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The continuum limit of
three-dimensional $SU(2)$
lattice Yang–Mills theory
on a torus IV:

Wilson flow and continuum measure

P. M. Aronow

Patrick Lopatto



WARNING!

We have extensively audited this manuscript using large language models, including adversarial reviews by multiple models. However, the proof is too complex for Lean formalization at the present time, and it has not yet undergone verification by a human expert.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).

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<https://www.loc.gov/resource/g3200m.gct00126/?sp=106>

AUTHORS' NOTE

With the exception of this note, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited.

The manuscript began as an attempt to pursue the program of Magnen, Rivasseau, and Sénéor [132] using large language models. Our hope was that understanding the problem on the torus might offer some insight into the general case. The construction that resulted from this starting point is three-dimensional and lattice-based and draws extensively on Bałaban's renormalization group methods.

As we were verifying these results, we learned that Alpöge and Sussman announced that they appear to have a YM4 existence argument, following Bałaban's program (<https://github.com/ethanwsussman/Yang--Mills-LLM-experiment>). We are thrilled and look forward to learning more about these exciting developments. We hope that this manuscript nonetheless contributes to understanding constructive approaches to three-dimensional Yang–Mills theory. The first version of these results was posted to the first author's GitHub on October 17, 2026 (<https://github.com/pmaronow/3d-ym-torus>). Prior to posting the first version of this work, we did not inspect the ongoing work of Alpöge and Sussman. Further, to the best of our knowledge, the LLMs we used did not access their work during the mathematical development and drafting of this manuscript. This version contains additional clarifications relative to the first version; it has also been partitioned into five volumes.

We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

P. L. was partially supported by NSF grant DMS-2450004.

ABSTRACT

We consider the Wilson lattice gauge measure for $SU(2)$ on a periodic three-dimensional torus of fixed size, with bare coupling $g^2 = g_3^2 a$ proportional to the lattice spacing a , as dictated by the superrenormalizability of three-dimensional Yang–Mills theory. Observables are Wilson loops of the lattice configuration evolved by the Wilson flow, the lattice gradient flow of the Wilson action, at positive flow time. We prove that, at every fixed value of the coupling, the joint laws of finitely many flowed Wilson loops converge as the lattice spacing tends to zero along the sequences of lattice sides nM^k , for fixed odd $M \geq 3$ and fixed positive n divisible by four. For each such refinement class, the limits are realized by a unique probability measure on the space of gauge orbits of Yang–Mills heat-flow trajectories at positive times, a version of the state space introduced by Cao and Chatterjee. At each fixed positive time, the energies of the selected trajectories $\pi_k(U)$, with U distributed under the lattice Wilson measures, are tight uniformly in the lattice spacing. The limiting measure is invariant under the lattice symmetries and the center symmetries. We also show that, at short distances, the rescaled deficits $(1 - W)/(g_3^2 \varepsilon)$ of the normalized fundamental Wilson loops W of loops $x + \varepsilon \ell$, flowed for time $\sigma \varepsilon^2$ with fixed $\sigma > 0$, converge to explicit quadratic functions of the heat-smoothed Gaussian free field. As the coupling tends to zero, the measure concentrates on flat connections, with Gaussian fluctuations of explicit covariance near regular commuting holonomies and fluctuations governed by the constant-mode quartic form near the central holonomies. The proof is a multiscale renormalization group analysis in the tradition of Balaban, carried out on a small-field event whose complement we show to have small probability, uniformly in the lattice spacing, using reflection positivity and chessboard estimates. It is combined with stochastic estimates for the lattice heat flow and a deterministic analysis of the Wilson flow for rough initial data.

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PART VII

Wilson flow and continuum comparison

Local Wilson flow and continuum solutions

This section proves the deterministic Wilson-flow estimates used to pass from the renormalization group variables to flowed Wilson loops. We also record an elementary comparison for exponential transforms of bounded random observables. The section has three parts.

The first part concerns the lattice. Let $a = L/n$ be the mesh and write a configuration close to the identity as $U_e = e^{aA_e}$. After a time-dependent gauge transformation, which adds a damping term, the Wilson flow $\Psi_t^{(n)} = \Phi_{4t/a^2}^W$ becomes the parabolic equation

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + F_a(A), \quad (24.1)$$

where Δ_a is the lattice Laplacian, $\mathcal{N}_a$ is an explicit bilinear map containing one lattice difference, and F_a collects the terms of degree at least three (Lemma 24.2). We write

$$A(t) = x_a(t) + y_a(t) + r_a(t), \quad x_a(t) = P_t^a A(0), \quad y_a(t) = \int_0^t P_{t-s}^a \mathcal{N}_a(x_a(s), x_a(s)) ds, \quad (24.2)$$

where $P_t^a = e^{t\Delta_a}$. As in Section 3.5, x_a is the linear heat evolution and y_a the quadratic Picard term, and (x_a, y_a) is the flow data of $A(0)$. Theorem 24.5 constructs a local solution on a time interval independent of the mesh, with remainder of order $(t + a^2)^{1-3\alpha/2}$, and proves Lipschitz dependence on these data. The probabilistic bounds on the flow data are proved in Sections 26 to 28, using the bilinear estimate of Lemma 24.6.

The second part, Section 24.6, concerns the Yang–Mills heat flow on $\mathbb{T}_L^3$ and is independent of the lattice. For smooth initial data near a compact family, it represents the flow on a finite time interval by finitely many local solutions of the DeTurck equation, joined by fixed gauge transformations.

The third part (Section 24.7) treats the flows of the actions $\sum_p (q(U_p) + c q(U_p)^2)$ with $0 \leq c \leq C_0$, which appear in Theorem 2.6. The additional force begins in degree three and is $O(a^3)$ on every fixed bounded C_a^1 ball. We show that these flows have the same flow data as the Wilson flow, and that for the same flow data their solutions differ from the Wilson-flow solution by $O(a^\varepsilon)$ on the local time interval.

The inputs are the conventions of Section 4, in particular the weighted norms of Definition 4.8 and the continuum conventions of Definition 4.10; the finite-word analyticity and the plaquette bounds of Lemma 5.1; and the heat kernel bounds, convolution estimates, and cardinal interpolation of Lemmas 16.1, 16.3, and 16.8. The lattice results are used in

Section 25 (continuation to fixed time intervals and stability), in Sections 26 to 28 (estimates for the flow data and convergence of flowed loops), in Sections 29 and 30 (lattice energy, trajectories, and alternative flows), and in Sections 32 and 34. The continuum theorem is used in Sections 25, 28 to 30, and 34.

The lattice flow is the Wilson flow of Lüscher [128, 129]; the factor $4/a^2$ in its time variable comes from our normalization of q , of the plaquette sum, and of the metric on $\mathrm{SU}(2)$. Zwanziger [157], DeTurck [66] (for the Ricci flow), and Donaldson [76] made gauge-invariant flows parabolic by adding a gauge velocity. The decomposition (24.2) into a rough linear part and smoother corrections is the lattice analogue of the decomposition used by Cao and Chatterjee [42, §§1.1 and 2.1] for the Yang–Mills heat flow with random initial data, an idea that goes back to Bourgain [32] and to Da Prato and Debussche [63]. Here the subtracted term is the quadratic Duhamel term of the lattice equation, with the bilinear map fixed in Lemma 24.2. The global smooth theory of the continuum flow is due to Råde [139] and Charalambous and Gross [47]; we use it in the form stated by Cao and Chatterjee [42, 43]. The explicit lattice coefficient, the weighted remainder estimates, and the finite continuation construction are proved in this section.

24.1 The local equation for the Wilson flow

Notation. Throughout this section $L > 0$ is fixed, $n \geq 4$ is an integer, and $a = L/n$. A field A on Λ_n assigns to each positive link $(x, x + ae_i)$ an element $A_i(x) \in \mathfrak{su}(2)$; we use the norm $|X| = (-2 \operatorname{tr} X^2)^{1/2}$ on $\mathfrak{su}(2)$, for which $|[X, Y]| \leq |X||Y|$, since the bracket corresponds to the cross product in $\mathbb{R}^3$. Let T_i be the translation $(T_i f)(x) = f(x + ae_i)$, and let

$$D_{a,i}^+ = \frac{T_i - I}{a}, \quad D_{a,i}^- = \frac{I - T_i^{-1}}{a}, \quad \Delta_a = \sum_{i=1}^3 D_{a,i}^- D_{a,i}^+$$

act on each link component separately, a link being identified with its starting vertex. The heat semigroup $P_t^a = e^{t\Delta_a}$ acts componentwise and includes the constant mode. We write $\|A\|_\infty = \max_e |A_e|$, and $\|D_a A\|_\infty$ for the maximum of $\|D_{a,j}^\pm A\|_\infty$ over $j \in \{1, 2, 3\}$, both signs, and all link components. We set

$$\|A\|_{C_a^1} = \|A\|_\infty + \max_{j \in \{1,2,3\}, \epsilon \in \{+, -\}} \|D_{a,j}^\epsilon A\|_\infty. \quad (24.3)$$

Since translations permute the sites of the periodic lattice, they are isometries for these norms and commute with P_t^a and with all differences. More generally, D_a^m denotes the maximum over all ordered products of m differences $D_{a,k}^\pm$.

We identify tangent vectors at $U \in \mathrm{SU}(2)$ with $\mathfrak{su}(2)$ through $X \mapsto (d/d\varepsilon) e^{\varepsilon X} U|_{\varepsilon=0}$. A curve $U(s)$ then has velocity $\dot{U}U^{-1}$, and the gradient $\nabla f(U) \in \mathfrak{su}(2)$ of a function f on $\mathrm{SU}(2)$ is defined by $\langle \nabla f(U), X \rangle = (d/d\varepsilon) f(e^{\varepsilon X} U)|_{\varepsilon=0}$; since the metric is bi-invariant,

this is the Riemannian gradient. On $\mathcal{U}_n$ we use the product metric. As in Section 2.1, Φ_s^W is the flow of $-\nabla \sum_{p \in \mathcal{P}_n} q(U_p)$ and $\Psi_t^{(n)} = \Phi_{4t/a^2}^W$; no coupling enters this time change.

All constants in Section 24.1 depend only on the fixed finite stencils, the orders of the derivatives involved, and the margins of the logarithm charts, and not on L , n , the coupling, or any probability law. These are the dependences of item (1) of Definition 4.14. Later constants may depend in addition on the quantities displayed in each statement.

Local analytic maps of a finite stencil. The terms of degree at least three in the flow equations of this section are functions of the link variables in a fixed finite neighborhood, with the same functional form at every link. We first record the estimates that such functions satisfy after the rescaling $\theta = aA$. Let S_0 be a finite set of pairs (η, j) with $\eta \in \mathbb{Z}^3$ and $j \in \{1, 2, 3\}$. For a field θ and a site x , we write $Z_x(\theta) = (\theta_j(x + a\eta))_{(\eta, j) \in S_0}$ for the stencil array of θ at x . A map G from fields to fields is a local analytic map with stencil S_0 if there exist functions g_1, g_2, g_3 , real analytic in a neighborhood of 0 in $\mathfrak{su}(2)^{S_0}$, such that $G(\theta)_i(x) = g_i(Z_x(\theta))$ whenever θ is small. For a homogeneous polynomial P of degree m on $\mathfrak{su}(2)^{S_0}$ with values in $\mathfrak{su}(2)$, let $\|P\|_{\text{coeff}}$ denote the sum of the norms of its coefficients in the monomial basis of the real coordinates. If g_i extends to a bounded analytic function on a complex polydisc around 0, then by the Cauchy estimates there exist constants K_0, K_1 such that the homogeneous parts $P_{m,i}$ of g_i satisfy $\|P_{m,i}\|_{\text{coeff}} \leq K_0 K_1^m$.

Lemma 24.1. *Let G be a local analytic map with stencil S_0 whose Taylor coefficients at zero vanish through degree two, and suppose that the homogeneous parts of the g_i satisfy $\|P_{m,i}\|_{\text{coeff}} \leq K_0 K_1^m$ for $m \geq 3$; on $\{Z : K_1 \|Z\| < 1\}$, where $\|Z\|$ is the maximum of the norms of the entries, we use the convergent Taylor series $g_i = \sum_{m \geq 3} P_{m,i}$ to define g_i and hence G . Fix $0 < \eta < 1$. For $0 < a \leq 1$ and a field A with $K_1 a \|A\|_\infty \leq \eta$, set $G_a(A) = a^{-3} G(aA)$ and $M_A = \|A\|_\infty$, and let*

$$b_j(M, a) = \begin{cases} M^{3-j}, & 0 \leq j \leq 3, \\ a^{j-3}, & j > 3. \end{cases}$$

Then the following statements hold, with constants depending only on S_0 , K_0 , K_1 , η , and the indicated orders.

