quadratic approximation, the input effective interaction, the Haar disintegration density, and the determinants of the triangular change of variables. The restrictions at coarser scales do not change this kernel once θ is fixed. On all of Ω_j ,

$$\|D_w^2 V_j - I\|_{\text{row+col}} \leq \vartheta_j = C[t_j p_j + t_j^2(1 + E_q)], \quad D_w^2 V_j \geq \tfrac{1}{2}I,$$

$$\text{Var}_{k_j} f \leq 2\mathbb{E}_{k_j} |\nabla f|^2, \quad \text{Ent}_{k_j}(f^2) \leq 4\mathbb{E}_{k_j} |\nabla f|^2.$$

The number of three-dimensional blocks satisfies $m_j \leq C_J(L/a_j)^3 \leq C_{J,L,\gamma} t_j^{-6}$, and the conditional means satisfy $\sup_\ell |\mathbb{E}_\theta w_{j,\ell}| \leq Ct_j \|\theta\|_\infty$; in particular they vanish at $\theta = 0$. Here $\mathbb{E}_\theta$ denotes expectation under $k_j(\theta, \cdot)$, for a coarse field with $e^\theta \in B_r$ as in Section 31.1. The threshold s_0 can be chosen so that, in addition, $p_j \geq 1$, $6m_j e^{-p_j^2/192} \leq 1/2$, and $|\mathbb{E}_\theta w_{j,\ell}^c| \leq p_j/(16\sqrt{3})$ at all these levels, for every color component c and every coarse field with $e^\theta \in B_r$.

2. For every θ in the reconstruction domain, with $b = \|\theta\|_\infty$,

$$\|\Gamma_j(\theta)\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq C\omega_j, \quad \|\Gamma_j(\theta)\|_{\ell^\infty \rightarrow \mathbb{R}^{3m}} \leq C\sqrt{\varepsilon/a_j},$$

$$\|\Gamma_j(\theta) - \Gamma_j(0)\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq Cb\omega_j, \quad \|\Gamma_j(\theta)\Gamma_j(\theta)^\top - \Gamma_j(0)\Gamma_j(0)^\top\| \leq Cb\omega_j^2,$$

where ℓ^2 and ℓ^∞ refer to the counting norms on the fluctuation variables.

3. Let $t = t_j$, and recall that the couplings $g_{j,\ell} \leq t_j$ of the intermediate scales have fixed ratios to t_j . Let $F_e(\theta, q)$ be an analytic function of the coarse field θ and the coupling-scaled fluctuation inputs q on the common larger coarse-free domain, with values in $\mathbb{R}^3$, equivariant under global color rotations, such that $F(0,0) = 0$, such that its absolute coefficient row sum at the fixed output e is at most a given constant $C_F < \infty$, and such that, after the substitution $q_\ell = g_{j,\ell} w_\ell$, every term of degree l in w carries a factor t^l , also after one derivative in θ . The constants in this part may depend on C_F and on the radius of the larger domain, but not on the output resolution. For $D_e(\theta, w) = F_e(\theta, g_{j,\cdot} w) - F_e(\theta, 0)$ and every coarse field θ with $e^\theta \in B_r$, we have

$$|\mathbb{E}_\theta D_e| \leq Ct^2 \|\theta\|_\infty, \quad \mathbb{E}_0 D_e = 0.$$

For $\theta_u = u\theta$, $0 \leq u \leq 1$, and the potential $V_{\theta_u} = V_j(\theta_u, \cdot)$,

$$\frac{d}{du} \mathbb{E}_{\theta_u} D_e(\theta_u, w) = \mathbb{E}_{\theta_u} \partial_u D_e - \text{Cov}_{\theta_u}(D_e, \partial_u V_{\theta_u}),$$

and the local estimates

$$\sup_l \|\partial_{w_l} \partial_u V_{\theta_u}\|_{L^2(k_j(\theta_u, \cdot))} \leq Ct \|\theta\|_\infty, \quad \sum_l \|\partial_{w_l} D_e\|_{L^2(k_j(\theta_u, \cdot))} \leq Ct$$

hold. The derivative of the normalization, the input effective interaction, the Haar disintegration density, and all determinants are included in these statements. At $\theta = 0$ the covariance of the fluctuation variables has the form $\mathbb{E}_0[w_l^a w_h^b] = S_{lh} \delta_{ab}$, with all spatial correlations S_{lh} retained, and its contraction with any equivariant bilinear coefficient with values in $\mathbb{R}^3$ is zero. This is a consequence of global color equivariance, not of parity.

4. With $b_{j-1} = \|\hat{\theta}_{j-1}\|_\infty$,

$$|\Gamma_j(\hat{\theta}_{j-1}) \bar{w}_j + \mathbb{E}[\mathbf{R}_j^{\text{cur}} \mid \mathcal{F}_{j-1}]| \leq Ct_j b_{j-1} \sqrt{\varepsilon/a_j},$$

$$\mathbb{E}[|\mathbf{R}_j^{\text{cur}} - \mathbb{E}(\mathbf{R}_j^{\text{cur}} | \mathcal{F}_{j-1})|^2 | \mathcal{F}_{j-1}] \leq Ct_j^2 p_j^2 \omega_j^2.$$

The same mean bound holds separately for $|\Gamma_j(\widehat{\theta}_{j-1})\bar{w}_j|$ and for $|\mathbb{E}[\mathbf{R}_j^{\text{cur}} | \mathcal{F}_{j-1}]|$, and both vanish when $\widehat{\theta}_{j-1} = 0$.

All constants in parts (1)–(4) are independent of $s \geq s_0$, K , j , d_j , the output resolution, the point x , and the lattice offset. They may depend on L , γ , M , J , Θ , the loops, σ , the analytic domains, and, in part (3), on C_F and the larger domain.

Proof. We first identify the conditional kernels and choose s_0 . We then bound the coefficients at a general coarse field (Step 2), prove the moment and coupling bounds needed for general outputs (Steps 3–5), prove part (3) (Step 6), and apply it to the circulation (Step 7).

Step 1: the kernels and the threshold. By Theorem 15.13, the reconstruction maps preserve the conditional law and its product of balls at every index d_j . We apply Theorem 13.8 to this law; its bound on the input effective interaction is E_q , and its largest coupling is t_j . For the selected law all cutoff radii within one RG step equal $p_j = A_w \log(1/t_j)$, as in Definition 4.4. Thus the common-radius hypotheses of the conditional estimates hold, and choosing the coarse side large bounds the denominator in the rational estimate away from zero. This gives the bound ϑ_j , and the convexity, Poincaré, and entropy statements on the closed product. Every factor coming from coarser restrictions is a function of the coarse field and cancels in the normalized kernel, as stated in the same theorem. Neither this cancellation nor the choice of gauge representatives removes from V_j a term that depends on w .

The count of free coordinates in Theorem 12.13, summed over the J intermediate scales, bounds the number of three-dimensional blocks by $C_J(L/a_j)^3$. Since $t_j = \lambda_B g_3 \sqrt{a_j}$, we have $(L/a_j)^3 = L^3 (\lambda_B g_3)^6 t_j^{-6}$. The mean bound of Theorem 13.10, applied to this product with its common radius p_j , implies that $\sup_\ell |\mathbb{E}_\theta w_{j,\ell}| \leq Ct_j b$; in particular the means vanish at $\theta = 0$.

At coarse index s the largest coupling is

$$t_1(s) = g_3 \sqrt{H_s/M}, \quad t_j(s) = t_1(s) B^{-(j-1)/2}, \quad (31.8)$$

and $t_1(s) \rightarrow 0$ as $s \rightarrow \infty$. All the domain, analytic, curvature, and moment conditions required here are finitely many fixed inequalities in t and in its cutoff radius $p(t) = A_w \log(1/t)$. We also require $n_s \geq n_\Gamma$ for every $s \geq s_0$. We increase s_0 until all these inequalities, including $Ct_1(s_0)r \leq 1$ for the constant in the mean bound, hold for every $0 < t \leq t_1(s_0)$. This is possible also for the condition $6m_j e^{-p_j^2/192} \leq 1/2$, which involves the number of blocks: with $u = \log(1/t)$ its left side is at most

$$C_{J,L,\gamma} \exp(6u - A_w^2 u^2/192) \rightarrow 0 \quad (u \rightarrow \infty).$$

The same reasoning bounds the quantity $(1 + 3m_j)(6m_j)^{1/2}e^{-p_j^2/384}$, which appears in the cutoff estimates of Lemma 32.6. As a function of $h = a_j$, we have

$$(1 + 3m_j)(6m_j)^{1/2}e^{-p_j^2/384} \leq C(1 + (L/h)^3)(L/h)^{3/2}e^{-c \log^2(1/(c'g_3\sqrt{h}))}, \quad (31.9)$$

and for every fixed $v > 0$ the right side is $O(h^v)$ as $h \downarrow 0$, since after taking logarithms the negative quadratic term dominates every fixed linear term. The bound on the mean in part (1) follows from $Ct_j r \leq p_j/(16\sqrt{3})$ after another increase of s_0 , since $\|\theta\|_\infty < r$ when $e^\theta \in B_r$. These inequalities persist at every later s and every level j . Finally, we choose $\varepsilon_0(s)$ as in the statement, with c small enough that $\eta < 1$ and the small lattice cycles have zero winding. None of these choices changes the reconstruction maps, their thresholds, or the small-field event at any side.

Step 2: coefficients at a general coarse field. We suppress a , ε , and x from $\mathcal{C} = \mathcal{C}_{a,\varepsilon,x}$, and define the map to the fine lattice

$$Z_j(\theta, w) = \widehat{\mathcal{E}}_j^a(\widehat{\Phi}_j(\theta, g_j, w)). \quad (31.10)$$

By the identity of Theorem 31.3(5), we have $G_j = \mathcal{C}(a_j^{-1}Z_j)$. We compose the joint coefficient series of Theorem 15.13 with the majorant (31.7). The column bound for compositions, proved in Step 4 of the proof of Theorem 31.3, applies with one fluctuation input distinguished, and we obtain

$$D_w Z_j(\theta, 0) = g_j T_j(\theta), \quad \|T_j(\theta)\|_{\text{row+col}} \leq C, \quad \|T_j(\theta) - T_j(0)\|_{\text{row+col}} \leq Cb. \quad (31.11)$$

Here the fluctuation variables are referred to the mesh a_j . Their meshes differ from a_j by the fixed factors $1, M, \dots, M^{J-1}$, and their couplings differ from g_j by the corresponding fixed square-root factors, so converting their columns and densities to the mesh a_j introduces only constants depending on J . A derivative in θ introduces a factor depending only on the fixed gap between the larger and the smaller domain, and integrating it from 0 to θ proves the last bound; no depth-dependent derivative bound or new domain is used. Since $\Gamma_j(\theta) = \mathcal{C}((g_3\sqrt{a_j}/a_j)T_j(\theta))$, Theorem 31.3(3), applied to $T_j(\theta)$ and to $T_j(\theta) - T_j(0)$ with $h = a_j$, proves the three bounds on Γ_j in part (2). For the Gram matrices we use, with $A = \Gamma_j(\theta)$ and $A_0 = \Gamma_j(0)$,

$$\|AA^T - A_0A_0^T\| \leq \|A - A_0\|(\|A\| + \|A_0\|),$$

which gives the bound $Cb\omega_j^2$.

Step 3: local moments. We fix one of these products of balls, set $\theta_u = u\theta$, and write $\mathbb{E}_u$ for expectation under $k_j(\theta_u, \cdot)$. By the mean bound and the inequalities of Step 1, we

have $\sup_l |\mathbb{E}_u w_l| \leq 1$. For a scalar coordinate let $\xi = w_l^c - \mathbb{E}_u w_l^c$. Applying the Poincaré inequality first to ξ and then to ξ^2 , we obtain

$$\mathbb{E}_u \xi^2 \leq 2, \quad \mathbb{E}_u \xi^4 - (\mathbb{E}_u \xi^2)^2 \leq 2\mathbb{E}_u |\nabla \xi^2|^2 = 8\mathbb{E}_u \xi^2 \leq 16.$$

Hence all fourth moments of scalar coordinates are bounded by a fixed constant, and the inequality $(x_1^2 + x_2^2 + x_3^2)^2 \leq 3 \sum_c x_c^4$ gives the same for each three-dimensional block. These bounds are uniform in u , the dimension of the product, and the output mesh.

Step 4: coupling powers in the potential. We read off the powers of the coupling in the potential from Theorem 12.13. The first derivative of each classical action with respect to the free coordinates vanishes at its conditional constrained minimizer. Its quadratic term becomes exactly $|w_i|^2/2$, because the normalization uses $K_i^{-1/2}$, the inverse square root of the constrained Hessian. Although the minimizer and the square root depend on the fluctuation variables of the coarser intermediate scales, this quadratic expression does not. After subtracting $|w|^2/2$, the classical part has total degree at least three in w . The coupling-scaled fluctuation inputs are $q_i = g_i w_i = t \lambda_i w_i$, with fixed λ_i , and the free logarithmic coordinates are $z_i = z_i^* + K_i^{-1/2} q_i$, where z_i^* and K_i depend on the coarser fields. We expand the analytic reconstruction in the inputs q_i , keeping the conditional constrained minimizers and the Hessian factors inside it. Each term of total degree r then contributes t^r after the substitution, and the classical divisor g_i^2 removes two powers of t . The differences of the Haar disintegration density and of the determinants have positive degree and no inverse power of the coupling. Writing H_r for the part of degree r in w of the interaction generated at the current integration step, there are tensors A_r and B_r (unrelated to the small-link domain B_r) such that

$$H_r(\theta, w; t) = t^{r-2} A_r(\theta)[w^r] + t^r B_r(\theta)[w^r], \quad A_1 = A_2 = 0. \quad (31.12)$$

The differences of the Haar density and of the determinants belong to the terms B_r , and their dependence on the fluctuation variables of the coarser intermediate scales is kept in this decomposition.

Step 5: the mixed derivative of the potential. Fix a level j and condition on $\mathcal{F}_{j-1}$. We apply the local clause of Theorem 13.10 with

$$t = t_j, \quad \theta = \hat{\theta}_{j-1}, \quad E_* = E_q, \quad \theta_u = u\theta \quad (0 \leq u \leq 1),$$

and check its hypotheses. The coarse field satisfies $e^\theta \in B_r$ on the support of $\nu_{K,s}$, as noted in Section 31.1; its radial segment remains in B_r and in the larger analytic domain. The ratios $g_{j,\ell}/t_j$ are the fixed ratios of that theorem. The product is exactly Ω_j , with its radii $p_{j,\ell}$, which depend on the level and the coupling but not on u . The conditions on the product, including the smallness of $t_j(p_j + C_j)^3$, were imposed in the choice

of s_0 , and the input effective interaction has norm at most E_q at every index d_j . By Theorem 15.13, the change of gauge representative preserves the fluctuation variables, the product of balls, the part of the classical action beyond its quadratic approximation, the Haar disintegration density, the determinants, the normalization constant, and the normalized conditional kernel; hence the theorem applies to this kernel. A difference between two choices of the potential that depends only on the coarse field has zero mixed derivative. We conclude that

$$\sup_l \|\partial_{w_l} \partial_u V_j(\theta_u, \cdot)\|_{L^2(k_j(\theta_u, \cdot))} \leq Ct_j \|\theta\|_\infty, \quad (31.13)$$

with a constant independent of the side, j , K , d_j , and u . This is a bound at each fluctuation index, with a summable spatial profile; no global norm of the extensive derivative $\partial_u V$ is used. The Gaussian part, the dependence of the determinants on the coarser fluctuation variables, and the derivatives of the input effective interaction through the chain rule are included.

For the mean bounds below the following weaker estimate suffices. Summing the pointwise clause for the mixed derivative in Theorem 13.10 against θ , we obtain

$$\sup_l \|\partial_{w_l} \partial_u V_{\theta_u}\|_{L^2} \leq \sup_l \|\partial_{w_l} \partial_u V_{\theta_u}\|_\infty \leq Ct(p^+)^2 b, \quad p^+ = p + C_j; \quad (31.14)$$

the exponential decay in the coarse index is summable since the number of coordinates per cell is bounded. We increase s_0 once more so that $t(p^+)^2 \leq 1$ at every level. Both estimates concern the same potential.

Step 6: general equivariant outputs. Fix C_F and the larger domain, and let F range over the class described in part (3). Its stated absolute coefficient row bound on that domain, together with the fixed radius margin to the smaller domain, bounds the differentiated absolute coefficient row sum in θ there: the degree factors from differentiation are absorbed by the fixed radius ratio. All constants in this step may depend on C_F , the radius, and the fixed ratios of radii, but not on the number of output sites or on the resolution. The part of degree r in w of $D_e = F_e(\theta, t\lambda w) - F_e(\theta, 0)$ has row coefficients bounded by $C_1 C_2^r t^r$, also after a derivative in θ . We split $D_e = D_e^{(1)} + D_e^{(\geq 2)}$ into its part of degree one and the sum of all higher degrees. By the mean bound and the absolute output row, we have

$$|\mathbb{E}_\theta D_e^{(1)}| \leq Ct \sup_l |\mathbb{E}_\theta w_l| \leq Ct^2 b.$$

For the terms of degree $r \geq 2$ in $\partial_u D_e^{(\geq 2)}$, we bound two of the fluctuation factors in L^4 and the others by p^+ . For a derivative in one fluctuation variable instead, one factor is bounded in L^2 and the remaining $r - 2$ by p^+ . Summing the absolute coefficient series, we obtain

$$|\mathbb{E}_u \partial_u D_e^{(\geq 2)}| \leq Cbt^2, \quad \sum_l \|\partial_{w_l} D_e^{(\geq 2)}\|_{L^2} \leq Ct^2; \quad (31.15)$$

for example, the second sum is at most $C \sum_{r \geq 2} r C_2^r t^r (p^+)^{r-2} \leq Ct^2$. The derivative of the part of degree one has output row at most Ct ; hence $\sum_l \|\partial_{w_l} D_e\|_{L^2} \leq Ct$ for the whole increment. All sums are at the single fixed output e , so the number of outputs does not appear.

The local covariance inequality of Theorem 13.8 has kernel $Ce^{-b' \text{dist}(l,h)}$. Its rows sum to a fixed constant, since the number of fluctuation coordinates per cell is bounded and $\sum_{q \geq 0} (q+1)^3 e^{-b'q} < \infty$. Combining it with (31.14) and (31.15), we find

$$|\text{Cov}_u(D_e^{(\geq 2)}, \partial_u V_{\theta_u})| \leq C \sum_{l,h} e^{-b' \text{dist}(l,h)} \|\partial_{w_l} D_e^{(\geq 2)}\|_{L^2} \|\partial_{w_h} \partial_u V_{\theta_u}\|_{L^2} \leq Ct^3 (p^+)^2 b \leq Ct^2 b.$$

For each finite product, differentiation under the integral is justified: the domain is compact and fixed, and the functions and their derivatives in the coarse field are continuous on an open neighborhood. Differentiating the numerator and the denominator separately gives exactly

$$\frac{d}{du} \mathbb{E}_u D_e = \mathbb{E}_u \partial_u D_e - \text{Cov}_u(D_e, \partial_u V_{\theta_u}).$$

Thus the derivative of the normalization is present, with the minus sign in the statement, and all Haar, determinant, and effective-interaction terms are those of the same V_j .

The same identity holds with $D_e^{(\geq 2)}$ in place of D_e . At $\theta = 0$ the conditional law is invariant under simultaneous color rotation of all blocks. Both $D_e(0, w)$ and $D_e^{(\geq 2)}(0, w)$ transform as vectors in the adjoint representation, since taking a homogeneous part of fixed degree commutes with color rotations. Each of their expectations is fixed by every rotation, and a vector in $\mathbb{R}^3$ fixed by the rotations by π about two perpendicular axes is zero. Hence both expectations vanish at $\theta = 0$. Integrating the identity for $D_e^{(\geq 2)}$ from $u = 0$ to $u = 1$ and using the two bounds above, we obtain $|\mathbb{E}_\theta D_e^{(\geq 2)}| \leq Ct^2 b$. Adding the bound for the part of degree one, we find $|\mathbb{E}_\theta D_e| \leq Ct^2 b$ for the whole increment. This argument does not use (31.13).

The same reasoning proves the cancellation at $\theta = 0$ without discarding spatial correlations. Let $S_{la,hb} = \mathbb{E}_0[w_l^a w_h^b]$. Invariance makes each 3×3 color matrix $(S_{la,hb})_{a,b}$ commute with every rotation, so $S_{la,hb} = S_{lh} \delta_{ab}$ by the argument used for E_j^a in Step 2 of the proof of Theorem 31.3. If B_e is an equivariant bilinear coefficient with values in $\mathbb{R}^3$, then $\sum_{l,h,a} S_{lh} B_e(e_{la}, e_{ha})$ is fixed by every color rotation and hence zero. Every homogeneous equivariant term with values in $\mathbb{R}^3$ has zero expectation for the same reason. This proves the cancellation for the whole series, which converges absolutely on the product.

Step 7: the remainder of the circulation. We write $\mathbb{E}_{j-1} = \mathbb{E}[\cdot \mid \mathcal{F}_{j-1}]$. The map Z_j in (31.10) belongs to one fixed class of outputs as in part (3). Indeed, we choose its larger coarse/free domain inside the domain of Theorem 15.13, with image in the reconstruction

domain on which $\widehat{\mathcal{E}}_j^a$ is defined, and we compose the joint coefficient series with (31.7). The composition bound fixes one constant C_F and one larger radius for all j , k , and fine output sites; a distinguished coarse input costs one fixed restriction of the domain. Equivariance and the value zero at $(0,0)$ are preserved. The substitution $q_{j,\ell} = g_{j,\ell} w_\ell$ gives one factor t_j for each fluctuation factor, also after a derivative in the coarse field. Thus neither C_F nor the radius depends on the remaining number of steps or on the output resolution. By part (3), the increment $Z_j(\theta, w) - Z_j(\theta, 0)$ has conditional mean at most $C t_j^2 b$ at each output site. Since $G_j(\theta, w) - G_j(\theta, 0) = \mathcal{C}(a_j^{-1}(Z_j(\theta, w) - Z_j(\theta, 0)))$, and since heat smoothing does not increase supremum norms while the cycle has length at most $C_\ell \varepsilon$, we obtain

$$|\Gamma_j \bar{w}_j + \mathbb{E}_{j-1} R_j^{\text{cur}}| \leq C \frac{\varepsilon}{\sqrt{\eta}} \cdot \frac{t_j^2}{a_j} b = C \lambda_B t_j b \sqrt{\varepsilon/a_j},$$

and we absorb λ_B into C . By the mean bound of part (1) and the ℓ^∞ bound of part (2), we have $|\Gamma_j \bar{w}_j| \leq C t_j b \sqrt{\varepsilon/a_j}$ separately, and subtracting it from the preceding bound gives the bound for $|\mathbb{E}_{j-1} R_j^{\text{cur}}|$. Both vanish at zero coarse field, by the color argument of Step 6 and the vanishing of the conditional mean of w_j .

Finally, every term of $D_w Z_j(\theta, w) - D_w Z_j(\theta, 0)$ has one differentiated fluctuation factor and at least one further fluctuation factor. In the joint coefficient series their sizes are t_j and at most $t_j p_j^+$, respectively. Summing all remaining factors on the fixed larger domain, and using the column bound with a fluctuation input distinguished, we find

$$\|D_w Z_j(\theta, w) - D_w Z_j(\theta, 0)\|_{\text{row+col}} \leq C t_j^2 p_j.$$

We divide this matrix by g_j and apply Theorem 31.3(3); since $t_j/g_j = \lambda_B$, we obtain

$$\|D_w R_j^{\text{cur}}\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq C t_j p_j \omega_j.$$

The conditional Poincaré inequality bounds the conditional variance of each of the $3m$ components of R_j^{cur} by twice its expected squared gradient. Summing over the components and using $\|D_w R_j^{\text{cur}}\|_{\text{HS}}^2 \leq 3m \|D_w R_j^{\text{cur}}\|_{\text{op}}^2$ proves the variance bound of part (4), with no dependence on the number of fluctuation variables. Every constant was fixed on the common larger domains or by the choice of s_0 ; none depends on a later s , on K , j , x , the lattice offset, or d_j . Together with (31.13), this proves all claims. $\square$

31.5 Covariance identity for the linearized reconstruction

The coefficients $\Gamma_j(0)$ at the zero field are derivatives of the reconstruction at the origin in all its coordinates. We show that, in the Gaussian theory obtained by linearizing the reconstruction, the fluctuation variables become independent standard Gaussian vectors.

Then the sum of the Gram matrices $\Gamma_j(0)\Gamma_j(0)^\top$ is the covariance of the fine circulation minus the covariance of its conditional mean given the coarse field. The calculation is Gaussian conditioning applied to the linearized reconstruction; the corresponding constrained Gaussian calculation for Balaban's averaging operation is in [10, Section D, (1.47)–(1.49), p. 26, and (1.64)–(1.67), p. 29].

Theorem 31.6. *Assume the setting of Section 31.1, and let $s \geq s_0$, $K \geq s$, $k = K - s$, $H_s = L/n_s$, and $a = L/N_K$. Let X_a be the real finite-dimensional space of link fields with three colors on the fine lattice modulo periodic gradients and constant fields, and let G_a be the Gaussian measure on X_a with density proportional to $\exp(-\|d\theta\|_{\ell^2}^2/(2g_a^2))$, where $g_a^2 = g_3^2 a$, d is the dimensionless plaquette coboundary, and the norm counts each positive plaquette once. Let $Q_{a,H_s} : X_a \rightarrow X_{H_s}$ be the derivative at the identity of the composed block averaging map from the fine lattice to the lattice of side n_s , on the corresponding quotients, and let $C_{s,a} = \sigma(Q_{a,H_s}\theta)$.*

The differential of the reconstruction at zero coarse field and zero fluctuation variables,

$$L(z, w) = L_0 z + \sum_{j=1}^k L_j^w w_j,$$

is a bijection from $X_{H_s} \times \prod_j \mathbb{R}^{\dim w_j}$ onto X_a , and it satisfies

$$\frac{1}{g_a^2} \|dL(z, w)\|_{\ell^2}^2 = \langle z, \mathbf{H}_{s,a} z \rangle + \sum_{j=1}^k |w_j|^2, \quad Q_{a,H_s} L(z, w) = z.$$

Here $\mathbf{H}_{s,a}$ is the Schur complement of the quadratic form on the coarse lattice given by the nested minimization of Theorem 10.10, including its coupling; it is positive definite on X_{H_s} , and its inverse is used only at this fixed s . For every linear map $F : X_a \rightarrow \mathbb{R}^d$,

$$\sum_{j=1}^k (FL_j^w)(FL_j^w)^\top = \text{Cov}_{G_a}(F\theta) - \text{Cov}_{G_a}(\mathbb{E}[F\theta \mid C_{s,a}]).$$

For $a \leq \varepsilon \leq \varepsilon_{\text{loop}}$ the circulation $F = \mathcal{I}_a$ is defined on X_a , and the identity reads

$$\sum_{j=1}^k \Gamma_j(0)\Gamma_j(0)^\top = C^{a,\varepsilon,L} \otimes I_3 - C_{s,a,\varepsilon}^{\text{ret}}, \quad C_{s,a,\varepsilon}^{\text{ret}} = \text{Cov}_{G_a}(\mathbb{E}[\mathcal{I}_a\theta \mid C_{s,a}]) \geq 0.$$

Uniformly in K , x , and the lattice offset,

$$\|C_{s,a,\varepsilon}^{\text{ret}}\| \leq C n_s^2 \varepsilon / H_s \quad (a \leq \varepsilon \leq \varepsilon_0(s)).$$

Together with Lemma 31.2, this gives, for every fixed $s \geq s_0$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x, \text{offset}} \left\| \sum_{j=1}^{K-s} \Gamma_j(0)\Gamma_j(0)^\top - \Sigma \otimes I_3 \right\| = 0.$$

The constant C in the bound for $C_{s,a,\varepsilon}^{\text{ret}}$ depends only on L, γ, M, J, Θ , the loops, and σ ; it is independent of s, K , and d_j . Constants in the positive-definiteness and inverse estimates at a fixed coarse side may additionally depend on n_s .

Remark 31.7. The Gaussian coordinates of this theorem are independent only in the linearized calculation; the conditional fluctuation kernels k_j are not Gaussian. The theorem concerns the differential of the reconstruction at the origin in all coordinates. It is not a statement about the nonlinear reconstruction conditioned on $\hat{\theta}_0 = 0$, whose later fields would remain random. Gradients and constant fields are removed only for this Gaussian identity; they are not removed from the Wilson measure.

Proof. We first set up coordinates on the quotients and prove the energy identity, then prove that L is bijective, and only then change variables in the Gaussian integral. Finally we bound the coarse contribution. All quotient spaces and Gaussian coordinates in this proof are real.

Step 1: coordinates. We choose linear coordinates on each quotient by taking the representatives with mean zero and zero lattice divergence. By the calculation of the kernel of the curl in Step 2 of the proof of Theorem 31.3, these representatives identify exactly the quotient by gradients and constants. The Maxwell form is positive definite on this space: a field of zero energy has zero curl, and its gradient and constant parts have been removed. All Gaussian integrals below are proper finite-dimensional integrals.

We differentiate the reconstruction at its origin and write the resulting linear map, with values in X_a , as $L(z, w) = L_0 z + \sum_j L_j^w w_j$. Here z uses the coordinates on X_{H_s} just described, and w_j ranges over the space of free fluctuation coordinates of Theorem 12.13; the factors $K_i^{-1/2}$ defining them exist by Theorem 10.10. We do not yet claim that L is injective. The gauge transformations of Theorem 15.13, which depend only on the coarse field and equal the identity at the tree roots, add gradients at the identity and therefore do not change this map with values in X_a . The linearized averaging of Lemma 8.1 maps gradients to gradients and constant fields to constant fields, so the differential of the block averaging map descends to the map Q_{a,H_s} between the quotients.

Step 2: the energy identity. We differentiate the exact classical decomposition of Theorem 12.13 at zero coarse field and zero fluctuation variables and take its homogeneous quadratic part. At every intermediate scale the normalized quadratic free action is $|w_i|^2/2$. All remaining classical terms have degree at least three in the fluctuation variables, as shown in (31.12), for every coarse field. They contribute nothing in total degree two, also after composition with the coarser fields. The value and the first derivative of the classical action at the identity vanish. It follows that the quadratic part of the fine action, evaluated on the nonlinear reconstruction, is its quadratic form evaluated on L , with no

contribution from second derivatives of the reconstruction. Iterating over all RG steps, we obtain

$$\frac{\|dL(z, w)\|_{\ell^2}^2}{2g_a^2} = \frac{1}{2}\langle z, \mathbf{H}_{s,a}z \rangle + \frac{1}{2}\sum_j |w_j|^2. \quad (31.16)$$

By Theorem 10.10, the form $\langle z, \mathbf{H}_{s,a}z \rangle$ is the nonnegative Schur-complement form, including its coupling. Differentiating the identity expressing that the reconstruction has the prescribed coarse field, we find

$$Q_{a,H_s}L(z, w) = z. \quad (31.17)$$

The Haar, Haar disintegration, determinant, and effective-interaction terms enter the nonlinear conditional law of Theorem 31.5; they are not part of this identity for the classical quadratic form.

Step 3: bijectivity and positivity. If $L(z, w) = 0$ in X_a , its Maxwell energy is zero. Both terms on the right of (31.16) are nonnegative, so every $w_j = 0$, and (31.17) then implies $z = 0$ in X_{H_s} . Thus L is injective. For a lattice of side n , the space of link fields with three colors has dimension $9n^3$, the gradients have dimension $3(n^3 - 1)$, and the constant fields have dimension nine, so the quotient has dimension $6n^3 - 6$. By Theorem 12.13, the number of free fluctuation coordinates from side Mn to side n is $6(M^3 - 1)n^3$. Summing these numbers over all averaging steps gives $6(N_K^3 - n_s^3)$, and adding the dimension $6n_s^3 - 6$ of X_{H_s} gives exactly $\dim X_a$. Injectivity in equal finite dimensions implies surjectivity. Further, for nonzero z , injectivity makes $L(z, 0)$ nonzero in X_a ; positivity of its Maxwell energy and (31.16) show that $\mathbf{H}_{s,a}$ is positive definite on X_{H_s} . We do not claim a lower bound on $\mathbf{H}_{s,a}$ independent of the side.

Step 4: the Gaussian change of variables. We make the linear change of variables $\theta = L(z, w)$ in the normalized Gaussian integral on X_a . Its Jacobian is constant and cancels between numerator and denominator. By (31.16), in these coordinates

$$z \sim N(0, \mathbf{H}_{s,a}^{-1}), \quad w_j \sim N(0, I),$$

and all these vectors are independent; the inverse exists on X_{H_s} at this fixed side. For a linear map F , (31.17) gives

$$\mathbb{E}[F\theta \mid \mathcal{C}_{s,a}] = FL_0z, \quad F\theta = FL_0z + \sum_j FL_j^w w_j.$$

By independence, expanding the covariance of $F\theta$ proves the general covariance identity, and the subtracted covariance is positive semidefinite.

For $F = \mathcal{I}_a$, the closed cycles and their zero winding make F a well-defined functional on X_a , by Lemma 31.2(3). By the chain rule at the origin of the reconstruction, we have

$$FL_j^w = D_w \mathcal{I}_a(\hat{\Pi}_{k \leftarrow j} \hat{\Phi}_j(0, g_j, w))|_{w=0} = \Gamma_j(0).$$

The finite-lattice Fourier calculation of Lemma 31.2(4) identifies $\text{Cov}_{\mathbb{G}_a}(\mathcal{I}_a\theta)$ as $C^{a,\varepsilon,L} \otimes I_3$, since the density of $\mathbb{G}_a$ is that of the Maxwell field of that lemma written in the variable $\theta = aA$. This proves the identity for $\sum_j \Gamma_j(0)\Gamma_j(0)^\top$.

Step 5: the coarse contribution. We write $H = H_s$ and $n = n_s$. At $j = 0$, the linear comparison of Theorem 31.3(2) implies that, on circulations, $FL_0 = \mathcal{C}_{a,\varepsilon,x}(H^{-1}E_0^a)$. Using the ℓ^2 circulation bound of Theorem 31.3(3) with input mesh H , dividing out the normalization $g_3\sqrt{H}$, and taking the branch $\sqrt{\varepsilon/H}$, we obtain

$$\|FL_0\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq \frac{C\sqrt{\varepsilon}}{g_3 H}. \quad (31.18)$$

In unitary Fourier coordinates on the coarse lattice, Theorem 8.3 bounds the covariance of the coarse link variables by $Cg_H^2/\lambda(\vartheta)$ on each nonzero transverse frequency ϑ , uniformly in the number of eliminated scales; the gradient term in the linearized block averaging map disappears on the quotient by Lemma 8.1. For a nonzero frequency on a lattice of side $n \geq 4$,

$$\lambda(\vartheta) \geq 4 \sin^2(\pi/n) \geq 16/n^2,$$

where the second inequality uses $\sin u \geq 2u/\pi$ for $0 \leq u \leq \pi/2$. Since $g_H^2 = g_3^2 H$, this proves

$$\|\mathbf{H}_{s,a}^{-1}\|_{\ell^2 \rightarrow \ell^2} \leq Cg_3^2 H n_s^2. \quad (31.19)$$

Only the factor n_s^2 of the fixed side has been used. Multiplying the square of (31.18) by (31.19), we obtain

$$\|C_{s,a,\varepsilon}^{\text{ret}}\| = \|(FL_0)\mathbf{H}_{s,a}^{-1}(FL_0)^\top\| \leq Cn_s^2 \varepsilon / H_s,$$

uniformly in the number of eliminated scales, x , and the lattice offset. The factor n_s^2 is allowed to grow with the fixed coarse side.

Step 6: the limits. The covariance identity implies

$$\left\| \sum_j \Gamma_j(0)\Gamma_j(0)^\top - \Sigma \otimes I_3 \right\| \leq \|C^{a,\varepsilon,L} - C^{\varepsilon,L}\| + \|C^{\varepsilon,L} - \Sigma\| + Cn_s^2 \varepsilon / H_s.$$

For each fixed $\varepsilon > 0$ the first term tends to zero as $K \rightarrow \infty$, uniformly in x and the lattice offset, by Lemma 31.2(5). After this limit, the second term tends to zero as $\varepsilon \downarrow 0$ by the same lemma, and the last term tends to zero at fixed s . This is the iterated limit in the statement. At $k = 0$ the same change of variables has no fluctuation variables and the whole covariance is the coarse covariance, so the identities also hold in that case. $\square$

31.6 The coarse-field contribution and summation over scales

We now prove the decomposition stated at the beginning of the section and sum the estimates of Theorem 31.5 over all scales. The sums are controlled by a deterministic

bound on the size b_j of the reconstructed fields, which we prove first; its constant does not depend on the coarse mesh H_s , which matters because s is later sent to infinity.

Lemma 31.8. *Assume the setting of Theorem 31.5, with s_0 enlarged if necessary depending only on the data fixed in Section 31.1. For every $s \geq s_0$ the following deterministic bounds hold on the support of $v_{K,s}$, uniformly in $K \geq s$:*

$$b_0 \leq r, \quad b_j \leq C[r\alpha_*^j + g_3\sqrt{a_j}p_j],$$

$$\sum_{j=1}^{K-s} b_{j-1} \leq D_s, \quad D_s = C\left[\frac{r}{1-\alpha_*} + g_3\sqrt{H_s}(p_1(s) + 1)\right], \quad \sup_{s \geq s_0} D_s < \infty,$$

where $p_1(s) = p_1$ is the cutoff radius of the first RG step. Put

$$T_s = \mathcal{I}_a(\widehat{\Pi}_{k \leftarrow 0} \widehat{\theta}_0), \quad \mathbf{N}_{K,s} = \sum_{j=1}^k [\Gamma_j(\widehat{\theta}_{j-1}) \bar{w}_j + \mathbf{R}_j^{\text{cur}}].$$

Then we have the exact decomposition

$$I^{K,s} = T_s + \sum_{j=1}^k \Gamma_j(\widehat{\theta}_{j-1})(w_j - \bar{w}_j) + \mathbf{N}_{K,s}.$$

For $a \leq \varepsilon \leq \varepsilon_0(s)$,

$$\sup_{\|\widehat{\theta}_0\|_\infty \leq r} |T_s| \leq \frac{Cr}{g_3 H_s} \sqrt{\varepsilon},$$

$$\sup_{x, \text{offset}} \mathbb{E}_{v_{K,s}} |\mathbf{N}_{K,s}| + \sup_{\|\widehat{\theta}_0\|_\infty \leq r} |T_s| \leq \frac{Cr}{g_3 H_s} \sqrt{\varepsilon} + Cg_3 D_s \sqrt{\varepsilon} + C\sqrt{\eta} [1 + \log(1/\eta)]^2.$$

The constants C and the threshold s_0 have the dependence stated in Theorem 31.5, and C does not depend on s ; the quantity D_s has the displayed additional dependence on s . The coarse-field term is allowed to grow like H_s^{-1} . For each fixed $s \geq s_0$, the right side of the last display tends to zero as $\varepsilon \downarrow 0$, uniformly in K ; no limit in which s and ε vary together is asserted.

Proof. We first prove the bounds on b_j (Steps 1–3) and their sum (Step 4), then the decomposition (Step 5), and finally the bounds on T_s and $\mathbf{N}_{K,s}$ (Steps 6 and 7).

Step 1: a recurrence. The inputs $q_j = g_j \cdot w_j$ of the j -th reconstruction map satisfy $\|q_j\|_\infty \leq t_j(p_j + C_J)$, since $g_{j,\ell} \leq t_j$ and $|w_{j,\ell}| \leq p_{j,\ell} \leq p_j + C_J$. On the fixed larger coarse/free domain let $C_{\text{fr},j}$ bound the derivative of $\widehat{\Phi}_j$ in the coupling-scaled fluctuation inputs; this bound does not depend on the side or on d_j . By the row majorant of

Theorem 31.3(1), we have $\|\widehat{P}_j(v)\|_\infty \leq \phi(\|v\|_\infty)$. Integrating this derivative along the segment from 0 to q_j , we obtain, with $\kappa = C_{\text{fr},J}(1 + C_J)$ and $p_j \geq 1$,

$$b_j \leq \frac{\alpha_* b_{j-1}}{1 - \beta_J b_{j-1}} + \kappa t_j p_j \leq a_{\text{map}} b_{j-1} + \kappa t_j p_j, \quad (31.20)$$

as long as $\widehat{\theta}_{j-1}$ lies in the reconstruction domain. Here $a_{\text{map}} = \alpha_*/(1 - \beta_J r_{\text{eff}})^2$ is the bound of Lemma 15.16 on the derivative of the zero-fluctuation maps on the reconstruction domain, which satisfies $a_{\text{map}} \sqrt{B} \leq 32/49 < 1$ by Theorem 23.5.

Step 2: a first bound and the sum. Let $\rho_B = B^{-1/2}$ and $d_p = A_w \log B/2$. Since $t_i = t_j B^{(j-i)/2}$ and $p_i \leq p_j$ for $i \leq j$, iterating the second inequality in (31.20), we obtain

$$b_j \leq r a_{\text{map}}^j + \kappa' t_j p_j, \quad \kappa' = \frac{\kappa}{1 - a_{\text{map}} \sqrt{B}}.$$

For small t the function $t \mapsto t p(t)$ is increasing, so a fixed upper bound on t_1 with $\kappa' t_1 p_1 \leq r_{\text{eff}}/4$ keeps this bound below $r + r_{\text{eff}}/4 < r_{\text{eff}}/2$, which justifies the use of (31.20) at every step by induction. Increasing s_0 enforces this bound for every later s , without changing any radius of the construction. The geometric series and its derivative give

$$\sum_{l=0}^{k-1} b_l \leq \frac{r_{\text{eff}}}{4(1 - a_{\text{map}})} + \kappa' t_1 \left[\frac{p_1}{1 - \rho_B} + \frac{d_p \rho_B}{(1 - \rho_B)^2} \right] \leq S_*.$$

Here we fix S_* by replacing $t_1(p_1 + 1)$ by its supremum over the allowed range $0 < t_1 \leq t_1(s_0)$; this supremum is finite since $t \log(1/t) \rightarrow 0$. Thus S_* depends neither on H_s nor on s .

Step 3: the rate α_*^j . For $0 \leq \beta_J b \leq \beta_J r_{\text{eff}} < 1/8$ we have

$$\frac{\alpha_*}{1 - \beta_J b} \leq \alpha_* \exp(8\beta_J b/7).$$

This implies that every product over h consecutive steps of the multiplicative coefficients $\alpha_*/(1 - \beta_J b_l)$ in the first recurrence of (31.20) is at most $C_{\text{mult}} \alpha_*^h$, where $C_{\text{mult}} = \exp(8\beta_J S_*/7)$. Unrolling that recurrence, we find

$$b_j \leq C_{\text{mult}} \left[r \alpha_*^j + \kappa \sum_{i=1}^j \alpha_*^{j-i} t_i p_i \right] \leq C_{\text{mult}} \left[r \alpha_*^j + \frac{\kappa t_j p_j}{1 - \alpha_* \sqrt{B}} \right],$$

where the denominator is positive because $\alpha_* \sqrt{B} = \sqrt{q} \leq 1/2$. Substituting $t_j = \lambda_B g_3 \sqrt{a_j}$, we obtain

$$b_j \leq C_{\text{cap}} (r \alpha_*^j + g_3 \sqrt{a_j} p_j), \quad (31.21)$$

with C_{cap} independent of H_s and s . This is the first bound of the lemma. The rate α_*^j does not follow from the first bound of Step 2, where the contraction factor is a_{map} ; the product argument is needed.

Step 4: the sum of the bounds. By the definition of p_j and $a_j = H_s B^{-j}$,

$$p_j = p_1(s) + (j-1)d_p. \quad (31.22)$$

Summing (31.21), including the separate term $b_0 \leq r$, and using $\sqrt{a_l} = \sqrt{H_s} \rho_B^l$, we obtain

$$\begin{aligned} \sum_{j=1}^k b_{j-1} &\leq \frac{Cr}{1-\alpha_*} + Cg_3\sqrt{H_s} \sum_{l \geq 1} \rho_B^l [p_1(s) + (l-1)d_p] \\ &\leq C \left[\frac{r}{1-\alpha_*} + g_3\sqrt{H_s} (p_1(s) + 1) \right]. \end{aligned}$$

In the last inequality we used $\sum_{l \geq 1} \rho_B^l = \rho_B / (1 - \rho_B)$ and $\sum_{l \geq 1} (l-1)\rho_B^l = \rho_B^2 / (1 - \rho_B)^2$. By (31.8), $p_1(s) = A_w \log(\sqrt{M}/(g_3\sqrt{H_s}))$ is affine in s , while $\sqrt{H_s} = \sqrt{L/n_*} B^{-s/2}$ decays geometrically, so their product is bounded for $s \geq s_0$. More directly, $g_3\sqrt{H_s}(p_1(s) + 1) = \sqrt{M} t_1(s)(p_1(s) + 1)$ is bounded by the supremum used in Step 2. Since the constant in (31.21) does not depend on s , this proves $\sup_s D_s < \infty$.

Step 5: the decomposition. For $0 \leq j \leq k$, let $F_j = \mathcal{I}_a(\widehat{\Pi}_{k \leftarrow j} \widehat{\theta}_j)$. Then $F_0 = T_s$ and $F_k = I^{K^s}$. Since $\widehat{\Pi}_{k \leftarrow j-1} = \widehat{\Pi}_{k \leftarrow j} \circ \widehat{\Phi}_j(\cdot, 0)$, and all fields involved lie in the reconstruction domain, we have

$$F_j - F_{j-1} = G_j(\widehat{\theta}_{j-1}, w_j) - G_j(\widehat{\theta}_{j-1}, 0) = \Gamma_j(\widehat{\theta}_{j-1}) w_j + R_j^{\text{cur}}.$$

Summing this finite identity over j , and subtracting and adding $\Gamma_j(\widehat{\theta}_{j-1}) \bar{w}_j$ in each summand, gives the decomposition. The coefficients $\Gamma_j(\widehat{\theta}_{j-1})$ are $\mathcal{F}_{j-1}$ -measurable and the increments $w_j - \bar{w}_j$ are conditionally centered, so the middle sum is a martingale for the filtration $(\mathcal{F}_j)$.

Step 6: the coarse-field contribution. By the row bound on $D\widehat{\mathcal{E}}_0^a$ in Theorem 31.3(5) and $\widehat{\mathcal{E}}_0^a(0) = 0$, we have

$$\|H_s^{-1} \widehat{\mathcal{E}}_0^a(\widehat{\theta}_0)\|_\infty \leq Cr/H_s.$$

By the identity of Theorem 31.3(5), the fact that heat smoothing does not increase supremum norms, and the length bound on the cycles, we obtain

$$|T_s| \leq \frac{C\epsilon r}{\sqrt{\eta} H_s} = \frac{Cr}{g_3 H_s} \sqrt{\epsilon}. \quad (31.23)$$

This bound is uniform over all coarse fields with $\|\widehat{\theta}_0\|_\infty \leq r$ and over K , including $k = 0$.

Step 7: conditional means and centered remainders. We write $R_j^{\text{cur}} = \mathbb{E}_{j-1} R_j^{\text{cur}} + \zeta_j$, where $\mathbb{E}_{j-1}$ denotes conditional expectation given $\mathcal{F}_{j-1}$. By Theorem 31.5(4), using $t_j \sqrt{\epsilon/a_j} = \lambda_B g_3 \sqrt{\epsilon}$, we have

$$\sum_{j=1}^k |\Gamma_j(\widehat{\theta}_{j-1}) \bar{w}_j + \mathbb{E}_{j-1} R_j^{\text{cur}}| \leq Cg_3\sqrt{\epsilon} \sum_{j=1}^k b_{j-1} \leq Cg_3 D_s \sqrt{\epsilon}.$$

For the centered terms we use the triangle inequality and the conditional Cauchy–Schwarz inequality, without any independence:

$$\mathbb{E} \left| \sum_{j=1}^k \zeta_j \right| \leq C \sum_{j=1}^k t_j p_j \omega_j. \quad (31.24)$$

We bound the right side uniformly in s . Shrinking the constant c in $\varepsilon_0(s)$ if necessary, we may assume $u = \log(1/\eta) \geq 1$. For a scale with $a_j > \varepsilon$,

$$t_j p_j \omega_j = \lambda_B \sqrt{\eta} p_j, \quad p_j \leq C(1+u);$$

the second inequality follows from $\log(1/t_j) = (1/2) \log(1/(g_3^2 a_j)) - \log \lambda_B$ and $a_j > \varepsilon$. The number of such scales is at most $1 + \log(L/\varepsilon)/\log B \leq C(1+u)$, with a constant depending only on L, g_3 , and B , since $H_s \leq L/4$. Their contribution is at most $C\sqrt{\eta}(1+u)^2$. If there exists a first index j_ε with $a_{j_\varepsilon} \leq \varepsilon$, let $h_* = a_{j_\varepsilon}$; then $\varepsilon/B < h_* \leq \varepsilon$, since $\varepsilon < cH_s$. Using (31.22), the contribution of the finer scales is at most

$$\begin{aligned} \sum_{r \geq 0} \frac{\lambda_B g_3}{\varepsilon} (h_* B^{-r})^{3/2} (p_{j_\varepsilon} + r d_p) &\leq C g_3 \sqrt{\varepsilon} (p_{j_\varepsilon} + 1) \\ &\leq C \sqrt{\eta} (1+u). \end{aligned}$$

In the last inequality we used $h_* > \varepsilon/B$ in the formula for $\log(1/t_{j_\varepsilon})$. Extending the finite sum to all $r \geq 0$ is allowed since all summands are nonnegative, and if there is no such index this contribution is zero. Hence

$$\sum_j t_j p_j \omega_j \leq C \sqrt{\eta} [1 + \log(1/\eta)]^2.$$

Combining this with (31.24), the bound on the conditional means, and (31.23) proves the last estimate of the lemma, uniformly in x , the lattice offset, and K in the stated range. For fixed s the first constant is finite and $\sqrt{\eta}(1 + \log(1/\eta))^2 \rightarrow 0$ as $\varepsilon \downarrow 0$ at fixed $g_3 > 0$. $\square$

Remark 31.9. The bound (31.21) also follows from the forced-recursion estimate of Lemma 15.16, which applies to inputs of size at most $tp(t)$, $t = \lambda_B g \sqrt{a_j}$, for all t up to a fixed threshold τ , with constants uniform in the coarse mesh and in the coupling g in this range. To apply it, we increase s_0 so that $p_1 \geq 1 + C_J + 2A_w \log 2$ and $2t_1 \leq \tau$, and set $\tilde{t}_j = 2t_j$ and $\tilde{p}_j = A_w \log(1/\tilde{t}_j) = p_j - A_w \log 2$. Then $\tilde{p}_j \geq 1$ and

$$t_j(p_j + C_J) \leq \tilde{t}_j \tilde{p}_j \quad \text{for every } j \geq 1, \quad (31.25)$$

since the difference divided by t_j is $p_j - C_J - 2A_w \log 2 \geq 1$. This is only a deterministic upper bound for the same inputs; the coupling, the measure, the maps, the fluctuation variables, and the cutoff radii are unchanged. Since the constant of that lemma is uniform in H and for the auxiliary coupling in the stated range, the lemma applies with $g = 2g_3$; as $\tilde{p}_j \leq p_j$, it gives (31.21). Steps 1–3 above verify the independence of H_s directly. No stochastic estimate with a constant depending on a fixed coarse mesh is used.

31.7 Validity at each fixed positive coupling

Lemma 31.10. *Let the parameters be as in Theorem 23.5. Fix $0 < \gamma \leq \Gamma$ and one sequence $N_K = n_* B^K$ with its reconstruction maps, and let n_s , H_s , a_j , and t_j be as in Section 31.1. There exists $s_0 < \infty$ such that every $s \geq s_0$ satisfies all conclusions of Lemmas 31.2 and 31.8 and Theorems 31.3, 31.5, and 31.6, with the same quantifiers and the same dependence of the constants. The constants in the bounds on coefficients, potential derivatives, conditional means, reconstructed fields, and conditional remainders do not depend on $s \geq s_0$, while the factor $1/(g_3 H_s)$ in the bound on the coarse-field contribution and the factor n_s^2/H_s in the bound on the coarse covariance remain explicit.*

Proof. We check that each argument of this section uses only the smallness of t_j and never an upper bound on γ such as $\gamma \leq 1$.

Deterministic estimates. The zero-fluctuation maps, their coefficient series, the identities expressing that they have the prescribed coarse field, and their column bounds are those of Theorem 23.5 and Lemma 15.16. The proof of Theorem 31.3 uses Lemmas 16.1 and 11.9 and the majorant $\alpha_* z/(1 - \beta_j z)$, with $B\alpha_*^2 < 1$; no step uses $\gamma \leq 1$. In each circulation coefficient the factor g_3 cancels against $\sqrt{g_3^2 \varepsilon}$, and the only remaining scale is $\omega_j = \min\{a_j/\varepsilon, \sqrt{\varepsilon/a_j}\}$. Thus the deterministic coefficient bounds and the expansion in nonlinear remainders hold on the same domains.

Probabilistic estimates. The conditional potential is that of Theorem 13.8, and Theorem 15.13 preserves its coordinates, its balls, and its density. Theorem 13.10, applied to this product and to the radial segment of the coarse field, gives the conditional means of order $t_j b$ and the local bound $\sup_l \|\partial_{w_l} \partial_u V\|_{L^2} \leq C t_j b$, with all effective-interaction, determinant, and Haar disintegration terms included. The class of outputs used for the circulation is fixed by the coefficient series on one larger domain (Step 7 of the proof of Theorem 31.5), so its constant C_F depends neither on the remaining number of steps, nor on the output resolution, nor on s . The differentiation formula and the local covariance bound therefore give the mean and remainder conclusions of Theorem 31.5. Global color equivariance at zero coarse field gives the vanishing mean and the cancellation of equivariant bilinear terms. The Poincaré and logarithmic Sobolev inequalities hold for the same conditional kernel, with dimension-free constants.

The threshold. We have exactly

$$t_1(s)^2 = \frac{\gamma}{M n_s}, \quad t_j = t_1(s) B^{-(j-1)/2},$$

so $t_1(s) \downarrow 0$ as $s \rightarrow \infty$ for fixed $\gamma > 0$. Every analytic, domain, Hessian, potential-derivative, and moment condition is a fixed inequality of the form $t[p(t) + c_j]^v \leq c'$

or $t^2(1 + E_q) \leq c'$, with a fixed finite v , and all of them hold on an interval $0 < t \leq \tau_*$. We also require $n_s \geq n_\Gamma$. There are at most $m_j \leq C_J(L/a_j)^3 = C_J L^3 (\lambda_B g_3)^6 t_j^{-6}$ three-dimensional blocks, and with $u = \log(1/t)$ the quantity $6m_j e^{-p_j^2/192}$ is at most $C_{J,L,\gamma} \exp(6u - A_w^2 u^2/192)$, which tends to zero as $u \rightarrow \infty$. Thus a single s_0 enforces $p_j \geq 1$, $6m_j e^{-p_j^2/192} \leq 1/2$, and $C t_j r \leq p_j/(16\sqrt{3})$ for all j and all later s , as in Step 1 of the proof of Theorem 31.5. The cutoff error $(1 + 3m_j)(6m_j)^{1/2} e^{-p_j^2/384}$, which retains the number of blocks, satisfies (31.9); as a function of $h = a_j$ it is $O(h^v)$ for every fixed $v > 0$, and in particular it is summable over the scales.

The bounds on the reconstructed fields. The recurrence (31.20) and Steps 1–4 of the proof of Lemma 31.8 use only the smallness of $t_1(s)$; equivalently, by Remark 31.9, the forced-recursion estimate of Lemma 15.16 applies with the auxiliary input bound (31.25), without changing the measure. They give

$$b_j \leq C[r\alpha_*^j + g_3\sqrt{a_j} p_j], \quad \sum_{j=1}^{K-s} b_{j-1} \leq D_s = C\left\{\frac{r}{1-\alpha_*} + g_3\sqrt{H_s}(p_1(s) + 1)\right\},$$

with a constant independent of H_s and of s . Since $p_1(s)$ is affine in s and $\sqrt{H_s}$ decays geometrically, $\sup_{s \geq s_0} D_s < \infty$.

The covariance identity. The identity of Theorem 31.6 is the exact quadratic part of Theorems 10.10 and 12.13 at the origin of the reconstruction, and the bound on $\mathbf{H}_{s,a}^{-1}$ comes from Theorem 8.3. The independent Gaussian coordinates and the coarse Schur complement therefore give

$$\sum_j \Gamma_j(0) \Gamma_j(0)^\top = C^{a,\varepsilon,L} \otimes I_3 - C_{s,a,\varepsilon}^{\text{ret}}, \quad \|C_{s,a,\varepsilon}^{\text{ret}}\| \leq C n_s^2 \varepsilon / H_s,$$

and the coupling cancels in the normalized covariance.

All results of this section have now been checked at every $s \geq s_0$. □

Gaussian short-distance limits and asymptotic independence

This section completes the proof of the short-distance theorem, Theorem 2.8. Fix smooth closed loops $\ell_1, \dots, \ell_m$ in $\mathbb{R}^3$ and $\sigma > 0$, set $g_3^2 = \gamma/L$, and for $x \in \mathbb{T}_L^3$ and $\varepsilon > 0$ let

$$Z_\varepsilon(x) = \left(\frac{1 - W_{x+\varepsilon\ell_i, \sigma\varepsilon^2}}{g_3^2\varepsilon} \right)_{i=1}^m,$$

where $W_{x+\varepsilon\ell_i, \sigma\varepsilon^2}$ is the normalized fundamental Wilson loop of the trajectory at time $\sigma\varepsilon^2$. We prove that under the continuum measure λ of Theorem 2.3, the law of $Z_\varepsilon(x)$ converges to the law of $Z_G = (|C_i|^2/8)_{i=1}^m$, uniformly in x (Theorem 32.13). Here the three color components of C are independent centered Gaussian vectors with the covariance Σ of Definition 31.1, the covariance of heat-smoothed circulations of a Gaussian Maxwell field; in the notation of that definition, $C_i = G_i$ and $(Z_G)_i = Z_i$. We deduce a lower bound on the variance of small Wilson loops (Corollary 32.14), and we prove that $Z_\varepsilon(x)$ is asymptotically independent of every fixed bounded Borel function of the trajectory (Theorems 32.16 and 32.17). All results are at a fixed positive coupling; no weak-coupling limit is taken.

The form of the limit has a short heuristic explanation. By the computation at the beginning of Section 31, the circulation of the heat-smoothed Gaussian Maxwell field around $x + \varepsilon\ell_i$ is a centered Gaussian vector with covariance of order $g_3^2\varepsilon\Sigma$ in each color. A Wilson loop sees only the squared norm of this circulation, since for $X \in \mathfrak{su}(2)$

$$1 - \frac{1}{2} \operatorname{tr} e^X = |X|^2/8 + O(|X|^4).$$

Dividing by $g_3^2\varepsilon$ therefore produces $|C_i|^2/8$. The proof shows that the nonlinear terms, the coarse field, and the conditioning on the small-field event do not change this picture.

The proof proceeds as follows. By Lemma 31.8, the heat-smoothed circulation $I^{K,s}$ is the sum of a coarse-field term, a martingale whose increments are linear in the centered fluctuation variables, and a small remainder. For each RG step we compare the conditional law of the linear increment with a Gaussian law by an integration by parts with a cutoff, in the spirit of Stein's method [147] (Lemma 32.4), and we telescope the conditional characteristic functions (Lemma 32.5). The cutoff error grows with the number of fluctuation variables, which is polynomial in $1/t_j$, but the probability that the

cutoff is active decays like $\exp(-c \log^2(1/t_j))$ (Lemma 32.6). We then replace the coefficient matrices by their values at the zero field, whose Gram sum converges to $\Sigma \otimes I_3$ by Theorem 31.6. This proves a Gaussian limit for the circulations under the conditioned measure (Theorem 32.8), and the same proof applies conditionally on the first q RG steps, uniformly in their values (Lemma 32.10). Next we compare flowed Wilson loops with circulations (Lemma 32.11), remove the conditioning using the bound of Theorem 23.10, and pass to the continuum measure using Section 29. For asymptotic independence we approximate macroscopic flowed Wilson loops by functions of the first q RG steps (Lemma 32.15).

Four length scales appear: the coarse mesh H_s , the mesh $a_q = H_{s+q}$ after q RG steps, the loop size ε , and the fine mesh a , with $a \leq \varepsilon \ll a_q \leq H_s$. The estimates split at the scales with $a_j \approx \varepsilon$; the meshes H_s and a_q stay fixed while $\varepsilon \rightarrow 0$. The limits are taken in the following order. For the Gaussian limit,

$$K \rightarrow \infty, \quad \varepsilon \downarrow 0 \quad (s, R \text{ fixed}), \quad R \rightarrow \infty, \quad s \rightarrow \infty,$$

where R is the radius of a ball for the linear and quadratic flow data; for asymptotic independence,

$$K \rightarrow \infty, \quad \varepsilon \downarrow 0 \quad (s, q, R, \delta \text{ fixed}), \quad q \rightarrow \infty, \quad \delta \downarrow 0, \quad R \rightarrow \infty, \quad s \rightarrow \infty,$$

where δ is a tolerance. No estimate allows these parameters to vary simultaneously.

Besides Section 31, we use the bound for conditioning of Lemma 5.5, the joint realization of the reconstruction and fine configuration following Lemma 15.10, and the bound on the small-field failure probability of Theorem 23.10. We also use the local theory of the Wilson flow and its stability, Theorem 24.5 and Corollary 25.11, the estimates on the linear and quadratic flow data of Lemmas 28.4 and 28.10, and the construction of the continuum measure in Section 29. The section ends with the proof of Theorem 2.8.

32.1 Setting and inputs

Definition 32.1. Fix $L > 0$, $\Gamma > 0$, $0 < \gamma \leq \Gamma$, and the parameters of Theorem 23.5, chosen before γ : an odd integer $M \geq 3$, the integer J , $B = M^J$, the reconstruction maps, and their analytic domains. Fix $n_* \in 4\mathbb{N}$, smooth closed loops $\ell_1, \dots, \ell_m$ in $\mathbb{R}^3$, and $\sigma > 0$. We use the notation of Section 31.1: $N_K = n_* B^K$, $n_s = n_* B^s$, $H_s = L/n_s$, $k = K - s$, $a_j = H_s B^{-j}$, $a = a_k$, $g_3^2 = \gamma/L$, $\eta = g_3^2 \varepsilon$, $t_j = M^{(J-1)/2} g_3 \sqrt{a_j}$, and $p_j = A_w \log(1/t_j)$. The probability measure is the conditioned Wilson measure

$$\nu_{K,s} = \nu_{N_K, n_s, \gamma} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, n_s, \gamma}).$$

We use the joint realization of Remark 15.11, with fine-configuration marginal $\nu_{K,s}$; the reconstructed terminal field is gauge equivalent to this fine configuration on every sample. Its reconstruction sequence is

$$\hat{\theta}_j = \hat{\Phi}_j(\hat{\theta}_{j-1}, g_j, w_j), \quad \mathcal{F}_j = \sigma(\hat{\theta}_0, w_1, \dots, w_j),$$

where $g_{j,\ell}$ are the couplings of the intermediate scales of the j -th RG step, and, conditionally on $\mathcal{F}_{j-1}$, the vector w_j has the conditional fluctuation kernel $k_j(\hat{\theta}_{j-1}, \cdot)$ of Theorem 31.5, whose potential includes the input effective interaction produced by the integrations at finer scales, the Haar disintegration density, and the determinants of the change of variables. We write $\bar{w}_j = \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$. All maps, zero-fluctuation continuations, and circulations use the same reconstruction maps and the same choice of representative of the coarse field. Put

$$\hat{\Pi}_{k \leftarrow j} = \hat{\Phi}_k(\cdot, 0) \circ \dots \circ \hat{\Phi}_{j+1}(\cdot, 0), \quad \hat{\Pi}_{k \leftarrow k} = \text{id}.$$

The circulation map $\mathcal{I}$ is the map denoted $\mathcal{I}_a$ in Definition 31.1, with lattice cycles of physical length at most $C_\ell \varepsilon$, zero winding, and swept area at most $C_\ell \varepsilon a$, which tends to zero at fixed ε as $a \rightarrow 0$. Thus $I^{K,s} = \mathcal{I}(\hat{\theta}_k) \in \mathbb{R}^{3m}$. As in Section 31.1 we put

$$G_j(v, w) = \mathcal{I}(\hat{\Pi}_{k \leftarrow j} \hat{\Phi}_j(v, g_j, w)), \quad \Gamma_j(v) = D_w G_j(v, 0), \quad \omega_j = \min\{a_j/\varepsilon, \sqrt{\varepsilon/a_j}\},$$

$$Q_j^{\text{Tay}} = G_j(\hat{\theta}_{j-1}, w_j) - G_j(\hat{\theta}_{j-1}, 0) - \Gamma_j(\hat{\theta}_{j-1})w_j.$$

The Taylor remainder Q_j^{Tay} is the quantity denoted R_j^{cur} in Section 31.1, and the vector C below is the vector G of Definition 31.1. Unless stated otherwise, constants may depend on $L, \gamma, M, J, n_*, \Theta$, the analytic domains, the loops, σ , and the coarse index s (the dependence described in item (5) of Definition 4.14); they are independent of K, j, x , the lattice offset, and the classical recursion index $d_j = J(K - s - j)$. A constant indexed by s may depend on H_s . All regions are the full periodic torus.

The threshold s_0 below depends on the fixed data and is chosen before the coarse index s ; the reconstruction maps do not change with s . The classical recursion index d_j is the depth $\mathfrak{d}_j$ of Section 26, not the increment $x_j - x_{j-1}$ of the heat path there.

Remark 32.2. The bounds of Theorem 31.5(4) on the circulation Taylor remainders Q_j^{Tay} come from the bound for general outputs in part (3) of that theorem, applied to

$$Z_j^a(\theta, q) = \hat{\mathcal{E}}_j^a(\hat{\Phi}_j(\theta, q)), \quad G_j(\theta, w) = \mathcal{C}_{a,\varepsilon,x}(a_j^{-1} Z_j^a(\theta, g_j, w)),$$

with the field $\hat{\mathcal{E}}_j^a$ of Theorem 31.3. That bound is useful only if the larger coarse/free radius, which we denote ρ_{joint}^+ , and the coefficient bound, which we denote C_F^{cur} , are fixed

before K , s , j , and the output resolution vary. We explain why. The only composition of growing length in Z_j^a is the field $\hat{\mathcal{E}}_j^a$. Let C_E bound the rows and volume-normalized columns of the matrices E_i^a , let C_{src} be the constant in Theorem 31.3(4), and recall $D = \beta_J/(1 - \alpha_*)$ and $q = B\alpha_*^2 < 1$. A common majorant for $\hat{\mathcal{E}}_j^a$ is

$$\mathcal{M}_{\text{ext}}(z) = C_E z + \frac{C_E C_{\text{src}} C_*}{1 - q} \frac{z^2}{(1 - Dz)^2}. \quad (32.1)$$

Indeed, a remainder at scale $i > j$, with $r' = i - 1 - j$, contributes at most $C_E C_{\text{src}} C_* q^{r'} z^2 / (1 - Dz)^2$, since $(a_j/a_i)\alpha_*^{2r'+1} = C_* q^{r'}$; for a distinguished input its column factors multiply to a^3/a_j^3 , so the same series and its derivatives bound the columns. We fix a radius ρ_E inside the larger domain, with $D\rho_E < 1$, such that the reconstruction images lie strictly inside the disk of radius ρ_E . Then $\mathcal{M}_{\text{ext}}(\rho_E) < \infty$, and its derivatives on the smaller domain are fixed constants. We then choose the larger coarse/free domain of radius ρ_{joint}^+ so that its image under $\hat{\Phi}_j$ lies in this disk, uniformly in the indices, as in Step 7 of the proof of Theorem 31.5. The majorant $\mathcal{M}_{\text{ext}}$ and its derivatives on the smaller domain, composed with the joint coefficient bound for $\hat{\Phi}_j$, then give the fixed constant C_F^{cur} . Both maps are equivariant and vanish at zero, so $Z_j^a(0,0) = 0$. After the substitution $q_{j,\ell} = g_{j,\ell} w_{j,\ell}$, each fluctuation factor contributes its coupling $g_{j,\ell} \leq t_j$, and one derivative in the coarse field costs a fixed factor on the larger domain, which we include in C_F^{cur} . Thus neither C_F^{cur} nor ρ_{joint}^+ depends on the remaining number of steps or on s ; the bound for general outputs is not applied to an output chosen anew at each cutoff. This is Step 7 of the proof of Theorem 31.5.

The following proposition collects the estimates of Section 31 that are used below, together with the dependence of their constants.

Proposition 32.3. *In the setting of Definition 32.1, let s_0 be the threshold of Theorem 31.5, enlarged as in Lemma 31.8. For every $s \geq s_0$, $K \geq s$, $1 \leq j \leq k$, and $a \leq \varepsilon \leq \varepsilon_0(s)$, the following statements hold. The constants do not depend on $s \geq s_0$, except through the factors in H_s and n_s that are displayed.*

1. *Conditionally on $\mathcal{F}_{j-1}$, the vector w_j has the law $k_j(\hat{\theta}_{j-1}, \cdot)$ on $\Omega_j = \prod_\ell \bar{B}_{\mathbb{R}^3}(0, p_{j,\ell})$, with $p_j \leq p_{j,\ell} \leq p_j + C_j$; this kernel does not depend on the restrictions at coarser scales once $\hat{\theta}_{j-1}$ is fixed. Its potential satisfies $D_w^2 V_j \geq I/2$ and $\|D_w^2 V_j - I\|_{\text{row+col}} \leq \vartheta_j \leq C[t_j p_j + t_j^2(1 + E_q)]$, and $\text{Var } f \leq 2\mathbb{E}|\nabla f|^2$ and $\text{Ent}(f^2) \leq 4\mathbb{E}|\nabla f|^2$ conditionally on $\mathcal{F}_{j-1}$. Further, $m_j \leq C_J(L/a_j)^3$, $|\mathbb{E}[w_{j,\ell} \mid \mathcal{F}_{j-1}]| \leq C t_j b_{j-1}$, $p_j \geq 1$, $6m_j e^{-p_j^2/192} \leq 1/2$, and $|\mathbb{E}[w_{j,\ell}^c \mid \mathcal{F}_{j-1}]| \leq p_j/(16\sqrt{3})$.*
2. *For θ in the reconstruction domain, $\|\Gamma_j(\theta)\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq C\omega_j$, $\|\Gamma_j(\theta)\|_{\ell^\infty \rightarrow \mathbb{R}^{3m}} \leq C\sqrt{\varepsilon/a_j}$, and $\|\Gamma_j(\theta) - \Gamma_j(0)\|_{\ell^2 \rightarrow \mathbb{R}^{3m}} \leq C\|\theta\|_{\infty}\omega_j$.*

3. We have $|\Gamma_j(\hat{\theta}_{j-1})\bar{w}_j + \mathbb{E}[Q_j^{\text{Tay}} | \mathcal{F}_{j-1}]| \leq Ct_j b_{j-1} \sqrt{\varepsilon/a_j}$, the same bound for $|\mathbb{E}[Q_j^{\text{Tay}} | \mathcal{F}_{j-1}]|$, and $\mathbb{E}[|Q_j^{\text{Tay}} - \mathbb{E}(Q_j^{\text{Tay}} | \mathcal{F}_{j-1})|^2 | \mathcal{F}_{j-1}] \leq Ct_j^2 p_j^2 \omega_j^2$.
4. Pathwise, $b_0 \leq r$, $b_j \leq C[r\alpha_*^j + g_3\sqrt{a_j} p_j]$, and $\sum_{j=1}^k b_{j-1} \leq D_s$, with D_s as in Lemma 31.8 and $\sup_{s \geq s_0} D_s < \infty$. With T_s as in Lemma 31.8 and $\tilde{N}$ the quantity denoted $N_{K,s}$ there, $\sup_{\|\hat{\theta}_0\|_\infty \leq r} |T_s| \leq Cr\sqrt{\varepsilon}/(g_3 H_s)$ and $\mathbb{E}|\tilde{N}| + \sup_{\|\hat{\theta}_0\|_\infty \leq r} |T_s| \leq Cr\sqrt{\varepsilon}/(g_3 H_s) + Cg_3 D_s \sqrt{\varepsilon} + C\sqrt{\eta}[1 + \log(1/\eta)]^2$.
5. We have $\sum_j \Gamma_j(0)\Gamma_j(0)^\top = C^{a,\varepsilon,L} \otimes I_3 - C_{s,a,\varepsilon}^{\text{ret}}$ with $\|C_{s,a,\varepsilon}^{\text{ret}}\| \leq Cn_s^2 \varepsilon/H_s$. As $a \downarrow 0$ at fixed ε , $C^{a,\varepsilon,L} \rightarrow C^{\varepsilon,L}$ uniformly in x and the lattice offset, and $C^{\varepsilon,L} \rightarrow \Sigma$ as $\varepsilon \downarrow 0$.

Proof. Parts (1)–(3) are parts (1), (2), and (4) of Theorem 31.5; the constants in part (3) are fixed by Remark 32.2. Part (4) is Lemma 31.8, and part (5) is Theorem 31.6 together with Lemma 31.2(5). By Lemma 31.10, all of these hold at the fixed coupling γ , with constants independent of $s \geq s_0$ apart from the displayed factors. $\square$

32.2 Integration by parts and conditional characteristic functions

We now prove two finite-dimensional lemmas. The first compares one conditional increment with a Gaussian vector by an integration by parts against the density of the conditional fluctuation kernel, with a cutoff that vanishes near the boundary of the product of balls; this is the characteristic-function form of Stein's Gaussian integration by parts [147]. The second lemma assembles the one-step comparisons by telescoping conditional characteristic functions.

Lemma 32.4. *Let $\mathcal{K} \subset \mathbb{R}^D$ be compact and convex with nonempty interior, let V be a C^2 function on a neighborhood of $\mathcal{K}$ with $D^2V \geq \kappa I$ for some $\kappa > 0$, and let $d\zeta = Z^{-1}\mathbf{1}_{\mathcal{K}}e^{-V}dw$. Assume the Poincaré inequality $\text{Var}_\zeta f \leq \kappa^{-1}\mathbb{E}_\zeta|\nabla f|^2$ for every real C^1 function f . Let Ψ be a real $d \times D$ matrix with rows ψ_b , and set*

$$\mu = \mathbb{E}_\zeta w, \quad \xi = \Psi(w - \mu), \quad S = \Psi\Psi^\top, \quad R_b = \langle \nabla V - (w - \mu), \psi_b \rangle.$$

Then for every $\chi \in C_c^1(\text{int } \mathcal{K}; [0, 1])$, with $\vartheta = \sup_{\text{supp } \chi} \|D^2V - I\|_{\text{op}}$, every $u \in \mathbb{R}^d$, and $1 \leq b \leq d$,

$$\begin{aligned} |\mathbb{E}_\zeta[(\xi_b - i(Su)_b)e^{iu \cdot \xi}]| &\leq \frac{\sqrt{2}}{\kappa} \|\Psi\|_{\text{op}} |u| (\vartheta |\psi_b| + \|R_b \nabla \chi\|_{L^2(\zeta)}) \\ &\quad + 2\mathbb{E}_\zeta |D_{\psi_b} \chi| + 2\mathbb{E}_\zeta [(1 - \chi)|\xi_b|] + |(Su)_b| \zeta\{\chi < 1\}. \end{aligned}$$

The numerical coefficients do not depend on D , on the shape of $\mathcal{K}$, or on its radii.

By Proposition 32.3(1), the conditional fluctuation kernels satisfy the Poincaré hypothesis with $\kappa = 1/2$.

Proof. We write $\mathbb{E}$ for $\mathbb{E}_\zeta$ and $e_u(w) = e^{iu \cdot \zeta(w)}$. Since χ has compact support in the interior of $\mathcal{K}$, we may integrate the directional derivative of $\chi e_u e^{-V}$, extended by zero, over $\mathbb{R}^D$; there is no boundary contribution. Since

$$D_{\psi_b}(u \cdot \zeta) = (Su)_b, \quad D_{\psi_b}V = \zeta_b + R_b,$$

the resulting identity is

$$\mathbb{E}[\chi(\zeta_b + R_b)e_u] = i(Su)_b \mathbb{E}[\chi e_u] + \mathbb{E}[(D_{\psi_b}\chi)e_u]. \quad (32.2)$$

Rearranging, we obtain

$$\mathbb{E}[(\zeta_b - i(Su)_b)e_u] = \mathbb{E}[(1 - \chi)(\zeta_b - i(Su)_b)e_u] - \mathbb{E}[\chi R_b e_u] + \mathbb{E}[(D_{\psi_b}\chi)e_u]. \quad (32.3)$$

At $u = 0$, the identity (32.2) and $\mathbb{E}\zeta_b = 0$ imply

$$\mathbb{E}[\chi R_b] = \mathbb{E}[D_{\psi_b}\chi] + \mathbb{E}[(1 - \chi)\zeta_b]. \quad (32.4)$$

Here it matters that R_b is defined with the mean μ of ζ .

For a real function F and a complex function H , we have $|\text{Cov}(F, H)| \leq (\text{Var } F)^{1/2}(\mathbb{E}|H - \mathbb{E}H|^2)^{1/2}$ by the Cauchy–Schwarz inequality. We apply the Poincaré inequality to $F = \chi R_b$ and separately to $\Re e_u$ and $\Im e_u$; the sum of the squared gradients of the latter two functions is $|\Psi^\top u|^2 \leq \|\Psi\|_{\text{op}}^2 |u|^2$. Further,

$$\nabla(\chi R_b) = \chi(D^2V - I)\psi_b + R_b \nabla \chi, \quad \|\nabla(\chi R_b)\|_{L^2} \leq \vartheta |\psi_b| + \|R_b \nabla \chi\|_{L^2},$$

and only the Hessian on $\text{supp } \chi$ enters the last bound. Thus

$$|\text{Cov}(\chi R_b, e_u)| \leq \kappa^{-1} \|\Psi\|_{\text{op}} |u| (\vartheta |\psi_b| + \|R_b \nabla \chi\|_{L^2}).$$

In (32.3) we write $\mathbb{E}[\chi R_b e_u] = \text{Cov}(\chi R_b, e_u) + \mathbb{E}[\chi R_b] \mathbb{E}e_u$ and use (32.4). The two occurrences of each cutoff term are bounded by the corresponding absolute expectations, and the remaining term with $(Su)_b$ is bounded by $|(Su)_b| \zeta \{\chi < 1\}$. This proves the stated inequality, even with 1 in place of $\sqrt{2}$ in the first coefficient. $\square$

Lemma 32.5. *Let $(\mathcal{F}_j)_{j=0}^k$ be a finite filtration, and let $\zeta_j \in \mathbb{R}^d$ be bounded, $\mathcal{F}_j$ -measurable, and conditionally centered, $\mathbb{E}[\zeta_j \mid \mathcal{F}_{j-1}] = 0$. Let $S_j^0 \geq 0$ be deterministic matrices, and let $\rho_j, \eta_j \geq 0$ be integrable $\mathcal{F}_{j-1}$ -measurable random variables. Suppose that for every $u \in \mathbb{R}^d$*

$$|\mathbb{E}[(\zeta_j - iS_j^0 u) e^{iu \cdot \zeta_j} \mid \mathcal{F}_{j-1}]| \leq \rho_j |u| + \eta_j (1 + |u|).$$

For $0 \leq q \leq k$ put $M_{q,k}^0 = \sum_{j=q+1}^k \zeta_j$. Then almost surely

$$\left| \mathbb{E}[e^{iu \cdot M_{q,k}^0} \mid \mathcal{F}_q] - \exp\left(-\frac{1}{2} u^\top \sum_{j=q+1}^k S_j^0 u\right) \right| \leq \sum_{j=q+1}^k \mathbb{E}\left[\frac{1}{2} \rho_j |u|^2 + \eta_j |u| (1 + |u|) \mid \mathcal{F}_q\right].$$

For every integrable random vector N , replacing $M_{q,k}^0$ by $M_{q,k}^0 + N$ adds at most $|u|\mathbb{E}[|N| \mid \mathcal{F}_q]$ to the bound. No independence or smallness of the increments is assumed, and the numerical constants do not depend on k, q , or the dimensions.

Proof. We set

$$\varphi_j(v) = \mathbb{E}[e^{iv \cdot \xi_j} \mid \mathcal{F}_{j-1}], \quad \Delta_j(v) = \mathbb{E}[(\xi_j - iS_j^0 v)e^{iv \cdot \xi_j} \mid \mathcal{F}_{j-1}].$$

Since ξ_j is bounded, we may differentiate along $v = \tau u$, and we find

$$\frac{d}{d\tau} \varphi_j(\tau u) = -\tau u^\top S_j^0 u \varphi_j(\tau u) + iu \cdot \Delta_j(\tau u).$$

The integrating factor from τ to 1 is $\exp[-(1 - \tau^2)u^\top S_j^0 u/2] \leq 1$. Since $\varphi_j(0) = 1$, integrating from 0 to 1, we obtain

$$|\varphi_j(u) - \varphi_j^0(u)| \leq e_j(u), \quad \varphi_j^0(u) = e^{-u^\top S_j^0 u/2}, \quad e_j(u) = \frac{1}{2}\rho_j|u|^2 + \eta_j|u|(1 + |u|). \quad (32.5)$$

Fix q , and for $q \leq j \leq k$ set

$$H_j = \exp\left(iu \cdot \sum_{l=q+1}^j \xi_l\right) \prod_{l=j+1}^k \varphi_l^0(u).$$

Then $H_q = \prod_{l=q+1}^k \varphi_l^0(u)$ is deterministic and $H_k = e^{iu \cdot M_{q,k}^0}$. Conditioning the difference $H_j - H_{j-1}$ first on $\mathcal{F}_{j-1}$ and then on $\mathcal{F}_q$, and using (32.5), we bound its conditional expectation given $\mathcal{F}_q$ in modulus by $\mathbb{E}[e_j(u) \mid \mathcal{F}_q]$; here we used that the exponential of the past increments has modulus one and that every remaining deterministic factor lies in $[0, 1]$. Telescoping proves the assertion. Finally, $|e^{iu \cdot (z+n)} - e^{iu \cdot z}| \leq |u||n|$ implies the statement about N ; no measurability or independence requirement on N is needed. $\square$

32.3 Telescoping circulations and dimension-dependent cutoffs

The next lemma applies Lemma 32.4 to the conditional fluctuation kernels. The cutoff χ_j must vanish near the boundary of the product of balls, and its gradient grows like the square root of the number D_j of fluctuation variables, so the error terms carry a polynomial factor in D_j . This factor is harmless because the cutoff radius grows logarithmically in $1/t_j$, so that the probability that the cutoff is active decays faster than any power of the mesh.

Lemma 32.6. *In the setting of Definition 32.1, for every finite $K \geq s$,*

$$I^{K,s} = T_s + \sum_{j=1}^k \Gamma_j(\hat{\theta}_{j-1})(w_j - \bar{w}_j) + \tilde{N},$$

where

$$T_s = \mathcal{I}(\widehat{\Pi}_{k \leftarrow 0} \widehat{\theta}_0), \quad \widetilde{N} = \sum_{j=1}^k \{\Gamma_j(\widehat{\theta}_{j-1}) \bar{w}_j + Q_j^{\text{Tay}}\}.$$

All zero-fluctuation continuations lie in the reconstruction domain, and the middle sum is a sum of martingale increments under $\nu_{K,s}$.

There is $s_0 < \infty$, depending only on $L, \gamma, M, J, n_*, \Theta$, the analytic domains and column bounds, the loops, and σ , and not on the later index s , such that for $s \geq s_0$ the following holds at each RG step. The kernel k_j lives on m_j balls in $\mathbb{R}^3$, with $D_j = 3m_j \leq 3C_J(L/a_j)^3$, and $p_j \geq 1$, $D_w^2 V_j \geq I/2$, and

$$\|D_w^2 V_j - I\|_{\text{op}} \leq \vartheta_j \leq C\{t_j p_j + t_j^2(1 + E_q)\} \leq \frac{1}{2}.$$

There is a smooth function $\chi_j : \Omega_j \rightarrow [0, 1]$, equal to one on the product of the balls of half the radii and to zero outside the product of the balls of three quarters of the radii, with $|\nabla \chi_j| \leq C\sqrt{D_j}/p_j$ and $\nabla \chi_j = 0$ on $\{\chi_j = 1\}$. The conditional probability that the cutoff is active,

$$q_j^{\text{cut}} = \mathbb{P}\{\chi_j(w_j) < 1 \mid \mathcal{F}_{j-1}\},$$

satisfies $q_j^{\text{cut}} \leq 6m_j e^{-p_j^2/192} \leq 1/2$. Put $\widehat{\tau}_j = (1 + 3m_j)(6m_j)^{1/2} e^{-p_j^2/384}$. Let Ψ_j be either $\Gamma_j(\widehat{\theta}_{j-1})$ or $\Gamma_j(0)$, and let $\xi_j = \Psi_j(w_j - \bar{w}_j)$ and $S_j = \Psi_j \Psi_j^T$. Then for every $u \in \mathbb{R}^{3m}$,

$$|\mathbb{E}[(\xi_j - iS_j u) e^{iu \cdot \xi_j} \mid \mathcal{F}_{j-1}]| \leq \rho_j |u| + \eta_j (1 + |u|),$$

with

$$\rho_j \leq C_{3m} \vartheta_j \|\Psi_j\|_{\text{op}}^2, \quad \eta_j \leq C_{3m} \|\Psi_j\|_{\text{op}} (1 + \|\Psi_j\|_{\text{op}}) \widehat{\tau}_j,$$

where C_{3m} depends only on m, J , and the ratios of the radii, and not on D_j, p_j , or the coarse field. There is a deterministic function T of the mesh such that

$$\widehat{\tau}_j \leq T(a_j) \leq C(1 + (L/a_j)^3)(L/a_j)^{3/2} \exp\left\{-\frac{A_w^2}{384} \log^2 \frac{1}{M^{(J-1)/2} g_3 \sqrt{a_j}}\right\}.$$

For every $r' > 0$ we have $T(h) \leq C_{r'} h^{r'}$ as $h \downarrow 0$. In particular, with $A_s^{\text{cut}} = \sum_{j \geq 1} a_j^{-1/2} T(a_j)$ and $\Theta_{\text{cut}}(v) = \sup_{0 < h \leq v} T(h)$,

$$\sum_{j=1}^k \omega_j (1 + \omega_j) \widehat{\tau}_j \leq C\sqrt{\varepsilon} A_s^{\text{cut}} + C_B \Theta_{\text{cut}}(\varepsilon) \longrightarrow 0 \quad (\varepsilon \downarrow 0),$$

uniformly in K, x , the lattice offset, and d_j , and $A_s^{\text{cut}} \leq A_{s_0}^{\text{cut}} < \infty$. The number of blocks is $m_j = O_{J,L,\gamma}(t_j^{-6})$.

Proof. The decomposition is that of Lemma 31.8, proved in Step 5 of its proof: for $F_j = \mathcal{I}(\widehat{\Pi}_{k \leftarrow j} \widehat{\theta}_j)$ we have $F_0 = T_s$, $F_k = I^{K,s}$, and $F_j - F_{j-1} = \Gamma_j(\widehat{\theta}_{j-1})w_j + Q_j^{\text{Tail}}$, since $\widehat{\Pi}_{k \leftarrow j-1} = \widehat{\Pi}_{k \leftarrow j} \circ \widehat{\Phi}_j(\cdot, 0)$ on the reconstruction domain, which contains all sampled fields and all zero-fluctuation continuations by Theorem 31.3. The coefficients are $\mathcal{F}_{j-1}$ -measurable and $\bar{w}_j$ is the conditional mean under the same law, so the middle sum is a martingale.

We next fix the threshold, uniformly for all later coarse indices; all data in the dependence list are fixed before this choice. The largest coupling of each RG step and the cutoff radii satisfy

$$t_1(s) = g_3 \sqrt{H_s/M}, \quad t_j(s) = t_1(s) B^{-(j-1)/2}, \quad p_j = p_1(s) + \frac{1}{2} A_w (j-1) \log B. \quad (32.6)$$

By Proposition 32.3(1), the kernel k_j lives on m_j balls with radii $p_j \leq p_{j,\ell} \leq p_j + C_J$ and has the stated Hessian, dimension, and mean bounds. The bound on rows and columns implies the bound on the operator norm, since for a finite matrix A

$$\|A\|_{\ell^2 \rightarrow \ell^2}^2 \leq \left(\sup_i \sum_l |A_{il}| \right) \left(\sup_l \sum_i |A_{il}| \right),$$

as follows by applying the weighted Cauchy–Schwarz inequality in each row and then summing. The Hessian error therefore tends to zero with $t_j p_j + t_j^2(1 + E_q)$. Since $t_1(s) \downarrow 0$, one enlargement of s_0 enforces $p_j \geq 1$ and $\vartheta_j \leq 1/2$ at every level. The number of blocks satisfies

$$m_j \leq C_J (L/a_j)^3 \leq C_{J,L,\gamma} t_j^{-6}, \quad a_j = t_j^2 / (\lambda_B^2 g_3^2),$$

with $\lambda_B = M^{(J-1)/2}$, so $6m_j e^{-p_j^2/192} \rightarrow 0$ uniformly over all sufficiently small t_j . The same s_0 enforces the bound $1/2$ on this quantity and the bound on the means

$$|\mathbb{E}[w_{j,\ell}^c \mid \mathcal{F}_{j-1}]| \leq p_j / (16\sqrt{3}), \quad 1 \leq c \leq 3. \quad (32.7)$$

These are restrictions on the coarse side only; the cutoff radii and the maps are not changed.

The cutoff. Fix a smooth function $\psi : [0, \infty) \rightarrow [0, 1]$, equal to one on $[0, 1/2]$ and to zero on $[3/4, \infty)$, and set

$$\chi_j(w) = \prod_{\ell=1}^{m_j} \psi(|w_\ell|/p_{j,\ell}).$$

It is smooth, since each factor is constant near $w_\ell = 0$, and it has compact support in the interior of the product. The gradient of each factor has norm at most C/p_j , so $|\nabla \chi_j| \leq C\sqrt{m_j}/p_j \leq C\sqrt{D_j}/p_j$, and $\nabla \chi_j$ vanishes wherever $\chi_j = 1$. These bounds depend only on ψ , not on the number of balls.

The probability that the cutoff is active. We use the logarithmic Sobolev inequality $\text{Ent}(f^2) \leq 4\mathbb{E}|\nabla f|^2$ of Proposition 32.3(1) on the product itself, by Herbst's argument (see [123]). For a scalar coordinate $F(w) = w_\ell^c$, let $H(\lambda) = \log \mathbb{E}e^{\lambda F}$. Applying the inequality to $f = e^{\lambda F/2}$, we find $\lambda H'(\lambda) - H(\lambda) \leq \lambda^2$. Integrating $(H/\lambda)' \leq 1$ from zero for $\lambda > 0$, and applying the same argument to $-F$, we obtain

$$\mathbb{E}e^{\lambda(F-\mathbb{E}F)} \leq e^{\lambda^2} \quad (\lambda \in \mathbb{R}), \quad \mathbb{P}\{|F - \mathbb{E}F| \geq v\} \leq 2e^{-v^2/4},$$

where the tail bound follows by taking $\lambda = v/2$ in $e^{\lambda^2 - \lambda v}$ for each of the two tails. If $\chi_j < 1$, then some $|w_\ell| > p_{j,\ell}/2 \geq p_j/2$, so some coordinate has absolute value larger than $p_j/(2\sqrt{3})$. By (32.7), its deviation from its conditional mean is at least $p_j/(4\sqrt{3})$. A union bound over the $3m_j$ coordinates gives

$$q_j^{\text{cut}} = \mathbb{P}\{\chi_j < 1 \mid \mathcal{F}_{j-1}\} \leq 6m_j e^{-p_j^2/192}. \quad (32.8)$$

The error terms with the dimension retained. We suppress the index j and write $D = 3m_j$, $p = p_j$, $q = \zeta\{\chi < 1\} \leq 1/2$, $\Psi = \Psi_j$, and ζ for the conditional kernel. Both choices of Ψ are linear maps whose rows are fixed once the coarse field is fixed; neither choice changes the law, its mean, the cutoff, or the score vector $\nabla V - (w - \mu)$. The contraction $R_b = (\nabla V - (w - \mu)) \cdot \psi_b$ retains its dependence on the chosen row ψ_b . The product of balls has diameter at most $C_J p \sqrt{D}$, and by convexity the mean $\mu = \mathbb{E}w$ lies in the product. Since $\nabla R_b = (D^2 V - I)\psi_b$, with $c_b = R_b(\mu)$ we have

$$\sup_w |R_b(w) - c_b| \leq C_J \vartheta p \sqrt{D} |\psi_b|. \quad (32.9)$$

By the Poincaré inequality, $\mathbb{E}|\xi_b|^2 \leq 2|\psi_b|^2$. Further,

$$\mathbb{E}|D\psi_b \chi| \leq C\sqrt{D} |\psi_b| q/p, \quad \mathbb{E}[(1 - \chi)|\xi_b|] \leq \sqrt{2} |\psi_b| \sqrt{q}.$$

Using (32.4), $\mathbb{E}\chi \geq 1/2$, and (32.9), we obtain

$$|c_b| \leq C_J |\psi_b| \{\vartheta p \sqrt{D} + (\sqrt{D}/p)q + \sqrt{q}\}.$$

Since $\nabla \chi$ vanishes on $\{\chi = 1\}$, multiplying the resulting bound on $\sup |R_b|$ by the gradient bound and by $\sqrt{q}$, and using $p \geq 1$ and $\vartheta \leq 1/2$, we find

$$\|R_b \nabla \chi\|_{L^2} \leq C_J (1 + D) |\psi_b| \sqrt{q}. \quad (32.10)$$

In particular, R_b is not bounded independently of D .

We insert these estimates into Lemma 32.4 with $\kappa = 1/2$, and use $|(Su)_b| \leq |\psi_b| \|\Psi\|_{\text{op}} |u|$ and $(\sum_b |\psi_b|^2)^{1/2} \leq \sqrt{3m} \|\Psi\|_{\text{op}}$. The Euclidean norm of the vector of errors is at most

$$C_{3m} \vartheta \|\Psi\|_{\text{op}}^2 |u| + C_{3m} \|\Psi\|_{\text{op}} (1 + \|\Psi\|_{\text{op}}) (1 + D) \sqrt{q} (1 + |u|).$$

All constants may depend on J and the ratios of the radii, but not on D , p , or the coarse field. By (32.8), $(1 + D)\sqrt{q} \leq \hat{\tau}_j$, which gives the stated ρ_j and η_j .

The deterministic envelope. On $0 < h \leq H_{s_0}/B$ we take

$$T(h) = C(1 + (L/h)^3)(L/h)^{3/2} \exp\left[-\frac{A_w^2}{384} \log^2 \frac{1}{\lambda_B g_3 \sqrt{h}}\right], \quad (32.11)$$

with one sufficiently large C ; then $\hat{\tau}_j \leq T(a_j)$, since $m_j \leq C_J(L/a_j)^3$ and $p_j = A_w \log(1/(\lambda_B g_3 \sqrt{a_j}))$. This definition uses the mesh itself, not the index j or the coarse index. To verify the decay, let $h_* = H_{s_0}/B$ and $z = \log(h_*/h) \geq 0$. The logarithm of the polynomial prefactor is at most $C + (9/2)z$. The exponent is a quadratic polynomial in z with leading coefficient $-A_w^2/1536$. For every $r' > 0$, subtracting $\log h^{r'}$ adds only $r'z$ and a constant, and the resulting quadratic polynomial is bounded above. This proves $T(h) \leq C_{r'} h^{r'}$ and $\Theta_{\text{cut}}(v) \rightarrow 0$, with constants of the stated dependence.

Taking $r' = 2$ shows that $\sum_{j \geq 1} a_j^{-1/2} T(a_j) < \infty$. For $s \geq s_0$ this series is a tail of the same series along the fixed sequence of meshes, so $A_s^{\text{cut}} \leq A_{s_0}^{\text{cut}}$. Finally $\omega_j \leq 1$, and

$$\sum_{a_j > \varepsilon} \omega_j(1 + \omega_j)T(a_j) \leq 2\sqrt{\varepsilon} A_s^{\text{cut}}, \quad \sum_{a_j \leq \varepsilon} \omega_j(1 + \omega_j)T(a_j) \leq \frac{2\Theta_{\text{cut}}(\varepsilon)}{1 - B^{-1}}.$$

For the second bound we used that the meshes form a geometric sequence, so that $\sum_{a_j \leq \varepsilon} a_j/\varepsilon \leq (1 - B^{-1})^{-1}$. Truncating either sum at k only decreases it. This proves the summability and uniformity assertions. $\square$

32.4 Gaussian approximation of the circulations

We first record how characteristic-function estimates are converted into estimates in the bounded-Lipschitz distance d_{BL} , which uses test functions bounded by one with Lipschitz constant at most one. The argument also covers a singular limiting covariance.

Lemma 32.7. *Let P' and P be probability measures on $\mathbb{R}^d$, with characteristic functions $\hat{P}'(u) = \int e^{iu \cdot z} P'(dz)$ and $\hat{P}$. For all $A > 0$ and $\delta > 0$,*

$$d_{\text{BL}}(P', P) \leq P'\{|z| > A\} + P\{|z| > A\} + 4\delta \mathbb{E}|Z_d| + \frac{|B_{\mathbb{R}^d}(0, A+1)|}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\delta^2|u|^2/2} |\hat{P}'(u) - \hat{P}(u)| du, \quad (32.12)$$

where Z_d is a standard Gaussian vector in $\mathbb{R}^d$. In particular, if probability measures P_l are uniformly tight and their characteristic functions converge pointwise to that of P , then $d_{\text{BL}}(P_l, P) \rightarrow 0$. If $d = 3m$, the same convergence holds after applying $q(v) = (|v_i|^2/8)_{i=1}^m$, provided the tails of the P_l and of P vanish uniformly as the radius tends to infinity; quantitatively, for every $A > 0$,

$$d_{\text{BL}}(q_* P', q_* P) \leq \max\{1, A/4\} d_{\text{BL}}(P', P) + 2P'\{|v| > A\} + 2P\{|v| > A\}. \quad (32.13)$$

The constants depend only on d and on the chosen radii A and δ .

Proof. Let f satisfy $\|f\|_\infty \leq 1$ and $\text{Lip } f \leq 1$, and let $\chi_A(z) = \min\{1, (A + 1 - |z|)_+\}$ and $h = f\chi_A$. Then h is supported in the ball of radius $A + 1$, is bounded by one, and has Lipschitz constant at most two. If φ_δ is the centered Gaussian density with covariance $\delta^2 I_d$, then $\|h * \varphi_\delta - h\|_\infty \leq 2\delta\mathbb{E}|Z_d|$. The Fourier transform of a Gaussian density gives $\varphi_\delta(z) = (2\pi)^{-d} \int e^{iu \cdot z} e^{-\delta^2|u|^2/2} du$. Inserting this formula into the convolution and using Fubini's theorem, which is justified since h and the Gaussian in u are integrable, we obtain, with $\hat{h}(u) = \int h(z) e^{-iu \cdot z} dz$,

$$P'(h * \varphi_\delta) - P(h * \varphi_\delta) = (2\pi)^{-d} \int \hat{h}(u) e^{-\delta^2|u|^2/2} (\hat{P}'(u) - \hat{P}(u)) du.$$

Since $|\hat{h}(u)| \leq |B_{\mathbb{R}^d}(0, A + 1)|$, the modulus of the right side is at most the last term of (32.12). Restoring h costs $4\delta\mathbb{E}|Z_d|$, and restoring f costs at most $P'\{|z| > A\} + P\{|z| > A\}$, since $|f - h| \leq \mathbf{1}_{\{|z| > A\}}$. The right side does not depend on f , which proves (32.12). Under the hypotheses on P_l , dominated convergence (the characteristic functions differ by at most two) shows that the last term tends to zero for fixed A and δ ; we then let $A \rightarrow \infty$ and $\delta \downarrow 0$.

For the quadratic map, let π_A be the nearest-point projection onto the closed ball of radius A . It is 1-Lipschitz: for $z, z' \in \mathbb{R}^d$, the inequalities $\langle z - \pi_A z, y - \pi_A z \rangle \leq 0$ for y in the ball, applied with $y = \pi_A z'$ and symmetrically, give $|\pi_A z - \pi_A z'|^2 \leq \langle z - z', \pi_A z - \pi_A z' \rangle$. On the ball,

$$|q(v) - q(w)| \leq \frac{1}{8}(|v| + |w|)|v - w| \leq (A/4)|v - w|.$$

Thus, for a test function f on $\mathbb{R}^m$ bounded by one with Lipschitz constant at most one, the function $f \circ q \circ \pi_A$ is bounded by one and has Lipschitz constant at most $A/4$. Further, $f \circ q$ differs from it only outside the ball, by at most two. This proves (32.13). For the convergence, take $P' = P_l$, let $l \rightarrow \infty$, and then let $A \rightarrow \infty$. $\square$

Theorem 32.8. *In the setting of Definition 32.1, there is an integer s_0 such that for every fixed integer $s \geq s_0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x \in \mathbb{T}_L^3, \text{offset}} d_{\text{BL}} \left(\text{Law}_{\nu_{K,s}} I^{K,s}(\varepsilon, x), N(0, \Sigma \otimes I_3) \right) = 0,$$

where Σ is the covariance of Definition 31.1; it may be singular. The underlying estimates hold for every $K \geq s$, $1 \leq j \leq K - s$, and $a \leq \varepsilon \leq \varepsilon_0(s)$, where one may take $\varepsilon_0(s) \leq \min\{cH_s, \varepsilon_{\text{loop}}, c/g_3^2\}$. The constants in the bounds on the coefficients, on the variation of the covariance, and on the conditional remainders can be chosen independently of $s \geq s_0$. More explicitly, let $\kappa_* = -\log \alpha_* / \log B$, which exceeds $1/2$ since $\alpha_* < B^{-1/2}$, and let

$$\Psi_\kappa(u) = \begin{cases} u^\kappa, & \kappa < 1, \\ u(1 + |\log u|), & \kappa = 1, \\ u, & \kappa > 1, \end{cases} \quad 0 < u \leq 1.$$

Then the variation of the coefficients and the remainder satisfy

$$\sum_j \|\widehat{\theta}_{j-1}\|_\infty \omega_j^2 \leq Cr \Psi_{\kappa_*}(\varepsilon/H_s) + C\sqrt{\eta} [1 + |\log \eta|],$$

$$\mathbb{E}|\widetilde{N}| + |T_s| \leq \frac{Cr}{g_3 H_s} \sqrt{\varepsilon} + Cg_3 D_s \sqrt{\varepsilon} + C\sqrt{\eta} [1 + |\log \eta|]^2,$$

where $D_s = C\{r/(1 - \alpha_*) + g_3 \sqrt{H_s}(p_1(s) + 1)\}$ and $\sup_{s \geq s_0} D_s < \infty$. The coarse-field term may grow like H_s^{-1} , $\varepsilon_0(s)$ may decrease with s , and s is fixed before both limits.

Proof. We first sum the one-step bounds over the scales (Steps 1–3), then replace the coefficients by their values at the zero field (Step 4), then bound the characteristic functions (Step 5), and finally pass to the bounded-Lipschitz distance (Step 6).

Step 0: the threshold. We take the threshold of Lemma 32.6 and enlarge it, if necessary, to that of Proposition 32.3. All constants in the coefficient, potential-derivative, and conditional variance bounds are independent of the later s . Indeed, the maps for coarse index s are those of the fixed sequence at the absolute levels $s + j$, with recursion indices $J(K - s - j)$, and the output class is that of Remark 32.2. We use only the mean and variance bounds of Proposition 32.3(3); the local bound (31.13) is not needed. The domain and zero-fluctuation statements come from Theorem 31.3. The law of the coarse field may depend on s ; since every estimate holds for every coarse field, no comparison of these laws is needed. We take the range of ε in the statement, shrinking c if necessary so that $\eta < 1$ and $p(\varepsilon) = A_w \log(1/(\lambda_B g_3 \sqrt{\varepsilon})) \geq 1$; this is possible with c depending only on M, J , and A_w .