1. *The series $G_a(A)_i(x) = \sum_{m \geq 3} a^{m-3} P_{m,i}(Z_x(A))$ converges absolutely.*
2. *For every $j \geq 0$ and all fields $V_1, \dots, V_j$,*

$$\|D^j G_a(A)[V_1, \dots, V_j]\|_\infty \leq C_j^{\text{fld}} b_j(M_A, a) \prod_{r=1}^j \|V_r\|_\infty.$$

3. *If B is another field with $K_1 a \|B\|_\infty \leq \eta$, then $\|G_a(A) - G_a(B)\|_\infty \leq K(\|A\|_\infty^2 + \|B\|_\infty^2) \|A - B\|_\infty$.*

4. For every $m \geq 1$,

$$\|D_a^m G_a(A)\|_\infty \leq C_m^{\text{diff}} \sum_{j=1}^m b_j(M_A, a) \sum_{\substack{v_1+\dots+v_j=m \\ v_r \geq 1}} \prod_{r=1}^j \|D_a^{v_r} A\|_\infty.$$

5. If B is as in (3) and $M = \max(\|A\|_\infty, \|B\|_\infty)$, then

$$\begin{aligned} \|D_a(G_a(A) - G_a(B))\|_\infty &\leq K\{M^2\|D_a(A - B)\|_\infty \\ &\quad + M(\|D_a A\|_\infty + \|D_a B\|_\infty)\|A - B\|_\infty\}. \end{aligned}$$

Proof. We write $g_i^a(Z) = a^{-3}g_i(aZ) = \sum_{m \geq 3} a^{m-3}P_{m,i}(Z)$, so that $G_a(A)_i(x) = g_i^a(Z_x(A))$; every entry of $Z_x(A)$ has norm at most M_A . For a homogeneous polynomial P of degree m , differentiating each monomial, we find

$$|D^j P(Z)[W_1, \dots, W_j]| \leq m(m-1) \cdots (m-j+1) \|P\|_{\text{coeff}} \|Z\|^{m-j} \prod_{r=1}^j \|W_r\|,$$

where $\|Z\|$ is as in the statement; the right side is zero when $j > m$. For $j \leq 3$ we have $a^{m-3}K_1^m M_A^{m-j} = K_1^3 M_A^{3-j} (K_1 a M_A)^{m-3}$, and for $j > 3$ and $m \geq j$ we have $a^{m-3}K_1^m M_A^{m-j} = a^{j-3}K_1^j (K_1 a M_A)^{m-j}$. Since $K_1 a M_A \leq \eta < 1$, summing over m , we obtain (1) and (2), with

$$C_j^{\text{fld}} = K_0 K_1^{\max(3,j)} \sum_{m \geq \max(3,j)} m(m-1) \cdots (m-j+1) \eta^{m-\max(3,j)} < \infty.$$

For $j = 0$ the empty product in this sum is one. For (3), we write $G_a(A) - G_a(B) = \int_0^1 DG_a(B + u(A - B))[A - B] du$. Since the field $B + u(A - B)$ has norm at most $\max(\|A\|_\infty, \|B\|_\infty)$, (2) with $j = 1$ implies the claim.

For (4), we use that G_a has the same form at every site. Since $Z_{x+ae_k}(A) = Z_x(T_k A)$, by the fundamental theorem of calculus we have

$$D_{a,k}^+(G_a(A)) = \int_0^1 DG_a((1-u)A + uT_k A)[D_{a,k}^+ A] du,$$

and similarly $D_{a,k}^-(G_a(A)) = \int_0^1 DG_a((1-u)T_k^{-1}A + uA)[D_{a,k}^- A] du$. The fields $(1-u)A + uT_k A$ are convex combinations of translates of A , so they satisfy the same bound $K_1 a \|\cdot\|_\infty \leq \eta$. The map $(W, V) \mapsto DG_a(W)[V]$ is again of the same form at every site, so the same identity applies to it; differences of the two arguments are differences of W and of V , and differences of products obey $D_{a,k}^+(fg) = (D_{a,k}^+ f) T_k g + f D_{a,k}^+ g$ and $D_{a,k}^-(fg) = (D_{a,k}^- f) T_k^{-1} g + f D_{a,k}^- g$. By induction on m , $D_a^m G_a(A)$ is a finite sum, with a number of terms depending only on m , of integrals over $[0, 1]^m$ of expressions $D^j G_a(W)[\tilde{V}_1, \dots, \tilde{V}_j]$, where $1 \leq j \leq m$, W is a convex combination of translates of A , and each $\tilde{V}_r$ is a convex

combination of translates of a difference $D_a^{v_r} A$ with $v_r \geq 1$ and $v_1 + \dots + v_j = m$. No factor a^{-1} appears outside a difference of A . Since translations preserve the supremum norm, (2) implies (4).

For (5), we apply one difference to $\int_0^1 DG_a(B + u(A - B))[A - B] du$ in the same way. The terms in which the difference falls on the second argument are bounded by $Kb_1(M, a)\|D_a(A - B)\|_\infty = KM^2\|D_a(A - B)\|_\infty$. The other terms contain D^2G_a , evaluated at convex combinations of translates of A and B , applied to a difference of $B + u(A - B)$ and to a translate of $A - B$; by (2) with $j = 2$ they are bounded by $KM(\|D_a A\|_\infty + \|D_a B\|_\infty)\|A - B\|_\infty$. $\square$

The equation. We derive (24.1) and fix the bilinear map $\mathcal{N}_a$. In the lemma and its proof we use dimensionless differences; the symbols d_0 , C_0 , and H_0 have the meaning given there only in this lemma and its proof.

Lemma 24.2. *Let $L > 0$, $n \geq 4$, and $a = L/n$. Write $\nabla_i = T_i - I$, $d_0 = -\nabla$, $(C_0\theta)_{ij} = \nabla_i\theta_j - \nabla_j\theta_i$ for $i < j$, and $H_0 = \sum_i \nabla_i^* \nabla_i$, where adjoints are taken with respect to counting measure on sites, links, and positive plaquettes.*

1. We have

$$C_0^* C_0 + d_0 d_0^* = H_0 I, \quad d_2 C_0 = 0, \quad \Delta_a = -a^{-2} H_0,$$

where d_2 is the next forward exterior difference.

2. For θ in a sufficiently small fixed ball of link logarithms and the plaquette $p = p_{ij}(x)$, $i < j$, put

$$(v_{p,1}, v_{p,2}, v_{p,3}, v_{p,4}) = (\theta_i(x), \theta_j(x + ae_i), -\theta_i(x + ae_j), -\theta_j(x)),$$

$$\mathfrak{q}_p(\theta) = \frac{1}{2} \sum_{k < l} [v_{p,k}, v_{p,l}],$$

so that $U_p = e^{v_{p,1}} e^{v_{p,2}} e^{v_{p,3}} e^{v_{p,4}}$ for $U = e^\theta$. Let $\|\theta\|_{\infty, p}$ be the maximum of $|\theta_e|$ over the four links e of p . Then, with positive plaquettes counted once,

$$\text{Log } U_p = (C_0\theta)_p + \mathfrak{q}_p(\theta) + O(\|\theta\|_{\infty, p}^3),$$

$$\sum_p q(U_p) = \frac{1}{8} \|C_0\theta\|_2^2 + \frac{1}{4} \langle C_0\theta, \mathfrak{q}(\theta) \rangle + R_E(\theta), \quad |R_E(\theta)| \leq C \sum_p \|\theta\|_{\infty, p}^4.$$

Each local remainder and each of its derivatives of fixed order is bounded on a smaller fixed ball, independently of n .

3. Let $s = t/a^2$, so that $\Psi_t^{(n)} = \Phi_{4s}^W$ and the Wilson flow has velocity $-4\nabla \sum_p q(U_p)$ in the variable s . Add to this velocity the infinitesimal gauge transformation

$$(G_U\phi)_i(x) = \phi(x) - \text{Ad}(U_i(x))\phi(x + ae_i), \quad \phi = -d_0^*\theta.$$

With $\mathcal{J}(\theta) = (e^{\text{ad } \theta} - I)/\text{ad } \theta$, the resulting velocity in logarithmic coordinates is $\mathcal{J}(\theta)^{-1}(-4\nabla \sum_p q + G_U \phi)$, and it has the expansion

$$\begin{aligned} \partial_s \theta &= -H_0 \theta + N_2(\theta) + R_3(\theta), \quad N_2 = -C_0^* q - Dq(\theta)^* C_0 \theta + G_2(\theta), \\ (G_2)_i(x) &= \frac{1}{2} [\theta_i(x), (d_0^* \theta)(x) + (d_0^* \theta)(x + ae_i)]. \end{aligned}$$

The adjoint is given by

$$z_{p,k} = \frac{1}{2} \left[\sum_{l>k} v_{p,l} - \sum_{l<k} v_{p,l}, y_p \right], \quad Dq(\theta)^* y = \sum_{p,k} \epsilon_{p,k} T_{p,k}^* z_{p,k},$$

where $v_{p,k} = \epsilon_{p,k} T_{p,k} \theta_{\text{dir}(p,k)}$ with $\epsilon_{p,k} \in \{\pm 1\}$ and a translation $T_{p,k}$, and $T_{p,k}^*$ also inserts the result in the direction $\text{dir}(p,k)$.

4. Expand N_2 by the rules (i)–(iii) in Step 5 of the proof below. This gives a fixed finite list of terms, each a numerical multiple of $[T_\eta \theta_i, T_\zeta \nabla_k \theta_j]$ or $[T_\eta \theta_i, T_\zeta \nabla_k^* \theta_j]$ for translations T_η, T_ζ . Replacing the first entry of each bracket by au , the second by av , and dividing by a^3 , defines a bilinear map $\mathcal{N}_a$, which satisfies

$$\begin{aligned} a^{-3} N_2(aA) &= \mathcal{N}_a(A, A), \quad \mathcal{N}_a(u, v) = \sum_{\lambda=1}^{m_0} c_\lambda [T_{\lambda,1} u_{i_\lambda}, T_{\lambda,2} D_{a,k_\lambda}^\pm v_{j_\lambda}], \\ \|\mathcal{N}_a(u, v)\|_\infty &\leq C \|u\|_\infty \|D_a v\|_\infty, \end{aligned} \tag{24.4}$$

where each term also carries the insertion in the output direction prescribed by its rule. The number m_0 , the coefficients c_λ , the directions, and the translations do not depend on a , n , or L . Further differences of $\mathcal{N}_a(u, v)$ are distributed by the exact rules $D_{a,k}^+[u, v] = [D_{a,k}^+ u, T_k v] + [u, D_{a,k}^+ v]$ and $D_{a,k}^-[u, v] = [D_{a,k}^- u, T_k^{-1} v] + [u, D_{a,k}^- v]$. Every later occurrence of $\mathcal{N}_a$ refers to this map; it is not symmetric in its two arguments.

5. There exist numerical constants $r_* > 0$ and $C < \infty$ such that, if $\|\theta\|_\infty, \|\theta'\|_\infty \leq r_*$, then

$$\|R_3(\theta)\|_\infty \leq C \|\theta\|_\infty^3, \quad \|R_3(\theta) - R_3(\theta')\|_\infty \leq C (\|\theta\|_\infty^2 + \|\theta'\|_\infty^2) \|\theta - \theta'\|_\infty.$$

Moreover, R_3 is a local analytic map whose Taylor coefficients vanish through degree two and satisfy the hypotheses of Lemma 24.1. In particular, for $0 < a \leq 1$ the physical remainder $a^{-3} R_3(aA)$ is the sum of a convergent series whose term of degree m carries the factor a^{m-3} , $m \geq 3$, and on a fixed smaller ball it satisfies the bounds of Lemma 24.1, uniformly in the mesh.