Step 1: sums over the scales. Since the meshes a_j decrease by the ratio B , splitting at $a_j = \varepsilon$ gives

$$\sum_{j=1}^k \omega_j^2 \leq \frac{1}{1 - B^{-1}} + \frac{1}{1 - B^{-2}} = C_B^{(2)}. \quad (32.14)$$

For $a_j > \varepsilon$ the summands are ε/a_j , and reading them from the last coarse scale backward bounds their sum by the first geometric series; for $a_j \leq \varepsilon$ they are a_j^2/ε^2 , bounded by the second series. The estimate holds also if either part is empty. For $h \leq \varepsilon$ write

$$p(h) = p(\varepsilon) + \frac{1}{2} A_w \log(\varepsilon/h),$$

so that $p_j = p(a_j)$. For each fixed $v > 0$, geometric summation gives

$$\sum_{a_j \leq \varepsilon} (a_j/\varepsilon)^v (1 + p(a_j)) \leq C_{B,v,A_w} (1 + p(\varepsilon)). \quad (32.15)$$

Indeed, the first ratio $z = a_{j_0}/\varepsilon \leq 1$ need not be one, and the sum is at most $z^v \sum_{l \geq 0} B^{-vl} [1 + p(\varepsilon) + (A_w/2)(\log(1/z) + l \log B)]$; we use $z^v \leq 1$,

$\sup_{0 < z \leq 1} z^v \log(1/z) < \infty$, and the convergence of $\sum_l B^{-vl}$ and $\sum_l l B^{-vl}$. For $a_j > \varepsilon$ we have $p(a_j) \leq p(\varepsilon)$ and $\sum_{a_j > \varepsilon} a_j^{-1/2} \leq C_B \varepsilon^{-1/2}$. It follows that

$$g_3 \sum_j \sqrt{a_j} (1 + p_j) \omega_j^2 \leq C g_3 \sqrt{\varepsilon} (1 + p(\varepsilon)) \leq C \sqrt{\eta} (1 + |\log \eta|), \quad (32.16)$$

where on the fine scales we used (32.15) with $v = 5/2$, and on the coarse scales the sum of inverse square roots. Here and below the constants may depend on γ and on H_{s_0} , but not on the later s . Since $p_j \geq 1$ and $t_j \leq t_1(s_0)$, the Hessian error satisfies $\vartheta_j \leq C g_3 \sqrt{a_j} (1 + p_j)$. Thus

$$\sum_j \vartheta_j \omega_j^2 \leq C \sqrt{\eta} (1 + |\log \eta|). \quad (32.17)$$

Step 2: the variation of the coefficients. We write $b_j = \|\widehat{\theta}_j\|_\infty$. The bound of Proposition 32.3(4) is pathwise, with a coefficient independent of H_s and of all later $s \geq s_0$; we use this uniform coefficient rather than a bound with a constant depending on H_s . Its conditions hold here: all fields and zero-fluctuation continuations lie in the reconstruction domain, and $t_1(s) \leq t_1(s_0)$ satisfies the required upper bound for every later s . With $b_0 \leq r$, it implies

$$b_{j-1} \leq C \{r(a_j/H_s)^{\kappa_*} + g_3 \sqrt{a_j} (1 + p_j)\}, \quad 1 \leq j \leq k. \quad (32.18)$$

For $j \geq 2$ we used $a_{j-1} = B a_j$, $p_{j-1} \leq p_j$, and $\alpha_*^{j-1} = \alpha_*^{-1} (a_j/H_s)^{\kappa_*}$; for $j = 1$ we used $b_0 \leq r$ and absorbed α_*^{-1} into C . The deterministic part of the weighted sum $\sum_j b_{j-1} \omega_j^2$ over the coarse scales is

$$C r \varepsilon H_s^{-\kappa_*} \sum_{a_j > \varepsilon} a_j^{\kappa_* - 1}.$$

If $\kappa_* < 1$, summing from the last coarse mesh bounds this by $C r (\varepsilon/H_s)^{\kappa_*}$. If $\kappa_* = 1$, every summand is one and there are at most $C_B (1 + \log(H_s/\varepsilon))$ of them, which gives the logarithmic case of Ψ_{κ_*} ; this case cannot be bounded by $C \varepsilon/H_s$. If $\kappa_* > 1$, summing from the largest mesh bounds it by $C r \varepsilon/H_s$. On the fine scales the sum is

$$C r H_s^{-\kappa_*} \varepsilon^{-2} \sum_{a_j \leq \varepsilon} a_j^{\kappa_* + 2} \leq C r (\varepsilon/H_s)^{\kappa_*}.$$

In all three cases this is bounded by $C r \Psi_{\kappa_*}(\varepsilon/H_s)$. For the fluctuation part of (32.18) we use (32.16). This proves, pathwise, the first estimate of the theorem:

$$\sum_j b_{j-1} \omega_j^2 \leq C r \Psi_{\kappa_*}(\varepsilon/H_s) + C \sqrt{\eta} (1 + |\log \eta|). \quad (32.19)$$

Step 3: the remainder. Let $\mathcal{D}_j = Q_j^{\text{Tay}} - \mathbb{E}[Q_j^{\text{Tay}} \mid \mathcal{F}_{j-1}]$. By the conditional mean and variance bounds of Proposition 32.3(3), the tower property, $\mathbb{E}|\mathcal{D}_j| \leq (\mathbb{E}|\mathcal{D}_j|^2)^{1/2}$, and the triangle inequality, we have, without any independence,

$$\mathbb{E}|\tilde{N}| \leq C \sum_j t_j \mathbb{E}b_{j-1} \sqrt{\varepsilon/a_j} + C \sum_j t_j p_j \omega_j. \quad (32.20)$$

In the first sum we apply the pathwise bound inside the expectation:

$$\sum_j t_j \mathbb{E}b_{j-1} \sqrt{\varepsilon/a_j} = \lambda_B g_3 \sqrt{\varepsilon} \mathbb{E} \left[\sum_j b_{j-1} \right] \leq \lambda_B g_3 \sqrt{\varepsilon} D_s.$$

On the fine scales the second sum is $\lambda_B g_3 \varepsilon^{-1} \sum_j a_j^{3/2} p_j$, which is at most $C \sqrt{\eta} (1 + p(\varepsilon))$ by (32.15). On the coarse scales each term is $\lambda_B \sqrt{\eta} p_j$; there are at most $C_B [1 + \log(H_{s_0}/\varepsilon)]$ such scales, and each $p_j \leq p(\varepsilon)$. For fixed g_3 and H_{s_0} both factors are at most $C(1 + |\log \eta|)$. Adding the bound on T_s from Proposition 32.3(4), we obtain

$$\mathbb{E}|\tilde{N}| + \sup_{\|\hat{\theta}_0\|_\infty \leq r} |T_s| \leq \frac{Cr}{g_3 H_s} \sqrt{\varepsilon} + C g_3 D_s \sqrt{\varepsilon} + C \sqrt{\eta} (1 + |\log \eta|)^2, \quad (32.21)$$

which is also the estimate of Lemma 31.8. The quantity D_s is the sum of the bounds on b_{j-1} , with a coefficient independent of s , and $\sup_{s \geq s_0} D_s < \infty$ follows from (32.6): $p_1(s)$ grows linearly in s , and $\sqrt{H_s}$ decreases by the factor $B^{-1/2}$ at each increment of s .

Step 4: coefficients at the zero field. We now replace the coefficient matrices by their values at the zero field, keeping the centered fluctuation variables $e_j = w_j - \bar{w}_j$ of the conditioned measure. Let

$$M_k = \sum_j \Gamma_j(\hat{\theta}_{j-1}) e_j, \quad M_k^0 = \sum_j \Gamma_j(0) e_j, \quad S_j^0 = \Gamma_j(0) \Gamma_j(0)^\top.$$

The reconstruction domain gives a fixed deterministic bound $b_j \leq b_{\max} < \infty$. By the conditional Poincaré inequality and the bound on $\Gamma_j(\theta) - \Gamma_j(0)$ in Proposition 32.3(2), we have

$$\mathbb{E}[|(\Gamma_j(\hat{\theta}_{j-1}) - \Gamma_j(0))e_j|^2 \mid \mathcal{F}_{j-1}] \leq C_{3m} b_{j-1}^2 \omega_j^2.$$

Since these differences are martingale increments for the same measure, conditioning on the earlier variables makes all cross terms vanish. With (32.19) we obtain

$$\mathbb{E}|M_k - M_k^0|^2 \leq C_{3m} \sum_j \mathbb{E}b_{j-1}^2 \omega_j^2 \leq C_{3m} b_{\max} \{Cr \Psi_{\kappa_*}(\varepsilon/H_s) + C \sqrt{\eta} (1 + |\log \eta|)\} \longrightarrow 0, \quad (32.22)$$

uniformly in K at fixed s .

Step 5: characteristic functions. We apply Lemma 32.6 with $\Psi_j = \Gamma_j(0)$, under the same conditional kernels at the coarse fields $\widehat{\theta}_{j-1}$; the kernel at the zero field is not used. The bound $\|\Gamma_j(0)\|_{\text{op}} \leq C\omega_j$ implies the deterministic bounds

$$\rho_j^0 \leq C\vartheta_j\omega_j^2, \quad \eta_j^0 \leq C\omega_j(1 + \omega_j)\widehat{\tau}_j,$$

whose sums tend to zero by (32.17) and Lemma 32.6. Lemma 32.5 with $q = 0$, followed by (32.22) and (32.21), bounds the error in the characteristic function of the circulation by

$$\begin{aligned} \left| \mathbb{E} e^{iu \cdot I^{K,s}} - e^{-u^T(\Sigma \otimes I_3)u/2} \right| &\leq |u| \left\{ \mathbb{E}|T_s + \widetilde{N}| + (\mathbb{E}|M_k - M_k^0|^2)^{1/2} \right\} \\ &\quad + \sum_j \left\{ \frac{1}{2} \rho_j^0 |u|^2 + \eta_j^0 |u| (1 + |u|) \right\} \\ &\quad + \frac{1}{2} |u|^2 \left\| \sum_j S_j^0 - \Sigma \otimes I_3 \right\|_{\text{op}}. \end{aligned} \quad (32.23)$$

In the last term we used $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$. The last term is deterministic. By Proposition 32.3(5),

$$\sum_j S_j^0 = C^{a,\varepsilon,L} \otimes I_3 - C_{s,a,\varepsilon}^{\text{ret}}, \quad \|C_{s,a,\varepsilon}^{\text{ret}}\| \leq Cn_s^2\varepsilon/H_s,$$

and the first covariance converges to $C^{\varepsilon,L} \otimes I_3$ as $K \rightarrow \infty$ and then to $\Sigma \otimes I_3$ as $\varepsilon \downarrow 0$, uniformly in x and the lattice offset. This identification concerns the linearization of the reconstruction at the origin and the linearized coarse field; the nonlinear reconstruction is never conditioned to have zero coarse field. At each fixed s , every term on the right of (32.23) therefore vanishes in the stated iterated limit, uniformly in x and the lattice offset. In particular,

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x, \text{offset}} \left| \mathbb{E} e^{iu \cdot I^{K,s}} - e^{-u^T(\Sigma \otimes I_3)u/2} \right| = 0. \quad (32.24)$$

No random covariance and no compensated exponential is needed.

Step 6: the bounded-Lipschitz distance. By the conditional Poincaré inequality, we have $\mathbb{E}[|\Gamma_j(0)e_j|^2 \mid \mathcal{F}_{j-1}] \leq 2 \text{tr } S_j^0$, and orthogonality together with (32.14) gives $\mathbb{E}|M_k^0|^2 \leq C_{3m}$. With (32.22) and (32.21), this gives a bound on $\mathbb{E}|I^{K,s}|$ that is uniform in K , x , the lattice offset, and ε in a sufficiently small range at the fixed s ; by Markov's inequality the tails of $I^{K,s}$ vanish uniformly. We apply (32.12) with $P' = \text{Law}_{\nu_{K,s}} I^{K,s}$ and $P = N(0, \Sigma \otimes I_3)$. Its last term is bounded by the integral against $e^{-\delta^2|u|^2/2}$ of $\min\{2, \mathcal{E}_{K,\varepsilon,s}(u)\}$, where $\mathcal{E}_{K,\varepsilon,s}$ is a deterministic bound on the right side of (32.23), uniform in x and the lattice offset. Such a bound is obtained by using (32.21), (32.22), the bounds in Step 5 on ρ_j^0 and η_j^0 , and the uniform covariance convergence at the end of that step. We choose it to be a polynomial in $|u|$ with nonnegative deterministic coefficients supplied by these bounds.

For each fixed u , it tends to zero in the iterated upper-limit sense $K \rightarrow \infty$, then $\varepsilon \downarrow 0$. The integrand is bounded by the integrable function $2e^{-\delta^2|u|^2/2}$; the dominated upper-limit inequality followed by dominated convergence therefore makes the integral vanish in that iterated limit, for fixed A and δ . Letting then $A \rightarrow \infty$ and $\delta \downarrow 0$ proves the theorem, also for singular Σ . The dependence on s of the coarse-field constant and of the range of ε is exactly the one in the statement; no choice of s depending on ε has been made. $\square$

Remark 32.9. Lemmas 32.4 and 32.5 are abstract finite-dimensional statements. Their application to the circulations fixes the coupling $\gamma > 0$ before s_0 is chosen, and gives estimates with the dependence described in item (5) of Definition 4.14; it does not give constants uniform as $\gamma \rightarrow 0$ at a fixed side, as would be needed for item (6). The proof of Theorem 32.8 uses no tightness of the linear and quadratic flow data; these enter in Lemma 32.11 and in the passage to the continuum measure.

The same proof applies conditionally on the first q RG steps, with error bounds that do not depend on the values of these steps. This uniformity is what makes the asymptotic independence in Section 32.8 possible; it does not follow from Theorem 32.8 itself.

Lemma 32.10. *In the setting of Definition 32.1, fix $s \geq s_0$. Write $\|h\|_{\text{BL}} = \max\{\|h\|_\infty, \text{Lip } h\}$. Fix an integer $q \geq 0$, and for $K - s \geq q$ let $\mathcal{F}_q = \sigma(\widehat{\theta}_0, w_1, \dots, w_q)$ be the σ -field generated by the coarse field and the fluctuation variables of the first q RG steps. Let $C \sim N(0, \Sigma \otimes I_3)$ and $\mathbf{q}(v) = (|v_i|^2/8)_{i=1}^m$. Regular conditional probabilities can be chosen so that*

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x, \text{offset}} \sup_{v_{K,s}} \text{ess sup } d_{\text{BL}}\left(\text{Law}(I^{K,s}(\varepsilon, x) \mid \mathcal{F}_q), \text{Law}(C)\right) = 0. \quad (32.25)$$

The same holds with $I^{K,s}$ and C replaced by $\mathbf{q}(I^{K,s})$ and $\mathbf{q}(C)$. In particular, if $|B_{K,q}| \leq 1$ is any real $\mathcal{F}_q$ -measurable random variable, then

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x, \text{offset}} \sup_{\|f\|_{\text{BL}} \leq 1} \left| \mathbb{E}_{v_{K,s}}[f(\mathbf{q}(I^{K,s}))B_{K,q}] - \mathbb{E}f(\mathbf{q}(C)) \mathbb{E}_{v_{K,s}} B_{K,q} \right| = 0. \quad (32.26)$$

The constants may depend on s and q and otherwise have the dependence of item (5) of Definition 4.14. The variable $B_{K,q}$ may depend on K , and no regularity in the values of the first q steps is required. Nothing is asserted uniformly in q .

Proof. We follow the proof of Theorem 32.8 conditionally on $\mathcal{F}_q$, and check that every error bound is deterministic and does not depend on the values of the first q steps.

Step 1: the decomposition after q steps. Since all spaces of coarse fields and fluctuation variables are finite-dimensional compact metric spaces, the explicit normalized kernels give regular conditional probabilities. Fix a realization of the first q steps. By Proposition 32.3(1), the later conditional kernels are the same kernels as before conditioning,

since the restrictions at coarser scales do not alter them once their coarse field is fixed. Every such field lies in the reconstruction domain and satisfies $e^{\hat{\theta}_{j-1}} \in B_r$, as noted in Section 31.1. The maps at the absolute levels $s + q + 1, \dots, K$ are the same reconstruction maps, with the same recursion indices. By Theorem 31.3, every zero-fluctuation continuation is defined. Thus the decomposition starting at q is

$$I^{K,s} = T_q + \sum_{j=q+1}^{K-s} \Gamma_j(\hat{\theta}_{j-1})(w_j - \bar{w}_j) + N_q,$$

where

$$T_q = \mathcal{I}(\hat{\Pi}_{K-s \leftarrow q} \hat{\theta}_q), \quad N_q = \sum_{j=q+1}^{K-s} \{\Gamma_j(\hat{\theta}_{j-1})\bar{w}_j + Q_j^{\text{Tay}}\},$$

and $\bar{w}_j$ is still the conditional mean under the conditioned Wilson measure. This is a restriction of the same telescoping sum, with no change of measure.

Step 2: bounds uniform in the first q steps. The field $\hat{\mathcal{E}}_q^a$ of Theorem 31.3 vanishes at zero and has bounded derivative on the reconstruction domain, so $\|\hat{\mathcal{E}}_q^a(\hat{\theta}_q)\|_\infty \leq Cb_{\max}$. By its identity for circulations, the contraction of the heat semigroup, and the length bound, we have, with $a_q = H_{s+q}$,

$$|T_q| \leq \frac{Cb_{\max}}{g_3 a_q} \sqrt{\varepsilon}.$$

Here $b_{\max}$ is the fixed bound for the reconstruction domain. We keep the pathwise bounds of Proposition 32.3(4) and (32.19), which hold after conditioning as well; they are not restarted at $\hat{\theta}_q$. Using the mean and variance bounds of Proposition 32.3(3), which hold for every coarse field, and the tower property, we obtain

$$\mathbb{E}[|N_q| \mid \mathcal{F}_q] \leq Cg_3 D_s \sqrt{\varepsilon} + C \sum_{j>q} t_j p_j \omega_j \leq Cg_3 D_s \sqrt{\varepsilon} + C\sqrt{\eta} (1 + |\log \eta|)^2.$$

For the means we used $t_j \sqrt{\varepsilon/a_j} = \lambda_B g_3 \sqrt{\varepsilon}$ and the deterministic bound on $\sum_j b_{j-1}$; for the centered part, the conditional variance and the conditional Cauchy–Schwarz inequality. These bounds vanish at fixed s and q and do not depend on the values of the first q steps. The cutoff estimates of Lemma 32.6 hold at every later coarse field. For the replacement of the coefficients by their values at the zero field, we set

$$B_{s,\varepsilon} = Cr\Psi_{\kappa_*}(\varepsilon/H_s) + C\sqrt{\eta} (1 + |\log \eta|).$$

By the conditional Poincaré inequality and the orthogonality of martingale increments, we have

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{j>q} (\Gamma_j(\hat{\theta}_{j-1}) - \Gamma_j(0))(w_j - \bar{w}_j)\right|^2 \mid \mathcal{F}_q\right] &\leq C \mathbb{E}\left[\sum_{j>q} b_{j-1}^2 \omega_j^2 \mid \mathcal{F}_q\right] \\ &\leq Cb_{\max} B_{s,\varepsilon}. \end{aligned}$$

The last inequality follows from the pathwise bound (32.19), which holds on every sample path; its right side is deterministic and tends to zero at fixed s . Lemma 32.5 applies to this tail of the sum.

Step 3: the covariance of the tail. The coefficients $\Gamma_j(0)$ for $j > q$ are exactly the coefficients at the zero field for the coarse index $s + q$, since their defining maps are the same restrictions of the fixed family. By Theorem 31.6 at index $s + q$,

$$\sum_{j>q} \Gamma_j(0) \Gamma_j(0)^T = C^{a,\varepsilon,L} \otimes I_3 - C_{s+q,a,\varepsilon}^{\text{ret}}, \quad \|C_{s+q,a,\varepsilon}^{\text{ret}}\| \leq C n_{s+q}^2 \varepsilon / H_{s+q},$$

and by Lemma 31.2 the first term has the same limit $\Sigma \otimes I_3$. For every u , the conditional error in the characteristic function is therefore bounded almost surely by the sum of these deterministic remainder, coefficient, cutoff, and covariance errors, as in (32.23). This bound does not depend on the first q steps, and it vanishes in the stated order at fixed s and q . The factors $1/a_q$ and n_{s+q}^2/H_{s+q} grow with q , which is why q is fixed before the limits.

Step 4: conditional laws. By the conditional Poincaré inequality, the conditional second moment of the tail $\sum_{j>q} \Gamma_j(0)(w_j - \bar{w}_j)$ also has a deterministic bound, and the conditional first-moment bounds of Step 2 then give uniform conditional tails for $I^{K,s}$. We apply (32.12) to these conditional laws: its Fourier integral is bounded by the same deterministic envelope for the characteristic functions, and its truncation terms by these conditional tails. Choosing one null set for all rational u and using the continuity of each conditional characteristic function, we obtain an almost sure bound that does not depend on the version of the conditional law or on the test function. This proves (32.25).

Step 5: the quadratic map. By (32.13) applied to the conditional laws, with the projection onto a ball of radius A on which q has Lipschitz constant at most $A/4$, and with the conditional tails outside the ball vanishing uniformly as $A \rightarrow \infty$, the statement for $q(I^{K,s})$ follows; we let $K \rightarrow \infty$ and $\varepsilon \downarrow 0$ before $A \rightarrow \infty$. Multiplying by $B_{K,q}$ and using the tower property proves (32.26). $\square$

32.5 Wilson flow and removal of the conditioning

We now compare the normalized Wilson-loop deficits of the flowed fine configuration with the squared circulations, and remove the conditioning on the small-field event. The comparison has two parts. The first is an expansion of the trace of a product of $SU(2)$ matrices to second order, with a remainder controlled by the total variation and independent of the number of links. The second is the local theory of the Wilson flow of Theorem 24.5, which writes the flowed field as the linear heat evolution plus the quadratic

Picard term plus a smaller remainder. The Wilson flow is the one introduced by Lüscher [128], with time measured in physical units as in Section 2.1.

Lemma 32.11. *The following statements hold.*

1. For every finite sequence $v_e \in \mathbb{R}^3$, with $D = \sum_e |v_e|$ and $Y = \sum_e v_e$, and every $s' > 0$,

$$\left| \frac{1 - \frac{1}{2} \operatorname{tr} \prod_e e^{s' v_e^a \tau_a}}{(s')^2} - \frac{|Y|^2}{8} \right| \leq \frac{s' D^3}{48} e^{s' D/2}.$$

The same bound holds for an ordered exponential along a path of finite length, with D the total variation of the integrated coefficient. The bound does not depend on the number of links.

2. Fix $1/2 < \alpha_{\text{fl}} < 2/3$ and a horizon $T_0 > 0$, and put $\beta_{\text{fl}} = \alpha_{\text{fl}} - 1/2$ and $\kappa_{\text{fl}} = 1 - 3\alpha_{\text{fl}}/2$. For the reconstruction of Definition 32.1, let $\mathfrak{x} = P^a A^{K,s}$ be the linear heat evolution of the initial potential $A^{K,s} = a^{-1} \widehat{\theta}_k$, let $\mathfrak{y} = \mathcal{Q}_a(\mathfrak{x}, \mathfrak{x})$ be its quadratic Picard term, and let $\mathcal{B}_{K,s} = \mathbf{X}_{\alpha_{\text{fl}}}^a(\mathfrak{x}) + \mathbf{Y}_{\alpha_{\text{fl}}}^a(\mathfrak{y})$ be the norm of the linear and quadratic flow data of Definition 4.8. For every $R < \infty$ there are $T_R \leq T_0$, $a_R > 0$, $\varepsilon_R > 0$, and C_R such that, on $\{\mathcal{B}_{K,s} \leq R\}$ and for $a \leq \varepsilon \leq \varepsilon_R$, $a \leq a_R$, and $\sigma \varepsilon^2 < T_R$, the representative A_{fl} of the Wilson flow in the damping gauge of Theorem 24.5 satisfies

$$A_{\text{fl}} = \mathfrak{x} + \mathfrak{y} + v, \quad \|v(t)\|_{\infty} \leq C_R(t + a^2)^{\kappa_{\text{fl}}}.$$

Write $Y_i = (a/\sqrt{\eta}) \sum_{e \in \ell_{i,\varepsilon,a,x}} \operatorname{sgn}(e) A_{\text{fl},e}(\sigma \varepsilon^2)$ and $D_i = (a/\sqrt{\eta}) \sum_{e \in \ell_{i,\varepsilon,a,x}} |A_{\text{fl},e}(\sigma \varepsilon^2)|$. Then, uniformly in x , the lattice offset, K , and s , whenever the same radius and domains apply,

$$\max_i |Y_i - I_i^{K,s}| \leq C_R \varepsilon^{3/2-2\alpha_{\text{fl}}}, \quad \max_i (D_i + |I_i^{K,s}|) \leq C_R \varepsilon^{-\beta_{\text{fl}}},$$

$$\max_i \left| \frac{1 - \frac{1}{2} \operatorname{tr} \operatorname{Hol}_{\ell_{i,\varepsilon,a,x}} \frac{\Psi^{(N_K)} U}{\sigma \varepsilon^2}}{\eta} - \frac{|I_i^{K,s}|^2}{8} \right| \leq C_R \varepsilon^{2-3\alpha_{\text{fl}}}.$$

The representative stays in the logarithmic chart on the whole time interval:

$$a \sup_{t \leq T_R} \|A_{\text{fl}}(t)\|_{\infty} \leq C_R \{a^{1-\alpha_{\text{fl}}} + a^{2-2\alpha_{\text{fl}}} + a(T_R + a^2)^{\kappa_{\text{fl}}}\} \longrightarrow 0 \quad (a \rightarrow 0).$$

3. Enlarge s_0 so that n_{s_0} also satisfies the requirement on the coarse side in Lemma 28.10 for the interval $\{\gamma\}$, the horizon T_0 , and the exponent α_{fl} . For each fixed $s \geq s_0$, $\rho_s(R) = \sup_{K \geq s} \nu_{K,s} \{\mathcal{B}_{K,s} > R\} \rightarrow 0$ as $R \rightarrow \infty$. Let

$$Z_{\varepsilon,K} = \left(\frac{1 - \frac{1}{2} \operatorname{tr} \operatorname{Hol}_{\ell_{i,\varepsilon,a,x}} \frac{\Psi^{(N_K)} U}{\sigma \varepsilon^2}}{\eta} \right)_{i=1}^m$$

be the vector of normalized Wilson-loop deficits on the fine lattice, and $\mathbf{q}(I) = (|I_i|^2/8)_{i=1}^m$. For $a \leq a_R$, $a \leq \varepsilon \leq \min\{\varepsilon_R, \varepsilon_0(s)\}$, and $\sigma \varepsilon^2 < T_R$, every $f : \mathbb{R}^m \rightarrow \mathbb{R}$ with $\|f\|_{\infty} \leq 1$ and $\operatorname{Lip} f \leq 1$ satisfies

$$|\mathbb{E}_{\mu_{N_K,\gamma}} f(Z_{\varepsilon,K}) - \mathbb{E}_{\nu_{K,s}} f(\mathbf{q}(I^{K,s}))| \leq 2\mu_{N_K,\gamma}(S_{N_K,n_s,\gamma}^c) + 2\rho_s(R) + C_R \varepsilon^{2-3\alpha_{\text{fl}}}.$$

The constants have the dependence described in items (5) and (7) of Definition 4.14, with additional dependence on R , T_0 , and α_{fl} . No expectation of an unbounded function is transferred between the two measures.

Proof. Part (1) is an algebraic expansion (Step 1). Part (2) follows from the local theory of the Wilson flow and part (1) (Steps 2–4), and part (3) combines part (2) with the tightness of the flow data and Lemma 5.5 (Step 5).

Step 1: the trace expansion. Let $B_e = v_e^a \tau_a$. Then $\|B_e\|_{\text{op}} = |v_e|/2$, and $|\text{tr } T|/2 \leq \|T\|_{\text{op}}$ for every 2×2 matrix T . In the product of the exponential series, the sum of the norms of all terms that are homogeneous of degree p is at most

$$\sum_{\Sigma_e} \prod_{p_e=p} \frac{(s'|v_e|/2)^{p_e}}{p_e!} = \frac{(s'D/2)^p}{p!},$$

by the multinomial identity; this bound has no constant depending on the number of links. The trace of the term of degree one vanishes. The identity $\text{tr}(B_e B_f)/2 = -v_e \cdot v_f/4$ gives the half-trace of the terms of degree two,

$$-\frac{(s')^2}{8} \sum_e |v_e|^2 - \frac{(s')^2}{4} \sum_{e < f} v_e \cdot v_f = -\frac{(s')^2}{8} |Y|^2.$$

The norms of all terms of degree at least three sum to at most $(s'D/2)^3 e^{s'D/2}/6$. Dividing by $(s')^2$ gives the first assertion. For an integrable connection coefficient along a path of finite length, the term of degree p of the ordered exponential series has norm at most $(s'D/2)^p/p!$, since the ordered integral is bounded by $1/p!$ times the integral of the symmetric product of the absolute coefficients. The linear term has trace zero, and the half-trace of the quadratic term is $-(s')^2 \int |v|^2/8$, since the dot product is symmetric on the two ordered halves of the square. The same remainder bound therefore proves the assertion for ordered exponentials, with $D = \int |v|$.

Step 2: the flow. Fix R , T_0 , and α_{fl} , and abbreviate $\alpha = \alpha_{\text{fl}}$, $\beta = \beta_{\text{fl}} = \alpha - 1/2$, and $\kappa = \kappa_{\text{fl}} = 1 - 3\alpha/2$. The event $\{\mathcal{B}_{K,s} \leq R\}$ controls exactly the linear heat evolution, the quadratic Picard term, and all first differences required in Theorem 24.5. Indeed, the linear part is $P^a A^{K,s}$, and the quadratic part is the ordered quadratic term of the damped Wilson equation on the same fine lattice. Restricting the horizon only decreases these norms, and the difference arrays include every forward and backward difference in the norm required by that theorem. We use its bound on the remainder of one solution and its identification with the Wilson flow, reducing its lifespan to $T_R \leq T_0$ if necessary. It gives the decomposition and the bound on v , and it identifies $e^{aA_{\text{fl}}}$ with the Wilson flow up to a time-dependent periodic lattice gauge transformation. The trace of a closed loop is unchanged by this gauge transformation.

The chart condition holds on the whole interval, not only at time zero. Indeed, by the two bounds on the flow data and the remainder bound, we have

$$a \sup_{t \leq T_R} \|A_{\text{fl}}(t)\|_{\infty} \leq C_R \{a^{1-\alpha} + a^{1-2\beta} + a(T_R + a^2)^{\kappa}\}.$$

Since $1 - 2\beta = 2 - 2\alpha > 0$ and $\kappa > 0$, all terms tend to zero as $a \rightarrow 0$. The last term takes its maximum at the final time, where it is $O_R(a)$ for fixed T_R . We choose a_R below the threshold of the chart and $\varepsilon_R \leq 1$ with $\sigma\varepsilon_R^2 < T_R$.

Step 3: the circulation of the flowed field. At $t_{\varepsilon} = \sigma\varepsilon^2$ and $a \leq \varepsilon$ we have $\sigma\varepsilon^2 \leq t_{\varepsilon} + a^2 \leq (\sigma + 1)\varepsilon^2$. By the definitions of the norms and the remainder bound, we have

$$\|\mathfrak{x}(t_{\varepsilon})\|_{\infty} \leq C_{\sigma} R \varepsilon^{-\alpha}, \quad \|\eta(t_{\varepsilon})\|_{\infty} \leq C_{\sigma} R \varepsilon^{-2\beta}, \quad \|v(t_{\varepsilon})\|_{\infty} \leq C_{R,\sigma} \varepsilon^{2\kappa}. \quad (32.27)$$

The physical length of the cycle divided by $\sqrt{\eta}$ is at most $C_{\ell} \sqrt{\varepsilon}/g_3$. Since $I_i^{K,s} = (a/\sqrt{\eta}) \sum_e \text{sgn}(e) \mathfrak{x}_e(t_{\varepsilon})$, multiplying the last two estimates by this factor, we obtain

$$\max_i |Y_i - I_i^{K,s}| \leq C_R \{\varepsilon^{1/2-2\beta} + \varepsilon^{1/2+2\kappa}\} \leq C_R \varepsilon^{3/2-2\alpha},$$

where we used $1/2 - 2\beta = 3/2 - 2\alpha$ and $1/2 + 2\kappa = 5/2 - 3\alpha > 3/2 - 2\alpha$. Multiplying all three estimates in (32.27) by the same factor, we find $D_i + |I_i^{K,s}| \leq C_R \varepsilon^{-\beta}$. These constants depend on the fixed $g_3 > 0$, as allowed.

Step 4: the Wilson-loop deficit. For each oriented link we take $v_e = \text{sgn}(e) a A_{\text{fl},e} / \sqrt{\eta}$ and $s' = \sqrt{\eta}$ in the trace expansion of Step 1. A link traversed against its orientation contributes the inverse of the positive link variable, whose logarithm is the negative of the positive one. The resulting D and Y are exactly D_i and Y_i . In particular,

$$\sqrt{\eta} D_i \leq C_R \varepsilon^{1-\alpha}, \quad \sqrt{\eta} D_i^3 \leq C_R \varepsilon^{2-3\alpha},$$

and the exponential in the trace bound is bounded for small ε . Further,

$$||Y_i|^2 - |I_i^{K,s}|^2| \leq (|Y_i| + |I_i^{K,s}|) |Y_i - I_i^{K,s}| \leq C_R \varepsilon^{2-3\alpha}.$$

This proves the bound on the deficit. It is deterministic on the event $\{\mathcal{B}_{K,s} \leq R\}$ and uses only the length bound on the cycles, so it is uniform over x , the lattice offset, K , and s for which the same radius and domains apply.

Step 5: radius tightness and removal of the conditioning. For each fixed $s \geq s_0$, we apply Lemma 28.10, in particular (28.18), with coarse side n_s , remaining depth $K - s$, and $I = \{\gamma\}$; the enlargement of s_0 in part (3) makes n_s an admissible coarse side for that lemma. Its fine potential is $A^{K,s} = a^{-1} \hat{\theta}_{K-s}$, by the definition of the reconstruction, and no interpolation is involved at the fine lattice. Its tightness statement on the horizon T_0 is exactly $\rho_s(R) \rightarrow 0$, at fixed s and fixed positive coupling.

Finally, in the joint realization of Remark 15.11, the fine configuration U has marginal $\nu_{K,s}$ and is gauge equivalent to the terminal reconstruction. The joint values of $Z_{\varepsilon,K}$ and all other gauge-invariant observables of that configuration therefore agree with those computed on the reconstruction. Lemma 5.5, applied to the bounded function $f(Z_{\varepsilon,K})$, shows that replacing $\mu_{N_K,\gamma}$ by $\nu_{K,s}$ costs at most $2\mu_{N_K,\gamma}(S_{N_K,n_s,\gamma}^c)$; the event has positive probability, since $\nu_{K,s}$ is defined. Under $\nu_{K,s}$, the difference between $f(Z_{\varepsilon,K})$ and $f(q(I^{K,s}))$ is at most $C_R\varepsilon^{2-3\alpha}$ on $\{\mathcal{B}_{K,s} \leq R\}$, where the dimension m is absorbed into C_R , and at most two on its complement. Adding these bounds proves the last inequality. $\square$

The next lemma restates the Gaussian limits in the form in which they are applied below. Since every input of the preceding proofs is available at each fixed coupling $0 < \gamma \leq \Gamma$ by Lemma 31.10, it requires no new argument.

Lemma 32.12. *In the setting of Lemma 31.10, every fixed $s \geq s_0$ satisfies the conclusion of Theorem 32.8. Moreover, for every fixed $q \geq 0$ the conclusions of Lemma 32.10 hold, including the essential supremum over the values of the first q RG steps, with $\mathcal{F}_q = \sigma(\hat{\theta}_0, w_1, \dots, w_q)$. In particular, with $C \sim N(0, \Sigma \otimes I_3)$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \rightarrow \infty} \sup_{x, \text{offset}} \text{ess sup } d_{\text{BL}}(\text{Law}(I^{K,s} \mid \mathcal{F}_q), \text{Law}(C)) = 0,$$

and the same holds after applying $q(v) = (|v_i|^2/8)_{i=1}^m$. The constants have the dependence described in item (5) of Definition 4.14, with additional dependence on q ; nothing is asserted for q growing with ε or K . Definition 32.1 and the conclusions of Lemmas 32.4 to 32.6 and 32.11 also hold in this setting; for the last one, only the radius tightness at a fixed coarse side is used.

Proof. We check the inputs of the local tools. Lemmas 32.4 and 32.5 are finite-dimensional statements; the conditional fluctuation kernels satisfy the Poincaré hypothesis by Proposition 32.3(1). The decomposition in Lemma 32.6 is an exact finite sum, and its cutoff estimates use the conditions on the number of blocks and on the means, and the decay of the cutoff error, which hold by Step 1 of the proof of Theorem 31.5 and (31.9). Hence every cutoff estimate holds with the same constants. Lemma 32.11 uses Theorem 24.5 for the linear and quadratic flow data on the fine lattice, the joint realization of Remark 15.11 of the reconstructed field and the fine configuration, and the trace expansion; these deterministic arguments need no bound on γ , and the removal of the radius uses (28.18). None of these checks uses a limiting random variable for the flow data.

By Lemma 31.10, all five inputs of Proposition 32.3 hold at this fixed coupling, with the stated independence of s . The proofs of Theorem 32.8 and Lemma 32.10 therefore apply with these inputs. In the conditional proof, the pathwise bounds of Proposition 32.3(4) are kept, not restarted; the coarse-field term after q steps is bounded by $Cb_{\max}\sqrt{\varepsilon}/(g_3a_q)$, and the coarse covariance by $Cn_{s+q}^2\varepsilon/H_{s+q}$. Thus q remains fixed before both limits. $\square$

32.6 The continuum limit

We now pass to the continuum measure, with the limits in the order displayed at the beginning of the section. The final limit removes the conditioning error, whose available bound $F_\Gamma(n_s)$ tends to zero as $s \rightarrow \infty$ and is independent of K . In the theorem below, the class $[n_*]$ in the subscript of $\lambda_{L,\gamma,[n_*]}$ is the class $[n_*]_M$ of Theorem 29.28.

Theorem 32.13. *Fix $L > 0$, an odd integer $M \geq 3$, a base side $n_* \in 4\mathbb{N}$ and its equivalence class $[n_*]_M$, and any $\gamma > 0$. Choose $\Gamma \geq \gamma$ and the parameters of Theorem 23.5, and let $\lambda_{L,\gamma,[n_*]}$ be the continuum measure of Theorem 2.3. For every finite family of smooth closed loops $\ell_1, \dots, \ell_m$ in $\mathbb{R}^3$ and every $\sigma > 0$, put $g_3^2 = \gamma/L$ and*

$$Z_\varepsilon(x) = \left(\frac{1 - W_{x+\varepsilon\ell_i, \sigma\varepsilon^2}}{g_3^2\varepsilon} \right)_{i=1}^m.$$

Then

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{T}_L^3} d_{\text{BL}}(\text{Law}_{\lambda_{L,\gamma,[n_*]}} Z_\varepsilon(x), \text{Law } Z_G) = 0, \quad Z_G = (|C_i|^2/8)_{i=1}^m,$$

where the three color components of C are independent centered Gaussian vectors with covariance

$$\Sigma_{ih} = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{e^{-2\sigma|k|^2}}{|k|^2} \overline{\widehat{J}_i(k)} \cdot \widehat{J}_h(k) dk, \quad \widehat{J}_i(k) = \int_0^1 \dot{\ell}_i(v) e^{-ik \cdot \ell_i(v)} dv.$$

Degenerate covariances and repeated or reversed loops are allowed. Every sufficiently large fixed coarse index satisfies the conclusion of Lemma 32.12 for the circulations. The limits are taken in the order described above. The constants have the dependence described in item (5) of Definition 4.14; no rate uniform in the coupling, in the coarse index, or as $\sigma \downarrow 0$ is asserted.

Proof. By Lemma 31.2(1), $k \cdot \widehat{J}_i(k) = 0$ and $|\widehat{J}_i(k)| \leq C \min\{|k|, 1\}$, so Σ is finite, real, symmetric, and positive semidefinite, and its positive square root constructs the three color vectors. This is the covariance of Definition 31.1, and Lemma 31.10 shows that the results of Section 31 apply at this coupling.

Let $F_\Gamma(n_s)$ be the supremum over $0 < \gamma' \leq \Gamma$ of the bound in Theorem 23.10 at coarse side n_s ; by that theorem, $F_\Gamma(n_s) \rightarrow 0$ as $s \rightarrow \infty$. At fixed s , let $\rho_s(R)$ be the radius tail of Lemma 32.11, which tends to zero by (28.18) with $I = \{\gamma\}$. Let $Z_{\varepsilon,K}(x)$ be the vector of normalized Wilson-loop deficits of Lemma 32.11 for the lattice cycles approximating $x + \varepsilon\ell_i$. By that lemma and the triangle inequality, for every function f with $\|f\|_\infty \leq 1$ and $\text{Lip } f \leq 1$,

$$\begin{aligned} |\mathbb{E}_{\mu_{N_K,\gamma}} f(Z_{\varepsilon,K}(x)) - \mathbb{E} f(Z_G)| &\leq 2F_\Gamma(n_s) + 2\rho_s(R) + C_R \varepsilon^{2-3\alpha_\Pi} \\ &\quad + d_{\text{BL}}(\text{Law}_{\nu_{K,s}} q(I^{K,s}), \text{Law } Z_G). \end{aligned}$$

For each fixed $\varepsilon > 0$ and x , the joint convergence of flowed Wilson loops, Corollary 29.26(1) and Theorem 29.23, with the class identification of Theorem 29.28(4), implies that $\mathbb{E}_{\mu_{N_K, \gamma}} f(Z_{\varepsilon, K}(x)) \rightarrow \mathbb{E}_{\lambda_{L, \gamma, [n_*]}} f(Z_\varepsilon(x))$ as $K \rightarrow \infty$; the lattice cycles are admissible approximations of the loops $x + \varepsilon \ell_i$. The bound above is uniform in x and the lattice offset. To see that the identification at fixed ε is compatible with this uniformity, we translate the lattice sequence by a lattice vector so that the base point is near a vertex, and use the allowed lattice approximation of the translated loops; the additional swept area is $O(\varepsilon a)$, a connecting path and its inverse do not change the trace, and the laws of the fine lattice and of the continuum are invariant under translations. At fixed ε all deficits lie in the compact cube $[0, 2/(g_3^2 \varepsilon)]^m$, so a finite net of test functions with $\|f\|_\infty \leq 1$ and $\text{Lip } f \leq 1$ makes this passage to the limit uniform over all such test functions. We let $K \rightarrow \infty$, and then $\varepsilon \downarrow 0$, using the statement for $q(I^{K, s})$ in Lemma 32.12 and $2 - 3\alpha_{\text{fl}} > 0$. What remains is $2\Gamma(n_s) + 2\rho_s(R)$. We let $R \rightarrow \infty$ at fixed s , and only then $s \rightarrow \infty$. This proves the theorem. For the full class $[n_*]_M$, the continuum measure was identified in Theorem 29.28; the argument above needs only one subsequence $N_K = n_* B^K$. $\square$

32.7 A uniform short-distance variance lower bound

Corollary 32.14. *In Theorem 32.13 take $m = 1$, write $\ell = \ell_1$ and $v = \Sigma_{11}$, and let $D_{\varepsilon, x} = (1 - W_{x+\varepsilon\ell, \sigma\varepsilon^2})/(g_3^2 \varepsilon)$. Then*

$$\liminf_{\varepsilon \downarrow 0} \inf_x \text{Var}_{\lambda_{L, \gamma, [n_*]}} D_{\varepsilon, x} = \liminf_{\varepsilon \downarrow 0} \inf_x \frac{\text{Var}_{\lambda_{L, \gamma, [n_*]}} W_{x+\varepsilon\ell, \sigma\varepsilon^2}}{(g_3^2 \varepsilon)^2} \geq \frac{3v^2}{32}. \quad (32.28)$$

If $\hat{J}_1 \not\equiv 0$, then $v > 0$, and for every sufficiently small fixed ε the observable $W_{x+\varepsilon\ell, \sigma\varepsilon^2}$ is not almost surely constant, for every x . The threshold for ε may depend on the data listed in item (5) of Definition 4.14.

Proof. For independent copies D and D' of a random variable with finite variance, $\text{Var } D = \mathbb{E}(D - D')^2/2$. At each fixed scale the variance is finite, since $0 \leq D_{\varepsilon, x} \leq 2/(g_3^2 \varepsilon)$. By Theorem 32.13, the laws of pairs of independent copies converge, uniformly in x , to the law of independent copies Z and Z' of $v\chi_3^2/8$, where χ_3^2 is a chi-square variable with three degrees of freedom: on compact sets, approximate a bounded continuous function of two variables by finite sums of products, and use the uniform tails of the marginals on the complement. Applying this to the bounded continuous function $\min\{A, (u - u')^2\}$ and then letting $A \rightarrow \infty$, we find that the left side of (32.28) is at least $\mathbb{E}(Z - Z')^2/2 = \text{Var } Z = 3v^2/32$. A nonzero continuous function $\hat{J}_1$ is nonzero on an open set, on which the positive weight in the integral defining v has positive integral, so $v > 0$. The strict lower bound then gives a common threshold below which the observable is not constant. No moment is transferred from the Wilson measure. $\square$

32.8 Asymptotic independence from macroscopic Wilson loops

The small-loop deficits are asymptotically independent of fixed macroscopic flowed Wilson loops. The reason is that a macroscopic flowed Wilson loop is approximated, on the same probability space, by a function of the first q RG steps, while the Gaussian limit of Lemma 32.10 holds uniformly conditionally on these steps. The approximation must have constants independent of the higher derivatives of the finite-scale reconstruction, which grow with q ; this is provided by the stability estimate Corollary 25.11.

Lemma 32.15. *In the setting of Definition 32.1, fix $s \geq s_0$, finitely many piecewise C^1 closed loops Γ_r in $\mathbb{T}_L^3$, times $t_r > 0$, and characters χ_r of finite-dimensional representations of $SU(2)$, $1 \leq r \leq b$. Use lattice approximations $\Gamma_{r,K}$ of bounded physical length and swept area $O(a)$, and let*

$$V_{K,r} = \chi_r(\text{Hol}_{\Gamma_{r,K}} \Psi_{t_r}^{(N_K)}(U)), \quad \mathbf{V}_K = (V_{K,r})_{r=1}^b$$

be the vector of flowed Wilson loops of the fine configuration under $\nu_{K,s}$. There are $\mathcal{F}_q$ -measurable random vectors $\mathbf{V}_K^{[q]}$ on the same probability space such that

$$\lim_{q \rightarrow \infty} \limsup_{K \rightarrow \infty} \mathbb{E}_{\nu_{K,s}} \min\{2, |\mathbf{V}_K - \mathbf{V}_K^{[q]}|\} = 0. \quad (32.29)$$

The vector $\mathbf{V}_K^{[q]}$ consists of the flowed Wilson loops of the finite-scale reconstruction $U_K^{[q]} = \exp(aI_{a_q \rightarrow a} A_q)$, with the same times and lattice paths, where $A_q = \widehat{\theta}_q / a_q$, $a_q = H_s B^{-q}$, and $I_{a_q \rightarrow a}$ is the cardinal interpolation of Lemma 28.10. The approximation does not depend on ε , x , or any test function of the small loops. The constants have the dependence described in item (7) of Definition 4.14, at fixed coupling and with s fixed.

Proof. We use the cardinal interpolation and the notation of Lemma 28.10. On the fine lattice the linear and quadratic flow data of the finite-scale reconstruction are

$$x_q = P^a I_{a_q \rightarrow a} A_q, \quad y_q = \mathcal{Q}_a(x_q, x_q).$$

This is the cardinal interpolation of the field at index q ; we do not claim that it equals the nonlinear zero-fluctuation continuation. Its link variables and their Wilson flow are defined without any small-field event of their own. At fixed q the field A_q ranges over a fixed compact set and its cardinal interpolation is smooth, as in Lemma 28.4. The construction uses only $\widehat{\theta}_q$, so it is $\mathcal{F}_q$ -measurable.

Fix $1/2 < \alpha < 2/3$ and a horizon $T_0 > 0$, and set

$$d_{K,q} = X_\alpha^a(x_{K-s} - x_q) + Y_\alpha^a(y_{K-s} - y_q), \quad \mathcal{B}_{K,s} = X_\alpha^a(x_{K-s}) + Y_\alpha^a(y_{K-s}).$$

Both norms include every first difference on the common lattice. The exact bilinear decomposition of $y_{K-s} - y_q$ and the estimates of Lemma 28.10 on the linear term and on the two mixed terms and the quadratic term imply, for every $\delta > 0$,

$$\lim_{q \rightarrow \infty} \sup_{K \geq s+q} \nu_{K,s} \{d_{K,q} > \delta\} = 0. \quad (32.30)$$

The linear part is bounded by Markov's inequality and the tail estimate for the linear term; the quadratic part is bounded by the sum of the tail estimates for the mixed terms and the quadratic term, each with threshold $\delta/3$. The radius tail $\rho_s(R) = \sup_{K \geq s} \nu_{K,s} \{\mathcal{B}_{K,s} > R\}$ tends to zero by (28.18).

Fix R and $0 < \delta \leq 1$. On the event $\{\mathcal{B}_{K,s} \leq R, d_{K,q} \leq \delta\}$, both pairs of flow data lie in the ball of radius $R + 1$. Let

$$0 < u_R < \frac{1}{4} \min\{T_{R+1}, T_0, \min_r t_r\}.$$

By the estimate of Theorem 24.5 for differences on the same lattice, we have, at this early time,

$$\|A_a(u_R) - A_a^{[q]}(u_R)\|_{C_a^1} \leq C_{R,u_R} d_{K,q}, \quad \|A_a(u_R)\|_{C_a^1} + \|A_a^{[q]}(u_R)\|_{C_a^1} \leq K_R < \infty.$$

Both damping gauges start at the identity. By the joint realization of Remark 15.11 and Theorem 24.5, the first damped solution represents the Wilson flow of the fine configuration, and the second represents the Wilson flow started from $U_K^{[q]}$. Since each character is gauge invariant, both flows may be restarted from these damped fields at time u_R for the purpose of evaluating their later traced loops.

We apply Corollary 25.11 with radius $R + 1$, input horizon T_0 , early time $u = u_R$, $\tau = \min_r t_r$, $T = \max_r t_r$, the given characters, and the fixed bounds on the lengths of the paths. Both pairs of flow data lie in that ball on the stated event, in the same initial gauge, and the discrepancy in that corollary is exactly $d_{K,q}$. It gives

$$|\mathbf{V}_K - \mathbf{V}_K^{[q]}| \leq C_{R,u_R,t,\Gamma,\chi} d_{K,q}, \quad (32.31)$$

for the whole vector of characters, including the normalized fundamental traces. The constant in (32.31) depends on the radius and on the fixed waiting time, but not on the higher derivatives of the finite-scale reconstruction, which grow with q ; the threshold for the mesh depends on R , u_R , and the loops, but not on q . For all sufficiently fine meshes it follows that

$$\mathbb{E} \min\{2, |\mathbf{V}_K - \mathbf{V}_K^{[q]}|\} \leq 2\rho_s(R) + 2\nu_{K,s} \{d_{K,q} > \delta\} + C_R \delta. \quad (32.32)$$

We let $q \rightarrow \infty$ at fixed s , R , and δ , using (32.30), then $\delta \downarrow 0$, and finally $R \rightarrow \infty$. This proves the claim on the same finite probability spaces, without coupling different cutoffs. $\square$

Theorem 32.16. *Assume the setting of Theorem 32.13. Fix, independently of ε and x , a finite vector $\mathbf{V} = (\chi_r(\text{Hol}_{\Gamma_r} X(t_r)))_{r=1}^b$ of characters of holonomies of the trajectory X around piecewise C^1 closed loops Γ_r in $\mathbb{T}_L^3$ at times $t_r > 0$. Then*

$$\limsup_{\varepsilon \downarrow 0} \sup_x \sup_{\|f\|_{\text{BL}}, \|g\|_{\text{BL}} \leq 1} |\mathbb{E}_{\lambda_{L,\gamma,[n_*]}} [f(Z_\varepsilon(x))g(\mathbf{V})] - \mathbb{E}f(Z_G) \mathbb{E}_{\lambda_{L,\gamma,[n_*]}} g(\mathbf{V})| = 0, \quad (32.33)$$

where $\|h\|_{\text{BL}} = \max\{\|h\|_{\infty}, \text{Lip } h\}$. The constants have the dependence described in items (5) and (7) of Definition 4.14. Nothing is asserted uniformly in a growing family of macroscopic loops, as $\gamma \downarrow 0$, as $\sigma \downarrow 0$, or as $\min_r t_r \downarrow 0$.

Proof. On the joint realization of Remark 15.11, whose fine-configuration marginal is $\nu_{K,s}$, let $\mathbf{V}_K$ be the vector of flowed Wilson loops of the fine configuration and $\mathbf{V}_K^{[q]}$ the $\mathcal{F}_q$ -measurable approximation of Lemma 32.15. By (32.32),

$$a_{K,s,q} = \mathbb{E}_{\nu_{K,s}} \min\{2, |\mathbf{V}_K - \mathbf{V}_K^{[q]}|\} \leq 2\rho_s(R) + 2\nu_{K,s}\{d_{K,q} > \delta\} + C_R\delta.$$

Let $\Delta_{K,s,q}(\varepsilon)$ be the deterministic bound on the conditional bounded-Lipschitz distance for the squared circulations from Lemma 32.12. Multiplying by $g(\mathbf{V}_K^{[q]})$, which is $\mathcal{F}_q$ -measurable and bounded by one, and using the tower property, we obtain an error of at most $\Delta_{K,s,q}(\varepsilon)$ against the independent Gaussian factor. Replacing $g(\mathbf{V}_K^{[q]})$ by $g(\mathbf{V}_K)$, in the product and in its separate expectation, costs at most $2a_{K,s,q}$, since $|g(\mathbf{V}_K) - g(\mathbf{V}_K^{[q]})| \leq \min\{2, |\mathbf{V}_K - \mathbf{V}_K^{[q]}|\}$. Replacing $f(q(I^{K,s}))$ by the deficits $f(Z_{\varepsilon,K}(x))$ costs at most $2\rho_s(R) + C_R\varepsilon^{2-3\alpha_{\text{fl}}}$, by Lemma 32.11. Finally, two applications of Lemma 5.5, to the product and to the expectation of $g(\mathbf{V}_K)$, transfer from $\nu_{K,s}$ to $\mu_{N_K,\gamma}$ at a cost of at most $4F_{\Gamma}(n_s)$. Thus the error under the Wilson measure is at most

$$4F_{\Gamma}(n_s) + 2\rho_s(R) + C_R\varepsilon^{2-3\alpha_{\text{fl}}} + 2a_{K,s,q} + \Delta_{K,s,q}(\varepsilon).$$

For fixed ε , x , f , and g , the joint law of the deficits and of $\mathbf{V}_K$ under $\mu_{N_K,\gamma}$ converges to the corresponding joint law under $\lambda_{L,\gamma,[n_*]}$, by Corollary 29.26(1), Theorem 29.23, and the class identification of Theorem 29.28(4), applied to the union of the two families of loops. The bound above does not depend on x , f , or g , so it also bounds their supremum after this identification. We let $K \rightarrow \infty$, then $\varepsilon \downarrow 0$ at fixed s , R , q , and δ . Next we let $q \rightarrow \infty$, using (28.17) to bound $\sup_{K \geq s+q} \nu_{K,s}\{d_{K,q} > \delta\}$, then $\delta \downarrow 0$, then $R \rightarrow \infty$ at this s , and finally $s \rightarrow \infty$. The bound vanishes. $\square$

32.9 Asymptotic independence from bounded Borel functions

The last theorem extends the asymptotic independence to every fixed bounded Borel function on the trajectory space. In the terminology of Rényi, as discussed by Aldous and Eagleson [4], it says that $Z_{\varepsilon}(x)$ converges mixingly relative to the Borel σ -field of $\mathcal{X}_L$, uniformly in x and over test functions with $\|f\|_{\text{BL}} \leq 1$: the limit is independent of that σ -field. This only names the mode of convergence.

Theorem 32.17. *Assume the setting of Theorem 32.16, with $Z_{\varepsilon}(x)$ and Z_G as there. For every bounded Borel function $H : \mathcal{X}_L \rightarrow \mathbb{R}$, chosen independently of ε and x ,*

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{T}_L^3} \sup_{\|f\|_{\text{BL}} \leq 1} |\mathbb{E}_{\lambda_{L,\gamma,[n_*]}}[f(Z_{\varepsilon}(x))H] - \mathbb{E}f(Z_G) \mathbb{E}_{\lambda_{L,\gamma,[n_*]}}H| = 0.$$

In particular this holds for $H = h \circ i^{1,2}$, for every bounded Borel function h on the space of Sobolev trajectories, where $i^{1,2}$ is the map of Definition 29.1. The approximations in the proof may depend on H and $\lambda_{L,\gamma,[n_*]}$. There is no supremum over H , and nothing is asserted for H depending on ε or x .

Proof. Let $\lambda = \lambda_{L,\gamma,[n_*]}$. By Theorem 29.3, the Borel σ -field of $\mathcal{X}_L$ is generated by the determining coordinates w_j , $j \geq 1$, which are normalized fundamental Wilson loops. We first approximate H in $L^1(\lambda)$ by a bounded Lipschitz function of finitely many of these coordinates.

The events A whose indicators can be approximated in $L^1(\lambda)$ by $[0,1]$ -valued Borel functions of finitely many coordinates form a σ -field. It contains the events determined by finitely many coordinates. It is closed under complements, and under finite unions by the Lipschitz map $(s_1, s_2) \mapsto \max\{s_1, s_2\}$ on $[0,1]^2$; for a countable union, we first truncate to a finite union whose indicator has small L^1 error, by continuity of λ from below. Hence it contains every Borel event, and approximating a bounded H by simple functions gives approximations bounded by $\|H\|_\infty$.

Next, a bounded Borel function of $(w_1, \dots, w_q)$ can be approximated by a bounded Lipschitz function. The law λ_q of $(w_1, \dots, w_q)$ is a Borel probability measure on the compact cube $[-1,1]^q$, so each Borel set A has a compact subset K_A and a relatively open superset O_A with $\lambda_q(O_A \setminus K_A)$ arbitrarily small; indeed, the sets with this property contain the compact sets, by continuity from above along decreasing open neighborhoods, and they form a σ -field, by choosing the outer errors summably and keeping finitely many inner compact sets. If the relative complement of O_A is nonempty, it has positive distance d_A from K_A , and $\max\{0, 1 - \text{dist}(\cdot, K_A)/d\}$ with $0 < d < d_A$ is Lipschitz, equals one on K_A , and vanishes outside O_A ; in the remaining cases we use constants. Its $L^1(\lambda_q)$ distance to $\mathbf{1}_A$ is at most $\lambda_q(O_A \setminus K_A)$. Applying this to the indicators of a simple function, extending the resulting Lipschitz function from the cube to $\mathbb{R}^q$ by the distance-cone extension, and truncating at $\|H\|_\infty$, we find for every $\eta' > 0$ a bounded Lipschitz function $g_{\eta'} : \mathbb{R}^q \rightarrow \mathbb{R}$ with

$$H_{\eta'} = g_{\eta'}(w_1, \dots, w_q), \quad \mathbb{E}_\lambda |H - H_{\eta'}| < \eta'.$$

Both q and the Lipschitz constant are fixed before the loops shrink.

Let $\mathbf{V} = (2w_1, \dots, 2w_q)$. Its components are fundamental characters, so $\mathbf{V}$ satisfies the hypotheses of Theorem 32.16. We apply that theorem to the function $z \mapsto g_{\eta'}(z/2)/A_{\eta'}$, where $A_{\eta'} = \max\{1, \|g_{\eta'}\|_\infty, \text{Lip } g_{\eta'}\}$. This function is bounded by one and has Lipschitz constant at most one, and its value at $\mathbf{V}$ is $H_{\eta'}/A_{\eta'}$. Multiplying by $A_{\eta'}$, we find that the expression in the statement with $H_{\eta'}$ in place of H tends to zero, uniformly in x and in f with $\|f\|_{\text{BL}} \leq 1$. Replacing $H_{\eta'}$ by H costs at most

$$\mathbb{E}_\lambda |H - H_{\eta'}| + |\mathbb{E}f(Z_G)| \mathbb{E}_\lambda |H - H_{\eta'}| \leq 2\eta',$$

uniformly in these variables and in ε . We let $\varepsilon \downarrow 0$ and then $\eta' \downarrow 0$. Finally, by Theorem 30.5 the map $i^{1,2}$ is Borel, so $h \circ i^{1,2}$ is an admissible H ; no continuity of $i^{1,2}$ is used. $\square$

Proof of Theorem 2.8. Let $\lambda = \lambda_{L,\gamma,[n]_M}$ with n divisible by four. Part (1) is Theorem 32.13, applied with $n_* = n$ and any $\Gamma \geq \gamma$; its covariance Σ is the matrix in (2.6). Part (2) is Corollary 32.14. Part (3) is Theorem 32.17, applied to the bounded Borel function G . $\square$

PART XI

Weak-coupling asymptotics near flat
connections

Laplace asymptotics near flat connections

This section begins the analysis of weak coupling. We fix the side n of the coarse lattice and let $\gamma \rightarrow 0$. Then every scale of the renormalization group is weakly coupled. The law of the coarsest field V^K under the conditioned Wilson measure is the stopped measure $\hat{\nu}$ of Theorem 14.8, and it concentrates near the flat configurations on Λ_n . We study $\hat{\nu}$ by Laplace's method on the finite-dimensional space $\mathcal{U}_n$, uniformly in the number of averaging steps that precede the coarse lattice. The results are used in Sections 34 to 36 to prove Theorems 2.9 to 2.11; nothing in Sections 31 and 32 depends on this section.

We prove four main results. Throughout, $g^2 = \gamma/n$ and $E_0 = 4 \sum_p q(V_p)$ is the Wilson action on Λ_n .

1. Theorem 33.10 shows that the partition function of $\hat{\nu}$ is comparable to γ^{3n^3-1} , uniformly in the depth, for sufficiently small $\gamma \in (0, \gamma_0]$, and that the integral of $e^{-\kappa E_0/g^2}$ over all of $\mathcal{U}_n$ is $O(g^{6n^3-2})$.
2. Theorem 33.8 constructs coordinates near the trivial connection in which the $6n^3 - 6$ nonconstant directions transverse to the gauge orbits are nondegenerate for the effective classical action E_d , uniformly in d , and integrates these directions out by Laplace's method. What remains is an integral over the nine constant coordinates in the Lie algebra, in which E_d has no positive quadratic part; this integral is analyzed in Section 36.
3. Theorem 33.11 computes, in the continuum and on the lattice, the covariance Σ_θ of the linearization of the flowed cycle logarithms at a regular flat connection, under the Gaussian measure defined by the quadratic part of the Yang–Mills action.
4. Proposition 33.12 shows that, transverse to a compact patch of regular flat configurations, $\hat{\nu}$ is Gaussian up to errors of order $\sqrt{\gamma}$. It also shows that the lattice covariance Σ_θ^N decomposes exactly into a contribution of the coarse lattice and contributions of the normalized fluctuation variables of each RG step.

The tools are an analysis of the cone of commuting triples in $\mathfrak{su}(2)^3$ (Lemma 33.2) and a Laplace estimate in normal coordinates that is uniform in an arbitrary tangential weight (Lemma 33.3 and Corollary 33.4).

The inputs from earlier sections are the stopped density and its estimates (Theorem 14.8), the description of flat configurations and of their refinements (Lemma 14.9), bounds and symmetries of the effective classical actions (Theorems 10.11 and 11.2

and Lemma 10.8), the triangular change of variables of one RG step (Theorem 12.13), and the reconstruction maps $\widehat{\mathcal{R}}$ (Theorem 15.13).

The exponent. We work modulo based gauge transformations, that is, lattice gauge transformations equal to the identity at a fixed root. This quotient still carries the simultaneous conjugation of the cycle holonomies; see Section 33.1. The exponent $6n^3 - 2$ in Theorem 33.10 comes from the following count of directions.

Directions	Dimension	Factor in $\widehat{Z}$
Based gauge quotient of $\mathcal{U}_n$	$6n^3 + 3$	ambient space
Regular flat configurations	$5 = 3 + 2$	tangential volume
Normal to a regular flat configuration	$6n^3 - 2$	g^{6n^3-2}
Nonconstant transverse directions near I	$6n^3 - 6$	g^{6n^3-6}
Normal to a nonzero commuting triple	4	g^4
Constant modes with $ X \leq \sqrt{g}$	9	$g^{9/2}$

The five flat directions are the three holonomy angles and the two directions in which simultaneous conjugation moves the common axis of the holonomies. Near a regular flat configuration the normal integral directly gives g^{6n^3-2} . Near the trivial connection the count is less direct. We write the constant modes as $X \in \mathfrak{su}(2)^3 \simeq \mathbb{R}^9$. On the constant configurations $e^{X/n}$ the Wilson action is comparable to $Q(X)/n$, where $Q(X) = 2^{-1} \sum_{i < j} |x_i \times x_j|^2$, and Q vanishes exactly on the cone $\mathcal{M}$ of commuting triples. Near a commuting triple of radius ϱ , the volume element of $\mathcal{M}$ is proportional to $\varrho^4 d\varrho$, and in each of the four normal directions Q grows like $\varrho^2 |b|^2/2$ (Lemma 33.2). Each normal Gaussian width is then of order g/ϱ , and the product $\varrho^4 (g/\varrho)^4 d\varrho = g^4 d\varrho$ integrates to $O(g^4)$ over a bounded range of radii. The ball of radius $\sqrt{g}$ contributes only $O(g^{9/2})$. With the factor g^{6n^3-6} from the nonconstant directions, we again obtain $g^{6n^3-2} = (\gamma/n)^{3n^3-1}$. The same count indicates that the measure $e^{-Q(Y)} dY$ on $\mathbb{R}^9$ has infinite total mass. This divergence of the bosonic three-matrix $SU(2)$ integral is known [7, 121, 122]; see also Section 36. In this section e^{-Q} is only integrated over bounded sets.

Two normalizations. Two normalizations of the exponent occur. In Section 33.5 we write the exponent of $\widehat{v}$ as E_d/g^2 and use the Laplace parameter $g^2 = \gamma/n$. In Sections 33.6 and 33.8 we write it as nE_d/γ and use the Laplace parameter γ . These are the same exponent, since

$$\frac{E_d}{g^2} = \frac{nE_d}{\gamma}.$$

Accordingly the normal Hessians H_d in Lemma 33.9 and Proposition 33.12 are Hessians of nE_d .

Constants. Constants in results about $\widehat{v}$ may depend on n, L, M, J , the parameter tuple Θ , the coupling bound γ_0 , and on fixed patches, charts, and test functions, all chosen before

γ ; this is the dependence described in item (6) of Definition 4.14. They do not depend on the depth d or on $\gamma \in (0, \gamma_0]$. All spectral gaps are proved at the fixed side n , and we prove no bound that is uniform in n . The general Laplace estimates of Section 33.4 state their own finite-dimensional dependence, as in item (1) of Definition 4.14.

Earlier work. Laplace's method in the directions normal to a manifold of critical points is rooted in Bott's theory of nondegenerate critical manifolds [31]; see Wong [155] for the asymptotic analysis of integrals. We use no specific theorem from these works. The estimates we need are uniform in the depth, the coupling, and an arbitrary tangential weight; they are proved in Section 33.4, and the singular tip of the cone of commuting triples is treated separately. The special role of the constant modes on a finite periodic torus in weak-coupling analysis is well known in the physics literature; see Lüscher [127] and the computation of the zero-momentum contribution to Wilson loops by Coste, González-Arroyo, Jurkiewicz, and Korthals Altes [62]. For the gradient flow on a four-dimensional periodic torus, Fodor, Holland, Kuti, Nógrádi, and Wong [91] separate constant and nonconstant modes and their flowed observables. These works do not treat the fixed-side three-dimensional normalization or the covariance of flowed based cycle logarithms. The decomposition of the covariance across scales in Proposition 33.12 is completion of squares for Gaussian measures; the work lies in matching it with the constrained coordinates of the renormalization group.

33.1 Preliminaries

Lie-algebra conventions. We use the conventions of Section 4. Let $\tau_a = -i\sigma_a/2$, where σ_a are the Pauli matrices, and write $X = x^a \tau_a \in \mathfrak{su}(2)$. Then $[\tau_a, \tau_b] = \epsilon_{abc} \tau_c$, $\langle X, Y \rangle = -2\operatorname{tr}(XY)$, and $|X| = |x|$, so that the bracket corresponds to the cross product in $\mathbb{R}^3$. We identify $\mathfrak{su}(2)^3$ with $\mathbb{R}^9$ by $X = (x_1, x_2, x_3)$ and use the Euclidean norm $|X|$. In exponential coordinates

$$dH(e^X) = \frac{1}{16\pi^2} \operatorname{sinc}^2(|X|/2) d^3X, \quad q(e^X) = 1 - \cos(|X|/2) = \frac{|X|^2}{8} + O(|X|^4),$$

by (4.2). The logarithm Log is the inverse of $\exp$ on $\{|X| < 2\pi\}$.

Based gauge transformations. Fix a root $x_* \in \Lambda_n$. A lattice gauge transformation k is *based* if $k_{x_*} = I$. The group of based gauge transformations acts freely on $\mathcal{U}_n$, and we call the quotient the based quotient. Constant gauge transformations are not divided out: they act on the based quotient, and on the holonomies around the three coordinate cycles based at x_* they act by simultaneous conjugation. In the proof of Theorem 33.8 we show, using a spanning tree, that the based quotient is diffeomorphic to $\operatorname{SU}(2)^{2n^3+1}$, of dimension $6n^3 + 3$, and that the based gauge transformations carry Haar measure of total mass one. A configuration V is flat if $V_p = I$ for every plaquette p .

Smooth charts. We use the following routine facts about local coordinates several times. We include the short proofs because the dependence of the constants matters below.

Lemma 33.1. 1. Let F be a smooth map from a neighborhood of $p \in \mathbb{R}^k$ into $\mathbb{R}^k$ such that $T = DF(p)$ is invertible. Then there exist a closed ball $\bar{B}$ about p and an open neighborhood W of $F(p)$ such that F maps an open subset of $\bar{B}$ diffeomorphically onto W . The radii of $\bar{B}$ and W and the bounds for every fixed derivative of the inverse depend only on $\|T^{-1}\|$ and on bounds for the derivatives of F on a fixed neighborhood of p .

2. Let $\mathcal{Q}$ be a smooth Riemannian manifold, for instance an open subset of $\mathbb{R}^k$ with the Euclidean metric or a compact Lie group with a bi-invariant metric, and let Exp be its exponential map. Let $\Sigma \subset \mathcal{Q}$ be a smooth embedded submanifold and let $P \subset \Sigma$ be compact. Then there exists $r_0 > 0$ such that the normal exponential map $(y, v) \mapsto \text{Exp}_y(v)$, defined for y in a compact neighborhood of P in Σ and $v \perp T_y \Sigma$ with $|v| < r_0$, is injective and is a local diffeomorphism with smooth positive Jacobian. Further, there exist finitely many open sets $W_l \Subset W_l^+ \subset \Sigma$ with $P \subset \bigcup_l W_l$ and smooth orthonormal frames $(v_{l,1}, \dots, v_{l,k'})$ of the normal bundle over W_l^+ such that all derivatives of fixed order of the maps $(y, v) \mapsto \text{Exp}_y(\sum_\alpha v_\alpha v_{l,\alpha}(y))$, $v \in \mathbb{R}^{k'}$, and of their inverses are bounded on the closures of $\{(y, v) : y \in W_l, |v| \leq r_0/2\}$. The transitions between these trivializations preserve y and are orthogonal in v , so integrals over the tube $\{\text{Exp}_y(v) : y \in P, |v| < r_0/2\}$ can be computed with a partition of unity depending only on y . In $\mathbb{R}^k$ with the Euclidean metric, $\text{Exp}_y(v) = y + v$.

Proof. For (1), we translate p to zero and write $T^{-1}(F(h) - F(0)) = h + r(h)$, where $Dr(0) = 0$. On a sufficiently small closed ball $\bar{B}$ we have $\|Dr\| \leq 1/2$. For $|z|$ less than half the radius of $\bar{B}$, the iterates $h_{k+1} = z - r(h_k)$, $h_0 = 0$, stay in $\bar{B}$, and their successive differences decrease by the factor $1/2$. Their limit is the unique solution h in $\bar{B}$ of $h + r(h) = z$. Differentiating this identity, we find that the inverse is smooth with $\|Dh\| \leq 2$, and repeated differentiation bounds every fixed derivative in terms of those of r .

For (2), we take a point of P , an orthonormal tangent frame of Σ near it, and fixed smooth vector fields of $\mathcal{Q}$ whose projections onto the normal spaces are linearly independent at this point. The Gram–Schmidt formulas applied to these projections give a smooth orthonormal normal frame, since their denominators stay positive near the point. Since the differential of Exp_y at 0 is the identity, the derivative of $(y, v) \mapsto \text{Exp}_y(\sum_\alpha v_\alpha v_{l,\alpha}(y))$ at $v = 0$ consists of the tangent and normal frames. It is invertible, with determinant one in orthonormal coordinates. By (1), applied in a chart of $\mathcal{Q}$, the map is a local diffeomorphism near $v = 0$, and its Jacobian is smooth and positive. Since P is compact, finitely many such charts cover a compact neighborhood P' of P in Σ , and we choose open sets $W_l \Subset W_l^+$ inside them, covering P , and a common normal radius. All maps extend smoothly past the closures of the smaller sets, and their derivatives of every fixed order are bounded there. It remains to prove injectivity for a common radius. If it failed

on $\{(y, v) : y \in P', |v| < 1/k\}$ for every k , there would be distinct points (y_k, v_k) and (y'_k, v'_k) of the normal bundle with $|v_k|, |v'_k| < 1/k$ and $\text{Exp}_{y_k}(v_k) = \text{Exp}_{y'_k}(v'_k)$. Passing to subsequences, $y_k \rightarrow y$ and $y'_k \rightarrow y'$ in P' , and the common image converges both to y and to y' . Then $y = y'$, and both preimages converge to $(y, 0)$, near which the map is a local diffeomorphism and hence injective; this is a contradiction. Thus, for a small r_0 , the base point y is a function on the tube, the transitions between the orthonormal frames act on v by orthogonal matrices depending on y , and a partition of unity in y handles overlaps. $\square$

33.2 The stopped measure at a fixed side

Fix $L > 0$, $n \in 4\mathbb{N}$, an odd integer $M \geq 3$, an integer $J \geq 1$, and the tuple Θ of construction parameters of Theorem 23.2. Let ρ , r , and E_* be the analytic radius, the radius of the small-field domains $B_r \subset D_r$, and the bound for the effective interactions in Theorem 14.8. For a depth $d = kJ$, $k \geq 0$, and $\gamma > 0$, we set $g^2 = \gamma/n$ and consider the stopped measure

$$d\hat{v}_{n,d,\gamma}(V) = \hat{Z}_{n,d,\gamma}^{-1} \mathbf{1}_{D_r}(V) \exp\left(-\frac{E_{d,\text{sat}}(V)}{g^2} - \bar{R}_{d,\gamma}(V)\right) dH_n(V), \quad \bar{R}_{d,\gamma} = R_{d,\gamma} - R_{d,\gamma}(I), \quad (33.1)$$

of Theorem 14.8, with the indicator of D_r and the Haar disintegration factors of the construction. Here $E_{d,\text{sat}}$ and the subscript sat in general denote the gauge-invariant extension from B_r to D_r , $R_{d,\gamma}$ is the effective interaction, $\bar{R}_{d,\gamma}$ is the centered effective interaction, and $\hat{Z}_{n,d,\gamma}$ is the partition function. The depths $d = kJ$ are those at which the recursion of Theorem 14.8 defines an effective interaction. Subtracting $R_{d,\gamma}(I)$ fixes the normalization constant. If Z^{raw} denotes the partition function before centering, that is, the integral with $R_{d,\gamma}$ in place of $\bar{R}_{d,\gamma}$, then $\hat{Z}_{n,d,\gamma} = e^{R_{d,\gamma}(I)} Z^{\text{raw}}$ (Theorem 33.10).

For $d = JK$ and $N = nB^K$, the measure $\hat{v}_{n,JK,\gamma}$ is the law of the stopping field V^{JK} under the conditioned Wilson measure $\nu_{N,n,\gamma}$ (Theorem 14.8). For a stopped measure at an intermediate side this identification fails, since the conditioned Wilson measure also imposes the remaining coarser restrictions (Definition 4.4). Our results concern the density (33.1) and make no identification of the latter kind.

We use the following properties. All constants and domains in them are independent of d and of $\gamma \in (0, \gamma_0]$.

- (P1) There exists $\kappa > 0$ such that $E_{d,\text{sat}} \geq \kappa E_0$ on D_r for every integer $d \geq 0$, whether or not an effective interaction is attached to d (Theorem 14.8; see also Theorem 11.2).
- (P2) The effective classical action satisfies $E_d \geq 0$ (Theorem 11.2). By Theorem 11.6, $E_d(c^k) = E_d(c)$ for every real periodic gauge transformation k for which both representatives lie in the common chart, and the saturated extension $E_{d,\text{sat}}$ is gauge

invariant on D_r . An arbitrary gauge transformation need not preserve the unsaturated chart. Translations, spatial inversion, and constant color rotations on their common charts have the exact invariances of Lemma 10.8. The action vanishes at constant commuting configurations satisfying the smallness condition of Lemma 33.7, as proved there. Its derivatives through order five are bounded uniformly in d on a fixed larger link chart (Theorem 10.11).

- (P3) The centered effective interaction satisfies $|\bar{R}_{d,\gamma}| \leq C_n$ on D_r (Theorem 14.8), and its first derivatives in fixed charts are bounded on the larger chart (Lemma 33.6).
- (P4) The coupling bound $\gamma_0 > 0$ satisfies $\sqrt{\gamma_0/n} \leq g_*$, where g_* is the coupling bound of Theorem 14.8, and, with $t_0 = \sqrt{\gamma_0/(Mn)}$ and $p(t) = A_w \log(1/t)$,

$$p(t) \geq p_0, \quad C_J t p(t) < \frac{1}{2}, \quad \frac{C_J (t p(t)^3 + t p(t))}{1 - C_J t p(t)} \leq \sigma_{\text{seq}} \quad (0 < t \leq t_0), \quad (33.2)$$

where p_0 , C_J , and σ_{seq} are the constants of Theorem 12.13 at the fixed support scale of the construction.

Since the largest coupling of an RG step with output side $m \geq n$ is $\sqrt{\gamma/(Mm)} \leq t_0$, the conditions (33.2) contain the coupling conditions of Theorem 12.13 for all these steps. Once n , M , J , and Θ are fixed, (P4) holds for all sufficiently small $\gamma_0 > 0$: the function p is decreasing, $t \log^a(1/t) \rightarrow 0$ as $t \rightarrow 0$ for $a = 1, 3$, and these two functions are increasing in t when $\log(1/t) > a$. Thus γ_0 is chosen after n , M , J , and Θ , and before γ , and no separate condition is imposed for an individual coupling or depth.

33.3 The cone of commuting triples

Since two elements of $\mathfrak{su}(2)$ commute if and only if they are parallel, the commuting triples in $\mathfrak{su}(2)^3 \simeq \mathbb{R}^9$ are the triples $(\vartheta_1 u, \vartheta_2 u, \vartheta_3 u)$ with $\vartheta \in \mathbb{R}^3$ and $u \in S^2$. If we regard $X \in \mathbb{R}^9$ as the real 3×3 matrix with columns x_1, x_2, x_3 , these are exactly the matrices of rank at most one. The following lemma collects the geometry of this cone and the behavior of the quartic Q near it. The distance to the set of matrices of rank at most one is computed from the singular values, as in the rank-one case of the theorem of Eckart and Young [80].

Lemma 33.2. *Set*

$$Q(X) = \frac{1}{2} \sum_{i < j} |x_i \times x_j|^2, \quad \mathcal{M} = \{(\vartheta_i u)_{i=1}^3 : \vartheta \in \mathbb{R}^3, u \in S^2\}.$$

Then the following statements hold.

1. The set $\mathcal{M} \setminus \{0\}$ is a smooth submanifold of $\mathbb{R}^9$ of dimension five. The map $(\boldsymbol{\vartheta}, u) \mapsto (\vartheta_i u)_i$ identifies exactly $(\boldsymbol{\vartheta}, u)$ with $(-\boldsymbol{\vartheta}, -u)$. At $y = (\vartheta_i u)_i$ with $\varrho = |\boldsymbol{\vartheta}| > 0$ and $\omega = \boldsymbol{\vartheta}/\varrho$, the normal space and the induced volume are

$$\mathcal{N}_y = \left\{ b = (b_1, b_2, b_3) : b_i \perp u, \sum_i \vartheta_i b_i = 0 \right\}, \quad \dim \mathcal{N}_y = 4,$$

$$d\text{vol}_{\mathcal{M}} = \frac{1}{2} \varrho^4 d\varrho d\Omega_\omega d\Omega_u.$$

Note that $|y| = \varrho$.

2. There exists an absolute constant $c \in (0, 1)$ such that $(y, b) \mapsto y + b$, for $y \in \mathcal{M} \setminus \{0\}$, $b \in \mathcal{N}_y$, and $|b| \leq c\varrho$, is injective and is a local diffeomorphism (a normal tube), with Euclidean Jacobian $1 + O(|b|/\varrho)$. The implicit constant is absolute; in particular it is uniform in the directions ω and u .
3. For $y \in \mathcal{M} \setminus \{0\}$ and $b \in \mathcal{N}_y$,

$$Q(y + b) = \frac{1}{2} \varrho^2 |b|^2 + \frac{1}{2} \sum_{i < j} |b_i \times b_j|^2,$$

and for every $X \in \mathbb{R}^9$,

$$Q(X) \geq \frac{1}{6} |X|^2 \text{dist}(X, \mathcal{M})^2.$$

4. Let f be a nonnegative C^5 function on a fixed ball about 0 that vanishes on $\mathcal{M}$ in that ball. Then $Df(0) = 0$, $D^2f(0) = 0$, and $D^3f(0) = 0$. On a smaller fixed ball,

$$\|D^2f(X)\| \leq C|X|^2, \quad \|D^3f(X)\| \leq C|X|,$$

and, for $y \in \mathcal{M} \setminus \{0\}$ and $b \in \mathcal{N}_y$ with $y + b$ in the smaller ball,

$$f(y + b) = \frac{1}{2} b^\top D_\perp^2 f(y) b + O(\varrho |b|^3 + |b|^4),$$

where $D_\perp^2 f(y)$ is the Hessian of f at y restricted to $\mathcal{N}_y$. If f_4 is the homogeneous quartic Taylor polynomial of f at 0 and $y = \varrho \hat{y}$ with $|\hat{y}| = 1$, then $D_\perp^2 f(\varrho \hat{y}) = \varrho^2 D_\perp^2 f_4(\hat{y}) + O(\varrho^3)$. If, in addition, $f \geq c_0 Q$ on the ball, then $D_\perp^2 f_4(\hat{y}) \geq c_0 I_4$. The constants depend only on the radii of the two balls, on $\|f\|_{C^5}$, and, in the last assertion, on c_0 .

The last hypothesis $f \geq c_0 Q$ cannot be omitted: the function Q^2 is nonnegative and vanishes on $\mathcal{M}$, but its normal Hessian vanishes on all of $\mathcal{M}$.

Proof. We begin with part (1). The tangent vectors at $y_i = \vartheta_i u \neq 0$ have the form

$$t_i = \eta_i u + \vartheta_i v, \quad \eta \in \mathbb{R}^3, \quad v \perp u.$$

These five parameters are independent, and orthogonality to every η and every v gives exactly the asserted normal space. In the parameters $(\boldsymbol{\vartheta}, u)$ the induced metric is

$$\sum_i |d(\vartheta_i u)|^2 = |d\boldsymbol{\vartheta}|^2 + \varrho^2 |du|^2.$$

Each nonzero matrix of rank one has exactly two preimages: a nonzero column determines the common line, and the two unit vectors on this line determine all three coefficients. We obtain the volume element $\varrho^2 d^3 \boldsymbol{\vartheta} d\Omega_u / 2$, which is the displayed formula after polar coordinates in $\boldsymbol{\vartheta}$.

For part (2), the tube is taken over the nonzero stratum, and its radius is proportional to $|y|$. We first restrict to $1/2 < |y| < 2$. By Lemma 33.1(2), finitely many coordinate patches of this compact piece of $\mathcal{M}$ carry smooth orthonormal normal frames with bounded first derivatives on smaller patches. In such a frame the derivative of $(y, b) \mapsto y + b$ at $b = 0$ is the orthogonal sum of the tangent and normal frames, and its volume determinant is one. At $b \neq 0$ its tangent columns differ by $O(|b|)$, and the determinant is $1 + O(|b|)$. We now check that the base point is unique. If $b \in \mathcal{N}_y$, the matrix $X = y + b$ satisfies

$$XX^\top = \varrho^2 uu^\top + bb^\top, \quad bb^\top u = 0,$$

since $\sum_i \vartheta_i b_i = 0$ and $b_i \perp u$. If $|b| < c\varrho$ with $c < 1$, the largest eigenvalue ϱ^2 of $XX^\top$ is simple, and its eigenline is exactly $\mathbb{R}u$. Projecting the columns of X onto this line recovers y , and the two choices of sign of u give the same matrix y . Together with the local invertibility of the derivative, this proves that the map is a normal tube. Since the map commutes with the dilations $(y, b) \mapsto (\lambda y, \lambda b)$, the bound on the unit stratum implies the Jacobian bound $1 + O(|b|/\varrho)$ with an absolute constant.

For part (3), let $b \in \mathcal{N}_y$ and expand

$$(\vartheta_i u + b_i) \times (\vartheta_j u + b_j) = u \times (\vartheta_i b_j - \vartheta_j b_i) + b_i \times b_j.$$

The first summand is perpendicular to u and the second is parallel to u , so they are orthogonal. Since $\vartheta_i b_j - \vartheta_j b_i \perp u$, we have $|u \times (\vartheta_i b_j - \vartheta_j b_i)| = |\vartheta_i b_j - \vartheta_j b_i|$, and the Lagrange identity gives

$$\sum_{i < j} |\vartheta_i b_j - \vartheta_j b_i|^2 = |\boldsymbol{\vartheta}|^2 |b|^2 - \left| \sum_i \vartheta_i b_i \right|^2 = \varrho^2 |b|^2.$$

This proves the formula for $Q(y + b)$. For the global bound, let $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$ be the singular values of X . Since $|x_i \times x_j|^2$ is the sum of the squares of the 2×2 minors in the columns i and j , the quantity $2Q(X)$ is the sum of the squares of all 2×2 minors of X . This sum is invariant under left and right multiplication by orthogonal matrices, and evaluating it on a diagonal matrix gives

$$2Q(X) = \sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2.$$

For a fixed unit vector u , the closest matrix with all columns parallel to u has columns $(u \cdot x_i)u$, and its squared distance to X is $|X|^2 - u^\top XX^\top u$. Diagonalizing $XX^\top$, we see that the minimum over u is $\sigma_2^2 + \sigma_3^2 = \text{dist}(X, \mathcal{M})^2$. Since $\sigma_1^2 \geq |X|^2/3$, we obtain

$$Q(X) \geq \frac{1}{2} \sigma_1^2 (\sigma_2^2 + \sigma_3^2) \geq \frac{1}{6} |X|^2 \text{dist}(X, \mathcal{M})^2.$$

For part (4), since the origin is a minimum of f , we have $Df(0) = 0$, and $D^2f(0)$ is positive semidefinite. Since $\mathcal{M}$ is a cone, the restriction of f to every line through a matrix of rank one vanishes near zero, and the quadratic form $D^2f(0)$ vanishes on every matrix of rank one. If a positive semidefinite form H satisfies $v^\top H v = 0$, then $Hv = 0$, as one sees by diagonalizing H . Since the matrices with a single nonzero entry have rank one and span $\mathbb{R}^9$, we conclude that $D^2f(0) = 0$. The cubic Taylor term vanishes as well, that is, $D^3f(0) = 0$: on each line through the origin it would otherwise take opposite signs on the two sides of zero, contradicting nonnegativity, and a cubic form that vanishes on every line is zero.

By Taylor's formula with integral remainder, we have, on a smaller concentric ball,

$$\|D^2f(X)\| \leq C|X|^2, \quad \|D^3f(X)\| \leq C|X|, \quad D^2f(X) = D^2f_4(X) + O(|X|^3).$$

For example, the last error is at most $\sup \|D^5f\| |X|^3/6$; the other constants use the bounds for the fourth and fifth derivatives on the larger ball. Since $f(y) = 0$ is a minimum at every $y \in \mathcal{M}$ in the ball, we have $Df(y) = 0$. Using Taylor's formula on the segment from y to $y + b$ and the bound for D^3f , we find a cubic remainder at most $C(\varrho + |b|)|b|^3$, which proves the normal expansion. The Hessian of the quartic f_4 is homogeneous of degree two, and $\mathcal{N}_{\varrho\hat{y}} = \mathcal{N}_{\hat{y}}$; this proves the expansion at $y = \varrho\hat{y}$. If $f \geq c_0Q$, the function $f - c_0Q$ is nonnegative and vanishes at y , and its Hessian at this point is positive semidefinite. By the exact normal formula in part (3), the normal Hessian of Q at y is $\varrho^2 I_4$, and we obtain

$$D_{\perp}^2 f(\varrho\hat{y}) \geq c_0 \varrho^2 I_4.$$

Dividing by ϱ^2 and letting $\varrho \downarrow 0$, we obtain $D_{\perp}^2 f_4(\hat{y}) \geq c_0 I_4$. This completes the proof. $\square$

33.4 Laplace asymptotics in normal coordinates

We now prove the Laplace estimates used in the normal directions of the regular flat configurations and in the nonconstant directions near the trivial connection. The main point is uniformity. The phase, the amplitude, and the tangential weight may depend on an arbitrary index, which in the applications records the depth and the coupling, and the tangential weight need not converge or have any regularity. We therefore prove the estimates for each fixed tangential point and then integrate them against the tangential marginal of the measure itself.

Lemma 33.3. *Let P be a compact patch of a smooth manifold with a smooth volume measure $d\text{vol}$, and let a Euclidean vector bundle of rank $d \geq 1$ over P be given with a finite atlas of smooth orthonormal trivializations. Fix $0 < c_0 \leq C_0 < \infty$, $0 < b_1 < b_0$, and a cutoff $\chi_y(b)$ with $0 \leq \chi \leq 1$, jointly Borel in every trivialization, equal to one for $|b| \leq b_1$ and supported in a compact subset of the tube $\{|b| < b_0\}$. Let ι range over an arbitrary index set and let*

$0 < \delta \leq \delta_0$. For each ι and δ let $V_\iota(y, b)$ and $a_{\iota, \delta}(y, b)$ be real functions on the tube, jointly Borel in every trivialization, and assume that the Hessian $L_\iota(y) = D_b^2 V_\iota(y, 0)$ exists and is a measurable function of y . Assume that on the support of χ

$$\begin{aligned} V_\iota(y, 0) = D_b V_\iota(y, 0) &= 0, & c_0 I &\leq L_\iota(y) \leq C_0 I, \\ |V_\iota(y, b) - \tfrac{1}{2} b^\top L_\iota(y) b| &\leq C_0 |b|^3, & V_\iota(y, b) &\geq c_0 |b|^2, \\ c_0 &\leq a_{\iota, \delta}(y, b) \leq C_0, & |a_{\iota, \delta}(y, b) - a_{\iota, \delta}(y, 0)| &\leq C_0 |b|. \end{aligned}$$

The amplitude $a_{\iota, \delta}$ includes the Jacobian of the tube coordinates, so that the reference measure is $d\text{vol}(y) db$, where db is Lebesgue measure in an orthonormal trivialization. Let $h_{\iota, \delta} \geq 0$ be any measurable function on P with $0 < \int_P h_{\iota, \delta} d\text{vol} < \infty$; no regularity in y and no convergence as ι or δ varies is assumed. Let

$$dP_{\iota, \delta} = Z_{\iota, \delta}^{-1} h_{\iota, \delta}(y) \chi_y(b) a_{\iota, \delta}(y, b) e^{-V_\iota(y, b)/\delta} d\text{vol}(y) db$$

be the normalized measure, and set

$$z = \delta^{-1/2} L_\iota(y)^{1/2} b, \quad \alpha_{\iota, \delta}(y) = \frac{a_{\iota, \delta}(y, 0)}{\sqrt{\det L_\iota(y)}}.$$

Let G_d be a standard Gaussian vector in $\mathbb{R}^d$ in each orthonormal trivialization. Let $(y, z) \mapsto P_y(z)$ be a measurable scalar function on the given vector bundle, with fiber restrictions that are polynomials of a fixed bounded degree and whose coefficients in the fixed finite atlas are bounded measurable functions of y . Its local polynomial expressions are required to transform with the fiber coordinates on overlaps, so that they define the same scalar function. Then for every such fiberwise polynomial and every bounded measurable function f on the base patch P ,

$$\begin{aligned} Z_{\iota, \delta} \mathbb{E}_{P_{\iota, \delta}}[f(y) P_y(z)] &= (2\pi\delta)^{d/2} \left[\int_P f h_{\iota, \delta} \alpha_{\iota, \delta} \mathbb{E} P_y(G_d) d\text{vol} + \mathcal{R}_{\iota, \delta}(f, P) \right], \\ |\mathcal{R}_{\iota, \delta}(f, P)| &\leq C_P \sqrt{\delta} \int_P |f| h_{\iota, \delta} d\text{vol}. \end{aligned} \tag{33.3}$$

The error is additive, also when $\mathbb{E} P_y(G_d) = 0$. For $y \in P$ let $P_{\iota, \delta}(\cdot \mid y)$ be the probability measure on the fiber with density proportional to $\chi_y(b) a_{\iota, \delta}(y, b) e^{-V_\iota(y, b)/\delta}$. It is defined for every y , also where $h_{\iota, \delta}(y) = 0$, and it is a version of the conditional law of b given y under $P_{\iota, \delta}$. Moreover, the following hold.

1. For every $y \in P$, $|\mathbb{E}_{P_{\iota, \delta}}[P_y(z) \mid y] - \mathbb{E} P_y(G_d)| \leq C_P \sqrt{\delta}$, and $\mathbb{E}_{P_{\iota, \delta}}[|z|^k \mid y] \leq C_k$ for every fixed real $k \geq 0$.
2. Let $d = 4$ and $q_\iota = b^\top L_\iota(y) b / 2$. Then for $j = 0, 1, 2$,

$$|\mathbb{E}_{P_{\iota, \delta}}[f(y) (q_\iota / \delta)^j] - m_j \mathbb{E}_{P_{\iota, \delta}} f(y)| \leq C_j \|f\|_\infty \sqrt{\delta}, \quad (m_0, m_1, m_2) = (1, 2, 6).$$

3. Let $d = 4$, let L_∞ be a measurable field of symmetric matrices with $c_0 I \leq L_\infty(y) \leq C_0 I$, and let $q_\infty = b^\top L_\infty(y) b / 2$. Then the estimate in (2) holds with q_∞ in place of q_ι , up to the additional error $C_j \|f\|_\infty \|L_\iota - L_\infty\|_\infty$.

The constants depend only on d , on b_0 , b_1 , and the atlas, on c_0 and C_0 , and on the degree and the coefficient bounds of P_y (respectively on k and j). They are independent of ι , δ , and δ_0 , and of $h_{\iota,\delta}$, its total mass, and its variation. In applications at a fixed side n , the geometric data may depend on n .

Corollary 33.4. Assume the hypotheses of Lemma 33.3. For $y \in P$ let $r_{\iota,\delta,y}$ be the Lebesgue density of $z = \delta^{-1/2} L_\iota(y)^{1/2} b$ under $P_{\iota,\delta}(\cdot | y)$, and let $\phi_d(z) = (2\pi)^{-d/2} e^{-|z|^2/2}$. For every real $s \geq 0$ there exists $C_s < \infty$ such that, for all $0 < \delta \leq \delta_0$,

$$\sup_{\iota,y} \int_{\mathbb{R}^d} (1 + |z|^s) |r_{\iota,\delta,y}(z) - \phi_d(z)| dz \leq C_s \sqrt{\delta}. \quad (33.4)$$

The density $r_{\iota,\delta,y}$ is defined by its normal integral at every y , also where the tangential weight vanishes, and assertions integrated against the tangential marginal of $P_{\iota,\delta}$ hold almost everywhere with respect to that marginal; no assumption on the behavior of this marginal as ι and δ vary is made. The constant C_s has the dependence described in Lemma 33.3, and in addition depends on s .

Suppose in addition that $V_\iota(y, b) \geq c|b|^2$ and $c \leq a_{\iota,\delta}(y, b) \leq C$ for all $|b| < b_0$, with $c, C > 0$ independent of ι , y , and δ . In the following whole-tube conclusions, the constants C_s may additionally depend on c and C ; they remain independent of ι , y , δ , δ_0 , and the tangential weight. Then there exists $c' > 0$, depending only on the data of Lemma 33.3 and on c and C , such that for $0 < \delta \leq 1$

$$\int_{|b| < b_0} (1 + |z|^s) (1 - \chi_y(b)) a_{\iota,\delta}(y, b) e^{-V_\iota(y,b)/\delta} db \leq C_s \delta^{d/2} e^{-c'/\delta}. \quad (33.5)$$

Let $r_{\iota,\delta,y}^{\text{tube}}$ be defined as $r_{\iota,\delta,y}$, but with χ_y replaced by the indicator of the tube $\{|b| < b_0\}$. Then $\int (1 + |z|^s) |r_{\iota,\delta,y}^{\text{tube}} - r_{\iota,\delta,y}| dz \leq C_s e^{-c'/\delta}$ for $0 < \delta \leq 1$, and (33.4) holds for $r_{\iota,\delta,y}^{\text{tube}}$ for all $0 < \delta \leq \delta_0$.

In the application of Proposition 33.12, the rank is $6n^3 - 2$, $\delta = \gamma$, the phase is $nE_{JK}(V(y, v))$, and $L = H_K^{\text{act}}(y) = D_v^2[nE_{JK}(V(y, v))]|_{v=0}$. In this application the additional hypotheses on the whole tube hold, and all constants are independent of K .

Both results follow from an estimate at a fixed tangential point, which we prove first.

Lemma 33.5. Assume the hypotheses of Lemma 33.3. Fix a trivialization of the atlas, a point y in its domain, and the parameters ι and δ . Write $L = L_\iota(y)$ and $a_0 = a_{\iota,\delta}(y, 0)$. Let

$$u_y(z) = \chi_y(b) a_{\iota,\delta}(y, b) e^{-V_\iota(y,b)/\delta}, \quad b = \delta^{1/2} L^{-1/2} z,$$

extended by zero outside the tube, and let $Z_y = \int u_y(z) dz$ and $r_y = u_y/Z_y$. Then $Z_y > 0$, and for every real $s \geq 0$ there exist $C_s < \infty$ and $c > 0$, with the dependence described in Lemma 33.3 and additionally on s , such that the following hold for all $0 < \delta \leq \delta_0$.

1. $\int (1 + |z|^s) |u_y(z) - a_0 e^{-|z|^2/2}| dz \leq C_s \sqrt{\delta}$.
2. If $\delta \leq 1$, then $Z_y \geq c$.
3. $\int (1 + |z|^s) |r_y(z) - \phi_d(z)| dz \leq C_s \sqrt{\delta}$.
4. $\int |z|^s r_y(z) dz \leq C_s$.

Proof. Let $\varepsilon = \sqrt{\delta}$. On the support of χ we set

$$A(z) = V_l(y, \varepsilon L^{-1/2} z) / \delta, \quad B(z) = |z|^2 / 2.$$

Since $|b| \leq \varepsilon |z| / \sqrt{c_0}$ and $|z|^2 = b^T L b / \delta$, the hypotheses imply, with constants determined by c_0 and C_0 , that

$$A(z) \geq c|z|^2, \quad B(z) \geq c|z|^2, \quad |A(z) - B(z)| \leq C\varepsilon|z|^3, \quad |a_{l,\delta}(y, b) - a_0| \leq C\varepsilon|z|.$$

The identity $e^{-A} - e^{-B} = -(A - B) \int_0^1 e^{-[(1-t)B + tA]} dt$ yields

$$|e^{-A} - e^{-B}| \leq |A - B| e^{-\min(A, B)}. \quad (33.6)$$

This bound does not require $|A - B|$ to be small. Here the lower bound $V_l \geq c_0|b|^2$ on the whole support of χ enters, since the cubic Taylor bound alone would not control the integrand far from $b = 0$. On the support of χ we obtain

$$|a_{l,\delta} e^{-A} - a_0 e^{-B}| \leq a_{l,\delta} |e^{-A} - e^{-B}| + |a_{l,\delta} - a_0| e^{-B} \leq C\varepsilon(|z| + |z|^3) e^{-c|z|^2}.$$

We have $u_y - a_0 e^{-B} = \chi(a_{l,\delta} e^{-A} - a_0 e^{-B}) - (1 - \chi)a_0 e^{-B}$, and the last term is supported where $|b| > b_1$, hence where $|z| \geq \sqrt{c_0} b_1 / \varepsilon$. For $0 < \delta \leq 1$ we have

$$\begin{aligned} \int (1 + |z|^s) |u_y - a_0 e^{-B}| dz &\leq C\varepsilon \int (1 + |z|^s) (|z| + |z|^3) e^{-c|z|^2} dz \\ &\quad + C_0 \int_{|z| \geq \sqrt{c_0} b_1 / \varepsilon} (1 + |z|^s) e^{-|z|^2/2} dz \leq C_s \varepsilon. \end{aligned}$$

In the last inequality, we used polar coordinates and $\int_0^\infty \varrho^m e^{-c\varrho^2} d\varrho < \infty$ for $m > -1$ for the first integral; for the second, we absorbed half of the exponential into the polynomial to obtain the bound $C_s e^{-c'/\delta}$, and we used that $u^{-1} e^{-c'/u^2}$ is bounded for $u \in (0, 1]$. This proves (1) for $\delta \leq 1$. If $\delta > 1$, then u_y is supported in $|z| \leq C_0^{1/2} b_0$ and $0 \leq u_y \leq C_0$, since $V_l \geq 0$ on the support of χ . Both $\int (1 + |z|^s) u_y dz$ and $a_0 \int (1 + |z|^s) e^{-|z|^2/2} dz$ are then bounded by a constant C_s , and $C_s \leq C_s \sqrt{\delta}$. This proves (1) in general. Note that no dependence on δ_0 enters.

For (2), let $0 < \delta \leq 1$ and set $\varrho_0 = \min(b_1/2, \sqrt{\delta})$. On the ball $|b| \leq \varrho_0$ we have $\chi = 1$, $a_{\iota, \delta} \geq c_0$, and $V_{\iota} \leq C_0|b|^2/2 + C_0|b|^3 \leq C_0(1/2 + b_0)\delta$. Changing variables, we obtain

$$Z_y = \delta^{-d/2}(\det L)^{1/2} \int \chi_y a_{\iota, \delta} e^{-V_{\iota}/\delta} db \geq \delta^{-d/2} c_0^{d/2} c_0 e^{-C_0(1/2+b_0)} \omega_d \varrho_0^d \geq c,$$

where ω_d is the volume of the unit ball and we used $\varrho_0^d \delta^{-d/2} \geq \min(b_1/2, 1)^d$. The same integral over $|b| \leq b_1/2$ shows that $Z_y > 0$ for every δ .

For (3) with $\delta \leq 1$, let $A_0 = a_0(2\pi)^{d/2}$, so that $a_0 e^{-|z|^2/2} = A_0 \phi_d(z)$. Then

$$r_y - \phi_d = \frac{u_y - a_0 e^{-|z|^2/2}}{Z_y} + a_0 e^{-|z|^2/2} \left(\frac{1}{Z_y} - \frac{1}{A_0} \right).$$

By (1) with $s = 0$ we have $|Z_y - A_0| \leq C\sqrt{\delta}$, and $Z_y \geq c$ by (2) and $A_0 \geq c_0(2\pi)^{d/2}$. Inserting these bounds and (1) into the last display, we obtain (3). For $\delta > 1$, we use that r_y and ϕ_d are probability densities and that r_y is supported in $|z| \leq C_0^{1/2}b_0$. Then the left side of (3) is at most $1 + (C_0^{1/2}b_0)^s + \int (1 + |z|^s) \phi_d \leq C_s \leq C_s \sqrt{\delta}$. Finally, (4) follows from (3) for $\delta \leq 1$, and from the bound on the support of r_y for $\delta > 1$. $\square$

Proof of Lemma 33.3. We first check that all densities and conditional laws are well defined. On the spectral interval $[c_0, C_0]$ the maps $L \mapsto L^{1/2}$ and $L \mapsto L^{-1/2}$ are continuous: their binomial series in $I - L/C_0$ converge uniformly because $\|I - L/C_0\| \leq 1 - c_0/C_0 < 1$, and all partial sums are polynomials in the matrix entries. Since the Hessian is measurable, so are z , $\det L_{\iota}$, and $\alpha_{\iota, \delta}$. Since V_{ι} , $a_{\iota, \delta}$, and χ are jointly Borel, every normal integrand below is jointly measurable. Integrals over the fiber of nonnegative integrands are measurable functions of y , by integration of nonnegative simple functions followed by monotone limits; signed polynomial integrands are treated through their positive and negative parts, once the bounds below establish absolute integrability. Thus Fubini's theorem applies to the weight $h_{\iota, \delta}$ without any continuity in y .

Next we prove the formula at a fixed tangential point. Fix a trivialization and y , and let u_y , Z_y , and r_y be as in Lemma 33.5. The change of variables $b = \delta^{1/2} L^{-1/2} z$ has Jacobian $\delta^{d/2}(\det L)^{-1/2}$, so with

$$I_y(P) = \int \chi_y(b) a_{\iota, \delta}(y, b) e^{-V_{\iota}(y, b)/\delta} P_y(z) db$$

we have $I_y(P) = \delta^{d/2}(\det L)^{-1/2} \int u_y P_y dz$. Since $|P_y(z)| \leq C_P(1 + |z|^{\deg P})$, part (1) of Lemma 33.5 implies that $|\int u_y P_y dz - a_0(2\pi)^{d/2} \mathbb{E} P_y(G_d)| \leq C_P \sqrt{\delta}$. Since $(\det L)^{-1/2} \leq c_0^{-d/2}$, we obtain, uniformly in y and ι ,

$$I_y(P) = (2\pi\delta)^{d/2} [\alpha_{\iota, \delta}(y) \mathbb{E} P_y(G_d) + \mathcal{E}_y(P)], \quad |\mathcal{E}_y(P)| \leq C_P \sqrt{\delta}. \quad (33.7)$$

We multiply (33.7) by $f(y) h_{\iota, \delta}(y)$ and integrate. By Fubini's theorem, the left side becomes $Z_{\iota, \delta} \mathbb{E}_{P_{\iota, \delta}} [f P_y(z)]$ on the domain of the trivialization, and the error is bounded by

$C_P\sqrt{\delta} \int |f| h_{l,\delta}$, also when the Gaussian mean vanishes. Orthogonal changes of the normal frame preserve db , the law of G_d , and the positive square root, and they act on z by orthogonal maps. The polynomial expressions transform by the same coordinate change, so $P_y(z)$ is unchanged as a scalar function. Hence the formula does not depend on the trivialization, and a finite nonnegative partition of unity on the base patch subordinate to the atlas proves (33.3). We obtain such a partition by choosing coordinate balls whose smaller concentric balls cover the patch, placing a smooth bump equal to one on each smaller ball, and dividing by the positive sum of the bumps; only the atlas enters its constants. No derivative or convergence of $h_{l,\delta}$ has been used.

For the conditional statements, the unnumbered positivity assertion of Lemma 33.5 and the change of variables above give $I_y(1) = \delta^{d/2}(\det L)^{-1/2}Z_y > 0$ for every y and every $0 < \delta \leq \delta_0$. The density $\chi_y a_{l,\delta} e^{-V_l/\delta} / I_y(1)$ of $P_{l,\delta}(\cdot | y)$ is jointly Borel, and its product with the tangential density $Z_{l,\delta}^{-1} h_{l,\delta}(y) I_y(1)$ is the joint density of $P_{l,\delta}$. Thus $P_{l,\delta}(\cdot | y)$ is a version of the conditional law, including on the set where $h_{l,\delta}$ vanishes. Since z has density r_y under this law, we have

$$|\mathbb{E}[P_y(z) | y] - \mathbb{E}P_y(G_d)| \leq C_P \int (1 + |z|^{\deg P}) |r_y - \phi_d| dz \leq C_P \sqrt{\delta}$$

by Lemma 33.5(3), and the moment bound in (1) is Lemma 33.5(4).

For (2), let $d = 4$. Then $q_l/\delta = |z|^2/2$ exactly. For independent one-dimensional standard Gaussians, integration by parts gives $\mathbb{E}G_i^2 = 1$ and $\mathbb{E}G_i^4 = 3$. This implies

$$\mathbb{E}(|G_4|^2/2) = 2, \quad \mathbb{E}(|G_4|^2/2)^2 = \frac{1}{4}(4 \cdot 3 + 12) = 6.$$

By (1), the conditional expectations of $(q_l/\delta)^j$ given y differ from m_j by at most $C_j\sqrt{\delta}$, uniformly in y . Integrating against the tangential marginal of $P_{l,\delta}$, multiplied by f , we obtain (2).

For (3), let $\eta = \|L_l - L_\infty\|_\infty$. Since $|b|^2 \leq \delta|z|^2/c_0$, we have pointwise

$$|q_l - q_\infty|/\delta \leq C\eta|z|^2, \quad |q_l|/\delta + |q_\infty|/\delta \leq C|z|^2,$$

and both constants depend only on c_0 and C_0 , because L_∞ obeys the same spectral bounds as L_l . The case $j = 0$ is trivial. For $j = 1$ the first bound and the bounded second moment give the additional error $C\|f\|_\infty\eta$. For $j = 2$ we write $x^2 - y^2 = (x - y)(x + y)$ and use the bounded fourth moment from (1). All constants have the stated dependence. This completes the proof. $\square$

Proof of Corollary 33.4. In a trivialization of the atlas, the density of z under $P_{l,\delta}(\cdot | y)$ is $r_{l,\delta,y} = r_y$ of Lemma 33.5, and (33.4) is part (3) of that lemma. Since orthogonal changes of the normal frame act on z by orthogonal maps and preserve ϕ_d and $|z|$, the left side of (33.4) does not depend on the trivialization. The density is jointly measurable by the

Borel hypotheses and integration over the fiber, as in the proof of Lemma 33.3. Since the estimate holds at every y , integrating it against the tangential marginal of $P_{i,\delta}$ requires no assumption on this marginal. We note that the constant in part (3) of Lemma 33.5 does not depend on δ_0 : for $\delta \leq 1$ the proof uses the lower bound $Z_y \geq c$, valid on all of $(0, 1]$, and for $\delta > 1$ it uses only that r_y is supported in a fixed ball.

We turn to the additional statements on the whole tube. Let $0 < \delta \leq 1$, and let Z^{tube} and Z^χ be the integrals of $a_{i,\delta} e^{-V_i/\delta}$ over $|b| < b_0$, without and with the factor χ_y . By the proof of Lemma 33.5(2) we have $Z^{\text{tube}} \geq Z^\chi \geq c\delta^{d/2}$. On the region where $\chi_y(b) \neq 1$ we have $|b| \geq b_1$, and the lower bound $V_i \geq c|b|^2$ on the whole tube bounds the integrand by $C(1 + |z|^s)e^{-c|b|^2/\delta}$, with $|z| \leq C|b|/\sqrt{\delta}$. Absorbing half of the exponential, we obtain

$$\begin{aligned} \int_{b_1 \leq |b| < b_0} (1 + C\delta^{-s/2}|b|^s) C e^{-c|b|^2/\delta} db &\leq C(1 + \delta^{-s/2}) e^{-cb_1^2/(2\delta)} \int e^{-c|b|^2/(2\delta)} db \\ &\leq C_s \delta^{d/2} e^{-c'/\delta}, \end{aligned}$$

which is (33.5). Let p^{tube} and p^χ be the two normalized fiber densities. Then

$$\int w |p^{\text{tube}} - p^\chi| \leq \frac{1}{Z^{\text{tube}}} \int w(1 - \chi_y) a_{i,\delta} e^{-V_i/\delta} + \frac{|Z^{\text{tube}} - Z^\chi|}{Z^{\text{tube}}} \int w p^\chi,$$

with $w = 1 + |z|^s$. By (33.5), the lower bound for Z^{tube} , and the bounded moments of p^χ from Lemma 33.5(4), the right side is at most $C_s e^{-c'/\delta}$. Since $e^{-c'/\delta} \leq C\sqrt{\delta}$ for $\delta \leq 1$, (33.4) holds for r^{tube} when $\delta \leq 1$. When $\delta > 1$, the density r^{tube} is supported in $|z| \leq C_0^{1/2} b_0$, and the argument for part (3) of Lemma 33.5 with $\delta > 1$ applies.

The final assertion concerns the stopped measures near a compact patch of regular flat configurations. We verify it at the end of Section 33.8, after constructing the normal coordinates, by checking the phase and amplitude hypotheses and applying the general estimates proved above. This verification does not use the deferred assertion. $\square$

33.5 Based coordinates near the trivial connection

In this section we construct coordinates on the based quotient near the class of the identity, separate the nine constant directions from the $6n^3 - 6$ nonconstant directions transverse to the gauge orbits, and integrate out the latter by Lemma 33.3. We first record the derivative bounds for the effective interaction and the domain conditions for small commuting configurations that the integration requires.

Lemma 33.6. *Let $R = R_{d,\gamma}$ be the effective interaction in (33.1), at a depth $d = kJ$ and with $0 < \gamma \leq \gamma_0$, and let $\bar{R} = R - R(I)$. Fix $r < r_1 < \min(\rho, \pi)$ and put $\eta = 1 - r_1/\rho$. There exist constants C_j , independent of d and γ , such that in link logarithm coordinates $\xi = (\xi_e)$ with $\max_e |\xi_e| \leq r_1$,*

$$\|D_\xi^j \bar{R}(e^\xi)\|_{\text{root}} \leq C_j E_* \rho^{2-j} \eta^{-j}, \quad j \geq 1, \quad (33.8)$$

where $\|\cdot\|_{\text{root}}$ is the weighted absolute coefficient norm of Definition 4.6 with one index fixed, followed by the supremum over the fixed index. In particular, let $K_1 = C_1 E_* \rho \eta^{-1}$. Then at every point of the coordinates $V(X, s) = e^{X/n+s}$ of Theorem 33.8 whose link logarithms lie in B_{r_1} ,

$$|D_X \bar{R}(e^{X/n+s})| \leq 3n^2 K_1, \quad |D_s \bar{R}(e^{X/n+s})| \leq 3n^{3/2} K_1, \quad (33.9)$$

where both norms are Euclidean and the derivative in s is taken in the subspace S_n .

Further, let $\mathcal{K} \subset D_r$ be compact. There exist finitely many based gauge transformations and an open cover of $\mathcal{K}$ such that each member of the cover, together with a larger neighborhood, is mapped into B_{r_1} by one of these gauge transformations. Let $W \subseteq W^+$ be open sets of parameters and let $\Psi : W^+ \rightarrow \mathcal{U}_n$ be smooth, extend smoothly to a neighborhood of $\bar{W}$, and satisfy $\Psi(\bar{W}) \subset \mathcal{K}$. Then for every $j \geq 1$,

$$\|D^j(\bar{R}_{\text{sat}} \circ \Psi)\|_{L^\infty(W)} \leq C_{n,j,\mathcal{K},\Psi,r_1,\rho,\eta} E_*, \quad (33.10)$$

where the constant depends on the suprema over $\bar{W}$ of the derivatives of Ψ through order j , which are finite by smooth extension and compactness, but not on d or γ . The gauge transformations, the subcharts, and the constants are chosen independently of d and γ . For charts of the based quotient, Ψ denotes a fixed smooth lift to configurations. These bounds concern only the effective interactions $R_{d,\gamma}$, $d = kJ$, of the stopped densities (33.1).

Proof. By Theorem 14.8, the effective interaction of (33.1) satisfies $\mathcal{B}_{\rho,a_{\text{sp}}}(R) \leq E_*$ and $DR(I) = 0$, and its local expressions in different gauges agree on the overlaps of their domains in the common outer ball. That theorem also contains the bounds (33.8) and their transfer to fixed charts. We recall the argument and verify the conditions on the charts in the form used below. The constant $R(I)$ does not enter derivatives of positive order. Set

$$A_k = \frac{\rho^{k-2}}{k!} \|D^k R(0)\|_{a_{\text{sp}},\text{root}}, \quad \sum_{k \geq 2} A_k \leq E_*, \quad q = r_1/\rho = 1 - \eta.$$

We fix one index of a differentiated coefficient, sum over the contracted indices, and use $|\xi_e| \leq r_1$. The spatial weights are at least one, and deleting contracted indices does not increase the length of a shortest connecting network of the remaining indices. By the absolute Taylor expansion, we have

$$\|D^j R(e^{\xi})\|_{\text{root}} \leq C_j \rho^{2-j} \sum_{k \geq \max(2,j)} A_k \frac{k!}{(k-j)!} q^{k-j},$$

where the fixed conventions for color coordinates contribute only the factor C_j . For $k \geq j$ the binomial identity implies

$$\binom{k}{j} \eta^j q^{k-j} \leq \sum_{l=0}^k \binom{k}{l} \eta^l q^{k-l} = 1,$$

so that $k!q^{k-j}/(k-j)! \leq j!\eta^{-j}$. This summable majorant also justifies differentiating the series term by term in the interior, and it proves (33.8).

There are $9n^3$ scalar link and color coordinates, and each first partial derivative of $\bar{R}$ is at most K_1 , by (33.8) with $j = 1$. Then the Euclidean gradient is at most $3n^{3/2}K_1$, and its restriction to the subspace $\mathcal{S}_n$ has no larger norm. In the coordinates (X, s) we have $\xi_i(x) = X_i/n + s_i(x)$. For each of the nine components of X , the chain rule expresses the derivative as a sum of n^3 partial derivatives with coefficient $1/n$, which is at most n^2K_1 in absolute value. The Euclidean norm of the gradient in X is then at most $3n^2K_1$. This proves (33.9).

For the transfer, let $U_0 \in \mathcal{K}$ and choose a lattice gauge transformation k_0 with $U_0^{k_0} \in B_r$. It can be taken based: replacing $k_0(x)$ by $k_0(x_*)^{-1}k_0(x)$ conjugates the transformed configuration by one constant matrix, which preserves every link logarithm norm. We keep this gauge transformation fixed. By continuity there exist neighborhoods $O_0 \subseteq O_0^+$ of U_0 , with closures inside an ambient smooth coordinate chart, whose images under k_0 lie in B_{r_1} even on $\overline{O_0^+}$. To find them, we take an open neighborhood with image in B_{r_1} and then two smaller coordinate balls with closures inside it. Finitely many of the sets O_0 cover $\mathcal{K}$. They depend on $\mathcal{K}$ and on $r < r_1$, but not on the effective interaction. The action of a fixed gauge transformation is smooth on all of $\mathcal{U}_n$ and an isometry for the product metric; in particular, it extends past each of these closures.

By the agreement of the local expressions, we have, on the larger neighborhood O_l^+ of a member of this cover,

$$\bar{R}_{\text{sat}}(U) = R(e^{\xi^{(l)}(U)}) - R(I), \quad \xi^{(l)}(U)_e = \text{Log}((U^{k_l})_e).$$

Since $r_1 < \pi$, the logarithm and all its derivatives of fixed order are bounded on $\{|\text{Log}| \leq r_1\}$. Indeed, the transverse eigenvalue of the differential of the exponential map has modulus $2 \sin(|\xi|/2)/|\xi|$, which is bounded away from zero on this closed ball. Repeated differentiation of the inverse function identity then bounds every higher derivative. We refine the parameter subcharts so that the closure of each is contained in $W^+ \cap \Psi^{-1}(O_l^+)$ for one l ; this is possible because the sets O_l cover $\Psi(\bar{W})$ and $\bar{W}$ is compact. The maps $\xi^{(l)} \circ \Psi$ then extend smoothly past these closures, and for every fixed j their derivatives through order j are bounded there. This verifies, at each use, the condition on compact subcharts in Theorem 14.8. By the chain rule, the derivative of order j of $\bar{R}_{\text{sat}} \circ \Psi$ is a finite sum of contractions of $D^a R$, $1 \leq a \leq j$, with derivatives of $\xi^{(l)} \circ \Psi$. The bound (33.8) and the conversion from the norm $\|\cdot\|_{\text{root}}$ to Euclidean norms at the fixed side n bound every term. Since the local expressions agree on overlaps, these bounds concern one smooth function $\bar{R}_{\text{sat}}$ on a neighborhood of $\mathcal{K}$. This proves (33.10). We note that no derivative of a gauge transformation chosen as a function of U was taken. $\square$

The next lemma is a local version of Lemma 14.9 for small constant commuting configurations. Unlike that result, it requires no condition on the side m . It is used for

the chart of Theorem 33.8 and for the lower bound in Theorem 33.10.

Lemma 33.7. *There exists $\epsilon_{\text{loc}} > 0$, depending only on the fixed domains of the classical background fields, of the local constraint coordinates and Haar disintegration densities, and of the based gauge transformations used in Theorem 12.13, and in particular independent of the preceding depth and of γ , with the following property. Let $m \geq 4$, let $H_1, H_2, H_3 \in \mathfrak{su}(2)$ commute, and suppose that $\epsilon = \max_i |H_i|/m < \epsilon_{\text{loc}}$. Let c be the configuration on Λ_m with $c_e = e^{H_i/m}$ on every link e in direction i . Then c and its constant commuting refinements with links $\exp(H_i/(mM^\ell))$ on Λ_{mM^ℓ} , $\ell \geq 1$, together with their representatives in the block axial gauge, lie in the interior of these domains for one RG step, with a margin independent of m , of the depth, and of γ . For every depth $d \geq 0$ we have $E_d(c) = 0$. In the free logarithmic coordinates relative to the coarse-field lift $U_0(c)$, the refinement has $z = 0$; this point is the conditional constrained minimizer $z_d^*(c)$, and the corresponding normalized fluctuation variable is $w = 0$. The lemma imposes no condition on the other flat configurations on Λ_m .*

Proof. We check the domain conditions directly, without using a neighborhood of the small-field event. We use the identities for commuting roots in Lemma 14.9, but not its condition on the minimum side. For one averaging step, let

$$a_i = e^{H_i/(Mm)}, \quad c_i = a_i^M,$$

where a denotes the constant configuration on Λ_{Mm} with these link variables. In the block averaging map (4.3), every translated path with endpoint connectors $F_{c,x}$ has the same transport a_i^M as the reference path C_c , since all factors commute and the paths have the same displacement. Hence the logarithms of all based loop holonomies vanish, $T_M a = c$, and no exceptional value of T_M occurs.

We write $M = 2R_M + 1$. In a centered block with center v , the based gauge transformation to the block axial gauge at $x = v + r$ is $k(x) = \prod_j a_j^{r_j}$, where $|r_j| \leq R_M$. By commutativity, we have

$$|\text{Log } k(x)| \leq \frac{3R_M}{M}\epsilon, \quad \max_e |\text{Log } a_e| \leq \epsilon/M, \quad \max_e |\text{Log } U_0(c)_e| \leq \epsilon. \quad (33.11)$$

A transformed link in the interior of a block is the identity. On a link between two blocks in direction i , the offsets r_i at its endpoints differ by $1 - M$, and the transformed link is $a_i^M = c_i$. Thus the configuration in the block axial gauge is exactly the coarse-field lift $U_0(c)$ of Section 4, and all free logarithmic coordinates and the logarithms on the eliminated edges e_c^* vanish. Since the derivative of the block averaging constraint with respect to the eliminated edge is $M^{-2}I_3$ there, the point lies in the domain of the local constraint coordinates, with positive Jacobian. The Haar factors are positive on the small link ball.

We now choose the margin. Let $b_* > 0$ be smaller than r , than the radii r_* and r_{sym} and the constant κ_{sym} of Theorem 10.11 and Lemma 10.8, than the radius r_{lift} of the C^1 ball

of fine potentials in Lemma 10.8, and than the fixed radii of the coarse-field, constraint, Haar disintegration, and gauge domains of one RG step in Theorem 12.13. These are the radii used in the choice of r_{flat} in Lemma 14.9. There are only finitely many types of words in the construction of one RG step. Let $C_{\text{geo},J} \geq 1 + 3R_M/M$ be a bound for their lengths and for the sums of the offsets in their based gauge transformations, measured in units of the output lattice of the RG step. This bound does not depend on the side, since the words are translates of the words in one fixed block, and there are only J successive block sizes. Finally, we require

$$4C_{\text{geo},J}\epsilon_{\text{loc}} < b_*.$$

The logarithm of a product of commuting small factors is bounded by its word length times the largest logarithm of a factor. Thus (33.11) and this choice verify every intermediate condition on coarse fields, tree gauges, and gauge transformations, with a fixed margin. At a finer side mM^l the link bound is ϵM^{-l} , and none of these inequalities becomes worse with the number of averaging steps. The gauge identity of Lemma 10.8 applies, because both representatives lie in its ball of radius r_{sym} and the logarithms of the gauge transformations are below κ_{sym} .

We prove that the effective classical action vanishes by induction on d , simultaneously for all sides $m \geq 4$ and all constant commuting configurations as in the statement. The comparison below concerns real configurations and real free coordinates. No minimum and no positive Hessian is asserted for complex coarse fields. The continuation in a complex parameter below concerns only the analytic identity for the minimizer, after it has been proved near real parameter values. For $d = 0$, the Wilson action vanishes at the flat configuration c . Suppose that the claim holds at depth d . The refinement a has side Mm and satisfies the smaller bound ϵ/M . By the induction hypothesis, we have $E_d(a) = 0$, and by the gauge invariance just checked, $E_d(U_0(c)) = E_d(a) = 0$. By the definition of the effective classical action in Section 4, for the actions and minimizers of Theorem 12.13, we have

$$E_{d+1}(c) = MF_d(c, z_d^*(c)) = M \min_z F_d(c, z), \quad F_d(c, z) = E_d(b(c, z)),$$

where the minimum is over the domain of the classical background field. The free point $z = 0$ lies in this domain by the margins verified above; this does not presuppose that $z = 0$ lies in a fluctuation cutoff domain centered at $z_d^*(c)$. Since $F_d(c, 0) = 0$, the minimum is at most zero, and nonnegativity (Theorem 11.2) implies that $E_{d+1}(c) = 0$. This completes the induction, and it also shows that $E_d(U_0(c)) = 0$ at every depth. In other words, the iteration uses exact commuting roots, whose potential in units of the coarse lattice is H_i/m at every side and whose first spatial differences vanish; no loss of domain accumulates.

It remains to identify $z = 0$ with the conditional constrained minimizer. In the coordinates of Theorem 12.13, z_d^* is the analytic branch of constrained minimizers through

$z_d^*(I) = 0$. Since the point $z = 0$ attains the minimum of $F_d(c, \cdot)$, we have $D_z F_d(c, 0) = 0$. At $c = I$ the constrained Hessian $K_d(I)$ is positive. By Lemma 33.1(1), applied to $D_z F_d$, the critical point near 0 is then locally unique for c near I . For each fixed side and depth, we find $z_d^*(e^{tH/m}) = 0$ for real t near zero. The minimizer is holomorphic on the common domain of coarse fields of Theorem 12.13, and the factor four in the choice of ϵ_{loc} keeps $e^{tH/m}$ in this domain for complex $|t| < 2$. Each component of $t \mapsto z_d^*(e^{tH/m})$ is therefore holomorphic on $|t| < 2$ and vanishes on an interval around zero. Its Taylor coefficients at zero vanish, and continuation through overlapping discs gives $z_d^*(e^{tH/m}) = 0$ for $|t| < 2$, in particular at $t = 1$. The radius of local uniqueness at the origin need not be uniform; the common domain and the margins are. Since the normalization $z - z_d^*(c) = g_d K_d^{-1/2} w$ is invertible, $z = z_d^*(c)$ corresponds to $w = 0$, which lies in the interior of every fluctuation cutoff domain. By Theorem 15.13, the reconstruction maps $\widehat{\mathcal{R}}$, whose gauge transformations are based at the coarse vertices, preserve these properties. This completes the proof. $\square$

Theorem 33.8. *Consider the stopped measures of Section 33.2, with $L, n \in 4\mathbb{N}$, M, J, Θ , and γ_0 fixed before γ , and let $g = \sqrt{\gamma/n}$. Let d_0 denote the forward lattice gradient on Λ_n (not a depth) and d_0^* its adjoint, the discrete divergence, both acting componentwise on $\mathfrak{su}(2)$ -valued functions and link arrays, and let*

$$\mathcal{S}_n = \left\{ s = (s_i(x)) \in \mathfrak{su}(2)^{\mathcal{E}_n} : d_0^* s = 0, \sum_x s_i(x) = 0 \ (i = 1, 2, 3) \right\},$$

and $d_n = \dim \mathcal{S}_n = 6n^3 - 6$. There exist $\rho_n, s_n, c_n, C_n > 0$ such that the following hold.

1. The based quotient of $\mathcal{U}_n$ is a compact manifold of dimension $6n^3 + 3$, and the based gauge transformations carry Haar measure of total mass one. The map $(X, s) \mapsto V(X, s) = e^{X/n+s}$, which assigns $\exp(X_i/n + s_i(x))$ to the link from x to $x + a_n e_i$, defined for $X \in \mathbb{R}^9$ with $|X| < \rho_n$ and $s \in \mathcal{S}_n$ with $|s| < s_n$, induces coordinates on a neighborhood of the class of the identity in the based quotient. In these coordinates the image of H_n is $\int J_n(X, s) dX ds$, where J_n is smooth and strictly positive. The configurations $V(X, s)$ lie in $B_r \subset D_r$. Simultaneous conjugation at the root is not divided out.
2. Let $F_d(X, s) = E_d(e^{X/n+s})$, $f_d(X) = F_d(X, 0)$, and $A_d(X) = D_s^2 F_d(X, 0)$. For every integer $d \geq 0$,

$$\begin{aligned} D_s F_d(X, 0) &= 0, & F_d(X, s) &= f_d(X) + \frac{1}{2} \langle s, A_d(X) s \rangle + O_n(|s|^3), \\ c_n I &\leq A_d(X) \leq C_n I, & F_d(X, s) &\geq f_d(X) + c_n |s|^2. \end{aligned}$$

3. For $d = kJ$ and $0 < \gamma \leq \gamma_0$, write $\bar{R}_{d,\gamma}(X, s) = \bar{R}_{d,\gamma}(V(X, s))$. Then, uniformly in X, d , and γ ,

$$\int_{|s| < s_n} e^{-F_d(X, s)/g^2 - \bar{R}_{d,\gamma}(X, s)} J_n(X, s) ds = (2\pi)^{d_n/2} g^{d_n} e^{-f_d(X)/g^2} a_{d,\gamma}(X) [1 + O_n(g)],$$

$$a_{d,\gamma}(X) = \frac{e^{-\bar{R}_{d,\gamma}(X,0)} J_n(X,0)}{\sqrt{\det A_d(X)}}.$$

The amplitude $a_{d,\gamma}$ has positive upper and lower bounds and a C^1 bound, uniformly in d and γ .

The constants may depend on $n, L, M, J, \Theta, \gamma_0$, and on the larger chart and the bounds in (P2) and (P3), but not on d , on the total number of averaging steps, or on $\gamma \in (0, \gamma_0]$.

No positive Hessian is asserted in the nine constant directions X : at the identity the quadratic part of E_0 vanishes on constant configurations.

Proof. The proof has five steps. We construct the based quotient and the coordinates (X, s) , fix the radii, prove the stationarity in s and the gap at the fixed side n , and then apply Lemma 33.3 in the variable s at each fixed X .

Step 1: inputs. All bounds in this proof are at the fixed side n . The lower bound (P1) holds at every integer $d \geq 0$, whether or not an effective interaction is attached to d . In the statements involving $\bar{R}_{d,\gamma}$, we have $d = kJ$, and $R_{d,\gamma}$ is the effective interaction of the density (33.1), with the value bound and the analytic norm of that density. Its first derivatives in the chart are obtained from Lemma 33.6, on the relatively compact subcharts chosen in Step 3 and with the identity gauge transformation, which extends past their closures. They do not follow from the value bound. The derivative bounds for the classical actions in fixed charts come from Theorem 10.11: restricted to the finitely many coordinates at side n , its coefficient bounds give bounds for ordinary derivatives, with constants depending on n . The exact translation invariance of E_d is Lemma 10.8. The vanishing of E_d and the domain conditions for small constant commuting configurations are proved in Lemma 33.7, with no condition on the minimum side. Finally, (P4) allows us to apply Theorems 12.13 and 14.8 for every $0 < \gamma \leq \gamma_0$.

Step 2: the based quotient. We fix a periodic spanning tree of the lattice graph Λ_n . There exists a unique based gauge transformation that makes every tree link the identity. It is obtained successively from the root, by prescribing its value at a new vertex from the tree link leading to that vertex. The remaining links are $3n^3 - (n^3 - 1) = 2n^3 + 1$ independent non-tree links. This change of variables has an exact interpretation for the measure. We condition on the preceding tree variables and use a left or right translation of Haar measure at the next tree link. This replaces that link by the value of the gauge transformation at the new vertex, with Haar measure. At each non-tree link the two endpoint factors are determined by the tree variables, and its change of variables is again a Haar translation. By Fubini's theorem, the image of H_n is the product of Haar probability measures in the $n^3 - 1$ values of the based gauge transformation and in the $2n^3 + 1$ non-tree link variables. Integrating out the gauge variables gives one. In

particular the based quotient is the compact product $\mathrm{SU}(2)^{2n^3+1}$, of dimension $6n^3 + 3$, and no conjugation at the root has been divided out from the non-tree link variables.

Step 3: the coordinates and the radii. The gradient is injective on $\mathfrak{su}(2)$ -valued vertex functions that vanish at the root: if the gradient of such a function vanishes, the function is constant on the connected graph and hence zero. The image of these gradients has dimension $3(n^3 - 1)$, and its orthogonal complement in the $9n^3$ -dimensional space of link arrays is $\ker d_0^*$, of dimension $6n^3 + 3$. The nine arrays that are constant in position lie in this complement, and their orthogonal complement within it is $\mathcal{S}_n$, of dimension $6n^3 - 6$. The orthogonal decomposition can be constructed explicitly. For an array A , the equation $d_0^* d_0 \xi = d_0^* A$ has a unique solution ξ with mean zero, by diagonalization of this positive matrix on mean-zero vertex functions. Subtracting $d_0 \xi$ and then the spatial means of the three components, we obtain the component in $\mathcal{S}_n$.

We consider the map that sends (X, s) to $e^{X/n+s}$ and then to the non-tree link variables of its representative in the tree gauge. At $(0, 0)$ its derivative is the projection along gradients onto the non-tree link coordinates. Indeed, the linearized tree gauge fixing sends a link logarithm array A to $A - d_0 \xi$, where ξ vanishes at the root and $A - d_0 \xi$ vanishes on the tree, and then reads off the non-tree links. The kernel of the derivative is the image of d_0 on functions vanishing at the root. The derivative is therefore injective on the complement $\mathbb{R}^9 \oplus \mathcal{S}_n$ constructed above, and invertible there, since the dimensions agree. By Lemma 33.1(1), the map is a diffeomorphism near $(0, 0)$. Pulling back the product of Haar measures on the non-tree links, whose density in exponential coordinates is smooth and positive near the identity, we obtain a smooth strictly positive density $J_n(X, s)$. Its bounds and the bounds for its derivatives of fixed order on smaller closed coordinate balls depend only on n and on these balls.

We now fix the radii, independently of d and γ . We shrink the balls so that $\max_{x,i} |X_i/n + s_i(x)| < r/2$, and choose slightly larger balls with the same property. We also require $2\rho_n/n < \epsilon_{\mathrm{loc}}$, with ϵ_{loc} from Lemma 33.7, and put the larger balls inside the charts of radii r_* and r_{sym} of Theorem 10.11 and Lemma 10.8. More precisely, we choose the product $\mathcal{C}$ of the two coordinate balls in the statement and a larger product $\mathcal{C}^+$ with $\overline{\mathcal{C}} \subset \mathcal{C}^+$, such that $\overline{\mathcal{C}^+}$ lies in the chart just constructed and satisfies all these strict bounds on link logarithms. These finitely many conditions are compatible with positive radii. The map $(X, s) \mapsto e^{X/n+s}$ and the quotient chart extend smoothly to a neighborhood of $\overline{\mathcal{C}^+}$, and the gauge transformation used here is the identity. Every derivative of fixed order of the chart is therefore bounded on this closure, with constants depending only on n . This verifies the condition on compact subcharts in Theorem 14.8 and Lemma 33.6, both for the integration in s and for the derivative of the amplitude in X . The configurations $V(X, s)$ lie in $B_r \subset D_r$, with link logarithms in the ball B_{r_1} of Lemma 33.6, for a fixed $r_1 \in (r, \min(\rho, \pi))$; in particular $\eta = 1 - r_1/\rho$ is fixed. Every commuting point X used here satisfies $\max_i |X_i|/n < \epsilon_{\mathrm{loc}}$, so E_d vanishes there at every

depth.

Step 4: stationarity and the gap. Fix a constant array of logarithms $A_i(x) = X_i/n$. Translation by any lattice vector fixes this array. Differentiating the exact translation invariance of E_d , we find that the gradient $DE_d(e^A)$ in link logarithm coordinates is constant in position for each direction and color. Its pairing with every array of spatial mean zero vanishes, and

$$D_s F_d(X, 0) = 0 \quad (33.12)$$

for every d and every X in the chart. This argument concerns E_d only, and it does not set any derivative of the effective interaction or of the Haar density to zero.

We now prove the gap at the fixed side. The logarithm of a plaquette at the identity has the expansion $d_1 A + O(|A|^2)$ in link logarithms, where

$$(d_1 A)_{ij}(x) = A_j(x + e_i) - A_j(x) - A_i(x + e_j) + A_i(x)$$

is the unscaled lattice curl. Since $4q(e^Y) = |Y|^2/2 + O(|Y|^4)$, it follows that

$$E_0(e^A) = \frac{1}{2} \sum_{x, i < j} |A_j(x + e_i) - A_j(x) - A_i(x + e_j) + A_i(x)|^2 + O_n(|A|^3),$$

where e_i denotes one lattice step. At the Fourier mode $m \in \mathbb{Z}^3/n\mathbb{Z}^3$, let $\eta_i(m) = e^{2\pi i m_i/n} - 1$. For each color,

$$\sum_{i < j} |\eta_i a_j - \eta_j a_i|^2 = |\eta|^2 |a|^2 - |\eta^* a|^2.$$

Since the zero mode is absent on $\mathcal{S}_n$ and $\eta^* a = 0$ there, the Hessian of E_0 at the identity is at least $\lambda_n I$ on $\mathcal{S}_n$, where $\lambda_n = 4 \sin^2(\pi/n) > 0$. All inequalities between functions and Hessians here concern real configurations and real tangent vectors. Since $E_d - \kappa E_0 \geq 0$ near the identity by (P1) and vanishes at the identity, its Hessian at this point is positive semidefinite. This implies

$$D_s^2 F_d(0, 0) \geq \kappa \lambda_n I.$$

By the uniform bound on third derivatives in (P2), we have

$$\|D_s^2 F_d(X, s) - D_s^2 F_d(0, 0)\| \leq C_n(|X| + |s|).$$

We choose the radii so that the right side is at most $\kappa \lambda_n/2$; the upper derivative bounds in (P2) give a uniform upper bound. Then $A_d(X) = D_s^2 F_d(X, 0) \geq \kappa \lambda_n I/2$, and Taylor's formula with integral remainder and (33.12) give

$$F_d(X, s) = f_d(X) + \frac{1}{2} \langle s, A_d(X) s \rangle + O(C_n |s|^3), \quad F_d(X, s) - f_d(X) \geq \frac{1}{4} \kappa \lambda_n |s|^2.$$

This proves part (2), after decreasing c_n .

Step 5: integration in s and the amplitude. On the larger s -ball we apply Lemma 33.3, at each fixed X , with

$$V(X, s) = F_d(X, s) - f_d(X), \quad \delta = g^2, \quad a(X, s) = e^{-\bar{R}_{d,\gamma}(X,s)} J_n(X, s),$$

the rank d_n , and the Hessian $L = A_d(X)$. All these functions and the Hessian field are smooth in (X, s) on the fixed real chart, hence jointly Borel, and we choose a fixed smooth cutoff in s , equal to one on a smaller fixed ball and supported in the ball $|s| < s_n$. The phase bounds were proved in Step 4. For the amplitude, let $B_n = \sup_{D_r} |\bar{R}_{d,\gamma}|$, which is bounded by (P3). By (33.9) and the mean value theorem on the convex s -ball,

$$|a(X, s) - a(X, 0)| \leq e^{B_n} (\|J_n\|_\infty 3n^{3/2} K_1 + \|D_s J_n\|_\infty) |s|.$$

The positive lower and upper bounds follow from $e^{-B_n} \leq e^{-\bar{R}_{d,\gamma}} \leq e^{B_n}$ and the positivity of J_n . Every hypothesis of Lemma 33.3 therefore holds with constants c_0, C_0 depending only on the fixed data. The case $P = 1$ of (33.7) gives the integral with the cutoff. The additive error $O(g)$ becomes a relative error because the amplitude $a(X, 0) / \sqrt{\det A_d(X)}$ has a fixed positive lower bound. The lower bound $F_d - f_d \geq c_n |s|^2$ holds on the whole ball $|s| < s_n$, and so do the amplitude bounds. By (33.5), for $g \leq 1$ and after factoring out $e^{-f_d(X)/g^2}$, the difference between the integral with the cutoff and the integral over the whole ball is at most $C g^{d_n} e^{-c'/g^2}$, which is $O_n(g^{d_n+1})$. If $\gamma_0/n > 1$, then for $1 < g^2 \leq \gamma_0/n$ both the integral and the main term are bounded above and below by positive constants depending only on the fixed data, and the asserted formula holds there after enlarging the constant. We obtain

$$(2\pi)^{d_n/2} g^{d_n} e^{-f_d(X)/g^2} \frac{e^{-\bar{R}_{d,\gamma}(X,0)} J_n(X, 0)}{\sqrt{\det A_d(X)}} [1 + O_n(g)].$$

For the C^1 bound of the amplitude, we have

$$D_X \log a_{d,\gamma}(X) = -D_X \bar{R}_{d,\gamma}(X, 0) + D_X \log J_n(X, 0) - \frac{1}{2} \operatorname{tr}(A_d(X)^{-1} D_X A_d(X)).$$

The first term is bounded by $3n^2 K_1$, by (33.9); the value bound alone would not give this estimate. The trace is bounded by $d_n \|A_d^{-1}\| \|D_X A_d\|$, and every factor is bounded uniformly, by the fixed chart, the lower bound for the Hessian, and the bound on third derivatives. This proves the bounds for the amplitude, with the phase f_d at finite depth. In particular, no limit of the classical actions as $d \rightarrow \infty$ at fixed coupling has been inserted into an exponent divided by g^2 . All constants have the dependence stated in the theorem. $\square$

33.6 Flat configurations and the partition function

We now describe the flat configurations on Λ_n away from the central triples, that is, the triples of central cycle holonomies. We then prove the two-sided bound for the partition

function. Recall from Lemma 14.9 that the flat configurations modulo all periodic gauge transformations are classified by their commuting cycle holonomies modulo simultaneous conjugation. In the based quotient, where simultaneous conjugation is not divided out, the flat set is the set of commuting triples $(H_1, H_2, H_3) \in \mathrm{SU}(2)^3$. We call a flat configuration regular if its cycle holonomies are not all central. A compact regular patch is a compact subset of the flat set in the based quotient containing none of the eight configurations with central cycle holonomies.

Lemma 33.9. 1. *In the based quotient at side $n \geq 4$, the set of regular flat configurations is a smooth embedded submanifold of dimension five. The Hessian of E_0 is positive definite in the normal directions at every regular flat configuration. On every compact regular patch it has a positive lower bound, and there is a finite atlas of normal coordinates $V(y, v)$, with y in the flat manifold and $v \in \mathbb{R}^{6n^3-2}$, in which the image of H_n has a smooth positive density $J(y, v)$ with respect to $d\mathrm{vol}(y) dv$.*

2. *Consider the stopped measures of Section 33.2 and a compact regular patch. Assume either that the patch consists of configurations gauge equivalent to constant commuting links $e^{H_i/n}$ with $\max_i |H_i|/n < \epsilon_{\mathrm{loc}}$, as in Lemma 33.7, or that $2\pi/n < r_{\mathrm{flat}}$, where r_{flat} is the radius of Lemma 14.9. Assume also that the patch lies in a common larger classical link chart, after fixed gauge transformations. Then there is a normal tube over the patch, fixed independently of d and γ , on which, for every integer $d \geq 0$,*

$$\begin{aligned} nE_d(V(y, v)) &= \frac{1}{2}v^T H_d(y)v + O(C_n|v|^3), \\ c_n I &\leq H_d(y) \leq C_n I, \quad nE_d(V(y, v)) \geq c_n|v|^2. \end{aligned}$$

The normal rank is $6n^3 - 2$. The constants may depend on n , the patch, the larger chart, the constant κ of (P1), the derivative bounds of (P2), and the margins of the tube.

Proof. We use the based quotient $\mathrm{SU}(2)^{2n^3+1}$ of non-tree link variables constructed in Step 2 of the proof of Theorem 33.8. By Lemma 14.9, a flat point of the based quotient is determined by its three commuting cycle holonomies. This identification and its inverse are locally smooth. Indeed, each holonomy is a finite product of non-tree link variables. Conversely, after fixing paths in the tree, the flat transport along every non-tree link is the word in the three commuting cycle transports given by the winding numbers of the closed path that it forms with the tree. Such a word is a smooth product of integer powers.

At a regular triple we choose a noncentral member. Its quaternion vector is nonzero and, after fixing its sign, determines a smooth unit axis on a small patch. Every member of a commuting triple has its vector parallel to that axis. We choose local lifts of the three angles on the corresponding circles and two local coordinates for the axis. Differentiation in the three angles and the two axis coordinates is injective, since the chosen member has nonzero vector. Projection onto five suitable ambient coordinates therefore has invertible

derivative. By Lemma 33.1(1), the regular flat set is a smooth embedded five-dimensional manifold, and its only possible singular points are the eight triples whose members are all central.

Next we determine the kernel of the Hessian, including the count of based gauge directions. By a periodic gauge transformation and a constant conjugation, we conjugate a flat representative to the constant commuting links $U_i^0 = e^{\theta_i \tau_3/n}$. Such transformations change neither the rank of the Hessian nor its quadratic form on the gauge quotient. We write a variation of the links in left coordinates as $e^{ta_i(x)} U_i^0$. The derivative of the plaquette logarithm is

$$(d_U a)_{ij}(x) = \text{Ad}_{U_i^0} a_j(x + e_i) - a_j(x) - \text{Ad}_{U_j^0} a_i(x + e_j) + a_i(x),$$

as one sees by differentiating its four ordered factors, since the background plaquette variable is the identity. Since the quadratic part of E_0 is $\sum |d_U a|^2/2$, the kernel of the Hessian is $\{d_U a = 0\}$.

We split the colors into $\mathbb{R}\tau_3$, the neutral color, and its orthogonal plane, the charged color, which we identify with $\mathbb{C}$ so that ad_{τ_3} acts as multiplication by i . In neutral color the Fourier difference vector is $\eta_i = e^{2\pi i m_i/n} - 1$. If $\eta \neq 0$, the equations $\eta_i a_j - \eta_j a_i = 0$ imply $a = \eta \phi$, with $\phi = a_j/\eta_j$ for an index j with $\eta_j \neq 0$. These are exactly the gradient directions, and the zero mode gives three real constant neutral directions. In charged color the difference vector is

$$\eta_i = e^{i(2\pi m_i + \theta_i)/n} - 1.$$

It cannot vanish in all three components, since that would force every θ_i into $2\pi\mathbb{Z}$ and all three holonomies to be central. The same algebra $a = \eta \phi$ shows that every charged mode in the kernel is a covariant gradient. This argument includes the cases in which some θ_i vanish, without division by a possibly vanishing component.

The covariant gradient on vertex functions has a one-dimensional kernel, consisting of the constant neutral functions, since its charged Fourier vectors never vanish and its neutral kernel consists of constants. Its real image has dimension $3n^3 - 1$. Its restriction to functions vanishing at the root is injective, since a covariantly constant function vanishing at one point vanishes everywhere by transport along links. Hence the image of the based gradients has dimension $3n^3 - 3$, and after dividing out the based gradients, the kernel has dimension

$$(3n^3 - 1) + 3 - (3n^3 - 3) = 5.$$

Two of these directions are infinitesimal simultaneous conjugations at the root, and three are the constant neutral directions. They are exactly the tangent directions of the five-dimensional manifold parametrized above. Since the Hessian is a nonnegative symmetric matrix with no further kernel, it is positive definite on a normal complement. Its least normal eigenvalue has a positive minimum on a compact regular patch, since continuous

unit normal vectors over that compact set form a compact set on which the quadratic form does not vanish.

For integration, we give the based quotient $\mathrm{SU}(2)^{2n^3+1}$ its product Riemannian metric, which is bi-invariant, and let $\mathcal{F}$ be the manifold of regular flat configurations. We use the normal exponential map

$$V(y, v) = \mathrm{Exp}_y(v), \quad y \in \mathcal{F}, \quad v \perp T_y \mathcal{F},$$

of Lemma 33.1(2). Over a compact neighborhood of a compact regular patch, it is injective for a common small normal radius. We choose finitely many smooth orthonormal trivializations of the normal bundle of $\mathcal{F}$, with working subcharts relatively compact in larger charts. Since their transitions preserve y and are orthogonal in v , a partition of unity on the base extends consistently to the tube. The maps $V(y, v)$, their lifts to configurations, and the normal frames extend smoothly past the closures of the working subcharts, so all derivatives of fixed order of the charts are bounded there. Since the Haar measure of the product group is a constant multiple of its Riemannian volume, its pullback $J(y, v)$ is smooth and positive with respect to $d\mathrm{vol}(y) dv$. This proves part (1).

For part (2), on patches of small commuting configurations, the vanishing of the classical actions is proved in Lemma 33.7. On a general patch at a side with $2\pi/n < r_{\mathrm{flat}}$, it is the conclusion of Lemma 14.9 for all flat configurations: every averaging step has output side $m \geq n$, and $2\pi/m \leq 2\pi/n < r_{\mathrm{flat}}$ verifies the condition on the side at each step. By gauge invariance, these zeros transfer to the smooth based representatives used here. Since $E_d \geq 0$, its first derivative vanishes on the flat manifold. Since $E_d - \kappa E_0 \geq 0$ by (P1) and vanishes there, the Hessian of $E_d - \kappa E_0$ is positive semidefinite on real normal vectors. The lower bound of part (1) then implies $H_d = D_v^2(nE_d)(y, 0) \geq c_n I$. The uniform derivative bounds of (P2) on the common larger chart give the upper bound and a common cubic Taylor error. If a quotient chart is not itself a representative with small links, we fix a gauge transformation taking its center into B_r and shrink the neighborhood inside the fixed larger classical chart. By compactness, finitely many such fixed gauge transformations suffice. Their derivatives and the bounds of Theorem 10.11 give the same bound by the chain rule, and no gauge transformation depending on the point is needed. We choose the radius of the tube so that the cubic error is at most one quarter of the lower bound for the Hessian times $|v|^2$. This proves the lower bound for nE_d on the whole smaller tube. All these choices depend only on the listed data. The normal rank is the dimension $6n^3 + 3$ of the based quotient minus five. $\square$

Theorem 33.10. *The following statements hold.*

1. *For every integer $n \geq 4$ and every $\kappa > 0$ there exists $C_{n,\kappa} < \infty$ such that for $0 < g \leq 1$*

$$\int_{\mathcal{U}_n} e^{-\kappa E_0(V)/g^2} dH_n(V) \leq C_{n,\kappa} g^{6n^3-2}, \quad E_0(V) = 4 \sum_{p \in \mathcal{P}_n} q(V_p).$$

This is an integral over the whole configuration space, including neighborhoods of all eight flat configurations with central cycle holonomies. Divisibility of n by four is not needed, and $C_{n,\kappa}$ depends only on n and κ .

2. Consider the stopped measures of Section 33.2. There exist $g_0(n) > 0$ and $0 < c_n \leq C_n < \infty$, with the dependence described in Theorem 33.8, such that for all depths $d = kJ$ and all $0 < \gamma \leq \gamma_0$ with $g = \sqrt{\gamma/n} \leq g_0(n)$,

$$c_n g^{d_n+4} \leq \widehat{Z}_{n,d,\gamma} \leq C_n g^{d_n+4}, \quad d_n = 6n^3 - 6.$$

Equivalently, $\widehat{Z}_{n,d,\gamma}$ is comparable to γ^{p_n} , $p_n = 3n^3 - 1$, uniformly in d and γ in this range, at the fixed side n . Here $\widehat{Z}_{n,d,\gamma}$ is the partition function of (33.1), with the centered effective interaction. If Z^{raw} is defined with $R_{d,\gamma}$ in place of $\bar{R}_{d,\gamma}$, then $\widehat{Z}_{n,d,\gamma} = e^{R_{d,\gamma}(I)} Z^{\text{raw}}$ exactly.

3. For every $\rho > 0$ and $c > 0$ there exists $C_{\rho,c}$ such that for $0 < g \leq 1$

$$\int_{|X| < \rho} e^{-cQ(X)/g^2} dX \leq C_{\rho,c} g^4.$$

Here ρ is an arbitrary radius, not the analytic radius. More precisely, for $g \leq \min(1, \rho^2)$ the ball $|X| \leq \sqrt{g}$ contributes $O(g^{9/2})$, and each annulus $\varrho/2 < |X| < \varrho$ with $\sqrt{g} \leq \varrho \leq \rho$ contributes $O(g^4 \varrho)$.

The lower bound in (2) comes from a fixed annulus of regular constant commuting configurations in the chart of Theorem 33.8, and it uses only the bound on the values of the centered effective interaction.

Proof. We prove the upper bound in (1) by covering the based quotient by neighborhoods of the eight central flat configurations, normal tubes over the remaining compact regular part of the flat set, and a compact set without flat configurations. Near the central configurations, we integrate the nonconstant directions with the gap of Theorem 33.8, and the constant directions with the estimate in (3). We then deduce (2) from (1), (P1), and (P3).

Step 1: constant configurations. We need an exact lower bound for the Wilson action on constant configurations. We write $U = aI + 2v^a \tau_a$ and $W = bI + 2w^a \tau_a$ with $a^2 + |v|^2 = b^2 + |w|^2 = 1$. The multiplication rule

$$(a, v)(b, w) = (ab - v \cdot w, aw + bv + v \times w)$$

follows directly from the products of the matrices τ_a . Conjugation by (a, v) sends w to $(a^2 - |v|^2)w + 2(v \cdot w)v + 2a(v \times w)$. Thus the scalar part of $UWU^{-1}W^{-1}$ is

$$b^2 + [(a^2 - |v|^2)w + 2(v \cdot w)v + 2a(v \times w)] \cdot w = 1 - 2|v \times w|^2,$$

which implies $q(UWU^{-1}W^{-1}) = 2|v \times w|^2$ exactly. For $U_i = e^{x_i/n}$ we have $v_i = \sin(|x_i|/(2n))\hat{x}_i$, and summing over the n^3 plaquettes in each of the three planes gives, continuously also when some x_i vanish,

$$\begin{aligned} E_0(e^{X/n}) &= 8n^3 \sum_{i < j} \sin^2(|x_i|/(2n)) \sin^2(|x_j|/(2n)) |\hat{x}_i \times \hat{x}_j|^2 \\ &= \frac{1}{2n} \sum_{i < j} |x_i \times x_j|^2 \operatorname{sinc}^2(|x_i|/(2n)) \operatorname{sinc}^2(|x_j|/(2n)). \end{aligned} \quad (33.13)$$

Since the factors sinc^2 have positive lower and upper bounds on a fixed small ball, $E_0(e^{X/n})$ is comparable to $Q(X)/n$ there. This is a relative bound, valid also in almost commuting directions.

Step 2: the neighborhood of the identity. Applying the proof of Theorem 33.8 to the Wilson action alone, we obtain, on a fixed neighborhood of the identity in the based quotient,

$$E_0(e^{X/n+s}) \geq E_0(e^{X/n}) + c_n |s|^2 \geq c'_n (Q(X) + |s|^2), \quad (33.14)$$

where the second inequality is (33.13). This uses only the translation invariance of E_0 , the gap at side n , and the smoothness of E_0 , all verified explicitly in Steps 3 and 4 of that proof. It does not use the stopped density or the divisibility of n by four, and it holds at every integer $n \geq 4$.

Step 3: the nine-dimensional estimate. We prove (3). The ball $|X| \leq \sqrt{g}$ has volume $C_9 g^{9/2}$, which bounds its contribution. We now consider an annulus $\varrho/2 < |X| < \varrho$ with $\varrho \geq \sqrt{g}$, and set $X = \varrho Y$ and $\epsilon = g/\varrho^2 \leq 1$. Since Q is homogeneous of degree four, the integral over the annulus is

$$\varrho^9 \int_{1/2 < |Y| < 1} e^{-cQ(Y)/\epsilon^2} dY.$$

On the fixed closed annulus $1/2 \leq |Y| \leq 1$, the zero set of Q is a compact piece of the nonzero cone $\mathcal{M}$. In a fixed small normal tube around it the radius of the base point is bounded below, and by Lemma 33.2 we have $Q(y+b) \geq c'|b|^2$ and a bounded Jacobian. The tangential volume is finite by the volume formula of Lemma 33.2. Integrating in the four normal directions, we see that the tube contributes at most $C\epsilon^4$. Off the tube, Q has a positive minimum q_* on the annulus: a sequence with $Q \rightarrow 0$ would have a limit point in the zero set, hence in the tube. Since $\sup_{u>0} u^2 e^{-cq_* u} < \infty$, the integrand off the tube is at most $C\epsilon^4$. The annulus therefore contributes at most

$$C\varrho^9 \epsilon^4 = Cg^4 \varrho. \quad (33.15)$$

We take the radii $\rho, \rho/2, \rho/4, \dots$ until the last annulus reaches $\sqrt{g}$. The sum of these radii is at most 2ρ , and the remaining ball has radius at most $2\sqrt{g}$ and contributes $O(g^{9/2})$.

This proves the bound $C_{\rho,c}g^4$ for $g \leq \min(1, \rho^2)$. If $\rho^2 \leq g \leq 1$, the volume of the ball, $C\rho^9$, is at most $C\rho g^4$. The statements about the ball and the annuli follow from the same estimates.

Step 4: the whole configuration space. Integrating (33.14) against the bounded density of the chart, first in s and then in X , we find that the neighborhood of the identity contributes at most

$$C_{n,\kappa}g^{d_n} \int_{|X| < \rho_n} e^{-c_{n,\kappa}Q(X)/g^2} dX \leq C_{n,\kappa}g^{d_n+4},$$

by Step 3. To cover the other seven central points, we multiply, for each of the three directions, the links crossing one wrapping coordinate plane in that direction by either I or $-I$, and call the resulting array of central link variables z . Since each plaquette contains either no such factor or two canceling ones, $z_p = I$ for every plaquette. The three cycle signs of z are arbitrary, and they give the eight central triples. The map $V \mapsto zV$ preserves every plaquette variable and the product Haar measure, and it commutes with based gauge transformations because z is central. It maps the identity point of the flat set in the based quotient to the corresponding central flat point. Transporting the chart just used, we obtain the same bound near every central point, without dividing by the volume of a stabilizer.

We choose smaller open neighborhoods inside these eight charts; the flat set outside them is a compact regular patch. By Lemma 33.9(1), it is covered by finitely many normal coordinate charts with bounded Haar density. By Taylor's formula as in the proof of part (2) of that lemma, with E_0 in place of E_d , we have $E_0 \geq c_n|v|^2$ on these tubes after shrinking them. Each such tube contributes at most Cg^{6n^3-2} , since its normal rank is $6n^3 - 2$. The complement of the union of these tubes and of the central neighborhoods is compact and contains no flat configuration. Since $E_0 \geq 0$ and its zero set consists exactly of the flat configurations, E_0 has a positive minimum e_n on this complement. The complement therefore contributes at most $e^{-\kappa e_n}/g^2 \leq C_{n,\kappa}g^{6n^3-2}$ for $0 < g \leq 1$. Summing over the finite cover, and recalling that the based gauge transformations have mass one, we obtain (1).

Step 5: the stopped measure. For the upper bound in (2), we use (P1), which holds at every integer $d \geq 0$, and the bound $|\bar{R}_{d,\gamma}| \leq C_n$ of (P3) at the depths $d = kJ$. Enlarging the domain of integration from D_r to $\mathcal{U}_n$ and applying (1), we obtain

$$\hat{Z}_{n,d,\gamma} \leq e^{C_n} \int_{\mathcal{U}_n} e^{-\kappa E_0/g^2} dH_n \leq C_n g^{6n^3-2}.$$

Only an upper integral has been enlarged, and the partition function is that of the centered density (33.1).

For the lower bound, we choose a small fixed annulus of constant commuting configurations $X = (\vartheta_i u)_i$, with $|\vartheta|$ between two fixed positive radii, strictly inside the chart

of Theorem 33.8, with positive tangential volume and with $\max_i |X_i|/n < \epsilon_{\text{loc}}/2$. Such an annulus exists at every fixed side, since its radii can be taken arbitrarily small. These prescribed constant commuting refinements and their representatives in the block axial gauge satisfy the domain conditions with the margins of Lemma 33.7, which are independent of the depth; no condition on the side is needed for this annulus. By compactness of the annulus and openness of the gauge-invariant set D_r , we shrink its fixed normal tube so that it stays inside D_r ; its points therefore have representatives in B_r . On this compact annulus, Lemma 33.9 implies that $E_d \leq C_n |v|^2$ and that the Haar density has positive upper and lower bounds. The lower bound for the density times e^{-C_n} bounds the amplitude $e^{-\bar{R}_{d,\gamma}} J$ from below. We integrate only over $|v| \leq g$ and over a fixed smaller tangential subpatch of positive volume. For g below the fixed radius of the tube, we obtain

$$\hat{Z}_{n,d,\gamma} \geq c_n \int_{|v| \leq g} e^{-C_n |v|^2 / g^2} dv \geq c'_n g^{6n^3 - 2}.$$

All the sets used here are subsets of D_r . Neither bound requires derivative bounds for the centered effective interaction or any convergence of its restriction to the flat directions as $\gamma \rightarrow 0$.

Since $d_n + 4 = 6n^3 - 2 = 2(3n^3 - 1)$ and $g^2 = \gamma/n$, the two bounds have the asserted form γ^{p_n} at the fixed side n . Finally,

$$e^{-E_d/g^2 - (R_{d,\gamma} - R_{d,\gamma}(I))} = e^{R_{d,\gamma}(I)} e^{-E_d/g^2 - R_{d,\gamma}}$$

on the same domain, which proves the relation between the two normalizations. All constants in the lower and upper bounds for $\hat{Z}_{n,d,\gamma}$ are uniform in d and γ ; their dependence on the side has not been bounded. $\square$

33.7 Linearized flowed cycle logarithms

We now compute the covariance Σ_θ of (4.12), which appears in Theorem 2.10. It is the covariance, divided by γ , of the linear response of the flowed holonomy logarithms at a regular flat connection, under the Gaussian measure defined by the quadratic part of the Yang–Mills action. The computation has three ingredients: the covariance of the transverse Gaussian field at the flat background; the derivative of the flowed cycle logarithms, including the differential of the logarithm; and the projection onto the normal space $\mathcal{N}_y$, which removes the longitudinal part. We carry out the computation in the continuum and on the lattice, and compare the two. The lattice flow is the Wilson flow of Lüscher [128], with the normalization of Section 4.

Theorem 33.11. *Fix $L > 0$, $\theta > 0$, and a compact regular patch of commuting triples $y = (\vartheta_i u)_{i=1}^3$ with $u \in S^2$ and $0 < r_0 \leq |\vartheta| \leq r_1 < \pi$. Work in the variables of the unit torus, $\tilde{A}(x) = LA(Lx)$, in which the flat background is $\tilde{A}_i^0 = \vartheta_i u$ and the time $t = \theta L^2$ becomes θ . Let*

$D_i = \partial_i + \vartheta_i \text{ad}_u$ be the covariant derivative of the background, acting on $\mathfrak{su}(2)$ -valued one-forms $a = (a_1, a_2, a_3)$ on $\mathbb{T}_1^3$, and consider the quadratic form

$$\frac{1}{2\gamma} \int_{\mathbb{T}_1^3} \sum_{i < j} |D_i a_j - D_j a_i|^2 dx.$$

Its kernel $\mathcal{K}$ consists of the covariant gradients $D\phi$ and the constant one-forms with values in $\mathbb{R}u$. The form is positive definite on the quotient by $\mathcal{K}$. Let a be the centered Gaussian field with distributional values on this quotient, realized on the L^2 -orthogonal complement of $\mathcal{K}$, whose covariance is γ times the inverse of the positive operator associated with $\int_{\mathbb{T}_1^3} \sum_{i < j} |D_i a_j - D_j a_i|^2 dx$ on this complement; it is constructed from its Fourier coefficients in Step 2 of the proof. Let $\mathcal{L}$ be the derivative at $\tilde{A}^0$ of the map that sends a connection to the logarithms of the holonomies of its Yang–Mills heat flow at time θ around the three coordinate cycles based at 0, followed by the orthogonal projection onto $\mathcal{N}_y$, initially defined on smooth perturbations. This positively flowed composition extends to the Gaussian distribution above. It vanishes on $\mathcal{K}$, and the covariance of $\mathcal{L}(a)$ is $\gamma \Sigma_\theta(y)$, where $\Pi_\vartheta = I - \vartheta \vartheta^\top / |\vartheta|^2$ and

$$k_m = 2\pi m + \vartheta, \quad \lambda_m = |k_m|^2, \quad R_i(m, \vartheta) = \begin{cases} 1, & m_i = 0, \\ \vartheta_i / (2\pi m_i + \vartheta_i), & m_i \neq 0, \end{cases}$$

$$d_i = \sum_{m \in \mathbb{Z}^3} \frac{e^{-2\theta \lambda_m}}{\lambda_m} R_i(m, \vartheta)^2, \quad D = \text{diag}(d_1, d_2, d_3), \quad \Sigma_\theta(y) = (\Pi_\vartheta D \Pi_\vartheta)|_{\vartheta^\perp} \otimes I_{u^\perp}.$$

Here $\mathcal{N}_y$ is identified with $\vartheta^\perp \otimes u^\perp$, and these formulas include the cases in which some coordinates of ϑ vanish. Moreover, the following hold.

1. On the patch, $cI_4 \leq \Sigma_\theta \leq CI_4$, and Σ_θ and its inverse are smooth, covariant under color rotations, and invariant under $(\vartheta, u) \mapsto (-\vartheta, -u)$. The inverse is

$$\left[D^{-1} - \frac{D^{-1} \vartheta \vartheta^\top D^{-1}}{\vartheta^\top D^{-1} \vartheta} \right] \Big|_{\vartheta^\perp} \otimes I_{u^\perp}.$$

The constant charged mode $m = 0$ alone contributes $e^{-2\theta|\vartheta|^2} |\vartheta|^{-2} I_4$.

2. For a smooth perturbation h , before applying the flow the Fréchet derivative of the cycle logarithms is

$$T_i h = J_i^{-1} \int_0^1 e^{v \vartheta_i \text{ad}_u} h_i(v e_i) dv, \quad J_i = \int_0^1 e^{v \vartheta_i \text{ad}_u} dv,$$

where J_i^{-1} is the differential of the logarithm. Its normal projection annihilates covariant gradients and the neutral color $\mathbb{R}u$, and the linearized flow multiplies the charged transverse Fourier mode m by $e^{-\theta \lambda_m}$.

3. For $N \geq 4$ let $\mathcal{B}_N$ be the set of centered representatives of $\mathbb{Z}^3 / N\mathbb{Z}^3$, and set

$$\delta_i^N = N(e^{i(2\pi m_i + \vartheta_i)/N} - 1), \quad \lambda_m^N = \sum_i |\delta_i^N|^2, \quad R_i^N = i\vartheta_i / \delta_i^N,$$

with $R_i^N = 1$ if $m_i = \vartheta_i = 0$ and $R_i^N = 0$ if $\vartheta_i = 0 \neq m_i$. On the lattice Λ_N of mesh $1/N$ in $\mathbb{T}_1^3$, consider the background links $e^{\vartheta_i u/N}$, the Gaussian measure defined by the quadratic part of the Wilson exponent $(4N/\gamma) \sum_p q(U_p)$ on the quotient by its kernel, and the derivative of the cycle logarithms after the Wilson flow at lattice time $4\theta N^2$, projected onto $\mathcal{N}_y$. The covariance of this derivative, divided by γ , is exactly

$$\Sigma_\theta^N(y) = \left(\Pi_\vartheta \operatorname{diag} \left(\sum_{m \in \mathcal{B}_N} \frac{e^{-2\theta \lambda_m^N}}{\lambda_m^N} |R_i^N|^2 \right) \Pi_\vartheta \right) \Big|_{\vartheta^\perp} \otimes I_{u^\perp},$$

and $\sup_y \|\Sigma_\theta^N(y) - \Sigma_\theta(y)\| \leq CN^{-2}$.

4. As $\varrho = |\vartheta| \downarrow 0$, with $\varrho < r_1 < \pi$,

$$\Sigma_\theta(y) = \varrho^{-2} I_{\mathcal{N}_y} + O_\theta(1), \quad \varrho^2 \Sigma_\theta(y) \rightarrow I_{\mathcal{N}_y},$$

uniformly in the directions of ϑ and u .

The constants, and the bounds for any fixed number of derivatives on the patch, depend on r_0, r_1, θ , the order of the derivatives, and the margins of the charts, but not on N , on γ , or on any depth. The bounds on the patch are not uniform as $|\vartheta| \rightarrow 0$.

The positive time is needed: for $\theta = 0$ the series defining d_i diverges logarithmically, since $R_i = 1$ on the plane $m_i = 0$. Part (4) is a statement about the explicit matrix Σ_θ ; it is used in Section 36 only for a comparison measure.

Proof. The proof follows the three ingredients described above: Steps 1 and 2 construct the Gaussian field, Steps 3–5 compute the response, and Step 6 performs the projection. Steps 7–10 prove the bounds, the lattice formula, the comparison with the continuum, and the central asymptotics.

Step 1: normalization. The map $\tilde{A}(x) = LA(Lx)$ sends the flat connection in physical units to $\tilde{A}_i^0 = \vartheta_i u$ and the time $t = \theta L^2$ to θ . Since the curvature is multiplied by L^2 , the volume by L^{-3} , and $g_3^2 L = \gamma$, the quadratic part of the exponent $E_L / (2g_3^2)$ becomes exactly the form in the statement.

Step 2: the Gaussian field. We fix an oriented orthonormal basis of $u^\perp$ in which $v \mapsto u \times v$ is multiplication by i , and regard the two real charged colors as one complex color. On the Fourier mode $e^{2\pi i m \cdot x}$, the operator D_i acts on charged fields as multiplication by $ik_{m,i}$, and the quadratic form on that mode is

$$\sum_{i < j} |k_{m,i} a_j - k_{m,j} a_i|^2 = \lambda_m |a|^2 - |k_m^\top a|^2. \quad (33.16)$$

Since $|\boldsymbol{\theta}| \geq r_0$ and $r_1 < \pi$,

$$|k_0| \geq r_0, \quad |k_m| \geq 2\pi|m| - r_1 \quad (m \neq 0),$$

so $k_m \neq 0$ for every m . By (33.16) the charged kernel consists of the multiples of k_m , which are the charged covariant gradients. In neutral color the form at mode m is $|2\pi m|^2|a|^2 - |2\pi m \cdot a|^2$, whose kernel consists of gradients for $m \neq 0$ and of all constants for $m = 0$. This proves the description of $\mathcal{K}$. The form is positive definite on the quotient by $\mathcal{K}$. We realize the quotient on the L^2 -orthogonal complement of $\mathcal{K}$, namely the covariantly transverse one-forms, $\sum_j D_j a_j = 0$, that are orthogonal to the constant neutral one-forms. We show in Step 4 that $\mathcal{L}$ vanishes on $\mathcal{K}$. Since the form vanishes on $\mathcal{K}$ as well, the projection along $\mathcal{K}$ identifies any two complements isometrically for the form and does not change $\mathcal{L}$. In particular, the law of $\mathcal{L}(a)$ does not depend on the choice of the complement. On the chosen complement we have $k_m^\top a = 0$ at each charged mode, and the covariance divided by γ is

$$\lambda_m^{-1} P(k_m), \quad P(k) = I - k k^\top / |k|^2. \quad (33.17)$$

This follows by diagonalizing the finite-dimensional form (33.16) and using $\int e^{-x^2/(2\sigma^2)} e^{i\tau x} dx = e^{-\sigma^2\tau^2/2} \int e^{-x^2/(2\sigma^2)} dx$. We construct all the Fourier Gaussians in this way and take their sum as a distribution: for every $s > 1/2$ the expected squared norm in the Sobolev space H^{-s} is finite, since $\sum_{m \neq 0} |m|^{-2-2s} < \infty$. Since $\theta > 0$, each flowed normal response considered below is an absolutely summable Gaussian series. The unflowed derivative T in part (2) and Steps 3–4 acts initially on smooth perturbations; its composition with the positive-time linearized flow extends to the Gaussian distribution just constructed.

The complex notation does not double the covariance. We regard the complex Hilbert space of charged fields as a real Hilbert space with inner product $\operatorname{Re} \int a^* b$. For each orthonormal complex Fourier vector e , the pair e, ie is real orthonormal, and a precision eigenvalue λ assigns the variance γ/λ to each of the two real coefficients. The realification of a Hermitian covariance C is therefore

$$\begin{pmatrix} \operatorname{Re} C & -\operatorname{Im} C \\ \operatorname{Im} C & \operatorname{Re} C \end{pmatrix},$$

with no extra scalar factor. When C is real, this is $C \otimes I_{u^\perp}$ in spatial and color notation.

Step 3: the derivative of the cycle logarithms. We first compute on a smooth perturbation h . Let $Y_i = \partial_i u$. With the convention $H'(v) = H(v)A(\ell(v))\dot{\ell}(v)$ of Definition 4.10, the transport along the i th cycle at the background is e^{vY_i} at the parameter v . Differentiating the transport equation, we find the right-trivialized derivative

$$\delta H_i H_i^{-1} = \int_0^1 e^{v\partial_i \operatorname{ad}_u} h_i(v e_i) dv.$$

For the exponential map the corresponding derivative is $\delta(e^{Y_i})e^{-Y_i} = J_i \delta Y_i$ with $J_i = \int_0^1 e^{v\vartheta_i \text{ad}_u} dv$. In neutral color $J_i = 1$, and in charged color $J_i = (e^{i\vartheta_i} - 1)/(i\vartheta_i)$, with value one at $\vartheta_i = 0$. It is invertible on the whole patch because $|\vartheta_i| < \pi$. The differential of the logarithm is therefore J_i^{-1} , which proves the formula for T_i in part (2). On the charged Fourier mode m this derivative is the scalar

$$\frac{\int_0^1 e^{i(2\pi m_i + \vartheta_i)v} dv}{\int_0^1 e^{i\vartheta_i v} dv}.$$

For $\vartheta_i \neq 0$, canceling $e^{i\vartheta_i} - 1$ gives $\vartheta_i/(2\pi m_i + \vartheta_i)$. If $m_i = 0$, the two integrals are equal, also when $\vartheta_i = 0$. If $\vartheta_i = 0$ and $m_i \neq 0$, the numerator vanishes and the denominator is one. These are exactly the cases in the definition of R_i .

Step 4: the normal projection. Let ϕ be a smooth periodic gauge parameter. Integrating the derivative of $v \mapsto e^{v\vartheta_i \text{ad}_u} \phi(v e_i)$ over $[0, 1]$ and using periodicity, we find

$$T_i(D\phi) = \vartheta_i \text{ad}_u \phi(0).$$

As a triple indexed by i , this is ϑ tensored with a vector of $u^\perp$, an infinitesimal simultaneous conjugation at the root, and the normal projection $\Pi_\vartheta \otimes (I - uu^\top)$ annihilates it. Every response in neutral color is parallel to u and is annihilated as well. Hence the constant neutral one-forms and all other flat tangent directions contribute zero, $\mathcal{L}$ vanishes on $\mathcal{K}$, and the normal response is well defined on the quotient, independently of the representative used to compute it.

Step 5: the linearized flow. At a flat background $[D_i, D_j] = 0$, and the linearization of the Yang–Mills heat flow $\partial_t A_i = \sum_j D_j F_{ji}$ is

$$\partial_\theta a_i = \sum_j D_j (D_j a_i - D_i a_j).$$

Covariant gradients are stationary for this equation. On covariantly transverse fields, $\sum_j D_j a_j = 0$, it reads $\partial_\theta a = \sum_j D_j^2 a$. The Fourier multiplier on charged modes is $-\lambda_m$, and the linearized flow multiplies the mode m by $e^{-\theta \lambda_m}$. A linearized gauge damping term can only change a covariant gradient, whose normal logarithmic response vanishes by Step 4. This derives the response of the Yang–Mills heat flow itself on the quotient, rather than replacing it by the scalar heat flow at a charged background.

Step 6: the covariance. Let $R_m = \text{diag}(R_i(m, \vartheta))$. Applying the response of Steps 3–5 to (33.17), we find the spatial covariance

$$\sum_m \frac{e^{-2\theta \lambda_m}}{\lambda_m} (\Pi_\vartheta R_m P(k_m) R_m \Pi_\vartheta) \big|_{\vartheta^\perp}.$$

For each i , including every case in which a coordinate vanishes, $R_i k_{m,i} = \vartheta_i$. Then $R_m k_m = \vartheta$, and the longitudinal term $R_m k_m k_m^\top R_m / \lambda_m$ is annihilated exactly by Π_ϑ . What remains is $\Pi_\vartheta D \Pi_\vartheta$ restricted to $\vartheta^\perp$. Since this matrix is real, the realification of Step 2 gives the factor $I_{u^\perp}$. This proves the formula for Σ_ϑ .

Step 7: bounds, inverse, and invariance. Every R_i is bounded on the patch. For any fixed number of derivatives in ϑ , the terms of the series are bounded by $C(1 + |m|)^q e^{-c\theta|m|^2}$, with constants depending on the order, on r_0 and r_1 , and on θ . The sum of these bounds is finite, as one sees by separating the three coordinates and comparing $\sum_{j \in \mathbb{Z}} (1 + |j|)^q e^{-c\theta j^2}$ with a finite initial segment and a convergent integral. Termwise differentiation is therefore valid. The term $m = 0$ has $R_i = 1$ and contributes

$$e^{-2\theta|\vartheta|^2} |\vartheta|^{-2} I_4.$$

This term gives a positive lower bound on the compact patch, and the summable series gives an upper bound; in particular, all d_i are positive. To invert the form, let $b, v \perp \vartheta$ with $\Pi_\vartheta Dv = b$. Then $Dv = b + \alpha \vartheta$ for a scalar α , and the condition $\vartheta^\top v = 0$ gives

$$\alpha = -\frac{\vartheta^\top D^{-1}b}{\vartheta^\top D^{-1}\vartheta}.$$

This proves the formula for the inverse. Its denominator is positive, and the bounds just proved give smoothness and the two-sided bound. Color rotations act by the corresponding orthogonal maps of $u^\perp$. The reversal $(\vartheta, u) \mapsto (-\vartheta, -u)$ is implemented in the series by $m \mapsto -m$ and complex conjugation of the charged color, since $R_i(m, -\vartheta) = R_i(-m, \vartheta)$ and $\lambda_m(-\vartheta) = \lambda_{-m}(\vartheta)$. Thus the covariance and its inverse have the asserted invariance. This proves part (1).

Step 8: the lattice covariance. We compute the lattice covariance directly, without using the continuum. At the background links $U_i^0 = e^{\vartheta_i u/N}$, we write a perturbation as $e^{a_i(x)/N} U_i^0$, where x is now an integer lattice coordinate. Differentiating the four factors of the plaquette variable, we find, in charged color,

$$\delta \text{Log } U_{p_{ij}(x)} = N^{-1} \{ e^{i\vartheta_i/N} a_j(x + e_i) - a_j(x) - e^{i\vartheta_j/N} a_i(x + e_j) + a_i(x) \}.$$

On the Fourier mode m this is $N^{-2}(\delta_i^N a_j - \delta_j^N a_i)$. The coefficient $4N/\gamma$ of the Wilson exponent and $q(e^v) = |v|^2/8 + O(|v|^4)$ give the quadratic exponent

$$\frac{1}{2\gamma N^3} \sum_x \sum_{i < j} |\delta_i^N a_j - \delta_j^N a_i|^2. \quad (33.18)$$

Fourier orthonormality uses the normalized volume $N^{-3} \sum_x$. With $\delta = (\delta_i^N)_i$, the corresponding transverse projection is $P_\delta = I - \delta \delta^* / \lambda_m^N$, and the covariance divided by γ is

$(\lambda_m^N)^{-1}P_\delta$. We treat the quotient by the kernel and its independence of the complement as in Step 2, using the description of the kernel in the proof of Lemma 33.9.

We check the flow factor in the same coordinates. In the Lie-algebra coordinates $b = a/N$ of the links, the quadratic part of $\sum_p q(U_p)$ is $\|d_U b\|^2/8$, and its Hessian is $d_U^* d_U/4$. Since the left group coordinates are orthonormal for the group metric at the critical flat configuration, the linearized negative gradient equation is $\partial_s b = -d_U^* d_U b/4$. On the transverse mode m its eigenvalue is $-\lambda_m^N/(4N^2)$, and at the lattice time $s = 4\theta N^2$ the multiplier is exactly $e^{-\theta\lambda_m^N}$.

The derivative of the cycle product, including the differential of its final logarithm, is

$$J_i^{-1} \frac{1}{N} \sum_{l=0}^{N-1} e^{i(2\pi m_i + \vartheta_i)l/N}.$$

For $\vartheta_i \neq 0$, the formula for a geometric sum and the formula for J_i reduce this to

$$\frac{i\vartheta_i}{N(e^{i(2\pi m_i + \vartheta_i)/N} - 1)} = R_i^N.$$

For $\vartheta_i = 0$ the sum is one if $m_i = 0$ and zero otherwise, which are the prescribed values. We conclude that, with $R_m^N = \text{diag}(R_i^N)$,

$$R_m^N \delta = i\vartheta, \quad \Pi_\vartheta R_m^N P_\delta (R_m^N)^* \Pi_\vartheta = \Pi_\vartheta \text{diag}(|R_i^N|^2) \Pi_\vartheta.$$

This equality uses the complex phases of R_i^N before the longitudinal part cancels; the phases can be discarded only afterwards. Summing over the modes in the real covariance, as in Step 2, we obtain the formula for Σ_ϑ^N .

Step 9: comparison with the continuum. For centered representatives and $N \geq 4$ we have $|k_{m,i}/(2N)| \leq \pi/2 + r_1/8 < 5\pi/8$. On this interval the function $\sin x/x$ has a positive minimum, and Taylor's formula gives $|4\sin^2(x/2) - x^2| \leq C|x|^4$. Since $\lambda_m^N = \sum_i 4N^2 \sin^2(k_{m,i}/(2N))$, we obtain

$$c|k_m|^2 \leq \lambda_m^N \leq |k_m|^2, \quad |\lambda_m^N - |k_m|^2| \leq CN^{-2}|k_m|^4. \quad (33.19)$$

Also

$$|R_i^N|^2 = R_i^2 \left(\frac{k_{m,i}/(2N)}{\sin(k_{m,i}/(2N))} \right)^2, \quad ||R_i^N|^2 - R_i^2| \leq CN^{-2} R_i^2 k_{m,i}^2 \leq CN^{-2},$$

where the continuous extensions at the vanishing cases are understood. In particular $R_i^2 k_{m,i}^2 = \vartheta_i^2$, also when $m_i = 0$, and all these multipliers are bounded. For $F(x) = e^{-2\theta x}/x$ we have $|F'(x)| = e^{-2\theta x}(2\theta/x + x^{-2})$. By the mean value theorem and (33.19), the difference of corresponding summands is at most

$$CN^{-2} e^{-c\theta|k_m|^2} (1 + |k_m|^2 + |k_m|^{-2}),$$

which is summable uniformly on the patch. The continuum modes not in $\mathcal{B}_N$ satisfy $|m_i| \geq N/2$ for some i , and their Gaussian tail is at most $Ce^{-c'\theta N^2} \leq C_\theta N^{-2}$. Summing, we obtain the uniform bound CN^{-2} in part (3). Fixed derivatives in the parameters obey the same bound. Indeed, differentiating the Taylor remainders in the sine expressions leaves a factor N^{-2} times a fixed polynomial in $|k_m|$. All inverse powers of λ_m^N and λ_m are bounded by the gap on the patch, and the differentiated exponentials add only polynomial factors. The same summable majorant applies, with constants depending also on the order, and smooth local normal frames introduce only constants depending on the margins of the patch.

Step 10: the central asymptotics. Let $\varrho = |\vartheta| \downarrow 0$ with $\varrho < r_1 < \pi$. The nonzero modes still satisfy $|k_m| \geq 2\pi|m| - r_1$, and their multipliers R_i are bounded uniformly in all directions, since $|2\pi m_i + \vartheta_i| \geq \pi$ for $m_i \neq 0$. Their total contribution is $O_\theta(1)$. The zero mode contributes

$$e^{-2\theta\varrho^2} \varrho^{-2} I_4 = \varrho^{-2} I_4 + O_\theta(1).$$

Since the projection onto the normal space has norm one, this proves both assertions of part (4), uniformly in the directions. It also shows why the bounds on the regular patch cannot be extended uniformly to the central point. $\square$

33.8 The stopped measure near regular flat configurations

We now combine the preceding results. The first part of the next proposition is the Laplace approximation of the stopped measure transverse to a compact patch of regular flat configurations. The second part is an exact decomposition of the covariance Σ_θ^N of Theorem 33.11 at the flat reconstruction, obtained by completion of squares for Gaussian measures in the constrained coordinates of Theorem 12.13. In Section 35 this decomposition identifies the covariance of the Gaussian approximation obtained at finitely many scales with Σ_θ .

Proposition 33.12. *Consider the stopped measures of Section 33.2 at the side n , a compact regular patch, and a fixed normal tube over a larger compact regular patch whose closure lies in D_r . Assume that $2\pi/n < r_{\text{flat}}$, where r_{flat} is the radius of Lemma 14.9, which lies below the fixed margins of all classical, constraint, Haar disintegration, and gauge domains. For a patch of small commuting configurations as in Lemma 33.7, the margins of that lemma may be used instead of this condition.*

1. *There is a finite atlas of normal coordinates $V(y, v)$ over the patch, of normal rank $\mathfrak{d}_n = 6n^3 - 2 = d_n + 4$, in which the image of H_n has a smooth positive density $J(y, v)$, and for*

every integer $d \geq 0$

$$\begin{aligned} nE_d(V(y, v)) &= \frac{1}{2}v^\top H_d(y)v + O_n(|v|^3), \\ c_n I &\leq H_d(y) \leq C_n I, \quad nE_d(V(y, v)) \geq c_n |v|^2. \end{aligned}$$

Let χ be a fixed normal cutoff equal to one near $v = 0$, let f be a bounded measurable function of y with compact support in the patch, and let P_y be a family of polynomials of a fixed degree with bounded measurable coefficients. Put

$$z_0 = H_d(y)^{1/2}v / \sqrt{\gamma}, \quad a_{d,\gamma}(y, v) = e^{-\bar{R}_{d,\gamma}(V(y, v))} J(y, v).$$

Then for $d = kJ$ and $0 < \gamma \leq \gamma_0$, with the partition of unity of the atlas understood and the insertion extended by zero outside the charts,

$$\begin{aligned} &\hat{Z}_{n,d,\gamma} \mathbb{E}_{\hat{v}_{n,d,\gamma}} [f(y)\chi(v)P_y(z_0)] \\ &= (2\pi\gamma)^{d_n/2} \left[\int f(y) \frac{a_{d,\gamma}(y, 0)}{\sqrt{\det H_d(y)}} \mathbb{E} P_y(G_{d_n}) d\text{vol}(y) + O_{n,f,P}(\sqrt{\gamma}) \right]. \end{aligned}$$

The error is uniform in d and γ . Given y , the moments of z_0 under the normal conditional law, normalized by its own normal integral, differ from those of G_{d_n} by $O_n(\sqrt{\gamma})$, and all fixed moments of z_0 are bounded. The tangential density may vary with γ within its bounds, without converging. Here $\hat{Z}_{n,d,\gamma}$ is the partition function of Theorem 33.10. For every nonnegative f as above with $\int f d\text{vol} > 0$ there exists $c_{n,f} > 0$ such that $\mathbb{E}_{\hat{v}_{n,d,\gamma}} [f(y)\chi(v)] \geq c_{n,f}$ for all depths $d = kJ$ and all sufficiently small γ .

2. Suppose in addition that the cycle holonomies of the patch have logarithms $y = (\vartheta_i u)_i$ with $0 < r_0 \leq |\vartheta| \leq r_1 < \pi$, and fix $\theta > 0$. Let $K \geq 0$, $N = nB^K$, $n_j = nB^j$, $d = JK$, and $H_K^{\text{act}}(y) = D_v^2(nE_{JK})(y, 0)$. Consider the reconstruction of a configuration on Λ_N from the stopping field V on Λ_n and the normalized fluctuation variables $w_j = (w_{j,l})_{0 \leq l < J}$ of the RG steps $j = 1, \dots, K$, given by the triangular coordinates

$$z_{j,l} = z_{j,l}^* + g_{j,l} K_{j,l}^{-1/2} w_{j,l}$$

of Theorem 12.13, computed in smooth local charts in which the expressions in different gauges agree, and satisfying $T_M^{JK} U = V$. At the flat reconstruction of the flat stopping field $V(y, 0)$, where all w_j vanish, differentiate the composition of this reconstruction with the map that sends a configuration on Λ_N to the logarithms of the cycle holonomies, based at the root, of its Wilson flow at physical time θL^2 , followed by the orthogonal projection onto $\mathcal{N}_y$. Normalize the normal variable of the stopping field by $v = \sqrt{\gamma} (H_K^{\text{act}})^{-1/2} z_0$. After division by $\sqrt{\gamma}$, denote the resulting linear maps by $A_{0,K}(y) : \mathbb{R}^{d_n} \rightarrow \mathcal{N}_y$ and $A_{j,K}(y) : \mathbb{R}^{D_j} \rightarrow \mathcal{N}_y$, $1 \leq j \leq K$, where $D_j = 6(n_j^3 - n_{j-1}^3)$ is the number of real normalized fluctuation variables of the j th RG step. Then for every finite K ,

$$A_{0,K} A_{0,K}^\top + \sum_{j=1}^K A_{j,K} A_{j,K}^\top = \Sigma_\theta^N(y), \quad \|A_{0,K}\|_{\text{HS}}^2 + \sum_{j=1}^K \|A_{j,K}\|_{\text{HS}}^2 \leq C.$$

Each $A_{j,K}$ includes the factors $g_{j,l}/\sqrt{\gamma}$, the inverse square roots of the constrained Hessians at the flat background, and the derivative of the composition with all finer reconstruction steps. If T_K denotes the response of the normal variable of the stopping field before normalization and $H_K^E = D_v^2 E_{JK}(y, 0)$ in the same normal coordinates, then

$$H_K^{\text{act}} = nH_K^E, \quad A_{0,K}A_{0,K}^T = n^{-1}T_K(H_K^E)^{-1}T_K^T.$$

The same identities hold for the reconstruction maps $\widehat{\mathcal{R}}$ of Theorem 15.13.

The constants depend on $n, L, \theta, M, J, \Theta$, the patch, the larger tube, the bounds for the polynomials, and the fixed analytic domains, with the dependence described in item (6) of Definition 4.14. They are independent of K , of the depth, and of $\gamma \in (0, \gamma_0]$.

The identities in part (2) concern the Gaussian quadratic model at the flat reconstruction; they do not assert that the normalized fluctuation variables are independent under the interacting measure. The normal responses of the stopping field, including those of the constant charged modes, are part of the identity: for $K = 0$ the block $A_{0,0}$ gives the whole covariance.

Proof of part (1) of Proposition 33.12. We first check that the flat configurations and their refinements lie in the domains of the construction, separately from the normal integral. Then we apply Lemma 33.3 in the normal coordinates of Lemma 33.9.

Step 1: flat refinements. First suppose that $2\pi/n < r_{\text{flat}}$. Fix the side n before the coupling, and let

$$\delta_{\text{flat}} = r_{\text{flat}} - 2\pi/n > 0. \quad (33.20)$$

The radius r_{flat} lies below all the classical, constraint, Haar disintegration, and gauge margins of one RG step, with the strict margin required in Lemma 14.9. These margins include the fixed conversions for the words of the block axial gauge. A flat triple of cycle holonomies has commuting logarithms H_i with $|H_i| \leq 2\pi$. At every averaging step with output side $m \geq n$, its constant representative and its refinement at the next finer side have link logarithms H_i/m and $H_i/(Mm)$, and

$$\max_i |H_i|/m \leq 2\pi/n = r_{\text{flat}} - \delta_{\text{flat}}.$$

By the computation for commuting products leading to (33.11), the representative in the block axial gauge is $U_0(c)$, and the free logarithmic coordinates and the logarithms on the eliminated edges vanish. The conclusion of Lemma 14.9 for all flat configurations, with smallest side n , verifies every intermediate classical, constraint, based gauge, and small-field condition, for every K . The classical action of the free coordinates, which is nonnegative, vanishes at this point. By the uniqueness of the conditional constrained minimizer in the same conclusion, the point is $z_i^*(c) = 0$, and each normalized fluctuation

variable vanishes. This verifies the hypothesis of part (2) at all the depths involved. The margins and the radii of the fluctuation cutoff domains are fixed by the parameters. No neighborhood of the flat set depending on the cutoffs of the small-field event is used as a normal tube. Under the alternative local hypothesis, we apply Lemma 33.7 on a slightly larger compact patch satisfying its strict inequality. It gives the same conclusions at every side $m \geq n$ and every depth, and its positive domain margins replace δ_{flat} below.

Step 2: phase and amplitude. The finite atlas and the estimates for the phase are those of Lemma 33.9, applied on the larger patch and then shrunk. The normal rank is $\mathfrak{d}_n = 6n^3 + 3 - 5 = 6n^3 - 2 = d_n + 4$. The vanishing of E_d on the flat set, the lower bound (P1), and the uniform bounds on third derivatives were verified there. The centered effective interaction of Theorem 14.8 is bounded on D_r by (P3). For its derivatives, we take the normal atlas on the larger tangential patch with a larger normal radius, and finitely many relatively compact parameter subcharts $W_l \Subset W_l^+$ covering the working tube. The maps $\Psi_l(y, v) = V(y, v)$ and their based lifts are smooth on W_l^+ , as constructed in Lemma 33.9, and extend past $\overline{W}_l$. Let $\mathcal{K}$ be the compact union of the images of the closures in the larger tube; it lies in D_r by hypothesis and does not depend on d or γ . On these compact subcharts we apply (33.10). Finitely many fixed based gauge transformations map neighborhoods of $\mathcal{K}$ into B_{r_1} , with $r < r_1 < \rho$ and the fixed margin η . We refine the subcharts so that each closure lies in one of the larger neighborhoods in the proof of Lemma 33.6. Each gauge transformation acts smoothly on all of $\mathcal{U}_n$. The chart composed with it extends to a neighborhood of the closure of the subchart, and its derivatives of every fixed order are bounded on these finitely many compact closures, independently of d and γ . These are the conditions on subcharts in Theorem 14.8, at each chart of the atlas. The chain rule then gives $|D_{y,v}\overline{R}_{d,\gamma}(V(y, v))| \leq C_n E_*$ on each compact parameter closure, uniformly in the depth and the coupling. The amplitude

$$a_{d,\gamma}(y, v) = e^{-\overline{R}_{d,\gamma}(V(y, v))} J(y, v)$$

therefore has fixed positive upper and lower bounds and a fixed Lipschitz bound in v ; it includes the Haar density of the based quotient. Since all insertions in the charts are supported in D_r , the indicator of D_r equals one where these coordinates are used. On each normal fiber, with $B_n = \sup_{D_r} |\overline{R}_{d,\gamma}|$, the mean value theorem gives

$$|a_{d,\gamma}(y, v) - a_{d,\gamma}(y, 0)| \leq e^{B_n} (\|J\|_\infty C_n E_* + \|D_v J\|_\infty) |v|.$$

This is the Lipschitz bound in the normal direction required by Lemma 33.3; a bound on the values of the effective interaction alone would not imply it.

Step 3: the Laplace approximation. We apply Lemma 33.3 with $\delta = \gamma$, the phase nE_d , the Hessian $L = H_d$, and the weight h given by f times the partition of unity of the atlas; the formula for general f follows by splitting f into its positive and negative parts.

The phase, the amplitude, and the Hessian are smooth in each real normal trivialization, hence jointly Borel, and we choose the normal cutoff smooth as well. The tests and the polynomial coefficients are measurable, as required in that lemma. We decrease the constant c_0 of that lemma and increase C_0 to include the bounds for the normal Hessian, the lower bound for the phase, the bounds for the amplitude, and the Lipschitz constant just verified. No matrix L_∞ is substituted in this application. We obtain

$$(2\pi\gamma)^{\mathfrak{d}_n/2} \left[\int f(y) \frac{a_{d,\gamma}(y,0)}{\sqrt{\det H_d(y)}} \mathbb{E} P_y(G_{\mathfrak{d}_n}) d\text{vol}(y) + O_{n,f,P}(\sqrt{\gamma}) \right].$$

Since the exponent of (33.1) is $E_d/g^2 = nE_d/\gamma$, the left side of (33.3) is exactly the integral of the insertion extended by zero against the unnormalized stopped density, namely $\hat{Z}_{n,d,\gamma} \mathbb{E}_{\hat{v}_{n,d,\gamma}} [f(y)\chi(v)P_y(z_0)]$. This proves the displayed identity, with the partition function of Theorem 33.10. The statements about conditional moments and bounded moments are part (1) of Lemma 33.3. If the conditional law is taken on the whole tube instead of with the cutoff, the additional part lies outside a fixed smaller normal ball. By Corollary 33.4 and the bounds on the whole tube verified below, it is exponentially small, and the conditional statements are unchanged.

Step 4: lower bounds for tangential tests. Let $f \geq 0$ with $\int f d\text{vol} > 0$. By the case $P = 1$ of the formula, with the error $C\sqrt{\gamma} \int f$, the numerator is at least $c_{n,f}\gamma^{\mathfrak{d}_n/2}$ for all sufficiently small γ . The upper bound of Theorem 33.10 is $\hat{Z}_{n,d,\gamma} \leq C_n g^{\mathfrak{d}_n} = C_n n^{-\mathfrak{d}_n/2} \gamma^{\mathfrak{d}_n/2}$, with n fixed. Dividing, we obtain the uniform positive lower bound for the expectation. For an event expressed by a bounded insertion, dividing this expectation by the supremum of the insertion gives a lower bound for the probability as well. The tangential amplitude may depend on γ arbitrarily within its uniform bounds. $\square$

Proof of the last assertion of Corollary 33.4. We verify the additional hypotheses on the whole tube in the setting of part (1) of Proposition 33.12, with $d = JK$. On the fixed working tube, Step 2 and Lemma 33.9 give $nE_{JK}(V(y,v)) \geq c_n|v|^2$, the quadratic expansion, and the bounds for the spectrum of H_K^{act} . The amplitude $a_{JK,\gamma}(y,v) = e^{-\bar{R}_{JK,\gamma}(V(y,v))} J(y,v)$ has uniform positive upper and lower bounds there: the centered effective interaction is bounded on D_r , and the Haar density of the based quotient is smooth and positive on the fixed compact subcharts. These bounds are used before the partition of unity of the atlas is absorbed into the tangential weight. Then the normal integral over the whole tube is at least $c_n\gamma^{\mathfrak{d}_n/2}$, uniformly in K and y , and the statements of Corollary 33.4 on the whole tube apply with $\delta = \gamma$, $L = H_K^{\text{act}}$, and $\text{rank } \mathfrak{d}_n = 6n^3 - 2$, with constants independent of K . $\square$

Proof of part (2) of Proposition 33.12. We first count the variables. Then we expand the classical action to second order along the flat reconstruction, which gives an identity

$S_K^\top \mathcal{H}_N S_K = I$ for the differential S_K of the reconstruction. Finally, we apply the derivative of the observable and Theorem 33.11.

Step 5: the count of variables. By Theorem 12.13, one averaging step from side Mm to side m has $6((Mm)^3 - m^3)$ real free coordinates. Summing over the J averaging steps of the j th RG step, from side n_j to side n_{j-1} , we find

$$D_j = 6(n_j^3 - n_{j-1}^3), \quad \mathfrak{d}_n + \sum_{j=1}^K D_j = 6N^3 - 2. \quad (33.21)$$

These are counts on based quotients. The five flat tangent directions belong to the stopping field, and they are held fixed in the quadratic computation. The right side $6N^3 - 2$ is the dimension of the based tangent space at side N , $6N^3 + 3$, minus the five-dimensional kernel of the Hessian of the Wilson action at a regular flat configuration (Lemma 33.9).

Step 6: the flat reconstruction. At a flat stopping field, Step 1 gives the flat refinement with vanishing normalized fluctuation variables at every averaging step, with the same based cycle holonomies and zero classical action. This uses Lemma 14.9 or, under the alternative local hypothesis, Lemma 33.7. The conditional constrained minimizers are these flat refinements. In particular, the flat zero-fluctuation reconstruction is critical for each classical action, and it is stationary for the Wilson flow on the finest lattice. The fixed domain conditions give smooth constraint coordinates and Haar disintegration densities near this reconstruction. By Theorem 15.13, the reconstruction maps $\hat{\mathcal{R}}$ preserve the coarse field, the orbit of the minimizer, the free coordinates, the square roots of the constrained Hessians, and the density in these coordinates. Their gauge transformations are smooth on these local charts and equal to the identity at the coarse vertices. No gauge transformation chosen measurably as a function of the configuration is differentiated. At each RG step we use the nested domains $\mathcal{D}_{\text{in}} \subseteq \mathcal{D}_{\text{mid}} \subseteq \mathcal{D}_{\text{out}}$ of Theorem 15.13. The flat coarse field lies below its margin by the domain margins verified in Step 1, and every free coordinate vanishes. Hence a joint neighborhood of the flat reconstruction lies in $\mathcal{D}_{\text{in}}$ before the outer domain of holomorphy is used. The fluctuation cutoff domains satisfy (33.2), and the reconstruction maps keep these domains, their kernels, and their identities at the coarse vertices. These checks use only the fixed parameters and the integer sides mM^i , $m \geq n \geq 4$, and no domain depending on the depth.

Step 7: completion of squares. We consider one averaging step with coarse field c and free coordinate z , and set

$$F(c, z) = E_{\text{old}}(b(c, z)), \quad z = z^*(c) + g_{\text{old}} K(c)^{-1/2} w, \quad K(c) = D_z^2 F(c, z^*(c)),$$

where E_{old} and g_{old} are the effective classical action and the coupling on the finer side. The minimizer satisfies $D_z F(c, z^*(c)) = 0$. By Taylor's formula and the definition of $K(c)$, we have

$$\frac{F(c, z)}{g_{\text{old}}^2} = \frac{F(c, z^*(c))}{g_{\text{old}}^2} + \frac{1}{2}|w|^2 + U^{\text{cl}}(c, w), \quad D_w^\alpha U^{\text{cl}}(c, 0) = 0 \quad (|\alpha| \leq 2).$$

The exact identity for the classical actions in Theorem 12.13 identifies the first term with $E_{\text{new}}(c)/g_{\text{new}}^2$, and these identities compose over the averaging steps. Since the displayed derivatives vanish for every coarse field c in the domain, U^{cl} has no joint Taylor terms of total degree at most two in the variation of the coarse field and the normalized fluctuation variables at the flat reconstruction. Hence, at the flat reconstruction, the quadratic part of the exponent is exactly

$$\frac{1}{2\gamma} \langle v, H_K^{\text{act}} v \rangle + \frac{1}{2} \sum_{j=1}^K |w_j|^2, \quad H_K^{\text{act}} = D_v^2(nE_{JK})(y, 0), \quad (33.22)$$

where we used that the exponent at the stopping side is $E_{JK}/g_n^2 = nE_{JK}/\gamma$. Second derivatives of the changes of coordinates do not affect this conclusion. In the second derivative of a composed scalar function, they are multiplied by the first derivative of the function, which vanishes at a flat configuration. Since the dependence of the positive square roots on the background is included in the differential of the coordinates, it creates no missing quadratic cross term. The Haar disintegration factors and the effective interaction remain in the amplitude and are not part of this identity for the classical quadratic form.

Step 8: Gaussian linear algebra. Let $\mathcal{H}_N$ be the Hessian of NE_0 at the finest flat configuration, on the based tangent space modulo its five-dimensional kernel; it is positive definite there by Lemma 33.9. By the count in the proof of that lemma, this quotient is the quotient of the full space of link variations by the kernel of the Hessian, which consists of all covariant gradients and the three constant neutral directions. Thus it is the space on which the lattice covariance of Theorem 33.11(3) is computed, and the exponent NE_0/γ is the Wilson exponent $(4N/\gamma) \sum_p q(U_p)$ used there. As in the statement, $v = \sqrt{\gamma}(H_K^{\text{act}})^{-1/2}z_0$. The normalized fluctuation variable $w_{j,l}$ of each averaging step enters through the differential $g_{j,l}K_{j,l}^{-1/2}w_{j,l}$ of the free coordinate. Composing with all finer reconstruction steps, we obtain the differential of the configuration on the finest lattice. After extracting $\sqrt{\gamma}$, we denote by S_K this linear map from $(z_0, w_1, \dots, w_K)$ to the quotient on the finest lattice. It does not depend on γ : the remaining coupling factors are $g_{j,l}/\sqrt{\gamma}$, and the classical maps and Hessians do not depend on the coupling. Since the quadratic part of NE_0/γ at the finest flat configuration, composed with the reconstruction, is (33.22), we have

$$S_K^T \mathcal{H}_N S_K = I. \quad (33.23)$$

This makes S_K injective. Since its domain and codomain have the same dimension by (33.21), S_K is invertible, and

$$S_K S_K^T = \mathcal{H}_N^{-1}.$$

Equivalently, consider independent standard real Gaussian variables in the normal coordinates of the stopping field and in all normalized fluctuation variables. In the auxiliary quadratic model, the differential of the reconstruction sends them to the Gaussian field on the finest lattice with covariance $\gamma \mathcal{H}_N^{-1}$.

Step 9: the observable. Let ℓ_N be the derivative of the flowed based cycle logarithms on the finest lattice, followed by the projection onto $\mathcal{N}_y$. By the computations for gauge directions and flat tangent directions in the proof of Theorem 33.11, ℓ_N vanishes on the kernel of the quotient. Applied to the columns of S_K , it gives exactly the response maps of the statement:

$$\ell_N S_K = (A_{0,K}, A_{1,K}, \dots, A_{K,K}).$$

The finest flat reconstruction has the same based cycle holonomies as the constant links $e^{\vartheta_{iu}/N}$. To justify the use of this Fourier representative, we fix a spanning tree and compare the flat transports of the reconstruction with those of the constant links. Their transports along the tree determine a periodic vertex gauge transformation, equal to the identity at the root, because the three cycle holonomies agree. This gauge transformation identifies the two flat configurations. The Wilson action and the Wilson flow commute with it, and the based holonomies are the same before and after it. If such a gauge transformation depends on parameters, its derivative contributes a covariant gradient, whose normal logarithmic response vanishes by the same computation. The covariance of ℓ_N is therefore exactly the one computed in Theorem 33.11(3), and

$$\sum_{j=0}^K A_{j,K} A_{j,K}^T = \ell_N S_K S_K^T \ell_N^T = \ell_N \mathcal{H}_N^{-1} \ell_N^T = \Sigma_\theta^N(y).$$

Taking traces and using the uniform upper bound for Σ_θ^N on the patch, which follows from Theorem 33.11(1) and (3), we obtain $\sum_{j=0}^K \|A_{j,K}\|_{\text{HS}}^2 \leq 4C$.

Step 10: the block of the stopping field. If T_K is the response of the normal variable of the stopping field before normalization, then

$$H_K^{\text{act}} = n H_K^E, \quad A_{0,K} = T_K (H_K^{\text{act}})^{-1/2} = n^{-1/2} T_K (H_K^E)^{-1/2}.$$

Multiplying by the transpose, we obtain the last identity of the statement. For $K = 0$ this block gives the whole covariance; in particular, it cannot be omitted. Since the gauge transformations of the reconstruction maps $\hat{\mathcal{R}}$ are equal to the identity at the coarse vertices and contribute only gauge variations, which are annihilated, the proof applies to these maps without a new choice of representatives. No growth in the number of scales enters the trace bound or the error of the Laplace approximation. $\square$

Remark 33.13. The convergence of the effective classical actions as the depth tends to infinity (Theorem 11.2) does not allow one to replace the phase at finite depth in an exponent divided by g^2 , uniformly as $\gamma \rightarrow 0$. None of the proofs above makes such a replacement. The Gaussian covariance computed here is the covariance of the quadratic model at a flat background; transferring its moments to the flowed observables under the interacting measures requires the additional estimates of Sections 34 and 35. The identification of the limiting observables uses Theorem 29.25, and Lemma 5.6 identifies the continuous functions of the cycle holonomies that are invariant under simultaneous conjugation as functions of the seven traces; neither identification replaces a comparison of normal moments.

33.9 An Abelian comparison measure

For later comparison with the measures λ_γ we record one explicit comparison measure in which all cycle holonomies commute: the Abelian Maxwell measure with Haar cycle holonomies of Definition 4.13. It is used, together with the other comparison measures, in Sections 35 and 36.