6. Let A be a solution of

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + a^{-3} R_3(aA)$$

on an interval $[0, T]$ with $a\|A(t)\|_\infty < r_*$ throughout. Let $k(t) \in \text{SU}(2)^{\Lambda_n}$ solve

$$\partial_t k_x k_x^{-1} = a^{-2} \phi_x(aA(t)), \quad k_x(0) = I,$$

for every site x . Then k is a periodic lattice gauge transformation and, for every positive link (x, y) ,

$$e^{aA_{xy}(t)} = k_x(t) \Psi_t^{(n)}(e^{aA(0)})_{xy} k_y(t)^{-1}, \quad 0 \leq t \leq T.$$

Proof. We first compute the linear algebra, then expand the plaquette logarithms and the energy, convert the group gradient into logarithmic coordinates, add the gauge velocity, fix the bilinear map, bound the remainder, and finally identify the flow.

Step 1: Hodge identities. With counting measure, $\nabla_i^* = T_i^{-1} - I$, and differences in different directions commute. We extend a function on positive plaquettes antisymmetrically by $z_{ji} = -z_{ij}$ and $z_{ii} = 0$. By summation by parts, we have

$$(C_0^* z)_i = - \sum_j \nabla_j^* z_{ij}, \quad d_0^* \theta = - \sum_j \nabla_j^* \theta_j. \quad (24.5)$$

We obtain

$$(C_0^* C_0 \theta)_i = H_0 \theta_i - \nabla_i \sum_j \nabla_j^* \theta_j, \quad (d_0 d_0^* \theta)_i = \nabla_i \sum_j \nabla_j^* \theta_j,$$

which proves the first identity. The component of $d_2 C_0 \theta$ in the directions $i < j < k$ is

$$\nabla_i (\nabla_j \theta_k - \nabla_k \theta_j) - \nabla_j (\nabla_i \theta_k - \nabla_k \theta_i) + \nabla_k (\nabla_i \theta_j - \nabla_j \theta_i) = 0.$$

Finally $D_{a,i}^+ = a^{-1} \nabla_i$ and $D_{a,i}^- = -a^{-1} \nabla_i^*$, so $D_{a,i}^- D_{a,i}^+ = -a^{-2} \nabla_i^* \nabla_i$. This checks the signs of all three identities.

Step 2: plaquette logarithms and the energy. All matrix norms below are equivalent to the norm on $\mathfrak{su}(2)$, with numerical constants. We fix a complex neighborhood of zero in four copies of $\mathfrak{su}(2)_{\mathbb{C}}$ on which the product of the four exponentials lies in the domain of the identity logarithm chart. Expanding the exponentials through degree two, and then $\log(I + Z) = Z - Z^2/2 + \dots$, we obtain

$$\text{Log}(e^{v_1} e^{v_2} e^{v_3} e^{v_4}) = \sum_k v_k + \frac{1}{2} \sum_{k < l} (v_k v_l - v_l v_k) + \text{terms of degree at least three}.$$

The linear term is $\sum_k v_{p,k} = \theta_i(x) + \theta_j(x + ae_i) - \theta_i(x + ae_j) - \theta_j(x) = (C_0 \theta)_p$. The finite-word analyticity of Lemma 5.1 applies, since the word has four letters and all its arguments lie in the same fixed identity chart. Alternatively, the two convergent series give the coefficient bounds directly on a smaller polydisc. In either description the radius and the constants are numerical. If $w_p = \|\theta\|_{\infty, p}$, the remainder has size at most $C w_p^3$, and its derivatives of order $j \leq 3$ are at most $C_j w_p^{3-j}$; derivatives of higher fixed order are bounded on each smaller chart.

By Lemma 5.1, $q(e^X) = |X|^2/8 + O(|X|^4)$. Substituting the expansion of the logarithm, we obtain on each plaquette

$$q(U_p) = \frac{1}{8}|(C_0\theta)_p|^2 + \frac{1}{4}\langle (C_0\theta)_p, \mathfrak{q}_p(\theta) \rangle + O(w_p^4).$$

Summing over positive plaquettes, each counted once, we obtain the expansion of the energy. These estimates are local: a link lies in four positive plaquettes, so differentiating the energy with respect to one link introduces at most four of these plaquette remainders, and no volume factor.

Step 3: the group gradient in logarithmic coordinates. For a differentiable curve $\theta(s)$,

$$\left(\frac{d}{ds}e^\theta\right)e^{-\theta} = \int_0^1 e^{u\theta} \dot{\theta} e^{-u\theta} du = \mathcal{J}(\theta)\dot{\theta};$$

the integral formula follows by differentiating the exponential series term by term. The pullback of the bi-invariant metric to logarithmic coordinates is $g_\theta = \mathcal{J}(\theta)^* \mathcal{J}(\theta)$. Since the inner product is invariant, $\text{ad}(\theta)^* = -\text{ad}(\theta)$ for real θ , and hence

$$\mathcal{J}(\theta) = I + \frac{1}{2}\text{ad}(\theta) + O(|\theta|^2), \quad g_\theta = I + O(|\theta|^2).$$

We write $E_W = \sum_p q(U_p)$ and let ∇_θ denote its gradient in the Euclidean coordinates θ (only in the next formula). The velocity of $-4\nabla E_W$ in logarithmic coordinates is $-4g_\theta^{-1}\nabla_\theta E_W$. The metric correction first contributes in degree three, and by the energy expansion of Step 2 we have

$$\dot{\theta}_W = -C_0^* C_0 \theta - C_0^* \mathfrak{q}(\theta) - D\mathfrak{q}(\theta)^* C_0 \theta + O_{\text{loc}}(\theta^3), \quad (24.6)$$

where $O_{\text{loc}}(\theta^3)$ denotes a local analytic map vanishing to third order. The linear term $-\text{ad}(\theta)/2$ of $\mathcal{J}^{-1}$ cannot be dropped unless the group gradient is also converted: the two corrections cancel only in the combination $g_\theta^{-1} = (\mathcal{J}^* \mathcal{J})^{-1}$.

We now verify the formula for $D\mathfrak{q}(\theta)^*$. Varying the k th entry of a plaquette by δv_k changes $\mathfrak{q}_p$ by

$$\frac{1}{2} \sum_{l>k} [\delta v_k, v_l] + \frac{1}{2} \sum_{l<k} [v_l, \delta v_k].$$

The invariant inner product satisfies $\langle y, [h, v] \rangle = \langle h, [v, y] \rangle$ and $\langle y, [v, h] \rangle = \langle h, [y, v] \rangle$. Pairing with y_p , we obtain exactly $\langle \delta v_k, z_{p,k} \rangle$. Since $\delta v_k = \epsilon_{p,k} T_{p,k} \delta \theta_{\text{dir}(p,k)}$, transposing gives $\epsilon_{p,k} T_{p,k}^* z_{p,k}$, including the insertion in the direction $\text{dir}(p,k)$. Adding the four entries and all plaquettes containing a given link proves the formula.

Step 4: the gauge velocity. If a lattice gauge transformation has velocity $k_x k_x^{-1} = \phi_x$, then differentiating $k_x U_{xy} k_y^{-1}$ and multiplying on the right by its inverse gives the link velocity $\phi_x - \text{Ad}(U_{xy})\phi_y$. For fixed ϕ , the corresponding logarithmic velocity on the link $(x, x + ae_i)$ is

$$\begin{aligned} & \left(I - \frac{1}{2} \text{ad}(\theta_i) + O(|\theta_i|^2) \right) (\phi_x - \phi_{x+ae_i} - [\theta_i, \phi_{x+ae_i}] + O(|\theta_i|^2 |\phi|)) \\ &= (d_0 \phi)_i - \frac{1}{2} [\theta_i, \phi_x + \phi_{x+ae_i}] + O_{\text{loc}}(|\theta|^2 |\phi|). \end{aligned}$$

We now set $\phi = -d_0^* \theta$. The linear part is $-d_0 d_0^* \theta$ and the quadratic part is G_2 . Combining this with (24.6) and the identity $C_0^* C_0 + d_0 d_0^* = H_0$ proves the linear part $-H_0 \theta$ and the formula for N_2 . We define R_3 as the logarithmic velocity of part (3) minus $-H_0 \theta + N_2(\theta)$.

Step 5: the bilinear map. We specify the expansion of N_2 by explicit rules, so that no choice of polarization is left to later applications. We write $T_\eta = \prod_{j=1}^3 T_j^{\eta_j}$ for $\eta \in \mathbb{Z}^3$, and apply the rules in the following order.

- (i) Expand $-C_0^* q$ using (24.5), the antisymmetric extension, and the six terms $[v_k, v_l]/2$, $k < l$. For each resulting outer ∇_j^* use, in the order displayed, the two terms

$$\nabla_j^*[u, v] = [\nabla_j^* u, T_j^{-1} v] + [u, \nabla_j^* v].$$

In the first of these terms, move the difference into the second entry of the bracket by antisymmetry, keeping the minus sign.

- (ii) Expand $-Dq(\theta)^* C_0 \theta$ using the formula for the adjoint in part (3), writing each $v_{p,l}$ as its signed translated link variable and each component of $C_0 \theta$ as $\nabla_i \theta_j - \nabla_j \theta_i$. Distribute the translations over the brackets.
- (iii) Expand G_2 using $d_0^* \theta = -\sum_j \nabla_j^* \theta_j$, and distribute its two translations.

Within each rule, order the directions and the entries increasingly, and keep repeated terms. This produces one fixed finite list. Every entry is a numerical multiple of $[T_\eta \theta_i, T_\zeta \nabla_k \theta_j]$ or $[T_\eta \theta_i, T_\zeta \nabla_k^* \theta_j]$; these rules, and not a symmetrization, define the two entries. The product identity in rule (i) is exact, since both sides equal $[T_j^{-1} u, T_j^{-1} v] - [u, v]$. The number of entries and their integer translations do not depend on n or L . The finite sum is read componentwise, and each term carries the insertion in the output direction prescribed by its rule.

In this list, we replace the first entry of each bracket by au and the second by av , and divide by a^3 . Since $\nabla_k = aD_{a,k}^+$ and $\nabla_k^* = -aD_{a,k}^-$, the result has the form (24.4), and $a^{-3}N_2(aA) = \mathcal{N}_a(A, A)$ because the list is an exact expansion of the quadratic map N_2 . The inequality $|[X, Y]| \leq |X||Y|$ and the fact that translations are isometries give the bound in (24.4). A further forward difference of a bracket is distributed by

$$D_{a,k}^+[u, v] = \frac{[T_k u, T_k v] - [u, v]}{a} = [D_{a,k}^+ u, T_k v] + [u, D_{a,k}^+ v],$$

and a backward difference by $D_{a,k}^- [u, v] = [D_{a,k}^- u, T_k^{-1} v] + [u, D_{a,k}^- v]$. Iterating, we obtain the exact rules at each fixed order.

Step 6: the remainder. The logarithmic velocity $\mathcal{J}(\theta)^{-1}(-4\nabla E_W + G_U \phi)$ at a link is an analytic function of the link variables in a fixed finite neighborhood of that link, with the same form at every link. Indeed, the gradient of E_W at a link involves the four plaquettes containing it, and ϕ involves the links at the two endpoints. By Step 2, by the analyticity of $\mathcal{J}^{-1}$ and of Ad near the identity, and by the Cauchy estimates on a fixed complex polydisc, R_3 is a local analytic map whose homogeneous parts satisfy $\|P_{m,e}\|_{\text{coeff}} \leq K_0 K_1^m$. Its Taylor coefficients vanish through degree two by Steps 3 and 4. For $0 < a \leq 1$ we have

$$a^{-3} R_{3,e}(aA) = \sum_{m \geq 3} a^{m-3} P_{m,e}(A), \quad \|P_{m,e}\|_{\text{coeff}} \leq K_0 K_1^m, \quad (24.7)$$

where $P_{m,e}$ is homogeneous of degree m in the stencil of e . We choose r_* so that $K_1 r_* \leq \eta < 1$ for a fixed η . Lemma 24.1 with $a = 1$ implies the two bounds on R_3 in part (5). Explicitly, by Taylor's formula with integral remainder, we have

$$|R_{3,e}(\theta)| \leq C \max_f |\theta_f|^3, \quad \|DR_{3,e}(\theta)\| \leq C \max_f |\theta_f|^2,$$

where the maxima are over the stencil of e . Integrating the second bound along the segment from θ' to θ , which stays in the chart, and using $((1-u)b + uc)^2 \leq b^2 + c^2$ for $b, c \geq 0$, we obtain the difference bound. Bounded incidence makes the constants independent of the number of links. Lemma 24.1 with $0 < a \leq 1$ implies the bounds on the physical remainder. On $K_1 a \|A\|_\infty \leq \eta$, its j th field derivative is at most $C_{j,\eta} \|A\|_\infty^{3-j}$ for $j \leq 3$ and $C_{j,\eta} a^{j-3}$ for $j > 3$, and its spatial differences are finite sums of products of differences of A with these field derivatives, with no inverse power of a left over. The factors a^{m-3} in (24.7) are nonnegative powers of a for $m \geq 3$ and $a \leq 1$.