Lemma 33.14. *Embed the subgroup $\{e^{v\tau_3} : v \in \mathbb{R}\}$ of $SU(2)$. Take a centered transverse Gaussian Maxwell field with one real color on $\mathbb{T}_L^3$, with covariance $g_3^2(-\Delta)^+ P_{\text{tr}}$, where $(-\Delta)^+$ is the inverse of $-\Delta$ on the nonconstant Fourier modes and P_{tr} the projection onto transverse fields, and three independent phases, uniformly distributed in $(\mathbb{R}/4\pi\mathbb{Z})^3$ and independent of the field. Smooth the field by the scalar heat flow at a fixed positive time, add the flat connection with these cycle phases, and form the three cycle holonomies in the embedded subgroup. Then the following hold.*

1. *The smoothed field is almost surely smooth, so the holonomies are defined. Conditionally on the Maxwell field, the three cycle angles are distributed according to the Haar probability measure on $(\mathbb{R}/4\pi\mathbb{Z})^3$; hence their joint law is exactly Haar. The three holonomies commute, and an independent common random conjugation does not change the value of any function invariant under simultaneous conjugation.*
2. *Let R_{ab} be the Euclidean norm of the three logarithms of the cycle holonomies, taken in the principal chart. For every nonnegative $\varphi \in C_c^\infty((1, 2))$ and every δ with $0 < 4\delta < 2\pi$,*

$$\mathbb{E}_{\text{ab}} \varphi(R_{\text{ab}}/\delta) = (4\pi)^{-3} 4\pi\delta^3 \int_1^2 \varphi(v) v^2 dv.$$

This lemma specifies a comparison probability measure and makes no assertion about the Wilson measure.

Proof. In the Fourier modes $k = 2\pi m/L$, $m \neq 0$, we use the transverse multiplier $g_3^2|k|^{-2}(I - kk^T/|k|^2)$ in one real color, with real sine and cosine bases orthonormal

in $L^2(\mathbb{T}_L^3, dx)$. Independent real Gaussian coefficients in a transverse orthonormal basis define this field. For every $s > 1/2$, the expected sum of the squared coefficients with the weight $(1 + |k|^2)^{-s}$ is finite, since $\sum_{m \neq 0} |m|^{-2-2s} < \infty$. The weighted sum of squared coefficients is then a nonnegative random variable with finite expectation, and the Fourier series converges almost surely in this weighted Hilbert norm. This constructs the field as a distribution. After any positive heat time, the expected sum of the absolute values of the differentiated Fourier coefficients is finite at every order of differentiation, because of the factor $e^{-t|k|^2}$. The smoothed field is therefore almost surely smooth, and the three cycle integrals are defined.

We add the independent flat connection whose three cycle integrals are the independent uniform phases $\alpha_i \in \mathbb{R}/4\pi\mathbb{Z}$. In the embedded Abelian subgroup, each smoothed cycle angle is $\alpha_i + l_i$ modulo 4π , where l_i is a function of the Maxwell field. Since a translation of a uniform variable on the circle preserves its law, the three angles are distributed, conditionally on the Maxwell field, according to the Haar probability measure. If $\beta_i \in [-2\pi, 2\pi)$ are their representatives, the distance of $e^{\beta_i \tau_3}$ from the identity is $|\beta_i|$, and $R_{ab} = |\beta|$. Since the support of the insertion lies in the Euclidean ball of radius $2\delta < 2\pi$, polar coordinates in $\mathbb{R}^3$ give

$$(4\pi)^{-3} \int_{\mathbb{R}^3} \varphi(|\beta|/\delta) d\beta = (4\pi)^{-3} 4\pi\delta^3 \int_1^2 \varphi(v) v^2 dv.$$

All group elements here belong to the same Abelian subgroup. An independent common conjugation therefore preserves the values of all functions invariant under simultaneous conjugation, and it preserves commutativity. This completes the proof. $\square$

Uniform fluctuation bounds and comparison of holonomies

In this section, we prove estimates for the conditional refinement of the coarse field to the fine field that are uniform as the dimensionless coupling $\gamma = g_3^2 L$ tends to zero, and we transfer them to the holonomy matrices of the Wilson flow at a positive time. The main application is Theorem 34.13, which is Theorem 2.9: every subsequential limit of the continuum measures λ_γ as $\gamma \rightarrow 0$ is supported on stationary flat trajectories. The holonomy estimates are also the inputs for the normal fluctuations near regular commuting triples in Section 35 and for the quartic asymptotics near the central triples in Section 36.

The basic comparison is the following. We fix the coarse field and the normalized fluctuation variables of the first m renormalization group steps from the coarse side, and we set the normalized fluctuation variables w_j of all later steps equal to zero. Applying the same reconstruction maps gives a configuration on the finest lattice, which we evolve by the same Wilson flow as the reconstructed field. We then compare the holonomies of the two flowed configurations around fixed loops, as matrices in one common initial gauge frame. The comparison configuration is neither the interpolation of the field on the m -th lattice nor a conditional mean: setting $w_j = 0$ is different from setting the centered variable $w_j - \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$ to zero.

We use three probability settings, summarized in Table 34.1.

Two estimates explain the exponent conditions used throughout. First, under the conditioned measure the reconstructed field at mesh ℓ_j has supremum norm at most of order $r\varrho^j + g_3\sqrt{\ell_j}p_j$, where $\varrho < B^{-1/2}$ is the contraction factor of the zero-fluctuation reconstruction maps and p_j is the logarithmic cutoff on the fluctuation variables. The signed difference between the linearized reconstruction and the interpolation of the coarser field costs a factor $\ell_j^{\alpha-1}$ in the norm of the linear heat evolution. Its random part therefore has size $g_3\ell_j^{\alpha-1/2}$, which is summable over the geometric sequence of meshes because $\alpha > 1/2$. A classical quadratic term has size ϱ^{2j}/ℓ_j , which is summable because $B\varrho^2 < 1$. The second point concerns maxima. The supremum over the finest lattice of a linear function of the fluctuation variables would cost a factor $\sqrt{\log(L/a)}$, which grows as the finest mesh $a \rightarrow 0$. Instead, we cover the coefficient arrays at the mesh ℓ_j of their inputs, in metrics of bounded diameter, and chain in these metrics. The loss is then

Setting	What is fixed	Random variables	Conclusion
Conditioned Wilson measure (Theorems 34.9 and 34.10)	The parameters; the coarse field is random	Fluctuation variables with their conditional fluctuation kernels	Holonomy differences have moments of order $\gamma^{p/2}$, and the leading term decays in m
Prescribed coarse field (Theorem 34.11)	Every initial field $v = e^{\theta_0}$ with $\ \theta_0\ _\infty \leq r$	The same kernels, iterated from v	Moments of order $\gamma^{p/2}$ of the supremum over a time interval, uniformly in v
Prescribed flat field (Lemma 34.12)	A regular flat coarse field y	Independent standard Gaussian vectors inserted into derivatives of the reconstruction	The omitted Gaussian covariance decays geometrically in m

Table 34.1. The three probability settings of this section.

$\sqrt{\log(2 + L/\ell_j)}$, which is summable against the positive powers of ℓ_j . Combined with dimension-free concentration for the conditional fluctuation kernels, this gives moment bounds of order $\sqrt{\gamma}$ without a factor $|\log \gamma|$ in the leading term.

The proofs use the following inputs. The concentration estimates for the conditional fluctuation kernels are those of Theorem 5.4, which adapt the methods of Brascamp and Lieb [33] and Bakry and Émery [8] to products of balls with reflecting boundary. The quadratic estimate there is of Hanson–Wright type [108], in the setting of dependent coordinates exemplified by Adamczak [1]. Neither independence nor Gaussianity of the fluctuation variables is used. The maximal estimates use increment chaining in the tradition of Dudley [79] and Dirksen [74, Theorems 3.2 and 3.5], in the elementary form of Lemma 16.4. The reconstruction maps, their coefficient bounds, and the conditional kernels come from Section 15; the zero-fluctuation maps are built from constrained minimizers in the tradition of Bałaban’s background fields [10, 11, 13, 16]. The parameters are those of Theorem 23.2. The coefficient bounds for the four parts of a reconstruction increment and for their ordered quadratic products are those of Sections 26 to 28. The decomposition of a rough initial datum into its linear heat evolution, its quadratic Picard term, and a smoother remainder follows Da Prato and Debussche [63] and Cao and Chatterjee [42]; the lattice flow theory is in Sections 24 and 25, and the flow is Lüscher’s Wilson flow [128]. The final theorem uses the continuum measures and the energy compactness of Section 29, which rest on the state space of Cao and Chatterjee [43] and on Uhlenbeck’s compactness theorem [151], and the stopped measures on the coarse lattice of Section 33.

Sections 34.1 and 34.2 fix the notation and collect the deterministic inputs: heat estimates, covering estimates for the coefficient arrays, the zero-fluctuation continuation, and two elementary probability facts. Sections 34.3 to 34.6 prove conditional bounds uniform in γ , recall the stability of based holonomy matrices, and compare the two reconstructions, first through their flow data and then through their holonomies, under the conditioned measure and for every prescribed coarse field. Sections 34.7 and 34.8 contain the applications: the Gaussian estimate at a prescribed flat field and the flat support of the weak-coupling limits.

34.1 Setting and notation

Throughout this section we fix $L > 0$, $\theta > 0$, a finite horizon $T \geq \theta L^2$, an odd integer $M \geq 3$, an integer $J \geq 1$, $B = M^J$, and a coupling ceiling $0 < \gamma_0 \leq 1$. All parameters of the small-field event, the analytic domains, and the coefficient bounds of the reconstruction are fixed as in Theorem 23.2, before $\gamma \in (0, \gamma_0]$ is chosen. We fix an integer $n \in 4\mathbb{N}$ in the lattice refinement sequence under consideration, at least the minimum coarse side of Theorem 23.2, and such that every flat coarse field has a representative well inside the small-field domain. Precisely, we fix $r_{\text{flat}} < r$ with $2\pi/n < r_{\text{flat}}/4$, so that Lemma 14.9 applies; here r is the radius of the small-link domain B_r . We use the exponents

$$\frac{1}{2} < \alpha = \alpha_{\text{fl}} < \frac{2}{3}, \quad \beta = \alpha - \frac{1}{2}, \quad \kappa_{\text{fl}} = 1 - \frac{3\alpha}{2}, \quad 0 < \sigma = \sigma_{\text{sm}} < 1 - \alpha,$$

and the weighted norms X_α^a , Y_α^a , H_σ^a , and V_σ^a of Definition 4.8 on the time interval $[0, T]$. These norms include every component, every physical first difference, and every shift of the Wilson stencil required there. For a random norm such as $X_\alpha^a(f)$, we write $\|X_\alpha^a(f)\|_p$ for the L^p norm of this scalar random variable, and similarly for conditional L^p norms. The moment order p is unrelated to the cutoff radii p_j defined below.

Definition 34.1. Fix an integer $K \geq 0$ and set $N = nB^K$. For $0 \leq j \leq K$ put

$$\ell_j = \frac{L}{n} B^{-j}, \quad a = \ell_K, \quad \mathfrak{g}_j^2 = \frac{\gamma}{nB^j} = g_3^2 \ell_j, \quad t_j = M^{(J-1)/2} \mathfrak{g}_j, \quad p_j = A_w \log(1/t_j).$$

1. *Conditional refinement.* Let $\nu = \nu_{N,n,\gamma}$ be the conditioned Wilson measure. Let $\hat{\theta}_0$ be the link logarithm of the coarse field on Λ_n in the gauge of the reconstruction maps, and for $1 \leq j \leq K$ let

$$\hat{\theta}_j = \hat{\Phi}_j(\hat{\theta}_{j-1}, (g_{j,r} w_{j,r})_r),$$

where $\hat{\Phi}_j$ is the reconstruction map of Theorem 15.13 for the j -th renormalization group step counted from the coarse side, from mesh ℓ_{j-1} to mesh ℓ_j . Here $w_j = (w_{j,r})_r$ are the normalized fluctuation variables of this step, r runs over its J intermediate scales, and $g_{j,r}$ are the couplings of these scales; their ratios to t_j

lie between $M^{-(J-1)/2}$ and 1. Let $\mathcal{F}_j = \sigma(\hat{\theta}_0, w_1, \dots, w_j)$ be the backward filtration. Conditionally on $\mathcal{F}_{j-1}$, the vector w_j has the law $k_j(\hat{\theta}_{j-1}, dw_j)$ of the conditional fluctuation kernel (4.8) of this step, on its fluctuation cutoff domain and at prior depth $J(K-j)$ (Lemma 15.10 and Theorem 15.9).

2. *Zero-fluctuation maps.* Let $P_j(v) = \hat{\Phi}_j(v, 0)$ and $L_j = DP_j(0)$. There exists a convex working ball, given by Lemma 15.16, which contains all the fields $\hat{\theta}_j$ and all the zero-fluctuation continuations below, and on which

$$\sup_j \|DP_j\| \leq a_{\text{map}}, \quad a_{\text{map}} \sqrt{B} \leq \frac{32}{49}, \quad \mathfrak{q} = Ba_{\text{map}}^2 < 1, \quad \kappa = -\frac{\log a_{\text{map}}}{\log B} > \frac{1}{2}.$$

We write $\varrho = a_{\text{map}}$. The absolute coefficient bounds of the reconstruction maps (on rows and on volume-normalized columns), their bounds in the weighted heat norms with first differences, the bounds on the three ordered quadratic forms of Theorem 27.5, the bounds on the conditional means, and the bounds on the derivatives of the normalized conditional log-densities with respect to the incoming field, $\partial \log k = -\partial V + \mathbb{E}_k \partial V$ (with the parameters and bounds of Theorems 13.10 and 15.9), are fixed on a larger domain before the working ball is chosen. Since $q_{\text{anc}} \leq \mathfrak{q}$, the ratio $\mathfrak{q}$ may be used in place of q_{anc} in all the estimates of Section 27.

3. *Zero-fluctuation continuation.* For $0 \leq m \leq K$ let $\hat{\theta}_j^0 = \hat{\theta}_j$ for $j \leq m$, and $\hat{\theta}_j^0 = P_j \cdots P_{m+1} \hat{\theta}_m$ for $m < j \leq K$.
4. *Flow data.* Let $I_{\ell_j \rightarrow a}$ be the cardinal interpolation of Lemma 16.3 from mesh ℓ_j to mesh a , with $I_{a \rightarrow a}$ the identity, and let P_t^a be the lattice heat semigroup. On the finest lattice define

$$\begin{aligned} x_j &= P^a I_{\ell_j \rightarrow a}(\hat{\theta}_j / \ell_j), & y_j &= \mathcal{Q}_a(x_j, x_j), \\ x_j^0 &= P^a I_{\ell_j \rightarrow a}(\hat{\theta}_j^0 / \ell_j), & y_j^0 &= \mathcal{Q}_a(x_j^0, x_j^0), \end{aligned}$$

where $\mathcal{Q}_a$ is the bilinear Duhamel term of the Wilson flow with gauge damping, in the order displayed in Section 4; it is not symmetric.

5. *Increments.* Let $A_j = \hat{\theta}_j / \ell_j$, $H = \ell_{j-1}$, and $\zeta_j = w_j - \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$. Then $d_j = x_j - x_{j-1}$ satisfies $d_j = s_j + z_j + r_j + u_j$, where

$$\begin{aligned} s_j &= \frac{\mathfrak{g}_j}{\ell_j} P^a I_{\ell_j \rightarrow a} E_{\hat{\theta}_{j-1}} \zeta_j, & z_j &= P^a I_{\ell_j \rightarrow a} (A_j - \mathbb{E}(A_j \mid \mathcal{F}_{j-1})) - s_j, \\ r_j &= P^a \left(I_{\ell_j \rightarrow a} \frac{L_j \hat{\theta}_{j-1}}{\ell_j} - I_{H \rightarrow a} \frac{\hat{\theta}_{j-1}}{H} \right), \\ u_j &= P^a I_{\ell_j \rightarrow a} U_j^0, & U_j^0 &= \mathbb{E}(A_j \mid \mathcal{F}_{j-1}) - \frac{L_j \hat{\theta}_{j-1}}{\ell_j}. \end{aligned} \tag{34.1}$$

Here E_θ is the normalized first Taylor coefficient of the reconstruction at the conditional mean, in the centered fluctuation variables, as in Lemma 26.2. Thus s_j is the

centered linear fluctuation term, z_j the centered nonlinear heat term, r_j the signed interpolation defect, and u_j the predictable term. For $j > m$ the continuation has increment $d_j^0 = x_j^0 - x_{j-1}^0 = r_j^0 + u_j^{\text{cl},0}$, where r_j^0 is r_j with $\hat{\theta}_{j-1}$ replaced by $\hat{\theta}_{j-1}^0$, and $u_j^{\text{cl},0} = P^a I_{\ell_j \rightarrow a} U_j^{\text{cl}}(\hat{\theta}_{j-1}^0)$ with $U_j^{\text{cl}}(v) = \ell_j^{-1}(P_j(v) - L_j v)$.

6. *Holonomies.* Let $\mathcal{H}^K = (\mathcal{H}_i^K)_{i=1}^3$ be the based holonomies around the three coordinate cycles of $\mathbb{T}_L^3$ of the configuration $\Psi_{\theta L^2}^{(N)}(\exp \hat{\theta}_K)$, that is, of $\exp \hat{\theta}_K$ after the Wilson flow for the lattice flow time $4\theta L^2/a^2$. Let $\Xi_m^K = (\Xi_{m,i}^K)_{i=1}^3$ be the same triple for $\exp \hat{\theta}_K^0$. Both triples are computed in the common initial frame of the reconstruction maps.

The two configurations on the finest lattice compared in this section are

$$U_K = \exp \hat{\theta}_K, \quad U_{K,m}^0 = \exp(P_K \cdots P_{m+1} \hat{\theta}_m).$$

Above level m , the first is refined with the random fluctuation variables and the second with zero fluctuation variables:

$$\begin{array}{ccccccc} \hat{\theta}_0 & \rightarrow & \cdots & \rightarrow & \hat{\theta}_m & \xrightarrow{\hat{\Phi}_{m+1}(\cdot, g^{w_{m+1}})} & \hat{\theta}_{m+1} \rightarrow \cdots \rightarrow \hat{\theta}_K, \\ \hat{\theta}_0 & \rightarrow & \cdots & \rightarrow & \hat{\theta}_m & \xrightarrow{P_{m+1}} & \hat{\theta}_{m+1}^0 \rightarrow \cdots \rightarrow \hat{\theta}_K^0. \end{array}$$

We have $U_j^0 = U_j^{\text{cl}}(\hat{\theta}_{j-1}) + m_j^{\text{ph}}(\hat{\theta}_{j-1})$, where the conditional mean correction $m_j^{\text{ph}}(v) = \mathbb{E}(A_j \mid \mathcal{F}_{j-1}) - \ell_j^{-1} P_j(v)$, at $\hat{\theta}_{j-1} = v$, is the difference between the conditional mean and the zero-fluctuation value. It enters $d_j - d_j^0$ and is estimated separately, never as part of a centered term.

We call $\mathcal{H}^K$ the flowed holonomy triple and Ξ_m^K the zero-fluctuation continuation triple. With $a_j = \ell_j$, $g_j = \mathfrak{g}_j$, the common mesh $\delta = a$, and the depth $\mathfrak{d}_j = J(K - j)$, the four terms in (34.1) are those of Lemma 26.2. There the logarithmic reconstruction map is $\mathcal{R}_j(\theta, w_j) = \hat{\Phi}_j(\theta, (g_{j,r} w_{j,r})_r)$; for an incoming link logarithm v , the zero-fluctuation reconstructed lattice potential is $C_j(v) = \ell_j^{-1} P_j(v)$; and the linearization on lattice potentials is $P_j = B L_j$, so that $\ell_j^{-1} L_j v = H^{-1} P_j v$. The expansion is taken at the conditional mean, not at zero (see Remark 26.3). In particular, the centered nonlinear heat term is $z_j = P^a I_{\ell_j \rightarrow a} Z_j$, the heat evolution of the conditionally centered nonlinear reconstruction residual Z_j of that lemma.

Constants. Unless stated otherwise, constants $C, c, \dots$ have the dependence described in item (6) of Definition 4.14, and may in addition depend on T, α, σ , the lengths of the loops considered, the margins of the reference neighborhoods of the flow, the fixed coefficient bounds of the reconstruction, and the fixed bounds on the conditional fluctuation kernels, their conditional means, and the derivatives of their log-densities. These data are fixed before the coupling is varied. The constants are independent of γ, K, N , the prior depths,

the indices m, j, i , and the finest mesh, unless an index is displayed. A subscript p allows dependence on a fixed moment order. All estimates concern the torus itself; no block region or frozen coordinate is varied. Recall that $g_3^2 = \gamma/L$, that $\mathfrak{g}_j = g_3\sqrt{\ell_j}$ is the coupling at mesh ℓ_j , that t_j is the largest coupling of the intermediate scales of the j -th step, and that p_j is the cutoff radius of its fluctuation variables.

Quadratic forms pulled back to earlier levels. The quadratic Picard term of the signed defect and of the predictable term is controlled through quadratic forms in the fields of earlier levels, as in Section 27. For $0 \leq i \leq j-1$ let $L_{j-1,i} = L_{j-1} \circ \cdots \circ L_{i+1}$, with $L_{j-1,j-1}$ the identity, and for an array v at level i we define

$$U_{j,i}v = P^a I_{\ell_j \rightarrow a} \frac{L_j L_{j-1,i} v}{\ell_j}, \quad V_{j,i}v = P^a \left(I_{\ell_j \rightarrow a} \frac{L_j L_{j-1,i} v}{\ell_j} - I_{\ell_{j-1} \rightarrow a} \frac{L_{j-1,i} v}{\ell_{j-1}} \right).$$

The three ordered quadratic forms are

$$\begin{aligned} T_{j,i}^{\text{UV}}(v, v) &= \mathcal{Q}_a(U_{j,i}v, V_{j,i}v), & T_{j,i}^{\text{VU}}(v, v) &= \mathcal{Q}_a(V_{j,i}v, U_{j,i}v), \\ T_{j,i}^{\text{VV}}(v, v) &= \mathcal{Q}_a(V_{j,i}v, V_{j,i}v), \end{aligned}$$

and we write $\omega \in \{\text{UV}, \text{VU}, \text{VV}\}$ for the label. Since $V_{j,j-1}\hat{\theta}_{j-1} = r_j$ and $U_{j,j-1}\hat{\theta}_{j-1} - V_{j,j-1}\hat{\theta}_{j-1} = x_{j-1}$, we have, at $(\hat{\theta}_{j-1}, \hat{\theta}_{j-1})$,

$$\mathcal{Q}_a(r_j, r_j) = T_{j,j-1}^{\text{VV}}, \quad \mathcal{Q}_a(x_{j-1}, r_j) = T_{j,j-1}^{\text{UV}} - T_{j,j-1}^{\text{VV}}, \quad \mathcal{Q}_a(r_j, x_{j-1}) = T_{j,j-1}^{\text{VU}} - T_{j,j-1}^{\text{VV}}. \quad (34.2)$$

The same identities hold for the continuation, with r_j^0 , x_{j-1}^0 , and $\hat{\theta}_{j-1}^0$. Let $S_{j,j-1}(v, v)$ be the quadratic part of $I_{\ell_j \rightarrow a} U_j^{\text{cl}}(v)$, and $S_{j,i}(v, v) = S_{j,j-1}(L_{j-1,i}v, L_{j-1,i}v)$; we call $S_{j,i}$ the quadratic initial forms, and we call H_σ^a the initial-array norm. By construction both kinds of forms are pulled back exactly by the linearized maps:

$$T_{j,i-1}^\omega(v, v) = T_{j,i}^\omega(L_i v, L_i v), \quad S_{j,i-1}(v, v) = S_{j,i}(L_i v, L_i v). \quad (34.3)$$

By Theorem 27.5, the coefficient norm of $T_{j,i}^\omega$ with output in the norm Υ_α^a is at most $C(1+j)^c q^{j-1-i}/\ell_i$, and by Theorem 27.7 the coefficient norm of $S_{j,i}$ with output in H_σ^a is at most Cq^{j-1-i}/ℓ_i . We write $K_{ji} = q^{j-1-i}/\ell_i$.

These coefficient bounds also hold on every compatible common mesh $a \leq \ell_j$, including meshes outside the geometric ladder. Here compatibility means that the ratios of the input and intermediate meshes to a are integers. We verify this extension using the absolute row and volume-normalized column norms R, C , and their bilinear counterparts R_2, C_2 , of Lemma 27.1 and Theorem 27.5. Set $H = \ell_{j-1}$, $G = \ell_i$, and $k = j-1-i$. Apply Lemma 17.2 with final mesh a ; its integer refinement hypothesis holds by compatibility.

The zero-mass and first-moment argument in the proof of Lemma 27.1(2)–(3), followed by the preceding linearizations, then gives, for $r = 0, 1$,

$$\begin{aligned} (R + C)(D_a^r U_{j,i}(s)) &\leq C_r A_{ji}(s + H^2)^{-r/2}, \\ (R + C)(D_a^r V_{j,i}(s)) &\leq C_r A_{ji}(s + H^2)^{-r/2} d_H(s), \end{aligned}$$

Here $A_{ji} = C_{CP}^k / G$ and $d_H(s) = H(s + H^2)^{-1/2}$. The matrices are the coefficients at time s of the displayed linear maps. Both column bounds use cancellation of each column before the earlier linearizations are composed; row bounds alone would not suffice. Each ordered quadratic Picard source $\mathcal{S}_s^\omega$ contains one spatial difference and at least one factor V . The pointwise-product bounds of Lemma 27.1(1) therefore give

$$(R_2 + C_2)(\mathcal{S}_s^\omega) \leq \frac{CA_{ji}^2 H}{s + H^2} \leq \frac{CK_{ji}}{s + H^2}, \quad \omega \in \{UV, VU, VV\},$$

because $A_{ji}^2 H = q_{anc}^k / G \leq q^k / G = K_{ji}$. One input uses a column bound and the other a row bound, so their intermediate volume factors cancel. The outer heat kernel has row and column norm one, and its first difference has both norms at most $C(t - s)^{-1/2}$. Consequently

$$\begin{aligned} (R_2 + C_2)(T_{j,i}^\omega(t)) &\leq CK_{ji} \log(1 + t/H^2), \\ (R_2 + C_2)(D_a T_{j,i}^\omega(t)) &\leq CK_{ji} \int_0^t (t - s)^{-1/2} (s + H^2)^{-1} ds. \end{aligned}$$

For $t \leq H^2$, multiplication by the two weights $(t + a^2)^\beta$ and $(t + a^2)^\alpha$ bounds both terms by $CK_{ji} H^{2\alpha-1}$, since $a \leq H$ and $2\beta = 2\alpha - 1 > 0$. For $t \geq H^2$, splitting the integral at $t/2$ gives the bound $C_{T,\alpha} K_{ji} [1 + \log(1 + T/H^2)]$. The factor $H^{2\alpha-1}$ is bounded for $H \leq H_0$, and (34.8) controls the logarithm by $C(1 + j)$ uniformly in the common mesh. This proves the asserted Y_α^a coefficient bound. For $S_{j,i}$, start with the static bound (27.31) before interpolation. Cardinal interpolation on any compatible mesh preserves its row and volume-normalized column bounds, and heat splitting gives the same bound in H_σ^a . No constant in this extension depends on H/a .

The identities (34.3) give a cancellation that we use repeatedly. For a bilinear form A and $P = L + E$ with L linear and $E(v)$ of order $|v|^2$, we have

$$A(Pv, Pv) - A(Lv, Lv) = A(Lv, Ev) + A(Ev, Lv) + A(Ev, Ev). \quad (34.4)$$

The right side has degree at least three in v . Its difference at two points v, v' is bounded by $(|v| + |v'|)^2 |v - v'|$ times the coefficient norm of A , and two undifferentiated factors remain. With $A = T_{j,i}^\omega$ or $S_{j,i}$ and $P = P_i$, $L = L_i$, this identity reduces the comparison of quadratic forms along the two reconstructions to centered fluctuation terms, classical terms of degree at least three, and conditional means.

The bilinear expansion of the quadratic Picard term. Since $\mathcal{Q}_a$ is bilinear, we have

$$y_l - y_m = \sum_{j=m+1}^l \{ \mathcal{Q}_a(x_{j-1}, d_j) + \mathcal{Q}_a(d_j, x_{j-1}) + \mathcal{Q}_a(d_j, d_j) \}. \quad (34.5)$$

Inserting $d_j = s_j + z_j + r_j + u_j$ produces, for each j , the sixteen ordered products $\mathcal{Q}_a(v_j, w_j)$ with $v, w \in \{s, z, r, u\}$, and the eight cross terms $\mathcal{Q}_a(x_{j-1}, v_j)$ and $\mathcal{Q}_a(v_j, x_{j-1})$ with the preceding field. We never estimate a product of two rough heat paths in the norm X_α^a directly. Instead, each term is handled by one of four tools: the estimates for the quadratic expression in the centered linear term in Theorem 26.12; the mixed linear rows of Theorem 26.10, for a product of s_j with a predictable heat path; the forms (34.2); and the ordered product inequality of Lemma 28.5, which states that for two heat paths v and $w = P^a w_0$,

$$Y_\alpha^a(\mathcal{Q}_a(v, w)) + Y_\alpha^a(\mathcal{Q}_a(w, v)) \leq C X_\alpha^a(v) H_\sigma^a(w_0), \quad (34.6)$$

separately for the two orders, with a constant independent of the coupling, the coarse side, and the depth. For differences between the two reconstructions we use the identity

$$\mathcal{Q}_a(v, u) - \mathcal{Q}_a(v^0, u^{\text{cl},0}) = \mathcal{Q}_a(v, u - u^{\text{cl},0}) + \mathcal{Q}_a(v - v^0, u^{\text{cl},0}), \quad (34.7)$$

where $v \in \{s, z, r, u\}$ and $v^0 = 0, 0, r^0, u^{\text{cl},0}$ respectively, together with the identity in which both arguments are interchanged; for the cross terms the same identity holds with $(v, v^0) = (x_{j-1}, x_{j-1}^0)$. Table 34.2 lists the tool used for each term in the three proofs where the expansion appears.

34.2 Heat estimates on a common mesh and fluctuation tails

We first record the elementary calculations needed to apply the coefficient bounds of Sections 15 and 26 to 28 uniformly in the coupling, and we specify which part of each earlier estimate at fixed coupling is used.

All uses below satisfy the following domain conditions. Each renormalization group step consists of J averaging steps; the lattice at level j has side nB^j , and the prior depth at level j is $J(K - j)$. Since $t_1^2 = \gamma/(Mn)$ and $t_j = t_1 B^{-(j-1)/2}$, the minimum-side condition of Theorem 23.2 puts every t_j below the upper bound on the couplings in the conditional kernel theorem. The inputs $g_{j,r} w_{j,r}$ of the reconstruction maps have size at most $t_j p_j$, and the larger coefficient domain and the working ball are those of Lemma 15.16, fixed before the coupling. In particular the deterministic amplitude bound of that lemma applies to these inputs, and the bounds with the factor a_{map} used below require no stochastic estimate uniform in the mesh. The heat horizon is always T . Every refinement ratio $\ell_i/a = B^{K-i}$ is an integer, and every order of spatial differences used below is an integer. The side n satisfies $2\pi/n < r_{\text{flat}}/4$, hence the condition on flat fields in Lemma 14.9.

These conditions let us evaluate the coefficients at zero fluctuation for a fixed incoming field, including a flat one; they do not identify a stopped measure with a marginal of the conditioned Wilson measure.

The linearized maps have the geometric contraction ratio $q_{\text{anc}} = C_{\text{CP}}^2/B$ of Lemma 15.16, and by the choice in Definition 34.1 we have $q_{\text{anc}} \leq q < 1$. All geometric losses below are taken from the positive gap $-\log q / \log B$ and the other positive exponents displayed in this section. The two ratios remain distinct in the normalization of differentiated coefficient rows; see (34.21).

In every use of the weighted coefficients of Theorem 27.5, the constants may depend on the coarse mesh $H_0 = L/n$. Indeed, for $H = \ell_{j-1} = H_0 B^{-(j-1)}$ and $j \geq 1$ we have

$$\begin{aligned} 1 + \log(1 + T/H^2) &\leq 1 + \log(1 + T/H_0^2) + 2(j-1) \log B \\ &\leq C_{T, H_0, B}(1+j), \end{aligned} \quad (34.8)$$

where in the first inequality we used $1 + B^{2(j-1)}T/H_0^2 \leq B^{2(j-1)}(1 + T/H_0^2)$. This constant is independent of the coupling, of all level indices, of the prior depth, and of the finest mesh. The coefficient rows without time weights keep the dependence stated in Theorem 27.5.

Lemma 34.2. *Put $H_0 = L/n$ and $\lambda = -\log q / \log B = 2\kappa - 1 > 0$. The following bounds hold.*

1. *For cardinal interpolation from any input mesh $H \geq a$ compatible with the common lattice, that is, with H/a an integer, and every fixed integer $q \geq 0$,*

$$\|D_a^q P_t^a I_{H \rightarrow a} v\|_\infty \leq C_q (t + H^2)^{-q/2} \|v\|_\infty. \quad (34.9)$$

2. *Let E be a coefficient array of a reconstruction map with respect to normalized fluctuation variables at input mesh H , whose output-fixed absolute coefficient row sums and volume-normalized input-fixed column sums satisfy the bounds of Lemma 15.16. Then*

$$\sup_x \|D_a^q P_t^a I_{H \rightarrow a} E(x, \cdot)\|_{\ell^2(w)} \leq C_q H^{3/2} (t + H^2)^{-3/4 - q/2}, \quad (34.10)$$

where $\ell^2(w)$ is the counting ℓ^2 norm in the fluctuation variables, and the operator norm of $D_a^q P_t^a I_{H \rightarrow a} E$ from $\ell^2(w)$ to the physical space L_a^2 , with measure a^3 times counting measure, is at most $C_q H^{3/2} (t + H^2)^{-q/2}$. The constants may depend on the fixed ratios of the reference volumes of the coordinate types within one renormalization group step.

3. *Let $H = \ell_{j-1}$, and let $\mathcal{D}_j v = I_{\ell_j \rightarrow a}(L_j v / \ell_j) - I_{H \rightarrow a}(v/H)$ be the initial value of the signed defect r_j when the incoming field is v . Then*

$$X_\alpha^a(P^a \mathcal{D}_j v) \leq C H^{\alpha-1} \|v\|_\infty. \quad (34.11)$$

4. Let $V_j(v)$ be the part of degree at least three of the initial value $I_{\ell_j \rightarrow a} U_j^{\text{cl}}(v)$ (this V_j is unrelated to the coarse configuration V of Section 34.8), so that $I_{\ell_j \rightarrow a} U_j^{\text{cl}}(v) = S_{j,j-1}(v, v) + V_j(v)$. On the working ball,

$$\|V_j(v) - V_j(v')\|_\infty \leq \frac{C}{H} (\|v\|_\infty + \|v'\|_\infty)^2 \|v - v'\|_\infty. \quad (34.12)$$

5. The conditional mean correction satisfies

$$\|m_j^{\text{ph}}(v)\|_\infty \leq C g_3^2 \|v\|_\infty, \quad U_j^0 = U_j^{\text{cl}}(\hat{\theta}_{j-1}) + m_j^{\text{ph}}(\hat{\theta}_{j-1}). \quad (34.13)$$

Every bound on the supremum norm of an initial array in items (4) and (5) implies the corresponding bound in the full norm H_σ^a , with a fixed factor.

6. If $x = P^a I_{H \rightarrow a}(v/H)$ and $h = H/B$, then

$$\sup_{t \leq T} \{ (t + h^2)^{\alpha/2} \|x(t)\|_\infty + (t + h^2)^{(\alpha+1)/2} \|D_a x(t)\|_\infty \} \leq C \{ X_\alpha^a(x) + \|v\|_\infty H^{\alpha-1} \}. \quad (34.14)$$

Each fixed higher difference $D_a^q x$ satisfies the corresponding bound with the weight $(t + h^2)^{(\alpha+q)/2}$.

7. For $b \geq 0$, $e > 0$, $0 < s < \min(e, \lambda)$, and $m \geq 0$,

$$\sum_{i>m} (1+i)^b B^{-ei} \leq C_{b,e} (1+m)^b B^{-em}, \quad (34.15)$$

$$\sum_{j>m} (1+j)^b \left(q^{j-1} + \sum_{i=1}^{j-1} q^{j-1-i} B^{-ei} \right) \leq C_{b,e,s} (1+m)^b B^{-sm}. \quad (34.16)$$

The corresponding fixed-level bound, including the initial contribution and every earlier ancestor, is

$$(1+j)^b \left(q^{j-1} + \sum_{i=1}^{j-1} q^{j-1-i} B^{-ei} \right) \leq C_{b,e,s} (1+j)^b B^{-sj}, \quad j \geq 1. \quad (34.16a)$$

These constants depend on b, e, s and the fixed B, λ , and not on m or j .

Proof. We begin with the cardinal interpolation. It has tensor-product basis functions $\psi((y-x)/H)$, where $\psi = \phi^{\otimes 3}$ and $\phi(u) = b_0(u)/\sum_{k \in \mathbb{Z}} b_0(u-k)$ for the fixed even smooth bump b_0 of Section 16.2, which is supported in $[-1, 1]$, positive inside, and flat at its endpoints; in the notation there, $b_0 = b$, $\phi = \zeta$, and $\psi = \zeta_3$. Its translates sum to one. Grouping the fine vertices according to their offset in a coarse cell, we find

$$a^3 \sum_{y \in \Lambda_N} \psi((y-x)/H) = H^3, \quad a^3 \sum_y \text{dist}(y, x) |\psi((y-x)/H)| \leq CH^4.$$

Taking q differences of the smooth basis costs $C_q H^{-q}$, independently of a/H . In particular these statements also hold for $H = a$, where the interpolation is the identity on arrays.

The scalar heat estimates follow from the symbol

$$\omega_a(k) = 4a^{-2} \sum_{i=1}^3 \sin^2(\pi a k_i / L), \quad k \in \mathbb{Z}^3, \quad |k_i| \leq L/(2a).$$

On this frequency cube $\omega_a(k) \geq c|k|^2/L^2$, and the symbol of D_a^q is at most $C_q(|k|/L)^q$. By Parseval's identity and comparison of each unit frequency cube with the corresponding Gaussian integral, we have

$$\|D_a^q P_t^a(x, \cdot)\|_{L_a^2} \leq C_{q,L,T} (t + a^2)^{-3/4 - q/2},$$

where the kernel is taken with respect to the physical measure. The corresponding L^1 bound is $C_{q,L,T} (t + a^2)^{-q/2}$. To prove it without discarding the constant mode, we periodize the heat kernel p_t of the infinite lattice. By the Cauchy–Schwarz inequality with the weight $(1 + |x|^2/(t + a^2))^2$ and Parseval's identity, the L^1 norm of $D_a^q p_t$ is at most

$$C(t + a^2)^{3/4} \|(1 + |x|^2/(t + a^2)) D_a^q p_t(x)\|_{L_a^2} \leq C_q (t + a^2)^{-q/2}.$$

In the last inequality, we used that multiplication by x_i corresponds to differentiation of the Fourier multiplier. Differentiating $e^{-t\omega_a}$ once or twice produces the factors $-t\partial_i\omega_a$ or $t^2(\partial_i\omega_a)^2 - t\partial_i^2\omega_a$, respectively, multiplying $e^{-t\omega_a}$, and differentiating a difference symbol lowers its frequency power by the number of derivatives. The bounds $|\partial_i\omega_a| \leq C\sqrt{\omega_a}$ and $|\partial_i^2\omega_a| \leq C$, together with the Gaussian integrals above, give the displayed powers. For $t \leq a^2$, the difference stencil and the facts that P_t^a is positive and has mass one (which follow from the Poisson series of the nearest-neighbor walk) give the bound $C_q a^{-q}$ instead. Periodization does not increase the L^1 norm, and the contribution of the constant mode to the L^2 norm is bounded on the horizon T by a constant depending on L and T . Splitting at $t = H^2$ now proves (34.9): for $t \leq H^2$ we put the differences on the cardinal basis and use that P_t^a has mass one, and for $t \geq H^2$ we use the L^1 bound and $t + a^2 \geq (t + H^2)/2$.

Next, we prove item (2) from its assumed output-fixed absolute coefficient row bounds and volume-normalized input-fixed column bounds. For the reconstruction coefficients used here, these joint bounds are supplied by Theorem 15.13(2) on the products checked in Lemma 26.1. Let $\mathcal{J} = I_{H \rightarrow a} E$. Its absolute row sums are at most C , and each of its columns has physical mass at most CH^3 . For an input ξ in counting ℓ^2 , by the weighted Cauchy–Schwarz inequality we have

$$a^3 \sum_y \left| \sum_\lambda \mathcal{J}_{y\lambda} \xi_\lambda \right|^2 \leq a^3 \sum_{y,\lambda} |\mathcal{J}_{y\lambda}| |\xi_\lambda|^2 \sum_\mu |\mathcal{J}_{y\mu}| \leq CH^3 \sum_\lambda |\xi_\lambda|^2.$$

For $t \leq H^2$, the absolute row of the differenced cardinal interpolation bounds the ℓ^2 row in (34.10) by CH^{-q} , and its volume-normalized column sum gives the same Schur

bound for the operator norm with the extra factor H^{-q} . For $t \geq H^2$, we compose $\mathcal{J}: \ell^2(w) \rightarrow L_a^2$ with the physical L_a^2 row of $D_a^q P_t^a$ computed above. The two norms multiply to $CH^{3/2}t^{-3/4-q/2}$, which proves (34.10). Composition with the operator $D_a^q P_t^a$ on L_a^2 , whose norm is at most $Ct^{-q/2}$, gives $CH^{3/2}t^{-q/2}$ and proves the operator bound. Conversion to the counting output norm would introduce a factor $a^{-3/2}$, which is canceled by the conversion of the physical heat-kernel coefficient row. Coordinate types at the other $J-1$ meshes of the step change these factors only by fixed powers of M .

For item (3), we use the signed mass identity of Theorem 15.13(4) and the column calculation of Lemma 27.1(2). The column of the linearized refinement and the column of the parent cardinal interpolation have equal physical mass, separately for every input orientation and color, and their difference has first absolute moment at most CH^4 about the input location. In a heat convolution we subtract the kernel value at the input location before summing such a column. Using the first difference of the kernel and this moment, we obtain, for every fixed integer $q \geq 0$,

$$\|D_a^q P_t^a \mathcal{D}_j v\|_\infty \leq \frac{C_q \|v\|_\infty}{H} (t + H^2)^{-q/2} \frac{H}{\sqrt{t + H^2}}.$$

For $t \leq H^2$ the same estimate follows from the unsigned cardinal rows. For $t \geq H^2$ the row bound requires a sampling estimate. On a cube of side H , repeated one-dimensional telescoping and averaging bounds the supremum of a sampled scalar function f by $CH^{-3} \sum_{|\eta| \leq 3} H^{|\eta|} \|D_a^\eta f\|_{L_a^1}$ on the enlarged cube, with at most one difference in each coordinate. In one dimension, we choose a vertex where $|f|$ is at most its average and telescope from that vertex; this is the discrete form of $|f(x)| \leq H^{-1} \int |f| + \int |Df|$. We apply it successively in the three coordinates. Applied to a differenced heat-kernel coefficient row, it shows that the sum over the cubes of H^3 times the local supremum is at most the L^1 bounds of its differences through three further orders. When $t \geq H^2$, these are at most $C(t + H^2)^{-(q+1)/2}$, since $H/\sqrt{t + H^2} \leq 1$. The first moment of the column gives the remaining factor H . For columns outside a fixed enlarged cube we apply this estimate on successive shells, whose polynomial volumes are summable against the exponential decay of the coefficients. This proves the row bound without counting output sites. Multiplying by the two weights of X_α^a and using $a \leq H$ proves (34.11); the weighted maxima are at most $CH^{\alpha-1}$ since $\alpha < 1$.

For item (4), we expand $P_j = L_j + P_{j,2} + P_{j,\geq 3}$ on the larger domain, where $P_{j,2}$ is quadratic and $P_{j,\geq 3}$ has degree at least three. The coefficientwise majorant series of Lemma 15.16 bounds the difference of the part of degree at least three by $C(\|v\| + \|v'\|)^2 \|v - v'\|$ on the working ball. Division by $\ell_j = H/B$ and interpolation prove (34.12), with B fixed.

For item (5), (27.41) directly gives $\|m_j^{\text{ph}}(v)\|_\infty \leq C_J(t_j^2/\ell_j)\|v\|_\infty$. Since $t_j^2/\ell_j = M^{J-1}g_3^2$, the fixed factor M^{J-1} is absorbed into the constant, proving the bound in (34.13). The

decomposition of U_j^0 is the conditional-mean identity of Theorem 27.7 and Lemma 28.5, together with Lemma 26.2. Since the heat semigroup contracts the supremum norm, the value part of $H_\sigma^a(f)$ for an initial array f is at most $C_{T,\sigma}\|f\|_\infty$; the bound $\|D_a P_t^a f\|_\infty \leq C(t+a^2)^{-1/2}\|f\|_\infty$ proved above gives the same bound for its first-difference part.

For (34.14), when $t \geq H^2$ we have $t+h^2 \leq 2(t+a^2)$, and the weights are bounded by those of X_α^a . When $t \leq H^2$, we use (34.9): for instance $(t+h^2)^{\alpha/2}\|x(t)\|_\infty \leq C(2H^2)^{\alpha/2}H^{-1}\|v\|_\infty$. For a higher difference, above H^2 we put the additional differences on a half heat step, and below H^2 we put them on the cardinal basis. This proves the assertion about higher differences.

To prove (34.15), we write $i = m + k$ and use $1 + m + k \leq (1 + m)(1 + k)$; the remaining series is $\sum_{k \geq 1} (1 + k)^b B^{-ek} < \infty$. For (34.16a), use $q = B^{-\lambda}$ and factor $B^{-s(j-1)}$ out of the convolution. Its remaining summands are

$$B^{-(\lambda-s)(j-1-i)} B^{-(e-s)i}, \quad 1 \leq i \leq j-1.$$

Their sum is bounded by the product of the convergent geometric series with ratios $B^{-(\lambda-s)}$ and $B^{-(e-s)}$. The initial term has the same bound because $\lambda > s$; the fixed factor B^s converts $B^{-s(j-1)}$ to B^{-sj} . This proves the fixed-level bound. Summing it over $j > m$ and applying (34.15) with exponent s proves (34.16), including all ancestors $i \leq m$ and the initial contribution. $\square$

The next lemma prepares the chaining arguments. Each component of a reconstruction increment is a random field indexed by time, position, and finitely many labels, and it is a linear, quadratic, or nonlinear function of the normalized fluctuation variables of one step, with deterministic coefficients. We divide each coefficient by the scale at which it is expected to be bounded, and we measure the distance between two indices by the distance between their normalized coefficients. In the lemma, we show that these coefficient metrics have bounded diameter and covering numbers polynomial in L/h_* , where h_* is the input mesh of the coefficients and not the finest mesh a . This is why the maximal estimates below contain $\log(2 + L/\ell_j)$ rather than $\log(L/a)$.

Lemma 34.3. *Fix the cutoff radii, an incoming field in the reconstruction domain, and a common mesh a no coarser than all input meshes in use. Each process below is indexed by the time $t \in [0, T]$, the position on the lattice of mesh a , and the finitely many labels (components, shifts, and the choice between value and first difference) required by its norm. Write $h = \ell_j$, $H = \ell_{j-1}$, and for $1 \leq i \leq j-1$ put $G = \ell_i = B^k H$ with $k = j-1-i$, and*

$$b_i(v) = C_J(a_{\text{map}}\|v\|_\infty + \mathfrak{g}_i p_i).$$

The quantity $b_i(v)$ bounds $\|\widehat{\theta}_i\|_\infty$ on the whole conditional fiber over the incoming field $v = \widehat{\theta}_{i-1}$. We consider the following families.

1. The coefficient of $w_j - \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$ in s_j , in the norm X_α^a , with scale A_s .
2. The quadratic part of the initial value of z_j , in the norm H_σ^a , with scale $A_{z,\sigma}$, and of the heat path z_j , in the norm X_α^a , with scale $A_{z,\alpha}$.
3. The quadratic part of $\mathcal{Q}_a(s_j, s_j)$, in the norm Y_α^a , with scale A_{ss} .
4. The two ordered terms $\mathcal{Q}_a(b, s_j)$ and $\mathcal{Q}_a(s_j, b)$, in the norm Y_α^a , where b is a predictable heat path with $\|D_a^q b(t)\|_\infty \leq C_q \mathcal{R}(t + \ell_j^2)^{-(\alpha+q)/2}$ for $0 \leq q \leq 4$; the scale is $Cg_3 \mathcal{R} h^{\alpha-1/2}$.
5. For each ω separately, the gradient in w_i of $T_{j,i}^\omega(\widehat{\theta}_i, \widehat{\theta}_i)$ on the whole conditional fiber, in the norm Y_α^a , with scale $C(1+j)^c K_{ji} g_i b_i(v) G^{2\alpha-1}$.
6. The gradient in w_i of $S_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i)$ on the whole conditional fiber, in the weighted value norm V_σ^a , with scale $CK_{ji} g_i b_i(v) G^\sigma$.

In items (1)–(3) the coefficients are real vectors or real symmetric matrices in the centered vector $w_j - \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$, before subtraction of any quadratic mean, and

$$\begin{aligned} A_s &= Cg_j h^{\alpha-1}, & A_{z,\sigma} &= Cg_3^2 h^\sigma, \\ A_{z,\alpha} &= Cg_3^2 h^\alpha, & A_{ss} &= Cg_3^2 h^{2\alpha-1}. \end{aligned} \quad (34.17)$$

In items (5)–(6) the gradients are taken before subtraction of the conditional mean. Put $h_* = H$ in item (5) and $h_* = h$ otherwise. Let S be the applicable scale, and after division by S let d_* be the counting ℓ^2 distance of coefficient rows for a linear family, the maximum of the Hilbert–Schmidt and operator distances for a quadratic family, and $\sup_w \|\nabla_w F_z - \nabla_w F_{z'}\|_2$ for a gradient family. If $S = 0$, the corresponding evaluations vanish and are omitted. If the fixed factor in S is large enough, then there exist a fixed dimensionless space-time distance d_0 , which separates the finitely many labels, and fixed $P_0 < \infty$, $\zeta > 0$, and $P, Q_1 < \infty$ such that

$$\begin{aligned} d_*(z, z') &\leq Ch_*^{-P_0} d_0(z, z')^\zeta, \\ \text{diam}(d_*) &\leq 2, \quad N(v, d_*) \leq C(1 + L/h_*)^P v^{-Q_1} \quad (0 < v \leq 1). \end{aligned} \quad (34.18)$$

At every fixed cutoff, the normalized coefficient or gradient images have continuous compact completions. The constants may depend also on the constants C_q of the background in item (4); they are independent of the coupling, the cutoff, the separation $j - i$, the incoming field, and the finer common mesh.

Proof. All assertions concern deterministic functions on a fixed conditional fiber, that is, for a fixed incoming field.

We begin with the scales. For the centered linear family, by (34.10) and the operator bound of Lemma 34.2, the rows have size

$$C(g_j/h)h^{3/2}(t+h^2)^{-3/4-\iota/2}, \quad \iota = 0, 1,$$

for the value and the first difference. Multiplication by $(t+a^2)^{(\alpha+\iota)/2}$ gives at most $Cg_j h^{\alpha-1}$: for $t \leq h^2$ we use $t+a^2 \leq 2h^2$, and for $t \geq h^2$ the remaining power $t^{\alpha/2-3/4}$

is decreasing. The quadratic coefficient rows of Theorem 26.11 have size $Cg_3^2 h^{3/2}(t + h^2)^{-3/4-\iota/2}$; the same two-range calculation, with norm exponent σ or α , gives the two scales of the centered nonlinear heat term. The Hilbert–Schmidt and operator rows of the Duhamel term in Theorem 26.12 contain the factor $t/(t + h^2)$, which vanishes at microscopic times, and give the scale A_{ss} in the norm Y_α^a . None of these steps uses a maximal estimate from those results.

Next we consider the covers. Every input of a coefficient lies on the input lattice of mesh h , which carries at most $C_J(L/h)^3$ scalar coordinates. The derivative bounds of the cardinal interpolation in Lemma 16.3 cost $h^{-\iota}$ at each fixed spatial order ι . We apply the exact shifted product rule of Lemma 24.2 to the source of a quadratic coefficient. Time derivatives are two physical differences of the smooth input, and the time derivative of a Duhamel term is its source plus its Laplacian, which we commute onto the source. Absolute coefficient rows bound the nuclear norm, hence the Hilbert–Schmidt and operator norms, of these differentiated matrices. In these crude estimates we may even sum over all inputs, since their number is a fixed power of h^{-1} . Thus the derivative bounds cost fixed powers of h^{-1} , keep the coupling prefactor, and contain no inverse volume of the finest mesh. Division by the scale cancels the prefactor. For a weight exponent $0 < \eta \leq 1$ we have

$$|(t + a^2)^\eta - (s + a^2)^\eta| \leq |t - s|^\eta.$$

Together with the differentiated rows, this gives for each normalized coefficient a modulus $Ch^{-P_0}(\text{dist}(x, y) + |t - s|^\zeta)$ with fixed P_0 and $\zeta > 0$, including at $t = 0$. Different scalar labels are covered by a finite union. Each normalized coefficient has norm at most one, which proves that the diameter is at most two, independently of the modulus.

For the mixed terms of item (4), the deterministic part of Theorem 26.10 and Lemma 26.7 applies with input mesh h and the four differences assumed for the background. It gives the scale $g_3 \mathcal{R} h^{\alpha-1/2}$, the normalized cover, and the diameter bound. Its two output row bounds are

$$Cg_3 \mathcal{R} h \int_0^t (s + h^2)^{-(\alpha+1)/2} (t - s + h^2)^{-\epsilon} ds, \quad \epsilon \in \{3/4, 5/4\}. \quad (34.19)$$

These are the deterministic ordered rows of that theorem, in its version for a predictable heat background with four differences. Splitting the integral at h^2 and at $t/2$, we find that the two weights of the norm give the stated scale. In later applications, the background is the signed defect or the field of the preceding level. The signed defect has the required four differences by the bound on $D_a^q P_t^a \mathcal{D}_j$ in the proof of Lemma 34.2, and the inequality

$$(t + H^2)^{\alpha/2} H(t + H^2)^{-1/2} \leq H^\alpha \quad (\alpha < 1)$$

gives the background bound $\mathcal{R} \leq C \|\widehat{\theta}_{j-1}\|_\infty H^{\alpha-1}$ on its two input lattices. The heat equation gives its time increments.

Integrating the bounds on the rows of the reconstruction maps with respect to the fluctuation inputs in Lemma 15.16 along the segment of fluctuation inputs, we obtain $\|\hat{\theta}_i\|_\infty \leq \varrho \|\hat{\theta}_{i-1}\|_\infty + C \mathfrak{g}_i p_i$. Thus $b_i(v)$ bounds $\|\hat{\theta}_i\|_\infty$ on the whole conditional fiber. Iterating, and using $p_l \leq p_i$ for $l \leq i$ and $\varrho \sqrt{B} < 1$, we obtain

$$\|\hat{\theta}_i\|_\infty \leq r \varrho^i + C \sum_{l=1}^i \varrho^{i-l} \mathfrak{g}_l p_l \leq C \{r \varrho^i + g_3 \sqrt{\ell_i} p_i\}. \quad (34.20)$$

In the last inequality we used $\mathfrak{g}_l = \mathfrak{g}_i B^{(i-l)/2}$, so that the sum is at most $\mathfrak{g}_i p_i \sum_{k \geq 0} (\varrho \sqrt{B})^k$.

For the gradients of item (5) we use the gradient rows and the differentiated gradient rows of Theorem 27.5, without time weights. Their weighted size is the scale of item (5). The crude differentiated rows there have the prefactor $B_* = C_{\text{CP}}^{2k} \mathfrak{g}_i b_i(v) G^{-2}$ and cost fixed powers of H^{-1} . Since $C_{\text{CP}}^2 = B q_{\text{anc}}$ and $G = B^k H$, both the coupling and the fiber bound cancel:

$$\frac{B_*}{S} \leq C \left(\frac{q_{\text{anc}}}{\mathfrak{q}} \right)^k H^{-1} G^{1-2\alpha} \leq C H^{-2\alpha}. \quad (34.21)$$

In the last inequality we used $q_{\text{anc}} \leq \mathfrak{q}$ and $G^{1-2\alpha} \leq H^{1-2\alpha}$, which holds since $G \geq H$ and $\alpha > 1/2$. The factor $(1+j)^c$ in S only decreases this ratio. The supremum over the fiber of the gradient differences therefore has the same polynomial modulus. Each normalized gradient has norm at most one on the whole fiber, and the diameter is at most two. This verifies the increment condition in the gradient metric directly, rather than deducing it from pointwise concentration.

For the gradients of item (6), the deterministic rows without time weights in Theorem 27.7 give the stated scale. Their crude derivatives have the prefactor $CK_{ji} \mathfrak{g}_i b_i(v)$ and cost fixed powers of ℓ_j^{-1} . After division by S their cost is at most $C \ell_j^{-P_0-\sigma}$, since $G \geq \ell_j$. The weight, with exponent $\sigma/2 > 0$, is continuous at time zero. This proves the modulus and the diameter bound for the weighted value process. The norm with first differences is obtained afterwards from the deterministic heat-splitting inequality of Theorem 27.7, and no additional cover is needed.

Finally, a modulus $Ch_*^{-P_0}(\text{dist}(x, y) + |t - s|^\zeta)$ is covered at radius v by the spatial step $cvh_*^{P_0}$ and the time step $(cvh_*^{P_0})^{1/\zeta}$. If the spatial step is smaller than a , we use every lattice point, whose number is then smaller than the proposed count. Otherwise we choose lattice representatives in spatial cubes; lattice paths between them give the same modulus. The resulting number of points is at most a fixed power of $1 + L/h_*$ times a fixed power of v^{-1} . All the coefficient maps are continuous in time and have finitely many spatial outputs. In the gradient case, since the fiber is compact, the supremum of the gradient norm keeps this continuity. The images therefore have compact completions with countable dense subsets. This proves every claimed cover with the stated dependence of the constants. $\square$

The following lemma controls the zero-fluctuation continuation from any coarse field. It is used for the comparison configuration in the rest of this section, and in Sections 35 and 36, where its holomorphic version gives Cauchy estimates.

Lemma 34.4. *Let θ_0 be a link logarithm on the lattice of mesh $H_0 = L/n$ in the reconstruction domain with $\|\theta_0\|_\infty \leq r$, and let $\theta_j = P_j \theta_{j-1}$ for $j \geq 1$. Let x_j and y_j be the flow data of θ_j as in Definition 34.1. For arbitrary compatible prior depths, every $l \geq m \geq 0$, and every compatible common mesh $a \leq \ell_l$, the following hold; here compatibility means that the refinement ratios ℓ_i/a , $i \leq l$, and the ratios of the intermediate meshes to a are integers.*

1. We have

$$\|\theta_j\|_\infty \leq r \varrho^j, \quad \sup_j \{X_\alpha^a(x_j) + Y_\alpha^a(y_j)\} \leq R_{cl}.$$

2. We have

$$X_\alpha^a(x_l - x_m) + Y_\alpha^a(y_l - y_m) \leq C(1+m)^c \vartheta^m, \quad \vartheta = \max\{\varrho B^{1-\alpha}, \mathfrak{q}\} < 1.$$

3. There exist radii $r < r_1 < r_2$, fixed together with the working ball, such that the maps $\theta_0 \mapsto \theta_j, x_j, y_j$ extend holomorphically to the polydisc $\mathbb{D}_{r_2}$ of complex link logarithms θ_0 whose $D = 9n^3$ real coordinates are complexified with modulus less than r_2 , and, with the norms applied to complex arrays,

$$\begin{aligned} \sup_{\theta_0 \in \mathbb{D}_{r_2}} \{X_\alpha^a(x_j) + Y_\alpha^a(y_j)\} &\leq C, \\ \sup_{\theta_0 \in \mathbb{D}_{r_2}} \{X_\alpha^a(x_l - x_m) + Y_\alpha^a(y_l - y_m)\} &\leq C(1+m)^c \vartheta^m. \end{aligned}$$

Write $x_j(\theta_0) = \sum_\mu f_{j,\mu} \theta_0^\mu$ and $y_j(\theta_0) = \sum_\mu h_{j,\mu} \theta_0^\mu$ for the Taylor expansions at zero in these D coordinates. Then

$$\begin{aligned} \sum_\mu r_1^{|\mu|} \{X_\alpha^a(f_{j,\mu}) + Y_\alpha^a(h_{j,\mu})\} &\leq C, \\ \sum_\mu r_1^{|\mu|} \{X_\alpha^a(f_{l,\mu} - f_{m,\mu}) + Y_\alpha^a(h_{l,\mu} - h_{m,\mu})\} &\leq C(1+m)^c \vartheta^m. \end{aligned}$$

These bounds are uniform in the prior depths, the length of the continuation, and the compatible common mesh, and the constants may depend in addition on r_1 and r_2 . If $\theta_0 = F(u)$ is the image of a parameter u under a fixed finite-scale reconstruction F , the same assertions hold on a complex neighborhood of the real parameters that may depend on F , but not on the length l of the continuation.

The constants R_{cl}, C, c depend on H_0, r , and the data of item (6) of Definition 4.14, and are independent of the coupling and of the length of the sequence.

Proof. We begin with item (1) and the first component in item (2). Since $P_j(0) = 0$, integrating DP_j along the segment from 0 to v , we obtain $\|P_j v\|_\infty \leq \varrho \|v\|_\infty$. Thus $b_j = r\varrho^j$ bounds the sequence. Its increments split as $d_j^0 = r_j^0 + u_j^{\text{cl},0}$, with no conditional mean, and

$$X_\alpha^a(r_j^0) \leq Cb_{j-1}\ell_{j-1}^{\alpha-1}, \quad H_\sigma^a(u_j^{\text{cl},0}(0)) \leq Cb_{j-1}^2/\ell_{j-1}. \quad (34.22)$$

The first inequality is (34.11). The second follows from the series of P_j in degrees at least two and from item (5) of Lemma 34.2. The two geometric ratios are $\varrho B^{1-\alpha} < 1$ and $B\varrho^2 = \mathfrak{q} < 1$. The initial field x_0 has bounded norm $X_\alpha^a(x_0)$ by (34.9). Summing these bounds, and using the inequality $X_\alpha^a(P^\alpha f) \leq CH_\sigma^a(f)$ (which holds since $\sigma < \alpha$ and T is finite) for the terms $u_j^{\text{cl},0}$, we find the uniform bound on $X_\alpha^a(x_j)$ and the tail $X_\alpha^a(x_l - x_m) \leq C\vartheta^m$.

We now turn to the quadratic component. We fix one of the three ordered forms and denote it by $T_{j,i}$. Its coefficient norm with output in Y_α^a is at most $C(1+j)^c K_{ji}$. By the pullback identity (34.3), we have the finite identity

$$\begin{aligned} T_{j,j-1}(\theta_{j-1}, \theta_{j-1}) &= T_{j,0}(\theta_0, \theta_0) \\ &\quad + \sum_{i=1}^{j-1} \{T_{j,i}(P_i\theta_{i-1}, P_i\theta_{i-1}) - T_{j,i}(L_i\theta_{i-1}, L_i\theta_{i-1})\}. \end{aligned} \quad (34.23)$$

Here $P_i - L_i$ starts at degree two. The coefficientwise majorant series gives $\|P_i v - L_i v\| \leq C\varrho\|v\|^2$ and $\|P_i v\| + \|L_i v\| \leq C\varrho\|v\|$ on the working ball. By (34.4) and the coefficient bound, we have

$$Y_\alpha^a(T_{j,j-1}(\theta_{j-1}, \theta_{j-1})) \leq C(1+j)^c \left\{ \frac{r^2 \mathfrak{q}^{j-1}}{H_0} + \sum_{i=1}^{j-1} \frac{\mathfrak{q}^{j-1-i}}{\ell_i} b_{i-1}^3 \right\}. \quad (34.24)$$

The sum is at most $Cr^3 H_0^{-1} \mathfrak{q}^{j-1}$. Indeed, since $\ell_i^{-1} = H_0^{-1} B^i$ and $\mathfrak{q} = B\varrho^2$, its i -th term divided by $\mathfrak{q}^{j-1}$ is $H_0^{-1} r^3 \varrho^{i-3}$; the factor ϱ^{-3} is fixed and $\sum_{i \geq 1} \varrho^i$ is finite.

By (34.2) for the continuation, $\mathcal{Q}_a(r_j^0, r_j^0)$, $\mathcal{Q}_a(x_{j-1}, r_j^0)$, and $\mathcal{Q}_a(r_j^0, x_{j-1})$ are T^{VV} , $T^{\text{UV}} - T^{\text{VV}}$, and $T^{\text{VU}} - T^{\text{VV}}$ at $(\theta_{j-1}, \theta_{j-1})$. Applying (34.24) to each order gives summable bounds for these three terms. Every remaining term of

$$y_l - y_m = \sum_{j=m+1}^l \{ \mathcal{Q}_a(x_{j-1}, d_j^0) + \mathcal{Q}_a(d_j^0, x_{j-1}) + \mathcal{Q}_a(d_j^0, d_j^0) \}$$

contains $u_j^{\text{cl},0}$ as one argument. We apply the two ordered inequalities (34.6), using the bounded norms $X_\alpha^a(x_j)$ just proved and (34.22). Their total tail is $C(1+m)^c \mathfrak{q}^m$ by (34.15), and the three forms above contribute at most $C(1+m)^c \mathfrak{q}^m$ as well. The initial term y_0 also has a fixed bound: its smooth cardinal input has bounded values and first two differences.

Heat contraction and the stencil of the quadratic term, which contains a single difference, then bound its Duhamel value and its first difference on $[0, T]$. This proves the bound R_{cl} and both tails.

For item (3), we note that the argument above uses only absolute values of coefficients. The coefficientwise majorant series of Lemma 15.16, evaluated on a slightly larger polydisc of complex inputs still inside the working domain, gives the same bounds for complex inputs. Indeed, by the strict choice of the working radius, its derivative series is bounded by ϱ there. No positivity of a real field was used in (34.23). The quadratic coefficient bounds include the volume-normalized columns and all output differences, and every absolute sum above remains valid. Since r is smaller than a fixed fraction of the working radius, we may choose $r < r_1 < r_2$ such that $\mathbb{D}_{r_2}$ lies inside that larger polydisc, independently of the subsequent number of steps. This proves the two bounds on $\mathbb{D}_{r_2}$, with r replaced by a multiple of r_2 in the constants. For the Taylor coefficients we apply the Cauchy estimates in each of the D coordinates on $\mathbb{D}_{r_2}$: for a holomorphic map g with values in a normed space and $\sup_{\theta_0 \in \mathbb{D}_{r_2}} \|g(\theta_0)\| \leq A$, its Taylor coefficients satisfy $\|g_\mu\| \leq Ar_2^{-|\mu|}$. Summing, we find

$$\sum_{\mu} r_1^{|\mu|} \|g_\mu\| \leq A \sum_{\mu} (r_1/r_2)^{|\mu|} = A(1 - r_1/r_2)^{-D}.$$

Applied to x_j, y_j , and the differences $x_l - x_m, y_l - y_m$ in the norms X_α^a and Y_α^a , this gives the coefficient bounds, with the fixed factor $(1 - r_1/r_2)^{-D}$. For a fixed finite-scale input, we first take a complex neighborhood whose image under the fixed finite reconstruction has positive distance from the boundary of the working domain, and then apply this argument to the composed map. This proves item (3). $\square$

The chaining estimate we use is the following specialization of Lemma 16.4.

Lemma 34.5. *Let $(F_z)_{z \in \mathcal{T}}$ be a centered real process indexed by a compact pseudometric space $(\mathcal{T}, d_*)$, continuous in d_* at each fixed cutoff. Let $z_0 \in \mathcal{T}$, $A > 0$, and $c_0 > 0$, and suppose that one of the following two sets of conditions holds for every $u > 0$ and $z, z' \in \mathcal{T}$:*

1. $\mathbb{P}(|F_z - F_{z'}| > Ad_*(z, z')u) \leq 2e^{-u^2/c_0}$ and $\mathbb{P}(|F_{z_0}| > Au) \leq 2e^{-u^2/c_0}$;
2. $\mathbb{P}(|F_z - F_{z'}| > Ad_*(z, z')(\sqrt{u} + u)) \leq 2e^{-u/c_0}$ and $\mathbb{P}(|F_{z_0}| > A(\sqrt{u} + u)) \leq 2e^{-u/c_0}$.

Assume that for some $H > 0$

$$\text{diam}(\mathcal{T}, d_*) \leq C_0, \quad N(v, d_*) \leq C(1 + L/H)^P v^{-Q_0} \quad (0 < v \leq 1).$$

Then for every $1 \leq p < \infty$,

$$\left\| \sup_z |F_z| \right\|_{L^p} \leq C_p A \sqrt{1 + \log(2 + L/H)}$$

under the conditions (1), and

$$\left\| \sup_z |F_z| \right\|_{L^p} \leq C_p A [1 + \log(2 + L/H)]$$

under the conditions (2). Here C_p depends only on p, c_0, C_0, C, P, Q_0 . The same holds under a conditional probability kernel when the hypotheses hold for each fixed value of the conditioning variables.

By Lemma 34.3, the normalized coefficient metrics for linear rows, for quadratic rows (with the maximum of the Hilbert–Schmidt and operator distances), and for gradients have diameter at most two and covering numbers of this form at their input mesh.

Proof. We apply Lemma 16.4 with the metric d_* , the diameter bound C_0 , the cover prefactor $C(1 + L/H)^P$, and the exponent Q_0 . Under the conditions (1) we take $s = A\sqrt{c_0}$ and $b = 0$ in that lemma, and under the conditions (2) we take $s = A\sqrt{c_0}$ and $b = Ac_0$. These choices correspond to the substitutions $u = \sqrt{c_0 v}$ and $u = c_0 v$ in the two tails, which turn them into the form $\mathbb{P}\{|F_z - F_{z'}| > d_*(z, z')(s\sqrt{v} + bv)\} \leq 2e^{-v}$ required there. The conditions at the reference point z_0 take the same form. Since

$$\log(2C(1 + L/H)^P) + Q_0 \leq C' [1 + \log(2 + L/H)],$$

the moment bound of Lemma 16.4 implies the two stated bounds, with the stated dependence of the constants. The conditional version is the corresponding clause of that lemma. $\square$

Lemma 34.6. *We have $p_j \leq C(1 + j + |\log \gamma|)$. Further, for every $\varepsilon > 0$ and $d \geq 0$,*

$$\sup_{0 < \gamma \leq \gamma_0} \gamma^\varepsilon p_j^d \leq C_{\varepsilon, d} (1 + j)^d,$$

where $C_{\varepsilon, d}$ depends in addition on ε and d .

In every later application a logarithm of the coupling is absorbed in this way only together with a positive power of γ .

Proof. By the definition of t_j , we have

$$p_j = \frac{A_w}{2} \{ \log(n/\gamma) + j \log B - (J - 1) \log M \},$$

which proves the first bound. Since $\gamma \leq 1$, we may write $v = |\log \gamma|$. For any $d \geq 0$ we have $(1 + j + v)^d \leq (1 + j)^d (1 + v)^d$, and $\sup_{v \geq 0} e^{-\varepsilon v} (1 + v)^d < \infty$, as one sees from the logarithmic derivative $-\varepsilon + d/(1 + v)$ of the last function. This proves the second bound. $\square$

34.3 Conditional bounds uniform in the coupling

We now bound the four parts of a reconstruction increment, conditionally on the preceding levels and with constants independent of γ . The leading terms carry no factor $|\log \gamma|$: the logarithm of the coupling enters only through the cutoff radii p_j , in terms of higher order in g_3 . The theorem also bounds the quadratic forms pulled back to earlier levels and the conditional means in the predictable term. For $1 \leq i \leq j-1$ we write

$$Z_{j,i}^\omega = T_{j,i}^\omega(\widehat{\theta}_i, \widehat{\theta}_i) - \mathbb{E}[T_{j,i}^\omega(\widehat{\theta}_i, \widehat{\theta}_i) \mid \mathcal{F}_{i-1}], \quad F_{j,i} = S_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i) - \mathbb{E}[S_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i) \mid \mathcal{F}_{i-1}]$$

for the centered pulled-back forms, and

$$e_{j,i}^\omega = \mathbb{E}[T_{j,i}^\omega(\widehat{\theta}_i, \widehat{\theta}_i) \mid \mathcal{F}_{i-1}] - T_{j,i-1}^\omega(\widehat{\theta}_{i-1}, \widehat{\theta}_{i-1})$$

for the conditional drifts, with the analogous drift $e_{j,i}^S$ for the quadratic initial forms. By (34.3), the drift $e_{j,i}^\omega$ is the sum of the classical part $T_{j,i}^\omega(P_i v, P_i v) - T_{j,i}^\omega(L_i v, L_i v)$ at $v = \widehat{\theta}_{i-1}$ and of the conditional-mean part $\mathbb{E}[T_{j,i}^\omega(\widehat{\theta}_i, \widehat{\theta}_i) \mid \mathcal{F}_{i-1}] - T_{j,i}^\omega(P_i v, P_i v)$.

Theorem 34.7. *Assume the setting of Definition 34.1. For every fixed $1 \leq p < \infty$, all $K \geq j \geq 1$, every incoming field $\widehat{\theta}_{j-1}$ in the reconstruction domain, and every finer common mesh $a \leq \ell_j$, the following conditional estimates hold. Let $h = \ell_j$ and let $\mathcal{Z}_j$ be the initial value of z_j . Then*

$$\|X_\alpha^a(s_j)\|_{L^p(\cdot \mid \mathcal{F}_{j-1})} \leq C_{\mathfrak{g}} h^{\alpha-1} [\sqrt{\log(2 + L/h)} + \sqrt{p}],$$

$$\|H_\sigma^a(\mathcal{Z}_j)\|_{L^p(\cdot \mid \mathcal{F}_{j-1})} \leq C_p \{g_3^2 h^\sigma (1+j) + g_3^3 h^{1/2} p_j^3\},$$

$$\|X_\alpha^a(z_j)\|_{L^p(\cdot \mid \mathcal{F}_{j-1})} \leq C_p \{g_3^2 h^\alpha (1+j) + g_3^3 h^{1/2} p_j^3\}.$$

The centered part of $\mathcal{Q}_a(s_j, s_j)$ has conditional $L^p(Y_\alpha^a)$ norm at most $C_p g_3^2 h^{2\alpha-1} (1+j)$, and its conditional mean has norm Y_α^a at most $C_p g_3^2 \|\widehat{\theta}_{j-1}\|_\infty$. Fix constants $C_0, \dots, C_4 < \infty$, independently of the coupling, the level indices, and the common mesh. Let b be an $\mathcal{F}_{j-1}$ -measurable heat path on the lattice of mesh a such that, for $0 \leq q \leq 4$ and $0 \leq t \leq T$,

$$\|D_a^q b(t)\|_\infty \leq C_q \mathcal{R} (t + \ell_j^2)^{-(\alpha+q)/2}.$$

Then each of the two ordered terms $\mathcal{Q}_a(b, s_j)$ and $\mathcal{Q}_a(s_j, b)$ has conditional $L^p(Y_\alpha^a)$ norm at most $C_p g_3 h^{\alpha-1/2} (1+j)^c \mathcal{R}$. The constant in this mixed estimate may also depend on $C_0, \dots, C_4$.

For each ω separately, each $1 \leq i \leq j-1$, and the incoming field $v = \widehat{\theta}_{i-1}$, put $b_i(v) = C_J(a_{\text{map}} \|v\|_\infty + \mathfrak{g}_i p_i)$. Then

$$\|Y_\alpha^a(Z_{j,i}^\omega)\|_{L^p(\cdot \mid \mathcal{F}_{i-1})} \leq C_p (1+j)^c \mathfrak{q}^{j-1-i} \mathfrak{g}_i b_i(v) \ell_i^{2\alpha-2},$$

$$\|H_\sigma^a(F_{j,i})\|_{L^p(\cdot|\mathcal{F}_{i-1})} \leq C_p(1+j)^c \frac{q^{j-1-i}}{\ell_i} \mathfrak{g}_i b_i(v) \ell_i^\sigma.$$

Along the history we write $b_i = b_i(\widehat{\theta}_{i-1})$ for $i \geq 1$ and set $b_0 = \|\widehat{\theta}_0\|_\infty \leq r$. Thus $\|\widehat{\theta}_i\|_\infty \leq b_i$, and the deterministic bound $b_i \leq C\{ra_{\text{map}}^i + g_3\sqrt{\ell_i} p_i\}$ holds. The conditional drifts satisfy

$$Y_\alpha^a(e_{j,i}^\omega) + H_\sigma^a(e_{j,i}^S) \leq C(1+j)^c \frac{q^{j-1-i}}{\ell_i} \{a_{\text{map}}^2 b_{i-1}^3 + \mathfrak{g}_i^2 b_{i-1} + \mathfrak{g}_i^3 p_i^3\},$$

and their conditional-mean parts are bounded by the last two terms alone. All norms include every required first difference and Wilson shift. Further, there exist $c, \epsilon > 0$ such that for every $z > 0$,

$$\sup_{0 < \gamma \leq \gamma_0} \sup_{K \geq m} \nu \{X_\alpha^a(x_K - x_m) + Y_\alpha^a(y_K - y_m) > z\} \leq C_z(1+m)^c B^{-\epsilon m}.$$

In particular this tail tends to zero in the order $K \rightarrow \infty$, then $\gamma \rightarrow 0$, then $m \rightarrow \infty$.

Here C_p depends in addition on p , and C_z on z ; no factor $\log(1/t_j)$ is hidden in the leading constants.

Proof. The proof has six steps. We first check that the concentration estimates of Theorem 5.4 apply to the conditional kernels. We combine them with the covers of Lemma 34.3 to bound the centered linear and quadratic terms, and we treat the remainder of higher degree and the conditional means separately. Finally, we sum over levels with explicit powers of ℓ_i and estimate the products in the bilinear expansion (34.5).

Step 1: concentration. All coefficient estimates are evaluated on the larger fixed domain, and only then restricted to a fixed incoming field. The conditional potential of the fluctuation variables is C^2 on a neighborhood of its convex product of balls. The Hessian condition of Theorem 23.2 implies that its Hessian is bounded below there by a fixed $m_* > 0$: the perturbation of the identity Hessian of the normalized variables is at most $8/27$. Then Theorem 5.4 applies with one constant m_* , independently of the radii of the balls, of the dimension, and of the prior depth.

Step 2: coefficient metrics and conditional maxima. We use Lemma 34.3 with $h = \ell_j$ and with its parameters for the pulled-back forms, followed by Theorem 5.4. Note that a difference of coefficient rows is linear in the centered vector ζ_j , and a difference of coefficient matrices is quadratic in it. By the Lipschitz concentration and the quadratic estimate of Theorem 5.4, their increments satisfy the tail conditions (1) and (2) of Lemma 34.5, respectively, with $A = CS$, where S is the scale. For the gradients of a nonlinear function of the fluctuation variables we have

$$\text{Lip}_w(F_z - F_{z'}) \leq S d_*(z, z'), \quad (F_z - \mathbb{E}F_z) - (F_{z'} - \mathbb{E}F_{z'}) = (F_z - F_{z'}) - \mathbb{E}(F_z - F_{z'}).$$

The Lipschitz concentration bound therefore applies to this difference, and also at the reference point of the chaining, with scale CS . This proves the increment conditions under the conditional fluctuation kernel. The continuity required in the gradient metric also follows directly: for $g = F_z - F_{z'}$ on the convex finite fiber Ω with conditional law μ , we have

$$|g(w) - \mathbb{E}g| \leq \int_{\Omega} |g(w) - g(v')| d\mu(v') \leq \text{diam}(\Omega) Sd_*(z, z').$$

The diameter of Ω is used only for continuity at a fixed cutoff, never in the constant of a maximal bound. The linear and quadratic coefficient images have the same continuity by the Cauchy–Schwarz inequality in finite dimension, applied to their centered evaluations. This verifies the countable separable realization required for chaining. Lemma 16.4, applied as in the proof of Lemma 34.5 but keeping track of the moment order, gives the losses

$$C[\sqrt{\log(2 + L/h_*)} + \sqrt{p}] \quad \text{and} \quad C[\log(2 + L/h_*) + p]$$

for linear and quadratic rows, respectively. Since $\log(2 + L/\ell_j) \leq C(1 + j)$, these losses imply the first bound of the theorem, the quadratic parts of the bounds on $\mathcal{Z}_j$ and z_j , the bound on the centered part of $\mathcal{Q}_a(s_j, s_j)$, the bound on the mixed terms with the background condition of item (4) of Lemma 34.3, and the bounds on $Z_{j,i}^\omega$ and $F_{j,i}$. For the last two, the scales of items (5) and (6) of that lemma are $C(1 + j)^c q^{j-1-i} \mathfrak{g}_i b_i(v) \ell_i^{2\alpha-2}$ and $Cq^{j-1-i} \ell_i^{-1} \mathfrak{g}_i b_i(v) \ell_i^\sigma$, since $K_{ji} = q^{j-1-i}/\ell_i$ and $G = \ell_i$.

Step 3: higher-degree remainder, conditional means, and first differences. The centered remainder of degree at least three in the fluctuation variables has supremum norm at most $Cg_3^3 h^{1/2} p_j^3$ by Theorem 26.11, and the heat-splitting inequality of Theorem 27.7 bounds its first difference. The conditional mean of $\mathcal{Q}_a(s_j, s_j)$ is the mean in Theorem 26.12, which is at most $Cg_3^2 \|\hat{\theta}_{j-1}\|_\infty$ in the norm Y_α^a .

To verify that the bound on $\mathcal{Z}_j$ holds in the norm H_σ^a , which includes first differences, let f be an initial array. If $t \geq a^2$, we have

$$\|D_a P_t^a f\|_\infty \leq Ct^{-1/2} \|P_{t/2}^a f\|_\infty \leq Ct^{-(\sigma+1)/2} V_\sigma^a(f).$$

If $t \leq a^2$, we use $\|f\|_\infty \leq a^{-\sigma} V_\sigma^a(f)$ and $\|D_a P_t^a\|_{\infty \rightarrow \infty} \leq C/a$. Multiplication by $(t + a^2)^{(\sigma+1)/2}$ cancels these factors. This proves $H_\sigma^a(f) \leq CV_\sigma^a(f)$, uniformly in the finer mesh, and we apply it to the centered initial field.

Step 4: the conditional drifts. The splitting of the conditional drift into its classical part and its conditional-mean part is given in Theorem 27.5 for the ordered forms and in Theorem 27.7 for the quadratic initial forms. We use $\|\hat{\theta}_{i-1}\|_\infty \leq b_{i-1}$, including the stated convention for $i = 1$. Its three bounds are

$$C(1 + j)^c K_{ji} (q^2 b_{i-1}^3 + \mathfrak{g}_i^2 b_{i-1} + \mathfrak{g}_i^3 p_i^3).$$

The first term bounds the classical part, which is the error of the zero-fluctuation pullback. The last two terms bound the conditional-mean part, which contains the conditional mean of the fluctuation variables and the covariance correction.

Step 5: summation over levels. We record the powers of the mesh needed to check summability. Fixed powers of H_0 are included in C . Using (34.20), $q^i = (\ell_i/H_0)^\kappa$, and $g_i = g_3\sqrt{\ell_i}$, we find that the centered pulled-back bounds and the conditional drifts split before the convolution in q as

$$\begin{aligned} g_i b_i \ell_i^{2\alpha-2} &\leq C \{g_3 r \ell_i^{\kappa+2\alpha-3/2} + g_3^2 \ell_i^{2\alpha-1} p_i\}, \\ g_i b_i \ell_i^{\sigma-1} &\leq C \{g_3 r \ell_i^{\kappa+\sigma-1/2} + g_3^2 \ell_i^\sigma p_i\}, \\ \frac{q^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3}{\ell_i} &\leq C \{r^3 \ell_i^{3\kappa-1} + g_3^2 r \ell_i^\kappa + g_3^3 \ell_i^{1/2} (p_i + 1)^3\}. \end{aligned}$$

In the last line we used $(u+v)^3 \leq 4(u^3 + v^3)$. Every exponent of ℓ_i is positive.

The initial value of the predictable term decomposes as $u_j(0) = S_{j,j-1}(\hat{\theta}_{j-1}, \hat{\theta}_{j-1}) + V_j(\hat{\theta}_{j-1}) + I_{\ell_j \rightarrow a} m_j^{\text{ph}}(\hat{\theta}_{j-1})$, and telescoping the first term by (34.3), we obtain

$$S_{j,j-1}(\hat{\theta}_{j-1}, \hat{\theta}_{j-1}) = S_{j,0}(\hat{\theta}_0, \hat{\theta}_0) + \sum_{i=1}^{j-1} (F_{j,i} + e_{j,i}^S).$$

The initial term is at most $Cr^2(1+j)^c q^{j-1}/H_0$. The other two terms of $u_j(0)$ are bounded by (34.12) and (34.13), and their bounds have the same powers as the third line above. Combining the chaining bounds of Step 2, the absorption of the logarithms of the coupling by Lemma 34.6, and the fixed-level convolution bound (34.16a), we obtain $\tau > 0$, a finite b , and constants independent of the coupling such that

$$\|H_\sigma^a(u_j(0))\|_p + \|X_\alpha^a(u_j)\|_p \leq C_p(1+j)^b B^{-\tau j}. \quad (34.25)$$

For instance, we may take τ strictly below $\min(\sigma, 1/2, \lambda, \kappa + \sigma - 1/2)$ and reduce it for the finitely many other positive exponents listed above. The bound on $X_\alpha^a(u_j)$ follows from the one on $H_\sigma^a(u_j(0))$, since $X_\alpha^a(P^a f) \leq CH_\sigma^a(f)$. All fixed moments follow from the conditional bounds by the conditional Minkowski inequality and then the ordinary Minkowski inequality under ν .

Combining the bounds on s_j and z_j with (34.11), (34.20), and (34.25), we find

$$\sup_{K,j} \|X_\alpha^a(x_j)\|_p \leq C_p, \quad \left\| X_\alpha^a \left(\sum_{j>m} d_j \right) \right\|_p \leq C_p(1+m)^b B^{-\tau_1 m} \quad (34.26)$$

for some $\tau_1 > 0$. Here the bound on s_j is of order $g_3 \ell_j^{\alpha-1/2} \sqrt{1+j}$, the bound on r_j is $CH^{\alpha-1} \|\hat{\theta}_{j-1}\|_\infty$ with $H = \ell_{j-1}$, and the initial field costs only $C_{H_0} r$. The correction in (34.14) also has bounded fixed moments: by (34.20) its powers of ℓ_{j-1} are $\kappa + \alpha - 1 > 0$ and $\alpha - 1/2 > 0$, with positive powers of the coupling on every remaining factor p_{j-1} .

Step 6: products and the probability tail. We now bound $y_K - y_m$ through (34.5), following the second column of Table 34.2. We write S_j for the bound on $X_\alpha^a(s_j)$, Z_j^- and Z_j^+ for the bounds on $H_\sigma^a(Z_j)$ and $X_\alpha^a(z_j)$, R_j for the bound on $X_\alpha^a(r_j)$ given by (34.11), and U_j for the bound in (34.25), always using moments of order $2p$ in a product. Up to a fixed factor, the sixteen ordered products $\mathcal{Q}_a(v_j, w_j)$, with v indexing rows and w indexing columns, are bounded in $L^p(Y_\alpha^a)$ by the entries of

	s	z	r	u
s	F_j	$S_j Z_j^-$	$S_j R_j$	$S_j U_j$
z	$S_j Z_j^-$	$Z_j^+ Z_j^-$	$R_j Z_j^-$	$U_j Z_j^-$
r	$R_j S_j$	$R_j Z_j^-$	W_j	$R_j U_j$
u	$U_j S_j$	$U_j Z_j^-$	$R_j U_j$	U_j^2

Here $F_j = C_p g_3^2 [\ell_j^{2\alpha-1}(1+j) + b_{j-1}]$ includes the conditional mean of $\mathcal{Q}_a(s_j, s_j)$, and W_j is the bound on $Y_\alpha^a(T_{j,j-1}^{\text{VV}}(\hat{\theta}_{j-1}, \hat{\theta}_{j-1}))$ obtained by telescoping as above: the initial term, the centered terms $Z_{j,i}^{\text{VV}}$, and both the classical and the conditional-mean parts of the conditional drifts $e_{j,i}^{\text{VV}}$. The entries with z or u use (34.6) and Hölder's inequality under ν . The entries (s, r) and (r, s) use (34.19), conditionally on $\mathcal{F}_{j-1}$, with the predictable background r_j , which has the four differences required there and $\mathcal{R} \leq CR_j$. The entry (r, r) is the form T^{VV} , not a difference of uncontrolled forms. Thus every entry is justified in L^p . The slowest powers of ℓ_j that occur include $2\alpha - 1$, $\kappa + 2\alpha - 3/2$, $\kappa + \alpha + \sigma - 1$, and $\sigma + \alpha - 1/2$; the entries with u also keep the decay $B^{-\tau j}$. All these exponents are positive. The parts of degree at least three in the fluctuation variables carry an additional factor $\ell_j^{1/2}$ and create no smaller exponent.

For the eight cross terms with x_{j-1} , we use (34.19) and the bounded norm (34.14) of x_{j-1} at its input mesh for s ; (34.6) and Z_j^- for z ; the forms $T^{\text{UV}} - T^{\text{VV}}$ and $T^{\text{VU}} - T^{\text{VV}}$ of (34.2) for r ; and (34.6) with (34.25) and (34.26) for u . This covers both orders of all four arguments. By (34.15) and (34.16), the sums over $j > m$ of these bounds and of the sixteen products are at most $C_p(1+m)^b B^{-\epsilon m}$ for some smaller $\epsilon > 0$. The expansion (34.5) then bounds $\|Y_\alpha^a(y_K - y_m)\|_p$ by the same expression. By Markov's inequality and (34.26), we obtain the probability tail, after enlarging C_z to cover the finitely many small indices. $\square$

34.4 The initial gauge frame along the flow

A based holonomy transforms by conjugation at its basepoint. Two configurations evolved with independent time-dependent gauge transformations can therefore have holonomies with equal traces but very different matrices. The Wilson flow is analyzed in a damped gauge, in which its linear part is the heat equation. The damping gauge transformations solve an ordinary differential equation started at the identity, and the difference of their

generators is integrable in time uniformly in the mesh. Hence the holonomy matrices of two configurations that start in the same frame can be compared directly. We use the following form of Theorem 25.10.

Remark 34.8. Fix $R < \infty$, $0 < t_- \leq t_+ \leq T$, and finitely many based piecewise C^1 loops $c_1, \dots, c_k$ with lattice approximations $c_{i,a}$ of uniformly bounded physical length. Let $U = e^{aA_0}$ and $U' = e^{aA'_0}$ be two configurations on the same lattice of mesh $a = L/N$, in the reconstruction domain and represented in the same initial vertex and basepoint frame, whose flow data (x, y) and (x', y') on $[0, T]$ have norms $X_\alpha^a + Y_\alpha^a$ at most R . Let $\mathcal{D} = X_\alpha^a(x - x') + Y_\alpha^a(y - y')$. By Theorem 25.10, applied with radius $\max(1, R)$ and input horizon T , there exists a flow stability constant $\Lambda_{R,t_-,T} < \infty$ such that

$$\sup_{t_- \leq t \leq t_+} \max_i \|\text{Hol}_{c_{i,a}} \Psi_t^{(N)}(U) - \text{Hol}_{c_{i,a}} \Psi_t^{(N)}(U')\|_{\text{op}} \leq \Lambda_{R,t_-,T} \mathcal{D},$$

where $\Psi_t^{(N)}$ is the Wilson flow run for the lattice flow time $4t/a^2$. The constant depends only on R, t_-, T , the loops, the fixed analytic domains, and the margins of the reference neighborhoods of the continuum flow in Theorem 24.10, with the dependence described in item (7) of Definition 4.14; it is independent of the mesh, the prior depth, and the coupling, and it covers the finitely many meshes that are too coarse for the small-mesh condition of the local theory. Let T_R and a_R be the constants of Theorem 24.5 for radius $\max(1, R)$, and put $h_R = \min(T_R, T)$. After decreasing a_R if necessary, for $a \leq a_R$ and $0 \leq s \leq h_R$ the damped potentials A, A' and their damping gauge transformations k, k' , with $k(0) = k'(0) = I$, satisfy the following estimates, where $v_a(s) = s + a^2$.

$$\|D_a(A - A')(s)\|_\infty \leq C_R \mathcal{D} \{v_a(s)^{-(\alpha+1)/2} + v_a(s)^{-\alpha} + v_a(s)^{\kappa_{\text{fl}}-1/2}\},$$

$$\|k(s, x_0) - k'(s, x_0)\|_{\text{op}} \leq C \int_0^s \|D_a(A - A')(u)\|_\infty du \leq C_R \mathcal{D}.$$

The three powers of $v_a(s)$ are integrable at zero uniformly in the mesh, since $(\alpha + 1)/2 < 1$, $\alpha < 1$, and $1/2 - \kappa_{\text{fl}} < 1$. For $a \leq a_*$ and $\mathcal{D} \leq \delta$, with a_* and δ supplied by Lemma 25.9 with $T_0 = T$ and $\tau = t_-$, both flows can use a common reference solution and the same fixed restarts on the later intervals. For other pairs or meshes, the matrix inequality follows from the unitary diameter bound or the finite-dimensional Wilson flow, as in Theorem 25.10. All matrices are expressed in their initial frames; no conjugation of either matrix is optimized.

34.5 Comparison of linear and quadratic flow data

We now compare the flow data of the reconstruction with those of its zero-fluctuation continuation after level m . The comparison is exact on the finest lattice: both fields are

interpolated by the identity there, and no interpolation error enters at level K . The two fields agree at level m .

Theorem 34.9. *Assume the setting of Definition 34.1. For every fixed $p \geq 1$ and all $K \geq j \geq m \geq 0$, under the conditioned Wilson measure ν ,*

$$\|\|\hat{\theta}_j - \hat{\theta}_j^0\|_\infty\|_p \leq C_p \mathfrak{g}_j (1+j)^{1/2}, \quad \|\|\hat{\theta}_j\|_\infty\|_p \leq C_p \{ra_{\text{map}}^j + \mathfrak{g}_j (1+j)^{1/2}\}.$$

On the finest lattice, writing $\Delta x_j = x_j - x_j^0$, we have

$$\Delta x_K = \sum_{j=m+1}^K (d_j - d_j^0), \quad y_K - y_K^0 = \sum_{j=m+1}^K (y_j - y_{j-1} - y_j^0 + y_{j-1}^0).$$

For each ordered form $T_{j,i}^\omega$ and each quadratic initial form $S_{j,i}$, the pullback (34.3) is exact. In particular, with $L_i = DP_i(0)$, the function

$$R_{j,i}^\omega(v) = T_{j,i}^\omega(P_i v, P_i v) - T_{j,i}^\omega(L_i v, L_i v)$$

has degree at least three and, on the working ball,

$$Y_\alpha^a(R_{j,i}^\omega(v) - R_{j,i}^\omega(v')) \leq C(1+j)^c \frac{\mathfrak{q}^{j-1-i}}{\ell_i} (\|v\|_\infty + \|v'\|_\infty)^2 \|v - v'\|_\infty.$$

The corresponding estimate for the quadratic initial forms holds in the full initial-array norm H_σ^a . There exist $s > 0$ and $c, d < \infty$, independent of K, m , and γ , such that for every fixed $p \geq 1$,

$$\begin{aligned} \sup_{m \leq l \leq K} \|X_\alpha^a(\Delta x_l)\|_p + \sum_{j=m+1}^K \|H_\sigma^a(u_j(0) - u_j^{\text{cl},0}(0))\|_p \\ \leq C_p [\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d] (1+m)^c B^{-sm}, \end{aligned}$$

$$\|X_\alpha^a(x_K - x_K^0) + Y_\alpha^a(y_K - y_K^0)\|_p \leq C_p \sqrt{\gamma} (1+m)^c B^{-sm} + C_p \gamma (1 + |\log \gamma|)^d.$$

Further, for each ω let

$$\mathcal{T}_{K,m}^\omega = \sum_{j=m+1}^K \{T_{j,j-1}^\omega(\hat{\theta}_{j-1}, \hat{\theta}_{j-1}) - T_{j,j-1}^\omega(\hat{\theta}_{j-1}^0, \hat{\theta}_{j-1}^0)\}.$$

Then $\|Y_\alpha^a(\mathcal{T}_{K,m}^\omega)\|_p \leq C_p [\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d] (1+m)^c B^{-sm}$. One may choose s strictly below each of the positive numbers

$$\alpha - \frac{1}{2}, \quad \sigma, \quad 2\alpha - 1, \quad \kappa + \sigma - \frac{1}{2}, \quad \kappa + 2\alpha - \frac{3}{2}, \quad 2\kappa - 1, \quad \kappa + \alpha + \sigma - 1, \quad -\frac{\log \mathfrak{q}}{\log B},$$

and below the fixed smaller exponents of the product estimates in Lemma 28.5.

Proof. We first bound the moments of the fields, before any product is formed. We then use (34.4) to write the difference of each pulled-back quadratic form along the two reconstructions as a sum of centered terms, classical terms of degree at least three, and conditional means. We record the size of each term, counting the classical part of degree at least three of the predictable term once. Summation over levels then gives the bounds on the predictable term, on the linear heat evolution, and on the three ordered forms. Finally, we bound both orders of every product in (34.5), following the third column of Table 34.2, with Hölder's inequality for higher moments under ν in place of independence. The only delicate product is that of Δx_{j-1} with the classical predictable term, whose sum over levels is not uniformly small in m before it is multiplied by the decaying factor. Throughout, we write $\varrho = a_{\text{map}}$ and $\delta_j = \widehat{\theta}_j - \widehat{\theta}_j^0$, and all conditional expectations are for the conditional kernels of ν .

Step 1: moments of the fields. We define the innovation about the zero-fluctuation value by

$$\epsilon_j = \widehat{\Phi}_j(\widehat{\theta}_{j-1}, g_j, w_j) - P_j(\widehat{\theta}_{j-1}).$$

By the coefficient bounds of the reconstruction maps, the Euclidean gradient of each scalar output with respect to the fluctuation variables is at most $C_J g_j$, and there are at most $C_J (nB^j)^3$ such outputs. Applying Theorem 5.4 for a fixed incoming field and taking a union bound over the outputs, we obtain

$$\mathbb{P}\{\|\epsilon_j - \mathbb{E}_{j-1}\epsilon_j\|_\infty > C g_j(\sqrt{1+j} + \sqrt{u}) \mid \mathcal{F}_{j-1}\} \leq Ce^{-u},$$

where $\mathbb{E}_{j-1} = \mathbb{E}(\cdot \mid \mathcal{F}_{j-1})$. Integrating the tail, we obtain the bound $C_p g_j \sqrt{1+j}$ in every fixed L^p , without a factor of the cutoff. Since $\mathbb{E}_{j-1}\epsilon_j = \ell_j m_j^{\text{ph}}(\widehat{\theta}_{j-1})$, (34.13) implies that $\|\mathbb{E}_{j-1}\epsilon_j\|_\infty \leq C g_j^2 \|\widehat{\theta}_{j-1}\|_\infty \leq C r g_j^2$, where in the last inequality we used that the working ball has a fixed radius. The exact recursion, with $\delta_m = 0$, is

$$\delta_j = P_j(\widehat{\theta}_{j-1}) - P_j(\widehat{\theta}_{j-1}^0) + \epsilon_j.$$

The segment between the two inputs lies in the convex working ball. By the Minkowski inequality and the bound on DP_j , we have

$$\|\|\delta_j\|_\infty\|_p \leq C_p \sum_{i=m+1}^j \varrho^{j-i} [g_i \sqrt{1+i} + r g_i^2].$$

Since $g_i = g_j B^{(j-i)/2}$, the first sum is at most $C_p g_j \sqrt{1+j} / (1 - \varrho \sqrt{B})$. For the second sum, after division by g_j , we use $g_i \leq g_{m+1}$ to bound it by $r g_{m+1} / (1 - \varrho \sqrt{B})$, and $g_{m+1} \leq g_1$ is bounded uniformly by the coupling ceiling. This proves the first estimate. The second follows by comparing the reconstruction with the zero-fluctuation continuation from the

same coarse field (the case $m = 0$), whose norm is at most $r\varrho^j$ by Lemma 34.4. By the same argument, for every $i \geq m$ we have

$$\begin{aligned} \|\|\widehat{\theta}_i\|_\infty\|_p + \|\|\widehat{\theta}_i^0\|_\infty\|_p &\leq C_p[r\varrho^i + \mathfrak{g}_i\sqrt{1+i}], \\ \|\|\delta_i\|_\infty\|_p &\leq C_p\mathfrak{g}_i\sqrt{1+i}. \end{aligned} \quad (34.27)$$

For the continuation in the first line we apply $\|P_i \cdots P_{m+1}v\|_\infty \leq \varrho^{i-m}\|v\|_\infty$ to the moment at level m just proved, and use $\varrho^{i-m}\mathfrak{g}_m \leq \mathfrak{g}_i$.

Step 2: exact telescopes and the classical cubic difference. At level m the two fields and their interpolations coincide, and $I_{a \rightarrow a}$ is the identity at the finest level. Subtracting the two finite telescopes proves the two identities for Δx_K and $y_K - y_K^0$ in the statement, without an interpolation error at level m .

Let $A_{j,i}$ denote any one of $T_{j,i}^{\text{UV}}$, $T_{j,i}^{\text{VU}}$, $T_{j,i}^{\text{VV}}$, or $S_{j,i}$. Its coefficient norm is at most $C(1+j)^c K_{ji}$, where the polynomial factor is needed only for the ordered forms, and the pullback (34.3) is exact by the construction in Theorems 27.5 and 27.7. Set $P_i = L_i + E_i$, where E_i starts at degree two. By (34.4), we have

$$C_{j,i}(v) = A_{j,i}(P_i v, P_i v) - A_{j,i}(L_i v, L_i v) = A_{j,i}(L_i v, E_i v) + A_{j,i}(E_i v, L_i v) + A_{j,i}(E_i v, E_i v).$$

By the coefficientwise majorant series of Lemma 15.16, we have $\|E_i v\| \leq C\|v\|^2$, $\|DE_i(v)\| \leq C\|v\|$, and $\|L_i\| \leq \varrho$ on the working ball. Applying the product rule to the last identity, we obtain

$$\|DC_{j,i}(v)[w]\|_{\text{out}} \leq C(1+j)^c K_{ji} \|v\|_\infty^2 \|w\|_\infty. \quad (34.28)$$

Here the output norm is Y_α^a , or, for a quadratic initial form, the initial-array norm H_α^a after cardinal interpolation. In the latter case, the coefficient row bound of the quadratic initial form controls the supremum of the initial value, and item (5) of Lemma 34.2 then controls the first differences. Equivalently, in the expansion in homogeneous coefficients one distinguishes one argument, keeps the other two, and sums the remaining degrees inside the fixed radius; the differentiated series converges there, and the constant is therefore independent of the length of the sequence. Integrating (34.28) along the segment from v' to v proves the difference estimate in the statement, including first differences. It also proves that $R_{j,i}^\omega = C_{j,i}$ has no quadratic part.

Step 3: decomposition of the pulled-back forms. We define

$$\begin{aligned} Z_{j,i} &= A_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i) - \mathbb{E}_{i-1} A_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i), \\ M_{j,i}(v) &= \mathbb{E}[A_{j,i}(\widehat{\Phi}_i(v, g_i, w), \widehat{\Phi}_i(v, g_i, w)) \mid \widehat{\theta}_{i-1} = v] - A_{j,i}(P_i v, P_i v). \end{aligned}$$

By the splitting of the conditional means in Theorems 27.5 and 27.7, we have

$$\|M_{j,i}(v)\|_{\text{out}} \leq C(1+j)^c K_{ji} [\mathfrak{g}_i^2 \|v\|_\infty + \mathfrak{g}_i^3 p_i^3]. \quad (34.29)$$

The classical part of degree three is contained in $C_{j,i}$ and does not appear again in (34.29). The pullback identity (34.3) implies that $A_{j,i}(\widehat{\theta}_i, \widehat{\theta}_i) - A_{j,i-1}(\widehat{\theta}_{i-1}, \widehat{\theta}_{i-1}) = Z_{j,i} + C_{j,i}(\widehat{\theta}_{i-1}) + M_{j,i}(\widehat{\theta}_{i-1})$. For the continuation, the same difference equals $C_{j,i}(\widehat{\theta}_{i-1}^0)$ when $i > m$. Summing over i , we obtain the identity

$$\begin{aligned} & A_{j,j-1}(\widehat{\theta}_{j-1}, \widehat{\theta}_{j-1}) - A_{j,j-1}(\widehat{\theta}_{j-1}^0, \widehat{\theta}_{j-1}^0) \\ &= \sum_{i=m+1}^{j-1} \{Z_{j,i} + C_{j,i}(\widehat{\theta}_{i-1}) - C_{j,i}(\widehat{\theta}_{i-1}^0) + M_{j,i}(\widehat{\theta}_{i-1})\}. \end{aligned} \quad (34.30)$$

The terms at level m cancel identically, since both arguments are $\widehat{\theta}_m$.

Step 4: sizes of the terms and summation. We give the quantitative bounds before suppressing positive powers of the mesh. In this proof we set $q_i = 1 + i + |\log \gamma|$; then $p_i \leq Cq_i$ by Lemma 34.6. Conditionally on the incoming field, the fiber bound in Theorem 34.7 is $C(\varrho \|\widehat{\theta}_{i-1}\|_\infty + \mathfrak{g}_i p_i)$. Taking its outer moments with (34.27), we obtain the following bounds, up to the factor $C_p(1+j)^c q^{j-1-i}$:

term	bound before the convolution in q
$Z_{j,i}$, quadratic initial form	$\sqrt{\gamma} r \ell_i^{\kappa+\sigma-1/2} + \gamma \ell_i^\sigma q_i$
$Z_{j,i}$, ordered form	$\sqrt{\gamma} r \ell_i^{\kappa+2\alpha-3/2} + \gamma \ell_i^{2\alpha-1} q_i$
$C_{j,i}(\widehat{\theta}_{i-1}) - C_{j,i}(\widehat{\theta}_{i-1}^0)$	$\sqrt{\gamma} r^2 \ell_i^{2\kappa-1/2} (1+i)^{1/2} + \gamma r \ell_i^\kappa q_i^2 + \gamma^{3/2} \ell_i^{1/2} q_i^3$
$M_{j,i}(\widehat{\theta}_{i-1})$	$\gamma r \ell_i^\kappa + \gamma^{3/2} \ell_i^{1/2} q_i^3$

(34.31)

Fixed factors depending on L , n , B , and r are allowed in these bounds. To verify the third line, we use (34.28) and Hölder's inequality with moments of order $3p$, which bound the difference by

$$\frac{C_p}{\ell_i} [r q^{i-1} + \mathfrak{g}_{i-1} \sqrt{1+i}]^2 \mathfrak{g}_{i-1} \sqrt{1+i}.$$

Expanding the square and using $q^i = H_0^{-\kappa} \ell_i^\kappa$ and $\mathfrak{g}_i^2 = \gamma \ell_i / L$ gives the listed powers. The first two lines follow by multiplying the conditional fiber bound by $\mathfrak{g}_i \ell_i^{\sigma-1}$ or by $\mathfrak{g}_i \ell_i^{2\alpha-2}$, respectively, as in Theorem 34.7. The last line follows from (34.29) and (34.27). Every power multiplying $\sqrt{\gamma}$ is free of $|\log \gamma|$, and every term that contains this logarithm has coupling order at least γ and a strictly positive power of the mesh.

We apply (34.30) to the quadratic initial term of $u_j(0) - u_j^{\text{cl},0}(0)$. The remaining classical part of this difference is $V_j(\widehat{\theta}_{j-1}) - V_j(\widehat{\theta}_{j-1}^0)$, whose bound by (34.12) is the third line of (34.31) without the factor q^{j-1-i} . The remaining mean is $I_{\ell_j \rightarrow a} m_j^{\text{ph}}(\widehat{\theta}_{j-1})$, which by (34.13) and (34.27) is at most $C_p[\gamma r \ell_{j-1}^\kappa + \gamma^{3/2} \ell_{j-1}^{1/2} \sqrt{1+j}]$. All these bounds hold in the initial-array norm H_σ^a ; for the centered quadratic initial terms, the first-difference part

was proved by heat splitting in Theorem 34.7. By (34.15) and (34.16), we obtain, for some $s_0 > 0$,

$$\sum_{j=m+1}^K \|H_\sigma^a(u_j(0) - u_j^{\text{cl},0}(0))\|_p \leq C_p [\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d] (1+m)^c B^{-s_0 m}. \quad (34.32)$$

This bounds the predictable term in the norm of its initial value, which is stronger than a bound in X_α^a .

For the linear heat evolution, the difference of the increments splits exactly as

$$d_j - d_j^0 = s_j + z_j + (r_j - r_j^0) + (u_j - u_j^{\text{cl},0}).$$

The bounds (34.11) and (34.27) imply that

$$\|X_\alpha^a(r_j - r_j^0)\|_p \leq C_p \sqrt{\gamma} (1+j)^{1/2} \ell_{j-1}^{\alpha-1/2}.$$

The centered linear term has the same leading power by Theorem 34.7, and the centered nonlinear heat term contributes $C_p [\gamma \ell_j^\alpha (1+j) + \gamma^{3/2} \ell_j^{1/2} p_j^3]$. The last part is bounded by (34.32) and the inequality $X_\alpha^a(P^a f) \leq CH_\sigma^a(f)$. Thus

$$\left\| \sup_{m \leq l \leq K} X_\alpha^a(\Delta x_l) \right\|_p \leq C_p [\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d] (1+m)^c B^{-s_0 m}. \quad (34.33)$$

By the triangle inequality, this bound holds with the supremum inside the expectation, and the sum over j of the $L^p(X_\alpha^a)$ norms of each of the four parts of $d_j - d_j^0$ is bounded by the same right side.

Applying (34.30) to each of the three ordered forms and using (34.31), we obtain

$$\begin{aligned} \sum_{j=m+1}^K \left\| Y_\alpha^a(T_{j,j-1}^\omega(\hat{\theta}_{j-1}, \hat{\theta}_{j-1}) - T_{j,j-1}^\omega(\hat{\theta}_{j-1}^0, \hat{\theta}_{j-1}^0)) \right\|_p \\ \leq C_p [\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d] (1+m)^c B^{-s_0 m}, \end{aligned} \quad (34.34)$$

separately for $\omega = UV, VU, VV$, with all conditional means included. We choose s_0 below the positive exponents listed in the statement and below the smaller exponents of the product estimates cited there. The other exponents in (34.31), such as $2\kappa - 1/2$ and κ , are positive and are accommodated by reducing s_0 further if necessary. The constants of all geometric sums depend on these fixed positive gaps, not on m .

Step 5: the cross products with the predictable term. There are two such products in each order. For the one containing $u_j - u_j^{\text{cl},0}$, we use (34.6), Hölder's inequality, (34.26), and (34.32); its summed norm is at most a fixed multiple of the right side of (34.32). For the other one, the continuation satisfies the deterministic bound

$$\sum_{j>m} H_\sigma^a(u_j^{\text{cl},0}(0)) \leq \frac{C \|\hat{\theta}_m\|_\infty^2}{\ell_m} \leq C [r^2 \ell_m^{2\kappa-1} + \gamma p_m^2], \quad (34.35)$$

given its input at level m . The first inequality sums the quadratic series of the zero-fluctuation maps with ratio $B\varrho^2 = \mathfrak{q}$, and the second uses (34.20). At $m = 0$ the last expression may be replaced by the constant Cr^2/H_0 . The right side is not bounded uniformly in m at fixed γ , because of the term γp_m^2 . We therefore multiply (34.35) by (34.33), using (34.6), before absorbing anything. The resulting term of order larger than γ is at most

$$C_p[\gamma^{3/2}p_m^2 + \gamma^2p_m^2(1 + |\log \gamma|)^d](1 + m)^c B^{-s_0 m}.$$

Since $p_m \leq C(1 + m)(1 + |\log \gamma|)$, its additional polynomial in m is absorbed by $B^{-(s_0-s)m}$ for any fixed $0 < s < s_0$, and after enlarging d the term has coupling order at least γ . The deterministic part $r^2\ell_m^{2\kappa-1}$ of (34.35) keeps the positive power $2\kappa - 1$. This bounds both orders of the other cross product with the predictable term. In particular γp_m^2 is never bounded uniformly in m before its decaying factor has been restored.

Step 6: the remaining products. We follow the third column of Table 34.2. Each term is a difference from the zero-fluctuation counterpart, which is zero whenever s or z occurs, and the factor $(1 + j)^c$ is suppressed in this step.

- (s, s) : the centered part is at most $C_p\gamma\ell_j^{2\alpha-1}(1 + j)$, and the conditional mean is at most $C_p\gamma[r\varrho^{j-1} + \mathfrak{g}_{j-1}\sqrt{1 + j}]$, by Theorem 34.7 with the field moments (34.27).
- (s, z) , (z, s) , (z, z) : the inequality (34.6) and Hölder's inequality under ν give the products of the $2p$ -moment bounds $S_j Z_j^-$ or $Z_j^+ Z_j^-$ of Theorem 34.7. Their coupling orders are at least $\gamma^{3/2}$ or γ^2 , and their powers of ℓ_j include $\beta + \sigma > 0$ and $\alpha + \sigma > 0$. The factors of higher degree carry an additional $\ell_j^{1/2}$.
- (s, r) , (r, s) : conditionally on $\mathcal{F}_{j-1}$ we use the two mixed linear rows with the predictable background r_j . Its background bound has moments $C_p[r\ell_{j-1}^{\kappa+\alpha-1} + \sqrt{\gamma}\ell_{j-1}^{\alpha-1/2}\sqrt{1 + j}]$. The resulting powers are $\sqrt{\gamma}r\ell_j^{\kappa+2\alpha-3/2}$ and $\gamma\ell_j^{2\alpha-1}$, each with a fixed polynomial factor.
- (z, r) , (r, z) : we put the norm H_σ^a of the initial value of z_j in (34.6) and use the moment bound for r_j just stated. The leading term is $\gamma r\ell_j^{\kappa+\alpha+\sigma-1}$; the rest has order at least $\gamma^{3/2}$ and the power $\alpha + \sigma - 1/2 > 0$.
- (r, r) : this is the difference of $T^{\vee\vee}$ in (34.34), including its classical and conditional-mean parts.
- The seven pairs containing u : we use (34.7) and the identity with both arguments interchanged; for (u, u) this gives $\mathcal{Q}_a(u, u - u^{\text{cl},0}) + \mathcal{Q}_a(u - u^{\text{cl},0}, u^{\text{cl},0})$. The individual norms of s_j , z_j , r_j , and u_j have bounded moments of order $2p$ by Theorem 34.7, and the sums of the moments of the differences $v - v^0$ are bounded by the statement following (34.33). We use (34.32) for the first term of (34.7), and (34.35) with the decaying factor of (34.33) for the second, as in Step 5.

There are also eight cross terms with x_{j-1} . Those with s are bounded by (34.19), the bounded moments of the norm (34.14) of x_{j-1} , and $\sqrt{\gamma}\ell_j^\beta$. Those with z use the norm H_σ^a of the initial value of z_j and $X_\alpha^a(x_{j-1})$; they have order at least γ and a positive power of the mesh. For those with r , the two exact identities $T^{\text{UV}} - T^{\text{VV}}$ and $T^{\text{VU}} - T^{\text{VV}}$ of (34.2) reduce the differences to (34.34). For the two orders with u we use the identity

$$\mathcal{Q}_a(x_{j-1}, u_j) - \mathcal{Q}_a(x_{j-1}^0, u_j^{\text{cl},0}) = \mathcal{Q}_a(x_{j-1}, u_j - u_j^{\text{cl},0}) + \mathcal{Q}_a(\Delta x_{j-1}, u_j^{\text{cl},0})$$

and its reversed order, which were bounded in Step 5. None of these terms requires a bilinear estimate of two rough heat paths in the norm X_α^a ; the products of z_j with r_j use only the initial-array norm of the centered nonlinear heat term, the norm of the signed defect, and (34.6).

By bilinearity, $y_K - y_K^0$ is the sum over $m < j \leq K$ of these differences, and we apply the Minkowski inequality to this finite sum. The terms of order $\sqrt{\gamma}$ listed above keep a positive power of the mesh and contain no logarithm of the coupling; their sum is at most $C_p \sqrt{\gamma} (1+m)^c B^{-sm}$. All terms containing a logarithm of the coupling have order at least γ , and the geometric sums above bound them by $C_p \gamma (1 + |\log \gamma|)^d$, where dropping a remaining decaying factor only weakens the bound. Together with (34.33), this proves the final estimate of the theorem, and (34.32), (34.33), and (34.34) prove the other estimates. Each application of Hölder's inequality used higher fixed moments under the same measure; for a final moment p it suffices to use moments up to order $6p$, since a cubic difference has three factors and a product has two. These moments were proved in Theorem 34.7 and in Step 1. The exponents and all constants are fixed before γ , and they are independent of K , m , and the prior depths. $\square$

34.6 Holonomy comparison at the weak-coupling scale

We now transfer the comparison of flow data to the holonomy matrices of the flowed configurations. For a pair $\mathcal{E} = (x, y)$ of heat paths on the lattice of mesh a we write $\|\mathcal{E}\|_{\text{fl}} = X_\alpha^a(x) + Y_\alpha^a(y)$ for the flow-data norm of Definition 24.3, with the fixed mesh a suppressed, and we abbreviate $\mathcal{L}_\gamma = 1 + |\log \gamma|$.

Theorem 34.10. *For the triples of Definition 34.1, let $D_{K,m} = \max_i \|\mathcal{H}_i^K - \Xi_{m,i}^K\|_{\text{op}}$. There exist $C, c, \epsilon, d > 0$ and, for every fixed $p \geq 1$, a constant $C_p < \infty$, all independent of $K \geq m \geq 0$ and $0 < \gamma \leq \gamma_0$, such that*

$$\mathbb{E}_v \min(1, D_{K,m})^2 \leq C\gamma\delta_m + C\gamma^2 \mathcal{L}_\gamma^{2d}, \quad \delta_m = (1+m)^c B^{-\epsilon m},$$

$$\mathbb{E}_v \min(1, D_{K,m})^p \leq C_p \gamma^{p/2}.$$

For every fixed m , the second term in the first bound is $o(\gamma)$ as $\gamma \rightarrow 0$, uniformly in K .

In particular the moments of order at most eight used in Section 35 have constants independent of m . The estimates hold under the conditioned measure, whose coarse field ranges over neighborhoods of all flat connections, regular and central.

Proof. Let $\mathcal{E} = (x_K, y_K)$, let $\mathcal{E}_m^0 = (x_K^0, y_K^0)$ be the flow data of the zero-fluctuation continuation after level m , and let $\mathcal{E}_{\text{cl}}$ be the flow data of the zero-fluctuation continuation from the same coarse field, that is, the case $m = 0$. All three live on the finest lattice. By Lemma 34.4, $\|\mathcal{E}_{\text{cl}}\|_{\text{fl}} \leq R_{\text{cl}}$ deterministically for every coarse field in the reconstruction domain, including every flat coarse field.

By Theorem 34.9, for every fixed p we have

$$\|\|\mathcal{E} - \mathcal{E}_m^0\|_{\text{fl}}\|_p \leq C_p \sqrt{\gamma} (1+m)^c B^{-sm} + C_p \gamma \mathcal{L}_\gamma^d. \quad (34.36)$$

The function $\gamma^{1/2} \mathcal{L}_\gamma^d$ is bounded on $(0, \gamma_0]$ by the calculation in the proof of Lemma 34.6, and $\sup_m (1+m)^c B^{-sm} < \infty$ by (34.15). Thus the right side of (34.36) is at most $C_p \sqrt{\gamma}$, uniformly in m . Its case $m = 0$ bounds $\mathcal{E} - \mathcal{E}_{\text{cl}}$ by the same quantity, and the triangle inequality then bounds $\mathcal{E}_m^0 - \mathcal{E}_{\text{cl}}$ by $C_p \sqrt{\gamma}$ as well.

Let $R = R_{\text{cl}} + 2$. For any real $q \geq 1$, the preceding bounds at moment $2q$ and Markov's inequality imply that

$$\nu\{\|\mathcal{E}\|_{\text{fl}} > R \text{ or } \|\mathcal{E}_m^0\|_{\text{fl}} > R\} \leq C_q \gamma^q. \quad (34.37)$$

This bound holds for each m with the same constant; there is no union over m . On the complementary event we apply Remark 34.8 with $t_- = t_+ = \theta L^2$ to the configurations $\exp \hat{\theta}_K$ and $\exp \hat{\theta}_K^0$. They have the common initial frame by construction, and their difference of flow data is exactly the one in (34.36). Hence

$$D_{K,m} \leq \Lambda_{R, \theta L^2, T} \|\mathcal{E} - \mathcal{E}_m^0\|_{\text{fl}}$$

on that event. On the exceptional event we use $\min(1, D_{K,m}) \leq 1$. Squaring (34.36), using $(u+v)^2 \leq 2u^2 + 2v^2$, and taking $q = 2$ in (34.37), we obtain

$$\mathbb{E}_\nu \min(1, D_{K,m})^2 \leq C \gamma (1+m)^{2c} B^{-2sm} + C \gamma^2 \mathcal{L}_\gamma^{2d} + C \gamma^2.$$

After increasing d to a positive value if necessary, the last term is absorbed into the second, and relabeling $2c$ and $2s$ as c and ϵ gives the first assertion. For a fixed $p \geq 1$ we use $(u+v)^p \leq 2^{p-1}(u^p + v^p)$, choose $q \geq \max(1, p/2)$ in (34.37), and use the bound $C_p \sqrt{\gamma}$ on the right side of (34.36); this gives $\mathbb{E}_\nu \min(1, D_{K,m})^p \leq C_p \gamma^{p/2}$. The stability estimate of Remark 34.8 covers the finitely many coarse meshes; no further event depending on the mesh has been discarded. Finally $\gamma \mathcal{L}_\gamma^{2d} \rightarrow 0$, again because $e^{-v}(1+v)^{2d} \rightarrow 0$ as $v = |\log \gamma| \rightarrow \infty$. Thus the second term in the bound on the mean square is $o(\gamma)$ uniformly in K for every fixed m ; in fact its constant is independent of m . $\square$

The next theorem gives the same comparison for every prescribed coarse field, uniformly over a time interval. It is used in Section 36, where the conditional estimate is multiplied by the small probability of a neighborhood of a central triple.

Theorem 34.11. *Fix the data of Definition 34.1 and $0 < \tau \leq T < \infty$, with the same three coordinate cycles. For each $K \geq 0$ and every coarse field $v = e^{\theta_0}$ in the reconstruction domain with $\|\theta_0\|_\infty \leq r$, define the probability*

$$\mathbb{Q}_v^K(dw_1 \cdots dw_K) = \prod_{j=1}^K k_j^K(\theta_{j-1}, dw_j), \quad \theta_j = \widehat{\Phi}_j(\theta_{j-1}, g_j, w_j),$$

as an iterated integral: w_1 has the law $k_1^K(\theta_0, \cdot)$, and given $w_1, \dots, w_{j-1}$ the vector w_j has the law $k_j^K(\theta_{j-1}, \cdot)$. Here k_j^K are the conditional fluctuation kernels of Theorem 15.9 and Lemma 15.10 at prior depth $J(K-j)$, written in the gauge of the reconstruction maps of Theorem 15.13; for $K=0$ the product is the unit point mass. Let $\theta_j^0 = P_j \cdots P_1 \theta_0$, and let $\mathcal{H}_t^K$ and $\mathcal{H}_{0,t}^K(v)$ be the based holonomy triples of $\Psi_t^{(N)}(\exp \theta_K)$ and $\Psi_t^{(N)}(\exp \theta_K^0)$, in their common initial frame. Set

$$d_{K, [\tau, T]} = \min \left(1, \sup_{\tau \leq t \leq T} \max_i \|\mathcal{H}_{t,i}^K - \mathcal{H}_{0,t,i}^K(v)\|_{\text{op}} \right).$$

There exists $\gamma_W \in (0, \gamma_0]$, independent of K and v , such that for every fixed $p \geq 1$ there exists $C_{p, \tau, T} < \infty$ with

$$\sup_{K \geq 0} \sup_v \mathbb{E}_{\mathbb{Q}_v^K} d_{K, [\tau, T]}^p \leq C_{p, \tau, T} \gamma^{p/2} \quad (0 < \gamma \leq \gamma_W).$$

The constants depend in addition on p, τ, T , and the loops, but not on the coarse field, which ranges over the set of initial fields specified in the theorem, including every flat coarse field. The ceiling γ_W is chosen before the moment order. The theorem is a statement about the probability $\mathbb{Q}_v^K$ itself, and it does not follow by conditioning Theorem 34.10 on the coarse field, which would only give a statement for almost every coarse field under the coarse marginal of v .

Proof. Given the coarse field, the conditioned Wilson measure has the same conditional structure as $\mathbb{Q}_v^K$. We first construct $\mathbb{Q}_v^K$ for every v and check that all conditional estimates of Theorem 34.7 hold under it. The proof of Theorem 34.9 then applies with $m=0$ and a deterministic coarse field; we repeat its steps in this setting, since the averaged statement cannot be quoted. The only change is that the sum of the classical predictable terms is now bounded by a deterministic constant. Finally we localize to a fixed ball of flow data and apply the matrix stability over the whole interval $[\tau, T]$.

Step 1: construction and coupling ceiling. The initial fields in the statement satisfy $\|\theta_0\|_\infty \leq r$. The kernel formula of Theorem 15.9 extends continuously to this closed ball: its product with the fluctuation cutoff domain lies strictly inside the common analytic domains, and its positive normalizer is the integral of a bounded positive density. The gauge change of Theorem 15.13 changes neither this density nor its product of balls. By item (2) of Theorem 15.13, with the closed-product image conditions verified in its proof, the reconstruction maps this product into B_r . Inductively, every later incoming field lies in B_r , and Theorem 15.9 gives each successive kernel. Iterated integration therefore defines $\mathbb{Q}_v^K$, including at initial fields to which the coarse marginal of v assigns no mass. By the amplitude bound in Lemma 15.10, all intermediate fields also stay in the working ball, and

$$\|\theta_j\|_\infty \leq C\{r\varrho^j + \mathfrak{g}_j p_j\}, \quad p_j \leq C(1 + j + |\log \gamma|).$$

We reduce γ_W so that the Hessian lower bound and the bounds on the conditional means of the kernels hold at all these fields. This is possible because $t_j \leq t_1 = \sqrt{\gamma/(Mn)}$ and $t^{a'} \log^{b'}(1/t) \rightarrow 0$ as $t \rightarrow 0$ for $a' > 0$. The hierarchy and the cutoffs of the fluctuation variables are not changed.

Step 2: transfer of conditional estimates. Let $\mathcal{F}_j = \sigma(w_1, \dots, w_j)$ on this probability space; the coarse field v is deterministic. Given $\mathcal{F}_{j-1}$, the law of the next vector is exactly $k_j^K(\theta_{j-1}, \cdot)$. Thus every estimate of Theorem 34.7 for a fixed incoming field holds here with the same constants. Explicitly, if $\mathbb{E}(|A|^q \mid \mathcal{F}_{i-1}) \leq C_q^q b(\theta_{i-1})^q$ is one of those estimates, then the tower property implies that

$$\|A\|_{L^q(\mathbb{Q}_v^K)} \leq C_q \|b(\theta_{i-1})\|_{L^q(\mathbb{Q}_v^K)}. \quad (34.38)$$

No coarse marginal enters this inequality.

Step 3: moments of the fields. We write $\delta_j = \theta_j - \theta_j^0$ and $\epsilon_j = \widehat{\Phi}_j(\theta_{j-1}, g_j, w_j) - P_j \theta_{j-1}$. The bound on the gradient in the fluctuation variables and Theorem 5.4, applied for a fixed incoming field to the at most $C_j(nB^j)^3$ scalar outputs, imply that

$$\|\epsilon_j - \mathbb{E}(\epsilon_j \mid \mathcal{F}_{j-1})\|_\infty \leq C_q \mathfrak{g}_j \sqrt{1+j}.$$

The conditional mean is bounded by (34.13): $\|\mathbb{E}(\epsilon_j \mid \mathcal{F}_{j-1})\|_\infty \leq C_r \mathfrak{g}_j^2$. Since the segment between the two incoming fields lies in the convex working ball, we have

$$\|\delta_j\|_\infty \leq \varrho \|\delta_{j-1}\|_\infty + C_q \mathfrak{g}_j \sqrt{1+j} + C_r \mathfrak{g}_j^2.$$

We start at $\delta_0 = 0$ and sum, using $\mathfrak{g}_i = \mathfrak{g}_j B^{(j-i)/2}$, $\varrho \sqrt{B} < 1$, and, for the quadratic term, $\mathfrak{g}_i^2 \leq \mathfrak{g}_1 \mathfrak{g}_i$. For every fixed $q < \infty$ we obtain

$$\|\delta_j\|_\infty \leq C_q \mathfrak{g}_j \sqrt{1+j}, \quad \|\theta_j\|_\infty \leq C_q \{r\varrho^j + \mathfrak{g}_j \sqrt{1+j}\}. \quad (34.39)$$

The continuation is deterministic and $\|\theta_j^0\|_\infty \leq r\varrho^j$.

Step 4: the pulled-back forms. Let $A_{j,i}$ be the quadratic initial form $S_{j,i}$ or one of the three ordered forms $T_{j,i}^\omega$, and recall $K_{ji} = q^{j-1-i}/\ell_i$; fixed polynomial factors in $1+j$ are allowed below. With $L_i = DP_i(0)$, we define

$$\begin{aligned} Z_{j,i} &= A_{j,i}(\theta_i, \theta_i) - \mathbb{E}[A_{j,i}(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}], \\ C_{j,i}(u) &= A_{j,i}(P_i u, P_i u) - A_{j,i}(L_i u, L_i u), \\ M_{j,i}(u) &= \mathbb{E}_{k_i^K(u, \cdot)} A_{j,i}(\theta_i, \theta_i) - A_{j,i}(P_i u, P_i u), \end{aligned}$$

where in the last line $\theta_i = \hat{\Phi}_i(u, g_i, w_i)$ with w_i distributed according to $k_i^K(u, \cdot)$. By the difference estimate of Theorem 34.9, which follows from (34.28), we have

$$\|C_{j,i}(u) - C_{j,i}(u')\|_{\text{out}} \leq C(1+j)^c K_{ji} (\|u\|_\infty + \|u'\|_\infty)^2 \|u - u'\|_\infty.$$

By the splitting into classical and conditional-mean parts in Theorems 27.5 and 27.7, we have

$$\|M_{j,i}(u)\|_{\text{out}} \leq C(1+j)^c K_{ji} (\mathfrak{g}_i^2 \|u\|_\infty + \mathfrak{g}_i^3 p_i^3). \quad (34.40)$$

Here the output of the quadratic initial form uses the initial-array norm H_σ^a and the ordered outputs use the norm Y_α^a ; there is no classical cubic term in $M_{j,i}$. As in Step 3 of the proof of Theorem 34.9, the exact pullback telescopes to

$$A_{j,j-1}(\theta_{j-1}, \theta_{j-1}) - A_{j,j-1}(\theta_{j-1}^0, \theta_{j-1}^0) = \sum_{i=1}^{j-1} \{Z_{j,i} + C_{j,i}(\theta_{i-1}) - C_{j,i}(\theta_{i-1}^0) + M_{j,i}(\theta_{i-1})\}. \quad (34.41)$$

The term at level zero cancels because both starting fields are θ_0 .

Step 5: sizes and summation. As before, let $q_i = 1 + i + |\log \gamma|$. Applying (34.38), (34.39), the conditional fiber bound $b_i(u) = C_J(q\|u\|_\infty + \mathfrak{g}_i p_i)$, and Hölder's inequality under $\mathbb{Q}_v^K$, we find that the four bounds of (34.31) hold in $L^q(\mathbb{Q}_v^K)$, up to the factor $C_q(1+j)^c q^{j-1-i}$, for $Z_{j,i}$ (for the quadratic initial form and each ordered form), for $C_{j,i}(\theta_{i-1}) - C_{j,i}(\theta_{i-1}^0)$, and for $M_{j,i}(\theta_{i-1})$. The verification is the one given after (34.31). For example, at moment $3q$ the difference of the classical terms is at most $C_q \ell_i^{-1} [r q^{i-1} + \mathfrak{g}_{i-1} \sqrt{1+i}]^2 \mathfrak{g}_{i-1} \sqrt{1+i}$; the bounds on $Z_{j,i}$ follow from the conditional bounds $\mathfrak{g}_i b_i \ell_i^{\sigma-1}$ and $\mathfrak{g}_i b_i \ell_i^{2\alpha-2}$; and the bound on $M_{j,i}$ is (34.40).

Since all the exponents of the mesh are positive, the geometric sums of Lemma 34.2 sum (34.41) separately for the quadratic initial form and for each of the ordered forms. The remaining local terms of the predictable part are the classical cubic term and the conditional mean of (34.12) and (34.13), which are bounded by the third and fourth lines of (34.31). The first differences in the initial-array norm are controlled by Theorem 27.7, including for $t \leq a^2$. We obtain

$$\sum_j \|H_\sigma^a(u_j(0) - u_j^{\text{cl},0}(0))\|_q + \sum_j \|X_\alpha^a(d_j - d_j^0)\|_q + \sum_{j,\omega} \|Y_\alpha^a(\Delta T_j^\omega)\|_q \leq C_q [\sqrt{\gamma} + \gamma \mathcal{L}_\gamma^d], \quad (34.42)$$

where ΔT_j^ω is the left side of (34.41) for the form T^ω , and every sum runs over $1 \leq j \leq K$. For the linear heat evolution, the bound on s_j is $C_q \sqrt{\gamma} \ell_j^{\alpha-1/2} \sqrt{1+j}$, the bound on z_j is $C_q(\gamma \ell_j^\alpha(1+j) + \gamma^{3/2} \ell_j^{1/2} p_j^3)$, the bound on $r_j - r_j^0$ follows from (34.11) and (34.39), and the inequality $X_\alpha^a(P^a f) \leq CH_\sigma^a(f)$ bounds $u_j - u_j^{\text{cl},0}$. Since these are the four parts of $d_j - d_j^0$, (34.42) also bounds $\|\sup_j X_\alpha^a(x_j - x_j^0)\|_q$.

Step 6: products. We use only the pointwise ordered inequality (34.6); the averaged conclusions at fixed coupling of Lemma 28.5 are not needed. We denote the q -moment bounds by

$$\begin{aligned} S_j &= C_q \sqrt{\gamma} \ell_j^{\alpha-1/2} \sqrt{1+j}, \\ Z_j^- &= C_q(\gamma \ell_j^\alpha(1+j) + \gamma^{3/2} \ell_j^{1/2} p_j^3), \\ Z_j^+ &= C_q(\gamma \ell_j^\alpha(1+j) + \gamma^{3/2} \ell_j^{1/2} p_j^3), \\ R_j &= C_q(r \ell_{j-1}^{\kappa+\alpha-1} + \sqrt{\gamma} \ell_{j-1}^{\alpha-1/2} \sqrt{1+j}), \end{aligned}$$

and use moments of order $2q$ in products. The norms $H_\sigma^a(u_j(0))$ and the norms (34.14) of x_{j-1} at its input mesh have uniformly bounded moments. This follows from the same estimates for the quadratic initial terms without subtraction, after adding the deterministic initial term and the classical cubic term, whose powers of the mesh are $2\kappa - 1 > 0$ and $3\kappa - 1 > 0$, besides the positive powers already listed. The initial cardinal field costs only $C_{H_0} r$. The sixteen ordered differences $\mathcal{Q}_a(v_j, w_j) - \mathcal{Q}_a(v_j^0, w_j^0)$ and the eight cross terms are then estimated as in Step 6 of the proof of Theorem 34.9, following the third column of Table 34.2, with the following bounds.

- (s, s) : the centered part is at most $C_q \gamma \ell_j^{2\alpha-1} (1+j)$ and the conditional mean at most $C_q \gamma [r \ell_{j-1}^{j-1} + \mathfrak{g}_{j-1} \sqrt{1+j}]$, by Theorem 34.7.
- (s, z) , (z, s) , (z, z) : respectively $CS_j Z_j^-$, $CS_j Z_j^-$, and $CZ_j^+ Z_j^-$, by (34.6) and Hölder's inequality under $\mathbb{Q}_v^K$.
- (s, r) , (r, s) : $CS_j R_j$, by the conditional mixed linear rows of Theorem 34.7; the powers are $\kappa + 2\alpha - 3/2$ and $2\alpha - 1$.
- (z, r) , (r, z) : $CZ_j^- R_j$ in each order, using the initial-array norm of the centered nonlinear heat term. The leading powers include $\kappa + \alpha + \sigma - 1$ and $\alpha + \sigma - 1/2$.
- (r, r) : the difference of the form T^{VV} in (34.41) and (34.42).
- The seven pairs containing u : we use (34.7) and the identity with both arguments interchanged; for (u, u) this gives $\mathcal{Q}_a(u, u - u^{\text{cl},0}) + \mathcal{Q}_a(u - u^{\text{cl},0}, u^{\text{cl},0})$. The first part is summed using the bound on the predictable term in (34.42) and the bounded norms.

For the second part, the quadratic series of the zero-fluctuation maps and $B\varrho^2 < 1$ give the deterministic bound

$$\sum_{j \geq 1} H_\sigma^a(u_j^{\text{cl},0}(0)) \leq C \|\theta_0\|_\infty^2 / H_0 \leq Cr^2 / H_0,$$

which we multiply by the part-by-part summed bounds for $X_\alpha^a(v_j - v_j^0)$, $v \in \{s, z, r, u\}$, established immediately after (34.42), with $u_j^0 = u_j^{\text{cl},0}$ in this comparison. Since $m = 0$ and the coarse field is deterministic, there is no term γp_m^2 here.

There are eight cross terms with x_{j-1} , two for each argument. For s , the conditional mixed linear rows give $C_q \sqrt{\gamma} \ell_j^{\alpha-1/2}$ times a bounded norm of x_{j-1} . For z , the two orders of (34.6) give $C_q Z_j^-$ times that norm. For r , the two exact differences $T^{\text{UV}} - T^{\text{VV}}$ and $T^{\text{VU}} - T^{\text{VV}}$ are bounded in (34.42). For u we use

$$\mathcal{Q}_a(x_{j-1}, u_j) - \mathcal{Q}_a(x_{j-1}^0, u_j^{\text{cl},0}) = \mathcal{Q}_a(x_{j-1}, u_j - u_j^{\text{cl},0}) + \mathcal{Q}_a(x_{j-1} - x_{j-1}^0, u_j^{\text{cl},0})$$

and its reversed order, which are bounded by the two estimates on the predictable terms just used. This accounts for all eight cross terms and all sixteen ordered products.

Step 7: the difference of flow data. The smallest exponents of the mesh in these sums are

$$\alpha - \frac{1}{2}, \quad \sigma, \quad 2\alpha - 1, \quad \kappa + \sigma - \frac{1}{2}, \quad \kappa + 2\alpha - \frac{3}{2}, \quad 2\kappa - 1, \quad \kappa + \alpha + \sigma - 1,$$

and $-\log q / \log B$, all strictly positive. Every term with a logarithm of the coupling has order at least γ , and Lemma 34.2 sums each fixed polynomial times its geometric decay. By bilinearity and the Minkowski inequality, under $\mathbb{Q}_v^K$,

$$\|\mathcal{E}_K - \mathcal{E}_K^0\|_{\text{fl}} \leq C_q (\sqrt{\gamma} + \gamma \mathcal{L}_\gamma^d) \leq C'_q \sqrt{\gamma}, \quad (34.43)$$

where $\mathcal{E}_K = (x_K, y_K)$ and $\mathcal{E}_K^0 = (x_K^0, y_K^0)$, and in the last inequality we used that $\sqrt{\gamma} \mathcal{L}_\gamma^d$ is bounded. The bookkeeping of moments uses at most order $6q$: three factors in a cubic difference and two in a product. Thus the final moments $q = 1, 2$ require only the finitely many moment bounds up to order 12, all of which were obtained for fixed incoming fields before the tower integration.

Step 8: localization and the interval estimate. By Lemma 34.4, $\|\mathcal{E}_K^0\|_{\text{fl}} \leq R_{\text{cl}}$ uniformly in v . With $R = R_{\text{cl}} + 2$, (34.43) at moment q and Markov's inequality imply that

$$\mathbb{Q}_v^K \{ \|\mathcal{E}_K\|_{\text{fl}} > R \} \leq C_q \gamma^{q/2}.$$

On the complementary event we apply Remark 34.8 directly with $t_- = \tau$ and $t_+ = T$. It bounds the supremum over $t \in [\tau, T]$ by $\Lambda_{R,\tau,T} \|\mathcal{E}_K - \mathcal{E}_K^0\|_{\text{fl}}$. On the exceptional event

we use $d_{K,[\tau,T]} \leq 1$. Taking $q \geq p$ and using (34.43) at moment p , we obtain the asserted bound $C_{p,\tau,T} \gamma^{p/2}$. The finitely many coarse meshes are included in the stability estimate. For $p = 1, 2$, the choice $q = 2$ suffices for the localization, and moments up to order 12 suffice for the whole argument. $\square$

34.7 Linear Gaussian response near a fixed flat connection

In this section, we prove the estimate used in Section 35 to calibrate the normal covariance near regular commuting triples. We fix a regular flat coarse field and differentiate the deterministic reconstruction and the Wilson flow at this field with respect to the fluctuation inputs of the steps after level m . We divide the coupling factors by $\sqrt{\gamma}$, and only then insert independent standard Gaussian vectors in place of the fluctuation variables. The lemma bounds the covariance, in the normal space, of the resulting Gaussian vector. Three points need care. First, the inputs $q_j = g_j, w_j$ are the arguments of the reconstruction maps. The normalization by the inverse square root of the constrained Hessian, which depends on the field, is already inside these maps, and the derivative therefore includes all normalizing matrices and their dependence on the field through the later reconstruction. Second, we recall from Definition 4.11 that at a point $y = (\vartheta_i u)_{i=1}^3$ of a compact regular patch of the cone of commuting triples, the normal space is $\mathcal{N}_y = \{b = (b_i)_{i=1}^3 : b_i \perp u, \sum_i \vartheta_i b_i = 0\}$, of dimension four, and $P_{\mathcal{N}_y}$ denotes the Euclidean orthogonal projection onto it in the fixed coordinates of cycle logarithms. Third, the identity $\mathbb{E}_G |\sum_j A_{j,K} G_j|^2 = \sum_j \text{tr}(A_{j,K} A_{j,K}^T)$ concerns only the contribution of the omitted steps. The covariance of the normal fluctuations of the coarse field is a separate contribution, which is not set to zero, and for this reason the lemma cannot be replaced by an averaged estimate for the conditioned measure.

Lemma 34.12. *Use the hierarchy, the side n , the time θL^2 , and the reconstruction maps of Definition 34.1. Fix a compact regular patch of the cone of commuting triples and margins for the logarithm and for the normal coordinates as in Theorem 33.11. For a flat coarse field V_y whose cycle holonomies lie in this patch, let $\mathcal{H}_K(y, q)$ be the based holonomy triple, around the three coordinate cycles, of the configuration on the finest lattice reconstructed from V_y with fluctuation inputs $q = (q_1, \dots, q_K)$, flowed for the time θL^2 . At $q = 0$ it equals $\text{Hol}(V_y)$, in the original frame. Put*

$$A_{j,K}(y) = P_{\mathcal{N}_y} D_{q_j} (\text{Log} \circ \mathcal{H}_K)(y, 0) \text{diag}_{r,e}(g_{j,r}/\sqrt{\gamma}), \quad 1 \leq j \leq K,$$

where the diagonal matrix has the entry $g_{j,r}/\sqrt{\gamma}$ at every scalar coordinate (r, e) of the fluctuation variables of step j ; these ratios do not depend on γ . Equivalently, $A_{j,K} = \gamma^{-1/2} P_{\mathcal{N}_y} D_{w_j} (\text{Log} \circ \mathcal{H}_K)$ at zero fluctuation, with the full zero-fluctuation continuation at the later levels and every

normalizing matrix included. Let G_j be independent standard real Gaussian vectors indexed like w_j . Then there exist $C, c, \epsilon > 0$ such that

$$\sup_y \mathbb{E}_G \left| \sum_{j=m+1}^K A_{j,K}(y) G_j \right|^2 \leq C(1+m)^c B^{-\epsilon m} \quad (K \geq m \geq 0).$$

The constants may depend on the fixed n, L, θ, M, J , the parameters Θ of Theorem 23.2, the patch, and the fixed bounds on the coefficients and the flow, but not on m, K , the prior depths, or $\gamma \in (0, \gamma_0]$. Before the flow is applied, the first-order variation of the flow data in these directions, measured in both full norms with their first differences, has a bound $C(1+m)^c B^{-sm}$ in $L^2(G)$ for some $s > 0$.

Only linear maps are evaluated at the unbounded Gaussian vectors; no nonlinear reconstruction is evaluated outside its cutoff domain. For $j \geq 1$, the matrices $A_{j,K}$ are the linear response matrices $A_{j,K}$ of Definition 35.7, on the same normal patch and with the same gauge and basepoint, as the second expression for $A_{j,K}$ shows; the coarse response $A_{0,K}$ there is a separate contribution.

Proof. The proof uses only deterministic coefficient estimates and independent Gaussian vectors; in particular, it does not use Theorems 34.9 and 34.10 as probability estimates. Throughout, we keep the zero-fluctuation base point and the direction of differentiation separate. We first set up and bound the linear recursion for the derivative of the reconstruction. We then write the first-order variation of the flow data as a sum over levels, bound each term by deterministic rows and Gaussian chaining, and sum. Finally, we pass to the holonomies through the derivative of the flow and the logarithm.

Step 1: the base point and the linear recursion. We write $\varrho = a_{\text{map}}$ and $H_0 = L/n$. In one of the finitely many fixed charts covering the compact patch, let $v_0(y)$ be the logarithmic representative of V_y in the reconstruction domain, and set

$$v_i = P_i v_{i-1}, \quad b_i^{\text{fl}} = r \varrho^i, \quad X_i = P_i^a I_{\ell_i \rightarrow a}(v_i / \ell_i), \quad Y_i = \mathcal{Q}_a(X_i, X_i). \quad (34.44)$$

The condition on flat fields was checked at the beginning of Section 34.2. By Lemma 14.9, at every side and prior depth, each constrained minimizer is an exact flat lift, its free coordinates vanish, and the gauges of the reconstruction maps preserve the prescribed cycle holonomies. The Wilson flow is stationary at each resulting flat field. It follows that $\mathcal{H}_K(y, 0) = \text{Hol}(V_y)$ exactly, in the original frame; this uses the refinement of flat fields, not a property of cardinal interpolation. By the bound on DP_i and Lemma 34.4,

$$\|v_i\|_\infty \leq b_i^{\text{fl}}, \quad \sup_{K,i,y} \{X_\alpha^a(X_i) + Y_\alpha^a(Y_i)\} \leq R_{\text{cl}}. \quad (34.45)$$

The same domains serve every chart of the finite cover.

Let D_i be the diagonal matrix with the entries $g_{i,r}/\sqrt{\gamma}$ at all scalar coordinates of the fluctuation variables of step i . We write

$$W_i = D_q \hat{\Phi}_i(v_{i-1}, 0) D_i, \quad \eta_m = 0, \quad \eta_i = DP_i(v_{i-1})\eta_{i-1} + W_i G_i \quad (i > m), \quad (34.46)$$

and we set $\eta_i = 0$ for $i < m$ as well. By the coefficient bounds of the reconstruction maps, we have

$$\|DP_i(v_{i-1})\|_{\infty \rightarrow \infty} \leq \varrho, \quad \sup_e \|W_i(e, \cdot)\|_2 \leq C_J d_i, \quad d_i = \mathfrak{g}_i / \sqrt{\gamma} = \sqrt{\ell_i / L}.$$

The ratios of the couplings within one step are kept in D_i ; replacing them by d_i costs only a constant depending on J . All these linear maps are independent of the coupling. Their derivatives contain the positive inverse square roots of the constrained Hessians used to normalize the fluctuation variables, including the dependence of these matrices on the field through the subsequent zero-fluctuation maps.

A scalar Gaussian row $c \cdot G$ has the Laplace transform $\mathbb{E}e^{tc \cdot G} = e^{t^2 |c|_2^2 / 2}$, by completing the square in each one-dimensional Gaussian integral. Hence its two-sided tail at the threshold $u|c|_2$ is at most $2e^{-u^2/2}$. There are at most $C_J(nB^i)^3$ scalar outputs at level i . By the finite-family clause of Lemma 16.4, with $N \leq C_J(nB^i)^3$, scale Cd_i , $b = 0$, and $e = 0$, we have $\|W_i G_i\|_{\infty} \|L^2(G)\| \leq Cd_i \sqrt{1+i}$. Applying the Minkowski inequality to (34.46), we obtain

$$\|\eta_i\|_{\infty} \|L^2(G)\| \leq C \sum_{l=m+1}^i \varrho^{i-l} d_l \sqrt{1+l} \leq C \sqrt{\ell_i} (1+i)^{1/2}, \quad (34.47)$$

where we used $d_l = d_i B^{(i-l)/2}$ and $\varrho \sqrt{B} < 1$. This bound is uniform in m , the cutoff, and the flat field.

Step 2: first-order identities on the common lattice. In this proof V_i and Z_i denote first variations of the flow data; they are unrelated to the flat coarse field V_y . Define

$$V_i = P^a I_{\ell_i \rightarrow a}(\eta_i / \ell_i), \quad Z_i = \mathcal{Q}_a(X_i, V_i) + \mathcal{Q}_a(V_i, X_i). \quad (34.48)$$

Both are linear in $G_{m+1}, \dots, G_i$. They are the first derivatives of the flow data (x_i, y_i) when only the fluctuation inputs of the steps after m vary, in the directions $D_l G_l$. This is an identity between finite-dimensional linear maps: we first differentiate at zero in deterministic directions, and only afterwards apply the derivatives to the Gaussian vectors.

We use the uncentered deterministic Taylor expansion in the fluctuation inputs,

$$\hat{\Phi}_i(v, q) = P_i v + D_q \hat{\Phi}_i(v, 0) q + O(q^2).$$

Along the deterministic reconstruction $\theta_i(q) = \hat{\Phi}_i(\theta_{i-1}(q), q_i)$ from $\theta_0 = v_0$, with flow data $x_i(q)$ and $y_i(q)$, the increment splits exactly into four uncentered summands,

$$x_i(q) - x_{i-1}(q) = \tilde{s}_i + \tilde{z}_i + \tilde{r}_i + \tilde{u}_i,$$

where, with $\theta = \theta_{i-1}(q)$,

$$\begin{aligned}\tilde{s}_i &= \ell_i^{-1} P^a I_{\ell_i \rightarrow a} D_q \hat{\Phi}_i(\theta, 0) q_i, & \tilde{z}_i &= \ell_i^{-1} P^a I_{\ell_i \rightarrow a} [\hat{\Phi}_i(\theta, q_i) - P_i \theta - D_q \hat{\Phi}_i(\theta, 0) q_i], \\ \tilde{r}_i &= P^a \mathcal{D}_i \theta, & \tilde{u}_i &= P^a I_{\ell_i \rightarrow a} U_i^{\text{cl}}(\theta).\end{aligned}$$

These are not the centered summands s_i and z_i of (34.1), which are defined under the conditioned measure by an expansion at the conditional mean; the present expansion is at zero, and no conditional mean enters. At $q = 0$ the four summands are $(0, 0, R_i, U_i)$, defined below, and their first variations in the directions $D_l G_l$ are $(\xi_i, 0, \rho_i, v_i)$. Thus the first-order part of the increment splits exactly as

$$V_i - V_{i-1} = \xi_i + \rho_i + v_i, \quad (34.49)$$

where

$$\xi_i = \ell_i^{-1} P^a I_{\ell_i \rightarrow a} W_i G_i, \quad \rho_i = P^a \mathcal{D}_i \eta_{i-1}, \quad v_i = P^a I_{\ell_i \rightarrow a} D U_i^{\text{cl}}(v_{i-1})[\eta_{i-1}].$$

Here $\mathcal{D}_i$ and U_i^{cl} are the signed defect and the classical predictable term of Lemma 34.2, including the quadratic terms and the terms of higher degree of the zero-fluctuation map. At order zero the same split is $X_i - X_{i-1} = R_i + U_i$, with $R_i = P^a \mathcal{D}_i v_{i-1}$ and $U_i = P^a I_{\ell_i \rightarrow a} U_i^{\text{cl}}(v_{i-1})$. Terms with two fluctuation factors have zero first derivative at zero. The zero-fluctuation continuation agrees with the base point through level m and has zero derivative afterwards, because $\eta_m = 0$. Thus

$$V_l = \sum_{i=m+1}^l (\xi_i + \rho_i + v_i), \quad Z_l = \sum_{i=m+1}^l (Z_i - Z_{i-1}),$$

and all initial terms at level m vanish exactly.

Step 3: linear rows and the background on two lattices. We apply Lemma 34.5 to the deterministic linear pullbacks, with the coefficient covers of Lemma 34.3. The values at the reference point have the Gaussian tail bounds just described. A difference of two linear Gaussian rows is a centered Gaussian variable whose variance is the squared coefficient distance, and it satisfies the conditions (1) of Lemma 34.5 exactly. We use only the deterministic coefficient part of Theorem 26.10, at zero fluctuation: we evaluate the coefficient at $(v_{i-1}, 0)$, divide its derivative in the fluctuation variables by $\sqrt{\gamma}$, and use the matrix D_i . Its weighted linear row scale is $C \ell_i^{\alpha-1/2}$. The normalized coefficient metric has bounded diameter and a cover at the input mesh, and the Gaussian Laplace transform gives the tails of the increments and at the reference point. Applying Lemma 34.5 under the Gaussian measure, we obtain

$$\|X_\alpha^a(\xi_i)\|_{L^2(G)} \leq C(1+i)^c \ell_i^{\alpha-1/2}, \quad \|X_\alpha^a(\rho_i)\|_{L^2(G)} \leq C(1+i)^{1/2} \ell_{i-1}^{\alpha-1/2}. \quad (34.50)$$

The second bound follows from (34.11) and (34.47).

We verify the stronger background conditions for the mixed rows used below. Let $H = \ell_{i-1} = B\ell_i$. Lemma 27.1 gives the signed heat estimate for difference orders 0, 1, and 2. For orders 3 and 4 and $t \geq H^2$, we put the additional differences on a half heat step in the estimate of order two, using Lemma 34.2. For $t \leq H^2$, we put all differences on the two smooth cardinal inputs, whose input meshes are H and H/B ; since their ratio is the fixed B , this costs only a constant $C_{J,q}$. For every $0 \leq q \leq 4$ this proves

$$\|D_a^q R_i(t)\|_\infty \leq C_q \frac{b_{i-1}^{\text{fl}}}{H} (t + H^2)^{-q/2} \frac{H}{\sqrt{t + H^2}} \leq C_{J,q} b_{i-1}^{\text{fl}} H^{\alpha-1} (t + \ell_i^2)^{-(\alpha+q)/2}. \quad (34.51)$$

The last inequality follows by maximizing $(t + H^2)^{\alpha/2} H / \sqrt{t + H^2}$, whose maximum is H^α since $\alpha < 1$. Since every refinement ratio of the common lattice is an integer, the statements about signed columns and cardinal interpolation apply. This background is a heat path built from two lattices, not a single cardinal interpolation of the parent field. The other background X_{i-1} has bounded norm at its input mesh by (34.14) and (34.45), and its differences through order four are bounded by the last assertion of item (6) of Lemma 34.2. Its correction term in (34.14) is $Cb_{i-1}^{\text{fl}} H^{\alpha-1}$, which is bounded since $\kappa + \alpha - 1 > 0$.

Thus both orders of the version of Theorem 26.10 with four differences apply, with the deterministic background bounds C for X_{i-1} and $Cb_{i-1}^{\text{fl}} H^{\alpha-1}$ for R_i . Its envelopes of order four give the normalized space-time cover, independently of a/ℓ_i , and the heat equation gives the time increments. Using its separate linear rows and Lemma 34.5, we obtain, for each order separately,

$$\|Y_\alpha^a(Q_a(X_{i-1}, \xi_i))\|_2 + \|Y_\alpha^a(Q_a(\xi_i, X_{i-1}))\|_2 \leq C(1+i)^c \ell_i^{\alpha-1/2}, \quad (34.52)$$

$$\|Y_\alpha^a(Q_a(R_i, \xi_i))\|_2 + \|Y_\alpha^a(Q_a(\xi_i, R_i))\|_2 \leq Cr(1+i)^c \ell_i^{\kappa+2\alpha-3/2}, \quad (34.53)$$

where $\|\cdot\|_2$ is the norm of $L^2(G)$. These are not the conditional maxima at fixed coupling of Theorem 26.10: we use only its deterministic rows and normalized covers, and the maximum is taken here for a linear Gaussian process.

Step 4: the pulled-back forms at first order. Let $T_{j,i}^\omega$ be the three ordered forms and $S_{j,i}$ the quadratic initial form, including its physical prefactor and its cardinal interpolation. For either type we write $F_{j,i}(v) = A_{j,i}(v, v)$, and we set

$$C_{j,i}(v) = F_{j,i}(P_i v) - F_{j,i}(L_i v), \quad B_{j,i} = DF_{j,i}(v_i) W_i.$$

The calculation that proved (34.28) implies that

$$\|DC_{j,i}(v)[w]\|_{\text{out}} \leq C(1+j)^c \frac{q^{j-1-i}}{\ell_i} \|v\|_\infty^2 \|w\|_\infty. \quad (34.54)$$

The output norm is Y_α^a for an ordered form and the initial-array norm H_σ^a for a quadratic initial form. Differentiating the finite telescope given by (34.3), or applying the chain rule term by term, we obtain exactly

$$DF_{j,j-1}(v_{j-1})[\eta_{j-1}] = \sum_{i=m+1}^{j-1} \{B_{j,i}G_i + DC_{j,i}(v_{i-1})[\eta_{i-1}]\}. \quad (34.55)$$

Indeed, the part of the chain rule that comes from P_i is $DF_{j,i}(v_i)DP_i(v_{i-1})\eta_{i-1}$; subtracting the derivative of the pullback, $DF_{j,i}(L_i v_{i-1})L_i\eta_{i-1}$, leaves $DC_{j,i}(v_{i-1})[\eta_{i-1}]$, and the remaining part is $DF_{j,i}(v_i)W_iG_i$. The derivative of $F_{j,m}(v_m)$ is zero because $\eta_m = 0$.

We now bound the Gaussian norm of $B_{j,i}G_i$ by deterministic rows at the base point. Let $H = \ell_{j-1}$, $G = \ell_i = B^k H$ with $k = j - 1 - i$, and $K_{ji} = q^k / G$. For the form T^{UV} the two terms in the derivative of the source are

$$\mathcal{N}_a(UW_iG_i, Vv_i) + \mathcal{N}_a(Uv_i, VW_iG_i),$$

where $U = U_{j,i}$, $V = V_{j,i}$, and $\mathcal{N}_a$ is the quadratic source of the Duhamel term. For T^{UV} we interchange U and V , and for T^{VV} we put V in both arguments of these two terms. Thus each of the three derivatives has one differentiated argument and one deterministic argument bounded by b_i^{fl} . By the product inequalities for rows and volume-normalized columns in Lemma 27.1, followed by the bounds on the time-indexed rows of heat-kernel coefficients in Theorem 27.5, we have

$$\begin{aligned} \sup_x \|B_{j,i}(t, x, \cdot)\|_2 &\leq CK_{ji}d_i b_i^{\text{fl}} \log(1 + t/H^2), \quad t \leq G^2, \\ \sup_x \|D_a B_{j,i}(t, x, \cdot)\|_2 &\leq CK_{ji}d_i b_i^{\text{fl}} J_H(t), \quad t \leq G^2, \\ \sup_x \|D_a^l B_{j,i}(t, x, \cdot)\|_2 &\leq CK_{ji}d_i b_i^{\text{fl}} G^{3/2} t^{-3/4-\iota/2} \ell_H(T), \quad t \geq G^2, \iota = 0, 1. \end{aligned}$$

Here J_H and ℓ_H are the functions defined in Theorem 27.5. The conversion of the Schur row-column estimate with volume-normalized columns uses the input volume G^3 and the output volume a^3 ; its factor $(G/a)^{3/2}$ cancels the factor $a^{3/2}$ of the row of heat-kernel coefficients and leaves $G^{3/2}$. The undifferentiated argument is evaluated at v_i , and only its deterministic size enters this calculation. We do not use b_i^{fl} as a bound on a conditional fiber with nonzero fluctuation variables, which is why no term $d_i p_i$ appears. The Jacobian at zero fluctuation has been substituted in the differentiated argument, with g_i replaced by d_i through the diagonal matrix D_i ; all other heat and signed-column operations are unchanged.

Taking the two weighted maxima of these early and late row bounds gives the scale $C(1+j)^c K_{ji} d_i b_i^{\text{fl}} G^{2\alpha-1}$. The crude space and time derivatives of the coefficients have the amplitude $C_{\text{CP}}^{2k} d_i b_i^{\text{fl}} / G^2$ and cost fixed powers of H^{-1} . Their ratio to the weighted

amplitude is bounded as in (34.21), with the same factor $(q_{\text{anc}}/q)^k \leq 1$. Therefore their normalized coefficient metrics have bounded diameter and polynomial covering numbers at the input mesh, including at time zero, and Lemma 34.5 proves, for each ordered form separately,

$$\|Y_\alpha^a(B_{j,i}G_i)\|_2 \leq Cr(1+j)^c q^{j-1-i} \ell_i^{\kappa+2\alpha-3/2}. \quad (34.56)$$

For the quadratic initial form we use the rows of Theorem 27.7 with respect to the fluctuation variables, with the same differentiated and deterministic arguments. Its value row is at most

$$CK_{j,i}d_i b_i^{\text{fl}} \min\{1, G^{3/2}(t+G^2)^{-3/4}\}.$$

After the weight with exponent $\sigma/2$ its scale is $CK_{j,i}d_i b_i^{\text{fl}} G^\sigma$. Its crude parameter derivatives keep the same factor $K_{j,i}d_i b_i^{\text{fl}}$, and division gives a fixed power of ℓ_j^{-1} and the cover at the input mesh. Lemma 34.5 for the value process, followed by the deterministic half-splitting of the heat semigroup for its linear initial field, proves

$$\|H_\sigma^a(B_{j,i}G_i)\|_2 \leq Cr(1+j)^c q^{j-1-i} \ell_i^{\kappa+\sigma-1/2}. \quad (34.57)$$

The half-splitting above a^2 and the bound by the initial value below a^2 are the two deterministic inequalities of Step 3 of the proof of Theorem 34.7, and this bound includes the first difference on every finer lattice.

Finally, (34.54) and (34.47) imply that, for both types of output,

$$\|\|DC_{j,i}(v_{i-1})[\eta_{i-1}]\|_{\text{out}}\|_2 \leq Cr^2(1+j)^c q^{j-1-i} \ell_i^{2\kappa-1/2}. \quad (34.58)$$

The part of degree at least three of the predictable term at level j satisfies the same bound with its own coefficient, by (34.12). There is no conditional-mean remainder in (34.55).

Step 5: summation of both components. Let $s_0 > 0$ be strictly smaller than each of

$$\alpha - \frac{1}{2}, \quad \kappa + \sigma - \frac{1}{2}, \quad \kappa + 2\alpha - \frac{3}{2}, \quad 2\kappa - \frac{1}{2}, \quad -\frac{\log q}{\log B}.$$

All these numbers are positive, and we reduce s_0 further if a fixed exponent of the product estimates requires it. For the predictable term, the initial value of its first-order variation is exactly

$$v_j(0) = DS_{j,j-1}(v_{j-1}, v_{j-1})[\eta_{j-1}] + I_{\ell_j \rightarrow a} DC_j^{\text{loc}}(v_{j-1})[\eta_{j-1}],$$

where C_j^{loc} is the part of degree at least three of U_j^{cl} at its input mesh, so that $I_{\ell_j \rightarrow a} C_j^{\text{loc}} = V_j$ in the notation of Lemma 34.2. The first term is given by the identity (34.55) for the quadratic initial form and the second by the bound on the part of degree at least three, with no conditional mean added to either. Summing both in the norm of the initial value, using (34.57), (34.58), and geometric summation, we obtain

$$\sum_{j=m+1}^K \|H_\sigma^a(v_j(0))\|_2 \leq C(1+m)^c B^{-s_0 m}. \quad (34.59)$$

Combining this with (34.50), we obtain

$$\left\| \sup_{m \leq l \leq K} X_\alpha^a(V_l) \right\|_2 + \sum_{j=m+1}^K [\|X_\alpha^a(\xi_j)\|_2 + \|X_\alpha^a(\rho_j)\|_2 + \|X_\alpha^a(v_j)\|_2] \leq C(1+m)^c B^{-s_0 m}. \quad (34.60)$$

For the ordered forms, (34.55), (34.56), and (34.58) imply the same summed tail, separately for each of the three forms. These estimates use the Minkowski inequality, not independence between an earlier field and a later coefficient.

We now treat the products, following the fourth column of Table 34.2, for the uncanceled summands of Step 2. Their values at zero fluctuation are $(0, 0, R_j, U_j)$ and their first variations are $(\xi_j, 0, \rho_j, v_j)$. Thus the product $(\tilde{s}, \tilde{s})$ and all seven products containing $\tilde{z}$ vanish at first order. The two products $(\tilde{s}, \tilde{r})$ and $(\tilde{r}, \tilde{s})$ are exactly (34.53). The product $(\tilde{r}, \tilde{r})$ and the two cross terms with $\tilde{r}$ are the derivatives of $T^{\vee\vee}$, $T^{\vee\vee} - T^{\vee\vee}$, and $T^{\vee\vee} - T^{\vee\vee}$, which have the separate tails just proved. For the seven products containing $\tilde{u}$, each nonzero derivative has one linear factor and one deterministic classical factor. For example, the derivative of $\mathcal{Q}_a(\tilde{r}, \tilde{u})$ is $\mathcal{Q}_a(\rho, U) + \mathcal{Q}_a(R, v)$, and the reversed order reverses both brackets; the derivative of $\mathcal{Q}_a(\tilde{u}, \tilde{u})$ is $\mathcal{Q}_a(v, U) + \mathcal{Q}_a(U, v)$; the derivatives for $(\tilde{s}, \tilde{u})$ and $(\tilde{u}, \tilde{s})$ pair ξ with U ; and those for $(\tilde{z}, \tilde{u})$ and $(\tilde{u}, \tilde{z})$ vanish, as already counted among the products with $\tilde{z}$. This accounts for all sixteen products.

We use the two ordered inequalities (34.6). The deterministic norms of X_j and R_j are bounded by (34.45) and (34.11), and the linear predictable terms are summed by (34.59). For the factors paired with a deterministic predictable term, the series of the zero-fluctuation maps gives the sharper deterministic sum

$$\sum_{j=m+1}^K H_\sigma^a(U_j(0)) \leq C \sum_{j>m} \frac{r^2 q^{2(j-1)}}{\ell_{j-1}} \leq C \frac{r^2 q^{2m}}{\ell_m} = Cr^2 H_0^{-2\kappa} \ell_m^{2\kappa-1}. \quad (34.61)$$

The ratio in this sum is $Bq^2 = q < 1$, and there is no term $d_m p_m$ at this base point. Multiplication by (34.60) bounds all the remaining products. It also bounds both orders of the cross terms with the predictable term,

$$\mathcal{Q}_a(V_{j-1}, U_j) + \mathcal{Q}_a(X_{j-1}, v_j) \quad \text{and} \quad \mathcal{Q}_a(U_j, V_{j-1}) + \mathcal{Q}_a(v_j, X_{j-1}).$$

The two cross terms with s are (34.52), and the two cross terms with z vanish. This covers both orders of every cross term. The constant in (34.6) is valid at every mesh $a = L/N$, including the finitely many coarse meshes at fixed L ; no condition $a \leq 1$ is needed.

Differentiating the finite telescope (34.5) and summing these bounds geometrically, we obtain, with $s < s_0$ if a polynomial factor is absorbed,

$$\|X_\alpha^a(V_K) + Y_\alpha^a(Z_K)\|_{L^2(G)} \leq C(1+m)^c B^{-sm}. \quad (34.62)$$

This proves the asserted bound on the first-order variation of the flow data, before the flow is applied.

Step 6: the derivative of the flow. We check the hypotheses of Theorem 25.13 without assuming a uniform bound on second derivatives for an unbounded Gaussian family. We fix a deterministic finite vector of inputs of the steps after m , let η_K be the solution of (34.46) with G replaced by this vector, and let $v = I_{a \rightarrow a}(\eta_K/a)$ be the corresponding direction of the initial potential. Let $V = P^a v$ and $Z = \mathcal{Q}_a(X_K, V) + \mathcal{Q}_a(V, X_K)$; these are V_K and Z_K when the deterministic vector is subsequently replaced by G . We write $\mathcal{H}_a(A_0)$ for the based cycle triple of the Wilson flow started from $\exp(aA_0)$. By the chain rule, the derivative of $\mathcal{H}_K(y, q)$ in these directions is $D\mathcal{H}_a(v_K/a)[\eta_K/a]$.

At a fixed cutoff, we consider the family of initial potentials $v_K/a + \varsigma\tau v$, where $\varsigma \in (-2, 2)$ and $\tau > 0$ is deterministic. Its flow data are

$$(X_K + \varsigma\tau V, Y_K + \varsigma\tau Z + \varsigma^2\tau^2\mathcal{Q}_a(V, V)).$$

We choose τ so that $\tau[X_\alpha^a(V) + Y_\alpha^a(Z)] \leq 1/16$, $\tau^2 Y_\alpha^a(\mathcal{Q}_a(V, V)) \leq 1/16$, and the perturbation of the initial link variables stays within one quarter of the fixed margin of the logarithmic chart. Such a $\tau > 0$ exists for each deterministic direction, and the zero direction needs no restriction. The flow data and their first two derivatives in ς are then bounded by $R_2 = R_{cl} + C$ on $U_1 = (-2, 2)$, uniformly in the mesh, and we take $U_0 = (-1, 1)$. The parameter space has dimension one, the parameter domains and chart margins are fixed, the loops and the positive observation time are the given ones, and all damping gauges start at the identity in the same frame. These are the hypotheses of Theorem 25.13. Its restart at a positive time uses the local chart and the C_a^1 bound already proved, and its fixed restarts include the conjugations at the basepoint. Thus its real differential is the differential of the based holonomy matrices of the Wilson flow used here.

To obtain a bound independent of the rescaling τ , we apply Remark 34.8 on the fixed ball of radius $R_{cl} + 1$ to the configurations of the family at ς and at 0, divide by $|\varsigma|\tau$, and let $\varsigma \rightarrow 0$. The difference of the flow data divided by $|\varsigma|\tau$ tends in both norms to (V, Z) ; the remaining term is bounded by $|\varsigma|\tau Y_\alpha^a(\mathcal{Q}_a(V, V))$ and tends to zero at this fixed cutoff. We obtain

$$\max_i |D\mathcal{H}_{a,i}(v_K/a)[\eta_K/a]| \leq C\{X_\alpha^a(V_K) + Y_\alpha^a(Z_K)\}. \quad (34.63)$$

The constant depends on R_{cl} and the fixed time and reference data, not on the rescaling, the direction, the cutoff, or m . The finitely many coarse meshes are included in both results on the flow. Differentiating the reconstruction instead of the straight line of initial potentials gives the same first derivative by the chain rule, and higher derivatives at the initial time do not enter (34.63).

Step 7: the logarithm and the conclusion. At the base point the cycle holonomies are exactly $\text{Hol}(V_y)$. The compact regular patch has the margin for the logarithm of Theorem 33.11, and the differential of Log is uniformly bounded there. In the notation

of that theorem, its differential on the right-trivialized variation $\delta H_i H_i^{-1}$ is J_i^{-1} , with $J_i = \int_0^1 e^{s\theta_i \text{ad}_u} ds$, where $u \in S^2$ is the color axis of the background. The projection $P_{\mathcal{N}_y}$ has norm at most one. Composing (34.63) with this differential and this projection, we obtain the pointwise bound

$$\left| \sum_{j=m+1}^K A_{j,K}(y) G_j \right| \leq C \{X_\alpha^a(V_K) + Y_\alpha^a(Z_K)\}.$$

By the definition of the diagonal matrices D_j in (34.46), these are exactly the blocks of the statement. Equivalently, the chain rule gives $D_{w_j}(\text{Log} \circ \mathcal{H}_K) = D_{q_j}(\text{Log} \circ \mathcal{H}_K) \text{diag}_{r,e}(g_{j,r})$, and multiplication by $\gamma^{-1/2} P_{\mathcal{N}_y}$ gives exactly the first expression for $A_{j,K}$ in the statement. Squaring, taking the Gaussian expectation, and using (34.62), we obtain the claim with $\epsilon = 2s$, after relabeling the polynomial exponent.

For comparison with the covariance notation of Section 35, independence and $\mathbb{E} G_j G_l^\top = \mathbf{1}_{j=l} I$ give exactly

$$\mathbb{E}_G \left| \sum_{j=m+1}^K A_{j,K} G_j \right|^2 = \sum_{j=m+1}^K \text{tr}(A_{j,K} A_{j,K}^\top).$$

The transpose here is the Euclidean transpose in the counting coordinates of the fluctuation variables, not an adjoint with respect to physical volumes; the physical volumes are already contained in the deterministic rows. Every bound above is uniform over the finite cover of the compact patch, the prior depths, K , m , and the coupling. This completes the proof. $\square$

34.8 Flat support of weak-coupling subsequential limits

In this section, we prove that the continuum measures concentrate on flat connections as $\gamma \rightarrow 0$. We fix the data of Definition 34.1 before varying $0 < \gamma \leq \gamma_0 \leq 1$; the hierarchy is the one fixed by Theorem 23.2. Here we choose the common input horizon $T = T_0 \geq \max\{1, L^2, \theta L^2\}$, and all flow-data norms and the preceding comparison estimates are evaluated on $[0, T_0]$. Later observation times need not lie in this interval: they are reached by the monotonicity of the energy after a short initial comparison. Let $n_* \in 4\mathbb{N}$ define the lattice refinement sequence $n_* B^K$, $K \geq 0$. We use a comparison side $n = n_* B^{k_0}$ of this sequence which satisfies the minimum-side condition, the condition on flat fields, and the conditions on the working radius above; in particular $2\pi/n < r_{\text{flat}}/4$. The sequence, the hierarchy, the side n , and all the bounds are fixed before the coupling and do not change as $\gamma \rightarrow 0$. We write $N_K = n B^K$, $a_K = L/N_K$, and $\nu_{K,\gamma} = \nu_{N_K, n, \gamma}$. The law of the coarse field $V = V^{JK}$ under $\nu_{K,\gamma}$ is the stopped measure $\hat{\nu}_{n, JK, \gamma}$, by item (9) of Theorem 14.8. Let $\lambda_\gamma = \lambda_{L, \gamma, [n_*]}$ be the continuum measure of Corollary 29.26. Constants have the dependence described in items (6) and (7) of Definition 4.14.

Theorem 34.13. Fix $L > 0$, the hierarchy, $n_* \in 4\mathbb{N}$, and a ceiling $0 < \gamma_0 \leq 1$, all before varying the coupling, and consider the lattice refinement sequence $n_* B^K$. Comparison sides $n = n_* B^{k_0}$ are required to satisfy the conditions of Definition 34.1. Let $\lambda_\gamma = \lambda_{L,\gamma,[n_*]}$ be the continuum measures of Corollary 29.26. Then the following statements hold.

1. For every $t > 0$ and $\delta > 0$,

$$\lambda_\gamma \{X : E_L(X(t)) > \delta\} \longrightarrow 0 \quad (\gamma \rightarrow 0). \quad (34.64)$$

More precisely, for every comparison side n as above, fixed before the coupling, there exist $\gamma_1 > 0$ and $C_{n,t,\delta} < \infty$ such that this probability is at most $C_{n,t,\delta}(\sqrt{\gamma} + \gamma)$ for $0 < \gamma \leq \gamma_1$.

2. For every $\eta > 0$ there exist finite $R_j \geq 0$ such that

$$\inf_{0 < \gamma \leq \gamma_0} \lambda_\gamma \{X : E_L(X(L^2 2^{-j})) \leq R_j \text{ for all } j \geq 1\} \geq 1 - \eta.$$

The set in this display is compact for $\mathcal{T}_W$ and for d_c . Every sequence $\gamma_k \rightarrow 0$ has a subsequence along which λ_{γ_k} converges, against all bounded $\mathcal{T}_W$ -continuous functions and weakly for d_c , to the same Borel probability measure. Along this subsequence the images of λ_{γ_k} in the Sobolev trajectory space of Definition 29.1 also converge against every bounded $\mathcal{T}_{1,2}$ -continuous function.

3. Every subsequential limit, in either of the topologies of item (2), is supported on the set

$$\mathcal{F}_L^{\text{stat}} = \{X : X(t) = [A] \text{ for all } t > 0, F_A = 0\}$$

of stationary flat trajectories. This set is compact, and the cycle holonomies identify it with

$$\mathcal{F}_L \simeq \text{Hom}(\mathbb{Z}^3, \text{SU}(2)) / \text{SU}(2) \simeq (\mathbb{R}/4\pi\mathbb{Z})^3 / \{\boldsymbol{\vartheta} \sim -\boldsymbol{\vartheta}\},$$

including the eight central triples. Every subsequential limit is the image of a Borel probability measure on this compact space.

The constants in item (1) have the dependence described in item (6) of Definition 4.14, with the displayed additional parameters, and the ultraviolet limit has been taken at each fixed coupling before $\gamma \rightarrow 0$. The theorem does not identify the probability measure on the moduli space: it asserts neither uniqueness of the subsequential limit nor a density on the moduli space, and it makes no assertion about unbounded moments of the energy or about ultraviolet convergence uniform down to zero coupling.

The proof combines four facts. The effective classical action of the coarse field has moments of order γ . The zero-fluctuation reconstruction from the coarse field has lattice energy exactly proportional to this action. The flow data of the reconstruction differ from those of the zero-fluctuation reconstruction by $O(\sqrt{\gamma})$ in every moment. Finally, a short-time comparison of the two Wilson flows, followed by the monotonicity of the energy, transfers this to the energy at positive times. We first prove two lemmas.

The coarse effective action and the zero-fluctuation reconstruction

Lemma 34.14. *Use the data above and the stopped densities of Section 33.2, at depths $d = kJ$. On the support of the stopped measure write $E_d(V) = E_{d,\text{sat}}(V)$, and put $g^2 = \gamma/n$ and $D_n = 6n^3 - 2$. There exists $\gamma_1 \in (0, \gamma_0]$, independent of the depth, such that for every fixed real $p > 0$ and every such depth d ,*

$$\mathbb{E}_{\widehat{\nu}_{n,d,\gamma}} E_d^p \leq C_{n,p} g^{2p}, \quad \mathbb{E}_{\widehat{\nu}_{n,d,\gamma}} E_0^p \leq C_{n,p} g^{2p} \quad (0 < \gamma \leq \gamma_1),$$

where $E_0 = S_W$ is the Wilson action of Definition 4.1 on the lattice of side n . At $d = JK$, start from the same representative $\widehat{\theta}_0(V)$ of the coarse field as the reconstruction, and set $\widehat{\theta}_j^0 = P_j \cdots P_1 \widehat{\theta}_0(V)$. Then the configuration $U_K^0(V) = \exp \widehat{\theta}_K^0$ on the finest lattice satisfies

$$E_0(U_K^0(V)) = B^{-K} E_{JK}(V), \quad E_{a_K}(U_K^0(V)) = \frac{2n}{L} E_{JK}(V), \quad (34.65)$$

where on the left E_0 is the Wilson action on the finest lattice and E_{a_K} is the lattice energy (2.2). In particular, the expected initial lattice energy of $U_K^0(V)$ under the stopped measure is at most $C_n \gamma$, and the expectation of its p -th power is at most $C_{n,p} \gamma^p$.

The constants depend in addition on p and on the fixed bounds on the coarse density, and they are independent of the coupling and of the depth. The identities (34.65) are exact for every finite K , including $K = 0$, with the prior depths $J(K - j)$ ending at depth zero.

Proof. By (33.1) and properties (P1), (P3) of Section 33.2, the stopped measure has the density

$$\widehat{Z}^{-1} \mathbf{1}_{D_r} e^{-E_d/g^2 - \overline{R}_{d,\gamma}} dH_n, \quad E_d \geq \kappa E_0 \geq 0, \quad |\overline{R}_{d,\gamma}| \leq C_n,$$

where $\kappa > 0$ is the constant in (P1); in this proof κ denotes this constant, not the contraction exponent of Definition 34.1. These bounds apply to the law of the coarse field under the conditioned measure itself, not to an interior stopped measure in its place. The lower bound of Theorem 33.10 is $\widehat{Z} \geq c_n g^{D_n}$ for $g \leq g_0(n)$. We choose $\gamma_1 \leq \min\{\gamma_0, n g_0(n)^2\}$, so that also $g \leq 1$. Since $u^p e^{-u} \leq C_p e^{-u/2}$ for $u \geq 0$, we have

$$\begin{aligned} \mathbb{E}_{\widehat{\nu}} E_d^p &\leq C_{n,p} g^{2p-D_n} \int_{D_r} e^{-E_d/(2g^2)} dH_n \\ &\leq C_{n,p} g^{2p-D_n} \int_{\mathcal{U}_n} e^{-\kappa E_0/(2g^2)} dH_n \leq C_{n,p} g^{2p}. \end{aligned}$$

In the first inequality we used the bound on $\overline{R}_{d,\gamma}$ and the lower bound on $\widehat{Z}$. In the last inequality we used the upper bound for the integral over the compact configuration space in Theorem 33.10, with the coefficient $\kappa/2$. The lower bound $E_d \geq \kappa E_0$ then implies the bound for E_0^p . In particular, the stopped measure concentrates near the set of all flat

coarse fields: outside any fixed open neighborhood of $\{E_0 = 0\}$ the continuous function E_0 has a positive minimum, and Markov's inequality applies.

We now check that the configuration in the identity is the zero-fluctuation continuation of Definition 34.1. At level j the prior depth is $d_j = J(K - j)$. By Theorems 15.13 and 15.14, P_j is the constrained minimization of the renormalization group step followed by its based gauge transformation, which depends only on the coarse field. It preserves the gauge orbit of the minimizer and the value of the effective classical action. Iterating the nested minimization identity $E_{d+1}(c) = M \min_z E_d(b(c, z))$ of Theorem 10.10 through the J averaging steps of the renormalization group step, we obtain, at every zero-fluctuation input in the reconstruction domain,

$$E_{d_j}(e^{\hat{\theta}_j^0}) = B^{-1} E_{d_j+J}(e^{\hat{\theta}_{j-1}^0}).$$

For the chosen representative of the coarse field, the initial value on the right is $E_{JK,\text{sat}}(V)$; the extension $E_{JK,\text{sat}}$ agrees with E_{JK} on the support and does not change it. All changes of representative are compact gauge transformations, which preserve this value. Iterating over $j = 1, \dots, K$ ends at $d_K = 0$ and proves the first identity. The continuation sets the fluctuation variables w to zero, not the centered variables $w - \mathbb{E}(w \mid \mathcal{F})$, and no conditional mean is substituted for the minimizer. Finally, by Lemma 29.14 we have $E_{a_K} = 2E_0/a_K$. Since $a_K = (L/n)B^{-K}$, this proves the second identity, and the moment bounds follow from the first part of the lemma and $g^2 = \gamma/n$. The identity also covers $K = 0$, without iteration. $\square$

Early-time comparison and energy monotonicity

Lemma 34.15. *Fix $L, R > 0$, $1/2 < \alpha < 2/3$, $T_0 \geq 1$, and $\tau > 0$. There exist $a_* > 0$ and $C < \infty$ with the following property. Let $a = L/N \leq a_*$, and let $U = e^{aA_0}$ and $U' = e^{aA'_0}$ be configurations on the lattice of mesh a , represented in the same initial frame, whose flow data $\mathcal{E}_a = (x_a, y_a)$ and $\mathcal{E}'_a = (x'_a, y'_a)$ as in Definition 24.3 have norms at most R on $[0, T_0]$. Put*

$$d_a = X_\alpha^a(x_a - x'_a) + Y_\alpha^a(y_a - y'_a),$$

with both full norms on this lattice and this horizon. Then, for the Wilson flow run for the lattice flow time $4t/a^2$,

$$\sup_{t \geq \tau} E_a(\Psi_t^{(N)} U) \leq E_a(U') + Cd_a, \quad \sup_{t \geq \tau} E_a(\Psi_t^{(N)} U) \leq C.$$

The second bound requires only the bound on the norm of $\mathcal{E}_a$.

The constants have the dependence described in item (7) of Definition 4.14 on the displayed parameters and on the fixed chart and stencil; they are independent of the coupling, of any probability measure, of the depth, and of the configurations.

Proof. Let $T_R \leq 1$ be the lifetime in Theorem 24.5, and choose $s = \min\{\tau, T_R\}/2 > 0$. Applied to the two flow data, that theorem yields local damped representatives A_a and A'_a with

$$\|A_a(s) - A'_a(s)\|_{C_a^1} \leq C_{R,s} d_a, \quad \|A_a(s)\|_{C_a^1} + \|A'_a(s)\|_{C_a^1} \leq K_{R,s}.$$

The second bound follows from $A = x + y + r$, the two weighted norms, and the bound on the remainder r in that theorem at the fixed positive time s . Its constants and its restriction on the mesh are independent of the coupling. The exponentials of $A_a(s)$ and $A'_a(s)$ are gauge representatives of the two Wilson flows at time s started from U and U' . By gauge invariance and the C_a^1 estimate for the lattice energy in Lemma 29.14,

$$E_a(\Psi_s^{(N)} U) \leq E_a(\Psi_s^{(N)} U') + C d_a \leq E_a(U') + C d_a,$$

where in the last inequality we used that the energy does not increase along the flow. The same monotonicity from s to every $t \geq \tau$ proves the first assertion. The uniform bound on the energy on a bounded C_a^1 ball in Lemma 29.14 gives $E_a(\Psi_s^{(N)} U) \leq C$, and monotonicity proves the second assertion. $\square$

Proof of the support theorem

Proof of Theorem 34.13. Step 1: the coarse action and the zero-fluctuation reconstruction. By Lemma 34.14, for $0 < \gamma \leq \gamma_1$ the expectation of $(2n/L)E_{JK}(V)$ under $\nu_{K,\gamma}$ is at most $C_n \gamma$, uniformly in K , and this quantity is exactly the initial lattice energy of the zero-fluctuation reconstruction $U_K^0(V)$.

Step 2: the distance between the flow data. We take $m = 0$ in Theorem 34.9 and denote by

$$D_K = X_\alpha^{a_K}(x_K - x_K^0) + Y_\alpha^{a_K}(y_K - y_K^0)$$

the distance between the flow data of the reconstruction and of the zero-fluctuation continuation from the coarse field. By that theorem, for every fixed $p \geq 1$ we have

$$\|D_K\|_{L^p(\nu_{K,\gamma})} \leq C_{n,p} \{\sqrt{\gamma} + \gamma(1 + |\log \gamma|)^d\} \leq C'_{n,p} \sqrt{\gamma},$$

uniformly in K and in the coupling; the last inequality uses the bound on $\sqrt{\gamma}(1 + |\log \gamma|)^d$ from the proof of Lemma 34.6. By Lemma 34.4, the norm of the flow data of the continuation is at most R_{cl} , deterministically for every coarse field in the reconstruction domain and independently of K and γ .

Step 3: comparison of energies. On the event $G_K = \{D_K \leq 1\}$ both flow data have norm at most $R_{cl} + 1$. The two configurations to which we apply Lemma 34.15 are $\exp \hat{\theta}_K$ and $U_K^0(V)$, in the same initial frame on the finest lattice. Indeed $I_{a_K \rightarrow a_K}$ is the identity in Definition 34.1, and both pairs consist of the linear heat evolution and the quadratic

Picard term of these fine arrays. The zero-fluctuation configuration is the one in (34.65), with the same representative $\hat{\theta}_0(V)$ of the coarse field; it is neither a flow started on the lattice of side n nor a separately chosen flat approximation. For every K large enough that $a_K \leq a_*$, a threshold independent of the coupling, Lemma 34.15 and (34.65) imply that

$$E_{a_K}(\Psi_t^{(N_K)} U) \leq \frac{2n}{L} E_{JK}(V) + C_{n,t} D_K \quad \text{on } G_K.$$

The configuration $\exp \hat{\theta}_K$ is a gauge transform of U by Theorem 15.14, and the inequality concerns the gauge-invariant lattice energy of U . By Step 1, Step 2, and Markov's inequality,

$$\nu_{K,\gamma} \{E_{a_K}(\Psi_t^{(N_K)} U) > \delta\} \leq \nu_{K,\gamma}(G_K^c) + \delta^{-1} \mathbb{E}_{\nu_{K,\gamma}} \left[\frac{2n}{L} E_{JK}(V) + C_{n,t} D_K \right] \leq C_{n,t,\delta}(\sqrt{\gamma} + \gamma).$$

Only the coarse action and the distance between flow data were integrated in this estimate; no unbounded quantity was transferred from the Wilson measure.

Step 4: transfer to the Wilson measure and to the continuum. For a Borel event A , the definition of the conditioned measure gives $\mu_{N_K,\gamma}(A) \leq \nu_{K,\gamma}(A) + \mu_{N_K,\gamma}(S_{N_K,n,\gamma}^c)$. By Corollary 23.11 with $q = 1$, at this fixed n , the last probability is at most $C_n \gamma$. We apply this to the event in Step 3. For the ultraviolet limit we use the bounded continuous function

$$f_\delta(v) = \min\{1, \max\{0, 2v/\delta - 1\}\}, \quad v \geq 0,$$

which equals one for $v \geq \delta$ and vanishes for $v \leq \delta/2$. Since $n = n_* B^a$, we have $nB^K = n_* B^{K+a}$, so the lattice energy sequence is a tail of the one in Theorem 30.4. That theorem gives, at each fixed γ , convergence in distribution of the lattice energy at time t to $E_L(X(t))$ under the same measure λ_γ identified in Corollary 29.26. Hence

$$\lambda_\gamma \{E_L(X(t)) > \delta\} \leq \mathbb{E}_{\lambda_\gamma} f_\delta(E_L(X(t))) = \lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K,\gamma}} f_\delta(E_{a_K}(\Psi_t^{(N_K)} U)) \leq C_{n,t,\delta}(\sqrt{\gamma} + \gamma),$$

where in the last inequality we used Step 3 with $\delta/2$ in place of δ and the transfer just described. The limit in the middle is taken at fixed γ , after the uniform estimates at finite K . This proves item (1).

Step 5: common compact containment. We argue directly on the whole interval of small couplings. For each fixed comparison side n , Step 2 implies that

$$\lim_{R \rightarrow \infty} \sup_{K \geq 0, 0 < \gamma \leq \gamma_0} \nu_{K,\gamma} \{\|(x_K, y_K)\|_{\text{fl}} > R\} = 0,$$

with constants that may depend on n . The second assertion of Lemma 34.15, applied on the event where this norm is at most R , bounds the lattice energy at each fixed $t > 0$ by a finite constant on all sufficiently fine lattices. The finitely many remaining lattices

have a deterministic finite maximum of the energy on their compact configuration spaces. Therefore, for each fixed n ,

$$\lim_{H \rightarrow \infty} \sup_{K \geq 0, 0 < \gamma \leq \gamma_0} \nu_{K, \gamma} \{E_{a_K}(\Psi_t^{(N_K)} U) > H\} = 0.$$

The comparison side may now be enlarged to reach a prescribed tolerance, always before the coupling is chosen. By Theorem 23.9 and Corollary 23.11,

$$\epsilon_n = \sup_{K \geq 0, 0 < \gamma \leq \gamma_0} \mu_{nB^K, \gamma}(S_{nB^K, n, \gamma}^c) \longrightarrow 0 \quad (n = n_* B^{k_0}, k_0 \rightarrow \infty).$$

Indeed, the bound of Theorem 23.9 is the sum of $C'n^3 e^{-\bar{c}n/\gamma}$ and $C_w n^3 e^{-a_w \log^2 \sqrt{Mn/\gamma}}$; both terms are largest at $\gamma = \gamma_0$ and tend to zero as $n \rightarrow \infty$. We choose n so that ϵ_n is small, and then the radius and the energy threshold for the conditioned measure. The transfer of bounded events gives uniform tails of the lattice energy under the Wilson measure along the sequence nB^K , and the finitely many initial terms of the sequence $n_* B^K$ below n have deterministic finite energy bounds as well. Thus

$$\lim_{H \rightarrow \infty} \sup_{K \geq 0, 0 < \gamma \leq \gamma_0} \mu_{n_* B^K, \gamma} \{E_{L/(n_* B^K)}(\Psi_t^{(n_* B^K)} U) > H\} = 0.$$

For each γ separately, the convergence of the energy at fixed coupling used in Step 4 and the portmanteau inequality for the open set $\{v > H\}$ give the same uniform tails for $\lambda_\gamma \{E_L(X(t)) > H\}$. No ultraviolet limit uniform in the coupling was used.

We now apply Theorem 29.6. We choose R_j such that the energy tail at time $s_j = L^2 2^{-j}$ is smaller than $\eta 2^{-j-1}$, uniformly in the coupling, and use the union bound. The compact set $\mathcal{K}(R)$ of that theorem, with $R = (R_j)_j$, gives the asserted common containment and the extraction of subsequences for $\mathcal{T}_W$ and for d_c . This proves item (2), except for the Sobolev images, which we treat in the next step.

Step 6: identification of the support. Let λ_* be a subsequential limit obtained from this extraction. All finite joint laws of regularized Wilson loops converge along the subsequence, since these coordinates are bounded and $\mathcal{T}_W$ -continuous. The general sequence clause of Theorem 30.5 applies: the energies at every positive time are tight, as just proved, and the joint limit of the Wilson loops is given by the Borel probability measure λ_* . That clause gives convergence of the Sobolev images to $(i^{1,2})_\# \lambda_*$. By Lemma 29.7, for any $s, \delta > 0$ the function

$$Z \longmapsto \min\{1, E_L(Z(s))/\delta\}$$

is bounded and continuous in the Sobolev trajectory topology. Its expectation under λ_γ tends to zero by item (1): for every $\varepsilon > 0$, it is at most $\varepsilon/\delta + \lambda_\gamma \{E_L(X(s)) > \varepsilon\}$. Its expectation under the limit is therefore zero. At each time s_j we obtain $E_L(X(s_j)) = 0$,

λ_* -almost surely. We take the countable intersection of these events of full measure. Since the energy does not increase along the flow (Theorem 29.6) and $s_j \downarrow 0$, we have $E_L(X(t)) = 0$ for every $t > 0$ on this event.

Since the energy is the sum of the squared norms of the curvature components, every smooth orbit at these times is flat. For a flat smooth connection the right side of the Yang–Mills heat equation vanishes, and the uniqueness of smooth solutions in Theorem 24.10 makes its future orbit constant. The trajectory relation then makes X a single stationary flat trajectory on $(0, \infty)$.

This proves the support property for every subsequential limit for $\mathcal{T}_W$. If a subsequence converges only weakly for d_c , we apply the common compact containment again to extract a further subsequence that converges for $\mathcal{T}_W$. Since the identity map from $\mathcal{T}_W$ to d_c is continuous by Theorem 29.3, the limit for d_c coincides with the limit for $\mathcal{T}_W$; equivalently, their finite-dimensional laws of the determining coordinates agree, and the same theorem identifies the Borel probability measures. The limit for d_c is therefore supported on stationary flat trajectories as well.

Step 7: the moduli space. We include the classification of flat connections on the torus; it is standard, and we check its normalization. A smooth flat connection lifted to the universal cover $\mathbb{R}^3$ has a path-independent transport $P_A(x)$ from a fixed basepoint, with $dP_A = P_A A$ and $P_A(x + Le_i) = h_i P_A(x)$. Flatness makes the three matrices h_i commute. Conversely, a commuting triple in $SU(2)$ is simultaneously diagonalizable, and it is realized by the constant flat connection $A_i = \vartheta_i \tau_3 / L$ with $h_i = e^{\vartheta_i \tau_3}$. If two flat connections A and B have the same holonomies at the basepoint, then $k(x) = P_B(x)^{-1} P_A(x)$ is smooth and periodic and gives $A^k = B$, by the conventions for gauge transformations and transport in Definition 4.10. Conjugate holonomies are handled by a constant gauge transformation at the basepoint. Two simultaneously diagonal triples are conjugate exactly when they agree or are related by the common Weyl inversion $\vartheta \mapsto -\vartheta$, and each angle has period 4π . This proves the two displayed identifications, including the central points.

The map from the compact quotient of angles to stationary trajectories is continuous for $\mathcal{T}_W$, since every holonomy of a constant commuting connection around a loop depends continuously on the three cycle holonomies, through the integer winding vector of the loop. It is a bijection onto the stationary flat trajectories. The target is Hausdorff, since d_c separates points by Theorem 29.3. A continuous bijection from a compact space onto a Hausdorff space is a homeomorphism, so the identification holds for $\mathcal{T}_W$ and also for d_c . Alternatively, the set of stationary flat trajectories is exactly $\mathcal{K}(0)$ in Theorem 29.6. Pulling back a probability measure supported on this set gives the asserted probability measure on the moduli space, which proves item (3). $\square$

Proof of Theorem 2.9. Fix $L > 0$, an odd integer $M \geq 3$, and n divisible by four, as in Theorem 2.9, and set $n_* = n$. Let the hierarchy, in particular the integer J , and a

ceiling $\gamma_0 \in (0, 1]$ be as in Theorem 23.2. We choose an integer $k_0 \geq 0$ such that the comparison side $\bar{n} = n_* B^{k_0}$ satisfies the minimum-side condition and the condition on flat fields of Definition 34.1, and we apply Theorem 34.13 with the original base n_* and this comparison side. All these choices precede the coupling. By item (4) of Theorem 29.28, for every $0 < \gamma \leq \gamma_0$ the measure $\lambda_{L, \gamma, [n]_M}$ of Theorem 2.3 coincides with the measure $\lambda_{L, \gamma, [n_*]}$ of Corollary 29.26 built from this hierarchy. Parts (1), (2), and (3) of Theorem 2.9 are therefore items (1), (2), and (3) of Theorem 34.13. $\square$

Terms	The terms themselves (Theorem 34.7)	Difference from the zero- fluctuation counterpart (Theorems 34.9 and 34.11)	First derivative at a flat base point, for the uncentered summands (Lemma 34.12)
(s, s)	Quadratic expression in s_j : centered part and conditional mean	The same; the counterpart is zero	Zero (two fluctuation factors)
$(s, z), (z, s),$ (z, z)	(34.6) with z_j in H_σ^a , and Hölder's inequality	The same; the counterpart is zero	Zero
$(s, r), (r, s)$	Mixed linear rows with background r_j , conditionally on $\mathcal{F}_{j-1}$	The same; the counterpart is zero	Mixed linear rows with the deterministic background
$(z, r), (r, z)$	(34.6) with z_j in H_σ^a	The same; the counterpart is zero	Zero
(r, r) The seven pairs contain- ing u	The form $T^{\vee\vee}$ (34.6) with u_j in H_σ^a	The difference of $T^{\vee\vee}$ (34.7) and (34.6)	The derivative of $T^{\vee\vee}$ Zero for $(z, u), (u, z)$; otherwise one linear and one determinis- tic classical factor, and (34.6)
x_{j-1} with s	Mixed linear rows with background x_{j-1}	The same; the counterpart is zero	Mixed linear rows
x_{j-1} with z	(34.6)	The same; the counterpart is zero	Zero
x_{j-1} with r	$T^{\vee\vee} - T^{\vee\vee}$ and $T^{\vee\vee} -$ $T^{\vee\vee}$	Differences of these forms	Derivatives of these forms
x_{j-1} with u	(34.6)	(34.7) with $x_{j-1} - x_{j-1}^0$, and (34.6)	One linear and one deterministic classical factor, and (34.6)

Table 34.2. The tools used for the sixteen ordered products and the eight cross terms in (34.5). Both orders of every pair are estimated separately. In the fourth column, s , z , r , and u stand for the uncentered summands $\tilde{s}_j$, $\tilde{z}_j$, $\tilde{r}_j$, and $\tilde{u}_j$ of the deterministic reconstruction in the proof of Lemma 34.12, whose values at zero fluctuation are $(0, 0, R_j, U_j)$ and whose first variations are $(\tilde{\zeta}_j, 0, \rho_j, v_j)$; they are not the centered summands of (34.1).

Normal fluctuations near regular commuting holonomies

This section studies the continuum measure λ_γ of Theorem 2.3 at weak coupling near a regular point of the space of flat connections, and proves Theorem 2.10. We observe the holonomies $H = (H_1, H_2, H_3)$ of a trajectory at the fixed time $t = \theta L^2$ around the three coordinate cycles of $\mathbb{T}_L^3$, and their logarithms $X = \text{Log } H \in \mathfrak{su}(2)^3 \simeq \mathbb{R}^9$ in a fixed identity chart. Commuting triples correspond to the cone $\mathcal{M}$. Away from the origin it is a five-dimensional manifold whose normal space $\mathcal{N}_y$ at y has dimension four, and near a regular point we write $X = y + b$ with $y \in \mathcal{M}$ and $b \in \mathcal{N}_y$. We prove three results. After localization to a fixed regular patch, the normalized normal component $\gamma^{-1/2} \Sigma_\theta(y)^{-1/2} b$ is compared, in the weighted normal-bundle sense, with a standard Gaussian vector in $\mathcal{N}_y$ and the same γ -dependent tangential marginal, where Σ_θ is the explicit covariance of (4.12) (Theorem 35.4). The localized quadratic form $q_*/\gamma = b^\top \Sigma_\theta(y)^{-1} b / (2\gamma)$ has the limit law with density se^{-s} on $[0, \infty)$ (Theorem 35.3). The normalized regression residual variance $\mathcal{C}_\gamma$ of Theorem 2.10 equals $2\gamma^2 + o(\gamma^2)$ (Theorem 35.2).

The following heuristic explains the coefficient. If $z = \gamma^{-1/2} \Sigma_\theta(y)^{-1/2} b$ were exactly a standard Gaussian vector in four dimensions, independent of y , then $q_*/\gamma = |z|^2/2$ would have mean 2, second moment 6, and variance 2. The observables of Theorem 2.10 are $B_{\text{obs}} = \psi$ and $A_{\text{obs}} = \psi q_*$, where $\psi = \psi_0(y) \chi(b)$ is a fixed tangential cutoff times a normal cutoff equal to one near $b = 0$. Omitting the normal cutoff, we set $B = \psi_0(y)$ and $A = \gamma B |z|^2/2$. If z is standard Gaussian and independent of y , the best affine predictor of A from B is $2\gamma B$, with mean square error $2\gamma^2 \mathbb{E} B^2$, and dividing by $\mathbb{E} B^2$ gives $2\gamma^2$. The proofs control the effect of the normal cutoff. The coefficient 2 is a property of four-dimensional Gaussian vectors, and a conditional normal Gaussian measure with the same covariance reproduces it (Lemma 35.17). The theorems show that the continuum measure has this Gaussian behavior in the normal directions, with the precise covariance $\gamma \Sigma_\theta(y)$, while its tangential marginal is not identified and may depend on γ . For comparison, the same statistic vanishes for the Abelian Maxwell measure with Haar cycle holonomies, and it is a positive constant independent of γ for Gaussian holonomies with independent Haar factors (Lemma 35.17).

Near a regular point the flat connections form a nondegenerate critical manifold of the action in the sense of Bott [31], and the $\sqrt{\gamma}$ scaling in the normal directions is

the usual Laplace asymptotics transverse to such a manifold (see [155]). The uniform versions on the coarse lattice that we use are proved in Section 33. The treatment of zero-momentum contributions to Wilson loops in finite periodic volume by Coste, González-Arroyo, Jurkiewicz, and Korthals Altes [62] is a related antecedent; the observables, normalization, and regression functional used here are defined below.

The proof works at a finite lattice cutoff, under the conditioned Wilson measure $\nu = \nu_{N,n,\gamma}$ with $N = nB^K$ and the coarse side n fixed before γ varies. It rests on two ideas. First, we compare with a Gaussian only finitely many scale variables: the coarse field and the normalized fluctuation variables of the first m renormalization group steps, counted from the coarse side. Their joint law is close to a Gaussian law in a space of fixed dimension, with error of order $\sqrt{\gamma}|\log \gamma|$ (Lemma 35.8). Second, the remaining steps are controlled directly at the scale $\sqrt{\gamma}$ by Section 34: Theorem 34.10 compares the flowed holonomies of the fine configuration with those of the finite-scale reconstruction, and Lemma 34.12 bounds the part of the Gaussian covariance coming from the omitted steps. The Gaussian covariance of all K steps is exactly the normal covariance of the finest lattice (Proposition 33.12), which converges to Σ_θ as $N \rightarrow \infty$ (Theorem 33.11). The resulting normalized moment errors have the form

$$C(\delta_m^{1/2} + \delta_m + N^{-2}) + e_m(\gamma), \quad e_m(\gamma) \rightarrow 0 \text{ as } \gamma \downarrow 0 \text{ for each fixed } m,$$

where $\delta_m \rightarrow 0$ is a fixed geometric envelope and C does not depend on m . We then let $K \rightarrow \infty$ at fixed γ and m , after removing the conditioning with Corollary 23.11 and passing to the continuum with Theorem 29.25; then $\gamma \downarrow 0$ at fixed m ; and finally $m \rightarrow \infty$. The constant C_m in the finite-dimensional comparison may be very large; this is harmless because m is fixed before $\gamma \rightarrow 0$. The other inputs are the conditional fluctuation kernels and their means (Theorems 13.8 and 13.10), the reconstruction maps (Theorem 15.13), the flat refinements (Lemma 14.9), the parameter derivatives of the Wilson flow (Theorem 25.13), the Laplace estimates and the normalization on the coarse lattice (Proposition 33.12, Corollary 33.4, and Theorem 33.10), and the trace coordinates and conditioning identities of Lemmas 5.5 and 5.6.

Besides Theorem 2.10, two results of this section are used in Section 36: the uniform regularity of the finite-scale flowed holonomy map near every flat configuration, including the central triples (Theorem 35.9), and the comparison measures (Lemmas 35.16 and 35.17).

The three main theorems are stated in Section 35.1 and proved in Sections 35.6 to 35.8, after the auxiliary results of Sections 35.2 to 35.5; the section ends with the comparison measures (Section 35.9) and the proof of Theorem 2.10.

35.1 Setting and main results

Fixed data. Throughout this section we fix $L > 0$, $\theta > 0$, a horizon $T \geq \theta L^2$, an odd integer $M \geq 3$, an integer $J \geq 1$, $B = M^J$, a coupling ceiling $0 < \gamma_0 \leq 1$, and the construction parameters of Definition 34.1: the cutoff amplitude A_w , the radii $r < \rho$, the reconstruction maps of Theorem 15.13, the flow exponents $1/2 < \alpha_{\text{fl}} < 2/3$ and σ_{sm} , and the contraction constant a_{map} with $a_{\text{map}}\sqrt{B} < 1$ and $\mathfrak{q} = Ba_{\text{map}}^2 < 1$. All of these are chosen before γ varies. We fix $n \in 4\mathbb{N}$ satisfying the minimum-side conditions of Definition 34.1; in particular

$$2\pi/n < r_{\text{flat}}/4,$$

where $r_{\text{flat}} < r$ is the flat margin of Lemma 14.9. For $K \geq 0$ we set $N = nB^K$, let $\nu = \nu_{N,n,\gamma}$ be the conditioned Wilson measure of Definition 4.4, and let $\mu = \mu_{N,\gamma}$ be the Wilson measure. The renormalization group steps of the hierarchy are indexed by $j = 1, \dots, K$, counted from the coarse side: the j -th step relates the lattices of sides nB^{j-1} and nB^j . We write $w_j = (w_{j,r,e})_{r,e}$ for the normalized fluctuation variables of the j -th step, indexed by its J intermediate scales r and the links e , and $g_{j,r}$ for the coupling at the intermediate scale r , so that $g_{j,r}^2/\gamma$ is the reciprocal of a fixed lattice side.

Let $\mathcal{H}^K = (\mathcal{H}_i^K)_{i=1}^3$ be the triple of coordinate-cycle holonomies, based at the origin, of the Wilson-flowed fine configuration at time θL^2 (lattice time $4\theta L^2/a_N^2 = 4\theta N^2$), computed for the reconstructed configuration $\exp \hat{\theta}_K$ of Definition 34.1 in the gauge frame at the basepoint fixed by the reconstruction maps. We call $\mathcal{H}^K$ the flowed holonomy triple. For $0 \leq m \leq K$ let Ξ_m^K be the same triple for the finite-scale reconstruction $\exp \hat{\theta}_K^0$, in which the coarse field and the normalized fluctuation variables of the first m steps are those of the same sample and all normalized fluctuation variables of the steps after the m -th are set to zero. Both triples are random variables under ν , defined in the same gauge frame at the basepoint, and both are flowed on the finest lattice.

Finally, $\lambda_\gamma = \lambda_{L,\gamma,[n_*]}$ denotes the continuum measure on the trajectory space associated with the sequence nB^K (Corollary 29.26); by Theorem 29.28 it is the measure $\lambda_{L,\gamma,[n]_M}$ of Theorem 2.3. For a trajectory $X \in \mathcal{X}_L$, the coordinate-cycle holonomies of $X(\theta L^2)$ are defined up to simultaneous conjugation, and so are the coordinates introduced below; all observables we evaluate are invariant under simultaneous conjugation.

Constants. Unless stated otherwise, constants c, C and thresholds in this section have the dependence described in item (6) of Definition 4.14. They may depend on n, L, θ, M, J , the construction parameters, the horizon, the flow exponents, the loop lengths, and the patches, tubes, test functions, and extensions fixed below, but not on K, N , the preceding classical depths, the index m , or γ below the relevant threshold. A subscript m , as in C_m or γ_m , allows additional dependence on m (through the first m steps and their finite dimensions); constants without this subscript do not depend on m . Further subscripts indicate dependence on moment orders, powers of γ , or test functions.