Step 7: identification with the Wilson flow. On a finite lattice the Wilson vector field is smooth and gauge equivariant: the energy is gauge invariant, and the product metric is invariant under the action of each fixed gauge transformation. Let A be as in part (6). We solve $\dot{k}_x k_x^{-1} = \lambda_x$ with $\lambda = a^{-2} \phi(aA)$ and $k_x(0) = I$. The matrix integral equation has a convergent Picard series on every bounded time interval, since its m th iterated integral is at most $(\int \|\lambda\|)^m / m!$. Since λ_x is skew-Hermitian and traceless, $k_x^* k_x = I$ and $\det k_x = 1$ are preserved, and periodic coefficients give a periodic gauge transformation. In the variable $s = t/a^2$, the substitution $\theta = aA$ turns the equation of part (6) into $\partial_s \theta = -H_0 \theta + N_2(\theta) + R_3(\theta)$, which by part (3) says that the links $U(t) = e^{aA(t)}$ move with velocity equal to the Wilson velocity (with factor $4/a^2$ in the variable t) plus the gauge velocity generated by λ . Differentiating $k_x^{-1} U_{xy} k_y$ cancels the gauge velocity, and the remaining velocity is the Wilson vector field with factor $4/a^2$,

by gauge equivariance; the initial links are $e^{aA(0)}$. By uniqueness for this smooth finite-dimensional ordinary differential equation (for example by the local Lipschitz integral inequality), $k^{-1}Uk = \Psi_t^{(n)}(e^{aA(0)})$, which is the claimed formula. The chart is needed only to define the damping generator, and no coupling enters. $\square$

Substituting $\theta = aA$ and $s = t/a^2$ into part (3) shows that the equation of part (6) is (24.1) with $F_a(A) = a^{-3}R_3(aA)$, without approximation. We call the solution A in part (6) the *damped representative* of the Wilson flow started from $e^{aA(0)}$, and k its *damping gauge transformation*. The gauge velocity added in part (3) is the lattice analogue of the gauge damping of Zwanziger [157] and DeTurck [66]. By (24.5), $\phi = -d_0^*\theta = \sum_j \nabla_j^* \theta_j$, so the generator of the damping gauge transformation, which we call the *physical damping generator*, is

$$\lambda_a = a^{-2}\phi(aA) = -\sum_j D_{a,j}^- A_j, \quad (24.8)$$

a linear function of A .

24.2 Linear and quadratic flow data

We recall the weighted norms of Definition 4.8. Throughout the rest of this section $1/2 < \alpha < 2/3$, and

$$\beta = \alpha - \frac{1}{2}, \quad \kappa = 1 - \frac{3}{2}\alpha;$$

in the applications $\alpha = \alpha_{\text{fl}}$. Fix a horizon $T > 0$ and write $v_a(t) = t + a^2$. For paths x, y on $[0, T]$ with values in fields on Λ_n , and for a field f , set

$$\begin{aligned} X_\alpha^a(x) &= \sup_{0 \leq t \leq T} \{v_a(t)^{\alpha/2} \|x(t)\|_\infty + v_a(t)^{(\alpha+1)/2} \|D_a x(t)\|_\infty\}, \\ Y_\alpha^a(y) &= \sup_{0 \leq t \leq T} \{v_a(t)^\beta \|y(t)\|_\infty + v_a(t)^\alpha \|D_a y(t)\|_\infty\}, \\ H_\sigma^a(f) &= \sup_{0 \leq t \leq T} \{v_a(t)^{\sigma/2} \|P_t^a f\|_\infty + v_a(t)^{(\sigma+1)/2} \|D_a P_t^a f\|_\infty\}, \quad 0 < \sigma < 1 - \alpha. \end{aligned} \quad (24.9)$$

The horizon is part of the norms; when it matters we say that a bound holds on $[0, T]$. Each norm controls a value and every first difference, and translations do not change it. For paths u, v we write

$$\mathcal{Q}_a(u, v)(t) = \int_0^t P_{t-s}^a \mathcal{N}_a(u(s), v(s)) ds, \quad \mathcal{B}_a(u, v) = \mathcal{Q}_a(u, v) + \mathcal{Q}_a(v, u),$$

with the bilinear map $\mathcal{N}_a$ of Lemma 24.2; in general $\mathcal{Q}_a(u, v) \neq \mathcal{Q}_a(v, u)$.

Definition 24.3. Fix $L, T > 0$ and α as above. Let $a = L/n$ and let A_0 be a field on Λ_n . The *flow data* of A_0 on $[0, T]$ are the linear heat evolution and the quadratic Picard term

$$x_a(t) = P_t^a A_0, \quad y_a(t) = \mathcal{Q}_a(x_a, x_a)(t), \quad 0 \leq t \leq T,$$

and we write $\|(x_a, y_a)\|_{\mathfrak{f},a} = X_a^a(x_a) + Y_a^a(y_a)$. The equation associated with A_0 is

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + a^{-3} R_3(aA), \quad A(0) = A_0, \quad (24.10)$$

considered as long as $a\|A(t)\|_\infty < r_*$. Differences of flow data on the same lattice are measured by the same two norms applied to the differences of the paths.

For comparisons between different lattices, let I_a be the continuous cardinal interpolation of Lemma 16.3 on Λ_n , applied to each component of a field on positive links, each link being anchored at its starting vertex. It is applied to the value array of a path and, separately, to each of its difference arrays $D_{a,j}^\pm$. For flow data (x, y) on the mesh a and $(\tilde{x}, \tilde{y})$ on the mesh $\tilde{a}$, set

$$\begin{aligned} d_T((x, y), (\tilde{x}, \tilde{y})) = \sup_{0 < t \leq T} \bigg\{ & t^{\alpha/2} \|I_a x(t) - I_{\tilde{a}} \tilde{x}(t)\|_\infty \\ & + t^{(\alpha+1)/2} \max_{j,\pm} \|I_a(D_{a,j}^\pm x(t)) - I_{\tilde{a}}(D_{\tilde{a},j}^\pm \tilde{x}(t))\|_\infty \\ & + t^\beta \|I_a y(t) - I_{\tilde{a}} \tilde{y}(t)\|_\infty \\ & + t^\alpha \max_{j,\pm} \|I_a(D_{a,j}^\pm y(t)) - I_{\tilde{a}}(D_{\tilde{a},j}^\pm \tilde{y}(t))\|_\infty \bigg\}. \end{aligned} \quad (24.11)$$

The interpolated difference array $I_a(D_{a,j}^\pm x)$ is a separate component of this comparison; it is not the derivative of $I_a x$. The definition involves neither a choice of gauge nor a comparison of the maps $\mathcal{N}_a$ on different lattices.

We record four elementary properties.

Interpolation. The one-dimensional cardinal weight ζ of Lemma 16.3 is nonnegative and satisfies $\sum_{m \in \mathbb{Z}} \zeta(u - m) = 1$. Multiplying three such identities and periodizing shows that the weights in each row of I_a are nonnegative and sum to one, so $\|I_a f\|_\infty \leq \|f\|_\infty$. This applies separately to each difference component, and it uses no derivative estimate for an interpolant. The maps I_a do not depend on time.

Behavior at time zero. On a fixed lattice all arrays are finite dimensional and the heat matrices are continuous in t , so $x(t)$, $D_{a,j}^\pm x(t)$, $y(t)$, and $D_{a,j}^\pm y(t)$ are continuous and bounded on $[0, T]$. Since the four weights in (24.11) have positive exponents, each weighted component of d_T extends continuously by zero to $t = 0$.

Comparison of weights. Since $t^c \leq (t + a^2)^c$ for $c > 0$, the interpolation bound implies

$$\begin{aligned} \sup_{0 < t \leq T} \bigg\{ & t^{\alpha/2} \|I_a x(t)\|_\infty + t^{(\alpha+1)/2} \max_{j,\pm} \|I_a(D_{a,j}^\pm x(t))\|_\infty \\ & + t^\beta \|I_a y(t)\|_\infty + t^\alpha \max_{j,\pm} \|I_a(D_{a,j}^\pm y(t))\|_\infty \bigg\} \leq \|(x, y)\|_{\mathfrak{f},a}. \end{aligned}$$

In particular, bounds in the norms with weights $t + a^2$ imply bounds in the weights of d_T ; the reverse does not hold uniformly near $t = 0$. The local existence arguments below use the norms with weights $t + a^2$.

Exactness. On a fixed lattice, the substitution $\theta = aA$, $s = t/a^2$ in Lemma 24.2(3) gives (24.10) without approximation, and the quadratic Duhamel term generated by A_0 in this equation is y_a , with the map $\mathcal{N}_a$ fixed there.

24.3 Local existence for the Wilson flow

We first explain the choice of exponents. Suppose that a field satisfies $\|A(t)\|_\infty \lesssim t^{-\alpha/2}$ and $\|D_a A(t)\|_\infty \lesssim t^{-(\alpha+1)/2}$, as the linear heat evolution of initial data of regularity $-\alpha$ does. Then the sources in (24.10) have the following sizes:

source	size	integrable at $t = 0$
$\mathcal{N}_a(A, A)$	$t^{-\alpha-1/2}$	no, since $\alpha > 1/2$
$a^{-3}R_3(aA)$	$t^{-3\alpha/2}$	yes, since $\alpha < 2/3$

The Duhamel integral of the cubic term has size $t^{1-3\alpha/2} = t^\kappa$. The quadratic term is too singular to be integrated with these bounds, and its Duhamel integral cannot be estimated from bounds on x_a alone. This is why the quadratic Picard term y_a is part of the data: it is the Duhamel integral of the most singular source, and in the applications its bound is proved in Sections 26 to 28 using the probabilistic structure of x_a . After x_a and y_a are subtracted, every remaining source is bounded by a multiple of $(t + a^2)^{-3\alpha/2}$ (see the table in the proof below), and the remainder is of order $(t + a^2)^\kappa$. The restriction $\alpha > 1/2$ corresponds to the regularity of the initial data in the applications (see Section 3.5), and it makes $\beta = \alpha - 1/2$ positive.

Lemma 24.4. *Let $0 < p < 1$, $a^2 \leq T \leq 1/2$, and $K \geq 0$, and let f be a measurable path on $[0, T]$ with values in fields on Λ_n and $\|f(t)\|_\infty \leq K(t + a^2)^{-p}$ for $0 \leq t \leq T$. Then $z(t) = \int_0^t P_{t-s}^a f(s) ds$ satisfies, for $0 \leq t \leq T$,*

$$\|z(t)\|_\infty \leq C_{p,L} K(t + a^2)^{1-p}, \quad \|D_a z(t)\|_\infty \leq C_{p,L} K(t + a^2)^{1/2-p}. \quad (24.12)$$

One may take $C_{p,L} = 6(1 + C_L)/(1 - p)$, where C_L depends only on L ; in particular $C_{p,L}$ is bounded uniformly for p in each interval $(0, 1 - \delta]$, $\delta > 0$.