Regular coordinates. In the fixed identity chart we write $X = (x_1, x_2, x_3) = \text{Log } H \in \mathbb{R}^9$, identifying $\mathfrak{su}(2)$ with $\mathbb{R}^3$ so that the bracket becomes the cross product. The cone of commuting triples is

$$\mathcal{M} = \{y = (\vartheta_1 u, \vartheta_2 u, \vartheta_3 u) : \vartheta \in \mathbb{R}^3, u \in S^2\}, \quad (\vartheta, u) \sim (-\vartheta, -u).$$

On the regular region $0 < r_0 \leq |\vartheta| \leq r_1 < \pi$ it is a smooth five-dimensional manifold, parametrized by the three angles ϑ and the direction u . Its normal space at $y = (\vartheta_i u)_i$ is

$$\mathcal{N}_y = \left\{ b = (b_1, b_2, b_3) : b_i \perp u, \sum_i \vartheta_i b_i = 0 \right\}.$$

The conditions $b_i \perp u$ leave six dimensions, and the relation $\sum_i \vartheta_i b_i = 0$ removes two of them, which gives $\dim \mathcal{N}_y = 4$. Simultaneous conjugation $H_i \mapsto g H_i g^{-1}$ acts on $\mathbb{R}^9$ by the same rotation $\text{Ad}(g)$ in each component; it preserves $\mathcal{M}$ and maps $\mathcal{N}_y$ isometrically onto $\mathcal{N}_{\text{Ad}(g)y}$. Two of the five tangential directions of $\mathcal{M}$ are orbits of this action, which is not removed by fixing a gauge at the basepoint.

We write $\Pi_\vartheta = I - \vartheta \vartheta^\top / |\vartheta|^2$ and let $\Sigma_\vartheta(y)$ be the positive definite matrix on $\mathcal{N}_y$ defined in (4.12). By Theorem 33.11, $\gamma \Sigma_\vartheta(y)$ is the covariance of the linearized flowed cycle logarithms, projected to $\mathcal{N}_y$, under the Gaussian measure of the Maxwell quadratic form around the flat connection with holonomy logarithms y . It does not depend on γ , depends smoothly on y , is equivariant under simultaneous conjugation, and satisfies $cI \leq \Sigma_\vartheta \leq CI$ on the regular region. For comparisons of vectors in $\mathbb{R}^9$ we extend the inverse covariance to the ambient matrix $P_{\mathcal{N}_y} \Sigma_\vartheta(y)^{-1} P_{\mathcal{N}_y}$, which is smooth on the regular region and has the same quadratic form on $\mathcal{N}_y$. Normal matrices are expressed in smooth orthonormal frames of a finite bundle atlas.

For a positive semidefinite matrix Σ we write G_Σ for a centered Gaussian vector with covariance Σ , realized as a linear image of a standard Gaussian vector, so that Σ may be singular. We write G_d for a standard Gaussian vector in a d -dimensional Euclidean space and Γ_y for the standard Gaussian measure on $\mathcal{N}_y$. For a bounded Lipschitz function f we write $\|f\|_{\text{BL}} = \|f\|_\infty + \text{Lip } f$. Its dual distance is not the bounded-Lipschitz distance of Section 4.10, which uses the maximum of the two terms, but differs from it by at most a factor two.

Definition 35.1. Fix compact regular patches $P \subset P^+ \subset P^{++}$ in $\mathcal{M} \cap \{r_0 \leq |\vartheta| \leq r_1\}$, invariant under simultaneous conjugation, each with nonempty relative interior and contained in the relative interior of the next, and radii $0 < b_0 < b^+ < b^{++}$. Put

$$\mathcal{U} = \{y + b : y \in P, b \in \mathcal{N}_y, |b| < b_0\},$$

and define $\mathcal{U}^+$ and $\mathcal{U}^{++}$ in the same way with (P^+, b^+) and (P^{++}, b^{++}) . The radius b^{++} is so small that $X = y + b$ are unique coordinates on $\mathcal{U}^{++}$ and that the closure of $\mathcal{U}^{++}$ lies in the identity chart at positive distance from its boundary. The following objects are fixed once, before γ varies.

1. On the whole tube $\mathcal{U}^{++}$ put

$$q_*(y+b) = \frac{1}{2} b^\top \Sigma_\theta(y)^{-1} b.$$

Outside $\mathcal{U}^{++}$, q_* is a fixed bounded continuous nonnegative function, invariant under simultaneous conjugation; on $\mathrm{SU}(2)^3$ it is evaluated through $X = \mathrm{Log} H$ and extended outside the identity chart by the same bounded function.

2. Let $\psi_0 : P \rightarrow [0, 1]$ be smooth and invariant, with compact support in the relative interior of P , and suppose that there exist nonempty open subsets O_0, O_+ of P , invariant under simultaneous conjugation, and $b_* > 0$ with $\psi_0 = 0$ on O_0 and $\psi_0 \geq 2b_*$ on O_+ . Let χ be a fixed smooth invariant normal cutoff with $0 \leq \chi \leq 1$, equal to one near the zero section and supported in $\{|b| < b_o\}$; it is a smooth function on the normal bundle. Put $\psi(y+b) = \psi_0(y)\chi(b)$ on $\mathcal{U}$ and $\psi = 0$ elsewhere.
3. Put $B_{\mathrm{obs}} = \psi$ and $A_{\mathrm{obs}} = \psi q_*$, regarded as functions on $\mathrm{SU}(2)^3$ through $X = \mathrm{Log} H$ and extended by zero outside the identity chart.
4. A localizing weight is a function $\tilde{w}(y+b) = w(y)\chi_w(b)$, extended by zero outside $\mathcal{U}$, where $w \geq 0$ is smooth and invariant with compact support in the relative interior of P , and $0 \leq \chi_w \leq 1$ is smooth, invariant, equal to one near $b = 0$, and supported in $\{|b| < b_o\}$.

For a probability measure P and bounded real random variables A, B with $\mathbb{E}_P B^2 > 0$ and $\mathrm{Var}_P B > 0$, define

$$\mathcal{C}(P; A, B) = \frac{\mathrm{Var}_P A - \mathrm{Cov}_P(A, B)^2 / \mathrm{Var}_P B}{\mathbb{E}_P B^2}, \quad (35.1)$$

and set $\mathcal{C}_\gamma = \mathcal{C}(\lambda_\gamma; A_{\mathrm{obs}}, B_{\mathrm{obs}})$, with $A_{\mathrm{obs}}, B_{\mathrm{obs}}$ evaluated at the coordinate-cycle holonomies of $X(\theta L^2)$.

Every localizing weight, ψ , and ψ^2 are supported in $\mathcal{U} \subset \mathcal{U}^{++}$, so the products $\tilde{w}q_*^l$, ψq_*^l , and $\psi^2 q_*^l$ involve only the exact quadratic form $b^\top \Sigma_\theta(y)^{-1} b / 2$, which is nonnegative. The extension of q_* outside $\mathcal{U}^{++}$ plays no role in any of the results. All further patches and tubes used in the proofs (the coarse tubes, the output tubes, and the auxiliary tubes) are chosen before γ varies, nested and with positive margins, between $\mathcal{U}$ and $\mathcal{U}^{++}$ (respectively over patches contained in P^{++}). By Lemma 35.12 below, $\mathcal{C}(P; A, B)$ is the minimal mean square error of an affine predictor of A from B , divided by $\mathbb{E}_P B^2$.

The envelope δ_m . We fix $c, \epsilon > 0$ such that

$$\delta_m = (1+m)^c B^{-\epsilon m}, \quad m \geq 0,$$

dominates the envelopes in Theorem 34.10 and Lemma 34.12, which have the same form. Then $\sup_m \delta_m < \infty$ and $\delta_m \rightarrow 0$ as $m \rightarrow \infty$. This choice does not change the dependence of any constant on m .

Main results.

Theorem 35.2. *Evaluate $A_{\text{obs}} = \psi q_*$ and $B_{\text{obs}} = \psi$ at the coordinate-cycle holonomies of $X(\theta L^2)$. These are bounded continuous functions of the seven normalized traces of Lemma 5.6, and they do not depend on γ . There exists $c > 0$ such that, for all sufficiently small $\gamma > 0$,*

$$\mathbb{E}_{\lambda_\gamma} B_{\text{obs}}^2 \geq c, \quad \text{Var}_{\lambda_\gamma} B_{\text{obs}} \geq c.$$

For $f = \psi$ and $f = \psi^2$,

$$\mathbb{E}_{\lambda_\gamma} [f q_*] = 2\gamma \mathbb{E}_{\lambda_\gamma} f + o(\gamma), \quad \mathbb{E}_{\lambda_\gamma} [f q_*^2] = 6\gamma^2 \mathbb{E}_{\lambda_\gamma} f + o(\gamma^2),$$

and the normalized regression residual variance satisfies

$$\mathcal{C}_\gamma = 2\gamma^2 + o(\gamma^2) \quad (\gamma \downarrow 0).$$

Further, there exists C such that for every fixed $m \geq 0$,

$$\limsup_{\gamma \downarrow 0} \left| \frac{\mathcal{C}_\gamma}{\gamma^2} - 2 \right| \leq C(\delta_m^{1/2} + \delta_m),$$

and the right side tends to zero as $m \rightarrow \infty$.

Theorem 35.3. *Let $\tilde{w} = w\chi_w$ be a localizing weight with w not identically zero, evaluated at the coordinate-cycle holonomies of $X(\theta L^2)$. There exist $c_w, \gamma_w > 0$ such that $D_\gamma = \mathbb{E}_{\lambda_\gamma} \tilde{w} \geq c_w$ for $0 < \gamma \leq \gamma_w$. For such γ define the probability measure*

$$\pi_\gamma(A) = \frac{\mathbb{E}_{\lambda_\gamma} [\tilde{w} \mathbf{1}_{\{q_*/\gamma \in A\}}]}{\mathbb{E}_{\lambda_\gamma} \tilde{w}}, \quad A \subset [0, \infty) \text{ Borel}.$$

As $\gamma \downarrow 0$, π_γ converges weakly to the law of $|G_4|^2/2$, which has density se^{-s} on $[0, \infty)$. Equivalently, for every bounded continuous $f : [0, \infty) \rightarrow \mathbb{R}$,

$$\frac{\mathbb{E}_{\lambda_\gamma} [\tilde{w} f(q_*/\gamma)]}{\mathbb{E}_{\lambda_\gamma} \tilde{w}} \longrightarrow \mathbb{E} f(|G_4|^2/2) = \int_0^\infty f(s) se^{-s} ds.$$

Further, there exist C_w , independent of m , and for each fixed $m \geq 0$ a function $e_{m,w}$ with $e_{m,w}(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$, such that for every bounded Lipschitz f and all sufficiently small γ , depending on m ,

$$\left| \int f d\pi_\gamma - \mathbb{E} f(|G_4|^2/2) \right| \leq \|f\|_{\text{BL}} [C_w(\delta_m^{1/2} + \delta_m) + e_{m,w}(\gamma)].$$

Since $q_* \geq 0$ on the support of $\tilde{w}$, the measure π_γ is a probability measure on $[0, \infty)$.

Theorem 35.4. Let $\tilde{w}$, D_γ , c_w , and γ_w be as in Theorem 35.3. On $\{\tilde{w} \neq 0\}$ write the logarithm of the coordinate-cycle holonomy triple of $X(\theta L^2)$ as $y_\gamma + b_\gamma$ and set

$$z_\gamma = \gamma^{-1/2} \Sigma_\theta(y_\gamma)^{-1/2} b_\gamma \in \mathcal{N}_{y_\gamma},$$

where $\Sigma_\theta(y)^{-1/2}$ is the positive inverse square root on $\mathcal{N}_y$. These coordinates are defined up to simultaneous conjugation, and any representative may be used in the invariant expressions below. Let $P' \subset P^{++}$ be a compact invariant regular patch containing the support of w in its relative interior. For a bounded continuous function F on the normal bundle $\{(y, z) : y \in P', z \in \mathcal{N}_y\} \subset \mathbb{R}^9 \times \mathbb{R}^9$, invariant under $(y, z) \mapsto (\text{Ad}(g)y, \text{Ad}(g)z)$ for $g \in \text{SU}(2)$, put

$$\bar{F}(y) = \int_{\mathcal{N}_y} F(y, z) \Gamma_y(dz).$$

Products with $\tilde{w}$ are defined to be zero where $\tilde{w} = 0$. Then

$$\frac{\mathbb{E}_{\lambda_\gamma}[\tilde{w} F(y_\gamma, z_\gamma)] - \mathbb{E}_{\lambda_\gamma}[\tilde{w} \bar{F}(y_\gamma)]}{D_\gamma} \longrightarrow 0 \quad (\gamma \downarrow 0). \quad (35.2)$$

Let $P'_{\text{orb}} = P' / \text{Ad}(\text{SU}(2))$, with its quotient Borel σ -algebra, and write $[y]$ for the class of y . The tangential marginal of the localized measure is the probability measure

$$\rho_\gamma(A) = D_\gamma^{-1} \mathbb{E}_{\lambda_\gamma}[\tilde{w} \mathbf{1}_{\{[y_\gamma] \in A\}}], \quad A \subset P'_{\text{orb}} \text{ Borel},$$

the function $\bar{F}(y)$ depends only on $[y]$, and

$$D_\gamma^{-1} \mathbb{E}_{\lambda_\gamma}[\tilde{w} \bar{F}(y_\gamma)] = \int_{P'_{\text{orb}}} \bar{F}(y) \rho_\gamma(d[y]).$$

If F is bounded Lipschitz for the Euclidean distance of $\mathbb{R}^9 \times \mathbb{R}^9$, then the absolute value of the left side of (35.2) is at most

$$\|F\|_{\text{BL}} [C_w(\delta_m^{1/2} + \delta_m) + e_{m,w}(\gamma)] \quad (35.3)$$

for every sufficiently large fixed m and all sufficiently small γ , depending on m , where C_w does not depend on m or F , and $e_{m,w}(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$ for each fixed m .

Remark 35.5. In the three theorems the continuum limit is taken first, at fixed γ . The proofs take the limits in the order $K \rightarrow \infty$ at fixed γ and m , then $\gamma \downarrow 0$ at fixed m , then $m \rightarrow \infty$. Only bounded observables, not divided by powers of γ , are compared between ν and μ , and no rate of convergence uniform in γ is used or asserted. The tangential marginal ρ_γ is not identified: Theorem 35.4 states that the normal fluctuations are Gaussian over this marginal, not that the pair (y_γ, z_γ) converges to a fixed limit law. Finally, Theorem 35.2 identifies no corrections to the coefficient 2, which is the coefficient of a four-dimensional Gaussian vector and is reproduced by the conditional normal Gaussian measure of Lemma 35.17(4).

Remark 35.6. Theorem 35.4 contains more information than the law of $|z_\gamma|^2/2 = q_*/\gamma$. For example,

$$F(y, z) = \frac{|z_1|^2 - |z_2|^2}{1 + |z|^2}$$

is bounded, Lipschitz, and invariant, and it requires no choice of a color frame. At $y = (0, 0, ru)$ with $r_0 < r < r_1$, its values at $(z_1, z_2, z_3) = (sv, 0, 0)$ and at $(0, sv, 0)$, with $v \perp u$, are opposite, although the two points have the same radius. In general, writing $R = |G_4|$ and $\Pi = \Pi_\theta$,

$$\bar{F}(y) = \frac{\Pi_{11} - \Pi_{22}}{2} \mathbb{E} \frac{R^2}{1 + R^2}.$$

Indeed, $\mathcal{N}_y = \theta^\perp \otimes u^\perp$, so the quadratic form $b \mapsto |b_i|^2$ has trace $2\Pi_{ii}$ on the four-dimensional space $\mathcal{N}_y$, and the conditional expectation of $|z_i|^2$ given $|z| = R$ under Γ_y is $R^2\Pi_{ii}/2$. In the theorem, both $\bar{F}$ and the normalization of b_γ are evaluated at y_γ , and the normalization is by D_γ .

35.2 The coarse marginal and the finite-dimensional Gaussian comparison

Definition 35.7. Let $\hat{\nu} = (T_M^{JK})_\# \nu$ be the law of the coarse field $V = T_M^{JK}U$ under ν . It has the density

$$d\hat{\nu} = \hat{Z}^{-1} \mathbf{1}_{D_r} e^{-nE_{JK,\text{sat}}/\gamma - \bar{R}_{K,\gamma}} dH_n, \quad \bar{R}_{K,\gamma} = R_{K,\gamma} - R_{K,\gamma}(I),$$

where $E_{JK,\text{sat}}$ is the gauge-invariant extension of the effective classical action E_{JK} from B_r to D_r of Theorem 11.6 (that is, $E_{JK,\text{sat}}(U) = E_{JK}(U^k)$ whenever $U^k \in B_r$), $R_{K,\gamma}$ is the effective interaction after K renormalization group steps, of classical depth JK , $\bar{R}_{K,\gamma}$ is the centered effective interaction, and $\hat{Z}$ is the normalizing constant of this density; it is the normalizing constant of Theorem 33.10.

Near the flat configurations of $\mathcal{U}_n$ whose holonomy logarithms lie in P^{++} we use the finite normal atlas $V = V(y, v)$ of Proposition 33.12. Here $y \in \mathcal{M}$ is the logarithm of the holonomy triple of the flat configuration $V(y, 0)$, and v is a normal variable of dimension

$$d_{\text{reg}} = 6n^3 - 2,$$

whereas $\dim \mathcal{N}_y = 4$. A coarse regular tube is a set $\{V(y, v) : y \in P_c, |v| \leq \varepsilon_c\}$, where $P_c \subset P^{++}$ is a compact invariant regular patch and $\varepsilon_c > 0$ is fixed. Put

$$H_K(y) = D_v^2(nE_{JK})(V(y, 0)), \quad z_0 = \gamma^{-1/2} H_K(y)^{1/2} v.$$

Let $\mathcal{H}_K$ denote the triple of coordinate-cycle holonomies of the Wilson-flowed fine configuration at time θL^2 , regarded as a function of the coarse coordinates (y, v) and of all normalized fluctuation variables $(w_1, \dots, w_K)$. Evaluate derivatives at the zero-fluctuation

reconstruction of the flat coarse field $V(y, 0)$, that is, at the sequence of flat refinements of $V(y, 0)$ obtained with $w = 0$. Let $T_{0,K}$ be the derivative in v of $P_{\mathcal{N}_y}(\text{Log} \circ \mathcal{H}_K)$ there, and define

$$A_{0,K} = T_{0,K} H_K^{-1/2}, \quad A_{j,K} = \gamma^{-1/2} P_{\mathcal{N}_y} D_{w_j} (\text{Log} \circ \mathcal{H}_K)|_{\text{flat}}, \quad 1 \leq j \leq K.$$

The Hessian normalization of each step at the flat background and the factors $g_{j,r}/\sqrt{\gamma}$ are included in $A_{j,K}$, exactly once. These deterministic matrices do not depend on γ . Put

$$\Sigma_m^K(y) = A_{0,K} A_{0,K}^T + \sum_{j=1}^m A_{j,K} A_{j,K}^T, \quad 0 \leq m \leq K.$$

The matrix $\Sigma_m^K(y)$ is the covariance of the image, under the linearized normal response, of independent standard Gaussian vectors placed in the coarse normal variable and in the first m steps. We call the matrices $A_{j,K}$ the linear response matrices. On a sample, $\mathcal{H}_K$ takes the value $\mathcal{H}^K$, the flowed holonomy triple of Definition 34.1, in the same gauge frame at the basepoint. In the notation of Proposition 33.12, $H_K = H_K^{\text{act}}$, the Hessian of the action phase nE_{JK} . For $j \geq 1$ the matrices $A_{j,K}$ coincide with the matrices $A_{j,K}$ of Lemma 34.12, as shown in the proof of Theorem 35.10.

Hypotheses of the earlier results. We record how the hypotheses of the results used below are met. None of these choices changes as γ varies.

Flat refinements. Since $2\pi/n < r_{\text{flat}}/4$ and $r_{\text{flat}} < r$ lies below every fixed margin of the constrained minimizers, of the inverse of the block averaging constraint, of the Haar disintegration densities, and of the gauge charts, every flat coarse field has a commuting representative with link logarithms of norm at most $2\pi/n$. By Lemma 14.9, its zero-fluctuation reconstruction is the sequence of flat refinements, and it lies inside all these coordinate domains with a positive margin at every depth. On the fixed coarse lattice the compact set of flat configurations has a neighborhood in D_r independent of the depth and of γ . This verifies the geometric hypothesis in every application of Proposition 33.12 below.

The coarse normal atlas. The dimension in the Laplace integral on the coarse lattice is $d_{\text{reg}} = \mathfrak{d}_n = 6n^3 - 2$, the regular normal rank of Proposition 33.12, not four, and $d_{\text{reg}}/2 = 3n^3 - 1$. We take the finite normal atlas on relatively compact smooth subcharts, in rooted spanning-tree gauge coordinates, whose charts and gauge transformations extend smoothly to neighborhoods of their closures. Their representatives and the closed normal tubes lie strictly inside the outer ball B_ρ , hence inside $B_{(1-\eta)\rho}$ for a fixed $0 < \eta < 1$, and all chart and gauge jets are bounded on these closures. We use the derivative bounds for the amplitude of the coarse density only on these subcharts; their constants may depend on n , η , the chart and gauge jets, and the outer domains, but not on the depth or on γ . In this atlas the density of $\hat{v}$ is

$$\hat{Z}^{-1} a_{K,\gamma}(y, v) e^{-nE_{JK}(V(y,v))/\gamma} dy dv, \quad a_{K,\gamma} = e^{-\bar{R}_{K,\gamma}} J(y, v),$$

where J is the density of the product Haar measure in these coordinates, including the partition of unity of the atlas. On each compact subpatch, before the partition is applied, $a_{K,\gamma}$ is bounded above and below by positive constants and is uniformly Lipschitz in v . These are hypotheses of the coarse family of Proposition 33.12, which also has $nE_{JK,\text{sat}} \geq \kappa_n E_0$ on its support and $|\bar{R}_{K,\gamma}| \leq C_n$. The phase and amplitude are smooth in the fixed charts, and the derivative bounds on the effective interaction in that family imply the amplitude bounds uniformly in the depth and in γ . In particular all phase, amplitude, and cutoff data are jointly Borel, H_K is measurable, and the phase is coercive on each chosen cutoff support. These facts verify the measurability, spectral, Lipschitz, and coercivity hypotheses of Lemma 33.3, on which the coarse Laplace formula rests. The partition of the atlas is absorbed into the measurable tangential weights, after the positive bounds on the amplitude have been used.

Coordinate domains of the reconstruction. Every coarse field in the support of $\hat{v}$ has a representative in B_r ; the reconstruction maps take values in the inner coordinate domain, and the larger complex images lie inside $B_{\rho/2}$. The products of coarse and free coordinates lie strictly inside the nested domains $\mathcal{D}_{\text{in}} \subseteq \mathcal{D}_{\text{mid}} \subseteq \mathcal{D}_{\text{out}}$ of Theorem 15.13, and every intervening constrained minimizer, inverse, logarithm, and Haar disintegration density is defined there. In terms of majorizing series, the radius of the coarse field plus the fixed radius of the free coordinates lies strictly below the radius of the outer domain and below the radius of convergence of the series. These are the hypotheses of Theorem 15.13 on the images of the reconstruction maps; no auxiliary integral is evaluated outside B_r .

Couplings and cutoffs. Let $t_* > 0$ be the common coupling ceiling in Theorems 13.8 and 13.10. At the j -th step we set

$$t_j = M^{(J-1)/2} \sqrt{\gamma/(nB^j)}, \quad p_j = A_w \log(1/t_j), \quad t_1^2 = \gamma/(Mn). \quad (35.4)$$

We keep the ceiling γ_0 for the bound on the small-field failure probability and for the continuum limit theorem. At the fixed side n we fix a threshold

$$0 < \gamma_W(n) \leq \gamma_0, \quad (35.5)$$

and all estimates below are restricted to $0 < \gamma \leq \gamma_W(n)$. The threshold may depend on n and the fixed data, but not on K , m , or γ . It is so small that $t_1 \leq t_*$, $p_1 \geq 1$, $Ct_j p_j \leq 1/2$, and the Hessian bound of Theorem 13.8 and the smallness condition on $t_j p_j^3$ of Theorem 13.10 hold for every $j \geq 1$. One choice serves all j : we have $t_j \leq t_1$, and each function $t^a \log^b(1/t)$ needed here is increasing on a sufficiently small fixed interval, since its derivative has the sign of $a \log(1/t) - b$. We also require that $t_1 p_1$ fit within the fixed margin of the coordinate domains for the free coordinates. No radius, cutoff, reconstruction threshold, or side changes as γ varies. Later reductions of the threshold concern only $\gamma_W(n)$, and every threshold γ_m below is taken at most $\gamma_W(n)$.

The reconstruction maps preserve the conditional potential, the product of cutoff balls, the normalized conditional kernels, and the coarse field. In particular, the two theorems of Section 13 quoted above apply to every conditional kernel in this section, uniformly in the coarse field, the side, and the preceding depth. We write E for their common bound E_* on the effective interaction.

Disintegration. At finite K we have the backward disintegration

$$d\nu = d\hat{\nu}(V) \prod_{j=1}^K k_j(V, w_1, \dots, w_{j-1}; dw_j), \quad (35.6)$$

where k_j is the conditional fluctuation kernel of the j -th step, given the coarse field and the earlier steps. Restricting ν to an event depending only on the coarse field multiplies the first factor by an indicator and leaves the kernels k_j unchanged. Each k_j still integrates to one, including its Haar disintegration density, its determinant, and the input effective interaction in its potential. Conditioning further on y changes the conditional law of ν in the first factor and leaves the same finer kernels. All conditional statements are understood for a version defined for almost every y ; their uniform bounds hold on the whole compact patch wherever that version is defined by the displayed smooth densities.

Lemma 35.8. *Use Definition 35.7, the disintegration (35.6), and a fixed coarse regular tube, and condition on the coarse tangential coordinate y . For every fixed $m \geq 0$ there exists $\gamma_m > 0$ such that the following holds uniformly in $K \geq m$ and $0 < \gamma \leq \gamma_m$. Let $\mathcal{F}_{j-1}$ be the σ -algebra generated by the coarse field and the normalized fluctuation variables of the first $j-1$ steps, let $\bar{w}_j = \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$ be the conditional mean under the conditional fluctuation kernel k_j , and put*

$$Z_{K,m} = A_{0,K}(y)z_0 + \sum_{j=1}^m A_{j,K}(y)(w_j - \bar{w}_j).$$

For every fixed $s \geq 0$ and every jointly Borel family of real functions F_y on $\mathcal{N}_y$ with $|F_y(z)| \leq C_F(1 + |z|^s)$ in the finite smooth bundle atlas,

$$|\mathbb{E}[F_y(Z_{K,m}) \mid y] - \mathbb{E}F_y(G_{\Sigma_m^K(y)})| \leq C_{m,s}C_F\sqrt{\gamma}(1 + |\log \gamma|).$$

All fixed moments of $Z_{K,m}$ and of its partial sums are bounded by constants $C_{m,s}$. More strongly, the same polynomially weighted comparison holds for the joint vector $(z_0, w_1 - \bar{w}_1, \dots, w_m - \bar{w}_m)$ against a standard Gaussian vector of the same fixed finite dimension. The restriction to the coarse tube changes only the first factor of (35.6); no conditional kernel and no nonlinear reconstruction is evaluated at a Gaussian argument.

Proof. We first bound each conditional kernel and the probability that its variables come close to the boundary of the cutoff product (Steps 1 and 2). We then control the linear term of the conditional potential (Step 3), compare each conditional density with a Gaussian

density in a weighted norm (Step 4), and treat the coarse variable (Step 5). Finally, a telescoping argument replaces the variables one at a time by Gaussian variables (Step 6), and we pass to the linear image (Step 7).

Step 1: the conditional kernels. Fix the coarse field and the first $j - 1$ steps in the coordinate domains. By Theorem 13.8, whose conclusions are preserved by Theorem 15.13, the kernel k_j on the product $\Omega_j \subset \mathbb{R}^{D_j}$ of cutoff balls satisfies

$$k_j(dw) = Z_j^{-1} \mathbf{1}_{\Omega_j} e^{-V_j(w)} dw, \quad \|D^2 V_j - I\| \leq \epsilon_j, \\ \text{Var } F \leq 2\mathbb{E}|\nabla F|^2, \quad \text{Ent}(F^2) \leq 4\mathbb{E}|\nabla F|^2,$$

where $\epsilon_j \leq C\{t_j p_j + t_j^2(1 + E)\}$ under the condition $Ct_j p_j < 1/2$, with t_j, p_j as in (35.4), and $D_j \leq 9J(nB^j)^3$. By Theorem 13.10, the conditional mean satisfies

$$\|\bar{w}_j\|_\infty \leq Ct_j \|\hat{\theta}_{j-1}\|_\infty \leq Ct_j r, \quad (35.7)$$

where $\hat{\theta}_{j-1}$ is the logarithm of the reconstructed field after $j - 1$ steps. For $j \leq m$ the dimension D_j is fixed, and we have $|\bar{w}_j| \leq C_m \sqrt{\gamma}$ and $\epsilon_j \leq C_m \sqrt{\gamma}(1 + |\log \gamma|)$. These bounds are uniform over all earlier variables in the coordinate domains and over K .

Step 2: concentration and an interior cutoff. Let f be a 1-Lipschitz function of w , and let $\Lambda(\lambda) = \log \mathbb{E} e^{\lambda(f - \mathbb{E}f)}$. Inserting $e^{\lambda f/2}$ into the logarithmic Sobolev inequality, we find $\lambda \Lambda' - \Lambda \leq \lambda^2$, which implies $\Lambda(\lambda) \leq \lambda^2$. Applying this also to $-f$ and choosing $\lambda = u/2$ in Markov's inequality, we obtain $\mathbb{P}\{|f - \mathbb{E}f| \geq u\} \leq 2e^{-u^2/4}$; this is the Herbst argument (see [123, Section 2.3]). Lipschitz functions are covered by smooth approximation on the compact product. Since $t_j = c_{n,M,J} \sqrt{\gamma} B^{-j/2}$ and $p_j = A_w \log(1/t_j)$, we may decrease γ_m so that $Ct_j r \leq p_j/(4\sqrt{3})$ and $p_j \geq 1$ for $j \leq m$. If a three-dimensional block w_a has $|w_a| \geq p_j/2$, then one of its scalar coordinates deviates from its conditional mean by at least $p_j/(4\sqrt{3})$. A union bound over the scalar coordinates yields

$$\mathbb{P}\left\{\max_a |w_a| \geq p_j/2\right\} \leq 2D_j e^{-p_j^2/192}.$$

Let $0 \leq \chi \leq 1$ be the product radial cutoff equal to one on the balls of half the cutoff radius and zero outside the balls of three quarters of it. Then $\chi \in C_c^\infty(\Omega_j)$, $|\nabla \chi| \leq C\sqrt{D_j}/p_j$, and $q_j^{\text{cut}} = \mathbb{P}(\chi < 1) \leq CD_j e^{-cp_j^2}$. At fixed m we have $p_j \geq c|\log \gamma| - C_m$. This implies that q_j^{cut} and the standard Gaussian mass outside Ω_j are $O_{m,q}(\gamma^q)$ for every fixed $q > 0$; polynomially weighted Gaussian tails have the same property, with the fixed moment order in the constant. Decreasing γ_m further, we may assume $q_j^{\text{cut}} \leq 1/2$, $|\bar{w}_j| \leq 1$, and $\epsilon_j \leq 1/4$ for $j \leq m$.

Step 3: the linear term. We write $D = D_j$, $\mu = \bar{w}_j$, $\epsilon = \epsilon_j$, and

$$\begin{aligned} V_j(w) - V_j(0) &= |w|^2/2 + \ell \cdot w + R(w), \quad \ell = \nabla V_j(0), \\ |\nabla R(w)| &\leq \epsilon|w|, \quad |R(w)| \leq \epsilon|w|^2/2. \end{aligned}$$

The bounds on R follow from Taylor's formula along radial segments, since the product is convex and contains zero. Applying the Poincaré inequality to each coordinate, we have $\mathbb{E}|w|^2 \leq 2D + |\mu|^2$. Integrating $\nabla(\chi e^{-V_j})$ over Ω_j , which gives zero because χ has compact support in the interior, even at the corners of the product, we obtain

$$\ell \mathbb{E}\chi = \mathbb{E}\nabla\chi - \mu + \mathbb{E}[(1 - \chi)w] - \mathbb{E}[\chi \nabla R(w)]. \quad (35.8)$$

Since $\nabla\chi$ is supported on $\{\chi < 1\}$, we have $|\mathbb{E}\nabla\chi| \leq C\sqrt{D} q_j^{\text{cut}}/p_j$. By the Cauchy–Schwarz inequality, $|\mathbb{E}[(1 - \chi)w]| \leq \sqrt{q_j^{\text{cut}}}(\mathbb{E}|w|^2)^{1/2}$, and $|\mathbb{E}[\chi \nabla R(w)]| \leq \epsilon(\mathbb{E}|w|^2)^{1/2}$. Since $\mathbb{E}\chi \geq 1/2$, we obtain

$$|\ell| \leq C_m \{ \sqrt{\gamma} + \epsilon + \sqrt{q_j^{\text{cut}}} \} \leq C_m \sqrt{\gamma} (1 + |\log \gamma|).$$

Here we used the bound on the conditional mean μ from Step 1, not that μ vanishes. A bound on $D^2 V_j - I$ alone would not control the linear coefficient ℓ .

Step 4: weighted density comparison. Let $U = V_j - V_j(0)$. When $\epsilon \leq 1/4$, the preceding bounds imply $U(w) \geq |w|^2/4 - C|\ell|^2$. After a further reduction of γ_m we have $|\ell| \leq 1$, and the exponential-difference identity (33.6), used in the proof of Corollary 33.4, gives on Ω_j

$$|e^{-U(w)} - e^{-|w|^2/2}| \leq C(|\ell||w| + \epsilon|w|^2)e^{-c|w|^2}.$$

We extend e^{-U} by zero outside Ω_j , integrate against $1 + |w|^s$, bound the Gaussian mass of the complement of Ω_j by Step 2, and compare the two normalizing constants. Since the Gaussian integral is $(2\pi)^{D/2}$, both normalizing constants have a common positive lower bound at this fixed D . Then the normalized conditional density r_j of w_j satisfies

$$\int (1 + |w|^s) |r_j(w) - \phi_D(w)| dw \leq C_{m,s} \sqrt{\gamma} (1 + |\log \gamma|), \quad (35.9)$$

where ϕ_D is the standard Gaussian density in $\mathbb{R}^D$. In particular all fixed uncentered conditional moments are bounded. The same bound holds for $x = w - \mu$ against $\phi_D(x)$. Indeed, translation changes the polynomial weights by a bounded factor when $|\mu| \leq 1$, and

$$\phi_D(x + \mu) - \phi_D(x) = \int_0^1 \mu \cdot \nabla \phi_D(x + u\mu) du$$

has weighted L^1 norm at most $C_{D,s}|\mu|$. No derivative of the mean in the earlier variables is needed.

Step 5: the coarse variable. We condition on y in the coarse tube. The weighted comparison of the conditional density of z_0 with $\phi_{d_{\text{reg}}}$ is given by Corollary 33.4, together with the extension to the whole tube established in its proof. That proof verifies the coercivity of the phase on the whole tube, the bounds on the amplitude, and the lower bound on the normalizing constant of the tube for the coarse family of Proposition 33.12, uniformly in K . The chart and amplitude checks recorded above supply the data on the subcharts. The comparison error for the coarse variable is $C_{n,s}\sqrt{\gamma}$. As noted after (35.6), the restriction to the coarse tube leaves the finer kernels unchanged; this also follows from the fixed-target statement of Theorem 13.8. All bounds of Steps 1–4 hold under this conditional probability.

Step 6: substitution of Gaussian variables. Let $X_0 = z_0$ and $X_j = w_j - \bar{w}_j$ for $1 \leq j \leq m$, and let F be a Borel function of $(X_0, \dots, X_m)$ bounded by $1 + |(X_0, \dots, X_m)|^s$. For $0 \leq j \leq m$ let I_j be the expectation of F when the arguments $0, \dots, j$ are $X_0, \dots, X_j$ and the arguments $j+1, \dots, m$ are replaced by independent standard Gaussian vectors, independent of everything else. For $j \geq 1$, integrating out these Gaussian vectors gives a function of the current variable bounded by

$$C_{m,s}(1 + |(X_0, \dots, X_{j-1})|^s + |X_j|^s).$$

For fixed values of the earlier variables, (35.9) in its translated form bounds the change from a Gaussian j -th argument to X_j by $C_{m,s}\sqrt{\gamma}(1 + |\log \gamma|)$ times $1 + |(X_0, \dots, X_{j-1})|^s$. Taking expectations, we find that $|I_j - I_{j-1}|$ is at most $C_{m,s}\sqrt{\gamma}(1 + |\log \gamma|)$ times $\mathbb{E}(1 + |(X_0, \dots, X_{j-1})|^s)$. This expectation is uniformly bounded: the moments of the coarse variable follow from the corollary of Step 5, each conditional variable has all fixed moments by Step 4, and finite-dimensional norm inequalities bound the whole vector. The centered variables are measurable functions of the earlier variables, so no inverse transformation of a Gaussian argument is required. Finally the coarse argument in I_0 is replaced by a Gaussian one using Step 5. Summing the finitely many differences proves the joint weighted comparison. The Gaussian arguments enter only F , never a reconstruction, a conditional mean, or a conditional kernel.

Step 7: the linear image. The exact Gaussian decomposition of Proposition 33.12 includes the bound

$$\|A_{0,K}\|_{\text{HS}}^2 + \sum_{j=1}^K \|A_{j,K}\|_{\text{HS}}^2 \leq C. \quad (35.10)$$

This implies that the linear map $(X_0, \dots, X_m) \mapsto A_{0,K}X_0 + \sum_{j=1}^m A_{j,K}X_j$ has operator norm bounded uniformly in K and m . Applying the joint comparison to F_y composed with this map, which has polynomial growth with a fixed bound, proves the stated estimate. The Gaussian image has covariance $\Sigma_m^K(y)$ by definition, and the argument covers singular covariances. The same norm bound and the moments of each argument give the bounds

on all moments of the partial sums. The kernels and the conditional means are Borel, as normal integrals of smooth conditional densities, and the Borel hypothesis on F_y together with Fubini's theorem justifies every substitution of a test function and every integration in y . $\square$

35.3 Finite-scale reconstructions and the normal covariance

The proofs in this subsection use coordinates $q_{j,r} = g_{j,r}w_{j,r}$ on fixed finite lattices; as coordinates they do not depend on γ , and all domains below are chosen independently of γ and of the number of later steps.

Theorem 35.9. *Fix $m \geq 0$, and express the triple Ξ_m^K as a function of the coarse coordinates and of the coordinates $q_{j,r} = g_{j,r}w_{j,r}$, $1 \leq j \leq m$, of the first m steps. There exist a neighborhood of its flat inputs (the inputs whose coarse field is flat and whose coordinates of the first m steps vanish) and a constant C_m such that, on this neighborhood,*

$$\sup_{K \geq m} \|\Xi_m^K\|_{C^{1,1}} \leq C_m.$$

The derivatives are taken in smooth rooted spanning-tree gauge coordinates; no derivative of a measurable choice of gauge representatives is taken. The neighborhood and C_m may depend on m and on the fixed lattices of the first m steps, but not on K , on γ , or on the preceding and later classical depths.

Moreover, Ξ_m^K extends to a fixed neighborhood of the set of all inputs $(V_0, 0)$ with $V_0 \in \mathcal{U}_n$ flat, and there exists a function ω_m with $\omega_m(s) \rightarrow 0$ as $s \downarrow 0$ such that, for every flat V_0 ,

$$\max_i \|\Xi_{m,i}^K(V, z_{\leq m}) - \text{Hol}_i(V_0)\|_{\text{op}} \leq \omega_m(d_{\text{in}}((V, z_{\leq m}), (V_0, 0))).$$

Here $z_{\leq m}$ denotes the physical coordinates $q_{j,r}$ of the first m steps. For this comparison with the zero section, $d_{\text{in}}((V, z_{\leq m}), (V_0, 0))$ is the sum of the distance between the based gauge classes of V and V_0 in one fixed product Riemannian metric on the compact coarse quotient in rooted spanning-tree coordinates and the Euclidean norm of $z_{\leq m}$. The triangular orthogonal transitions preserve this norm. This distance to a flat input is uniformly equivalent, on the finite cover used below, to the corresponding local product distances, with constants independent of K . Every chart uses the same gauge frame at the basepoint. Finally, $\Xi_m^K(V_0, 0) = \text{Hol}(V_0)$ for every finite $K \geq m$ and every flat V_0 , including those whose holonomy triple is one of the central triples $(\pm I, \pm I, \pm I)$.

Proof. Fix m . All parameter norms in this proof are norms on the coarse lattice and on the finite lattices of the first m steps; their dimensions and the constants comparing different norms may depend on m, n, M, J . We write u for these parameters. As arguments of the successive Hessian normalization we use the coordinates $q_{j,r} = g_{j,r}w_{j,r}$, which do not depend on γ . The changes of variables between these coordinates and the free logarithmic

coordinates have uniformly bounded charts and inverse charts on each fixed set of first m steps. On the coarse lattice we use analytic link coordinates or rooted spanning-tree gauge coordinates, keeping the initial gauge frame at the basepoint. The smooth normal atlas is composed with these coordinates only after the real derivative bounds have been obtained.

Step 1: a domain for the inputs. By Lemma 14.9 and the condition $2\pi/n < r_{\text{flat}}/4$, each of the first m zero-fluctuation steps at a flat input is flat and lies strictly inside every margin of the constrained minimizers, of the inverse of the block averaging constraint, of the Haar disintegration densities, and of the gauge charts. Each fixed finite composition of reconstruction maps has uniformly bounded weighted analytic coefficients on the nested domains of Theorem 15.13; in particular its first derivative is bounded there independently of the later depth. Let $F_m^K(u)$ be the field at the end of the m -th step, which is the input of the zero-fluctuation reconstruction of the later steps. At each flat chart center u_0 we start from a small commuting representative with link logarithm of norm at most $2\pi/n$, keeping the gauge frame at the basepoint. The reconstruction maps preserve flatness and the based cycle holonomies. Their zero-fluctuation maps P_j satisfy $P_j(0) = 0$ and $\|DP_j\| \leq a_{\text{map}} < 1$ on the convex working ball, by Lemma 15.16 and Definition 34.1. Integrating along segments and iterating therefore gives

$$\|F_m^K(u_0)\|_\infty \leq a_{\text{map}}^m \frac{2\pi}{n} \leq \frac{2\pi}{n} < \frac{r_{\text{flat}}}{4} < \frac{r}{4}.$$

This bound applies to the actual reconstructed representative, which may differ from the constant commuting refinement by a based gauge transformation. The derivative bound just obtained gives a constant $C_{\text{in},m}$ for $D_u F_m^K$, uniform in K and in the depths, before any of the later steps is applied. Let U_{larger} be a real neighborhood of the flat inputs, of radius ϵ_m , such that $C_{\text{in},m}\epsilon_m < r/4$ and the derivative bound times ϵ_m is less than half of every image margin of the first m steps. By the mean value formula, we have, in these fixed charts,

$$\sup_{K \geq m, u \in U_{\text{larger}}} \|F_m^K(u)\|_\infty \leq r/4 + C_{\text{in},m}\epsilon_m < r/2 < r, \quad (35.11)$$

and every intermediate image stays in its coordinate domain. The same holomorphic coefficient bounds allow a larger complex polydisc, on which the complex image has norm below $3r/4$, and two smaller polydiscs. Indeed, the complex derivative bound times the added radius may be chosen below $r/4$ and below half of the remaining image margins. A finite cover of the compact set of flat inputs gives common positive radii and margins. These choices depend on m and on the coefficient bounds, but not on K , on γ , or on the number of later steps. In particular, (35.11) verifies the bound $\|\theta_0\|_\infty \leq r$ on the initial field required in Lemma 34.4, independently of the larger radius and of the cutoffs.

Step 2: the flow data of the zero-fluctuation reconstruction. Let $a = \ell_K = (L/n)B^{-K}$. Let $\mathcal{E}_K(u)$ be the linear and quadratic flow data, in the sense of Definition 4.8, of the zero-fluctuation reconstruction from $F_m^K(u)$ down to the mesh a , including both weighted components and their first differences. We apply parts (1)–(3) of Lemma 34.4, including the analytic coefficient estimates and the finite-scale composition clause in part (3), starting at the mesh ℓ_m with this finite input. Its hypotheses are the bound (35.11), the contraction condition $a_{\text{map}}\sqrt{B} < 1$, the complex domain of Step 1, and zero fluctuation variables at all later steps; they hold by Step 1 and Definition 34.1. We obtain a fixed smaller complex domain, a bound R_m , and a number $0 < \tau < 1$ such that, for every fixed $l \geq m$,

$$\sup_{K \geq l, u} \|\mathcal{E}_K(u)\|_{\text{fl},a} \leq R_m, \quad \sup_{K \geq l, u} \|\mathcal{E}_K(u) - \mathcal{E}_{K,l}(u)\|_{\text{fl},a} \leq C_m(1+l)^c \tau^{l-m}. \quad (35.12)$$

Here $\mathcal{E}_{K,l}$ is the flow data of the field at the end of the l -th step, embedded on the same finest lattice, and $\|\cdot\|_{\text{fl},a}$ is the sum of the four weighted value and first-difference norms of Definition 4.8. One can take $\tau = \max\{a_{\text{map}}B^{1-\alpha_{\text{fl}}}, \mathfrak{q}\}$; both terms are less than one, since $\alpha_{\text{fl}} > 1/2$, $a_{\text{map}}\sqrt{B} < 1$, and $\mathfrak{q} = Ba_{\text{map}}^2 < 1$. Starting at ℓ_m only changes C_m and R_m , and the smaller complex domain does not shrink with the number of later steps.

Step 3: parameter derivatives of the flow data. Let $U_0 \Subset U_1$ be real bounded parameter domains inside that complex domain, and let $s_m > 0$ be a common complex distance from U_1 to its boundary. Applying Cauchy's integral formula for Banach-valued holomorphic functions in each input coordinate, we obtain

$$\sup_{K, u \in U_1} \sum_{r=0}^2 \|D_u^r \mathcal{E}_K(u)\|_{\text{fl},a} \leq C_{\dim u} R_m (1 + s_m^{-1} + s_m^{-2}) = R_{2,m}. \quad (35.13)$$

For a mixed second derivative we use two coordinate circles of radius at most $s_m/2$, which only changes the finite-dimensional constant. The estimate covers the first-difference components as well as the values. Complex continuation is used only for the reconstruction and the flow data, not for the nonlinear Wilson flow.

Step 4: real variations of the Wilson flow. We apply Theorem 25.13 with the parameter domains U_0, U_1 , dimension $\dim u$, bound $R_{2,m}$ on the flow data, horizon T , and smallest observation time $t_* = \theta L^2/2$. The flow data are twice continuously differentiable by (35.13) and are bounded in the norms required there. Every damping gauge transformation starts at the identity in the same initial gauge frame at the basepoint. The early parameter variations given by that theorem satisfy the first-difference bound

$$C_m \{(s + a^2)^{-(\alpha_{\text{fl}}+1)/2} + (s + a^2)^{-\alpha_{\text{fl}}} + (s + a^2)^{\kappa_{\text{fl}}-1/2}\},$$

which is integrable in s near zero, uniformly in a , since $(\alpha_{\text{fl}} + 1)/2 < 1$, $\alpha_{\text{fl}} < 1$, and $\kappa_{\text{fl}} > 0$. The condition keeping the links in the compact chart is

$$a \sup_{s \leq T_{R_{2,m}}} \|A(s)\|_{\infty} \leq C_m \{a^{1-\alpha_{\text{fl}}} + a^{2-2\alpha_{\text{fl}}} + a(T_{R_{2,m}} + a^2)^{\kappa_{\text{fl}}}\} \longrightarrow 0 \quad (a \rightarrow 0).$$

At a positive early restart time the same theorem gives smooth bounds at the restart and a finite reference family of smooth DeTurck solutions, joined by fixed gauge transformations, on an open neighborhood of the compact family of smooth restart configurations. We check its hypothesis that this family has a compact smooth closure, using the gauge transformations at the restart and the bounds on all fixed spatial differences given by the smoothing at positive time. For each restart array, the coefficients of its periodic trigonometric interpolant are bounded by $C_q(1 + |k|)^{-q}$ for every fixed q . To see this, we apply q differences in a coordinate with maximal $|k_i|$ and use $|a^{-1}(e^{2\pi i a k_i/L} - 1)| \geq c_L |k_i|$ on the centered Brillouin representatives, and at $k = 0$ we use the bound on the values. The tails of the coefficients therefore tend to zero uniformly in each C^s norm, taking $q > s + 3$. Finite Fourier truncations have compact sets of coefficients, and successive finite nets in these truncations, followed by a diagonal subsequence, give a compact smooth closure, uniformly over the compact parameter domains. Taking real parts at the unpaired Nyquist modes preserves real arrays and the estimates. This verifies the compactness hypothesis. We use the fixed gauge transformations of that theorem at the restart for the varying input, keeping their conjugations at the basepoint. The theorem then bounds the first and second real parameter derivatives of the three cycle matrices at time θL^2 by C_m . Since the initial configurations are the finest configurations of the zero-fluctuation reconstruction that define Ξ_m^K , and the lattice flow time is $4\theta L^2/a^2$, this bounds the derivatives of the map Ξ_m^K itself. The finitely many meshes for which the smallness conditions of that theorem fail are covered by its clause on compact smooth ordinary differential equations. Integrating the second derivative on smaller convex coordinate balls, we obtain $\sup_{K \geq m} \|\Xi_m^K\|_{C^{1,1}} \leq C_m$.

Step 5: all flat configurations. We repeat Steps 1–4 in ambient charts given by rooted spanning-tree gauges, covering the set of flat configurations, which is compact on the coarse lattice. Its quotient by gauge transformations fixed at the basepoint is parametrized by finitely many matrices on the non-tree links, because rooted spanning-tree gauges are unique and have no residual stabilizer. Hence smooth ambient charts exist also at flat configurations with central holonomies, although the set of flat configurations is singular there.

For each flat chart center V_* , fix a periodic coarse gauge transformation h_* with $h_*(0) = I$ taking V_* to a small constant commuting representative, as in Lemma 14.9, and use this same fixed transformation on its whole ambient neighborhood. The strict margins at the center from Step 1 and the uniform coefficient bounds for the first m

steps give a neighborhood independent of K . On an overlap, transport both the coarse field and the first m fluctuation coordinates by the triangular transitions $q'_{j,r} = O_{j,r}q_{j,r}$ of Theorems 13.4 and 14.3, where $O_{j,r}$ depends on the coarser inputs. By (15.28), the corresponding reconstructions are related by a fine gauge transformation whose restriction to the coarse vertices is the original coarse transformation. It therefore equals I at the common basepoint. Iterating this identity transports the whole prefix, and every subsequent zero fluctuation coordinate remains zero. The finest zero-fluctuation configurations and their Wilson flows thus have identical based cycle matrices on the overlap. These transitions also preserve the cutoff balls, the normalized conditional kernels, and the selected event. The uniform coefficient bounds on the first m fixed lattices control the finite input charts; exact equality of the based matrices requires no derivative of a gauge transformation composed through the later steps.

We take finitely many larger charts covering the compact flat set and smaller charts with compact closures inside them. Steps 1–4 then give a common constant C_m and a common positive neighborhood on this finite cover. This construction chooses each center gauge once and uses neither a logarithm of a varying central matrix nor a regular normal Hessian.

Step 6: exact values and the modulus. At a flat input with zero fluctuation variables, Lemma 14.9 identifies the successive constrained minimizers up to based gauge transformations with commuting flat refinements, which preserve the cycle holonomies. By Theorem 15.13, the gauge transformations of the reconstruction maps are the identity at the roots and therefore preserve these values. The gradient of the Wilson action vanishes on the refinements, so the Wilson flow does not move them. We conclude that $\Xi_m^K(V_0, 0) = \text{Hol}(V_0)$ for every finite K , for central as well as regular flat configurations.

It remains to obtain the modulus from the uniform first-derivative bounds. We take coordinate balls whose doubled closures lie in the larger charts, cover the set of flat configurations by finitely many of the corresponding half-balls, and let $\varrho_m > 0$ be a common smaller radius. An input and a flat zero-section input at distance at most ϱ_m lie in one of these charts, where the local product distance is bounded by a fixed multiple of d_{in} . The norm of the transported fluctuation input is unchanged, so the first-derivative estimate and the fixed gauge frame at the basepoint bound their matrix difference by $C'_m d_{\text{in}}$. For larger distances the difference is at most two. After increasing C'_m so that $C'_m \geq 2/\varrho_m$, the function $\omega_m(s) = \min(2, C'_m s)$ is a common modulus. On overlaps the gauge transformations keep the same initial gauge frame at the basepoint. This proves the statement near all flat configurations. $\square$

Theorem 35.10. *For y in the regular patch P^{++} and $N = nB^K$ we have the exact identity*

$$\Sigma^K(y) = \Sigma_\theta^N(y),$$

where $\Sigma_\theta^N(y)$ is the normal covariance, divided by γ , of the linearized flowed cycle logarithms under the Gaussian measure of the lattice Maxwell form of the finest lattice around the flat configuration y (Theorem 33.11 and Proposition 33.12). There exists C , independent of m , K , and y , such that

$$0 \leq \Sigma_\theta^N(y) - \Sigma_m^K(y) \leq C\delta_m I_{\mathcal{N}_y} \quad (K \geq m).$$

Put $\tilde{\Sigma}_m^K(y) = \Sigma_\theta(y)^{-1/2} \Sigma_m^K(y) \Sigma_\theta(y)^{-1/2}$. With constants independent of m , K , and y ,

$$\sup_y \|\tilde{\Sigma}_m^K(y) - I\|_{\text{op}} \leq C\delta_m + C(nB^K)^{-2}, \quad K \geq m. \quad (35.14)$$

The continuum covariance satisfies $cI \leq \Sigma_\theta \leq CI$ and $\Sigma_\theta(y) \geq |\boldsymbol{\theta}|^{-2} e^{-2\theta|\boldsymbol{\theta}|^2} I_{\mathcal{N}_y}$. For every fixed m , in the finite smooth normal atlas,

$$\sup_{K \geq m} \sum_{j=0}^m (\|A_{j,K}\|_\infty + \text{Lip}_y A_{j,K}) \leq C_m. \quad (35.15)$$

Finally, if $H_K^E = D_v^2 E_{JK}(V(y, 0))$, then $H_K = nH_K^E$ and

$$A_{0,K} A_{0,K}^\top = n^{-1} T_{0,K} (H_K^E)^{-1} T_{0,K}^\top.$$

Proof. The exact identity is the Gaussian quadratic identity of Proposition 33.12, applied to the linear response matrices of Definition 35.7. Its hypotheses concern the same reconstruction maps, the Hessians at the flat background, the coarse normal variable, and the derivative of the flowed cycle logarithms. In particular,

$$\Sigma_\theta^N - \Sigma_m^K = \sum_{j=m+1}^K A_{j,K} A_{j,K}^\top \geq 0. \quad (35.16)$$

If G_j are independent standard Gaussian vectors, this positive matrix has trace

$$\text{tr}(\Sigma_\theta^N - \Sigma_m^K) = \mathbb{E} \left| \sum_{j=m+1}^K A_{j,K} G_j \right|^2. \quad (35.17)$$

To bound this trace we identify our matrices with those of Lemma 34.12. Its flat coarse field V_y is $V(y, 0)$ here. Its coordinates $q_{j,r}$ are our $g_{j,r} w_{j,r}$, taken before the Hessian normalization of each step at the flat background; that normalization and the zero-fluctuation reconstruction of all finer steps are included in the differentiated reconstruction on both sides. The two response matrices therefore agree:

$$A_{j,K} = P_{\mathcal{N}_y} D_{q_j} (\text{Log} \circ \mathcal{H}_K) \text{diag}_{g_{r,e}}(g_{j,r} / \sqrt{\gamma}) = \gamma^{-1/2} P_{\mathcal{N}_y} D_{w_j} (\text{Log} \circ \mathcal{H}_K) \big|_{\text{flat}} = A_{j,K}.$$

The diagonal contains the coupling of every intermediate scale and every scalar color component. Its entries do not depend on γ , since each $g_{j,r}^2 / \gamma$ is the reciprocal of a fixed

lattice side. Both response maps use the Wilson flow at time θL^2 (lattice time $4\theta N^2$), the same based coordinate cycles, and the same initial gauge frame at the basepoint. Our patch is regular and has the margins of the identity chart and of the normal charts required by that lemma. Its flat refinements satisfy the coordinate-domain conditions by the condition $2\pi/n < r_{\text{flat}}/4$ and Lemma 14.9, and $D \text{Log}$ is the bounded differential of the identity chart. Thus every hypothesis of Lemma 34.12 holds, and we obtain

$$\sup_{K \geq m, y} \mathbb{E} \left| \sum_{j=m+1}^K A_{j,K}(y) G_j \right|^2 \leq C\delta_m. \quad (35.18)$$

Only a linear map acts on the independent Gaussian vectors, and the bound holds at each fixed flat y ; it is not deduced from an averaged estimate for the interacting measure. A positive semidefinite matrix is bounded above by its trace times the identity, as one sees by diagonalizing it. Combining (35.16), (35.17), and (35.18), we obtain the bound $0 \leq \Sigma_\theta^N - \Sigma_m^K \leq C\delta_m I$.

We next prove the derivative bounds at finite cutoff. The spectral bounds $c_n I \leq H_K \leq C_n I$ follow from Proposition 33.12. By the uniform bounds on the derivatives of the effective classical action in Theorem 10.11, composed with the fixed compact normal atlas, we have $\|H_K\|_{C^1(y)} \leq C_n$; only three derivatives of the action are needed. For the steps $j \geq 1$ we use the coordinates $q_{j,r} = g_{j,r} w_{j,r}$ of Theorem 12.13. The Hessian normalization of each step is part of the map in these coordinates and is preserved by Theorem 15.13; these are the coordinates used in the proof of Theorem 35.9, and no limit of a normalization matrix or of its derivative is needed.

For the inverse square root, let S be a symmetric matrix with $cI \leq S \leq CI$, choose $C' > C$ and write $S = C'(I - Q)$, so that $\|Q\| \leq 1 - c/C' < 1$. The binomial series for $(I - Q)^{-1/2}$ and its term-by-term derivative converge uniformly: the derivative of Q^k is a sum of k terms bounded by $\|Q\|^{k-1} \|DQ\|$, and the coefficients of the series are bounded. This bounds $S^{-1/2}$ and its first tangential derivative without constructing any limit. Applied to $S = H_K$, it bounds $H_K^{-1/2}$ and its first tangential derivative.

At the flat input, the derivative of $\text{Log} \circ \Xi_m^K$ in the variables of a step $j \leq m$ equals the corresponding derivative of $\text{Log} \circ \mathcal{H}_K$ with all other fluctuation variables set to zero. Indeed, setting the later variables to zero does not change the derivative in an earlier variable at the flat refinements. For $j \geq 1$ this gives the identity

$$A_{j,K} = P_{\mathcal{N}_y} D_{q_j} (\text{Log} \circ \Xi_m^K) |_{\text{flat}} \text{diag}_{r,e}(g_{j,r}/\sqrt{\gamma}),$$

with a fixed diagonal independent of y , K , and γ . Using the $C^{1,1}$ bounds of Theorem 35.9 in these coordinates, the bounds on the first two derivatives of the logarithm on the identity chart, and the smoothness of the embedding of the flat configurations, we bound each $A_{j,K}$ and its first tangential derivative. For $A_{0,K}$ we compose the corresponding coarse derivative with $H_K^{-1/2}$ and use the preceding bounds. Each step carries exactly one

Hessian normalization, already included in the differentiated map, and the derivatives of the orthonormal frames and of the projections in the finite atlas are bounded. This proves (35.15), and hence a Lipschitz bound for Σ_m^K at each fixed m .

By Theorem 33.11 we have $\|\Sigma_\theta^{nB^K} - \Sigma_\theta\| \leq C(nB^K)^{-2}$, together with the spectral bounds $cI \leq \Sigma_\theta \leq CI$. Adding this lattice error to the bound on the positive remainder proved above and conjugating by $\Sigma_\theta^{-1/2}$, we obtain (35.14). By the same theorem, the constant off-diagonal color modes alone contribute $e^{-2\theta|\theta|^2}|\theta|^{-2}I_{\mathcal{N}_y}$ to $\Sigma_\theta(y)$; since all other contributions are positive semidefinite, this gives the stated lower bound.

Finally, $H_K = D_v^2(nE_{JK}) = nH_K^E$, so

$$A_{0,K}A_{0,K}^\top = T_{0,K}H_K^{-1}T_{0,K}^\top = n^{-1}T_{0,K}(H_K^E)^{-1}T_{0,K}^\top.$$

In particular the contribution of the coarse normal variable is included in Σ_m^K for every m . $\square$

35.4 Finite-scale moments and localization

This subsection combines the Gaussian comparison of Lemma 35.8 with the deterministic results of Section 35.3. Two facts about the coarse lattice drive the argument. First, the normalizing constant $\hat{Z}$ is of order γ^{3n^3-1} (Theorem 33.10). This is also the order of the Laplace mass of any fixed regular patch, so a fixed regular neighborhood has probability bounded below, uniformly in γ . Second, the action has a positive gap at a fixed distance from the flat configurations. Together with the uniform modulus of Theorem 35.9, this shows that, with overwhelming probability, an output in a fixed regular tube comes from a coarse field near the corresponding regular flat configurations, and not from a neighborhood of other flat configurations, such as those with central holonomies.

Theorem 35.11. *Fix nested tubes as in Section 35.1. For every fixed $m \geq 0$ there exist $K_0(m)$ and $\gamma_m > 0$ such that the following statements hold uniformly for $K \geq K_0(m)$ and $0 < \gamma \leq \gamma_m$.*

1. *Fix bounded Borel functions on $\mathrm{SU}(2)^3$ that agree with the local maps $H \mapsto b(\mathrm{Log} H)$ and $H \mapsto y(\mathrm{Log} H)$ on a fixed regular output tube containing, with a positive margin, all flat outputs $\mathrm{Hol}(V(y, 0))$ of the selected coarse conditioning patch; we denote them again by $b(\mathrm{Log} H)$ and $y(\mathrm{Log} H)$. Under the restriction of ν to the corresponding coarse regular tube, normalized by its own mass, we have for every fixed finite r'*

$$\frac{b(\mathrm{Log} \Xi_m^K)}{\sqrt{\gamma}} = A_{0,K}(y)z_0 + \sum_{j=1}^m A_{j,K}(y)w_j + \mathcal{R}_{K,m},$$

$$\|\mathcal{R}_{K,m}\|_{L^{r'}} \leq C_{m,r'}\sqrt{\gamma}(1 + |\log \gamma|)^2, \quad \|y(\mathrm{Log} \Xi_m^K) - y\|_{L^{r'}} \leq C_{m,r'}\sqrt{\gamma}(1 + |\log \gamma|).$$

The same integrated estimates hold for the expectation under ν with any bounded measurable multiplicative factor supported in the coarse tube. Replacing w_j by $w_j - \bar{w}_j$ changes $\mathcal{R}_{K,m}$ by a term of norm $O_m(\sqrt{\gamma})$ in $L^{r'}$.

2. Let w be any fixed real smooth tangential weight compactly supported in the relative interior of P , and let χ_w be a fixed smooth normal cutoff supported in $\{|b| < b_0\}$ and equal to one near the zero section. Set $\tilde{w}(y+b) = w(y)\chi_w(b)$, with the fixed zero extension; for this part, nonnegativity and invariance of w and χ_w are not required. Choose a smooth normal cutoff ζ equal to one near the support of χ_w and supported in $\{|b| < b^+\}$, and set $w^+(H) = w(y(\text{Log } H))\zeta(b(\text{Log } H))$, extended by zero. For $l = 0, 1, 2$,

$$\begin{aligned} \mathbb{E}_\nu \left[\tilde{w}(\text{Log } \Xi_m^K) \left(\frac{q_*(\text{Log } \Xi_m^K)}{\gamma} \right)^l \right] \\ = \mathbb{E}_\nu \left[w^+(\Xi_m^K) m_l(\tilde{\Sigma}_m^K(y(\text{Log } \Xi_m^K))) \right] + O_{m,w}(\sqrt{\gamma}(1+|\log \gamma|)^{k_m}), \end{aligned}$$

where k_m is a fixed exponent, $\tilde{\Sigma}_m^K = \Sigma_\theta^{-1/2} \Sigma_m^K \Sigma_\theta^{-1/2}$, and

$$m_0(C) = 1, \quad m_1(C) = \frac{1}{2} \text{tr } C, \quad m_2(C) = \frac{1}{4} \{ (\text{tr } C)^2 + 2 \text{tr}(C^2) \}. \quad (35.19)$$

Only values in the support of w^+ are used on the right.

3. For every fixed r' , the normal moments of the coarse variable satisfy $\mathbb{E}_\nu[|v|^{r'} \mathbf{1}_{\{V \text{ in the coarse tube}\}}] \leq C_{r'} \gamma^{r'/2}$, and for every fixed larger regular output tube $\mathcal{V}$,

$$\mathbb{E}_\nu[|b(\text{Log } \Xi_0^K)|^{r'} \mathbf{1}_{\{\Xi_0^K \in \mathcal{V}\}}] \leq C_{r'} \gamma^{r'/2},$$

where $C_{r'}$ depends on $\mathcal{V}$ but not on m .

4. There exists $c_* > 0$, independent of K , m , and γ , such that, after possibly decreasing γ_m ,

$$\nu\{\psi(\text{Log } \Xi_m^K) = 0\} \geq c_*, \quad \nu\{\psi(\text{Log } \Xi_m^K) \geq b_*\} \geq c_*.$$

5. For every compact regular output tube $\mathcal{O}$ over a patch in the relative interior of P^{++} there exists a larger coarse regular tube $\mathcal{T}^+$ such that, for every $q > 0$,

$$\nu\{\Xi_m^K \in \mathcal{O}, V \notin \mathcal{T}^+\} \leq C_{m,q} \gamma^q.$$

This excludes contributions from neighborhoods of all other flat configurations, including those with central holonomies, as well as from nonflat coarse fields.

Throughout, the normalizing constant satisfies $c_n \gamma^{3n^3-1} \leq \hat{Z} \leq C_n \gamma^{3n^3-1}$.

Proof. The cases of finitely many small K are covered by the bounds of Section 35.3, so we may take $K_0(m) = m$; any larger choice is also allowed. We first record the Laplace estimates on the coarse lattice (Step 1) and prove the localization (Step 2), then construct the bounded output coordinates and a good event (Steps 3 and 4), prove the expansion of part (1) (Steps 5–7), and finally prove parts (2), (3), and (4) (Steps 8–10).

Step 1: Laplace estimates on the coarse lattice. By the first part of Proposition 33.12, with its exact conditional normalizing constant, we have

$$\mathbb{E}[|v|^r \mid y, V \text{ in the coarse tube}] \leq C_r \gamma^{r/2} \quad (r < \infty). \quad (35.20)$$

To obtain the mass of a patch, we apply the additive Laplace formula of the same lemma with a nonnegative smooth tangential bump h , supported strictly inside the patch, and a normal cutoff equal to one on a smaller ball. Positivity and the bounds on the amplitude and on the Hessian give

$$c_h \gamma^{3n^3-1} \leq \hat{Z} \mathbb{E}_{\hat{\nu}}[h(y)\chi(v)] \leq C_h \gamma^{3n^3-1}$$

for all sufficiently small γ , uniformly in K ; here the additive error $O_h(\sqrt{\gamma})$ has been absorbed into the positive leading integral, and we used $(2\pi\gamma)^{d_{\text{reg}}/2} = (2\pi\gamma)^{3n^3-1}$. By Theorem 33.10, $\hat{Z}$ is bounded above and below by constant multiples of the same power γ^{3n^3-1} . Dividing the two bounds, we find that every fixed nonempty open regular subpatch has coarse probability bounded below by a positive constant, uniformly in K and γ . No convergence of the tangential amplitude is used. Integrating (35.20) over the coarse tube proves the first assertion of part (3), both under the normalized restriction to the tube and with any bounded multiplicative factor under ν .

Step 2: localization (part (5)). We work on the compact quotient of the coarse configuration space by gauge transformations fixed at the basepoint. The zero set $\mathcal{F}_n$ of E_0 is the set of flat configurations, as in Section 36. For any fixed $\eta > 0$, compactness and continuity give

$$c_\eta = \inf\{\kappa_n E_0(V) : \text{dist}(V, \mathcal{F}_n) \geq \eta\} > 0,$$

since otherwise a convergent sequence in this closed set would have a flat limit in the set. Since the centered effective interaction is bounded, $nE_{JK,\text{sat}} \geq \kappa_n E_0$, and the product Haar measure has mass one, the lower bound on $\hat{Z}$ implies

$$\hat{\nu}\{\text{dist}(V, \mathcal{F}_n) \geq \eta\} \leq C_n \gamma^{-(3n^3-1)} e^{-c_\eta/\gamma} \leq C_{n,\eta,q} \gamma^q \quad (q > 0). \quad (35.21)$$

The last bound follows by maximizing $x^{3n^3-1+q} e^{-c_\eta x}$ over $x \geq 1/\gamma_0$.

Let $\mathcal{O}$ be the given compact output tube. The flat holonomies meeting a sufficiently small neighborhood of $\mathcal{O}$ stay away from the central triples and from the boundary of the identity chart. We may therefore choose a strictly larger compact set of regular commuting holonomies in the relative interior of P^{++} , and a coarse regular tube $\mathcal{T}^+$ over it. On the compact complement, within the set of flat configurations, of the flat configurations with holonomies in this larger patch, the distance of the flat holonomies from $\mathcal{O}$ is at least some $a_0 > 0$. By the modulus of Theorem 35.9, there exists $\eta_m > 0$ such that a change of the input by less than $2\eta_m$ changes every flat output by less than

$a_0/2$. For the finitely many steps $j \leq m$, the coordinates of the first m steps satisfy deterministically

$$|z_{\leq m}| \leq C_m \sqrt{\gamma} (1 + |\log \gamma|). \quad (35.22)$$

Indeed the coordinates collected in $z_{\leq m}$ are exactly $q_{j,r} = g_{j,r} w_{j,r}$; for the finitely many fixed scales $j \leq m$, each $g_{j,r}$ is a fixed ratio times $\sqrt{\gamma}$, and $|w_{j,r}| \leq p_j$. For small enough γ_m the right side of (35.22) is less than η_m . Decreasing η_m if necessary, we may also assume that a coarse field within distance η_m of a flat configuration with holonomies in the larger patch belongs to $\mathcal{T}^+$ whenever the holonomies of that flat configuration are within $a_0/2$ of $\mathcal{O}$. Now suppose that $\Xi_m^K \in \mathcal{O}$, $V \notin \mathcal{T}^+$, and $\text{dist}(V, \mathcal{F}_n) < \eta_m$, and let V_0 be flat with $\text{dist}(V, V_0) < \eta_m$. The input distance between $(V, z_{\leq m})$ and $(V_0, 0)$ is less than $2\eta_m$, so $\text{Hol}(V_0)$ is within $a_0/2$ of $\Xi_m^K \in \mathcal{O}$. By the choice of a_0 , the holonomies of V_0 lie in the larger patch, and then $V \in \mathcal{T}^+$, a contradiction. We conclude that an output in $\mathcal{O}$ coming from outside $\mathcal{T}^+$ forces $\text{dist}(V, \mathcal{F}_n) \geq \eta_m$, and (35.21) proves part (5). The flat configurations with central holonomies lie in the excluded part; the argument does not use that small curvature forces trivial holonomy.

Step 3: bounded output coordinates. We fix regular output tubes $\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}_3$ with positive margins, such that $\mathcal{O}_1$ contains the flat outputs of the larger coarse patch being used. Let ϕ_{out} be a smooth function equal to one on $\mathcal{O}_2$ and supported in $\mathcal{O}_3$. On $\mathcal{O}_3$ we multiply the local coordinate maps by ϕ_{out} , and we set both products equal to zero elsewhere on $\text{SU}(2)^3$. This gives the bounded Borel maps

$$b_{\text{ext}}(H) = \phi_{\text{out}}(H) b(\text{Log } H), \quad y_{\text{ext}}(H) = \phi_{\text{out}}(H) y(\text{Log } H), \quad (35.23)$$

where the right sides are used only on $\mathcal{O}_3$. They agree with the normal and tangential coordinates on $\mathcal{O}_2$, and they are the extensions used in part (1). They do not alter the invariant observables, which were extended by zero.

Step 4: a good event. We consider the restriction of ν to the coarse tube, normalized by its own mass. By the uniform first-derivative bound and the exact flat values of Theorem 35.9, there exists $\varepsilon_m > 0$, inside the input neighborhood of that theorem, such that $|v| + |z_{\leq m}| \leq 2\varepsilon_m$ implies $\Xi_m^K \in \mathcal{O}_2$, uniformly in K . By (35.22), after decreasing γ_m we have $|z_{\leq m}| \leq \varepsilon_m$ deterministically. Let $G_m = \{|v| \leq \varepsilon_m\}$ inside the coarse tube. Then

$$\mathbb{P}(G_m^c \mid V \text{ in the coarse tube}) \leq C_m \gamma^{-(3n^3-1)} e^{-c_m/\gamma}. \quad (35.24)$$

Indeed, the normal phase is at least $c_n |v|^2$ throughout the coarse tube, the amplitude is bounded, and the conditional normalizing constant at each y is bounded below by $c\gamma^{3n^3-1}$ by the coarse Laplace formula; we then integrate over the tangential distribution. The same estimate holds conditionally at each y in the compact subpatch. Since no departure from $\mathcal{O}_2$ occurs on G_m , (35.24) also bounds the probability of such a departure.

Step 5: Taylor expansion on the good event. On G_m we use the local coordinates and write $u = (v, z_{\leq m})$. By the uniform $C^{1,1}$ bound and the smoothness of the normal projection, Taylor's formula along the segment su , $0 \leq s \leq 1$, gives

$$\left| b(\text{Log } \Xi_m^K) - T_{0,K}v - \sqrt{\gamma} \sum_{j=1}^m A_{j,K} w_j \right| \leq C_m (|v| + |z_{\leq m}|)^2. \quad (35.25)$$

The derivatives here are those of the flowed triple $\mathcal{H}_K$ of the fine configuration, since setting the later fluctuation variables to zero does not change the derivative in an earlier variable at the flat refinements. Substituting $v = \sqrt{\gamma} H_K^{-1/2} z_0$ gives the coarse term $\sqrt{\gamma} A_{0,K} z_0$. The normal moments (35.20) and the deterministic bound (35.22) then bound the remainder divided by $\sqrt{\gamma}$, on G_m , by $C_{m,r'} \sqrt{\gamma} (1 + |\log \gamma|)^2$ in $L^{r'}$. Using the first-order bound for the tangential coordinate, we similarly obtain the bound $C_{m,r'} \sqrt{\gamma} (1 + |\log \gamma|)$ for its displacement.

Step 6: the exceptional event. On G_m^c we define $\mathcal{R}_{K,m}$ by subtracting the displayed linear terms from $b_{\text{ext}}(\Xi_m^K) / \sqrt{\gamma}$. Both extended coordinate maps are bounded, the coarse tube has bounded v , and the fluctuation terms are bounded by $C_m (1 + |\log \gamma|)$ there. This remainder is at most $C_m \gamma^{-1/2}$ for small γ , and (35.24) implies, for every fixed $r' < \infty$ and every $q > 0$,

$$\|\mathbf{1}_{G_m^c} \mathcal{R}_{K,m}\|_{L^{r'}} \leq C_{m,r',q} \gamma^q, \quad \|\mathbf{1}_{G_m^c} (y_{\text{ext}}(\Xi_m^K) - y)\|_{L^{r'}} \leq C_{m,r',q} \gamma^q;$$

here we took the r' -th root of the exponential bound and absorbed the fixed negative power of γ . Together with Step 5 this proves both estimates of part (1). Multiplying the r' -th powers by the mass of the coarse tube, and then by the norm of a bounded multiplicative factor, proves the integrated versions. Any other fixed bounded Borel extensions agreeing with the coordinates on $\mathcal{O}_2$ satisfy the same estimates, with their own bound in the constant.

Step 7: recentering. For $j \leq m$ the field at the start of the j -th step is a coarse field in B_r satisfying the coordinate-domain conditions. Since the conditional fluctuation kernel and its cutoff are preserved by Theorem 15.13, Theorem 13.10 applies with $t = t_j$. By (35.7),

$$|\bar{w}_j| \leq C \sqrt{D_j} t_j r \leq C_{m,n,M,J,r} \sqrt{\gamma} \quad (1 \leq j \leq m).$$

The inequality holds for every value of the earlier variables, hence also in $L^{r'}$ under the restriction to the coarse tube. Using (35.10) and the Cauchy–Schwarz inequality, we obtain

$$\left\| \sum_{j=1}^m A_{j,K} \bar{w}_j \right\|_{L^{r'}} \leq C \left(\sum_{j=1}^m D_j t_j^2 r^2 \right)^{1/2} \leq C_{m,r'} \sqrt{\gamma}. \quad (35.26)$$

The finite sum may grow with m , as allowed. This is the mean under the conditional fluctuation kernel, not under an enclosing product measure. Replacing w_j by $w_j - \bar{w}_j$ changes the scaled remainder by exactly this quantity, which completes the proof of part (1).

Step 8: the moment identities (part (2)). At fixed m all uncentered moments of the normalized fluctuation variables are bounded, since their centered moments were bounded in Lemma 35.8 and their means are $O_m(\sqrt{\gamma})$. Let $Z = Z_{K,m}$, $b' = b_{\text{ext}}(\Xi_m^K)$, and $y' = y_{\text{ext}}(\Xi_m^K)$. On G_m these are the local coordinates, so $X' = y' + b' = \text{Log } \Xi_m^K$ there. By Taylor's formula, the bound on the exceptional event, and (35.26), we have, under the whole restriction to the coarse tube,

$$b' / \sqrt{\gamma} = Z + R, \quad \|R\|_{L^r} \leq C_{m,r} \sqrt{\gamma} (1 + |\log \gamma|)^2.$$

Every fixed moment of Z is bounded, by Lemma 35.8, and the error R has bounded fixed moments for small γ . The inverse covariance Σ_θ^{-1} has a fixed Lipschitz norm on the larger regular tube. Expanding the quadratic form in $Z + R$, and using $\|y' - y\|_{L^{2r}} \leq C_{m,r} \sqrt{\gamma} (1 + |\log \gamma|)$ and Hölder's inequality with higher fixed moments of Z and R for each product, we obtain on G_m

$$\frac{q_*(X')}{\gamma} = \frac{1}{2} Z^\top \Sigma_\theta(y)^{-1} Z + O_{L^r, m}(\sqrt{\gamma} (1 + |\log \gamma|)^2).$$

For the square we use $a^2 - b^2 = (a - b)(a + b)$ and Hölder's inequality, which gives the same order with a possibly larger power of the logarithm. On G_m^c , each bounded weighted observable divided by γ^l has expectation at most $C\gamma^{-l}\mathbb{P}(G_m^c)$, which is smaller than any power of γ by (35.24). The corresponding polynomial in Z has a bounded second moment, so its contribution from G_m^c is at most $C_m \mathbb{P}(G_m^c)^{1/2}$; for $l \leq 2$ this uses moments of Z only up to order eight. Thus the comparison extends to the weighted expectations, without evaluating a logarithm or the quadratic form at a point where they are not defined.

The normal cutoff $\chi_w(b')$ differs from one only at a fixed positive normal radius. Using Taylor's formula, the coarse Laplace estimates, and the moments of the first m steps, we find $\mathbb{E}[|b'|^r \mathbf{1}_{\{\Xi_m^K \text{ in the larger tube}\}}] \leq C_{m,r} \gamma^{r/2}$ for every fixed r . Markov's inequality with $r > 2(q + l)$ bounds the effect of replacing $\chi_w(b')$ by one, even after division by γ^l , by $C_{m,q} \gamma^q$, and part (5) controls the complement of the larger tube. Next we replace the smooth tangential weight $w(y')$ by $w(y)$; the error is bounded by its Lipschitz norm times $|y' - y|$, multiplied by a power of the scaled quadratic form that is bounded in L^2 . We then apply Lemma 35.8 to the resulting polynomial of degree $2l$, conditionally on y , and integrate over the tangential distribution of the coarse field. The coefficient fields of this polynomial are smooth on the fixed compact bundle subcharts, hence bounded and Borel,

as required by that lemma. If $Z \sim G_{\Sigma_m^K(y)}$, then $Z^\top \Sigma_\theta(y)^{-1} Z / 2$ has the law of $G^\top C G / 2$ with $C = \tilde{\Sigma}_m^K(y)$ and G standard Gaussian in four dimensions, and diagonalizing the nonnegative symmetric matrix C gives

$$\mathbb{E} \left(\frac{1}{2} G^\top C G \right) = \frac{1}{2} \operatorname{tr} C, \quad \mathbb{E} \left(\frac{1}{2} G^\top C G \right)^2 = \frac{1}{4} \{ (\operatorname{tr} C)^2 + 2 \operatorname{tr} C^2 \}.$$

Here we used $\mathbb{E} G_i^2 = 1$, $\mathbb{E} G_i^4 = 3$, and, by independence, $\mathbb{E} G_i^2 G_j^2 = 1$ for $i \neq j$ (special cases of Isserlis' theorem [112]); a zero eigenvalue simply deletes the corresponding summand. This identifies the polynomials m_l of (35.19).

Step 9: return to the output coordinates. By (35.15), Σ_m^K has a uniformly bounded Lipschitz norm at each fixed m . The field Σ_θ and its inverse square root are smooth, with fixed positive spectral bounds, on the larger regular patch by Theorem 33.11, and the binomial series used in the proof of Theorem 35.10 also bounds the derivative of the inverse square root. Then $\tilde{\Sigma}_m^K$ has a uniform Lipschitz bound at each fixed m , and the tangential test $w(y) m_l(\tilde{\Sigma}_m^K(y))$ can be moved back to y' with error $C_{m,w} \sqrt{\gamma} (1 + |\log \gamma|)^{k_m}$. Finally we multiply by the normal cutoff $\zeta(b')$; as before, $1 - \zeta(b')$ is supported at a fixed positive normal radius, so this costs $C_{m,q} \gamma^q$, and part (5) controls the complement of the larger tube. The test on the right side is then exactly $w^+(\Xi_m^K) m_l(\tilde{\Sigma}_m^K(y'))$, and the tube coordinates are never evaluated outside the support of w^+ . This proves part (2).

Step 10: normal moments for $m = 0$ and the two value ranges of ψ (parts (3) and (4)).

For $m = 0$ there are no fluctuation variables. By the exact flat values and the uniform first derivative, we have $|b(\operatorname{Log} \Xi_0^K)| \leq C|v|$ in the larger coarse tube on G_0 , and (35.20) gives the claimed moment bound on G_0 . On its complement inside the larger coarse tube, the normal radius of an output in the given regular tube is bounded and (35.24) is smaller than any power of γ . Outside the coarse tube, part (5) gives the same bound. These three cases prove the second assertion of part (3); the constant depends on $\mathcal{V}$ but, since only $m = 0$ is used, not on m .

For part (4), let $h_0, h_+ \leq 1$ be nonnegative smooth tangential bumps, with positive integrals, supported strictly inside the open sets O_0 (where $\psi_0 = 0$) and O_+ (where $\psi_0 \geq 2b_*$), and fix a smaller normal ball on which $\chi = 1$. By Step 1, the coarse probability of each of the two corresponding sets of coarse fields is at least some $a > 0$. These compact supports have a fixed positive margin to the boundaries of O_0 and O_+ and to the boundary of $\{\chi = 1\}$. Markov's inequality, applied to the tangential and normal displacement bounds of part (1), makes the probability of crossing these margins less than $a/2$, after decreasing γ_m . On the first remaining event $\psi(\operatorname{Log} \Xi_m^K) = 0$, and on the second $\psi(\operatorname{Log} \Xi_m^K) \geq 2b_* \geq b_*$. Thus $c_* = a/2$ works, independently of m , K , and γ ; only the required threshold γ_m depends on m . This completes the proof. $\square$

35.5 Regression identities, moment comparison, and removal of the conditioning

This subsection collects the regression identities (Lemma 35.12), a comparison estimate for localized moments at nearby triples (Lemma 35.13), the removal of the conditioning together with the passage to the continuum (Lemma 35.14), and lower bounds for fixed nonnegative observables (Lemma 35.15).

Lemma 35.12. *Let P be a probability measure and let A, B be bounded real random variables with $s = \mathbb{E}_P B^2 > 0$ and $v = \text{Var}_P B > 0$. Then*

$$\mathcal{C}(P; A, B) = \frac{\text{Var}_P A - \text{Cov}_P(A, B)^2/v}{s} = \frac{\inf_{\alpha, c \in \mathbb{R}} \mathbb{E}_P(A - \alpha B - c)^2}{s}.$$

If $0 \leq B \leq 1$, $P\{B = 0\} \geq p_0 > 0$, and $P\{B \geq b_*\} \geq p_1 > 0$, then $s \geq p_1 b_*^2$ and $v \geq p_0 p_1 b_*^2$. Further, put $a = \mathbb{E}_P B$ and define e_1, e_2, e_3 by

$$\mathbb{E}_P A = \gamma(2a + e_1), \quad \mathbb{E}_P[AB] = \gamma(2s + e_2), \quad \mathbb{E}_P A^2 = \gamma^2(6s + e_3)$$

for some $\gamma > 0$. Then

$$\frac{\mathcal{C}(P; A, B)}{\gamma^2} = 2 + \frac{e_3 - 4e_2 - e_1^2 - (e_2 - ae_1)^2/v}{s}. \quad (35.27)$$

In particular, if $P = P_\gamma$ depends on γ , the bounds p_0, p_1 , and b_* hold uniformly, and $e_1, e_2, e_3 \rightarrow 0$ as $\gamma \downarrow 0$, then $\mathcal{C}(P_\gamma; A, B) = 2\gamma^2 + o(\gamma^2)$, even if $\mathbb{E}_{P_\gamma} B$ and $\mathbb{E}_{P_\gamma} B^2$ have no limits.

Proof. We minimize first over c . At a given α the unique minimizer is $c = \mathbb{E}A - \alpha \mathbb{E}B$, and the remaining quadratic is $\text{Var} A - 2\alpha \text{Cov}(A, B) + \alpha^2 v$. Since $v > 0$, its minimizer is $\alpha = \text{Cov}(A, B)/v$, and its minimum is the numerator in the first expression. This proves the regression identity.

For an independent copy B' of B we have $\text{Var} B = \mathbb{E}(B - B')^2/2$. The two events $\{B = 0, B' \geq b_*\}$ and $\{B' = 0, B \geq b_*\}$ each have probability at least $p_0 p_1$, and $(B - B')^2 \geq b_*^2$ on each of them, which implies $v \geq p_0 p_1 b_*^2$. Integrating B^2 over $\{B \geq b_*\}$ gives $s \geq p_1 b_*^2$.

For (35.27), note that $v = s - a^2$, that $\text{Cov}(A, B)/\gamma = 2v + e_2 - ae_1$, and that $\text{Var} A/\gamma^2 = 6s + e_3 - (2a + e_1)^2$. Subtracting the square of the first expression divided by v from the second, the terms linear in ae_1 cancel, and we obtain $2s + e_3 - 4e_2 - e_1^2 - (e_2 - ae_1)^2/v$; dividing by s gives (35.27). Uniform positive lower bounds on s and v , together with $|a| \leq 1$, make the correction in (35.27) tend to zero when $e_1, e_2, e_3 \rightarrow 0$, without any limit for a or s . $\square$

Lemma 35.13. *Let $\mathcal{U} \Subset \mathcal{U}^+ \Subset \mathcal{U}^{++}$ be the tubes of Definition 35.1, and let $\tilde{f}$ be a smooth invariant weight supported strictly inside $\mathcal{U}$ (for example ψ , ψ^2 , or a localizing weight $\tilde{w}$).*

Assume that the inverse covariance Σ_θ^{-1} is bounded and locally Lipschitz on $\mathcal{U}^{++}$ (it is in fact smooth there). Put $f_l = \tilde{f}q_*^l$, with the fixed bounded zero extensions. For two random triples H, H' in a common gauge frame at the basepoint let

$$D = \max_i \|H_i - H'_i\|_{\text{op}}, \quad d = \min(1, D), \quad \beta = |b(\text{Log } H')| \mathbf{1}_{\{H' \in \mathcal{U}^+\}}.$$

There exist $C < \infty$ and $0 < \eta_0 < 1$, depending only on the tubes and their margins, on $\|\Sigma_\theta^{-1}\|_\infty + \text{Lip}(\Sigma_\theta^{-1})$ on $\mathcal{U}^{++}$, and on the weight, such that for every $p \geq 1$ and $l = 1, 2$,

$$|\mathbb{E}f_l(H) - \mathbb{E}f_l(H')| \leq C[\mathbb{E}(\beta + d)^{4l-2}]^{1/2} [\mathbb{E}d^2]^{1/2} + C\eta_0^{-p} \mathbb{E}d^p,$$

where the constant in front of η_0^{-p} does not depend on p . For $l = 0$ the bound is $C\mathbb{E}d$. Applied under ν to $H = \mathcal{H}^K$ and $H' = \Xi_m^K$, with $p = 2l + 2 \leq 6$, these bounds give

$$\limsup_{K \rightarrow \infty} \gamma^{-l} |\mathbb{E}_\nu f_l(\mathcal{H}^K) - \mathbb{E}_\nu f_l(\Xi_m^K)| \leq C\delta_m^{1/2} + r_m(\gamma), \quad l = 1, 2,$$

where $r_m(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$ for each fixed m , and C does not depend on m .

Proof. We first prove the comparison inequality without any probabilistic assumption on H, H' . In the coordinates $X = y + b$ write $\mathcal{L}(y) = \Sigma_\theta(y)^{-1}$. The boundedness and local Lipschitz continuity of $\mathcal{L}$, and the smoothness of y and b on the tubes, give

$$|q_*(X) - q_*(X')| \leq C(|b(X)| + |b(X')|)|X - X'|.$$

Indeed, we expand the difference of the two quadratic forms, first changing b with $\mathcal{L}$ fixed and then changing $\mathcal{L}$; the second term is at most $C|b(X')|^2|y(X) - y(X')|$, which is bounded by the displayed expression since the tube radius is bounded. For $l = 1, 2$, the factorization of a difference of l -th powers and the Lipschitz bound on $\tilde{f}$ give

$$|f_l(X) - f_l(X')| \leq C(|b(X')| + |X - X'|)^{2l-1}|X - X'| \quad (35.28)$$

whenever the segment between the two points lies in $\mathcal{U}^{++}$. The derivative of the weight contributes a term of order $|b|^{2l}$, which we absorbed using the fixed upper bound on $|b|$.

We choose $\eta_0 < 1$ so small that any two triples at matrix distance at most η_0 , one of which lies in the support of f_l , both lie in $\mathcal{U}^+$ and have their logarithmic segment in $\mathcal{U}^{++}$. Such a choice exists: we cover the compact support by finitely many coordinate balls whose doubled closures lie in $\mathcal{U}^{++}$ and use the positive margins to the complement of $\mathcal{U}^+$. The logarithm and the tube coordinates have bounded derivatives on these balls, so $|X - X'| \leq CD$ there. On $\{D \leq \eta_0\}$, if both values of f_l vanish there is no difference; otherwise (35.28) gives

$$|f_l(H) - f_l(H')| \leq C(\beta + d)^{2l-1}d.$$

On $\{D > \eta_0\}$ the difference is at most $2\|f_l\|_\infty \leq 2\|f_l\|_\infty \eta_0^{-p} d^p$. Taking expectations and applying the Cauchy–Schwarz inequality on the first event proves the inequality; the constant in front of η_0^{-p} does not depend on p . For $l = 0$, the fixed smooth zero extension of $\tilde{f}$ is globally Lipschitz for the bounded distance d , and the same argument gives $C \mathbb{E} d$.

We now prove the application, and first bound the moments of β uniformly in m . Let $d_m = \min(1, D_{K,m})$ and $d_0 = \min(1, D_{K,0})$, where $D_{K,m} = \max_i \|\mathcal{H}_i^K - \Xi_{m,i}^K\|_{\text{op}}$ as in Theorem 34.10, and let $e_m = \min(1, \max_i \|\Xi_{m,i}^K - \Xi_{0,i}^K\|_{\text{op}})$. By the triangle inequality, $e_m \leq d_m + d_0$. On $\{e_m \leq \eta_1\}$, with η_1 fixed sufficiently small, an endpoint $\Xi_m^K \in \mathcal{U}^+$ has Ξ_0^K in a fixed larger regular tube $\mathcal{V} \subset \mathcal{U}^{++}$, and $|b(\text{Log } \Xi_m^K)| \leq |b(\text{Log } \Xi_0^K)| + C e_m$. On $\{e_m > \eta_1\}$, the bounded normal radius is at most $C \eta_1^{-1} e_m$. Combining the two cases, we find, for every fixed $r \leq 8$,

$$\mathbb{E} \beta^r \leq C_r \mathbb{E} [|b(\text{Log } \Xi_0^K)|^r \mathbf{1}_{\{\Xi_0^K \in \mathcal{V}\}}] + C_r \mathbb{E} (d_m + d_0)^r \leq C_r \gamma^{r/2}. \quad (35.29)$$

In the last step we used only part (3) of Theorem 35.11 with $m = 0$ and the bounds on high moments in Theorem 34.10. Both are estimates under ν , at the fixed side and in the same gauge frame at the basepoint. The tube $\mathcal{V}$ is chosen before m , and both constants for the distance moments are independent of m , so the constant in (35.29) does not depend on m .

For $l = 1, 2$ we take $r = 4l - 2$, which is two or six. By (35.29) and the moment bound for d_m , we have $\mathbb{E}(\beta + d_m)^{4l-2} \leq C \gamma^{2l-1}$. The second-moment bound of Theorem 34.10 gives

$$(\mathbb{E} d_m^2)^{1/2} \leq C \sqrt{\gamma} \delta_m^{1/2} + C \gamma (1 + |\log \gamma|)^k,$$

where k denotes the exponent written d in that theorem. With $p = 2l + 2$, the exceptional term is at most $C_p \gamma^{l+1}$ by the same theorem. The comparison inequality therefore gives, at every finite $K \geq m$,

$$\gamma^{-l} |\mathbb{E} f_l(\mathcal{H}^K) - \mathbb{E} f_l(\Xi_m^K)| \leq C \delta_m^{1/2} + C \sqrt{\gamma} (1 + |\log \gamma|)^k + C \gamma. \quad (35.30)$$

This proves the claim, with $r_m(\gamma) = C \sqrt{\gamma} (1 + |\log \gamma|)^k + C \gamma$. For $l = 2$ the sixth normal moment is used in an essential way. $\square$

Lemma 35.14. Fix $0 < \gamma \leq \gamma_W(n)$, and let $F : \text{SU}(2)^3 \rightarrow \mathbb{R}$ be bounded, continuous, and invariant under simultaneous conjugation; F may depend on γ . Write $e = \mu_{N,\gamma}(S_{N,n,\gamma}^c)$, with $N = nB^K$. Then, for every $q > 0$,

$$|\mathbb{E}_{\mu_{N,\gamma}} F(\mathcal{H}^K) - \mathbb{E}_\nu F(\mathcal{H}^K)| \leq 2\|F\|_\infty e \leq 2\|F\|_\infty C_{n,q} \gamma^q,$$

uniformly in K , and

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N,\gamma}} F(\mathcal{H}^K) = \mathbb{E}_{\lambda_\gamma} F(H),$$

where H denotes the coordinate-cycle holonomy triple of $X(\theta L^2)$.

Proof. The value $F(\mathcal{H}^K)$ depends only on the gauge orbit of the fine configuration, since F is invariant, the reconstruction maps produce a gauge-equivalent configuration, and the Wilson flow commutes with lattice gauge transformations. Hence $F(\mathcal{H}^K)$ equals F evaluated at the coordinate-cycle holonomy triple of $\Psi_{\theta L^2}^{(N)} U$, which is a bounded measurable function on the whole configuration space; this is how $F(\mathcal{H}^K)$ is understood under $\mu_{N,\gamma}$. For any bounded F and any event S with $e = \mu(S^c)$ we have

$$\mathbb{E}_\mu F = (1 - e)\mathbb{E}_\nu F + e \mathbb{E}_\mu(F \mid S^c), \quad |\mathbb{E}_\mu F - \mathbb{E}_\nu F| \leq 2\|F\|_\infty e, \quad (35.31)$$

where the last term of the identity is set to zero when $e = 0$; see also Lemma 5.5. By Corollary 23.11, at the fixed side n we have $e \leq C_{n,q}\gamma^q$ for every fixed $q > 0$, uniformly in K . This proves the first assertion.

For the second, Lemma 5.6 writes $F = f \circ \mathbf{p}$, where $\mathbf{p}$ is the vector of the seven normalized traces of $H_1, H_2, H_3, H_1H_2, H_1H_3, H_2H_3$, and $H_1H_2H_3$, and f is continuous on $[-1, 1]^7$ with $\|f\|_\infty \leq \|F\|_\infty$. These traces are the normalized fundamental Wilson loops of the three coordinate cycles and of their concatenations, which are fixed piecewise C^1 based loops of bounded length. On each lattice of the sequence they are exact lattice loops, which form admissible approximations. By Theorem 29.25, in the form of Corollary 29.26, the joint law of these seven traces under $\mu_{N,\gamma}$, at the flow time θL^2 , converges as $K \rightarrow \infty$ to their joint law under λ_γ . At fixed γ the corollary may discard finitely many initial terms of the sequence nB^K ; this does not affect the limit, and the side n is not changed. Since f is bounded and continuous, the expectations converge. $\square$

Lemma 35.15. *Let $K_1 \subset P^{++}$ be a compact invariant set with nonempty relative interior, contained in the relative interior of P^{++} , and let $r_c > 0$ be such that $\mathcal{T}_c = \{y + b : y \in K_1, b \in \mathcal{N}_y, |b| \leq r_c\} \subset \mathcal{U}^{++}$. Let $\varphi : \mathrm{SU}(2)^3 \rightarrow [0, \infty)$ be bounded, invariant under simultaneous conjugation, and Lipschitz for the matrix distance, and assume $\varphi(H) \geq c_1 > 0$ whenever $\mathrm{Log} H \in \mathcal{T}_c$. Then there exist $a, \gamma_1 > 0$ such that, for $0 < \gamma \leq \gamma_1$ and $K \geq K_0(0)$,*

$$\mathbb{E}_\nu \varphi(\Xi_0^K) \geq c_1 a, \quad \mathbb{E}_\nu \varphi(\mathcal{H}^K) \geq c_1 a / 2, \quad \mathbb{E}_{\lambda_\gamma} \varphi(H) \geq c_1 a / 4.$$

Proof. Let K'_1 be a compact invariant subset of the relative interior of K_1 with nonempty relative interior. At the flat coarse field $V(y, 0)$ we have $\mathrm{Log} \Xi_0^K = y$, by the exact flat values of Theorem 35.9. By the uniform first-derivative bound of that theorem with $m = 0$, there exists $\varepsilon' > 0$ such that, for every coarse field $V = V(y, v)$ with $y \in K'_1$ and $|v| \leq \varepsilon'$, the tangential component of $\mathrm{Log} \Xi_0^K$ lies in K_1 and its normal component has norm at most r_c ; hence $\mathrm{Log} \Xi_0^K \in \mathcal{T}_c$, uniformly in K . Taking in Step 1 of the proof of Theorem 35.11 a tangential bump supported in K'_1 and a normal cutoff supported in $\{|v| \leq \varepsilon'\}$, we find that the set $\{V(y, v) : y \in K'_1, |v| \leq \varepsilon'\}$ has $\hat{\nu}$ -probability at least some $a > 0$, uniformly in K , for all sufficiently small γ . This gives $\mathbb{E}_\nu \varphi(\Xi_0^K) \geq c_1 a$.

Since φ is Lipschitz for the bounded matrix distance, Theorem 34.10 with $m = 0$ gives

$$|\mathbb{E}_\nu \varphi(\mathcal{H}^K) - \mathbb{E}_\nu \varphi(\Xi_0^K)| \leq 2 \mathrm{Lip}(\varphi) (\mathbb{E}_\nu \min(1, D_{K,0}^2))^{1/2} \leq C\sqrt{\gamma},$$

so $\mathbb{E}_\nu \varphi(\mathcal{H}^K) \geq c_1 a/2$ for small γ , uniformly in K . Finally, by Lemma 35.14 with a fixed $q > 0$, we have $\mathbb{E}_{\mu_{N,\gamma}} \varphi(\mathcal{H}^K) \geq c_1 a/2 - 2C_{n,q} \gamma^q$ for every K , and letting $K \rightarrow \infty$ gives $\mathbb{E}_{\lambda_\gamma} \varphi \geq c_1 a/2 - 2C_{n,q} \gamma^q \geq c_1 a/4$ for small γ . $\square$

35.6 Proof of the regression coefficient

Proof of Theorem 35.2. We first check that the observables are admissible continuum observables (Step 1) and prove the lower bounds on the denominators (Step 2). We then prove the normalized moment estimate at finite cutoff (Steps 3 and 4), take the limits in the stated order (Step 5), and conclude with Lemma 35.12 (Step 6).

Step 1: the observables. The inverse covariance Σ_θ^{-1} , the tube coordinates, and the cutoffs are invariant under simultaneous conjugation. The supports of ψ and ψ^2 have positive distance from the boundary of the identity chart, so their zero extensions, and their products with q_* and q_*^2 , are bounded and continuous on $\text{SU}(2)^3$; they contain no coupling parameter. The continuous invariant factorization of Lemma 5.6 applies on the whole compact space of triples, including central triples and triples of deficient rank. In particular, these observables are continuous functions of the seven normalized traces, as stated.

Step 2: lower bounds on the denominators. We choose compact invariant sets $K_0 \subset O_0$ and $K_+ \subset O_+$ with nonempty relative interiors, and a radius $r_c > 0$ such that $\chi(b) = 1$ whenever $|b| \leq 2r_c$. Let $\rho_0, \rho_+ : \text{SU}(2)^3 \rightarrow [0, 1]$ be smooth invariant functions, equal to one on the tubes of radius r_c over K_0 and K_+ respectively, with ρ_0 supported in the interior of $\{\psi = 0\}$ and ρ_+ supported inside $\{\psi > b_*\}$. The latter set has room for such a choice because $\psi_0 \geq 2b_*$ on O_+ and $\chi = 1$ on the normal ball of radius $2r_c$. In finite coordinates such functions are obtained from a smooth function of the squared distance in finitely many small balls, by adding these bumps and truncating smoothly to one on their cores. Averaging over simultaneous conjugation with respect to the Haar probability measure makes them invariant; since the relevant sets are invariant, the support and core properties are preserved. Both functions are Lipschitz for the matrix distance. By Lemma 35.15, there exists $a' > 0$ such that $\mathbb{E}_{\lambda_\gamma} \rho_0 \geq a'$ and $\mathbb{E}_{\lambda_\gamma} \rho_+ \geq a'$ for all sufficiently small γ . By the support properties, $\lambda_\gamma\{B_{\text{obs}} = 0\} \geq a'$ and $\lambda_\gamma\{B_{\text{obs}} \geq b_*\} \geq a'$. Since $0 \leq B_{\text{obs}} \leq 1$, Lemma 35.12 implies $\mathbb{E}_{\lambda_\gamma} B_{\text{obs}}^2 \geq a' b_*^2$ and $\text{Var}_{\lambda_\gamma} B_{\text{obs}} \geq (a')^2 b_*^2$, which proves the first assertion.

Step 3: the Gaussian moment polynomials. On any bounded set of nonnegative 4×4 matrices we have

$$|\mathfrak{m}_1(C) - 2| + |\mathfrak{m}_2(C) - 6| \leq C' \|C - I\|.$$

Indeed, $|\text{tr}(C - I)| \leq 4\|C - I\|$, $|\text{tr} C^2 - 4| = |\text{tr}((C - I)(C + I))| \leq 4\|C - I\|(\|C\| + 1)$, and $(\text{tr} C)^2 - 16 = (\text{tr} C - 4)(\text{tr} C + 4)$. By (35.16), $0 \leq \Sigma_m^K \leq \Sigma_\theta^N$, and together with

the spectral bounds of Theorem 33.11 this bounds the norms of all $\tilde{\Sigma}_m^K$ by a constant independent of m . Then (35.14) compares the moment polynomials at finite K directly with $m_1(I) = 2$ and $m_2(I) = 6$.