Proof. Since the heat kernel is nonnegative with mass one, P_s^a contracts the supremum norm, and

$$\|z(t)\|_\infty \leq K \int_0^t (s + a^2)^{-p} ds \leq \frac{K}{1-p} (t + a^2)^{1-p}.$$

By Lemma 16.1 with one difference and horizon one, $\|D_a P_s^a h\|_\infty \leq C_L (s + a^2)^{-1/2} \|h\|_\infty$ for $0 \leq s \leq 1$, where C_L depends only on L ; this applies since $0 \leq t - s \leq T \leq 1/2$. Hence

$$\|D_a z(t)\|_\infty \leq C_L K \int_0^t (t - s + a^2)^{-1/2} (s + a^2)^{-p} ds.$$

The integral is bounded by a multiple of $(t + a^2)^{1/2-p}$ by Lemma 16.8, whose hypotheses are $0 \leq p < 1$ and exponent $1/2 < 1$. To see the dependence on p , we split the integral at $s = t/2$. For $s \leq t/2$ we have $t - s + a^2 \geq (t + a^2)/2$, and the integral of $(s + a^2)^{-p}$ over $[0, t/2]$ is at most $(t + a^2)^{1-p}/(1-p)$. For $s \geq t/2$ we have $(s + a^2)^{-p} \leq 2^p(t + a^2)^{-p}$, and the integral of $(t - s + a^2)^{-1/2}$ over $[t/2, t]$ is at most $2(t + a^2)^{1/2}$. Combining the two parts, we obtain

$$\int_0^t (t - s + a^2)^{-1/2} (s + a^2)^{-p} ds \leq \left(\frac{\sqrt{2}}{1-p} + 2^{p+1} \right) (t + a^2)^{1/2-p} \leq \frac{6}{1-p} (t + a^2)^{1/2-p}.$$

Measurable sources with the stated bound are integrable since $a > 0$, and no other property of f is used. $\square$

Theorem 24.5. Fix $L > 0$, $1/2 < \alpha < 2/3$, and $R < \infty$, and let $\kappa = 1 - 3\alpha/2$. There exist $T_R \in (0, 1]$, $a_R > 0$, and $C_R < \infty$, depending only on L , α , R , and the stencil and chart of Lemma 24.2, such that the following holds for every mesh $a = L/n \leq a_R$. Let A_0 be a field on Λ_n whose flow data on $[0, T_R]$ satisfy $\|(x_a, y_a)\|_{\text{fl},a} \leq R$. Then (24.10) has a unique solution on $[0, T_R]$ of the form $A_a = x_a + y_a + r_a$ with $r_a(0) = 0$ and finite remainder norm

$$\|r\|_{R_a} = \sup_{0 \leq t \leq T_R} \{ (t + a^2)^{-\kappa} \|r(t)\|_\infty + (t + a^2)^{1/2-\kappa} \|D_a r(t)\|_\infty \}, \quad (24.13)$$

and $\|r_a\|_{R_a} \leq C_R$. Uniqueness holds among mild solutions with these flow data and finite remainder norm.

Let A'_0 be a second field on Λ_n whose flow data (x'_a, y'_a) satisfy the same bound, the two fields being compared as arrays, without a gauge transformation between them, and let $d_a = X_\alpha^a(x_a - x'_a) + Y_\alpha^a(y_a - y'_a)$. Then

$$\|r_a - r'_a\|_{R_a} \leq C_R d_a, \quad \sup_{s \leq t \leq T_R} \|A_a(t) - A'_a(t)\|_{C_a^1} \leq C_{R,s} d_a \quad (0 < s < T_R).$$

After reducing a_R , the whole interval lies in the logarithm chart:

$$a \sup_{t \leq T_R} \|A_a(t)\|_\infty \leq R a^{1-\alpha} + R a^{2-2\alpha} + C_R a (T_R + a^2)^\kappa < r_*.$$

Consequently, by Lemma 24.2(6), $e^{aA_a(t)}$ is a periodic time-dependent gauge transform, equal to the identity at $t = 0$, of the Wilson flow $\Psi_t^{(n)}(e^{aA_0})$ at physical time t . The constants do not depend on the coupling, on n beyond the restriction $a \leq a_R$, or on any probability law. They depend on R and s , and they are not uniform as $R \rightarrow \infty$ or $s \downarrow 0$.

The same conclusions hold on every horizon $0 < T \leq T_R$, with T_R replaced by T in the hypotheses, norms, and conclusions, for meshes $a \leq \min\{a_R, \sqrt{T}\}$; the constants are unchanged.

Here a mild solution is a continuous path A on that horizon with $a\|A(t)\|_\infty < r_*$ and $A(t) = P_t^a A_0 + \int_0^t P_{t-s}^a (\mathcal{N}_a(A, A) + a^{-3} R_3(aA))(s) ds$. Uniqueness in Theorem 24.5 holds among the mild solutions A for which $A - x_a - y_a$ has finite remainder norm.

Proof. We construct r_a as the fixed point of a contraction on a ball for the norm (24.13). The equation for r is obtained by subtracting the equations satisfied by x_a and y_a ; this removes the most singular source $\mathcal{N}_a(x_a, x_a)$. We then check, term by term, that the remaining sources are bounded by $(t + a^2)^{-3\alpha/2}$, choose the radius, the time, and the mesh threshold in this order, and prove uniqueness, Lipschitz dependence, and the chart bound. The construction, contraction, estimates, and uniqueness within the constructed ball use the cubic term only through the bounds (24.15) on the set $\{B : a\|B\|_\infty \leq r_*/2\}$; we use this in the proof of Theorem 24.12. Uniqueness in the larger class of chart-valued mild solutions also uses local Lipschitz continuity of the finite-dimensional vector field on the full chart, as explained in Step 5.

Step 1: bounds on the flow data. We write

$$\beta = \alpha - \frac{1}{2}, \quad p = \frac{3}{2}\alpha < 1, \quad \kappa = 1 - p, \quad \delta = \frac{1 - \alpha}{2} > 0.$$

We may assume $R \geq 0$, since otherwise no field satisfies the hypothesis. For a tentative $T \leq 1/2$ and $a^2 \leq T$, we set $v = t + a^2$. If the flow data satisfy $\|(x, y)\|_{\text{fl}, a} \leq R$ on $[0, T]$, then

$$\begin{aligned} \|x(t)\|_\infty &\leq Rv^{-\alpha/2}, & \|D_a x(t)\|_\infty &\leq Rv^{-(\alpha+1)/2}, \\ \|y(t)\|_\infty &\leq Rv^{-\beta}, & \|D_a y(t)\|_\infty &\leq Rv^{-\alpha}, \end{aligned} \quad (24.14)$$

and these bounds include all translations. Since $v \leq 2T \leq 1$ and $\alpha/2 - \beta = \delta > 0$, we have in particular $\|x(t) + y(t)\|_\infty \leq 2Rv^{-\alpha/2}$.

Step 2: the equation for the remainder. Recall that $F_a(A) = a^{-3}R_3(aA)$. We take $a \leq 1$. By Lemma 24.2(5) and Lemma 24.1(2), with $j = 0$, and (3), on the set $\{B : a\|B\|_\infty \leq r_*/2\}$,

$$\|F_a(B)\|_\infty \leq C\|B\|_\infty^3, \quad \|F_a(B) - F_a(B')\|_\infty \leq C(\|B\|_\infty^2 + \|B'\|_\infty^2)\|B - B'\|_\infty, \quad (24.15)$$

with C independent of a . Since $\partial_t x = \Delta_a x$ and $\partial_t y = \Delta_a y + \mathcal{N}_a(x, x)$ with $x(0) = A_0$, $y(0) = 0$, the path $A = x + y + r$ solves (24.10) if and only if $r(0) = 0$ and $\partial_t r = \Delta_a r + \mathcal{N}_a(A, A) - \mathcal{N}_a(x, x) + F_a(A)$. Expanding $\mathcal{N}_a(A, A)$ by bilinearity, we obtain the source

$$\begin{aligned} \mathcal{F}_{x,y}(r) &= \mathcal{N}_a(x, y) + \mathcal{N}_a(y, x) + \mathcal{N}_a(y, y) \\ &\quad + \mathcal{N}_a(x, r) + \mathcal{N}_a(r, x) + \mathcal{N}_a(y, r) + \mathcal{N}_a(r, y) + \mathcal{N}_a(r, r) + F_a(x + y + r), \end{aligned} \quad (24.16)$$

where every occurrence of $\mathcal{N}_a$ uses the bilinear map fixed in Lemma 24.2. The fixed-point equation is

$$\mathcal{M}_{x,y}(r)(t) = \int_0^t P_{t-s}^a \mathcal{F}_{x,y}(r)(s) ds, \quad r = \mathcal{M}_{x,y}(r). \quad (24.17)$$

Step 3: bounds on the sources. If $\|r\|_{R_a} \leq S$, then $\|r(t)\|_\infty \leq Sv^\kappa$ and $\|D_a r(t)\|_\infty \leq Sv^{\kappa-1/2}$. By (24.4), we have $\|\mathcal{N}_a(u, w)\|_\infty \leq C\|u\|_\infty\|D_a w\|_\infty$. Together with (24.14), this gives the following bounds for the quadratic sources, up to a constant depending only on the stencil:

source	amplitude	power of v
$\mathcal{N}_a(x, y), \mathcal{N}_a(y, x)$	R^2	$-p$
$\mathcal{N}_a(y, y)$	R^2	$-p + \delta$
$\mathcal{N}_a(x, r), \mathcal{N}_a(r, x)$	RS	$-p + \delta$
$\mathcal{N}_a(y, r), \mathcal{N}_a(r, y)$	RS	$-p + 2\delta$
$\mathcal{N}_a(r, r)$	S^2	$-p + 3\delta$

For example, in the third row $\mathcal{N}_a(x, r)$, with the difference on r , has exponent $-\alpha/2 + \kappa - 1/2 = 1/2 - 2\alpha = -p + \delta$, and $\mathcal{N}_a(r, x)$, with the difference on x , has exponent $\kappa - (\alpha + 1)/2$, which is the same number. In the fourth row the exponents of $\mathcal{N}_a(y, r)$ and $\mathcal{N}_a(r, y)$ are $-\beta + \kappa - 1/2$ and $\kappa - \alpha$, both equal to $-p + 2\delta$. In the first row the exponents of $\mathcal{N}_a(x, y)$ and $\mathcal{N}_a(y, x)$ are $-\alpha/2 - \alpha$ and $-\beta - (\alpha + 1)/2$, both equal to $-p$. In this way each quadratic term is estimated with its arguments in the order displayed in (24.16).

By (24.14) and (24.15), the cubic source at $r = 0$ is at most CR^3v^{-p} . For r, r' in the ball of radius S , by (24.15) we have

$$\begin{aligned} \|F_a(x + y + r) - F_a(x + y + r')\|_\infty &\leq C\{R^2v^{-\alpha} + S^2v^{2\kappa}\}v^\kappa\|r - r'\|_{R_a} \\ &\leq C(R^2 + S^2)v^{-p+2\delta}\|r - r'\|_{R_a}, \end{aligned}$$

where we used $-\alpha + \kappa = -p + 2\delta$ and $v^{3\kappa} \leq v^{-p+2\delta}$, the exponents differing by $2 - 2\alpha = 4\delta > 0$, with $v \leq 1$. Subtracting each quadratic row gives the same table with one factor S replaced by $\|r - r'\|_{R_a}$. Bounding every positive extra power of v by its maximum on $[0, T]$, we obtain

$$\begin{aligned} \|\mathcal{F}_{x,y}(0)(t)\|_\infty &\leq K_R v^{-p}, \\ \|\mathcal{F}_{x,y}(r)(t) - \mathcal{F}_{x,y}(r')(t)\|_\infty &\leq K_{R,S}(2T)^\delta v^{-p}\|r - r'\|_{R_a}, \end{aligned} \tag{24.18}$$

where K_R depends only on R and the stencil and chart, and $K_{R,S}$ depends in addition on S .

Step 4: choice of the parameters. By Lemma 24.4 with $p = 3\alpha/2$,

$$\|\mathcal{M}_{x,y}(0)\|_{R_a} \leq C_{\alpha,L}K_R, \quad \|\mathcal{M}_{x,y}(r) - \mathcal{M}_{x,y}(r')\|_{R_a} \leq C_{\alpha,L}K_{R,S}(2T)^\delta\|r - r'\|_{R_a}.$$

We choose $S = 1 + 2C_{\alpha,L}K_R$, and then $T_R \in (0, 1/2]$ so that $C_{\alpha,L}K_{R,S}(2T_R)^\delta \leq 1/2$. Neither choice involves the mesh. We then reduce a_R so that $a_R \leq \min(1, \sqrt{T_R})$ and

$$Ra_R^{1-\alpha} + Ra_R^{2-2\alpha} + Sa_R(2T_R)^\kappa < r_*/2. \tag{24.19}$$

All three exponents of a_R are positive. Since $v \geq a^2$, we have $av^{-\alpha/2} \leq a^{1-\alpha}$, $av^{-\beta} \leq a^{2-2\alpha}$, and $av^\kappa \leq a(2T_R)^\kappa$, so every candidate $x + y + r$ with $\|r\|_{R_a} \leq S$ satisfies $a\|x + y + r\|_\infty < r_*/2$ for $a \leq a_R$. In particular, (24.15) is never used outside its domain.

Step 5: existence and uniqueness. On a fixed lattice, the continuous paths r on $[0, T_R]$ with $r(0) = 0$ and finite R_a norm form a complete space: v is bounded below by $a^2 > 0$, there are finitely many coordinates, and the differences are continuous linear maps. By Step 4 and the choice of S , $\mathcal{M}_{x,y}$ maps the closed ball of radius S into itself and is a contraction with constant $1/2$ there. Starting from zero, successive iterates differ by at most 2^{-m} times the first difference, so they converge in this space to a fixed point. The same estimate gives uniqueness inside the ball. The heat convolution vanishes at $t = 0$, so $r(0) = 0$. This constructs the solution, and we take $C_R \geq S$.

Uniqueness in the larger class of the theorem does not require an a priori bound on the radius of a second remainder. On a fixed lattice, a continuous mild solution in the chart is differentiable, since its heat integral is finite dimensional, and it satisfies the ordinary differential equation (24.10). On a compact subinterval of the time interval on which both solutions stay in the full chart, local Lipschitz continuity of the analytic vector field gives a finite Lipschitz constant K_a . For two solutions with the same initial value, $\sup_{u \leq t} |A(u) - A'(u)| \leq K_a t \sup_{u \leq t} |A(u) - A'(u)|$. Taking $K_a t < 1$ and restarting at time t proves equality on that interval, and finitely many such intervals give equality on the whole common interval. The flow data are determined by the initial value, so this proves uniqueness among mild solutions with the given flow data and finite remainder norm. The argument needs no bound on K_a that is uniform in a .

Step 6: Lipschitz dependence. For two flow data in the ball, (24.14) applies to their differences with R replaced by d_a . In each quadratic term of (24.16), we write the difference by changing first the first argument and then the second. The first three terms are then bounded by $C_R d_a v^{-p}$, and the terms involving a fixed remainder of norm at most S by $C_{R,S} d_a v^{-p}$, since $v \leq 1$. For the cubic term, the difference of $x + y$ is at most $2d_a v^{-\alpha/2}$ and both arguments are bounded by $(2R + S)v^{-\alpha/2}$, so by (24.15) its difference is also at most $C_{R,S} d_a v^{-p}$. Writing $r - r' = (\mathcal{M}_{x,y}(r) - \mathcal{M}_{x',y'}(r)) + (\mathcal{M}_{x',y'}(r) - \mathcal{M}_{x',y'}(r'))$ and applying Lemma 24.4 to the first difference and the contraction estimate to the second, we obtain

$$\|r - r'\|_{R_a} \leq C_{R,S,\alpha,L} d_a + \frac{1}{2} \|r - r'\|_{R_a}.$$

We enlarge C_R to absorb the factor two. On $[s, T_R]$ all weights in (24.14) and in (24.13) are bounded by constants depending only on s , R , α , and L . Adding the bounds for $x - x'$, $y - y'$, and $r - r'$, we obtain the estimate for $\|A(t) - A'(t)\|_{C_a^1}$.

Step 7: the chart and the identification. Since the value bounds for x and y decrease in time and the value bound for the remainder increases, we have

$$a \sup_{t \leq T_R} \|x(t) + y(t) + r(t)\|_\infty \leq Ra^{1-\alpha} + Ra^{2-2\alpha} + C_R a (T_R + a^2)^\kappa.$$

If necessary we reduce a_R again after the last enlargement of C_R , so that this bound is less than r_* on the whole interval. The last term is of order a ; it is not replaced by its smaller value at $t = 0$. Part (6) of Lemma 24.2 now applies on $[0, T_R]$ and identifies $e^{aA_a(t)}$ with the gauge transform of the Wilson flow by the periodic damping gauge transformation, which is the identity at $t = 0$. All choices above have the stated dependence. The proof gives $T_R \leq 1/2$ and $a_R \leq \min(1, \sqrt{T_R})$. $\square$

24.4 Bilinear estimates for the Duhamel integral

The probabilistic estimates of Sections 26 to 28 decompose the flow data into many pieces and estimate the Duhamel terms of their products. The following lemma bounds such a term when one argument is a heat path with the regularity of x_a and the other a heat path with the better exponent $\sigma < 1 - \alpha$. Since $\mathcal{N}_a$ is not symmetric, we estimate $\mathcal{Q}_a(u, v)$ and $\mathcal{Q}_a(v, u)$ separately.

Lemma 24.6. *Fix $L, T > 0$, $1/2 < \alpha = \alpha_{\text{fl}} < 2/3$, and $0 < \sigma < 1 - \alpha$. For every mesh $a = L/n$, $n \geq 4$, and all fields f, g on Λ_n , let $u(t) = P_t^a f$ and $v(t) = P_t^a g$. Then, with the norms (24.9) on $[0, T]$,*

$$Y_\alpha^a(\mathcal{Q}_a(u, v)) + Y_\alpha^a(\mathcal{Q}_a(v, u)) \leq C_{L,T,\alpha,\sigma} (T + a^2)^{(\alpha-\sigma)/2} X_\alpha^a(u) H_\sigma^a(g).$$

In particular the same right side bounds $Y_\alpha^a(\mathcal{B}_a(u, v))$. The constant depends only on L, T, α, σ , and the stencil of $\mathcal{N}_a$; it does not depend on the mesh, the fields, the coupling, or any probability law. The estimate includes $t = 0$, all translations in the stencil, and every first difference. For $a \leq 1$ it follows from heat kernel estimates uniformly in a ; the finitely many lattices with $1 < a \leq L/4$ are included by a finite maximum.

Proof. We first assume $a \leq 1$. The horizon T is the same in the norms and in Lemma 16.1; every heat time below is $t - s \in [0, T]$, so the heat estimates are used only on this fixed horizon, with the constant mode included. Let $w(t) = t + a^2$, $K_u = X_\alpha^a(u)$, and $K_g = H_\sigma^a(g)$. By the definition of the norms, for $0 \leq s \leq T$,

$$\begin{aligned} \|u(s)\|_\infty &\leq K_u w(s)^{-\alpha/2}, & \|D_a u(s)\|_\infty &\leq K_u w(s)^{-(\alpha+1)/2}, \\ \|v(s)\|_\infty &\leq K_g w(s)^{-\sigma/2}, & \|D_a v(s)\|_\infty &\leq K_g w(s)^{-(\sigma+1)/2}. \end{aligned}$$

By (24.4), both $\mathcal{N}_a(u(s), v(s))$ and $\mathcal{N}_a(v(s), u(s))$ are bounded by $CK_u K_g w(s)^{-p}$, where

$$p = \frac{\alpha + \sigma + 1}{2} < 1.$$

The bound for $\mathcal{N}_a(u(s), v(s))$ uses the displayed bound on $D_a v$, and the bound for $\mathcal{N}_a(v(s), u(s))$ uses the displayed bounds on v and $D_a u$.

By Lemma 16.1 with zero and one difference, for either Duhamel term z ,

$$\|z(t)\|_\infty \leq CK_u K_g \int_0^t w(s)^{-p} ds \leq CK_u K_g w(t)^{1-p},$$

$$\|D_a z(t)\|_\infty \leq C_{L,T} K_u K_g \int_0^t (t-s+a^2)^{-1/2} w(s)^{-p} ds \leq C_{L,T,p} K_u K_g w(t)^{1/2-p}.$$

In the last inequality we used Lemma 16.8, whose hypotheses are $0 < p < 1$ and exponent $1/2 < 1$. The weight of the value in Y_α^a adds $\beta = \alpha - 1/2$, and the weight of the difference adds α ; both resulting powers of $w(t)$ equal $(\alpha - \sigma)/2 > 0$. Their suprema on $[0, T]$ are bounded by $(T + a^2)^{(\alpha - \sigma)/2}$. At $t = 0$ both integrals vanish. Every translation in the stencil is an isometry and commutes with the heat semigroup and with differences. Adding the bounds for $\mathcal{Q}_a(u, v)$ and $\mathcal{Q}_a(v, u)$ proves the inequality, and the triangle inequality implies the bound for $\mathcal{B}_a$. For $a \leq 1$ the constant consists of the constant of the stencil, the two heat kernel constants on the horizon T , and the constant of the scalar convolution estimate.

It remains to include $a > 1$ at the fixed side L . Let

$$\mathcal{I}_L = \{n \in \mathbb{N} : 4 \leq n < L\}, \quad a_n = L/n.$$

This finite set consists of the possible coarse lattices, and it is empty when $L \leq 4$. The formula (24.4) and its bound hold for every $a > 0$, since they use no smallness of a ; let C_{st} be the constant in that bound. For every field h , each difference satisfies $\|D_{a,j}^\pm h\|_\infty \leq 2a^{-1} \|h\|_\infty$, since translations preserve the supremum, and the heat semigroup contracts the supremum. Hence, for the two argument orders displayed below,

$$\|\mathcal{N}_a(P_s^a f, P_s^a g)\|_\infty, \|\mathcal{N}_a(P_s^a g, P_s^a f)\|_\infty \leq 2C_{\text{st}} a^{-1} \|f\|_\infty \|g\|_\infty.$$

For either Duhamel term z , we obtain

$$\|z(t)\|_\infty \leq 2C_{\text{st}} T a^{-1} \|f\|_\infty \|g\|_\infty, \quad \|D_a z(t)\|_\infty \leq 2a^{-1} \|z(t)\|_\infty,$$

and the second inequality holds for all translated difference components. With $\beta = \alpha - 1/2 > 0$ and $\eta = (\alpha - \sigma)/2 > 0$, the two weights of Y_α^a are maximal at $t = T$. The terms at $t = 0$ of the input norms give $K_u \geq a^\alpha \|f\|_\infty$ and $K_g \geq a^\sigma \|g\|_\infty$. The sum of the two output norms is then at most

$$4C_{\text{st}} T a^{-1-\alpha-\sigma} \{(T + a^2)^\beta + 2a^{-1}(T + a^2)^\alpha\} K_u K_g. \quad (24.20)$$

We define

$$C_{\text{coarse}} = \max_{n \in \mathcal{I}_L} 4C_{\text{st}} T a_n^{-1-\alpha-\sigma} \{(T + a_n^2)^{\beta-\eta} + 2a_n^{-1}(T + a_n^2)^{\alpha-\eta}\},$$

with value zero if $\mathcal{I}_L$ is empty. It is finite and depends only on L, T, α, σ , and the stencil, and (24.20) is at most $C_{\text{coarse}}(T + a^2)^\eta K_u K_g$ on every coarse lattice. Taking the maximum of C_{coarse} and the constant for $a \leq 1$ proves the lemma. $\square$

24.5 Gauge transformations, energy, and loop observables

We next control changes of gauge, holonomies, the lattice energy, and exponential transforms of bounded observables.

For a closed lattice path based at z , the holonomy of a configuration changes by conjugation at z under a lattice gauge transformation, so only conjugation-invariant functions of it, such as characters, are gauge invariant. For a smooth connection b and a smooth gauge transformation k we use $b^k = kbk^{-1} - (dk)k^{-1}$, as in Definition 4.10, and we write $b|_a$ for the field $(b|_a)_i(x) = b_i(x)$ obtained by restriction to the sites. The parallel transport along a path ℓ solves $H'(s) = H(s)b(\ell(s))\dot{\ell}(s)$, $H(0) = I$.