Step 4: the normalized moments at finite cutoff. We apply part (2) of Theorem 35.11 with the tangential weights ψ_0 and ψ_0^2 and the normal cutoffs χ and χ^2 , so that $\tilde{w} = \psi$ or ψ^2 . On its right side, w^+ and $\tilde{w}$ differ only at a fixed positive normal radius. The normal moment bounds of Step 8 of its proof, Markov's inequality, and the boundedness of $m_l(\tilde{\Sigma}_m^K)$ show that replacing w^+ by $\tilde{w}$ changes the expectation by $O_{m,q}(\gamma^q)$ for every fixed q ; the polynomial comparison in that theorem is supplied by Lemma 35.8. We take $K_0(m) \geq \max\{m, K_0(0)\}$, as allowed. For $f = \psi$ and $f = \psi^2$ and $l = 1, 2$, we compare $\gamma^{-l}\mathbb{E}_\nu[fq_*^l(\mathcal{H}^K)]$ with $\gamma^{-l}\mathbb{E}_\nu[fq_*^l(\Xi_m^K)]$ by (35.30), then with $\mathbb{E}_\nu[f(\Xi_m^K)m_l(\tilde{\Sigma}_m^K(y'))]$ by part (2), then with $m_l(I)\mathbb{E}_\nu f(\Xi_m^K)$ by Step 3 and (35.14), and finally with $m_l(I)\mathbb{E}_\nu f(\mathcal{H}^K)$ by the case $l = 0$ of Lemma 35.13, which costs $C\mathbb{E}d_m \leq C\sqrt{\gamma}$. We obtain

$$\begin{aligned} & |\gamma^{-l}\mathbb{E}_\nu[fq_*^l(\mathcal{H}^K)] - m_l(I)\mathbb{E}_\nu[f(\mathcal{H}^K)]| \\ & \leq C\{\delta_m^{1/2} + \delta_m + (nB^K)^{-2}\} + e_m(\gamma), \quad K \geq K_0(m), \end{aligned} \quad (35.32)$$

where $e_m(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$ for each fixed m ; one may take for e_m a finite sum of terms $C_m\sqrt{\gamma}(1 + |\log \gamma|)^{k_m}$, $C\sqrt{\gamma}(1 + |\log \gamma|)^k$, and $C_m\gamma$. The constants in front of $\delta_m^{1/2}$, δ_m , and the lattice error do not depend on m .

Step 5: the limits. We apply Lemma 35.14 to each bounded product fq_*^l and to f , before dividing by γ^l . For a fixed $q > 2$, the cost of replacing ν by $\mu_{N,\gamma}$ is $O(\gamma^{q-1}) = o(1)$ after division by γ^l , for every $l \leq 2$. Then (35.32) holds for $\mu_{N,\gamma}$, with this additional error. At each fixed γ , Lemma 35.14 gives the limits as $K \rightarrow \infty$ of these expectations under $\mu_{N,\gamma}$, which are the corresponding expectations under λ_γ . When $K \rightarrow \infty$, the lattice error disappears, without any division by γ . Next we let $\gamma \downarrow 0$ at the fixed m , which gives

$$\limsup_{\gamma \downarrow 0} |\gamma^{-l}\mathbb{E}_{\lambda_\gamma}[fq_*^l] - m_l(I)\mathbb{E}_{\lambda_\gamma}f| \leq C(\delta_m^{1/2} + \delta_m).$$

Letting $m \rightarrow \infty$, we obtain the two moment identities. The side n and the sequence nB^K are fixed throughout; only finitely many initial terms of the sequence may be discarded in the inner limit $K \rightarrow \infty$, as allowed by Corollary 29.26.

Step 6: the statistic. With $A_{\text{obs}} = \psi q_*$ and $B_{\text{obs}} = \psi$, the three errors of Lemma 35.12 are

$$e_1 = \gamma^{-1}\mathbb{E}[\psi q_*] - 2\mathbb{E}\psi, \quad e_2 = \gamma^{-1}\mathbb{E}[\psi^2 q_*] - 2\mathbb{E}\psi^2, \quad e_3 = \gamma^{-2}\mathbb{E}[\psi^2 q_*^2] - 6\mathbb{E}\psi^2,$$

with all expectations under λ_γ . By Step 5 they tend to zero. Combining the lower bounds of Step 2 with (35.27), we obtain $\mathcal{C}_\gamma = 2\gamma^2 + o(\gamma^2)$. Further, Step 5 shows that at each fixed m the upper limits of $|e_1|, |e_2|, |e_3|$ are at most $C(\delta_m^{1/2} + \delta_m)$, and the exact identity

(35.27) then implies the stated bound on $\mathcal{C}_\gamma/\gamma^2 - 2$, with a constant independent of m . The quadratic errors e_1^2 and $(e_2 - ae_1)^2$ are absorbed using that δ_m is bounded in m and that $|a| \leq 1$. $\square$

35.7 A weighted chi-square limit with four degrees of freedom

Proof of Theorem 35.3. Only the values of q_* on the support of $\tilde{w}$ enter, where q_* is the exact nonnegative quadratic form; the extension of q_* has no effect on π_γ . We first prove a finite-scale version of the statement for bounded Lipschitz f (Steps 1–3) and replace the finite-scale triple by the triple of the fine configuration (Step 4). We then prove the lower bound on D_γ (Step 5), pass to the continuum and take the limits (Step 6), and finally extend the result to bounded continuous f and identify the limit law (Step 7).

Step 1: the finite-scale expansion. Fix m , let $K \geq K_0(m) \geq \max\{m, K_0(0)\}$, let γ be below the threshold of Theorem 35.11, and write $H_m = \Xi_m^K$ and $a_f = \mathbb{E}f(|G_4|^2/2)$. We choose a larger coarse regular tube before m : its flat holonomies cover a fixed larger regular patch and are separated from the compact support of $\tilde{w}$ on the complementary part of the set of flat configurations. The modulus of Theorem 35.9 has an input margin depending on m , which changes the coupling threshold but not this tube. By part (5) of Theorem 35.11, there exists a larger coarse regular tube $\mathcal{T}^+$ such that an output in the support of $\tilde{w}$ coming from outside $\mathcal{T}^+$ has probability $O_{m,q}(\gamma^q)$ for each fixed $q > 0$. Inside the coarse tube we use the restriction of ν normalized by its own mass, and write y for the tangential coordinate of the coarse field. Part (1) of the same theorem, with its bounded output coordinates, gives

$$b(H_m)/\sqrt{\gamma} = Z_{K,m} + \mathcal{R}_{K,m}, \quad \|\mathcal{R}_{K,m}\|_{L^r} \leq C_{m,r}\sqrt{\gamma}(1 + |\log \gamma|)^2,$$

and $\|y(H_m) - y\|_{L^r} \leq C_{m,r}\sqrt{\gamma}(1 + |\log \gamma|)$ for every fixed finite r , where the recentering by the conditional means is included in $\mathcal{R}_{K,m}$. These formulas are local coordinate identities only on the good event G_m of Step 4 of that proof, whose complement has probability $O_{m,q}(\gamma^q)$ for every fixed $q > 0$ by (35.24). By Lemma 35.8, all required moments of $Z_{K,m}$ are bounded at this fixed m .

Step 2: comparison with a Gaussian quadratic form. The ambient inverse covariance $P_{\mathcal{N}_y}\Sigma_\theta(y)^{-1}P_{\mathcal{N}_y}$ is smooth on the larger patch by Theorem 33.11. Expanding the quadratic form and using Hölder's inequality, we obtain on the good event

$$\left\| \frac{q_*(H_m)}{\gamma} - \frac{1}{2} Z_{K,m}^\top \Sigma_\theta(y)^{-1} Z_{K,m} \right\|_{L^1} \leq C_m \sqrt{\gamma} (1 + |\log \gamma|)^{k_m}.$$

Explicitly, the difference is bounded by a constant times $|\mathcal{R}|(|Z| + |\mathcal{R}|) + |y(H_m) - y|(|Z| + |\mathcal{R}|)^2$, and moments of order at most six suffice. Since f is Lipschitz, the same

bound times $\text{Lip } f$ controls the change of $f(q_*(H_m)/\gamma)$. The smooth weight $w(y(H_m))$ can be moved to $w(y)$, and back, with an error of the same order. The normal cutoff χ_w differs from one only at a fixed positive normal radius, so arbitrarily high fixed moments of the normal coordinate of H_m (Step 8 of the proof of Theorem 35.11) make the effect of replacing χ_w by one $O_{m,q}(\gamma^q)$. On the exceptional events we use only that f and the weight are bounded, without taking a logarithm there. These comparisons hold after multiplication by the mass of the coarse tube, which is at most one.

Step 3: the Gaussian value. Conditionally on y , we apply Lemma 35.8 to the bounded Borel function $z \mapsto f(z^\top \Sigma_\theta(y)^{-1} z/2)$. This gives the Gaussian value

$$\kappa_f(\tilde{\Sigma}_m^K(y)) = \mathbb{E} f\left(\frac{1}{2} G_4^\top \tilde{\Sigma}_m^K(y) G_4\right),$$

with error $C_m \|f\|_\infty \sqrt{\gamma}(1 + |\log \gamma|)$. The equality of the laws of the two Gaussian quadratic forms follows by diagonalizing the nonnegative covariance matrix; no inverse of Σ_m^K is used. Coupling the two quadratic forms through the same G_4 , we obtain

$$|\kappa_f(C) - a_f| \leq \frac{1}{2} \text{Lip } f \|C - I\| \mathbb{E}|G_4|^2 = 2 \text{Lip } f \|C - I\|.$$

Integrating against the tangential law of the coarse field and using (35.14), we obtain

$$|\mathbb{E}_\nu[\tilde{w}(H_m)f(q_*(H_m)/\gamma)] - a_f \mathbb{E}_\nu \tilde{w}(H_m)| \leq C_w \|f\|_{\text{BL}} \{\delta_m + (nB^K)^{-2}\} + \|f\|_{\text{BL}} e_{m,w}(\gamma), \quad (35.33)$$

with $e_{m,w}(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$. One may take for $e_{m,w}$ a finite sum of terms $C_{m,w} \sqrt{\gamma}(1 + |\log \gamma|)^{k_m}$ and of arbitrarily high fixed powers of γ . The constant C_w does not depend on m : only the bound on the weight and the uniform covariance error enter it.

Step 4: comparison with the triple of the fine configuration. We now replace H_m by $H = \mathcal{H}^K$ in the bounded test $F_\gamma(H) = \tilde{w}(H)f(q_*(H)/\gamma)$. This needs a separate estimate, because the Lipschitz constant of $f(q_*/\gamma)$ grows as $\gamma \downarrow 0$. We use the nested tubes of Lemma 35.13 and set

$$d_m = \min\left(1, \max_i \|H_i - H_{m,i}\|_{\text{op}}\right), \quad \beta_m = |b(\text{Log } H_m)| \mathbf{1}_{\{H_m \in \mathcal{U}^+\}}.$$

The normal coordinate is used only on the indicated tube. For a fixed $0 < \eta < 1$, if $d_m \leq \eta$ and at least one of the two weights is nonzero, then both triples lie in $\mathcal{U}^+$ and their logarithmic segment lies in $\mathcal{U}^{++}$. On this event the smooth coordinates and the Lipschitz bound on Σ_θ^{-1} give

$$|q_*(H) - q_*(H_m)| \leq C(\beta_m + d_m)d_m;$$

indeed we expand the two quadratic forms and use $|b(\text{Log } H)| \leq |b(\text{Log } H_m)| + Cd_m$. Therefore, on this event,

$$|F_\gamma(H) - F_\gamma(H_m)| \leq C_w \|f\|_\infty d_m + C_w \text{Lip } f \gamma^{-1}(\beta_m + d_m)d_m.$$

If both weights vanish, the difference is zero. On $\{d_m > \eta\}$ the boundedness of F_γ gives the bound $C_w \|f\|_\infty \eta^{-4} d_m^4$.

We bound the second normal moment with a constant independent of m . We compare H_m with $H_0 = \Xi_0^K$ and let $d_0 = \min(1, \max_i \|H_i - H_{0,i}\|_{\text{op}})$; the truncated distance between H_m and H_0 is at most $d_m + d_0$. When this distance is small and $H_m \in \mathcal{U}^+$, H_0 lies in a fixed larger regular tube $\mathcal{V}$ and $\beta_m \leq |b(\text{Log } H_0)| + C(d_m + d_0)$. When the distance exceeds a fixed margin, the boundedness of the normal radius gives $\beta_m \leq C(d_m + d_0)$. Combining the two cases, we obtain

$$\mathbb{E}_\nu \beta_m^2 \leq C \mathbb{E}_\nu [|b(\text{Log } H_0)|^2 \mathbf{1}_{\{H_0 \in \mathcal{V}\}}] + C \mathbb{E}_\nu (d_m + d_0)^2 \leq C\gamma,$$

where we used part (3) of Theorem 35.11 with $m = 0$ and the moment bounds of Theorem 34.10, which hold for all m with the same constants. The tube $\mathcal{V}$ is fixed before m . By the Cauchy–Schwarz inequality and Theorem 34.10, we have, uniformly for $K \geq K_0(m)$,

$$|\mathbb{E}_\nu F_\gamma(H) - \mathbb{E}_\nu F_\gamma(H_m)| \leq C_w \|f\|_{\text{BL}} \{ \delta_m^{1/2} + \sqrt{\gamma}(1 + |\log \gamma|)^k + \gamma^2 \}. \quad (35.34)$$

Indeed $(\mathbb{E}(\beta_m + d_m)^2)^{1/2} \leq C\sqrt{\gamma}$ and $(\mathbb{E}d_m^2)^{1/2} \leq C\sqrt{\gamma} \delta_m^{1/2} + C\gamma(1 + |\log \gamma|)^k$, so the factor γ^{-1} is compensated by the size of the quadratic form; the fourth moment on the far event is at most $C\gamma^2$. For the weight alone, its global Lipschitz continuity gives the simpler bound $|\mathbb{E}\tilde{w}(H) - \mathbb{E}\tilde{w}(H_m)| \leq C_w \sqrt{\gamma}$.

Step 5: the lower bound on D_γ . Since w is continuous, nonnegative, and not identically zero, there exist a compact invariant set K_1 with nonempty relative interior in the relative interior of P and $c_1 > 0$ with $w \geq c_1$ on K_1 ; let $r_c > 0$ be such that $\chi_w = 1$ on $\{|b| \leq r_c\}$. Then $\tilde{w} \geq c_1$ on the corresponding tube $\mathcal{T}_c$, and $\tilde{w}$ is smooth, invariant, and Lipschitz for the matrix distance. By Lemma 35.15, $\mathbb{E}_\nu \tilde{w}(H) \geq c'_w$ uniformly in K and $D_\gamma = \mathbb{E}_{\lambda_\gamma} \tilde{w} \geq c_w > 0$ for all sufficiently small γ .

Step 6: the continuum and the limits. For each fixed $\gamma > 0$, F_γ is a bounded continuous invariant function on the compact space of triples, with its fixed zero extension, and its supremum $\|\tilde{w}\|_\infty \|f\|_\infty$ does not depend on γ . By Lemma 35.14, the expectations of $F_\gamma(\mathcal{H}^K)$ under ν and under $\mu_{N,\gamma}$ differ by at most $C_{w,q} \|f\|_\infty \gamma^q$, whereas the corresponding expectations of $\tilde{w}(\mathcal{H}^K)$ differ by at most $C_{w,q} \gamma^q$, for any fixed $q > 0$. Both expectations under $\mu_{N,\gamma}$ converge to their expectations under λ_γ as $K \rightarrow \infty$. Writing $a_f = \mathbb{E}f(|G_4|^2/2)$, we have $|a_f| \leq \|f\|_\infty$, so the transfer error for the combination $\mathbb{E}F_\gamma - a_f \mathbb{E}\tilde{w}$ is at most $C_{w,q} \|f\|_\infty \gamma^q$. This uses only the supremum of the test, never its Lipschitz constant, so no negative power of γ appears. Combining (35.33) and (35.34) with the bound for the weight alone, letting $K \rightarrow \infty$ at fixed γ and m , and dividing by $D_\gamma \geq c_w$, we obtain the quantitative bound in the statement, after absorbing the terms from Step 4 and from the conditioning into $e_{m,w}$. The lattice error vanishes in this first limit. The estimates

at finite cutoff were needed only for $K \geq K_0(m)$, which suffices for this limit. Letting $\gamma \downarrow 0$ at fixed m , and then $m \rightarrow \infty$, we obtain $\int f d\pi_\gamma \rightarrow \mathbb{E}f(|G_4|^2/2)$ for every bounded Lipschitz f .

Step 7: bounded continuous tests and the limit law. We extend the convergence to bounded continuous f by tightness, without using convergence of moments. Let η_R be zero on $[0, R]$, linear on $[R, R+1]$, and one on $[R+1, \infty)$. It is bounded and Lipschitz, so the convergence already proved gives

$$\limsup_{\gamma \downarrow 0} \pi_\gamma([R+1, \infty)) \leq \mathbb{E}\eta_R(|G_4|^2/2) \leq \mathbb{P}\{|G_4|^2/2 > R\} \rightarrow 0 \quad (R \rightarrow \infty).$$

On each compact interval a bounded continuous f has uniformly accurate piecewise linear approximations, which we extend by constants outside the interval; they are bounded Lipschitz functions. The two tail probabilities bound the error outside the interval by a fixed multiple of $\|f\|_\infty$. Increasing first the interval and then decreasing the approximation error, we obtain the convergence for f . Finally, each of $(G_1^2 + G_2^2)/2$ and $(G_3^2 + G_4^2)/2$ has density $e^{-u}\mathbf{1}_{\{u \geq 0\}}$, by polar coordinates in the plane. They are independent, and their convolution has density $ue^{-u}\mathbf{1}_{\{u \geq 0\}}$, which identifies the law of $|G_4|^2/2$ and completes the proof. $\square$

35.8 Gaussian approximation on the regular normal bundle

Proof of Theorem 35.4. We first verify the geometric and measurability statements (Step 1) and extend a Lipschitz test to a larger bundle (Step 2). We then prove the quantitative bound for bounded Lipschitz F at finite cutoff, first for the finite-scale triple (Steps 3–5) and then for the triple of the fine configuration (Step 6), and pass to the continuum (Step 7). Finally we extend the result to bounded continuous F (Step 8).

Step 1: geometry of the quotient. All vector comparisons below use the Euclidean embedding of the normal spaces in $\mathbb{R}^9$; no global frame is introduced. On a finite smooth orthonormal bundle atlas, Σ_θ and its inverse square root are smooth with bounded derivatives, by Theorems 33.11 and 35.10, and they are equivariant under simultaneous conjugation, as are the normal coordinates. For $g \in \mathrm{SU}(2)$ the isometry $\mathrm{Ad}(g) : \mathcal{N}_y \rightarrow \mathcal{N}_{\mathrm{Ad}(g)y}$ therefore carries Γ_y to $\Gamma_{\mathrm{Ad}(g)y}$, and the invariance of F gives $\bar{F}(\mathrm{Ad}(g)y) = \bar{F}(y)$. The quotient of the compact invariant patch P' is compact and metrizable, for example by

$$d_{\mathrm{orb}}([y], [y']) = \min_{g \in \mathrm{SU}(2)} |y - \mathrm{Ad}(g)y'|.$$

Since the group is compact, the minimum is attained and zero distance identifies exactly one orbit; the isometric action gives the triangle inequality. Its topology is the quotient topology, since the continuous bijection from the compact quotient to this metric space is

a homeomorphism. On the compact support of $\tilde{w}$, the tangential projection followed by the quotient map is continuous and invariant. By Lemma 5.6, the seven traces separate the classes of simultaneous conjugacy; their map on this compact support is a continuous surjection onto its Hausdorff image, hence a quotient map, and the tangential class factors continuously through this image. In particular $[y_\gamma]$ is a Borel function on the support of $\tilde{w}$; elsewhere we may assign any fixed point of the quotient, since the weight vanishes there. Thus ρ_γ is a probability measure on the Borel quotient, and the identity $D_\gamma^{-1} \mathbb{E}[\tilde{w} \bar{F}(y_\gamma)] = \int \bar{F} d\rho_\gamma$ holds and does not depend on the choice of representatives. For the regularity of the fiber integral, let $E_y : \mathbb{R}^4 \rightarrow \mathcal{N}_y$ be a smooth orthonormal frame in one chart and write $\bar{F}(y) = \mathbb{E}F(y, E_y G_4)$. The bounded derivative of the frame gives

$$|\bar{F}(y) - \bar{F}(y')| \leq C \text{Lip } F (1 + \mathbb{E}|G_4|) |y - y'|,$$

and a finite cover with positive interior margins gives the same bound on the compact patch. Hence $\bar{F}$ is bounded Lipschitz with $\|\bar{F}\|_{\text{BL}} \leq C\|F\|_{\text{BL}}$. The same formula and dominated convergence show that $\bar{F}$ is continuous when F is only continuous.

Step 2: extension of a Lipschitz test. Let F be bounded Lipschitz. In the proof the test is evaluated at the tangential coordinate y of the coarse field and at triples where the weight may vanish, and these tangential coordinates may lie outside P' . We therefore first extend F . Let $\hat{F}(x) = \inf_s \{F(s) + \text{Lip } F |x - s|\}$, the infimum over s in the normal bundle over P' , be the McShane extension to $\mathbb{R}^9 \times \mathbb{R}^9$; it has the same Lipschitz constant. Truncating it at $\pm\|F\|_\infty$ preserves the Lipschitz constant and gives the bound $\|F\|_\infty$. Averaging the result with respect to the Haar probability measure over the diagonal action $(y, z) \mapsto (\text{Ad}(g)y, \text{Ad}(g)z)$, which acts by isometries of $\mathbb{R}^9 \times \mathbb{R}^9$, preserves the bound and the Lipschitz constant and gives an invariant function. Since F is invariant and the bundle over P' is invariant, the averaged function agrees with F there. We denote this extension, restricted to the normal bundle over P^{++} , again by F , and define $\bar{F}$ by the same formula on P^{++} ; Step 1 applies on P^{++} as well. Since $\tilde{w}$ vanishes unless the tangential coordinate lies in P' , the quantities in the theorem are unchanged.

Step 3: the finite-scale expansion. We work under ν , with $H = \mathcal{H}^K$ and $H_m = \Xi_m^K$, the triple of the finite-scale reconstruction from the first m steps on the same sample. All estimates are for $K \geq K_0(m)$, as allowed by Theorem 35.11. Under the restriction of ν to its larger coarse regular tube, normalized by its own mass, let y be the tangential coordinate of the coarse field. That theorem and Lemma 35.8 give

$$\begin{aligned} \gamma^{-1/2} b(H_m) &= Z_{K,m} + \mathcal{R}_{K,m}, & \|\mathcal{R}_{K,m}\|_{L^r} &\leq C_{m,r} \sqrt{\gamma} (1 + |\log \gamma|)^2, \\ \|y(H_m) - y\|_{L^r} &\leq C_{m,r} \sqrt{\gamma} (1 + |\log \gamma|), & Z_{K,m} &= A_{0,K} z_0 + \sum_{j=1}^m A_{j,K} (w_j - \bar{w}_j), \end{aligned}$$

where $\bar{w}_j$ is the conditional mean of w_j under the conditional fluctuation kernel and the error of the recentering is included in $\mathcal{R}_{K,m}$. All fixed moments of $Z_{K,m}$ are bounded uniformly in K at this m . These are local coordinate identities only on the output tube, with the bounded extensions of part (1) of Theorem 35.11. By part (5) of that theorem, outputs in the support of the weight coming from outside the larger coarse tube have probability at most $C_{m,q}\gamma^q$ for every fixed $q > 0$; inside, the exceptional event (35.24) has the same bound. Bounded tests on these exceptional events therefore contribute only $C_{m,q}\|F\|_\infty\gamma^q$.

Step 4: comparison of the normalized normal vectors. On the good event, using the bounded derivatives of the inverse square root, Hölder's inequality, and the preceding moments, we obtain

$$\mathbb{E}|\gamma^{-1/2}\Sigma_\theta(y(H_m))^{-1/2}b(H_m) - \Sigma_\theta(y)^{-1/2}Z_{K,m}| \leq C_m\sqrt{\gamma}(1 + |\log \gamma|)^{k_m};$$

indeed the difference is bounded by $C|\mathcal{R}_{K,m}| + C|y(H_m) - y||Z_{K,m}|$. Since F is Lipschitz in both arguments, $F(y(H_m), z(H_m))$ may be replaced by $F(y, \Sigma_\theta(y)^{-1/2}Z_{K,m})$ with an error $\text{Lip } F$ times this bound plus $\text{Lip } F \mathbb{E}|y(H_m) - y|$. Replacing $w(y(H_m))$ by $w(y)$ gives an error of the same type. The normal cutoff χ_w differs from one only at a fixed positive distance from zero, and arbitrarily high fixed normal moments from the same theorem bound the effect of replacing it by one by $C_{m,q}\gamma^q$.

Step 5: the Gaussian value. Conditionally on y , we apply Lemma 35.8 to the function $v \mapsto F(y, \Sigma_\theta(y)^{-1/2}v)$. Since it is a jointly Borel bounded function of the whole vector, the lemma applies without any radial restriction, and the Gaussian value is $\mathbb{E}F(y, (\tilde{\Sigma}_m^K(y))^{1/2}G_y)$ with $G_y \sim \Gamma_y$. For every nonnegative self-adjoint C we have

$$\|C^{1/2} - I\|_{\text{op}} \leq \|C - I\|_{\text{op}}, \quad |\mathbb{E}F(y, C^{1/2}G_y) - \bar{F}(y)| \leq 2\text{Lip } F \|C - I\|_{\text{op}}.$$

The first inequality follows on each eigenvalue from $|\sqrt{r} - 1| = |r - 1|/(\sqrt{r} + 1)$, including $r = 0$; the second uses $\mathbb{E}|G_y| \leq (\mathbb{E}|G_y|^2)^{1/2} = 2$. Moving $\bar{F}(y)$ back to $\bar{F}(y(H_m))$ with the Lipschitz bound of Step 1 and the weight back to $\tilde{w}(H_m)$, and using the covariance error (35.14), we obtain

$$\begin{aligned} & \left| \mathbb{E}_v \{ \tilde{w}(H_m) [F(y(H_m), z(H_m)) - \bar{F}(y(H_m))] \} \right| \\ & \leq C_w \|F\|_{\text{BL}} \{ \delta_m + (nB^K)^{-2} \} + \|F\|_{\text{BL}} e_{m,w}^1(\gamma), \end{aligned} \tag{35.35}$$

where $z(H') = \gamma^{-1/2}\Sigma_\theta(y(H'))^{-1/2}b(H')$ and $e_{m,w}^1(\gamma) \rightarrow 0$ as $\gamma \downarrow 0$ at fixed m . Multiplying by the mass of the coarse tube, which is at most one, justifies the use of the normalized conditional estimates. The tangential integration in this step is always against the law of the coarse field, and the constant in front of the covariance error does not depend on m .

Step 6: comparison with the triple of the fine configuration. We replace H_m by H on the same sample. Let $d_m = \min(1, \max_i \|H_i - H_{m,i}\|_{\text{op}})$, and fix nested regular output tubes with positive margins. There exists $\eta > 0$, independent of m , such that if $d_m \leq \eta$ and at least one of the two weights is nonzero, both triples lie in the larger identity chart. On that event the normal coordinate, the tangential coordinate, and the map $H' \mapsto \Sigma_\theta(y(H'))^{-1/2}b(H')$ have bounded derivatives, as one sees by differentiating the product on the fixed bounded tube. Hence, for $J_\gamma(H') = \tilde{w}(H')F(y(H'), z(H'))$,

$$|J_\gamma(H) - J_\gamma(H_m)| \leq C_w \|F\|_{\text{BL}} (1 + \gamma^{-1/2}) d_m.$$

If both weights vanish, the difference is zero; on $\{d_m > \eta\}$ we use the bound $2\|\tilde{w}\|_\infty \|F\|_\infty$. By Theorem 34.10,

$$\gamma^{-1/2} \mathbb{E}_\nu d_m \leq C \delta_m^{1/2} + C \sqrt{\gamma} (1 + |\log \gamma|)^k, \quad \nu\{d_m > \eta\} \leq C_\eta \gamma^2.$$

The same argument for $\tilde{w}(H')\bar{F}(y(H'))$ involves no factor $\gamma^{-1/2}$. Combining these bounds with (35.35), we obtain the same estimate for H in place of H_m , with an additional term $C_w \|F\|_{\text{BL}} \delta_m^{1/2}$ and an additional error at fixed m that tends to zero with γ .

Step 7: the continuum and the limits. For each fixed $\gamma > 0$, both tests J_γ and $\tilde{w}\bar{F}(y(\cdot))$ are bounded continuous invariant functions on $\text{SU}(2)^3$, with the stated zero extensions. By Lemma 35.14, the expectations under ν and $\mu_{N,\gamma}$ differ by at most $C_{w,q} \|F\|_{\text{BL}} \gamma^q$ for any fixed $q > 0$, and the expectations under $\mu_{N,\gamma}$ converge to those under λ_γ as $K \rightarrow \infty$. This uses only the supremum of the tests, not their Lipschitz constants, which depend on γ . Letting $K \rightarrow \infty$ at fixed γ and m and dividing by $D_\gamma \geq c_w$ (Theorem 35.3), we obtain (35.3). Letting $\gamma \downarrow 0$ at fixed m , and then $m \rightarrow \infty$, proves (35.2) for bounded Lipschitz F .

Step 8: bounded continuous tests. Let F be bounded and continuous. On the support of the weight we have $|z_\gamma|^2/2 = q_*/\gamma$, so Theorem 35.3 shows that the localized probabilities of $\{|z_\gamma| > R\}$ satisfy $\limsup_{\gamma \downarrow 0} \pi_\gamma((R^2/2, \infty)) \rightarrow 0$ as $R \rightarrow \infty$. For the comparison integral, the inner probability $\Gamma_y\{|z| > R\} = \mathbb{P}\{|G_4| > R\}$ does not depend on y or on the representative, and integrating it against ρ_γ gives the same tail bound. The set $\{(y, z) : y \in P', |z| \leq R\}$ is compact, and a continuous function on it admits uniformly accurate bounded Lipschitz approximations. To construct them, we take a sufficiently fine finite net, form the upper and lower cones of a sufficiently large finite slope through the values on the net, and truncate at the supremum bound. The uniform continuity of F gives the approximation as the net is refined and the slope is increased, and these cone functions are defined on the whole ambient space. Averaging them over simultaneous conjugations with respect to the Haar probability measure makes them invariant, preserves their bound and their Lipschitz constant, and preserves the approximation on this invariant compact set. We apply the bounded Lipschitz case to each approximant; the two tail probabilities control the errors outside the compact set. Letting first the approximation error tend to zero and then $R \rightarrow \infty$, we obtain (35.2). $\square$

35.9 Comparison probability measures

This subsection evaluates the observables of Definition 35.1 under the four comparison probability measures of Definition 4.13: the Abelian Maxwell measure with Haar cycle holonomies (item (1) there, constructed in Lemma 33.14), Gaussian holonomies with independent Haar factors (item (2)), Gaussian holonomies with Gaussian constant modes (item (3)), and the conditional normal Gaussian measure (item (4)). These measures are different constructions, and their hypotheses are part of their definitions. The comparisons show how to read the conclusion of Theorem 35.2. The conditional normal Gaussian measure has the same coefficient $2\gamma^2$. The two holonomy models without a non-Abelian flat background give a vanishing statistic or a statistic of order one, and the Gaussian holonomies with Gaussian constant modes give the regular patch an exponentially small mass.

For the third measure we use Gaussian concentration. Its classical antecedents are the Gaussian isoperimetric inequalities of Borell [30] and of Sudakov and Tsirelson [148], and the sample-function formulation of Cirel'son, Ibragimov, and Sudakov [60]; we use the formulation of Adler and Taylor [3].

Lemma 35.16. *Let $d \geq 1$, let G be a standard Gaussian vector in $\mathbb{R}^d$, and let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be Lipschitz with constant ℓ for the Euclidean distance. Then for every $\lambda \in \mathbb{R}$,*

$$\log \mathbb{E} e^{\lambda(F(G) - \mathbb{E}F(G))} \leq \lambda^2 \ell^2 / 2.$$

Let $\mathcal{T}$ be a nonempty compact metric space and $(Z_t)_{t \in \mathcal{T}}$ a centered real Gaussian process, whose finite-dimensional covariances may be degenerate, with a separable realization having almost surely continuous paths, and assume $\sigma_Z^2 = \sup_{t \in \mathcal{T}} \mathbb{E} Z_t^2 < \infty$. For any fixed countable dense set $\mathcal{T}_0 \subset \mathcal{T}$, the random variable $S_Z = \sup_{t \in \mathcal{T}} Z_t = \sup_{t \in \mathcal{T}_0} Z_t$ is measurable, almost surely finite, and integrable. If $\sigma_Z^2 > 0$, then, writing $m_Z = \mathbb{E} S_Z$,

$$\log \mathbb{E} e^{\lambda(S_Z - m_Z)} \leq \lambda^2 \sigma_Z^2 / 2 \quad (\lambda \in \mathbb{R}), \quad \mathbb{P}\{S_Z - m_Z \geq u\} \leq e^{-u^2 / (2\sigma_Z^2)} \quad (u > 0).$$

If $\sigma_Z^2 = 0$, the process and its supremum vanish almost surely. In particular this applies to $\mathcal{T} = \mathbb{T}_L^3 \times \{1, 2, 3\} \times S^2$ and $Z_{x,i,v} = \langle v, G_i(x) \rangle$ for a continuous centered Gaussian field $G = (G_i)$ with this realization; its supremum is $\max_i \sup_x |G_i(x)|$. The only constants are the displayed Lipschitz constant and supremum variance; no dimension, cardinality of the index set, or parameter of the Yang–Mills measure enters.

Proof. We use two statements of Adler and Taylor. By [3, Theorem 2.1.1, p. 50, (2.1.5)], the supremum of an almost surely bounded centered Gaussian process has finite mean and satisfies an upper tail bound. By [3, Lemma 2.1.5, p. 54, (2.1.13)], with the boundedness assumption added in the authors' correction list (entry 54-6), we have $\mathbb{E} e^{th(G)} \leq e^{t^2/2}$ for $t > 0$, every bounded C^2 function h with $\text{Lip}(h) \leq 1$ and $\mathbb{E} h(G) = 0$, and a vector G of

independent standard real Gaussian variables. The reference to Lemma 2.1.4 specifies this law of G .

The case $\lambda = 0$ is immediate, and if $\ell = 0$ then F is constant. Suppose $\ell > 0$ and $\lambda \neq 0$. Let $F_R = \max(-R, \min(F, R))$ and let $F_{R,\varepsilon}$ be its convolution with a nonnegative smooth probability kernel supported in a ball of radius ε . Then $F_{R,\varepsilon}$ is bounded and smooth with bounded derivatives, $\text{Lip}(F_{R,\varepsilon}) \leq \ell$, and $|F_{R,\varepsilon} - F_R| \leq \ell\varepsilon$. Applying the corrected lemma to $h = \pm(F_{R,\varepsilon} - \mathbb{E}F_{R,\varepsilon}(G))/\ell$, with $t = |\lambda|\ell$ and the sign of λ , gives the asserted exponential moment bound for $F_{R,\varepsilon}$. At fixed R , uniform convergence as $\varepsilon \downarrow 0$ gives the bound for F_R ; the centered functions differ by at most $2\ell\varepsilon$. Next $F_R \rightarrow F$ pointwise as $R \rightarrow \infty$ and $|F_R| \leq |F|$. The bound $|F(g)| \leq |F(0)| + \ell|g|$ implies integrability of $F(G)$ and every $e^{c|F(G)|}$, since $c\ell|g| \leq |g|^2/4 + c^2\ell^2$. In particular,

$$e^{\lambda(F_R(G) - \mathbb{E}F_R(G))} \leq e^{|\lambda|(|F(G)| + \mathbb{E}|F(G)|)},$$

an integrable bound independent of R . Dominated convergence for the expectations and centered exponentials proves the first assertion.

For the process, we set all values equal to zero on the exceptional null set where the paths are not continuous; this does not change the finite-dimensional laws. Continuity on the compact metric space identifies S_Z with the supremum over $\mathcal{T}_0$ and makes it measurable and almost surely finite. These properties, the centered Gaussian finite-dimensional laws, and separability are the hypotheses of the cited theorem, which gives a finite mean. Since $S_Z \geq Z_{t_1}$ for any fixed t_1 , the negative part of S_Z is bounded by $|Z_{t_1}|$, so S_Z is integrable. We enumerate $\mathcal{T}_0 = \{t_1, t_2, \dots\}$, repeating points if it is finite, and let $M_N = \max_{i \leq N} Z_{t_i}$. Let C_N be the positive semidefinite covariance of $(Z_{t_i})_{i \leq N}$, take a square root $C_N^{1/2}$, and denote its rows by v_i . Then $(Z_{t_i})_{i \leq N}$ has the law of $(v_i \cdot G)_{i \leq N}$, since both characteristic functions equal $\exp(-\xi^T C_N \xi / 2)$. Since $|v_i|^2 = (C_N)_{ii} \leq \sigma_Z^2$, the map $g \mapsto \max_i v_i \cdot g$ is σ_Z -Lipschitz, and the first assertion gives its centered exponential moment bound uniformly in N ; no inverse covariance is used, so singular covariances are included. Since $M_N - Z_{t_1} \uparrow S_Z - Z_{t_1} \geq 0$, we have $\mathbb{E}M_N \rightarrow m_Z$ by monotone convergence, and Fatou's lemma yields, for each $\lambda \in \mathbb{R}$,

$$\mathbb{E}e^{\lambda(S_Z - m_Z)} \leq \liminf_{N \rightarrow \infty} \mathbb{E}e^{\lambda(M_N - \mathbb{E}M_N)} \leq e^{\lambda^2 \sigma_Z^2 / 2}.$$

If $\sigma_Z > 0$, Markov's inequality with $\lambda = u/\sigma_Z^2$ gives the stated upper tail. If $\sigma_Z = 0$, all coordinates on $\mathcal{T}_0$ vanish on one event of probability one, and continuity extends this to every index. Finally, the directional index space $\mathbb{T}_L^3 \times \{1, 2, 3\} \times S^2$ is compact, and $\sup_{|v|=1} \langle v, G_i(x) \rangle = |G_i(x)|$. $\square$

Lemma 35.17. *Let ψ , q_* , A_{obs} , and B_{obs} be as in Definition 35.1, evaluated at the coordinate-cycle holonomy triple H at the fixed observation time, under each of the following probability measures of Definition 4.13.*

1. Under the Abelian Maxwell measure with Haar cycle holonomies, all three holonomies commute, $A_{\text{obs}} = 0$, the statistic vanishes exactly, $\mathcal{C}_{\text{Ab}} = 0$, and $\mathbb{E}B_{\text{obs}}^2$, $\text{Var } B_{\text{obs}} \geq c_{\text{Ab}} > 0$, independently of the coupling.
2. Under Gaussian holonomies with independent Haar factors, the triple H has the law $H^{\otimes 3}$ for every coupling, and its statistic is the fixed number $\mathcal{C}(H^{\otimes 3}; A_{\text{obs}}, B_{\text{obs}}) > 0$. More precisely, let K_1 be a compact tangential subset of the regular patch P with positive relative volume and $\psi_0 \geq b_* > 0$ on K_1 , let $r_c > 0$ with $\chi(b) = 1$ whenever $|b| \leq r_c$, and put $\mathcal{T}_c = \{y + b : y \in K_1, b \in \mathcal{N}_y, |b| \leq r_c\}$, a set contained in the tube $\mathcal{U}$ on which q_* is the exact quadratic form. If P is a probability measure on $\text{SU}(2)^3$ with $\text{Var}_P B_{\text{obs}} > 0$ whose law of $X = \text{Log } H$ dominates $c_0 d^9 X$ on $\mathcal{T}_c$, then $\mathcal{C}(P; A_{\text{obs}}, B_{\text{obs}}) \geq c_0 \kappa_c > 0$, where κ_c depends only on $\mathcal{T}_c$, the Jacobian of the tube coordinates, Σ_θ^{-1} , and the cutoffs.
3. Under Gaussian holonomies with Gaussian constant modes, assume the realization of Definition 4.13(3): with $S_t a$ the smoothed field at the observation time, the centered scalar Gaussian process $\langle v, (S_t a)_i(x) / \sqrt{\gamma} \rangle$, indexed by $(x, i, v) \in \mathbb{T}_L^3 \times \{1, 2, 3\} \times S^2$, has a separable realization with continuous paths and

$$\sigma_*^2 = \sup_{x, i, |v|=1} \text{Var} \langle v, (S_t a)_i(x) / \sqrt{\gamma} \rangle < \infty.$$

Put $Z_* = \sup_{x, i, |v|=1} \langle v, (S_t a)_i(x) / \sqrt{\gamma} \rangle$ and $m_* = \mathbb{E}Z_*$. Then $m_* < \infty$ and, by Lemma 35.16, $\mathbb{P}(Z_* - m_* > u) \leq e^{-u^2/(2\sigma_*^2)}$ for $u > 0$ if $\sigma_* > 0$. Let $r_* = \inf_{\text{supp } \psi} |\text{Log } H| > 0$ and $a_* = r_*/(\sqrt{3}L)$. If $\sqrt{\gamma}m_* \leq a_*/2$ and $\sigma_* > 0$, then

$$\mathbb{P}\{\text{Log } H \in \text{supp } \psi\} \leq \exp\left\{-\frac{(a_* - \sqrt{\gamma}m_*)^2}{2\gamma\sigma_*^2}\right\} \leq e^{-a_*^2/(8\gamma\sigma_*^2)},$$

and if $\sigma_* = 0$ the regular patch has mass zero. In particular $\mathbb{E}\psi = O(e^{-c/\gamma})$. The constants may depend on the fixed covariance and smoothing of this measure, as well as on the fixed data.

4. Under the conditional normal Gaussian measure, a fixed smooth positive probability density on the regular patch, charging O_0 and O_+ , is followed, given y , by the normalized Gaussian law $b \sim N(0, \gamma\Sigma_\theta(y))$ on $\mathcal{N}_y$ and by the map $X = y + b$; a fixed tube cutoff equal to one near $b = 0$, followed by normalization, changes all moments below only by $O(e^{-c/\gamma})$. This measure has the moment identities of Theorem 35.2, with errors $O(e^{-c'/\gamma})$, and $\mathcal{C} = 2\gamma^2 + O(e^{-c'/\gamma})$.

Proof. Item (1). In the embedded Abelian theory every value of the connection lies in the same one-dimensional subalgebra, so path ordering reduces to exponentiation of the integral and the three cycle matrices commute. Their logarithms in the identity chart lie in $\mathcal{M}$, so their normal coordinate is zero and $q_* = 0$ on the tube; off the chart the observables vanish by definition. Thus $A_{\text{obs}} = 0$ everywhere for this measure. Conditionally on the

field of the Abelian Maxwell measure, the three cycle angles are translates of the three independent uniformly distributed phases on $\mathbb{R}/4\pi\mathbb{Z}$, and their joint law is therefore the Haar probability measure on this three-dimensional torus, independently of the coupling (Lemma 33.14). The invariant open sets O_0 and O_+ contain smaller open sets of angles of positive Haar measure, and the invariance of ψ_0 makes these probabilities independent of the color axis of the embedded subgroup. At $b = 0$ the normal cutoff equals one. If $a_0, a_+ > 0$ are the probabilities of these two sets of angles, then $\mathbb{P}\{B_{\text{obs}} = 0\} \geq a_0$ and $\mathbb{P}\{B_{\text{obs}} \geq b_*\} \geq a_+$, and Lemma 35.12 implies $\mathbb{E}B_{\text{obs}}^2 \geq a_+b_*^2$ and $\text{Var } B_{\text{obs}} \geq a_0a_+b_*^2$. This proves item (1).

Item (2): the law of the triple. We condition on the Gaussian field. Each factor $\text{Hol}_i(S_i a)$ is then deterministic, right multiplication of the Haar distributed U_i^0 by it preserves the Haar measure, and the three U_i^0 remain conditionally independent. For a bounded product test $\prod_i f_i(H_i)$ the conditional expectation is therefore $\prod_i \int f_i dH$, independently of the field. Taking indicators of rectangles, we find that the triple has the law $H^{\otimes 3}$ at every coupling.

Item (2): the lower bound. On $\mathcal{T}_c$ the density of $d^9 X$ in the tube coordinates, $d^9 X = J(y, b) d\text{vol}(y) db$, has a positive lower bound J_c . For $y \in K_1$ we define

$$\bar{q}_y = \frac{1}{|B_{r_c}^4|} \int_{|b| \leq r_c} q_*(y + b) db, \quad V_c(y) = \int_{|b| \leq r_c} (q_*(y + b) - \bar{q}_y)^2 db,$$

where $B_{r_c}^4$ is the ball of radius r_c in $\mathcal{N}_y$. The positive definite quadratic form $q_*(y + \cdot)$ is not constant on this ball, so $V_c(y) > 0$. Since its coefficients vary continuously and the ball is fixed in any orthonormal bundle trivialization, V_c is continuous on K_1 and has a positive minimum by compactness. On $\mathcal{T}_c$ we have $B_{\text{obs}} = \psi_0(y) \geq b_*$ and $A_{\text{obs}} = \psi_0(y)q_*$. For all real α and c , subtracting the normal average shows

$$\int_{|b| \leq r_c} (\psi_0(y)(q_*(y + b) - \alpha) - c)^2 db \geq \psi_0(y)^2 V_c(y).$$

If the law of X under P dominates $c_0 d^9 X$ on $\mathcal{T}_c$, the infimum in Lemma 35.12 is therefore at least $c_0 \kappa_c$, where one choice is

$$\kappa_c = J_c b_*^2 \int_{K_1} V_c(y) d\text{vol}(y) > 0.$$

Since $\mathbb{E}_P B_{\text{obs}}^2 \leq 1$, the same lower bound holds for the statistic, whenever its denominators are positive, as assumed. In the identity chart the density of $H^{\otimes 3}$ is $(16\pi^2)^{-3} \prod_{i=1}^3 \text{sinc}^2(|x_i|/2)$ by (4.2), which is smooth and strictly positive on $\mathcal{T}_c$ for any choice of K_1 as above, for instance a compact subset of O_+ with positive relative volume. The open sets of triples over O_0 and O_+ near $b = 0$ have positive Haar measure, so $\text{Var } B_{\text{obs}} > 0$ and $\mathbb{E}B_{\text{obs}}^2 > 0$ under $H^{\otimes 3}$, and the lower bound just proved shows that the statistic of $H^{\otimes 3}$ is a strictly positive number. This proves item (2).

Item (3). We consider the scalar process $f_{x,i,v} = \langle v, (S_t a)_i(x) / \sqrt{\gamma} \rangle$ on the nonempty compact metric space $\mathbb{T}_L^3 \times \{1, 2, 3\} \times S^2$. Since the smoothing is linear and fixed, every finite vector of this process is centered Gaussian, possibly with singular covariance. Its continuous separable realization and the finiteness of σ_*^2 are the hypotheses imposed in the statement. By Lemma 35.16, we obtain measurability, almost sure finiteness, $m_* < \infty$, and the displayed tail bound above the mean; the supremum variance is $\sup \text{Var } f_{x,i,v} = \sigma_*^2$ and the supremum is $Z_* = \max_i \sup_x |(S_t a)_i(x)| / \sqrt{\gamma}$. The covariance before smoothing is γC with C fixed, so the law of the normalized field, m_* , and σ_*^2 do not depend on γ .

We check the deterministic bound on the holonomies. Along a coordinate cycle ℓ_i , the transport equation $H' = H(S_t a)_i$ has speed $|(S_t a)_i|$ for the invariant metric with the norm $|\cdot|$ on $\mathfrak{su}(2)$. The distance of its endpoint from the identity is at most the length of this path, which is at most $L \sup_{x,j} |(S_t a)_j(x)|$, and on the identity chart this distance is $|\text{Log } H_i|$. Taking the Euclidean norm of the three components, we obtain

$$|\text{Log } H| \leq \sqrt{3} L \sup_{x,i} |(S_t a)_i(x)| = \sqrt{3} L \sqrt{\gamma} Z_*,$$

where the equality uses the supremum over all unit vectors v , which includes both signs. Thus a triple with $\text{Log } H \in \text{supp } \psi$ forces $Z_* \geq a_* / \sqrt{\gamma}$. If $\sqrt{\gamma} m_* \leq a_*/2$, this threshold lies strictly above the mean, and the tail bound of Lemma 35.16 with $u = a_* / \sqrt{\gamma} - m_* > 0$ gives both displayed exponential bounds; then $\mathbb{E} \psi \leq \mathbb{P}\{\text{Log } H \in \text{supp } \psi\}$. If $\sigma_*^2 = 0$, the zero-variance clause of Lemma 35.16 shows that the directional process and Z_* vanish almost surely; the field vanishes, the holonomies are the identity, which lies outside the regular patch, and the patch has mass zero.

Item (4). In an orthonormal frame of the normal space let G be a standard Gaussian vector in four dimensions, independent of y , and let $b = \sqrt{\gamma} \Sigma_\theta(y)^{1/2} G$. The frame may change between charts, but the conditional law and the following quadratic form on the normal bundle are intrinsic. We define, on the whole normal bundle over the patch,

$$q_{\text{coord}}(y, b) = \frac{1}{2} b^\top \Sigma_\theta(y)^{-1} b = \gamma Q_0, \quad Q_0 = |G|^2 / 2. \quad (35.36)$$

We choose $b_t > 0$ so that $\{|b| < b_t\}$ lies in a fixed smaller normal tube, where the tube coordinates are unique and the normal cutoff of ψ equals one. Only on this event do we identify $q_*(y + b) = q_{\text{coord}}(y, b)$ and $\psi(y + b) = \psi_0(y)$; outside it, q_* and ψ denote their fixed bounded extensions, not the unbounded quadratic form q_{coord} . Gaussian integration by parts in one dimension gives $\mathbb{E} G_i^2 = 1$ and $\mathbb{E} G_i^4 = 3$, and expanding the square and using independence yields

$$\mathbb{E} Q_0 = 2, \quad \mathbb{E} Q_0^2 = 6, \quad \text{Var } Q_0 = 2.$$

The compact patch has spectral bounds $cI \leq \Sigma_\theta(y) \leq CI$, by Theorem 33.11. We fix $b_1 > 0$ smaller than b_t and then every radius on which a tube cutoff equals one. All

departures from the preceding local identities, or from these cutoffs being one, occur on $\{|b| \geq b_1\}$, which requires $|G| \geq b_1/\sqrt{C\gamma}$. For every fixed r , the Gaussian polynomial tail on this event is at most $C_r e^{-c/\gamma}$: the function $|G|^r e^{-|G|^2/8}$ is bounded, and the remaining Gaussian density integrates to a finite constant after extracting $e^{-b_1^2/(8C\gamma)}$. These bounds are uniform in y . For $k = 1, 2$ and $j = 0, 1, 2$, the difference between $\psi(y+b)^k q_*(y+b)^j$ and $\psi_0(y)^k q_{\text{coord}}(y, b)^j$ vanishes on $\{|b| < b_1\}$; on its complement the first is bounded by a fixed constant and the second by $C\gamma^j |G|^{2j}$. The polynomial tail bound therefore makes the expectation of the difference exponentially small, uniformly in y . Inserting the fixed tube cutoff also changes these moments by exponentially small errors, and its total mass is $1 + O(e^{-c/\gamma})$, so dividing by this mass preserves the errors; any polynomial factor in γ^{-1} is absorbed by decreasing c .

Up to these errors, $B_{\text{obs}} = \psi_0(y)$ is independent of Q_0 . The positive tangential density charges O_0 and O_+ , and together with the normal ball near $b = 0$ this gives fixed positive lower bounds on $\mathbb{E}B_{\text{obs}}^2$ and $\text{Var } B_{\text{obs}}$ for small γ , by Lemma 35.12. For $f = \psi$ and $f = \psi^2$ we obtain

$$\mathbb{E}[fq_*] = 2\gamma\mathbb{E}f + O(e^{-c/\gamma}), \quad \mathbb{E}[fq_*^2] = 6\gamma^2\mathbb{E}f + O(e^{-c/\gamma}).$$

The identity (35.27) of Lemma 35.12 now implies $\mathcal{C} = 2\gamma^2 + O(e^{-c'/\gamma})$ for a smaller $c' > 0$. $\square$

35.10 Proof of the theorem on regular commuting triples

Proof of Theorem 2.10. Let L, M, θ , and $n \in 4\mathbb{N}$ be as in Theorem 2.10, fix J and the construction parameters as in Section 35.1, and choose $a \geq 0$ so large that $n' = nB^a$ satisfies the minimum-side conditions of Section 35.1, including $2\pi/n' < r_{\text{flat}}/4$. Since $[n']_M = [n]_M$, the measure $\lambda_\gamma = \lambda_{L, \gamma, [n]_M}$ of Theorem 2.10 is the measure of Section 35.1 for the side n' (Corollary 29.26 and Theorem 29.28), so the results of this section apply with n' in place of n . The function ψ of Theorem 2.10, with a sufficiently small tube, is a function as in Definition 35.1(2). If necessary, we replace its normal cutoff by the average over simultaneous conjugation, which does not change ψ because ψ and its tangential factor are invariant. The tangential factor ψ_0 vanishes on a nonempty open set and is positive on another, hence at least $2b_*$ on a smaller nonempty open set for some $b_* > 0$; these sets may be taken invariant since ψ_0 is invariant. On the tube, q_* is the quadratic form of Theorem 2.10. Part (1) of Theorem 2.10 is Theorem 35.2. Part (2) is Theorem 35.3, since w and χ_w satisfy the invariance and support conditions of Definition 35.1(4); these invariant normal cutoffs, like the normal cutoff χ of ψ , need not be radial. Part (3) is Theorem 35.4, applied to the restriction of F to the normal bundle over a compact invariant regular patch containing the support of w in its relative interior. The constants depend on L, M, n, θ , and the test functions, but not on γ , and in each case the continuum limit is taken first, at fixed γ . $\square$

Quartic asymptotics near central holonomies

This section describes the continuum measures λ_γ near the eight central triples $(\pm I, \pm I, \pm I)$ as $\gamma \rightarrow 0$, and proves Theorem 2.11. We observe the holonomies $H = (H_1, H_2, H_3)$ around the three coordinate cycles, based at the origin, at a fixed flow time $t = \theta L^2$. Near the identity we write their logarithms as $X \in \mathfrak{su}(2)^3 \simeq \mathbb{R}^9$. Near the trivial flat connection, the Hessian of the Wilson action vanishes in the nine constant directions, and the leading term in these directions is the constant-mode quartic form

$$Q(Y) = \frac{1}{2} \sum_{i < j} |y_i \times y_j|^2, \quad Y = (y_1, y_2, y_3) \in \mathbb{R}^9,$$

where the cross product corresponds to the Lie bracket of $\mathfrak{su}(2)$. We study central windows of radius $r_\gamma = \gamma^{1/4}$ and show that compactly supported tests of X/r_γ have leading expectations proportional to their integrals against the quartic reference measure $e^{-Q(Y)} dY$. This measure has infinite total mass, and all statements concern compactly supported tests. A continuous, nonnegative, nonzero, compactly supported invariant test has expectation comparable to r_γ . Ratios of expectations converge to ratios of integrals against $e^{-Q(Y)} dY$, with an error $O(\gamma^{1/4})$ for Lipschitz tests, and the dilation statistic tends to zero. We also treat the eight central triples, finitely many flow times, and holonomy paths on a compact time interval, and we compute the dilation statistic for four comparison measures.

The block averaging constraint and the effective classical action come from Sections 6, 9, and 10, and the flat refinements, the reconstruction maps, and the disintegration of the conditioned Wilson measure from Sections 14 and 15. The coordinates near the trivial flat connection, the transverse Hessian, and the order of the normalization constant come from Section 33, and the regularity of the flowed zero-fluctuation reconstruction and the comparison of flowed holonomies from Sections 25, 34, and 35. Finally, we use the bound on the small-field failure probability from Section 23 and the continuum limit at fixed coupling from Section 29. The covariance of Theorem 33.11 enters only in the last comparison measure. The results of this section are not used in later sections; they prove Theorem 2.11 and the further statements about central triples described after it in Section 2.

Section 36.1 fixes the setting and explains the scaling. The main steps are the exact quartic term of the effective classical action (Section 36.2), the comparison of flowed

holonomies (Section 36.3), and the integration over the transverse modes (Section 36.5). The passage to the continuum measure and the further results occupy Sections 36.6 to 36.9, and Section 36.10 deduces Theorem 2.11.

The separation of constant modes, with a quartic commutator interaction, from the nonconstant modes on a torus is an established idea in finite-volume Yang–Mills theory; see Lüscher [127] and Coste, González-Arroyo, Jurkiewicz, and Korthals Altes [62]. For a direct comparison in four-dimensional perturbation theory in finite volume, see Fodor, Holland, Kuti, Nógrádi, and Wong [91, Section 2, (2.1)–(2.7)], who separate the constant modes, treat their quartic interaction separately, and integrate the nonconstant modes. What we add is the exact coefficient Q/n for the constrained minimizers of our block averaging constraint at every finite depth, with bounds uniform in the depth, and a uniform comparison on windows of size $\gamma^{1/4}$. Up to normalization, Q is the action of the bosonic reduced $SU(2)$ Yang–Mills matrix integral with three matrices, whose divergence is known: see Krauth, Nicolai, and Staudacher [122, arXiv version 3, (24), p. 6], Krauth and Staudacher [121, (2)], and, for a rigorous treatment, Austing and Wheeler [7, Section 3 and Appendix A]. The integration over the transverse modes is a Laplace argument with degenerate directions (see Wong [155] for background); the main work is to verify its hypotheses uniformly. The center symmetry of Lemma 36.11 is the standard center symmetry of periodic lattice gauge theory, which goes back to 't Hooft [149].

36.1 Setting and notation

We fix the comparison side n and all parameters of the construction before the coupling varies. Below, K counts RG steps, and the underlying classical depth is JK . The quantity a_{map} below is the bound on the derivatives of the zero-fluctuation reconstruction maps fixed in Definition 34.1.

Definition 36.1. Fix $L > 0$, $\theta > 0$, an odd integer $M \geq 3$, an integer $J \geq 1$ with $B = M^J$, and the remaining parameters of the construction, such that $a_{\text{map}}\sqrt{B} \leq 32/49$. Fix $n_* \in 4\mathbb{N}$ and a comparison side $n = n_*B^a$ on the sequence $(n_*B^K)_{K \geq 0}$, large enough for the lower bounds on the coarse side in Corollaries 23.11 and 29.26 and Definition 34.1, and in particular for (36.1) below. These choices are made before γ varies in $(0, \gamma_0]$, where $0 < \gamma_0 \leq 1$ is a fixed ceiling. For $K \geq 0$ set

$$N = nB^K, \quad g^2 = \frac{\gamma}{n}, \quad \nu = \nu_{N,n,\gamma}, \quad \hat{\nu} = (T_M^{JK})_{\#}\nu, \quad r_\gamma = \gamma^{1/4},$$

and let Q be the quartic form on $\mathbb{R}^9$ displayed at the beginning of this section.

1. The stopped measure $\hat{\nu}$ is the law of the coarse configuration $V = T_M^{JK}U$ on Λ_n when U has law ν . It has density

$$d\hat{\nu} = \hat{Z}^{-1} \mathbf{1}_{D_r} e^{-E_{JK,\text{sat}}/g^2 - \bar{R}_{K,\gamma}} dH_n, \quad \bar{R}_{K,\gamma} = R_{K,\gamma} - R_{K,\gamma}(I),$$

where $\widehat{Z}$ is the normalization constant, $R_{K,\gamma}$ is the effective interaction, and $E_{JK,\text{sat}}$ is the gauge-invariant extension of the effective classical action E_{JK} from B_r to D_r of Theorem 11.6, that is, $E_{JK,\text{sat}}(U) = E_{JK}(U^k)$ whenever $U^k \in B_r$. In particular $E_{JK,\text{sat}} = E_{JK}$ on B_r .

2. Near the trivial orbit modulo based gauge transformations (gauge transformations equal to the identity at the origin) we use the coordinates of Theorem 33.8,

$$V(X, s) = e^{X/n+s}, \quad X \in \mathbb{R}^9, \quad d_0^* s = 0, \quad \sum_x s_i(x) = 0,$$

with $d_{\text{st}} = \dim s = 6n^3 - 6$, where X/n denotes the constant configuration and d_0^* is the lattice divergence on Λ_n . We write $J_n(X, s) dX ds$ for the image of the product Haar measure in the based quotient, and set

$$F_K(X, s) = E_{JK}(V(X, s)), \quad f_K(X) = F_K(X, 0), \quad A_K(X) = D_s^2 F_K(X, 0).$$

We write $\overline{R}_{K,\gamma}(X, s)$ for $\overline{R}_{K,\gamma}(V(X, s))$.

3. Let $\mathcal{H}^K = (\mathcal{H}_i^K)_{i=1}^3$ be the holonomies, around the three coordinate cycles based at the origin, of the fine configuration flowed by the Wilson flow $\Psi_{\theta L^2}^{(N)}$, where the fine configuration is represented through the reconstruction maps of Theorem 15.13. Let $\mathcal{H}_0^K(V)$ be the same triple for the flowed zero-fluctuation reconstruction of V (the map Ξ_0^K of Definition 34.1), in the same gauge frame at the basepoint. Let $\mathcal{H}_n(V) = \text{Hol}(V)$ be the triple of coordinate-cycle holonomies of V itself, without flow.
4. Fix a neighborhood $\mathcal{V}$ of (I, I, I) , invariant under simultaneous conjugation and with compact closure in $(\text{SU}(2) \setminus \{-I\})^3$, on which Log denotes the componentwise principal logarithm. For a bounded Borel function $\varphi : \mathbb{R}^9 \rightarrow \mathbb{R}$ with compact support, define

$$\mathcal{I}_{\gamma, \varphi}(H) = \begin{cases} \varphi(\text{Log } H / r_\gamma), & H \in \mathcal{V}, \\ 0, & H \notin \mathcal{V}. \end{cases}$$

5. A radial cutoff is a function $\chi \in C_c^\infty([0, \infty))$ that is nonnegative, not identically zero, and constant near zero. For a radial cutoff χ we write $\mathcal{I}_{\gamma, \chi}(|\cdot|)$ for the observable of the test $Y \mapsto \chi(|Y|)$, and we set

$$W_\chi(Y) = (4Q(Y) - 9)\chi(|Y|) - |Y|\chi'(|Y|), \quad \mathcal{S}_{\gamma, P}(\chi) = \frac{\mathbb{E}_P \mathcal{I}_{\gamma, W_\chi}}{\mathbb{E}_P \mathcal{I}_{\gamma, \chi}(|\cdot|)}$$

for a probability P under which the denominator is positive.

6. For a probability P of a random vector $X \in \mathbb{R}^9$ we use instead $\mathcal{I}_{\gamma, \varphi}^{\text{coord}}(X) = \mathbf{1}_{\{X \in \text{Log } \mathcal{V}\}} \varphi(X / r_\gamma)$, and define $\mathcal{S}_{\gamma, P}(\chi)$ with $\mathcal{I}^{\text{coord}}$ in place of $\mathcal{I}$.

We call $\mathcal{S}_{\gamma,P}(\chi)$ the dilation statistic; the name refers to the dilation integration-by-parts identity $\int W_\chi e^{-Q} dY = 0$ of Lemma 36.4. For each fixed bound on the support of φ and all sufficiently small γ , the map $Y \mapsto r_\gamma Y$ sends the support of $Y \mapsto \varphi(Y)$ into a compact subset of $\text{Log } \mathcal{V}$. Then $\mathcal{I}_{\gamma,\varphi}$ is a continuous function on $\text{SU}(2)^3$ for continuous φ , and the indicator in $\mathcal{I}_{\gamma,\varphi}^{\text{coord}}$ equals one on the support of $\varphi(\cdot/r_\gamma)$. The observables are defined by extension by zero because we never use a logarithm outside $\mathcal{V}$.

We apply Theorem 33.8 at depth JK ; in the depth notation of that theorem, $F_K = F_{JK}$, $A_K = A_{JK}$, and $d_{\text{st}} = d_n$. The coordinates $V(X, s)$ are defined on a fixed neighborhood $\{|X| < \rho_n, |s| < s_n\}$ whose image lies in $B_r \subset D_r$, and the density J_n is smooth and strictly positive. The gauge directions along a based spanning tree integrate to one, while the conjugation at the root remains in the measure and is not divided out. The theorem also gives

$$D_s F_K(X, 0) = 0, \quad a_n I \leq A_K(X) \leq C_n I,$$

where the Hessian lower bound $a_n > 0$ is a fixed constant (in this section a_n does not denote the mesh L/n). The first derivative of $\bar{R}_{K,\gamma}$ remains in the density. The measure $\hat{\nu}$ is the stopped measure of Definition 4.4 at side n and depth JK . Since no coarser restriction remains at the side n , it is the law of V under ν by Theorem 14.8(9).

Constants. Unless stated otherwise, constants $C, c, C_n, c_n, \dots$ in this section may depend on n, L, θ, M, J , the remaining parameters, γ_0 , the charts, the neighborhoods, and the gauge frames fixed in this section, and the test functions, but not on K, N , the number of preceding averaging steps, or $\gamma \in (0, \gamma_0]$. This is the dependence described in item (6) of Definition 4.14. Additional dependence is indicated by subscripts. All estimates are on the periodic torus of side L .

Margins. Fix a radius $r_{\text{flat}} < r$ below the margins of the classical, block-averaging, and gauge domains of the construction. The lower bound on n in Definition 36.1 includes

$$\frac{2\pi}{n} < \frac{r_{\text{flat}}}{4}. \quad (36.1)$$

By Lemma 14.9, every flat configuration on Λ_n is gauge equivalent to constant commuting links $e^{H_i/n}$ with commuting $H_i \in \mathfrak{su}(2)$, $|H_i| \leq 2\pi$. Its flat refinements to side nM^ℓ have links $e^{H_i/(nM^\ell)}$, and $|H_i|/(nM^\ell) \leq 2\pi/n < r_{\text{flat}}/4$ for every $\ell \geq 0$. The flat refinements, their vanishing conditional constrained minimizers, and their vanishing normalized fluctuation variables therefore lie in all coordinate domains of the construction with the same margins at every depth. This holds near each of the eight central triples.

We also fix nested coordinate balls $\mathcal{C} \Subset \mathcal{C}^+$ in the chart of Theorem 33.8, with

$$\frac{\rho_c^+}{n} + s_c^+ < \frac{1}{2} \min(r, r_{\text{flat}}), \quad \mathcal{C}^+ = \{|X| < \rho_c^+, |s| < s_c^+\}, \quad (36.2)$$

where the radii may be reduced, and such that the closure of $\mathcal{C}^+$ lies in the chart. Since $\max_{x,i} |X_i/n + s_i(x)| \leq |X|/n + |s|$, the bound (36.2) gives a fixed chart of link logarithms on which the derivative bounds for $\bar{R}_{K,\gamma}$ in Lemma 33.6 hold. On $\{s = 0\}$, it also places the commuting constant configurations in the domains above. The coordinate map extends analytically past the closure of $\mathcal{C}^+$, and by compactness its derivatives of each fixed order are bounded there. These choices concern only the coordinates used for integration; the conditioned Wilson measure and the restriction to D_r in $\hat{v}$ are unchanged.

Heuristics. In the coordinates $V(X, s)$, the stopped measure has density proportional to $e^{-F_K(X, s)/g^2}$ times the smooth positive factor $e^{-\bar{R}_{K,\gamma} J_n}$. By Theorem 36.2 below, $f_K(X) = Q(X)/n + O(|X|^5)$, and by Theorem 33.8, $F_K(X, s) = f_K(X) + (1/2)s^\top A_K(X)s + O(|s|^3)$. Since $ng^2 = \gamma = r_\gamma^4$, the substitution

$$X = r_\gamma Y, \quad s = gz, \quad g = \frac{r_\gamma^2}{\sqrt{n}},$$

gives

$$\frac{F_K(X, s)}{g^2} = Q(Y) + \frac{1}{2}z^\top A_K(0)z + (\text{an error of order } r_\gamma \text{ after integration}).$$

The Jacobian of the substitution is $r_\gamma^9 g^{d_{\text{st}}}$. By Theorem 33.10, the normalization constant is of order $g^{d_{\text{st}}+4}$, where the extra power g^4 comes from a neighborhood of the cone of commuting triples, along which Q vanishes. Since $r_\gamma^9/g^4 = n^2 r_\gamma$, the expectation of a compactly supported test of X/r_γ is of order r_γ . More precisely, for a bounded test φ supported in $\{|Y| \leq R\}$ and either output map of Theorem 36.6, its expectation is $r_\gamma \kappa_{K,\gamma} \left\{ \int \varphi(Y) e^{-Q(Y)} dY + O_{n,R}(\|\varphi\|_\infty r_\gamma) \right\}$, where $\kappa_{K,\gamma}$ is positive and bounded above and below. The factor $\kappa_{K,\gamma}$ does not depend on the test and cancels in ratios, so we never need its limit. The measure $e^{-Q(Y)} dY$ has infinite mass because Q vanishes on the five-dimensional cone of commuting triples and grows only quadratically in the four normal directions, with a coefficient proportional to the square of the distance from the origin (Lemma 36.4). After rescaling, most of the mass near the cone lies at commuting triples far from the central triple. In the unscaled variables these triples give the dominant contribution to the normalization constant, but tests supported on the scale r_γ do not see them. This is why only compactly supported tests appear.

To pass from the coarse configuration to the flowed holonomies of the fine configuration, we compare $\mathcal{H}^K$ with the deterministic triple $\mathcal{H}_0^K(V)$. The matrix difference is of order $\sqrt{\gamma} = r_\gamma^2$, while the window $\{|\text{Log } \mathcal{H}_0^K(V)| \leq Rr_\gamma\}$ has probability of order r_γ . A direct unconditional L^1 comparison loses the window factor. Averaged higher moments, combined with this small-window bound and Hölder's inequality, suffice for the limit and give every rate $\gamma^{1/4-\varepsilon}$, $\varepsilon > 0$. For the stated sharp rate we use the estimate

conditional on the coarse configuration V of Theorem 34.11. Suppose that the window is measurable with respect to V and has probability $O(r_\gamma)$, and that the conditional expected difference is $O(r_\gamma^2)$ uniformly in V . Then the expected difference on the window is $O(r_\gamma^3)$; after the loss r_γ^{-1} from rescaling the logarithm and division by the window mass, the error is $O(r_\gamma) = O(\gamma^{1/4})$.

Finally, we distinguish three quotients. The integration over V takes place in the quotient by based gauge transformations, so that the constant coordinates $X \in \mathbb{R}^9$ include the simultaneous conjugation at the basepoint, which is kept in the nine-dimensional integral. Observables of the continuum measure are invariant under simultaneous conjugation of the triple, and at several flow times they are invariant under an independent conjugation at each time. On a time interval, the observable is a path in the quotient $\mathfrak{su}(2)^3 / \text{Ad } \text{SU}(2)$; no gauge frame common to different times is needed.

36.2 The quartic term of the effective classical action

The following theorem identifies the fourth Taylor polynomial of the effective classical action on constant configurations. Here E_d is the effective classical action at depth d on the lattice of side n , that is, the value of the constrained minimizer of Theorem 10.6 (the background field, following Balaban [16]), with $E_0 = S_W$.

Theorem 36.2. *For every side n as in Definition 36.1 there exist $\rho_0(n) > 0$ and $C_n < \infty$ such that, for every depth $d \geq 0$, the function $f_d(X) = E_d(e^{X/n})$ satisfies*

$$D^j f_d(0) = 0 \quad (0 \leq j \leq 3), \quad \frac{1}{4!} D^4 f_d(0)[X, X, X, X] = \frac{Q(X)}{n},$$

and

$$\left| E_d(e^{X/n}) - \frac{Q(X)}{n} \right| \leq C_n |X|^5 \quad (|X| < \rho_0(n)). \quad (36.3)$$

The constants depend only on n , M , and the domains and derivative bounds of Theorem 10.11; they do not depend on d or on the coupling. In particular this applies to f_K of Definition 36.1, with $d = JK$.

Moreover, for every fixed d and X there exist constant fine links $e^{a_i(u)}$, analytic in u for $|u| < u_{d,X}$, such that

$$a_i(u) = u M^{-d} x_i / n + O_{d,X}(u^2), \quad T_M^d(e^{a(u)}) = e^{uX/n},$$

and these links lie in the domain of the variational comparison of Theorem 9.7 at depth d . The radius $u_{d,X}$ need not be uniform in d .

We identify the fourth Taylor polynomial exactly, at each finite depth, by comparing the minimizer with an exactly constrained trial configuration. The proof uses two different

tools. To identify the Taylor coefficients at a fixed depth d , we use an inverse of the block averaging constraint on constant configurations whose radius of definition may shrink as d grows. This is harmless, because Taylor coefficients at the origin are determined by arbitrarily small neighborhoods. For the uniformity of the remainder in d , we use instead the bounds on the fifth derivative of E_d from Theorem 10.11, which are uniform in the depth. We also distinguish two constraints: the block averaging constraint $T_M^d U = e^\theta$ itself, and the gauge-modified constraint satisfied by the fields in the projected Landau gauge slice of Theorem 9.9. The trial configurations and the representative of the minimizer constructed in Step 2 satisfy the former, and their transforms into the slice in Step 1 satisfy the latter. The gauge transformation used in Step 5 need not preserve either constraint; we use it only to compute the action, which it leaves unchanged.

Proof. All variational comparisons use real targets, fields, and parameters; holomorphic extensions supply analytic representatives and Taylor bounds.

Step 1: the variational comparison. We set $\varepsilon_d = M^{-d}$ and measure lengths in units of the lattice of side n , so that the fine lattice has side n/ε_d and the action is

$$S_{\varepsilon_d}(U) = \frac{4}{\varepsilon_d} \sum_p q(U_p). \quad (36.4)$$

We take the minimizer and its value from Theorem 10.6, in the notation of Definition 10.1. We write Ξ_d for the constraint coordinates P_d^{cl} and H_d for the minimizing linear extension H_d , as in Theorem 9.7. The critical point is $v_d^*(\theta) \in \mathcal{V}_d$, the minimizing field is $A_d(\theta) = \Xi_d(v_d^*(\theta) + H_d\theta)$, and

$$E_d(e^\theta) = S_{\varepsilon_d}(e^{\varepsilon_d A_d(\theta)}). \quad (36.5)$$

Fix $0 < r_{\text{ph}} < r_0$ within the common margins, where r_0 and C_* are the constants of Theorem 9.7. By the bound on v_d^* in Theorem 10.6, there exists a constant $C_{\min}$, which includes the conversion from first differences to the curl, such that

$$\|v_d^*(\theta)\|_\infty + \|\mathbf{d}_{\varepsilon_d} v_d^*(\theta)\|_\infty \leq C_{\min} \|\theta\|_\infty^2.$$

Let $\tau > 0$ be a target radius with

$$\tau < \min\{r_\theta/2, r_{\text{ph}}/(2C_*)\}, \quad C_{\min}\tau^2 < r_{\text{ph}}/4, \quad (36.6)$$

where r_θ is the radius of Theorem 10.6. For $\|\theta\|_\infty < \tau$ the critical point then lies strictly inside the convex domain of potentials and curls of Theorem 9.7, which verifies the critical point hypothesis of that theorem.

Next we describe the trial configurations, which are needed only in a small neighborhood that may depend on d . Let $U = e^{\varepsilon_d A}$ satisfy $T_M^d U = e^\theta$ with $\|\theta\|_\infty < \tau$, $\|A\|_X = \|A\|_\infty + \|\text{div}_{\varepsilon_d} A\|_\infty < r_X$, and small curl. We apply Theorem 9.9 with prescribed

coarse gauge datum zero. The transformed field has vanishing projected divergence and gauge-modified target exactly e^θ . It has the same action, and its potential A' and its curl tend to zero with those of A . The transformed field need not satisfy the block averaging constraint $T_M^d U = e^\theta$ itself; for this reason, Step 2 constructs a separate representative of the minimizer that does.

To check that A' lies in the chart of Theorem 9.7, we write the constraint coordinates as $\Xi_d(w) = w - H_d \mathcal{D}_d(w)$, with $\mathcal{D}_d(0) = D\mathcal{D}_d(0) = 0$; here the argument w is a fine field, not a vector of normalized fluctuation variables. Since H_d is a fixed linear map on this finite lattice, the first-derivative estimate of Theorem 10.11 makes $w \mapsto A' + H_d \mathcal{D}_d(w)$ a contraction on a sufficiently small ball, which implies that Ξ_d is locally invertible at A' . By the defining identities of the constraint coordinates, the gauge-modified constraint of $\Xi_d(w)$ is the linear target, and the correction $H_d \mathcal{D}_d(w)$ has vanishing projected divergence. Hence the inverse parameter has the form $w = a + H_d \theta$ with $a \in \mathcal{V}_d$, and on this finite lattice the norm of a in potential and curl tends to zero with (A', θ) . In particular, a sufficiently small trial configuration satisfies

$$\|a\|_\infty + \|d_{\varepsilon_d} a\|_\infty < r_{\text{ph}}/2. \quad (36.7)$$

By (36.6) and (36.7), the trial parameter and the critical point lie in the same convex domain with target e^θ , and so does the segment joining them. Since the transformation into the slice preserves the action, the convexity inequality of Theorem 9.7, together with the minimum of Theorem 10.6, implies that

$$E_d(e^\theta) \leq S_{\varepsilon_d}(U). \quad (36.8)$$

The smallness required in this step may depend on d and on the direction of the trial configuration.

Step 2: an analytic representative of the minimizer. The field $A_d(\theta)$ is holomorphic, and its value is (36.5). It satisfies the gauge-modified constraint of the projected Landau slice, and its accumulated gauge transformation is a finite composition of the analytic gauge maps of that slice. We take the logarithm of this gauge transformation at the vertices of the coarse lattice, extend it by any fixed linear extension to all fine vertices, and exponentiate. The result is an analytic periodic fine gauge transformation with the prescribed values at the coarse vertices. Since T_M is gauge covariant, applying this transformation changes the target of the block averaging constraint to e^θ , and it leaves (36.4) unchanged. We obtain an analytic configuration $U_*(\theta)$ with

$$U_*(0) = I, \quad T_M^d U_*(\theta) = e^\theta, \quad S_{\varepsilon_d}(U_*(\theta)) = E_d(e^\theta).$$

Only analyticity at the fixed depth d is used: even if the derivative bounds of this extension grow with d , it tends to the identity as $\theta \rightarrow 0$ at fixed d .

Step 3: an exactly constrained trial and the upper bound. Fix d and X . On constant configurations, the map $G_d(a) = \text{Log } T_M^d(e^a)$ is analytic near zero and $DG_d(0) = \varepsilon_d^{-1} I_9$. Indeed, the derivative at the identity of the formula of Definition 6.1 is the average of the integrals of a over the block paths. On a constant configuration each of these paths has integral Ma_i , and d iterations give $M^d a_i$. To invert G_d without a radius uniform in d , we solve

$$a = \varepsilon_d u \frac{X}{n} - \varepsilon_d \{G_d(a) - \varepsilon_d^{-1} a\}.$$

Since the derivative of the term in braces vanishes at zero, the right side is a contraction with factor at most $1/2$ on a small fixed complex ball. For small $|u|$, it also maps the ball into itself. The iterates converge uniformly on a smaller complex disc in u , and the limit is analytic in u and real for real u . A fixed point satisfies $G_d(a) = uX/n$, and we obtain

$$a_i(u) = u\varepsilon_d x_i/n + O_{d,X}(u^2), \quad T_M^d(e^{a(u)}) = e^{uX/n}. \quad (36.9)$$

The potential $a(u)/\varepsilon_d$ is constant, has vanishing divergence and curl, and tends to zero as $u \rightarrow 0$. We reduce $u_{d,X}$ until $\|uX/n\|_\infty < \tau$, the input bound of the projected Landau slice holds, and the trial parameter satisfies (36.7). By (36.6), the critical point for the same target lies in the same convex chart, and (36.8) applies to $e^{a(u)}$. The logarithm of a plaquette of the constant configuration $e^{a(u)}$ in the (i, j) plane is $u^2[\varepsilon_d x_i/n, \varepsilon_d x_j/n] + O_{d,X}(u^3)$. Using $q(e^v) = |v|^2/8 + O(|v|^4)$ and the fact that there are $(n/\varepsilon_d)^3$ plaquettes in each plane, we obtain

$$\begin{aligned} E_d(e^{uX/n}) &\leq \frac{4}{\varepsilon_d} \left(\frac{n}{\varepsilon_d}\right)^3 \frac{u^4}{8} \sum_{i < j} |[\varepsilon_d x_i/n, \varepsilon_d x_j/n]|^2 + O_{d,X}(u^5) \\ &= \frac{u^4}{n} Q(X) + O_{d,X}(u^5). \end{aligned} \quad (36.10)$$

Step 4: the first-order term of the minimizer. We write the link logarithms of $U_*(uX/n)$ from Step 2 as $ua + O(u^2)$, where a is a cochain on the fine torus. By Step 2, its action equals $E_d(e^{uX/n})$, which is nonnegative and, by (36.10), of order u^4 . The coefficient of u^2 in its action therefore vanishes. This coefficient is

$$\frac{1}{2\varepsilon_d} \sum_x \sum_{i < j} |(da)_{ij}(x)|^2,$$

where d denotes the unscaled cochain differential on the fine torus. We conclude that $da = 0$.

We record the periodic decomposition of a closed cochain. A closed cochain has the same integral over every translate of a coordinate cycle, because the difference of two such cycles is a sum of plaquette boundaries. We subtract the constant c_i equal to this integral divided by n/ε_d . The resulting cochain is closed and has vanishing integrals over

the coordinate cycles. Its integrals over all closed paths vanish: neighboring coordinate steps can be exchanged using plaquette boundaries, adjacent inverse steps cancel, and a remaining closed path reduces to a combination of the three coordinate cycles. Hence its integral from a fixed vertex to a vertex x does not depend on the path. Applying this to a and writing $\phi(x)$ for the integral of the modified cochain from the fixed vertex to x , we have

$$a_i(x) = c_i + \phi(x + \varepsilon_d e_i) - \phi(x). \quad (36.11)$$

Each statement holds separately in each real component of $\mathfrak{su}(2)$.

At the identity, the derivative of the block averaging map is the average of the integrals over the block paths. For a closed cochain, a translated path with endpoint connectors and the reference path differ by a contractible closed path and have equal integrals. It follows that d averaging steps send the constant part of (36.11) to $\varepsilon_d^{-1} c_i$, and the gradient to the gradient of the restriction of ϕ to the coarse lattice. The linearization of the constraint $T_M^d U_*(uX/n) = e^{uX/n}$ states that this image equals x_i/n on every coarse link in direction i . When we average over all coarse links in direction i , the gradient terms cancel by periodicity, and we obtain

$$c_i = \varepsilon_d x_i / n. \quad (36.12)$$

Step 5: the lower bound. The periodic fine gauge transformation $e^{u\phi(x)}$, with ϕ normalized to vanish at the root, removes the gradient in (36.11) to first order, and it preserves the action. In this gauge the link logarithms are $uc + u^2b + O(u^3)$ for some cochain b , and the plaquette logarithms are

$$u^2 \{ (db)_{ij}(x) + [c_i, c_j] \} + O(u^3).$$

For each $i < j$, we have $\sum_x (db)_{ij}(x) = 0$ by periodicity, and therefore

$$\sum_x |(db)_{ij}(x) + [c_i, c_j]|^2 = \sum_x |(db)_{ij}(x)|^2 + \left(\frac{n}{\varepsilon_d}\right)^3 |[c_i, c_j]|^2.$$

Using (36.12) and $q(e^v) = |v|^2/8 + O(|v|^4)$, we find that the coefficient of u^4 in $E_d(e^{uX/n})$ is at least

$$\frac{1}{2\varepsilon_d} \left(\frac{n}{\varepsilon_d}\right)^3 \sum_{i < j} |\varepsilon_d x_i / n, \varepsilon_d x_j / n|^2 = \frac{Q(X)}{n}.$$

Together with (36.10), this proves that the fourth Taylor coefficient of $u \mapsto E_d(e^{uX/n})$ equals $Q(X)/n$ for every direction X . The vanishing of the lower coefficients also follows from $0 \leq f_d(uX) \leq C_{d,X} u^4$ for both signs of small real u . By analyticity and polarization, we obtain the stated identities for the multilinear derivatives, so the fourth Taylor polynomial of f_d at zero is Q/n at every finite depth. We make no claim about the fifth Taylor polynomial.

Step 6: the uniform remainder. Fix the side n . On a common smaller ball, we restrict the bound of Theorem 10.11 on the derivatives of order five of $\theta \mapsto E_d(e^\theta)$ to the linear substitution $\theta_{x,i} = X_i/n$. At the fixed side n there are finitely many coordinates $\theta_{x,i}$. Summing the tensor entries and including the factors $1/n$, we obtain

$$\sup_{d \geq 0} \sup_{|X| < \rho_0(n)} \|D^5 f_d(X)\| \leq C_n.$$

The ball and the bound do not depend on the radii $u_{d,X}$ of the trial configurations. Since the Taylor polynomial of order four of f_d at zero is Q/n by Step 5, Taylor's formula on the segment from 0 to X gives

$$f_d(X) - \frac{Q(X)}{n} = \frac{1}{4!} \int_0^1 (1-t)^4 D^5 f_d(tX)[X, X, X, X, X] dt.$$

The absolute value of the right side is at most $C_n |X|^5 / 5!$, which proves (36.3). $\square$

36.3 Holonomies near the trivial flat configuration

This section proves two facts about the flowed holonomies. The first is deterministic. In the central coordinates, the logarithm of the flowed zero-fluctuation holonomy triple equals X up to errors of order $|X|^2 + |s|$. Further, a configuration that is close to the set of flat configurations and whose zero-fluctuation holonomies are close to the identity lies close to the trivial flat configuration. The second fact is probabilistic: under the conditioned Wilson measure, the flowed holonomies of the configuration differ from those of its zero-fluctuation reconstruction by $O(\sqrt{\gamma})$ in mean square. Let $\mathcal{F}_n$ denote the set of flat configurations on Λ_n , regarded in the quotient by based gauge transformations.

Theorem 36.3. *After a fixed reduction of the central coordinate neighborhood and of $\mathcal{V}$, the map $Z_K(X, s) = \text{Log } \mathcal{H}_0^K(V(X, s))$ is defined on that neighborhood for every K , and*

$$\sup_K \|Z_K\|_{C^{1,1}} \leq C_n, \quad |Z_K(X, s) - X| \leq C_n(|X|^2 + |s|), \quad |Z_K(X, s)| \geq \frac{3}{4}|X| - C_n|s|,$$

$$\|D_X Z_K(X, s) - I\|_{\text{op}} \leq C_n(|X| + |s|).$$

The maps $\mathcal{H}_0^K$ are equicontinuous, in a fixed finite family of smooth charts of the based quotient, on a neighborhood of $\mathcal{F}_n$, and $\mathcal{H}_0^K = \mathcal{H}_n$ on $\mathcal{F}_n$ for every finite K . Fix a continuous distance d on the based quotient. For every open neighborhood $\mathcal{U}$ of the trivial based orbit there exists $\delta_{\mathcal{U}} > 0$, independent of K and γ , such that

$$d(V, \mathcal{F}_n) \leq \delta_{\mathcal{U}} \quad \text{and} \quad \left(\sum_{i=1}^3 \rho_G(\mathcal{H}_{0,i}^K(V))^2 \right)^{1/2} \leq \delta_{\mathcal{U}} \quad \text{imply} \quad V \in \mathcal{U}.$$

The constant $\delta_{\mathcal{U}}$ may depend on d and $\mathcal{U}$.

Let $D_K = \max_i \|\mathcal{H}_i^K - \mathcal{H}_{0,i}^K(V)\|_{\text{op}}$. Then, uniformly in K and $0 < \gamma \leq \gamma_0$,

$$\mathbb{E}_v \min(1, D_K)^2 \leq C_n \gamma.$$

For every neighborhood $\mathcal{V}'$ with compact closure in $\mathcal{V}$,

$$\mathbb{E}_v \left[\mathbf{1}_{\{\mathcal{H}^K, \mathcal{H}_0^K(V) \in \mathcal{V}'\}} \min(1, |\text{Log } \mathcal{H}^K - \text{Log } \mathcal{H}_0^K(V)|)^2 \right] \leq C_{n, \mathcal{V}'} \gamma.$$

The estimates on D_K are averages over the whole conditioned Wilson measure; they are not conditioned on the configuration being near a central triple.

Proof. Steps 1–3 prove the deterministic statements, and Step 4 the probabilistic ones.

Step 1: deterministic regularity. We apply Theorem 35.9 with $m = 0$ to the parameters and the side n of Definition 36.1. With $m = 0$ there are no fluctuation variables among the inputs. The input is the coarse configuration alone, and the map of that theorem is $\Xi_0^K = \mathcal{H}_0^K$, with the same gauge frame at the basepoint and the same flow time. The part of Theorem 35.9 concerning a neighborhood of all flat configurations applies, because (36.1) places every commuting constant representative strictly inside the classical, block-averaging, and gauge margins, as checked in Section 36.1. We obtain a neighborhood of the trivial flat configuration on which $\sup_K \|\mathcal{H}_0^K\|_{C^{1,1}} \leq C_n$, a modulus of continuity of $\mathcal{H}_0^K$ near $\mathcal{F}_n$ that is common to all K , and the identity

$$\mathcal{H}_0^K(V_0) = \text{Hol}(V_0) \quad (V_0 \in \mathcal{F}_n) \quad (36.13)$$

at every finite K . Independently, Lemma 14.9 explains (36.13): the zero-fluctuation refinements of a flat configuration are flat refinements, and the Wilson flow is stationary on flat configurations. These are identities of matrices; the conjugation at the basepoint has not been divided out. The neighborhoods and the modulus are those of the deterministic theorem, on its finite cover of charts, and they do not depend on K or γ . In particular, they are not chosen using the small-field event.

Step 2: the logarithm. At $V = I$ the output is (I, I, I) . By the common modulus, for any prescribed smaller neighborhood of the identity in $\mathcal{V}$, there exists a neighborhood of the input whose outputs lie in it. The logarithm and its first two derivatives are bounded on the closure of that neighborhood. Composing with the maps of Step 1, we obtain $\sup_K \|Z_K\|_{C^{1,1}} \leq C_n$. Let $b_K(X, s) = Z_K(X, s) - X$. If $s = 0$ and the three components of X are parallel, the links of $V(X, 0)$ commute, the configuration is flat, and (36.13) implies that $b_K(X, 0) = 0$. Each of the nine coordinate directions of $\mathbb{R}^9$ is of this form, and these directions span $\mathbb{R}^9$. Differentiating along them, we find

$$b_K(0, 0) = 0, \quad D_X b_K(0, 0) = 0.$$

Integrating the Lipschitz bound on $D_X b_K$ along $v \mapsto (vX, 0)$, we obtain $|b_K(X, 0)| \leq C_n |X|^2$, and the bound on the derivative in s gives $|b_K(X, s) - b_K(X, 0)| \leq C_n |s|$. The Lipschitz bound on the derivative also implies $\|D_X Z_K(X, s) - I\| \leq C_n(|X| + |s|)$. We reduce the radius in X so that $C_n |X| \leq 1/4$. Then, by the triangle inequality, $|Z_K(X, s)| \geq |X| - C_n |X|^2 - C_n |s| \geq (3/4)|X| - C_n |s|$. The same reduction works for every K and every coupling.

Step 3: localization near the trivial flat configuration. The based quotient is compact, since it is the quotient of a finite product of copies of $SU(2)$ by a compact group, and $\mathcal{F}_n$ is a closed subset. On a flat configuration, the transport along a path is unchanged by elementary plaquette moves and depends only on the homotopy class of the path. Then, on the universal covering lattice, the transports from the root define a periodic gauge transformation to the trivial configuration if and only if all three cycle holonomies are the identity. Thus the trivial orbit is the only flat based orbit whose cycle triple is (I, I, I) ; this is the based version of Lemma 14.9.

Let $\mathcal{U}$ be given, and let $\mathcal{U}_1$ be an open set containing the trivial orbit with $\overline{\mathcal{U}_1} \subset \mathcal{U}$. Suppose first that $\mathcal{F}_n \setminus \mathcal{U}_1$ is nonempty. Since it is compact and contains no flat configuration with identity cycle triple,

$$m = \min_{V_0 \in \mathcal{F}_n \setminus \mathcal{U}_1} \left(\sum_i \rho_G(\text{Hol}_i(V_0))^2 \right)^{1/2} > 0.$$

By the common modulus of Step 1, (36.13), and the continuity of ρ_G , there exists $\delta_1 > 0$ such that, for every K , if V_0 is flat and $d(V, V_0) < \delta_1$, then the quantities $(\sum_i \rho_G(\mathcal{H}_{0,i}^K(V))^2)^{1/2}$ and $(\sum_i \rho_G(\text{Hol}_i(V_0))^2)^{1/2}$ differ by less than $m/2$. Let $\delta_2 > 0$ be smaller than the distance from $\overline{\mathcal{U}_1}$ to the complement of $\mathcal{U}$, and let $\delta_{\mathcal{U}}$ be smaller than $\delta_1/2$, $\delta_2/2$, $m/4$, and the radius of the charts of Step 1. Suppose that V satisfies the two hypotheses, and let $V_0 \in \mathcal{F}_n$ be a closest flat point, which exists by compactness. Then the cycle radius of V_0 , that is, the second of the two quantities above, is less than $\delta_{\mathcal{U}} + m/2 < m$. Hence $V_0 \notin \mathcal{F}_n \setminus \mathcal{U}_1$, that is, $V_0 \in \mathcal{U}_1$, and since $d(V, V_0) < \delta_2$, we have $V \in \mathcal{U}$. If $\mathcal{F}_n \setminus \mathcal{U}_1$ is empty, then $\mathcal{F}_n$ has positive distance from the complement of $\mathcal{U}$, and the first hypothesis alone implies the conclusion. This argument includes all flat configurations, in particular those near the other seven central triples.

Step 4: the probabilistic estimates. We apply Theorem 34.10 with $m = 0$ and $p = 2$. Its comparison triple is $\mathcal{H}_0^K(V)$, and its probability measure is the whole conditioned Wilson measure ν . By the moment bound of that theorem, we have

$$\mathbb{E}_{\nu} \min(1, D_K)^2 \leq C_n \gamma.$$

This uses the condition $a_{\text{map}} \sqrt{B} \leq 32/49$ imposed in Definition 36.1. That theorem includes the conditional means, the matrices at the basepoint, and all flat configurations.

Finally, Log is Lipschitz, as a map of the three matrices, on the compact closure of $\mathcal{V}'$. To obtain a single constant for all pairs, we cover the closure by finitely many neighborhoods on which Log has bounded derivatives. Pairs closer than a Lebesgue number of this cover satisfy the derivative estimate in one of these neighborhoods. For the remaining pairs, we use the boundedness of Log and the positive lower bound on their distance. We obtain

$$\min(1, |\text{Log } H - \text{Log } H'|) \leq C_{\mathcal{V}'} \min(1, \max_i \|H_i - H'_i\|_{\text{op}}) \quad (H, H' \in \mathcal{V}').$$

Squaring, restricting to the event in the statement, and using the bound on $\mathbb{E}_\nu \min(1, D_K)^2$, we obtain the last estimate. All reductions used fixed neighborhoods, the fixed side, and the bounds of the cited theorems, so the constants have the stated dependence. $\square$

36.4 The quartic reference measure

We write m_Q for the quartic reference measure $dm_Q(Y) = e^{-Q(Y)} d^9Y$ on $\mathbb{R}^9$. The following lemma records its basic properties. The divergence of its total mass is known in the matrix-integral literature cited at the beginning of this section; we give a direct proof in the coordinates of Lemma 33.2.

Lemma 36.4. *The measure m_Q is locally finite and has infinite total mass. For every $\chi \in C_c^\infty([0, \infty))$ that is constant near zero and every $\lambda > 0$,*

$$\int \chi(\lambda|Y|) dm_Q(Y) = \lambda^{-9} \int \chi(|Y|) e^{-\lambda^{-4}Q(Y)} d^9Y,$$

and

$$\int \{(4Q(Y) - 9)\chi(|Y|) - |Y|\chi'(|Y|)\} dm_Q(Y) = 0.$$

If $\chi \geq 0$ and χ is not identically zero, then $\int \chi(|Y|) dm_Q(Y)$ is finite and strictly positive.

We call the second identity the dilation integration-by-parts identity. It states that $\int W_\chi dm_Q = 0$ for every radial cutoff χ , and it uses only the compact support of the test; no normalization of m_Q is involved.

Proof. Since $Q \geq 0$ is a polynomial, $0 < e^{-Q} \leq 1$ everywhere, which proves local finiteness.

Infinite mass. We use Lemma 33.2, with its homogeneous cone, its injective normal tube, and its bound on the Jacobian, at every radius. We write the commuting triples as $y_i = \rho a_i u$ with $\rho \geq 1$ and $a, u \in S^2$, where (a, u) and $(-a, -u)$ give the same triple, and we set $\zeta_i = a_i u$. The normal space at ζ is

$$\mathcal{B}_{a,u} = \left\{ b = (b_1, b_2, b_3) : b_i \perp u, \sum_i a_i b_i = 0 \right\}, \quad \dim \mathcal{B}_{a,u} = 4.$$

We claim that

$$Q(\zeta + b) = \frac{1}{2}|b|^2 + Q(b) \quad (b \in \mathcal{B}_{a,u}). \quad (36.14)$$

Indeed, $(\zeta_i + b_i) \times (\zeta_j + b_j) = u \times (a_i b_j - a_j b_i) + b_i \times b_j$. The first term lies in $u^\perp$, while $b_i \times b_j$ is parallel to u because $b_i, b_j \perp u$. The two terms are therefore orthogonal. Since $|u \times w| = |w|$ for $w \perp u$, and

$$\sum_{i < j} |a_i b_j - a_j b_i|^2 = |a|^2 \sum_i |b_i|^2 - \left| \sum_i a_i b_i \right|^2 = |b|^2,$$

we obtain (36.14). The same computation with $y = \rho \zeta$ gives $Q(y + b) = \rho^2 |b|^2 / 2 + Q(b)$.

Let $\eta > 0$ be smaller than the relative radius of the tube in Lemma 33.2 and so small that the Jacobian of the tube is at least $1/2$ when $|b|/\rho \leq \eta$. We restrict to $|b| \leq \eta/\rho$. For $\rho \geq 1$, this set lies in the tube, and $|b|/\rho \leq \eta$. Since

$$Q(b) \leq \frac{1}{2} \sum_{i < j} |b_i|^2 |b_j|^2 \leq \frac{1}{4} |b|^4,$$

we have $Q(y + b) \leq \eta^2/2 + \eta^4/4$, which we denote by C_η , on this whole unbounded tube. By Lemma 33.2, the volume of the cone is $(1/2)\rho^4 d\rho d\Omega_a d\Omega_u$, where the factor $1/2$ accounts for the double parametrization, and the tube is injective in the point y of the cone. The normal ball has volume $v_4 \eta^4 \rho^{-4}$, where v_4 is the volume of the unit ball in $\mathbb{R}^4$. For every $T > 1$, we obtain

$$\int_{\mathbb{R}^9} e^{-Q(X)} dX \geq \frac{1}{4} e^{-C_\eta} (4\pi)^2 v_4 \eta^4 \int_1^T \rho^4 \rho^{-4} d\rho = c_\eta (T - 1).$$

Letting $T \rightarrow \infty$ proves that the total mass is infinite. All radii and constants in this argument are absolute.

Scaling, the dilation integration-by-parts identity, and positivity. The change of variables $Z = \lambda Y$, together with $Q(Z/\lambda) = \lambda^{-4} Q(Z)$, gives the scaling formula. For λ in a compact neighborhood of 1, the integrands and their derivatives in λ are supported in a common compact set and have a common integrable bound. We may therefore differentiate at $\lambda = 1$, which gives the dilation integration-by-parts identity. Equivalently, by the homogeneity relation $Y \cdot \nabla Q = 4Q$, we have

$$\operatorname{div}\{Y\chi(|Y|)e^{-Q(Y)}\} = \{9\chi(|Y|) + |Y|\chi'(|Y|) - 4Q(Y)\chi(|Y|)\}e^{-Q(Y)}.$$

The vector field on the left is smooth at zero because χ is constant there, and it vanishes outside a compact set. The integral of each of its coordinate derivatives over a box containing its support is zero, which proves the dilation integration-by-parts identity with the stated sign. Finally, a nonnegative smooth radial test that is not identically zero is positive on an interval of radii, which corresponds to a subset of $\mathbb{R}^9$ of positive volume. Since the density e^{-Q} is positive, the integral is positive, and it is finite by compact support. $\square$

Remark 36.5. For $F_Q(\rho) = m_Q\{|Y| \leq \rho\}$ and $f_Q = F'_Q$ we have, as $\rho \downarrow 0$,

$$f_Q(\rho) = \frac{32\pi^4}{105}\rho^8[1 + O(\rho^4)], \quad F_Q(\rho) = \frac{32\pi^4}{945}\rho^9[1 + O(\rho^4)],$$

with absolute constants for the Euclidean norm and the normalization of Q used here. Indeed, polar coordinates give

$$f_Q(\rho) = \rho^8 \int_{S^8} e^{-\rho^4 Q(\omega)} d\omega. \quad (36.15)$$

Since Q is bounded on S^8 , the exponential is $1 + O(\rho^4)$ uniformly. The area of S^8 is $2\pi^{9/2}/\Gamma(9/2) = 32\pi^4/105$, by $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(v+1) = v\Gamma(v)$. Integrating the density from zero gives the asymptotics of F_Q . These formulas are not used below.

36.5 Integration over the transverse modes

The next theorem, our main integration result, computes the expectation, under the stopped measure, of a compactly supported test of the rescaled logarithm of either the unflowed coarse holonomy triple $\mathcal{H}_n(V)$ or the flowed zero-fluctuation holonomy triple $\mathcal{H}_0^K(V)$.

Theorem 36.6. *For every $R < \infty$ there exist $C_{n,R} < \infty$ and $\gamma_R > 0$ such that, for every bounded Borel function φ supported in $\{|Y| \leq R\}$, every $K \geq 0$, and every $0 < \gamma \leq \gamma_R$, both $T = \mathcal{H}_n(V)$ and $T = \mathcal{H}_0^K(V)$ satisfy*

$$\mathbb{E}_{\hat{\nu}} \mathcal{I}_{\gamma, \varphi}(T) = r_\gamma \kappa_{K, \gamma} \left\{ \int \varphi dm_Q + \mathcal{E}_{K, \gamma, T, \varphi} \right\}, \quad |\mathcal{E}_{K, \gamma, T, \varphi}| \leq C_{n,R} \|\varphi\|_\infty r_\gamma,$$

where the factor $\kappa_{K, \gamma}$ is the same for both maps and all tests, namely

$$\kappa_{K, \gamma} = (2\pi)^{d_{\text{st}}/2} n^2 \frac{e^{-\bar{R}_{K, \gamma}(0,0)} J_n(0,0)}{\sqrt{\det A_K(0)}} \frac{g^{d_{\text{st}}+4}}{\hat{Z}}, \quad 0 < c_n \leq \kappa_{K, \gamma} \leq C_n.$$

Here $\hat{Z}$ is the normalization constant of $\hat{\nu}$ in Definition 36.1, and $c_n g^{d_{\text{st}}+4} \leq \hat{Z} \leq C_n g^{d_{\text{st}}+4}$.

Further, for $T_K(X, s) = \text{Log } T(V(X, s))$ and both maps, the change of variables $Z = T_K(X, s)$ is invertible on a fixed smaller ball of outputs, and its inverse $X = H_K(Z, s)$ satisfies

$$|H_K(Z, s) - Z| \leq C_n(|Z|^2 + |s|), \quad |\det D_Z H_K(Z, s) - 1| \leq C_n(|Z| + |s|).$$

The error bound depends on φ only through $\|\varphi\|_\infty$, and no modulus of continuity of φ enters. Both deterministic maps use the based-quotient chart frames of Theorem 36.3; their comparison with the selected frame of the conditional reconstruction below uses invariant tests. The bound accounts for the density factor $e^{-\bar{R}_{K, \gamma} J_n}$, including the first

derivatives of $\bar{R}_{K,\gamma}$, for the Gaussian determinant of the transverse modes, and for the contributions of configurations away from the trivial flat configuration. We do not assert that $\kappa_{K,\gamma}$ has a limit. Since $\kappa_{K,\gamma}$ depends only on the stopped measure, it does not depend on the flow time at which $\mathcal{H}_0^K$ is observed.

Proof. Throughout the proof we write $D = d_{\text{st}}$ and $\epsilon = r_\gamma$, and we set

$$a_{K,\gamma}(X, s) = e^{-\bar{R}_{K,\gamma}(X, s)} J_n(X, s), \quad a_{K,\gamma}^0 = a_{K,\gamma}(0, 0).$$

Steps 1–3 localize to the central coordinates and invert the output map, Step 4 rescales so that the test becomes exactly $\varphi(Y)$, and Steps 5–6 compare the density with its Gaussian–quartic model and integrate.