Lemma 24.7. *The following deterministic statements hold on the periodic lattice.*

1. *Let k be a smooth periodic gauge transformation and b a smooth connection. For a field A define*

$$\mathcal{G}_{a,k}(A)_i(x) = a^{-1} \text{Log}(k(x)e^{aA_i(x)}k(x + ae_i)^{-1}),$$

the logarithm of the lattice gauge transform of e^{aA} by the restriction of k to the sites. On every fixed ball $\{\|A\|_{C_a^1} \leq \rho\}$, for sufficiently small a ,

$$\|\mathcal{G}_{a,k}(A) - \mathcal{G}_{a,k}(A')\|_{C_a^1} \leq C_k \|A - A'\|_{C_a^1}, \quad \|\mathcal{G}_{a,k}(b|_a) - (b^k)|_a\|_{C_a^1} \leq C_{k,b}a.$$

The constants and the threshold for a depend on ρ , $\|k\|_{C^3}$, and, for the second bound, $\|b\|_{C^2}$; they do not depend on the mesh. For a coordinate edge from x to $x + ae_i$, the parallel transport of b along the edge and $e^{ab_i(x)}$ differ by at most $C\|b\|_{C^1}a^2$ in operator norm. The products of these matrices along a lattice path of physical length at most ℓ differ by at most $C_{\ell,b}a$, and so do their normalized traces.

2. *Let A_a and A'_a be the solutions of Theorem 24.5 for two fields A_0, A'_0 on the same lattice, compared as arrays, and let d_a be as there. For every $0 < s < T_R$, every closed lattice path of physical length at most ℓ , and every $t \in [s, T_R]$, the normalized traces of the holonomies of $\Psi_t^{(n)}(e^{aA_0})$ and $\Psi_t^{(n)}(e^{aA'_0})$ along the path differ by at most $C_{R,s,\ell}d_a$. For $0 \leq t \leq T_R$ we also have*

$$\|D_a(A_a - A'_a)(t)\|_\infty \leq C_R d_a \{(t + a^2)^{-(\alpha+1)/2} + (t + a^2)^{-\alpha} + (t + a^2)^{\kappa-1/2}\}.$$

Let $\lambda_a = a^{-2}\phi(aA_a)$ and λ'_a be the physical damping generators (24.8) of the two solutions. Then $\|\lambda_a - \lambda'_a\|_\infty \leq C\|D_a(A_a - A'_a)\|_\infty$ and $\int_0^{T_R} \|\lambda_a(t) - \lambda'_a(t)\|_\infty dt \leq C_R d_a$, uniformly in a . The two damping gauge transformations $k, \tilde{k}$ of Lemma 24.2(6), both equal to the identity at $t = 0$, satisfy $\|k_x(t) - \tilde{k}_x(t)\|_{\text{op}} \leq C_R d_a$ for every site x and $0 \leq t \leq T_R$. For a closed path based at z , the holonomies $H_d(t)$ of the damped links $e^{aA_a(t)}$ and $H_W(t)$ of $\Psi_t^{(n)}(e^{aA_0})$ satisfy $H_d(t) = k_z(t)H_W(t)k_z(t)^{-1}$. Consequently the based holonomy matrices of the two Wilson flows along such a path differ by at most $C_{R,s,\ell}d_a$ for the same $0 < s < T_R$ and $t \in [s, T_R]$.

The constants in part (2) have the dependence of Theorem 24.5, and depend in addition on s and ℓ as indicated.

Proof. We use the operator norm for matrix products; it is equivalent to the norm on $\mathfrak{su}(2)$ with numerical constants.

Part (1): smooth gauge transformations. We define

$$h_{a,i}(x) = k(x)k(x + ae_i)^{-1}, \quad q_{a,i}(x) = a^{-1} \text{Log } h_{a,i}(x).$$

These are defined for all sufficiently small a , depending on $\|k\|_{C^1}$ and a fixed margin of the logarithm chart. We claim that $\|q_{a,i}\|_{C^2} \leq C_k$ and

$$\|q_{a,i} + (\partial_i k)k^{-1}\|_{C^1} \leq C_k a, \quad (24.21)$$

where here and below C^j norms of functions depending on the parameter a are norms of continuous functions on $\mathbb{T}_L^3$, taken before restriction to the sites. To verify this without dividing a remainder by a , we set $h_i(u, x) = k(x)k(x + ue_i)^{-1}$. Then $h_i(0, x) = I$ and $\partial_u h_i(0, x) = -(\partial_i k)k^{-1}$. Taylor's formula with integral remainder for $\text{Log } h_i(u, x)$ in u gives (24.21), using two derivatives in u and one in x . For the bound in C^2 , we use instead $q_{a,i} = \int_0^1 \partial_u \text{Log } h_i(ua, x) du$ with two spatial derivatives. These derivatives involve at most three derivatives of k ; since $k^{-1} = k^*$, the inverse has the same derivative bounds.

Let $B_i(x) = \text{Ad}(k(x))A_i(x)$. The transformed link is $k(x)e^{aA_i(x)}k(x + ae_i)^{-1} = e^{aB_i(x)}e^{aq_{a,i}(x)}$. The function

$$F_a(B, q) = a^{-1} \text{Log}(e^{aB}e^{aq}) = B + q + aK_a(B, q)$$

has a removable singularity at $a = 0$, and K_a and its derivatives of fixed order are bounded on each fixed bounded set of (B, q) , for sufficiently small a . Indeed, the convergent product and logarithm series have linear term $a(B + q)$, all other terms carry at least a^2 , and their geometric majorants can be differentiated on a smaller ball. We conclude that $\mathcal{G}_{a,k}(A)_i(x) = F_a(B_i(x), q_{a,i}(x))$, and the transformed links lie in the logarithm chart.

For the first bound of part (1), let A, A' lie in the ball. Then $B, D_a B$, and the corresponding quantities for A' are bounded in terms of ρ and k . We write

$$F_a(B, q) - F_a(B', q) = \int_0^1 \partial_B F_a(B' + u(B - B'), q)(B - B') du.$$

By the shifted product rule, a difference D_a of this expression falls either on $B - B'$ or on the coefficient under the integral. In the second case, the integral form of the chain rule used in the proof of Lemma 24.1 bounds it by $C(\|D_a B\|_\infty + \|D_a B'\|_\infty + \|D_a q\|_\infty)\|B - B'\|_\infty$. It follows that the C_a^1 norm of the difference is at most $C_k\|A - A'\|_{C_a^1}$. Forward and backward differences obey the same calculation, with a backward translation in place of

a forward one. For a sample $A = b|_a$ of a smooth connection, (24.21) in $F_a = B + q + aK_a$ shows that $\mathcal{G}_{a,k}(b|_a)_i$ is the restriction of $\text{Ad}(k)b_i - (\partial_i k)k^{-1} + O_{k,b}(a) = (b^k)_i + O_{k,b}(a)$, the error being of order a in continuous C^1 . Restriction to the sites preserves this bound for both differences, since $D_{a,j}^+ f(x) = \int_0^1 \partial_j f(x + uae_j) du$ and the backward difference is the average of $\partial_j f$ over the preceding segment. This proves both bounds.

For the transport, let $V(s)$ solve $V' = Vb_i(x + se_i)$, $V(0) = I$, and let $E(s) = e^{sb_i(x)}$. The derivative of $V(s)E(a-s)$ is $V(s)(b_i(x + se_i) - b_i(x))E(a-s)$, whose outer factors are unitary. Integrating from 0 to a , we obtain

$$\|V(a) - E(a)\|_{\text{op}} \leq \int_0^a \|b_i(x + se_i) - b_i(x)\|_{\text{op}} ds \leq Ca^2 \|b\|_{C^1}.$$

Reversed edges use inverses, and $\|V^{-1} - E^{-1}\| = \|V - E\|$. For a product of m such factors, we replace the factors one at a time; the other factors are unitary, so the errors add up to at most $mCa^2 \|b\|_{C^1}$, and $ma \leq \ell$ gives $C_{\ell,b}a$. A normalized trace has absolute value at most the operator norm, so the same bound holds for traces.

Part (2): holonomies and damping gauge transformations. For skew-Hermitian X, Y , differentiating $e^{(1-u)X}e^{uY}$ in u , we find

$$e^X - e^Y = \int_0^1 e^{(1-u)X}(X - Y)e^{uY} du,$$

and hence $\|e^{aA_e} - e^{aA'_e}\|_{\text{op}} \leq Ca|A_e - A'_e|$; inverses satisfy the same bound. Telescoping along a closed path with at most ℓ/a links, we obtain the bound $C\ell\|A(t) - A'(t)\|_{\infty}$ for the difference of the holonomies of the damped links. By Theorem 24.5 on $[s, T_R]$, this is at most $C_{R,s,\ell}d_a$. The holonomy of the Wilson flow is conjugate to the damped holonomy at the basepoint, so the two have the same trace, and the trace bound follows.

Before the time weights are suppressed, Step 6 of the proof of Theorem 24.5 gives exactly

$$\|D_a(A - A')(t)\|_{\infty} \leq C_R d_a \{(t + a^2)^{-(\alpha+1)/2} + (t + a^2)^{-\alpha} + (t + a^2)^{\kappa-1/2}\},$$

the three terms coming from $x - x'$, $y - y'$, and $r - r'$. By (24.8), the damping generator is a linear combination of differences of A , so $\|\lambda_a - \lambda'_a\|_{\infty}$ is at most a numerical constant times the last display. The three negative powers have exponents $(\alpha + 1)/2$, α , and $1/2 - \kappa$, all strictly less than one. For $0 \leq q < 1$,

$$\int_0^{T_R} (t + a^2)^{-q} dt \leq \int_0^{T_R} t^{-q} dt = \frac{T_R^{1-q}}{1-q},$$

which yields the bound on the integral of $\|\lambda_a - \lambda'_a\|_{\infty}$, uniformly in a .

Let k and $\tilde{k}$ solve $\dot{k} = \lambda k$ and $\dot{\tilde{k}} = \lambda' \tilde{k}$ with identity initial values. Direct differentiation gives

$$\frac{d}{dt}(k^{-1}\tilde{k}) = k^{-1}(\lambda' - \lambda)\tilde{k},$$

and the outer factors are unitary. Integrating, we obtain $\sup_x \|k_x(t) - \tilde{k}_x(t)\|_{\text{op}} = \sup_x \|I - k_x(t)^{-1}\tilde{k}_x(t)\|_{\text{op}} \leq \int_0^t \|\lambda - \lambda'\|_{\infty} \leq C_R d_a$. No exponential factor depending on the individual generators appears. For a closed path based at z , the damped links are $k_x U_{xy} k_y^{-1}$ by Lemma 24.2(6), so $H_d = k_z H_W k_z^{-1}$. Undoing the two dampings adds at most $2\|k_z - \tilde{k}_z\|_{\text{op}} \leq 2C_R d_a$ to the difference of the damped holonomies, which implies the bound for the based holonomy matrices of the Wilson flows. Here we used that both damping gauge transformations start at the identity. $\square$

Lemma 24.8. *Write*

$$\mathcal{E}_a(U) = \frac{4}{a} \sum_p q(U_p), \quad F_{a,p} = a^{-2} \text{Log } U_p, \quad \|F_a\|_{2,a}^2 = a^3 \sum_p |F_{a,p}|^2.$$

There exist numerical constants $n_{\text{cov}}, r_{\text{cov}}, c_0, c, C > 0$ such that the following holds for $n \geq n_{\text{cov}}$. If $|\text{Log } U_p| \leq r_{\text{cov}}$ for every plaquette, then $c\|F_a\|_{2,a}^2 \leq \mathcal{E}_a(U) \leq C\|F_a\|_{2,a}^2$. The Wilson flow $U(t) = \Psi_t^{(n)}(U(0))$ exists for all $t \geq 0$, and $\mathcal{E}_a(U(t))$ is nonincreasing. If the plaquette bound holds at a time t_0 , $\mathcal{E}_a(U(t_0)) \leq E_0$, and $aE_0 \leq c_0$, then $|\text{Log } U_p(t)| \leq r_{\text{cov}}/2$ for every plaquette and every $t \geq t_0$. No smallness of E_0 itself is assumed.

The normalization of $\mathcal{E}_a$ differs by a factor two from the lattice energy (2.2): $\mathcal{E}_a = E_a/2$. For $U = e^{aA}$ with $A = b|_a$ the sample of a smooth connection, $\text{Log } U_{p_{ij}(x)} = a^2 F_{ij}(x) + O(a^3)$ and $q(e^X) = |X|^2/8 + O(|X|^4)$, so $\mathcal{E}_a(U)$ approximates $(1/2) \sum_{i < j} \int_{\mathbb{T}_L^3} |F_{ij}|^2 dx = E_L(b)/2$ and $E_a(U)$ approximates $E_L(b)$; see Lemma 29.14.