Step 1: bounds on the density. We apply Theorem 33.8 to the stopped measure at depth JK , with the side, the parameters, and the upper bound γ_0 on the coupling from Definition 36.1. The chart $\mathcal{C}^+$ of (36.2) is a fixed chart of link logarithms, independent of the depth and the coupling. On it, Lemma 33.6, with $d = JK$ and link logarithms $X/n + s$, together with the residual value bound (P3) in Section 33.2, gives

$$\sup_{K, 0 < \gamma \leq \gamma_0} \sup_{\mathcal{C}^+} (|\bar{R}_{K,\gamma}| + \|D_{X,s} \bar{R}_{K,\gamma}\|) \leq C_n.$$

The smooth positive density J_n has positive upper and lower bounds and bounded first derivatives on the closure of a slightly smaller chart that still contains $\bar{\mathcal{C}}$, and we replace $\mathcal{C}^+$ by this chart. The identity

$$D_{X,s} a_{K,\gamma} = e^{-\bar{R}_{K,\gamma}} (D_{X,s} J_n - J_n D_{X,s} \bar{R}_{K,\gamma})$$

then gives, on this fixed chart,

$$0 < c_n \leq a_{K,\gamma}(X, s) \leq C_n, \quad \|D_{X,s} a_{K,\gamma}(X, s)\| \leq C_n. \quad (36.16)$$

This uses the first derivatives of the centered effective interaction in both X and s . Together with the uniform classical derivative bounds of Theorem 10.11, Theorem 33.8 also gives

$$\begin{aligned} c_n I &\leq A_K(X) \leq C_n I, & \|A_K(X) - A_K(0)\| &\leq C_n |X|, \\ F_K(X, s) &= f_K(X) + \frac{1}{2} s^\top A_K(X) s + O_n(|s|^3), & F_K(X, s) &\geq f_K(X) + c_n |s|^2. \end{aligned} \quad (36.17)$$

The density in Theorem 33.8 at depth JK is the stopped measure of Definition 36.1, so its normalization constant is $\hat{Z}$. By Theorem 33.10, after reducing the upper bound on the coupling if necessary, we have

$$c_n g^{D+4} \leq \hat{Z} \leq C_n g^{D+4}. \quad (36.18)$$

Step 2: reduction to a neighborhood of the trivial flat configuration. Fix a neighborhood $\mathcal{U}$ of the trivial based orbit whose closure lies in the coordinate chart of (36.17). By Theorem 36.3, applied to the map $\mathcal{H}_0^K$, there exists $\delta > 0$ such that a configuration within distance δ of $\mathcal{F}_n$ whose cycle radius $(\sum_i \rho_G(\mathcal{H}_{0,i}^K(V))^2)^{1/2}$ is at most δ lies in $\mathcal{U}$. For the map $\mathcal{H}_n = \text{Hol}$, the same statement follows from the argument of Step 3 of that proof, with the continuous map Hol in place of $\mathcal{H}_0^K$. On the complement of $\{d(V, \mathcal{F}_n) \leq \delta\}$, we have $E_0 \geq c_{n,\delta} > 0$ by the compactness of the configuration space and $E_0^{-1}(0) = \mathcal{F}_n$. The lower bound $E_{JK,\text{sat}} \geq \kappa E_0$ on D_r and the bound $|\bar{R}_{K,\gamma}| \leq C_n$ on D_r , properties (P1) and (P3) in Section 33.2, together with (36.18), give

$$\hat{\nu}\{d(V, \mathcal{F}_n) > \delta\} \leq C_n g^{-(D+4)} e^{-c_n/g^2}. \quad (36.19)$$

Since the product Haar measure has total mass one, no volume factor depending on K appears in this bound. Suppose that $R\epsilon < \delta$ and that the image of $\{|Y| \leq R\}$ under $Y \mapsto \epsilon Y$ lies in $\text{Log } \mathcal{V}$. If $\mathcal{I}_{\gamma,\varphi}(T) \neq 0$, then $|\text{Log } T| \leq R\epsilon$, and the cycle radius is at most $R\epsilon < \delta$, since $\rho_G(H_i) \leq |\text{Log } H_i|$. Every contribution to either observable from $V \notin \mathcal{U}$ is therefore bounded by (36.19). The flat configurations near the other seven central triples, and all other flat configurations, still contribute to $\hat{Z}$.

Step 3: the output coordinates. For $T = \mathcal{H}_0^K$, Theorem 36.3 gives

$$|T_K(X, s) - X| \leq C_n(|X|^2 + |s|), \quad \|D_X T_K(X, s) - I\| \leq C_n(|X| + |s|). \quad (36.20)$$

For $T = \text{Hol}$, each cycle holonomy is a product of n link matrices of $V(X, s)$, which are analytic functions of (X, s) at the fixed side n , and at $s = 0$ the i -th holonomy is exactly e^{x_i} . Then $T_K(X, 0) = X$ and $D_X T_K(X, 0) = I$. Since these finite products and Log have bounded derivatives, (36.20) holds for this map as well, with constants independent of K .

Fix a convex ball $\{|X| \leq \rho\}$ and a ball in s on which $\|D_X T_K - I\| \leq 1/4$ for both maps. We set $b_K = T_K - X$ and restrict $|s|$ further so that $|b_K(0, s)| \leq \rho/8$. For $|Z| < \rho/4$, the map $X \mapsto Z - b_K(X, s)$ sends the closed ball of radius $\rho/2$ into itself, since $|Z - b_K(X, s)| \leq |Z| + |b_K(0, s)| + |X|/4 \leq \rho/2$, and it is a contraction with factor at most $1/4$. Its iterates converge to a unique fixed point $X = H_K(Z, s)$, which solves $T_K(X, s) = Z$. The inequality

$$|T_K(X, s) - T_K(X', s)| \geq \frac{3}{4}|X - X'|$$

on the larger convex ball excludes any other preimage there. Differentiating the fixed point equation, or taking difference quotients and using the uniformly convergent Neumann series for $(I + D_X b_K)^{-1}$, we obtain

$$D_Z H_K = (I + D_X b_K(H_K, s))^{-1}.$$

The fixed point equation first gives $|H_K| \leq |Z| + |b_K(0, s)| + |H_K|/4$, which implies $|H_K| \leq C_n(|Z| + |s|)$, and then

$$|H_K(Z, s) - Z| \leq C_n(|Z|^2 + |s|), \quad |\det D_Z H_K(Z, s) - 1| \leq C_n(|Z| + |s|). \quad (36.21)$$

For the determinant, we expand about I_9 and use the bound on $D_X b_K$; the determinant is positive since $D_Z H_K$ stays close to I_9 . This proves the second part of the theorem. We may take the neighborhood $\mathcal{U}$ of Step 2 inside this domain, so that points outside it whose output lies in the compact window are controlled by (36.19). Inside $\mathcal{U}$, after a further fixed reduction of ρ , (36.20) implies the lower bound $|T_K(X, s)| \geq (3/4)|X| - C_n|s|$ for both maps. Because of its term in s , we use it together with the tail bound of Step 4.

Step 4: the transverse tail and the change of variables. Since $f_K \geq 0$, (36.17) and the bounds on the density factor give

$$\widehat{v}(\{|s| > \epsilon\} \cap \{V \text{ in the central coordinates}\}) \leq C_n g^{-4} e^{-c_n \epsilon^2 / g^2} = C_n g^{-4} e^{-c'_n / \sqrt{\gamma}}. \quad (36.22)$$

To see this, we split $e^{-c_n |s|^2 / g^2}$ into two equal factors and bound one of them by $e^{-c_n \epsilon^2 / (2g^2)}$ on $\{|s| > \epsilon\}$. Integrating the other over $\mathbb{R}^D$ gives $C_n g^D$. The X -integral is over a bounded set, and we divide by (36.18). In the last equality, we used $\epsilon^2 / g^2 = n / \sqrt{\gamma}$. Both (36.19) and (36.22) are smaller than every fixed power of ϵ : with $v = 1 / \sqrt{\gamma}$, each is a fixed power of v times e^{-cv} or e^{-cv^2} .

On $\{|s| \leq \epsilon\}$, an output with $|T_K| \leq R\epsilon$ satisfies $|X| \leq C_{n,R}\epsilon$ by the lower bound of Step 3. For γ_R small, this lies in the domain of the inverse. Conversely, the inverse is defined for all $|Z| \leq R\epsilon$ and $|s| \leq \epsilon$, and the bound $|H_K| \leq C_{n,R}\epsilon$ places its values in $\mathcal{U}$ for γ_R small. The part of the unnormalized integral of the observable over $\mathcal{U} \cap \{|s| \leq \epsilon\}$ is therefore transformed exactly by

$$Z = \epsilon Y, \quad s = gz, \quad X = H_K(\epsilon Y, gz), \quad dX ds = \epsilon^9 g^D \det D_Z H_K(\epsilon Y, gz) dY dz. \quad (36.23)$$

In these variables the test is exactly $\varphi(Y)$, and no regularity of φ is needed. The variable z ranges over the ball $\{|z| \leq \epsilon/g\}$, and the test restricts Y to $\{|Y| \leq R\}$. Since $g = \epsilon^2 / \sqrt{n}$, (36.21) gives on this domain

$$\left| \frac{X}{\epsilon} - Y \right| \leq C_{n,R}\epsilon(1 + |z|), \quad |\det D_Z H_K - 1| \leq C_{n,R}\epsilon(1 + |z|), \quad \frac{|X|}{\epsilon} \leq C_{n,R}. \quad (36.24)$$

The last bound uses $|s| \leq \epsilon$, so it holds on the whole truncated ball in z , even when $|z|$ is large.

Step 5: comparison of the density. By Theorem 36.2 and $ng^2 = \epsilon^4$, we have $f_K(X) / g^2 = Q(X/\epsilon) + O(|X|^5 / g^2)$, and therefore

$$\left| \frac{f_K(X)}{g^2} - Q(X/\epsilon) \right| \leq C_n \frac{|X|^5}{g^2} \leq C_{n,R}\epsilon.$$

Since X/ϵ and Y lie in a fixed ball, on which Q has bounded first derivatives, the first bound in (36.24) gives

$$\left| \frac{f_K(X)}{g^2} - Q(Y) \right| \leq C_{n,R}\epsilon(1 + |z|). \quad (36.25)$$

This uses only $|X| \leq C_{n,R}\epsilon$, which holds on the whole truncated domain. In particular, it bounds the absolute error on the compact rescaled window, including the directions of commuting triples, where Q vanishes. The expansion in (36.17) gives

$$\left| \frac{F_K(X, gz) - f_K(X)}{g^2} - \frac{1}{2}z^\top A_K(0)z \right| \leq C_{n,R}(\epsilon|z|^2 + g|z|^3), \quad (36.26)$$

where we used $\|A_K(X) - A_K(0)\| \leq C_n|X| \leq C_{n,R}\epsilon$ and the cubic remainder in $s = gz$. We also use the elementary inequality

$$|e^{-A} - e^{-B}| \leq |A - B|(e^{-A} + e^{-B}) \quad (A, B \in \mathbb{R}), \quad (36.27)$$

which holds because the integral of the derivative of $t \mapsto e^{-t}$ between A and B is at most $|A - B| \max(e^{-A}, e^{-B})$.

Let $P_K(Y, z) = F_K(H_K(\epsilon Y, gz), gz)/g^2$ and $P_K^0(Y, z) = Q(Y) + z^\top A_K(0)z/2$. Both are at least $c_n|z|^2$, by $f_K \geq 0$, $Q \geq 0$, and the lower bounds in (36.17). Combining (36.25) and (36.26), and using $g \leq C_n\epsilon^2 \leq C_n\epsilon$, we find

$$|P_K - P_K^0| \leq C_{n,R}\epsilon(1 + |z| + |z|^2 + |z|^3).$$

By (36.27), $|e^{-P_K} - e^{-P_K^0}|$ is at most $C_{n,R}\epsilon$ times a fixed polynomial in $|z|$ times $e^{-c_n|z|^2}$, which is integrable over $\mathbb{R}^D$ uniformly in K and γ ; the phase error itself need not be uniformly small on the whole truncated ball in z . For the density factor and the Jacobian, we write $b_{K,\gamma}(Y, z) = a_{K,\gamma}(X, gz) \det D_Z H_K(\epsilon Y, gz)$ with $X = H_K(\epsilon Y, gz)$. By (36.16) and (36.24), these quantities are bounded on our domain and

$$|b_{K,\gamma}(Y, z) - a_{K,\gamma}^0| \leq C_{n,R}\epsilon(1 + |z|).$$

Splitting the difference as $(b_{K,\gamma} - a_{K,\gamma}^0)e^{-P_K} + a_{K,\gamma}^0(e^{-P_K} - e^{-P_K^0})$ and using that $t \mapsto (1 + t + t^2 + t^3)e^{-c_n t^2/2}/(1 + t)$ is bounded on $[0, \infty)$, we obtain

$$\left| b_{K,\gamma}(Y, z)e^{-P_K(Y, z)} - a_{K,\gamma}^0 e^{-P_K^0(Y, z)} \right| \leq C_{n,R}\epsilon(1 + |z|)e^{-c'_n|z|^2} \quad (|Y| \leq R, |z| \leq \epsilon/g), \quad (36.28)$$

with $c'_n = c_n/2$.

Step 6: integration and the common factor. The Gaussian integral of $a_{K,\gamma}^0 e^{-P_K^0}$ over $\{|z| > \epsilon/g\}$ is exponentially small, since $\epsilon/g = \sqrt{n}/\epsilon \rightarrow \infty$. Integrating (36.28) first in z and then over $\{|Y| \leq R\}$, we obtain

$$\widehat{\mathbb{Z}} \mathbb{E}_{\widehat{\nu}} \mathcal{I}_{\gamma, \varphi}(T) = g^D \epsilon^9 \left\{ a_{K,\gamma}^0 \frac{(2\pi)^{D/2}}{\sqrt{\det A_K(0)}} \int \varphi(Y) e^{-Q(Y)} dY + O_{n,R}(\|\varphi\|_\infty \epsilon) \right\}. \quad (36.29)$$

The contributions discarded in Steps 2 and 4 satisfy the same bound after reducing γ_R : after normalization by $\widehat{Z}$ they are $O_{n,R}(\|\varphi\|_\infty \epsilon^2)$, while the prefactor of the claimed expansion is bounded above and below by fixed multiples of ϵ .

Finally, since $\epsilon^4 = \gamma = ng^2$, we have the identity

$$\frac{g^D \epsilon^9}{\widehat{Z}} = \epsilon n^2 g \frac{g^{D+4}}{\widehat{Z}}.$$

Thus the coefficient of $\int \varphi dm_Q$ in (36.29), divided by $\widehat{Z}$, is exactly $\epsilon \kappa_{K,\gamma}$. The bounds $c_n \leq \kappa_{K,\gamma} \leq C_n$ follow from (36.18), $c_n \leq a_{K,\gamma}^0 \leq C_n$, and $c_n I \leq A_K(0) \leq C_n I$ in the fixed dimension D . Dividing by $\epsilon \kappa_{K,\gamma}$, we obtain the stated bound on $\mathcal{E}_{K,\gamma,T,\varphi}$. The leading term involves only $a_{K,\gamma}^0$ and $A_K(0)$, which are the same for both maps; the factor is therefore the same for both maps and every test. Every reduction used a radius or an upper bound on the coupling depending only on the fixed data and R . $\square$

36.6 Passage to the continuum measure

We now pass from the stopped measure to the conditioned Wilson measure, then to the Wilson measure, and finally to the continuum measure. The order of limits is as follows: at each fixed γ we let $K \rightarrow \infty$ with the comparison side n fixed, and only afterwards do we let $\gamma \rightarrow 0$. Throughout, $\lambda_\gamma = \lambda_{L,\gamma,[n_*]_M}$ denotes the continuum measure for the sequence of sides $(n_* B^K)_{K \geq 0}$ (Theorem 29.25 and Corollary 29.26). Since $n = n_* B^a$, the classes $[n]_M$ and $[n_*]_M$ coincide. We observe the holonomies of a trajectory around the three coordinate cycles based at the origin, at the time $t = \theta L^2$. Under λ_γ this triple is defined up to simultaneous conjugation, and we use invariant tests.

Theorem 36.7. *Fix the data of Definition 36.1 before varying γ . For every continuous compactly supported function $\varphi : \mathbb{R}^9 \rightarrow \mathbb{R}$ that is invariant under simultaneous conjugation and every radial cutoff χ ,*

$$\lim_{\gamma \downarrow 0} \frac{\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma,\varphi}}{\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma,\chi(|\cdot|)}} = \frac{\int \varphi dm_Q}{\int \chi(|Y|) dm_Q(Y)}.$$

For every such φ that is nonnegative and not identically zero, there exist $0 < c_\varphi \leq C_\varphi < \infty$ and $\gamma_\varphi > 0$ such that

$$c_\varphi r_\gamma \leq \mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma,\varphi} \leq C_\varphi r_\gamma \quad (0 < \gamma \leq \gamma_\varphi).$$

In particular the denominator above is positive for small γ , and

$$\lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma,\lambda_\gamma}(\chi) = 0.$$

Further, for any two smooth compactly supported functions χ_1, χ_2 on $[0, \infty)$ that are not identically zero, with $A_i = \mathcal{I}_{\gamma,\chi_i}(|\cdot|)$,

$$\frac{\text{Var}_{\lambda_\gamma}(A_1)}{\text{Var}_{\lambda_\gamma}(A_2)} \longrightarrow \frac{\int \chi_1(|Y|)^2 dm_Q(Y)}{\int \chi_2(|Y|)^2 dm_Q(Y)} \quad (\gamma \downarrow 0).$$

The tests χ_1, χ_2 in the variance statement need not be nonnegative. The loops and the flow time are fixed, while the window shrinks with γ . Each expectation under λ_γ is the limit, as $K \rightarrow \infty$ at fixed γ , of the expectation of the corresponding observable of the flowed lattice holonomies under the Wilson measure $\mu_{nB^K, \gamma}$. By Corollary 23.11, the small-field failure probability satisfies $\sup_K \mu_{nB^K, \gamma}(S_{nB^K, n, \gamma}^c) = O_{n, q}(\gamma^q)$ for every fixed $q > 0$, which is $o(r_\gamma)$.

Proof. For a compactly supported bounded test, we write $m(\varphi) = \int \varphi dm_Q$, and we set $\epsilon = r_\gamma$. Until Step 4, where we pass to the continuum measure, we work at finite K , uniformly in K .

Step 1: comparison under the conditioned Wilson measure. Let the support of φ lie in the ball of radius R , and let $0 < \eta \leq 1$. For $\eta\epsilon < 1$, Markov's inequality and the bound on $\mathbb{E}_\nu \min(1, D_K)^2$ in Theorem 36.3 give

$$\nu\{D_K > \eta\epsilon\} \leq C_n \eta^{-2} \frac{\gamma}{\epsilon^2} = C_n \eta^{-2} \gamma^{1/2}. \quad (36.30)$$

On the complementary event, if either of $\mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K)$ and $\mathcal{I}_{\gamma, \varphi}(\mathcal{H}_0^K(V))$ is nonzero, the corresponding triple is within distance $C_R \epsilon$ of (I, I, I) as a triple of matrices, and the other triple is within an additional distance $\eta\epsilon$. For γ small, both triples therefore lie in a fixed smaller neighborhood of the identity, independently of K . Since Log is Lipschitz there, their rescaled logarithms differ by at most $C_n \eta$, and the rescaled logarithm of $\mathcal{H}_0^K(V)$ lies in the ball of radius $R + C_n$.

Let

$$\omega_\varphi(v) = \sup\{|\varphi(Y) - \varphi(Y')| : |Y - Y'| \leq v\},$$

which tends to zero as $v \downarrow 0$ since φ is uniformly continuous. Theorem 36.6, applied to the indicator function of the ball of radius $R + C_n$, shows that the event that the rescaled logarithm of $\mathcal{H}_0^K(V)$ lies in this ball has $\hat{\nu}$ -probability at most $C_{n, R} \epsilon$. Since $\mathcal{H}_0^K(V)$ depends only on V , and the law of V under ν is $\hat{\nu}$, we obtain

$$\left| \mathbb{E}_\nu \mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K) - \mathbb{E}_{\hat{\nu}} \mathcal{I}_{\gamma, \varphi}(\mathcal{H}_0^K) \right| \leq C_{n, R} \epsilon \omega_\varphi(C_n \eta) + C_n \|\varphi\|_\infty \eta^{-2} \gamma^{1/2}. \quad (36.31)$$

Here the first term bounds the difference on $\{D_K \leq \eta\epsilon\}$, and the second term bounds it on the event in (36.30). After division by ϵ , the right side tends to zero if we first let $\gamma \downarrow 0$ at fixed η and then let $\eta \downarrow 0$, uniformly in K . The factor ϵ in front of the modulus of continuity is essential; it comes from the small probability of the window, and uniform continuity alone would not give it.

Step 2: removing the conditioning. Let $e_{K, \gamma} = \mu_{nB^K, \gamma}(S_{nB^K, n, \gamma}^c)$. At our fixed side n , we have $e_{K, \gamma} \leq C_{n, q} \gamma^q$ for every fixed $q > 0$, uniformly in K , by Corollary 23.11. For any bounded observable F , conditioning on the small-field event gives

$$\mathbb{E}_\mu F = (1 - e_{K, \gamma}) \mathbb{E}_\nu F + e_{K, \gamma} \mathbb{E}_{\mu(\cdot | S^c)} F, \quad |\mathbb{E}_\mu F - \mathbb{E}_\nu F| \leq 2 \|F\|_\infty e_{K, \gamma},$$

where $\mu = \mu_{nBK, \gamma}$; if $e_{K, \gamma} = 0$ the inequality holds without the second conditional expectation. For $q > 1/4$, this error is $o(\epsilon)$ uniformly in K . Combining this with (36.31) and Theorem 36.6, we obtain, for each fixed continuous compactly supported test,

$$\mathbb{E}_{\mu_{nBK, \gamma}} \mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K) = \epsilon \kappa_{K, \gamma} m(\varphi) + \epsilon \delta_{K, \gamma}^\varphi, \quad \sup_K |\delta_{K, \gamma}^\varphi| \longrightarrow 0 \quad (\gamma \downarrow 0). \quad (36.32)$$

The signed error $\delta_{K, \gamma}^\varphi$ converges by the explicit bound with two parameters above. Given a tolerance, we first choose η to control the term with the modulus of continuity, and then γ to control the remaining terms. No choice of the comparison side or of the parameters depends on γ .

Step 3: bounds and ratios at finite K . If $\varphi \geq 0$ is continuous and not identically zero, it is positive on a nonempty open ball, and since e^{-Q} is positive everywhere, $m(\varphi) > 0$. By the bounds on $\kappa_{K, \gamma}$ in Theorem 36.6 and (36.32), there exist constants $0 < c_\varphi \leq C_\varphi$ with

$$c_\varphi \epsilon \leq \mathbb{E}_{\mu_{nBK, \gamma}} \mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K) \leq C_\varphi \epsilon \quad (36.33)$$

for all K and all sufficiently small γ . For a general φ and the fixed radial cutoff χ , let $b = m(\varphi)/m(\chi(|\cdot|))$. We apply (36.32) to $\varphi - b\chi(|\cdot|)$, using $\mathcal{I}_{\gamma, \varphi - b\chi(|\cdot|)} = \mathcal{I}_{\gamma, \varphi} - b\mathcal{I}_{\gamma, \chi(|\cdot|)}$. The leading term vanishes, and the expectation of this observable is $o(\epsilon)$ uniformly in K . Since (36.33) gives a lower bound $c\epsilon$ for the denominator, uniformly in K , the ratio of the expectations under the Wilson measure tends to b as $\gamma \downarrow 0$, uniformly in K .

Step 4: passage to the continuum measure. We fix γ and identify the limit as $K \rightarrow \infty$. For small γ , the support of the observable lies strictly inside the logarithmic chart, and $\mathcal{I}_{\gamma, \varphi}$, extended by zero, is a continuous function of the triple. It is invariant under simultaneous conjugation because $\mathcal{V}$ and φ are invariant and the logarithm is conjugation covariant in its chart. By Lemma 5.6, $\mathcal{I}_{\gamma, \varphi}$ is therefore a continuous function of the normalized traces of $H_1, H_2, H_3, H_1H_2, H_1H_3, H_2H_3$, and $H_1H_2H_3$, and convergence in law of these seven traces implies convergence of the expectation of $\mathcal{I}_{\gamma, \varphi}$. The seven traces are characters, divided by two, of the holonomies of loops obtained by concatenating the three based coordinate cycles, observed at the same time. These loops and their lattice versions are covered by Theorem 29.25.

To apply that theorem at a given γ , we may have to drop finitely many terms of the sequence $n_* B^K$. By Corollary 29.26, the shortened sequence has the same tail and hence the same limit of the joint law of the flowed loops at fixed coupling, and neither the comparison side in (36.32) nor its constants change.

We now let $K \rightarrow \infty$ at fixed γ . The bounds (36.33) pass to the limit, and the expectation of the observable of $\varphi - b\chi(|\cdot|)$ remains $o(\epsilon)$, since that bound was uniform in K . Dividing by the limiting denominator and then letting $\gamma \downarrow 0$, we obtain the ratio limit and the two-sided bounds for λ_γ .

Step 5: the dilation statistic and the variances. The function W_χ is continuous, compactly supported, and invariant, and $m(W_\chi) = 0$ by Lemma 36.4. Applying the ratio limit to W_χ , we obtain $\mathcal{S}_{\gamma, \lambda_\gamma}(\chi) \rightarrow 0$.

For the variances, let $m_i = m(\chi_i(|\cdot|)^2) > 0$ and $b = m_1/m_2$. Since $A_i^2 = \mathcal{I}_{\gamma, \chi_i(|\cdot|)^2}$, (36.32) applied to $\chi_1(|\cdot|)^2 - b\chi_2(|\cdot|)^2$ shows that $\mathbb{E}A_1^2 - b\mathbb{E}A_2^2 = o(\epsilon)$ under the Wilson measure, uniformly in K . By the same formula, each mean $\mathbb{E}A_i$ is $O(\epsilon)$, and subtracting the squared means changes this difference by $O(\epsilon^2)$. Further, since $\mathbb{E}A_2^2 \geq c\epsilon m_2$ and $(\mathbb{E}A_2)^2 = O(\epsilon^2)$, we have $\text{Var } A_2 \geq c'\epsilon$ for small γ , uniformly in K . We conclude that $\text{Var } A_1 / \text{Var } A_2 - b \rightarrow 0$ uniformly in K . At fixed γ , the first and second moments pass to λ_γ by the identification of Step 4, and the ratio of the variances tends to b . $\square$

36.7 Rates, central triples, and several times

This section strengthens Theorem 36.7 in three directions: a rate $\gamma^{1/4}$ for Lipschitz tests, the extension to the eight central triples, and joint statements at finitely many flow times.

Definition 36.8. For a bounded Borel function φ on $\mathbb{R}^9$ with compact support we write $m(\varphi) = \int_{\mathbb{R}^9} \varphi(Y) e^{-Q(Y)} d^9Y$. For a bounded function f on $(\mathbb{R}^9)^\ell$, $\ell \geq 1$, we write

$$\|f\|_{\text{BL}} = \|f\|_\infty + \text{Lip } f, \quad \|(Y^{(1)}, \dots, Y^{(\ell)})\|_{\max} = \max_{1 \leq j \leq \ell} |Y^{(j)}|,$$

where $\text{Lip } f$ is the global Lipschitz constant with respect to the metric of $\|\cdot\|_{\max}$; for $\ell = 1$ this is the Euclidean metric on $\mathbb{R}^9$. We write B_R for the closed Euclidean ball of radius R about the origin in $\mathbb{R}^9$. The coupling-dependent scale is always written r_γ ; the radius r in the domains B_r and D_r of configurations keeps its fixed meaning from Definition 36.1.

The norm $\|f\|_{\text{BL}}$ is a sum. Its unit ball is contained in the set of functions with $\|f\|_\infty \leq 1$ and $\text{Lip } f \leq 1$ that defines the bounded-Lipschitz distance d_{BL} used in Lemma 28.9, and it contains that set scaled by $1/2$. The distance dual to $\|f\|_{\text{BL}}$ is therefore equivalent to d_{BL} , within a factor two, but not identical to it.

Rates for compactly supported Lipschitz tests

The next lemma is the quantitative version of Step 1 of the proof of Theorem 36.7. It uses the comparison of holonomies conditional on the coarse configuration, Theorem 34.11. We work on the reconstruction probability space of Lemma 15.10; its gauge-orbit law is that of ν , so expectations of invariant observables agree. For a coarse configuration v , represented in B_r by the Borel selection of that lemma, let $\mathbb{Q}_v^K$ be the product of the conditional fluctuation kernels of all K renormalization group steps. Both cycle triples are computed in the basepoint frame of this selected representative. Their zero-fluctuation values and the chart values used in Theorem 36.6 differ by simultaneous conjugation

when the coarse representative changes. We identify their expectations across these frames only for invariant tests. On this reconstruction space the law is $\int Q_v^K(d \cdot) \hat{v}(dv)$.

Lemma 36.9. *For every $R < \infty$ there exist $C_R, \gamma_R > 0$, independent of K and of the test, such that every globally Lipschitz function $\varphi : \mathbb{R}^9 \rightarrow \mathbb{R}$ supported in B_R and invariant under simultaneous conjugation satisfies*

$$\left| \mathbb{E}_v \mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K) - \mathbb{E}_{\hat{v}} \mathcal{I}_{\gamma, \varphi}(\mathcal{H}_0^K) \right| \leq C_R \|\varphi\|_{\text{BL}} r_\gamma^2 \quad (36.34)$$

for all K and $0 < \gamma \leq \gamma_R$.

The proof is the mechanism described in Section 36.1: an event determined by V of probability $O(r_\gamma)$, a conditional matrix difference $O(r_\gamma^2)$, and a loss r_γ^{-1} from the rescaling of the logarithm.

Proof. Let $d = \min(1, \max_i \|\mathcal{H}_i^K - \mathcal{H}_{0,i}^K(v)\|_{\text{op}})$. By Theorem 34.11, applied with $\tau = T = \theta L^2$ and $p = 1, 2$, we have, for every coarse configuration v in the support of $\hat{v}$ and every K ,

$$\mathbb{E}_{Q_v^K} d \leq C r_\gamma^2, \quad \mathbb{E}_{Q_v^K} d^2 \leq C r_\gamma^4,$$

for γ below the upper bound on the coupling in that theorem. On the event $\{d \leq r_\gamma\}$, with $r_\gamma < 1$, if either observable is nonzero, then both triples lie in a fixed compact subset of the logarithmic chart, and their rescaled logarithms differ by at most Cd/r_γ . In this case, the zero-fluctuation triple also lies in the event

$$E_{R'} = \{ \mathcal{H}_0^K(v) \in \mathcal{V}, |\text{Log } \mathcal{H}_0^K(v)| \leq R' r_\gamma \},$$

where R' and the upper bound on γ depend only on R and the chart. The event $E_{R'}$ is determined by v . Theorem 36.6, applied to the indicator function of $B_{R'}$, implies that $\hat{v}(E_{R'}) \leq C_R r_\gamma$. By the disintegration of Lemma 15.10,

$$\mathbb{E}_v[d \mathbf{1}_{E_{R'}}] = \int_{E_{R'}} \mathbb{E}_{Q_v^K} d \hat{v}(dv) \leq C r_\gamma^2 \hat{v}(E_{R'}) \leq C_R r_\gamma^3.$$

The contribution of $\{d \leq r_\gamma\}$ to the left side of (36.34) is therefore at most $C \text{Lip } \varphi r_\gamma^{-1} \mathbb{E}_v[d \mathbf{1}_{E_{R'}}] \leq C_R (\text{Lip } \varphi) r_\gamma^2$. On $\{d > r_\gamma\}$ we use the bound $2\|\varphi\|_\infty$ and

$$\nu(d > r_\gamma) \leq r_\gamma^{-2} \mathbb{E}_v d^2 = r_\gamma^{-2} \int \mathbb{E}_{Q_v^K} d^2 \hat{v}(dv) \leq C r_\gamma^2.$$

This proves (36.34). □

Theorem 36.10. *Fix the data and the measures λ_γ of Theorem 36.7, and a radial cutoff χ , and set $b_\chi = \int \chi(|Y|) dm_Q(Y) > 0$. For every $R < \infty$ there exist $C_{R, \chi}, \gamma_{R, \chi} > 0$ such that every*

globally Lipschitz function φ supported in B_R and invariant under simultaneous conjugation satisfies

$$\left| \frac{\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma, \varphi}}{\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma, \chi(|\cdot|)}} - \frac{\int \varphi dm_Q}{b_\chi} \right| \leq C_{R, \chi} (\|\varphi\|_\infty + \text{Lip } \varphi) \gamma^{1/4} \quad (0 < \gamma \leq \gamma_{R, \chi}),$$

and the denominator is at least $c_\chi r_\gamma$. The same bounds hold for the expectations of $\mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K)$ and $\mathcal{I}_{\gamma, \chi(|\cdot|)}(\mathcal{H}^K)$ under the Wilson measures $\mu_{nB^K, \gamma}$, uniformly in K . In particular $|\mathcal{S}_{\gamma, \lambda_\gamma}(\chi)| \leq C_\chi \gamma^{1/4}$. For any two fixed smooth compactly supported functions χ_1, χ_2 on $[0, \infty)$ that are not identically zero, there exist $C_{\chi_1, \chi_2} < \infty$, $c_{\chi_2} > 0$, and $\gamma_{\chi_1, \chi_2} \in (0, \gamma_0]$ such that, whenever $0 < \gamma \leq \gamma_{\chi_1, \chi_2}$, the ratio of variances in Theorem 36.7 differs from $m(\chi_1(|\cdot|)^2)/m(\chi_2(|\cdot|)^2)$ by at most $C_{\chi_1, \chi_2} \gamma^{1/4}$, and $\text{Var } A_2 \geq c_{\chi_2} r_\gamma$. These assertions hold for λ_γ and, uniformly in K , for $\mu_{nB^K, \gamma}$.

Proof. We write $m(\varphi) = \int \varphi dm_Q$ and, in this proof only,

$$A_K(\varphi) = \mathbb{E}_{\mu_{nB^K, \gamma}} \mathcal{I}_{\gamma, \varphi}(\mathcal{H}^K).$$

We combine Lemma 36.9, Theorem 36.6, and the bound of Corollary 23.11 with the exponent $q = 1$. Since $\gamma = r_\gamma^4$, the error from removing the conditioning is at most $2\|\varphi\|_\infty C_{n,1} \gamma = 2C_{n,1} \|\varphi\|_\infty r_\gamma^4$, and we obtain

$$A_K(\varphi) = r_\gamma \kappa_{K, \gamma} m(\varphi) + O_R(\|\varphi\|_{\text{BL}} r_\gamma^2). \quad (36.35)$$

Applying (36.35) to $\chi(|\cdot|)$, which is globally Lipschitz, and using $\kappa_{K, \gamma} \geq c_n$, we find $A_K(\chi(|\cdot|)) \geq c_\chi r_\gamma$ for small γ . When we subtract $m(\varphi) A_K(\chi(|\cdot|))/b_\chi$ from $A_K(\varphi)$, the leading terms cancel at finite K . Since $|m(\varphi)| \leq m(\mathbf{1}_{B_R}) \|\varphi\|_\infty$, the difference is $O_{R, \chi}(\|\varphi\|_{\text{BL}} r_\gamma^2)$. Dividing by $A_K(\chi(|\cdot|)) \geq c_\chi r_\gamma$ proves the assertion at finite cutoff, with one upper bound on γ for the whole class of tests.

At fixed γ , these observables are continuous functions of the triple, supported strictly inside the logarithmic chart, and invariant under simultaneous conjugation. By Lemma 5.6, they factor continuously through the seven traces, which are traces of holonomies of concatenations of the three based cycles. We apply Theorem 29.25 and Corollary 29.26 to the combination $A_K(\varphi) - m(\varphi) A_K(\chi(|\cdot|))/b_\chi$ and to the denominator; the comparison side remains valid on the sequence $n_* B^K$, as in Step 4 of the proof of Theorem 36.7. Letting $K \rightarrow \infty$ at fixed γ proves the assertion for λ_γ .

The function W_χ is globally Lipschitz with a fixed support, and $m(W_\chi) = 0$ by Lemma 36.4. The ratio bound then gives the bound on the dilation statistic. For the variances, the argument is the same before and after the limit $K \rightarrow \infty$. The two means are $O(r_\gamma)$, and their squares are $O(r_\gamma^2)$. Applying (36.35) to the Lipschitz function $\chi_1(|\cdot|)^2 - b\chi_2(|\cdot|)^2$, we obtain

$$\mathbb{E} A_1^2 - b \mathbb{E} A_2^2 = O(r_\gamma^2), \quad b = \frac{m(\chi_1(|\cdot|)^2)}{m(\chi_2(|\cdot|)^2)} > 0.$$

By (36.35) and the uniform bound $\kappa_{K,\gamma} \geq c_n$, the leading coefficient of $\mathbb{E}A_2^2$ is at least $c_n m(\chi_2(|\cdot|)^2) > 0$. Choose γ_{χ_1, χ_2} so that the $O_{\chi_2}(r_\gamma^2)$ error in this second moment and the squared mean together are at most half this leading lower bound times r_γ , uniformly in K . This gives $\text{Var } A_2 \geq c_{\chi_2} r_\gamma$ throughout the stated range. Subtracting the squared means in the combination above and dividing by this lower bound proves the rate for the variances. The same bound passes to the continuum limit at each fixed coupling. $\square$

The eight central triples

The next lemma is the center symmetry of the periodic lattice Wilson measure [149], with a verification that it commutes with the Wilson flow. Here $\mu_{N,\gamma}$ is the Wilson measure of Definition 4.1, and the winding number $w_i(\ell) \in \mathbb{Z}$ of a closed lattice loop ℓ is the signed number of its crossings of the hyperplane $\{x_i = N - 1/2\}$ in lattice units.

Lemma 36.11. *Let $z = (z_1, z_2, z_3) \in \{I, -I\}^3$. There exists a central link configuration c^z , with all plaquette products equal to I and coordinate-cycle products equal to z_i , such that the map $\Gamma_z U = (c_e^z U_e)_e$ preserves $\mu_{N,\gamma}$ and commutes with the Wilson flow at every time. For every based closed lattice loop ℓ ,*

$$\text{Hol}_\ell(\Psi_t^{(N)}(\Gamma_z U)) = \left(\prod_{i=1}^3 z_i^{w_i(\ell)} \right) \text{Hol}_\ell(\Psi_t^{(N)} U).$$

In particular, for any finite list of times, the vector of coordinate-cycle holonomies of the flowed configuration has the same law under $\mu_{N,\gamma}$ after the i -th holonomy is multiplied by z_i at every listed time. These are exact identities for every $N \geq 4$ and $\gamma > 0$; they do not concern the small-field event.

Proof. We represent the vertices by $x \in \{0, \dots, N-1\}^3$ and define, on the positive link from x in direction i ,

$$c_{x,i}^z = \begin{cases} z_i, & x_i = N-1, \\ I, & x_i \neq N-1, \end{cases} \quad c_e^z = (c_e^z)^{-1}.$$

In an (i, j) plaquette, the two links in direction i carry the same factor, because a displacement in direction j does not change x_i , and similarly for the two links in direction j . Since the factors are central, they cancel against the oppositely oriented factors. Hence every plaquette product of c^z is I , including the plaquettes that cross a periodic seam. A positive coordinate cycle in direction i crosses the seam in its direction once, and its product is z_i . For a general closed loop, the signed number of crossings of the i -th seam is $w_i(\ell)$. Since the factors are central, they can be moved past every link matrix, and the product of all factors is $\prod_i z_i^{w_i(\ell)}$.

The map Γ_z is a smooth involution of the configuration space, and the product Haar measure is invariant under it, since Haar measure is invariant under multiplication by a fixed element (Lemma 5.1). Every plaquette matrix is unchanged, and so are every $q(U_p)$, the Wilson action, and the normalization constant. This proves that $\mu_{N,\gamma}$ is invariant.

We use the product of the bi-invariant Riemannian metrics, as in the definition of the Wilson flow. Since multiplication by a fixed central element is an isometry, so is Γ_z . Let $S(U) = \sum_p q(U_p)$, so that $S \circ \Gamma_z = S$. For every tangent vector v at U ,

$$\langle \nabla S(\Gamma_z U), D\Gamma_z v \rangle_{\Gamma_z U} = dS_{\Gamma_z U}(D\Gamma_z v) = dS_U(v) = \langle D\Gamma_z \nabla S(U), D\Gamma_z v \rangle_{\Gamma_z U},$$

which implies $\nabla S(\Gamma_z U) = D\Gamma_z \nabla S(U)$. The smooth vector field $-\nabla S$ on the compact configuration space has a unique global flow, and Γ_z maps its trajectories to trajectories. This proves commutation at every lattice time s , and hence for $\Psi_t^{(N)} = \Phi_{4t/a_N^2}^W$ at every physical time $t \geq 0$. The identity for the holonomies follows from the identity for the products of the factors. Integrating any bounded Borel function of the finite vector of holonomies, we obtain the identity in law. $\square$

Theorem 36.12. *Choose $\mathcal{V}$ in Definition 36.1 invariant under simultaneous conjugation. For $z \in \{I, -I\}^3$ let $\mathcal{V}_z = z\mathcal{V}$, and set*

$$\mathcal{I}_{\gamma,\varphi}^z(H) = \begin{cases} \varphi((\text{Log}(z_i^{-1}H_i))_{i=1}^3/r_\gamma), & H \in \mathcal{V}_z, \\ 0, & H \notin \mathcal{V}_z, \end{cases}$$

so that $\mathcal{I}_{\gamma,\varphi}^{(I,I,I)} = \mathcal{I}_{\gamma,\varphi}$. For every $\gamma > 0$, every positive observation time, and every bounded Borel compactly supported test φ invariant under simultaneous conjugation,

$$\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma,\varphi}^z = \mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma,\varphi}.$$

The corresponding identity holds under $\mu_{n\text{BK},\gamma}$ for the flowed lattice holonomies at every finite K , for every bounded Borel compactly supported test. Consequently the ratio limits, the two-sided bounds, the limits of the ratios of variances, and the limit of the dilation statistic in Theorem 36.7, and the rates of Theorem 36.10, hold at each of the eight central triples, with expectations equal to those at (I, I, I) and with constants and ceilings on γ independent of z .

Proof. Let H denote the triple of flowed coordinate-cycle holonomies. At finite cutoff, Lemma 36.11 shows that H and $z^{-1}H$ have the same law. By definition, $\mathcal{I}_{\gamma,\varphi}^z(H) = \mathcal{I}_{\gamma,\varphi}(z^{-1}H)$ pointwise, including on the complements of the charts, since $H \in z\mathcal{V}$ if and only if $z^{-1}H \in \mathcal{V}$. This proves the identity at finite cutoff, without any assumption of regularity or invariance on the test.

For the continuum measure, let F be any continuous function on $\text{SU}(2)^3$ invariant under simultaneous conjugation. Since z is central, $F(z^{-1}\cdot)$ is also continuous and invariant,

and both factor continuously through the seven traces by Lemma 5.6. Passing the identity $\mathbb{E}F(H) = \mathbb{E}F(z^{-1}H)$ at finite cutoff to the limit at every fixed $\gamma > 0$ by Theorems 29.28 and 30.6, we obtain $\mathbb{E}_{\lambda_\gamma}F(H) = \mathbb{E}_{\lambda_\gamma}F(z^{-1}H)$ for every such F . By the same lemma, the compact quotient of $\mathrm{SU}(2)^3$ by simultaneous conjugation is homeomorphic to the image of the seven traces. The law of the conjugacy class of H under λ_γ , a Borel probability on this compact metric space, is therefore invariant under the continuous map induced by multiplication by z^{-1} . Indeed, continuous functions determine Borel probabilities on a metric space. For a closed set C , the continuous functions $\max(0, 1 - j \operatorname{dist}(\cdot, C))$ decrease to $\mathbf{1}_C$, so the two probabilities agree on closed sets, and hence on all Borel sets by the monotone class theorem. The identity then extends to bounded Borel functions on the quotient. Since $\mathcal{V}$ is invariant, the logarithm is conjugation covariant on it, and φ is invariant, the observable $\mathcal{I}_{\gamma, \varphi}^z$ is a bounded Borel function on the quotient. This proves the identity for the continuum measure, even if φ is not continuous.

Applying the identity to each numerator and denominator in Theorems 36.7 and 36.10, and to the squares of the observables for the variances, we obtain the remaining assertions. Since all eight expectations are equal, so are the constants. $\square$

Joint asymptotics at finitely many times

We now observe the holonomies at finitely many flow times. The observables are invariant under an independent conjugation at each time; no relation between the gauge frames at different times is used or asserted.

Definition 36.13. Fix $\ell \geq 1$, positive times $t_j = \theta_j L^2$ for $1 \leq j \leq \ell$, and the same three based coordinate cycles at each time, before varying γ . Choose the parameters, the side n , and the margins as in Definition 36.1, so that the finitely many lower bounds required at each of the times t_j hold. Write $H^{(j)}$ for the triple of cycle holonomies at time t_j . Let $\mathcal{V}$ be a neighborhood of (I, I, I) as in Definition 36.1, invariant under simultaneous conjugation and small enough for all listed times. For $\Phi \in C_c((\mathbb{R}^9)^\ell; \mathbb{R})$ define

$$\mathcal{J}_{\gamma, \Phi}(H^{(1)}, \dots, H^{(\ell)}) = \begin{cases} \Phi(\operatorname{Log} H^{(1)}/r_\gamma, \dots, \operatorname{Log} H^{(\ell)}/r_\gamma), & H^{(j)} \in \mathcal{V} \text{ for every } j, \\ 0, & \text{otherwise.} \end{cases}$$

The class of tests consists of the functions Φ that are invariant under independent conjugations at the different times:

$$\Phi(\operatorname{Ad}(k_1)Y^{(1)}, \dots, \operatorname{Ad}(k_\ell)Y^{(\ell)}) = \Phi(Y^{(1)}, \dots, Y^{(\ell)}) \quad \text{for all } k_1, \dots, k_\ell \in \mathrm{SU}(2),$$

where each $\operatorname{Ad}(k_j)$ acts simultaneously on the three components of $Y^{(j)}$. Put $\Phi_\Delta(Y) = \Phi(Y, \dots, Y)$ and $m_\Delta(\Phi) = \int \Phi_\Delta dm_Q$. We write $\mathcal{H}_t^K$ and $\mathcal{H}_{0,t}^K(V)$ for the triples of Definition 36.1 observed at time t .

The following lemma localizes the transverse coordinates on the event that the flowed zero-fluctuation holonomies at the first time are in a window of size r_γ .

Lemma 36.14. *Use the stopped measure $\hat{\nu}$, the coordinates $V(X, s)$, and the zero-fluctuation holonomies at the first time t_1 of Definition 36.13. Set*

$$E_R = \{\mathcal{H}_{0,t_1}^K \in \mathcal{V}, |\text{Log } \mathcal{H}_{0,t_1}^K| \leq Rr_\gamma\}, \quad D = d_{\text{st}}, \quad g^2 = \gamma/n.$$

There is a fixed neighborhood $\mathcal{U}$ of the trivial based orbit, contained in the central coordinates, such that, for every fixed $R < \infty$, all sufficiently small γ , and every K ,

$$\hat{\nu}(E_R \setminus \mathcal{U}) \leq Cg^{-(D+4)}e^{-c/g^2}, \quad \mathbb{E}_{\hat{\nu}}[\mathbf{1}_{E_R \cap \mathcal{U}}|s|^a] \leq C_{R,a}r_\gamma g^a \quad (a = 0, 1, 2).$$

Further, $\hat{\nu}(\mathcal{U} \cap \{|s| > r_\gamma\}) \leq Cg^{-4}e^{-cr_\gamma^2/g^2}$.

Proof. We choose $\mathcal{U}$ with closure strictly inside the chart of Theorem 33.8 and inside the domains of the deterministic results at all listed times. By the localization statement of Theorem 36.3, applied with θ_1 in place of θ , there exists a fixed $\delta > 0$ such that a coarse configuration within distance δ of $\mathcal{F}_n$ whose holonomies at time t_1 have cycle radius at most δ lies in $\mathcal{U}$. For $Rr_\gamma < \delta$, every point of $E_R \setminus \mathcal{U}$ therefore lies outside the fixed neighborhood $\{d(V, \mathcal{F}_n) \leq \delta\}$. There $E_0 \geq c_{n,\delta} > 0$, by compactness of the configuration space at side n and $E_0^{-1}(0) = \mathcal{F}_n$, which follows from the classification of flat configurations in Lemma 14.9. The first estimate follows as in (36.19). We use properties (P1) and (P3) in Section 33.2, namely the lower bound $E_{JK,\text{sat}} \geq \kappa E_0$ on D_r and the bound on the centered effective interaction, together with the normalization of H_n and the lower bound on the normalization constant in Theorem 33.10.

On $\mathcal{U}$, Theorem 36.3 at time t_1 gives $|\text{Log } \mathcal{H}_{0,t_1}^K| \geq (3/4)|X| - C|s|$, which implies

$$|X| \leq C(Rr_\gamma + |s|) \quad \text{on } E_R \cap \mathcal{U}.$$

The density factor $e^{-\bar{R}_{K,\gamma}J_n}$ is bounded above, and $F_K(X, s) \geq f_K(X) + c|s|^2 \geq c|s|^2$ by Theorem 33.8. For fixed s , the possible values of X form a set of volume at most $C_R(r_\gamma + |s|)^9$. Substituting $s = gz$, we find

$$\begin{aligned} \mathbb{E}_{\hat{\nu}}[\mathbf{1}_{E_R \cap \mathcal{U}}|s|^a] &\leq \frac{C_R}{\hat{Z}} \int_{\mathbb{R}^D} |s|^a (r_\gamma + |s|)^9 e^{-c|s|^2/g^2} ds \\ &= \frac{C_R g^{D+a} r_\gamma^9}{\hat{Z}} \int_{\mathbb{R}^D} |z|^a \left(1 + \frac{g}{r_\gamma}|z|\right)^9 e^{-c|z|^2} dz. \end{aligned}$$

Since $g/r_\gamma = r_\gamma/\sqrt{n} \leq 1$, the Gaussian integral is bounded in the fixed dimension D . Since $\hat{Z} \geq cg^{D+4}$ and $r_\gamma^9 g^{-4} = n^2 r_\gamma$, the right side is at most $C_{R,a} r_\gamma g^a$. The conjugation at the basepoint has not been divided out: the nine-dimensional volume in X and the Haar density are kept.

For the last estimate there is no restriction on the output, and the volume in X is bounded by a fixed constant. We split the Gaussian factor into two factors, bound one of them on $\{|s| > r_\gamma\}$ by $e^{-cr_\gamma^2/(2g^2)}$, and integrate the other over $\mathbb{R}^D$. After division by $\hat{Z} \geq cg^{D+4}$, we obtain the bound $Cg^{-4}e^{-c'r_\gamma^2/g^2}$, and renaming c' proves the claim. Since $g = r_\gamma^2/\sqrt{n}$, the first and last bounds are smaller than every fixed power of r_γ . $\square$

Theorem 36.15. *Use Definition 36.13, and fix a radial cutoff χ . For every continuous compactly supported test Φ in the class of Definition 36.13,*

$$\lim_{\gamma \downarrow 0} \frac{\mathbb{E}_{\lambda_\gamma} \mathcal{J}_{\gamma, \Phi}}{\mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma, \chi(|\cdot|)}(H^{(1)})} = \frac{\int_{\mathbb{R}^9} \Phi(Y, \dots, Y) e^{-Q(Y)} dY}{\int_{\mathbb{R}^9} \chi(|Y|) e^{-Q(Y)} dY},$$

and the denominator is at least $c_\chi r_\gamma$ for small γ . If $\Phi \geq 0$ and $m_\Delta(\Phi) > 0$, the numerator is bounded above and below by positive multiples of r_γ ; if $m_\Delta(\Phi) = 0$, the numerator is $o(r_\gamma)$.

More precisely, with the factor $\kappa_{K, \gamma}$ of Theorem 36.6,

$$\mathbb{E}_{\mu_{nB^K, \gamma}} \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{t_1}^K, \dots, \mathcal{H}_{t_\ell}^K) = r_\gamma \kappa_{K, \gamma} \{m_\Delta(\Phi) + \delta_{K, \gamma}^\Phi\}, \quad \sup_K |\delta_{K, \gamma}^\Phi| \longrightarrow 0 \quad (\gamma \downarrow 0),$$

and the expectation of $\mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{0, t_1}^K, \dots, \mathcal{H}_{0, t_\ell}^K)$ under $\hat{v}$ has the same expansion with the same factor. For every $R < \infty$ there exist $C_{R, \chi, t}$ and $\gamma_{R, \chi, t} > 0$ such that, for every globally Lipschitz test Φ in the class with $\text{supp } \Phi \subset (B_R)^\ell$ and $0 < \gamma \leq \gamma_{R, \chi, t}$, the difference between the ratio above (for λ_γ , or for $\mu_{nB^K, \gamma}$ at any finite K) and $m_\Delta(\Phi)/m(\chi(|\cdot|))$ is at most $C_{R, \chi, t}(\|\Phi\|_\infty + \text{Lip } \Phi)\gamma^{1/4}$. Here Lip refers to the metric of $\|\cdot\|_{\max}$, and $\mathbf{t} = (t_1, \dots, t_\ell)$.

A nonnegative test that is not identically zero but vanishes near the diagonal has $m_\Delta(\Phi) = 0$, and its numerator is $o(r_\gamma)$: to leading order, the rescaled logarithms at the different times coincide. The diagonal is described through tests that are invariant under independent conjugations at the different times. Uniform statements on a time interval are proved in Theorem 36.18. The constants depend additionally on the list of times, on ℓ , R , and χ .

Proof. Step 1 reduces the joint observable of the zero-fluctuation holonomies to the single-time observable of Φ_Δ , Step 2 compares with the holonomies of the configuration, and Steps 3–5 treat Lipschitz tests, continuous tests, and the continuum measure. All estimates use the same coarse configuration V and the same reconstruction.

Step 1: the zero-fluctuation holonomies near the diagonal. For each listed time we apply Theorem 35.9 with $m = 0$. By its bound in $C^{1,1}$ at fixed time and its identity on flat configurations, we have, on a common smaller central neighborhood,

$$Z_j(X, s) = \text{Log } \mathcal{H}_{0, t_j}^K(V(X, s)) = X + O_t(|X|^2 + |s|), \quad \|D_X Z_j - I\| \leq C_t(|X| + |s|). \quad (36.36)$$

Indeed, each of the nine coordinate directions in X consists of commuting triples. The identity on flat configurations therefore gives $Z_j(ue_a, 0) = ue_a$, and hence $D_X Z_j(0, 0) = I$. Integrating the Lipschitz bound on the derivative along $v \mapsto (vX, 0)$, and then using the bounded derivative in s , we obtain (36.36). This is the statement of Theorem 36.3 at each listed time. We intersect finitely many domains and take the maximum of finitely many constants.

Let Φ be globally Lipschitz with support in $(B_R)^\ell$. We compare $\mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{0, t_1}^K, \dots, \mathcal{H}_{0, t_\ell}^K)$ with $\mathcal{I}_{\gamma, \Phi_\Delta}(\mathcal{H}_{0, t_1}^K)$. If either is nonzero, then the event E_R of Lemma 36.14 holds. Outside the fixed neighborhood $\mathcal{U}$ of that lemma its probability is smaller than every power of r_γ , and inside $\mathcal{U}$ we discard $\{|s| > r_\gamma\}$, again at a cost smaller than every power of r_γ . On the remaining set, the lower bound for Z_1 implies $|X| \leq C_R r_\gamma$. By (36.36), all outputs lie in $\mathcal{V}$ for small γ , and

$$\max_j \frac{|Z_j - Z_1|}{r_\gamma} \leq C_R \left(r_\gamma + \frac{|s|}{r_\gamma} \right).$$

On this set, both observables are given by their formulas in coordinates, $\Phi(Z_1/r_\gamma, \dots, Z_\ell/r_\gamma)$ and $\Phi(Z_1/r_\gamma, \dots, Z_1/r_\gamma)$, including when one of them vanishes. Lemma 36.14 gives, for any fixed $A \geq 2$,

$$\begin{aligned} & \left| \mathbb{E}_{\hat{\nu}} \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{0, t_1}^K, \dots, \mathcal{H}_{0, t_\ell}^K) - \mathbb{E}_{\hat{\nu}} \mathcal{I}_{\gamma, \Phi_\Delta}(\mathcal{H}_{0, t_1}^K) \right| \\ & \leq C_R (\text{Lip } \Phi) \left\{ r_\gamma \hat{\nu}(E_R \cap \mathcal{U}) + r_\gamma^{-1} \mathbb{E}_{\hat{\nu}}[|s| \mathbf{1}_{E_R \cap \mathcal{U}}] \right\} + O_A(\|\Phi\|_\infty r_\gamma^A) \\ & \leq C_R \|\Phi\|_{\text{BL}} r_\gamma^2, \end{aligned} \quad (36.37)$$

where in the last step we used $g = r_\gamma^2/\sqrt{n}$. The diagonal test Φ_Δ is globally Lipschitz, supported in B_R , and invariant under simultaneous conjugation. Theorem 36.6 at time t_1 therefore implies that

$$\mathbb{E}_{\hat{\nu}} \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{0, t_1}^K, \dots, \mathcal{H}_{0, t_\ell}^K) = r_\gamma \kappa_{K, \gamma} m_\Delta(\Phi) + O_R(\|\Phi\|_{\text{BL}} r_\gamma^2). \quad (36.38)$$

The factor is the same $\kappa_{K, \gamma}$ as at a single time, and no factor depending on the times appears.

Step 2: comparison at all listed times. Let $[\tau, T]$ be an interval containing all listed times, and let

$$d = \min \left(1, \sup_{\tau \leq t \leq T} \max_i \|\mathcal{H}_{t, i}^K - \mathcal{H}_{0, t, i}^K(v)\|_{\text{op}} \right).$$

By Theorem 34.11 with $p = 1, 2$, we have $\mathbb{E}_{Q_v^K} d \leq C r_\gamma^2$ and $\mathbb{E}_{Q_v^K} d^2 \leq C r_\gamma^4$ for every coarse configuration v and every K , where Q_v^K is the conditional law of Lemma 36.9. On $\{d \leq r_\gamma\}$, if either joint observable is nonzero, then all flowed holonomies of the configuration and of the zero-fluctuation reconstruction at the listed times lie in a fixed

compact subset of the logarithmic chart, and their rescaled logarithms differ by at most Cd/r_γ in the metric of $\|\cdot\|_{\max}$. In this case, the zero-fluctuation holonomies at time t_1 also lie in the enlarged window $E_{R'}$ of Lemma 36.14, which is determined by v and has probability $O_R(r_\gamma)$ by Theorem 36.6. The argument of Lemma 36.9, with the metric of $\|\cdot\|_{\max}$, gives

$$\left| \mathbb{E}_\nu \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{t_1}^K, \dots, \mathcal{H}_{t_\ell}^K) - \mathbb{E}_{\hat{\nu}} \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{0, t_1}^K, \dots, \mathcal{H}_{0, t_\ell}^K) \right| \leq C_{R, t} \|\Phi\|_{\text{BL}} r_\gamma^2. \quad (36.39)$$

The same argument at the time t_1 alone, applied to a Lipschitz function equal to one on B_{R+1} , shows that the event that $\mathcal{H}_{t_1}^K$ lies in $\mathcal{V}$ and its rescaled logarithm lies in B_{R+1} has ν -probability $O_R(r_\gamma)$.

Step 3: Lipschitz tests at finite cutoff. We remove the conditioning with Corollary 23.11 at $q = 1$, which costs at most $2\|\Phi\|_\infty C_{n,1} \gamma = 2C_{n,1} \|\Phi\|_\infty r_\gamma^4$. Combining this with (36.38) and (36.39), we obtain

$$\left| \mathbb{E}_{\mu_{nBK, \gamma}} \mathcal{J}_{\gamma, \Phi}(\mathcal{H}_{t_1}^K, \dots, \mathcal{H}_{t_\ell}^K) - r_\gamma \kappa_{K, \gamma} m_\Delta(\Phi) \right| \leq C_{R, t} \|\Phi\|_{\text{BL}} r_\gamma^2. \quad (36.40)$$

For the denominator, (36.35) at time t_1 holds with the same factor $\kappa_{K, \gamma}$, and the denominator is at least $c_\chi r_\gamma$. When we subtract $m_\Delta(\Phi)/m(\chi(|\cdot|))$ times the denominator, the factor $\kappa_{K, \gamma}$ cancels at finite K . Dividing by $c_\chi r_\gamma$, we obtain the rate $\gamma^{1/4} = r_\gamma$ for the ratio at finite cutoff, with one upper bound on γ for the whole class of tests supported in $(B_R)^\ell$.

Step 4: continuous tests. Let Φ be continuous, compactly supported, and in the class, with support in $(B_R)^\ell$. There exist smooth compactly supported functions Φ_η , $\eta > 0$, with support in $(B_{R+1})^\ell$ and $\|\Phi_\eta - \Phi\|_\infty \rightarrow 0$. For example, we may convolve Φ with the product of ℓ copies of a nonnegative radial smooth approximate identity on $\mathbb{R}^9$ of total mass one, and use the uniform continuity of Φ . Radial convolution preserves the invariance under independent conjugations, as we see by changing each integration variable by the orthogonal map $\text{Ad}(k_j)$. Each Φ_η is globally Lipschitz. The difference of the joint observables of Φ_η and Φ is nonzero only if the first output lies in a fixed enlarged compact window, which has probability $O_R(r_\gamma)$, both for the holonomies of the configuration under ν (by Step 2) and for the zero-fluctuation holonomies under $\hat{\nu}$ (by Theorem 36.6). The corresponding expectations therefore differ by at most $C_R r_\gamma \|\Phi_\eta - \Phi\|_\infty$. For the Wilson measure, we add at most $2C_{n,a} \gamma^a \|\Phi_\eta - \Phi\|_\infty$, with any fixed $a > 1/4$, from Corollary 23.11. Further,

$$|m_\Delta(\Phi_\eta) - m_\Delta(\Phi)| \leq m(\mathbf{1}_{B_{R+1}}) \|\Phi_\eta - \Phi\|_\infty.$$

At fixed η we apply (36.40) and (36.38) to Φ_η and let $\gamma \downarrow 0$ uniformly in K ; then we let $\eta \downarrow 0$. Since $c \leq \kappa_{K, \gamma} \leq C$, this proves $\sup_K |\delta_{K, \gamma}^\Phi| \rightarrow 0$, both for the Wilson measure and for the zero-fluctuation expectation. For continuous tests we do not obtain a rate.

Step 5: the continuum measure. At fixed small γ , the joint observable is a bounded continuous function on $(\mathrm{SU}(2)^3)^\ell$, since its support lies strictly inside the logarithmic charts. By the invariance under independent conjugations, it factors through the product of the ℓ maps $\mathfrak{p}$ of seven traces of Lemma 5.6: two points with the same vector of traces at every time differ by one conjugation at each time. The factored function f is continuous on the compact image C of this product map, since every convergent sequence of trace vectors has convergent subsequences of representatives, and all possible limits give the same value. On C , the functions $\inf_{u \in C} \{f(u) + k|v - u|\}$, truncated to the range of f , are bounded Lipschitz functions on the ambient Euclidean space that converge uniformly to f on C as $k \rightarrow \infty$. This follows from the uniform continuity of f , by considering separately the points u with $|v - u|$ small and large. Joint convergence in law of the traces therefore suffices for these observables. The seven words at each time are concatenations of the three based cycles at that time, and the whole finite vector of their traces is covered by Theorem 29.25, with the comparison side justified by Corollary 29.26. Letting $K \rightarrow \infty$ at fixed γ in the lower bound on the denominator and in the combination in which $\kappa_{K,\gamma}$ cancels proves the ratio limit for continuous tests and the rate for Lipschitz tests under λ_γ . The expansions and $c \leq \kappa_{K,\gamma} \leq C$ imply that the numerator is comparable with r_γ when $m_\Delta(\Phi) > 0$ and $\Phi \geq 0$, and that it is $o(r_\gamma)$ when $m_\Delta(\Phi) = 0$. These bounds are uniform in K and pass to the limit. $\square$

36.8 Holonomy paths on a time interval

We now observe the holonomies on a whole compact time interval $[\tau, T] \subset (0, \infty)$. The observable is the path of rescaled holonomy triples modulo simultaneous conjugation. After weighting with a compactly supported cutoff at the initial time, we show that this path converges to a constant path whose value has density proportional to the cutoff times e^{-Q} . The first ingredient is a version of Theorem 36.3 that is uniform in time.

Lemma 36.16. *Fix the data of Definition 36.1 and an interval $0 < \tau < T < \infty$ before varying γ , choosing the side n and the margins for the whole interval. There exist a central coordinate neighborhood $\mathcal{U}$ and a neighborhood of the identity in $\mathcal{V}$ such that*

$$Z_{K,t}(X, s) = \mathrm{Log} \mathcal{H}_{0,t}^K(V(X, s))$$

is defined on $\mathcal{U}$ for every K and $t \in [\tau, T]$, is continuous in t , and satisfies

$$\sup_{K,t} \|Z_{K,t}\|_{C^{1,1}(\mathcal{U})} \leq C_{\tau,T}, \quad \sup_t |Z_{K,t}(X, s) - X| \leq C_{\tau,T}(|X|^2 + |s|),$$

$$\sup_t \|D_X Z_{K,t}(X, s) - I\| \leq C_{\tau,T}(|X| + |s|).$$

For every flat coarse configuration, the flowed zero-fluctuation holonomy triple equals the holonomy triple of the coarse configuration, for every K and t . The neighborhoods and constants do not depend on K , γ , or the number of preceding averaging steps.

Proof. We work in the smooth chart $V(X, s)$ of the based quotient, not with a Borel selection of representatives; the parameters (X, s) have a fixed dimension at the side n . On a neighborhood of the trivial flat configuration, the zero-fluctuation reconstructions stay within the input and image bounds of the reconstruction maps of Theorem 15.13. Indeed, they vanish at zero input, and the uniform bounds on the coefficients together with the contraction condition $a_{\text{map}}\sqrt{B} \leq 32/49 < 1$ of Definition 36.1 give one smaller neighborhood, independent of the number of steps. More precisely, items (1) and (3) of Lemma 34.4 bound the linear and quadratic flow data $\mathcal{E}_K(u)$ of the zero-fluctuation reconstruction, in the weighted norms X_α^a and Y_α^a of Definition 4.8, by a constant R on a fixed complex domain of inputs. Here $a = L/(nB^K)$ is the finest mesh, and α is the fixed regularity exponent of Definition 34.1; the bound is uniform in K with this mesh. We compose with the analytic map $u = (X, s) \mapsto V(X, s)$ of finitely many variables and shrink the domain once. Let $U_0 \Subset U_1$ be real domains whose complex neighborhoods of radius $\delta > 0$ lie in this domain. The Cauchy integral formula on one coordinate circle, or on two circles for mixed derivatives, gives

$$\sup_{K, u \in U_1} \sum_{j=0}^2 \|D_u^j \mathcal{E}_K(u)\|_{\mathfrak{f}} \leq C_{n,\delta} R (1 + \delta^{-1} + \delta^{-2}),$$

where $\|\cdot\|_{\mathfrak{f}}$ is the flow-data norm at the finest mesh a , which we suppress from the notation. It is the sum of the two weighted norms, which include the first differences of both components. This bound is deterministic and does not depend on the coupling.

We apply Theorem 25.13 on $U_0 \subset U_1$, with waiting time $t_* = \tau$, horizon T , and damping gauge transformations started at the identity in the common gauge frame at the basepoint. Its hypotheses are the displayed bound on derivatives and the margins of the based charts. It yields bounds, uniform in K , on the first and second derivatives in the parameters of the flowed cycle holonomies, simultaneously for all $t \in [\tau, T]$, including the finitely many remaining meshes. Continuity in t follows from the smoothness of the flow on a finite lattice. At $u = 0$ every output is the identity. By the uniform bound on the first derivatives, the outputs on a smaller central neighborhood lie in one compact subset of the logarithmic chart, simultaneously for all K and t . Since Log has bounded first and second derivatives there, we obtain the bound in $C^{1,1}$.

By Lemma 14.9, the zero-fluctuation refinements of a flat configuration are flat and have the same based cycle holonomies, and the Wilson flow is stationary on flat configurations. The flowed holonomies therefore equal the coarse holonomies at every time, in the given gauge frame. For $s = 0$ and X in one of the nine coordinate directions,

the configuration consists of commuting links, and the identity on flat configurations gives $Z_{K,t}(ue_a, 0) = ue_a$. Since these directions span $\mathbb{R}^9$, we have $Z_{K,t}(0, 0) = 0$ and $D_X Z_{K,t}(0, 0) = I$ for all K and t . Integrating the uniform Lipschitz bound on the derivative along $v \mapsto (vX, 0)$ gives the remainder $C|X|^2$ and the error $C|X|$ for the derivative. The bounded derivative in s and the Lipschitz bound on the first derivative add $C|s|$ to these two errors. All choices were made before the coupling and are uniform on the interval. $\square$

Definition 36.17. Let

$$\mathfrak{C} = \mathfrak{su}(2)^3 / \text{Ad } \text{SU}(2), \quad \rho([Y], [Y']) = \min_{k \in \text{SU}(2)} |Y - \text{Ad}(k)Y'|,$$

$$\mathscr{P} = C([\tau, T]; \mathfrak{C}), \quad \rho_\infty(z, z') = \sup_{\tau \leq t \leq T} \rho(z(t), z'(t)).$$

Fix a smooth map $\eta : \text{SU}(2)^3 \rightarrow \mathfrak{su}(2)^3$, equivariant under simultaneous conjugation, that agrees with the componentwise logarithm on a fixed smaller neighborhood of the identity. For the triple $H(t)$ of cycle holonomies of a trajectory, defined up to simultaneous conjugation, set $Z_\gamma(t) = [\eta(H(t)) / r_\gamma]$. For a radial cutoff χ let

$$D_\gamma = \mathbb{E}_{\lambda_\gamma} \mathcal{I}_{\gamma, \chi(|\cdot|)}(H(\tau)), \quad \Pi_\gamma(F) = \frac{\mathbb{E}_{\lambda_\gamma} [\mathcal{I}_{\gamma, \chi(|\cdot|)}(H(\tau)) F(Z_\gamma)]}{D_\gamma}.$$

For small γ , $D_\gamma > 0$ by Theorem 36.10, and Π_γ is a probability on $\mathscr{P}$. Define $\Pi_{K, \gamma}$ by the same formula under the Wilson measure $\mu_{nBK, \gamma'}$ with the flowed lattice holonomies and with its own denominator.

Only the conjugacy class of $H(t)$ is needed to define Z_γ ; we never choose representatives depending on t .

Theorem 36.18. Fix L , an interval $0 < \tau < T < \infty$, the parameters, a sufficiently large side n on the sequence, and the central margins for that interval, all before varying γ , and use Definition 36.17. Let $b_\chi = m(\chi(|\cdot|))$, and define the probability

$$\Pi_0 = (Y \mapsto (t \mapsto [Y]))_\# \left(\frac{\chi(|Y|) e^{-Q(Y)} d^9 Y}{b_\chi} \right)$$

on $\mathscr{P}$. There exist $C_{\tau, T, \chi, \eta}$ and $\gamma_{\tau, T, \chi, \eta} > 0$ such that, for every bounded function $F : \mathscr{P} \rightarrow \mathbb{R}$ that is globally Lipschitz for ρ_∞ , and every $0 < \gamma \leq \gamma_{\tau, T, \chi, \eta}$,

$$|\Pi_\gamma(F) - \Pi_0(F)| \leq C_{\tau, T, \chi, \eta} (\|F\|_\infty + \text{Lip } F) \gamma^{1/4}.$$

The same bound holds for $\Pi_{K, \gamma}$, uniformly in K , and every denominator is at least $c_\chi r_\gamma$. At each fixed small γ , $\Pi_{K, \gamma} \rightarrow \Pi_\gamma$ weakly as $K \rightarrow \infty$. In particular $\Pi_\gamma \rightarrow \Pi_0$ weakly on $(\mathscr{P}, \rho_\infty)$ as $\gamma \downarrow 0$, after the continuum limit, and the limit does not depend on the choice of η .

Proof. The core of the proof is Steps 2–3, which reduce the weighted path to the single-time test $\varphi_F(Y) = \chi(|Y|) F(c_{[Y]})$ defined below.

Step 1: quotients and paths. The minimum defining ρ is attained because $\mathrm{SU}(2)$ is compact, and it vanishes exactly when the two orbits coincide. The symmetry of ρ follows from the invariance of the Euclidean norm under the orthogonal maps $\mathrm{Ad}(k)$, and composing minimizers gives the triangle inequality. The quotient map is 1-Lipschitz. Since $\rho([Y], [0]) = |Y|$, closed bounded balls of $\mathfrak{C}$ are continuous images of compact Euclidean balls. They are therefore compact, and $\mathfrak{C}$ is complete. The classes of vectors with rational coordinates are dense, and $\mathfrak{C}$ is separable. Since uniform limits of continuous paths are continuous, $\mathcal{P}$ is complete. For separability, let $\varepsilon > 0$. Given a path, by uniform continuity there exist a rational subdivision of $[\tau, T]$ on whose intervals the path oscillates by at most ε , and balls of radius ε , centered in a fixed countable dense set, containing its values at the subdivision points. We consider the classes of paths determined by a rational subdivision and a finite list of such balls, with oscillation at most ε on each interval. There are countably many classes, each of diameter at most 4ε , and we choose one path in each nonempty class. For $\varepsilon = 1/j$, $j \geq 1$, the chosen paths form a countable dense set, and $\mathcal{P}$ is Polish.

To see that a map η exists, we multiply the componentwise logarithm by a smooth cutoff, invariant under conjugation, supported strictly inside the logarithmic chart, and equal to one on a smaller neighborhood, and we extend the product by zero. The result is smooth and equivariant, and its global Lipschitz constant is finite by compactness of $\mathrm{SU}(2)^3$, for the maximum of the operator norms of the three components. By Lemma 5.6, the map $\mathfrak{p}$ of seven traces identifies the quotient of $\mathrm{SU}(2)^3$ by simultaneous conjugation homeomorphically with $C = \mathfrak{p}(\mathrm{SU}(2)^3)$. The map

$$f_\eta : C \rightarrow \mathfrak{C}, \quad f_\eta(\mathfrak{p}(H)) = [\eta(H)],$$

is therefore well defined, by the equivariance of η , and continuous. Composing with this uniformly continuous map at each time carries a continuous path of traces to a continuous path in $\mathfrak{C}$, continuously for the uniform metrics, and scaling by $1/r_\gamma$ is continuous at fixed γ . This defines Z_γ under λ_γ .

Step 2: comparison with the zero-fluctuation path. We abbreviate $I(H) = \mathcal{I}_{\gamma, \chi(|\cdot|)}(H(\tau))$, and we write Z and Z^0 for the paths of the flowed holonomies of the configuration and of the zero-fluctuation reconstruction. Let $d = d_{K, [\tau, T]}$ be the quantity of Theorem 34.11, that is, the minimum of 1 and the supremum over $t \in [\tau, T]$ of $\max_i \|\mathcal{H}_{t,i}^K - \mathcal{H}_{0,t,i}^K(v)\|_{\mathrm{op}}$. Since η is globally Lipschitz and the quotient map is 1-Lipschitz,

$$\rho_\infty(Z, Z^0) \leq C_\eta \frac{d}{r_\gamma} \quad \text{on } \{d \leq r_\gamma\}, \quad r_\gamma < 1.$$

On this event, if either $I(\mathcal{H}^K)$ or $I(\mathcal{H}_0^K)$ is nonzero, the zero-fluctuation holonomies at time τ lie in an enlarged window E_R as in Lemma 36.14, and

$$|I(\mathcal{H}^K) - I(\mathcal{H}_0^K)| \leq C_\chi \frac{d}{r_\gamma}.$$

These two assertions use the logarithm only at time τ , in its fixed compact subchart; at the other times we use η . Writing the difference of products as $(I(\mathcal{H}^K) - I(\mathcal{H}_0^K))F(Z) + I(\mathcal{H}_0^K)(F(Z) - F(Z^0))$, we find, on the same event,

$$|I(\mathcal{H}^K)F(Z) - I(\mathcal{H}_0^K)F(Z^0)| \leq C_{\chi,\eta}(\|F\|_\infty + \text{Lip } F) \frac{d}{r_\gamma} \mathbf{1}_{E_R}.$$

Integrating with the disintegration of Lemma 15.10, we have $\mathbb{E}_\nu[d\mathbf{1}_{E_R}] \leq Cr_\gamma^2 \hat{\nu}(E_R) \leq C_R r_\gamma^3$, exactly as in the proof of Lemma 36.9. On $\{d > r_\gamma\}$ we use the bound $2\|\chi\|_\infty \|F\|_\infty$ and $\nu(d > r_\gamma) \leq Cr_\gamma^2$. We conclude that

$$\left| \mathbb{E}_\nu I(\mathcal{H}^K)F(Z) - \mathbb{E}_{\hat{\nu}} I(\mathcal{H}_0^K)F(Z^0) \right| \leq C_{\chi,\eta} \|F\|_{\text{BL}} r_\gamma^2. \quad (36.41)$$

Step 3: the weighted zero-fluctuation path is nearly constant. We use Lemma 36.14 with $t_1 = \tau$, choosing its neighborhood $\mathcal{U}$ with closure in the domain of Lemma 36.16; the localization argument in the proof of that lemma allows any such fixed reduction. Outside $\mathcal{U}$ the contribution of the window at time τ is smaller than every power of r_γ , and so is the contribution of $\{|s| > r_\gamma\}$. On the remaining part of the weighted set, the lower bound on the output at time τ implies $|X| \leq C_\chi r_\gamma$. By Lemma 36.16, for small γ every output lies in the region where $\eta = \text{Log}$, and

$$\rho_\infty(Z^0, c_{[Z_{K,\tau}(X,s)/r_\gamma]}) \leq \sup_{\tau \leq t \leq T} \frac{|Z_{K,t} - Z_{K,\tau}|}{r_\gamma} \leq C_\chi \left(r_\gamma + \frac{|s|}{r_\gamma} \right),$$

where $c_{[Y]}$ denotes the constant path with value $[Y]$. The localized moments give $\hat{\nu}(E_R) \leq C_R r_\gamma$ and $\mathbb{E}_{\hat{\nu}}[|s| \mathbf{1}_{E_R \cap \mathcal{U}}] \leq C_R r_\gamma g$, with $g = r_\gamma^2 / \sqrt{n}$. Since $I(\mathcal{H}_0^K)F(c_{[Z_{K,\tau}/r_\gamma]}) = \mathcal{I}_{\gamma, \varphi_F}(\mathcal{H}_{0,\tau}^K)$ on this set, we obtain

$$\left| \mathbb{E}_{\hat{\nu}} I(\mathcal{H}_0^K)F(Z^0) - \mathbb{E}_{\hat{\nu}} \mathcal{I}_{\gamma, \varphi_F}(\mathcal{H}_{0,\tau}^K) \right| \leq C_\chi \|F\|_{\text{BL}} r_\gamma^2, \quad (36.42)$$

where $\varphi_F(Y) = \chi(|Y|)F(c_{[Y]})$. This test is invariant under simultaneous conjugation, supported in the fixed ball containing the support of $\chi(|\cdot|)$, and globally Lipschitz, with

$$\|\varphi_F\|_\infty \leq \|\chi\|_\infty \|F\|_\infty, \quad \text{Lip } \varphi_F \leq \text{Lip}(\chi(|\cdot|)) \|F\|_\infty + \|\chi\|_\infty \text{Lip } F,$$

since the quotient map and the map $[Y] \mapsto c_{[Y]}$ are 1-Lipschitz. We apply Theorem 36.6 to φ_F at time τ . Combining (36.41) and (36.42), and removing the conditioning with Corollary 23.11 at $q = 1$, we obtain

$$A_K(F) = \mathbb{E}_{\mu_{nBK,\gamma}} I(\mathcal{H}^K)F(Z) = r_\gamma \kappa_{K,\gamma} m(\varphi_F) + O(\|F\|_{\text{BL}} r_\gamma^2). \quad (36.43)$$

Here $A_K(F)$, notation local to this proof, is the numerator of $\Pi_{K,\gamma}(F)$. The same factor appears in $A_K(1)$, with leading coefficient $b_\chi > 0$, which implies $A_K(1) \geq c_\chi r_\gamma$. Since $m(\varphi_F) = b_\chi \Pi_0(F)$, subtracting $\Pi_0(F)A_K(1)$ from $A_K(F)$ cancels the leading terms. Dividing by $A_K(1)$ gives the bound for $\Pi_{K,\gamma} = A_K(\cdot)/A_K(1)$, uniformly in K .

Step 4: the continuum limit in the uniform topology. At fixed γ we apply Theorem 30.4 to the seven loops, concatenations of the three based coordinate cycles, that define $\mathfrak{p}$. Each loop has a single lattice approximation independent of time, of uniformly bounded length. The comparison side is covered by Corollary 29.26; at fixed coupling only finitely many terms of the sequence of sides are dropped. The paths of traces converge jointly in law in $C([\tau, T]; \mathbb{R}^7)$. All their values lie in the compact set C , and $C([\tau, T]; C)$ is a closed subset. Applying the continuous map of paths induced by f_η , followed by scaling, we obtain convergence in law of the lattice paths Z to Z_γ , jointly with the vector of traces at time τ . For small fixed γ , $I(H(\tau))$ is a continuous invariant function of the triple at time τ , and by Lemma 5.6 it is a continuous function of its seven traces. For every bounded continuous F on $\mathcal{P}$, the product $I(H(\tau))F(Z_\gamma)$ is then a bounded continuous function on the space of trace paths with values in C . By the continuous mapping theorem on this closed subspace, which carries all the measures involved, $A_K(F)$ converges to the numerator of $\Pi_\gamma(F)$, and $A_K(1)$ converges to D_γ . Dividing by the positive limit of the denominators proves $\Pi_{K,\gamma} \rightarrow \Pi_\gamma$ weakly, and the bound of Step 3 passes to the limit.

Step 5: weak convergence. The bound for bounded Lipschitz functions implies weak convergence on $\mathcal{P}$. Indeed, for a closed set $A \subset \mathcal{P}$, the functions $f_j(z) = (1 - j \operatorname{dist}(z, A))_+$, $j \geq 1$, are bounded and Lipschitz, and $\Pi_\gamma(f_j) \rightarrow \Pi_0(f_j)$ for each fixed j . This implies $\limsup_{\gamma \downarrow 0} \Pi_\gamma(A) \leq \Pi_0(f_j)$, and letting $j \rightarrow \infty$, by monotone convergence, $\limsup_{\gamma \downarrow 0} \Pi_\gamma(A) \leq \Pi_0(A)$. By the portmanteau theorem, $\Pi_\gamma \rightarrow \Pi_0$ weakly. The same argument applies to any sequence of finite cutoffs with $\gamma \downarrow 0$, since the bound is uniform in K . The argument works for each such η and gives the same Π_0 ; the limit therefore does not depend on η . $\square$

36.9 Dilation statistics of comparison measures

To show that the value zero in Theorem 36.7 reflects the structure of λ_γ near a central triple, we compute the dilation statistic for the comparison probabilities of Definition 4.13, observed at the same cycles and the same flow time. For $k \in \{1, 2, 3, 5\}$ we write P_k for the probability of item (k) of Definition 4.13. Here P_1 is the Abelian Maxwell measure with Haar cycle holonomies, P_2 the Gaussian holonomies with independent Haar factors, P_3 the Gaussian holonomies with Gaussian constant modes, and P_5 the mixture of normalized conditional normal Gaussian measures over the cone of commuting triples. For P_5 we use the observables $\mathcal{I}^{\text{coord}}$ of Definition 36.1. The local mass scales differ. A compact

window has probability of order r_γ^3 under P_1 , r_γ^9 under P_2 , and r_γ^5 under P_5 , while under P_3 the rescaled logarithm tends to zero. For λ_γ the scale is r_γ .

Recall that $\mathcal{M} = \{(\vartheta_1 u, \vartheta_2 u, \vartheta_3 u) : \vartheta \in \mathbb{R}^3, u \in S^2\}$ is the cone of commuting triples, that $\mathcal{N}_y = \{b : b_i \perp u, \sum_i \vartheta_i b_i = 0\}$ is the normal space at $y = (\vartheta_i u)_i \neq 0$, and that Σ_θ is the normal covariance of Theorem 33.11.

Theorem 36.19. *Let χ be a radial cutoff with $\chi(0) > 0$. Under each of P_1, P_2, P_3 , with the hypotheses of Definition 4.13 and Lemma 35.17, the denominator of the dilation statistic is positive for all sufficiently small γ , and*

$$\lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma, P_1}(\chi) = -6, \quad \lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma, P_2}(\chi) = 4 \frac{\int_{\mathbb{R}^9} Q(Y) \chi(|Y|) dY}{\int_{\mathbb{R}^9} \chi(|Y|) dY} > 0, \quad \lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma, P_3}(\chi) = -9.$$

For P_3 the hypotheses are that the field is a centered Gaussian field with covariance γC for a fixed covariance C , smoothed by a fixed linear operator, and that the smoothed field divided by $\sqrt{\gamma}$ has a centered separable realization with continuous paths, a measurable and almost surely bounded supremum, and finite supremum variance, as specified there.

For P_5 , let the prior be a fixed probability $\alpha(y) d\text{vol}_{\mathcal{M}}(y)$ on a small neighborhood of the tip of $\mathcal{M}$, with α continuous at the tip and $\alpha(0) > 0$. Conditionally on $y \neq 0$, let b be the normalized Gaussian $N(0, \gamma \Sigma(y))$ on $\mathcal{N}_y$, and put $X = y + b$. Assume that Σ is measurable and positive definite, with

$$c|y|^{-2}I \leq \Sigma(y) \leq C|y|^{-2}I, \quad |y|^2 \Sigma(y) \longrightarrow I_{\mathcal{N}_y} \quad (|y| \downarrow 0),$$

uniformly in the angular directions. The covariance $\Sigma = \Sigma_\theta$ has these properties; in fact $\Sigma_\theta(y) = |y|^{-2}I + O(1)$ uniformly for small $|y|$. Then, for every continuous compactly supported φ ,

$$r_\gamma^{-5} \mathbb{E}_{P_5} \varphi(X/r_\gamma) \longrightarrow \alpha(0) \int \varphi(y+b) d\mu_{13}(y,b),$$

where

$$d\mu_{13}(y,b) = (2\pi)^{-2} |y|^4 e^{-|y|^2 |b|^2 / 2} d\text{vol}_{\mathcal{M}}(y) db.$$

The weighted normal-bundle measure μ_{13} is locally finite, and y and b are bounded on compact sets of outputs $y+b$ because $y \perp b$. The denominator of $\mathcal{S}_{\gamma, P_5}(\chi)$ is positive for all sufficiently small γ . With the weighted bracket

$$\langle f \rangle_{\chi, \mu_{13}} = \frac{\int f(y,b) \chi(|y+b|) d\mu_{13}(y,b)}{\int \chi(|y+b|) d\mu_{13}(y,b)},$$

we have

$$\lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma, P_5}(\chi) = 4 + 2 \left\langle \sum_{i < j} |b_i \times b_j|^2 \right\rangle_{\chi, \mu_{13}} > 4.$$

The constants in these comparisons may depend on the fixed priors, the bounds on the covariances, the smoothing, and the test, but not on γ .

The factor $|y|^4$ in μ_{13} comes from the normalization of the four-dimensional conditional Gaussian. The weighted volume element $(2\pi)^{-2}|y|^4 d\text{vol}_{\mathcal{M}}(y) db$ is homogeneous of degree $13 = 5 + 4 + 4$ under dilations: five from the volume of the cone, four from the normal fibers, and four from the factor $|y|^4$. The remaining factor of μ_{13} is e^{-Q_2} , where $Q_2(y, b) = |y|^2|b|^2/2$ is homogeneous of degree four. The degree 13, compared with the degree 9 of Lebesgue measure on $\mathbb{R}^9$, produces the constant 4. Without the normalization of the conditional Gaussians, both the mixture and the value would change. The conclusions concern only these specified priors and covariances.

Proof. We write $\epsilon = r_\gamma$. Each numerator is the observable of the bounded compactly supported function W_χ .

The Abelian Maxwell measure with Haar cycle holonomies. We condition on the smoothed Abelian field and on the color direction $u \in S^2$ of the embedded subgroup. The circulations of the field translate the three independent uniformly distributed phases in $(\mathbb{R}/4\pi\mathbb{Z})^3$, and conditionally the cycle phases are still independent and uniformly distributed. In a small neighborhood of the identity their logarithms have the form (x_1u, x_2u, x_3u) , with density $(4\pi)^{-3}d^3x$. For radial tests, a further common conjugation does not change the integrals. The triple commutes, and $Q = 0$ on it. Once the compact support lies in the logarithmic chart, the denominator is exactly

$$(4\pi)^{-3}\epsilon^3 \int_{\mathbb{R}^3} \chi(|x|) dx > 0.$$

The divergence of $x\chi(|x|)$ on $\mathbb{R}^3$ has integral zero, which gives $\int |x|\chi'(|x|) dx = -3 \int \chi(|x|) dx$. The dilation statistic therefore equals $-9 + 3 = -6$ for every sufficiently small γ .

Gaussian holonomies with independent Haar factors. By Lemma 35.17, the triple has the law of three independent Haar distributed matrices for every γ . We write $H = \cos(|x|/2)I + 2\sin(|x|/2)(x/|x|)^a\tau_a$. The Haar probability is the normalized volume of S^3 , and its polar factor $\sin^2(|x|/2) d(|x|/2)/(2\pi^2)$ gives $dH(e^x) = (16\pi^2)^{-1} \text{sinc}^2(|x|/2) d^3x$, as in (4.2). For every bounded test ψ supported in $\{|Y| \leq R\}$, we obtain

$$\mathbb{E}_{P_2} \mathcal{I}_{\gamma, \psi} = \epsilon^9 (16\pi^2)^{-3} \left\{ \int \psi(Y) dY + O_R(\|\psi\|_\infty \epsilon^2) \right\}.$$

The denominator has a positive limiting coefficient. In $\mathbb{R}^9$ the divergence of $Y\chi(|Y|)$ has integral zero, that is, $\int (9\chi(|Y|) + |Y|\chi'(|Y|)) dY = 0$. The limit of the statistic is therefore $4 \int Q\chi / \int \chi$. To see that it is strictly positive, take a radius at which $\chi > 0$ and, at that radius, a triple with two nonzero orthogonal components. Then $Q > 0$ there, and a neighborhood of this point has positive measure, $\chi > 0$, and $Q > 0$.

Gaussian holonomies with Gaussian constant modes. At the observation time $t = \theta L^2$, let $G_i(x) = (S_t a)_i(x) / \sqrt{\gamma}$ be the smoothed field divided by $\sqrt{\gamma}$, and consider the real process

$$Z_{x,i,v} = \langle v, G_i(x) \rangle, \quad (x, i, v) \in \mathcal{T} = \mathbb{T}_L^3 \times \{1, 2, 3\} \times S^2.$$

The index set $\mathcal{T}$ is nonempty, compact, and metrizable, and the process is centered Gaussian since the smoothing is fixed and linear. By the hypotheses of Definition 4.13 and Lemma 35.17, there exists a separable realization with almost surely continuous paths on $\mathcal{T}$, and $\sigma_*^2 = \sup_{x,i,v} \mathbb{E} Z_{x,i,v}^2 < \infty$. These are the hypotheses of Lemma 35.16, which allows degenerate covariances; that lemma is the Borell–TIS inequality in the formulation of Adler and Taylor [3, Theorem 2.1.1]. It provides a measurable, almost surely finite, and integrable supremum

$$Z_* = \sup_{\mathcal{T}} Z_{x,i,v} = \max_i \sup_x |G_i(x)|, \quad m_* = \mathbb{E} Z_* < \infty,$$

where the equality follows by taking the supremum over all unit vectors v . The law of G , and hence m_* and σ_*^2 , do not depend on γ , since the field has covariance γC and the covariance and the smoothing are fixed. When $\sigma_*^2 > 0$, the same lemma implies that

$$\mathbb{P}\{Z_* - m_* \geq u\} \leq e^{-u^2/(2\sigma_*^2)} \quad (u > 0). \quad (36.44)$$

The group distance of the transport along a path from the identity is at most the integral of the norm of its generator. This follows by partitioning the path, using the triangle inequality for the bi-invariant metric, and passing to the limit in the transport equation. Since each coordinate cycle has length L , we obtain

$$\left(\sum_i \rho_G(H_i)^2 \right)^{1/2} \leq \sqrt{3} L \sqrt{\gamma} Z_*.$$

Since Z_* is almost surely finite and its law does not depend on γ , all three holonomies lie in the logarithmic chart with probability tending to one. There the rescaled logarithm has norm at most $\sqrt{3} L \epsilon Z_*$, which tends to zero in probability. Quantitatively, fix $\delta > 0$ and suppose that $\sigma_*^2 > 0$. For small γ we have $\sqrt{3} L \epsilon m_* \leq \delta/2$, and the ball of radius $\delta \epsilon$ about the identity, for the product of the group distances, lies in $\mathcal{V}$. By the transport bound and (36.44), we have

$$\begin{aligned} \mathbb{P}\{H \notin \mathcal{V} \text{ or } (H \in \mathcal{V}, |\text{Log } H|/\epsilon > \delta)\} &\leq \exp\left\{-\frac{(\delta/(\sqrt{3} L \epsilon) - m_*)^2}{2\sigma_*^2}\right\} \\ &\leq \exp\left\{-\frac{\delta^2}{24 L^2 \sigma_*^2 \sqrt{\gamma}}\right\}, \end{aligned} \quad (36.45)$$

with constants depending only on δ , L , the covariance, and the smoothing. If $\sigma_*^2 = 0$, Lemma 35.16 shows that the continuous process vanishes almost surely, and then $G = 0$

and $H = (I, I, I)$ almost surely. Since the two tests are bounded and continuous at zero, we obtain

$$\mathbb{E}_{P_3} \mathcal{I}_{\gamma, \chi}(|\cdot|) \longrightarrow \chi(0) > 0, \quad \mathbb{E}_{P_3} \mathcal{I}_{\gamma, W_\chi} \longrightarrow W_\chi(0) = -9\chi(0);$$

to see this, we split each expectation according to whether the rescaled logarithm has norm at most δ , use continuity at zero on this event, and bound the expectation on its complement by twice the supremum of the test times its probability. This gives the value -9 without any statement about densities of the nonlinear cycle logarithms.

The mixture over the cone of commuting triples. Here P_5 is the law of the vector $X = y + b \in \mathbb{R}^9$, and the statistic uses $\mathcal{I}^{\text{coord}}$. Fix $\zeta > 0$ with $\{|X| < \zeta\} \subset \text{Log } \mathcal{V}$. For a bounded test ψ supported in $\{|Y| \leq R\}$ and $R\epsilon < \zeta$, we have the identity on all of $\mathbb{R}^9$

$$\mathcal{I}_{\gamma, \psi}^{\text{coord}}(X) = \psi(X/\epsilon). \quad (36.46)$$

Indeed, if the right side is nonzero, then $|X| \leq R\epsilon < \zeta$ and the indicator equals one; if it is zero, both sides vanish. This applies to $\psi = \chi(|\cdot|)$ and $\psi = W_\chi$ simultaneously, with one bound on the supports. The observables therefore agree with the expectations computed below, even though the conditional Gaussians have unbounded support.

We record the geometry and the normalization. Away from the tip, we write $y_i = \vartheta_i u$ and $s = |\vartheta| = |y| > 0$; in this proof and in Remark 36.20, s denotes this radius, not a transverse coordinate. For variations of ϑ and u , the induced metric is $|d\vartheta|^2 + s^2|du|^2$, since $u \cdot du = 0$, and the parametrization is two-to-one under the simultaneous change of sign. The volume of the cone is therefore

$$d\text{vol}_{\mathcal{M}}(y) = \frac{1}{2}s^4 ds d\omega du, \quad \omega = \vartheta/s \in S^2, \quad u \in S^2, \quad (36.47)$$

which is homogeneous of degree five and gives zero volume to the tip. The normal space $\mathcal{N}_y$ has dimension four, and $\langle y, b \rangle = \sum_i \vartheta_i u \cdot b_i = 0$, which gives

$$|y + b|^2 = |y|^2 + |b|^2. \quad (36.48)$$

The computation proving (36.14), applied with $a = \vartheta/s$, gives

$$Q(y + b) = Q_2(y, b) + \frac{1}{2} \sum_{i < j} |b_i \times b_j|^2, \quad Q_2(y, b) = \frac{1}{2} |y|^2 |b|^2. \quad (36.49)$$

These are identities on the normal bundle; they do not require uniqueness of the decomposition $X = y + b$.

The conditional density in the fiber $\mathcal{N}_{y_0}$ is

$$(2\pi\gamma)^{-2} \det \Sigma(y_0)^{-1/2} \exp\{-b_0^\top \Sigma(y_0)^{-1} b_0 / (2\gamma)\} db_0.$$

We set $y_0 = \epsilon y$, $b_0 = \epsilon b$, and $C_\epsilon(y) = \epsilon^2 \Sigma(\epsilon y)$, using that the normal spaces are constant along rays. Then $d\text{vol}(y_0) = \epsilon^5 d\text{vol}(y)$ and $db_0 = \epsilon^4 db$, and since $\gamma = \epsilon^4$ we obtain the exact formula

$$\epsilon^{-5} \mathbb{E}_{P_5} \varphi(X/\epsilon) = \int \varphi(y+b) \alpha(\epsilon y) (2\pi)^{-2} \det C_\epsilon(y)^{-1/2} e^{-b^\top C_\epsilon(y)^{-1} b/2} d\text{vol}_{\mathcal{M}}(y) db, \quad (36.50)$$

where the prior is set to zero outside its neighborhood; on a compact set of outputs, this neighborhood eventually contains every ϵy that occurs. By the assumptions on the covariance, we have, uniformly in the angular directions and for $y \neq 0$,

$$C_\epsilon(y) \longrightarrow |y|^{-2} I, \quad c|y|^{-2} I \leq C_\epsilon(y) \leq C|y|^{-2} I.$$

Hence the normalized Gaussian density in (36.50) tends to $(2\pi)^{-2} |y|^4 e^{-Q_2(y,b)}$ and is bounded by $C|y|^4$. If φ is supported in the ball of radius R , then (36.48) bounds both $|y|$ and $|b|$ by R . The continuity of α at the tip bounds $\alpha(\epsilon y)$ there and gives the limit $\alpha(0)$. The dominating integral is finite, since by (36.47) its radial factor near zero is $s^8 ds$, and each fiber is restricted to a bounded ball in four dimensions. In finitely many angular trivializations of the normal bundle, or with the intrinsic Gaussian density on each fiber, the dominated convergence theorem applies. Then (36.50) gives the claimed limit with the normalization $(2\pi)^{-2}$ of μ_{13} .

For $\Sigma = \Sigma_\theta$, the explicit formula (4.12) gives $\Sigma_\theta(y) = |y|^{-2} I + O_\theta(1)$ uniformly in the angular directions: the term $m = 0$ of each d_i equals $e^{-2\theta|\vartheta|^2} |\vartheta|^{-2}$, since $R_i(0, \vartheta) = 1$, and the sum of the terms with $m \neq 0$ is bounded for small $|\vartheta|$, since then $\lambda_m \geq c|m|^2$ and $|R_i| \leq 1$. After a fixed reduction of the neighborhood carrying the prior, this implies both bounds on the covariance and the relative limit, and Theorem 33.11 shows that Σ_θ is positive definite and measurable there.

By the same domination and (36.48), μ_{13} is locally finite as a measure of the output $y + b$. Its integral against $\chi(|y + b|)$ is finite and strictly positive, for instance by considering an open set of the bundle with $|y| > 0$ and output radius in an interval on which $\chi > 0$. By (36.46), the denominator of the statistic satisfies

$$\mathbb{E}_{P_5} \mathcal{I}_{\gamma, \chi(|\cdot|)}^{\text{coord}}(X) \sim \epsilon^5 \alpha(0) \int \chi(|y + b|) d\mu_{13} > 0.$$

Under the dilation $(y, b) \mapsto (\lambda y, \lambda b)$, the measure $d\text{vol}_{\mathcal{M}} db$ is homogeneous of degree $5 + 4 = 9$, the factor $|y|^4$ of degree four, and Q_2 of degree four. Changing variables in the compactly supported integral of $\chi(\lambda|y + b|)$ and differentiating at $\lambda = 1$, we obtain

$$\int \{(4Q_2 - 13)\chi(\rho) - \rho\chi'(\rho)\} d\mu_{13} = 0, \quad \rho = |y + b|. \quad (36.51)$$

The differentiation is justified on a common compact set of outputs by (36.48), with the integrable radial bound $Cs^8 ds$. This change of variables includes the tip of the cone and

needs no boundary term there. We apply the limit of the rescaled expectations to W_χ and $\chi(|\cdot|)$, using (36.46), and divide by the positive limit of the denominators. Combining (36.49) and (36.51), we find

$$\lim_{\gamma \downarrow 0} \mathcal{S}_{\gamma, P_5}(\chi) = 4 + 2 \frac{\int \sum_{i < j} |b_i \times b_j|^2 \chi(\rho) d\mu_{13}}{\int \chi(\rho) d\mu_{13}}.$$

The second term is strictly positive. To see this, take ϑ in the third coordinate direction, a fixed color direction u , linearly independent $b_1, b_2 \in u^\perp$, and $b_3 = 0$. Then $b \in \mathcal{N}_y$ and $b_1 \times b_2 \neq 0$. Scaling y and b together, we place the output radius in an interval on which $\chi > 0$. A neighborhood of this point in the bundle has positive measure, and on it $\chi > 0$ and the polynomial $\sum_{i < j} |b_i \times b_j|^2$ is positive. This proves the strict inequality. $\square$

Remark 36.20. With the normalization of μ_{13} in Theorem 36.19,

$$\mu_{13}\{|y + b| \leq \rho\} = \frac{8\pi^2}{1287} \rho^{13} [1 + O(\rho^4)] \quad (\rho \downarrow 0).$$

Indeed, on the ball $\{|y + b| \leq \rho\}$ we have $0 \leq Q_2 \leq \rho^4/8$ by (36.48), and $e^{-Q_2} = 1 + O(\rho^4)$ uniformly there. The angular integral in (36.47) is $(4\pi)^2/2 = 8\pi^2$, and the ball of radius $(\rho^2 - s^2)^{1/2}$ in the four-dimensional fiber has volume $(\pi^2/2)(\rho^2 - s^2)^2$. Hence

$$\begin{aligned} \mu_{13}\{|y + b| \leq \rho\} &= \pi^2 \int_0^\rho s^8 (\rho^2 - s^2)^2 ds [1 + O(\rho^4)] \\ &= \pi^2 \rho^{13} \left(\frac{1}{9} - \frac{2}{11} + \frac{1}{13} \right) [1 + O(\rho^4)] = \frac{8\pi^2}{1287} \rho^{13} [1 + O(\rho^4)]. \end{aligned}$$

This formula is not used elsewhere.

36.10 The main result on central holonomies

We conclude by deducing Theorem 2.11 from the results of this section.

Proof of Theorem 2.11. Fix $L > 0$, an odd integer $M \geq 3$, an integer n divisible by four, and $\theta > 0$. By Theorem 2.3 (part 5), proved in Theorem 29.28, the measure $\lambda_\gamma = \lambda_{L, \gamma, [n]_M}$ depends only on L , γ , and the class $[n]_M$. We choose J and the remaining parameters as in Corollary 29.26 and Definition 34.1, so that $a_{\text{map}} \sqrt{B} \leq 32/49$, and take $n_* = n$ and a comparison side nB^a satisfying the lower bounds of Definition 36.1. Then λ_γ is the continuum measure of the sequence $(nB^K)_{K \geq 0}$, and the results of this section apply with this comparison side in place of n ; all choices depend only on L , M , n , and θ . The observable $\mathcal{I}_{\gamma, \varphi}$ of Theorem 2.11 is the observable of Definition 36.1, with the extension by zero outside the fixed neighborhood $\mathcal{V}$ of the identity. With $\rho = |X|/r_\gamma$, the statistic of part (3) of Theorem 2.11 is the dilation statistic $\mathcal{S}_{\gamma, \lambda_\gamma}(\chi)$.

Part (1) of Theorem 2.11 is the ratio limit and the two-sided bound of Theorem 36.7. Parts (2) and (3) are the rates of Theorem 36.10, for invariant Lipschitz tests supported in the ball of radius R and for the dilation statistic. Part (4) follows from Theorem 36.12, with $\mathcal{V}$ chosen invariant under simultaneous conjugation: at each fixed γ the expectations near the eight central triples are equal, and hence parts (1)–(3) hold near each of them with the same constants. In all these results, the continuum limit is taken first, at fixed γ , and then $\gamma \rightarrow 0$. The constants depend on L, M, n, θ , and the test functions, but not on γ . The further statements after Theorem 2.11 are Theorems 36.15, 36.18, and 36.19. $\square$

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