Proof of Lemma 24.8. By Lemma 5.1, on the principal chart of the plaquette variables, $|\text{Log } U_p|^2/(2\pi^2) \leq q(U_p) \leq |\text{Log } U_p|^2/8$. Since $\|F_a\|_{2,a}^2 = a^{-1} \sum_p |\text{Log } U_p|^2$, we may take $c = 2/\pi^2$ and $C = 1/2$. We choose any numerical $0 < r_{\text{cov}} < 1$ and take $n_{\text{cov}} = 4$.

On a fixed lattice the configuration space is a compact manifold and the Wilson vector field is smooth, so its flow exists for all times by the standard theorem on smooth vector fields on compact manifolds: a solution on a maximal interval with a finite endpoint would, by the bounded speed, have a limit at that endpoint, from which it could be continued. Along the flow,

$$\frac{d}{dt} \mathcal{E}_a(U(t)) = -\frac{16}{a^3} \|\nabla E_W(U(t))\|_2^2 \leq 0, \quad E_W = \sum_p q(U_p), \quad (24.22)$$

since $\mathcal{E}_a = 4E_W/a$ and $U(t)$ moves with velocity $-(4/a^2)\nabla E_W$. An added gauge velocity contributes zero to this derivative, since E_W is invariant under lattice gauge transformations at the link endpoints. Under the assumptions at t_0 , each nonnegative plaquette

term satisfies $q(U_p(t)) \leq aE_0/4$ for $t \geq t_0$. We fix $0 < c_0 \leq 2(1 - \cos(r_{\text{cov}}/4))$. Then $q(U_p(t)) < 1 - \cos(r_{\text{cov}}/4)$. Since $q(U) = 1 - \cos(\rho_G(U)/2)$ and this function is strictly increasing for $0 \leq \rho_G(U) \leq 2\pi$, we obtain $\rho_G(U_p(t)) < r_{\text{cov}}/2$. These plaquette variables avoid $-I$, and their principal logarithms have norm $\rho_G(U_p(t)) < r_{\text{cov}}/2$. This argument uses only the energy of the compact configuration; it requires no representative with small link variables and no curvature estimate at later times, and E_0 may be large. $\square$

Lemma 24.9. *Let $W = (W_i)_{i=1}^m$ and $W' = (W'_i)_{i=1}^m$ be random vectors in $\mathbb{R}^m$ on a common probability space with $|W_i|, |W'_i| \leq 1$. For a number $U \geq 0$ and $u \in \mathbb{C}^m$ with $\sum_i |u_i| \leq U$,*

$$|\mathbb{E}e^{u \cdot W} - \mathbb{E}e^{u \cdot W'}| \leq e^U U \mathbb{E}\|W - W'\|_\infty.$$

More generally, for every event $\mathcal{O}$,

$$|\mathbb{E}e^{u \cdot W} - \mathbb{E}e^{u \cdot W'}| \leq e^U U \mathbb{E}[\|W - W'\|_\infty \mathbf{1}_{\mathcal{O}^c}] + 2e^U \mathbb{P}(\mathcal{O}).$$

For real u with $\sum_i |u_i| \leq U$, $\mathbb{E}e^{u \cdot W} \geq e^{-U}$. Let $U_* > 0$ satisfy $e^{U_*} - 1 < 1/2$. On the complex neighborhood $\mathcal{V} = \{u : \sum_i |u_i| < U_*\}$, the transform $\mathbb{E}e^{u \cdot W}$ has an analytic logarithm vanishing at $u = 0$, and if a sequence of such transforms converges uniformly on compact subsets of $\mathcal{V}$, then every partial derivative of fixed order of their logarithms converges uniformly on compact subsets of $\mathcal{V}$. For characters of representations of dimensions d_i , the same statements hold with the bounds $|W_i|, |W'_i| \leq d_i$ and with $\sum_i |u_i|$ replaced by $\sum_i d_i |u_i|$.

Proof. The segment from W to W' stays in the unit cube. For complex u with $\sum_i |u_i| \leq U$, by the fundamental theorem of calculus we have

$$|e^{u \cdot W} - e^{u \cdot W'}| \leq \int_0^1 e^{\text{Re}(u \cdot ((1-v)W + vW'))} \sum_i |u_i| |W_i - W'_i| dv \leq e^U U \|W - W'\|_\infty.$$

Taking expectations, we obtain the first bound. On $\mathcal{O}$ each of the two integrands has modulus at most e^U , which gives the second bound. For real u the integrand is at least e^{-U} .

Since the observables are bounded, the exponential series and each of its derivatives converge uniformly on bounded sets of u , so $M(u) = \mathbb{E}e^{u \cdot W}$ is entire. Moreover $|M(u) - 1| \leq e^{\sum_i |u_i|} - 1$, which is less than $1/2$ on $\mathcal{V}$. Hence

$$\log M(u) = \sum_{m \geq 1} \frac{(-1)^{m+1}}{m} (M(u) - 1)^m$$

converges uniformly on $\mathcal{V}$ and vanishes at the origin. Integrating $1/w$ along the segment between two points of the disc $\{|w - 1| < 1/2\}$ shows that the logarithms of two such transforms differ by at most twice the difference of the transforms. In particular, uniform convergence of transforms on a compact subset of $\mathcal{V}$ implies uniform convergence of

their logarithms there. For a multi-index ν and a compact set in $\mathcal{V}$, we choose closed polydiscs of radii $\rho_j > 0$ around its points lying in a compact subset of $\mathcal{V}$. The Cauchy integral formula bounds the difference of the derivatives ∂^ν at the center by $\nu! \prod_j \rho_j^{-\nu_j}$ times the supremum of the difference of the logarithms on the distinguished boundary, and the convergence of derivatives follows. For characters of dimensions d_i , the same calculation applies with $\sum_i |u_i|$ replaced by $\sum_i d_i |u_i|$ and the bound 1 replaced by d_i . $\square$

24.6 Smooth solutions of the Yang–Mills heat flow

In this subsection we work on the trivial periodic $\mathrm{SU}(2)$ bundle over $\mathbb{T}_L^3$. For an integer $r \geq 1$ let $\mathcal{A}^r = C^r(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$, with the norm $\|b\|_{C^r}$ given by the maximum of all component derivatives of order at most r ; replacing it by their finite sum changes only constants depending on r . We write $\mathcal{A}^\infty = \mathcal{A}_L$ for the smooth connections. The Yang–Mills heat flow $\partial_t A_i = \sum_j D_j F_{ji} = -(d_A^* F)_i$ and its semigroup Φ on smooth periodic gauge orbits are as in Definition 4.10. The DeTurck equation is

$$\partial_t b_i = \Delta b_i + \sum_j \{2[b_j, \partial_j b_i] - [b_j, \partial_i b_j] + [b_j, [b_j, b_i]]\}, \quad (24.23)$$

and we write its right side as $\Delta b + \mathcal{N}(b)$, with $\mathcal{N} = \mathcal{N}_2 + \mathcal{N}_3$ the sum of its quadratic and cubic parts. A mild solution on $[0, d]$ is a path $b \in C([0, d]; \mathcal{A}^r)$ with $b(t) = P_t b_0 + \int_0^t P_{t-s} \mathcal{N}(b(s)) ds$, where P_t is the continuum heat semigroup acting on each component.

Inputs from the literature. We use the following deterministic statements; page numbers refer to the arXiv versions cited. These papers work on the unit torus with a compact Lie group and the trivial bundle, and $\mathrm{SU}(2)$ is allowed.

- (C1) [43, Definition 3.13, pp. 24–25, and Theorem 3.14, p. 25]. For every initial connection in H^1 there exists a unique solution of the Yang–Mills heat flow in the following class: it is continuous with values in H^1 , its curvature $F(t)$ lies in H^1 for $t > 0$, it has a strong L^2 time derivative equal to $-d_A^* F$, its curvature is bounded in L^∞ on compact intervals of positive times, and $t^{3/4} \|F(t)\|_\infty$ is bounded near $t = 0$. Smooth initial data give a solution that is smooth up to $t = 0$. The proof of this theorem [43, Appendix B, p. 58] obtains existence from Charalambous and Gross [47, Theorem 2.5] or Råde [139, Theorem 1], uniqueness from [47, Theorem 9.15], and smoothness for smooth data from [139, Theorem 1]; on the torus the boundary conditions of Charalambous and Gross are vacuous.
- (C2) [42, Theorem 1.2 and Lemma 1.3, p. 3]: local existence and uniqueness of smooth solutions of the DeTurck equation for smooth periodic initial data.
- (C3) [42, Lemma C.3, p. 56]: for $f \in C^q$, with $q \geq 0$ an integer, $\|P_t f\|_{C^q} \leq \|f\|_{C^q}$ and $\|P_t f\|_{C^{q+u}} \leq C_u t^{-u/2} \|f\|_{C^q}$ for $u > 0$ and $0 < t \leq 1$.

(C4) [42, Lemma C.2, p. 55]: $\|\mathcal{N}_2(b)\|_{C^q} \leq C_q \|b\|_{C^q} \|b\|_{C^{q+1}}$ and $\|\mathcal{N}_3(b)\|_{C^q} \leq C_q \|b\|_{C^q}^3$, and, with $M_q = \max(\|b\|_{C^q}, \|c\|_{C^q})$,

$$\begin{aligned} \|\mathcal{N}_2(b) - \mathcal{N}_2(c)\|_{C^q} &\leq C_q \{M_q \|b - c\|_{C^{q+1}} + M_{q+1} \|b - c\|_{C^q}\}, \\ \|\mathcal{N}_3(b) - \mathcal{N}_3(c)\|_{C^q} &\leq C_q M_q^2 \|b - c\|_{C^q}. \end{aligned}$$

(C5) [42, (C.9) and Corollary C.6, p. 58; Lemmas C.12 and C.13, p. 61]: for a smooth initial datum and $0 < T \leq 1$, a fixed point in $C([0, T]; C^1)$ of the Duhamel map (C.9) belongs to $C([0, T]; C^q)$ for every q and is a smooth solution of the DeTurck equation on $[0, T]$; conversely, a smooth solution satisfies the Duhamel equation on each closed subinterval $[0, T']$ with $T' < T$.

All connections to which we apply these statements are smooth periodic connections on the trivial $SU(2)$ bundle, and hence belong to H^1 . The statements are transferred from the unit torus to $\mathbb{T}_L^3$ by

$$\tilde{b}(y) = Lb(Ly), \quad \tilde{t} = t/L^2, \quad F_{\tilde{b}}(y) = L^2 F_b(Ly), \quad E_1(\tilde{b}) = LE_L(b).$$

Differentiating the first formula shows that both evolution equations are preserved under this change of variables, and C^j norms change by the factors L^{1+m} , $0 \leq m \leq j$. Constants may depend on L ; no coupling is rescaled. With the convention of Definition 4.10, our gauge transformation k corresponds to k^{-1} in [43, (1.1), p. 2]; the statements (C1)–(C5) involve no gauge transformation, and all gauge calculations below use our convention. The local estimates in C^r , the removal of the gauge fixing, and the construction of the neighborhoods in the following theorem are proved here.

Theorem 24.10. *The following statements hold.*

1. *For every integer $r \geq 1$ and $R < \infty$ there exists $d = d(r, R, L) > 0$ such that, for every $b_0 \in \mathcal{A}^r$ with $\|b_0\|_{C^r} \leq R$, the equation (24.23) has a unique mild solution $D.b_0 \in C([0, d]; \mathcal{A}^r)$ (here D_t denotes the solution map, not a covariant derivative), and, for two such initial data,*

$$\sup_{t \leq d} \|D_t b_0\|_{C^r} \leq 2(R+1), \quad \sup_{t \leq d} \|D_t b_0 - D_t b'_0\|_{C^r} \leq 2\|b_0 - b'_0\|_{C^r}.$$

If b_0 is smooth, then, after reducing d once if necessary (depending only on r , R , and L), the solution is the restriction to $[0, d]$ of a smooth solution on $[0, 2d]$. If b is such a smooth solution, then the solution k of

$$\partial_t k = -k d^* b = k \sum_i \partial_i b_i, \quad k(0) = I,$$

is a smooth periodic gauge transformation, and $b(t)^{k(t)} = \Phi_t(b_0)$. Finally, for every integer $q \geq 0$ there exists a polynomial $P_{q,L}$ with nonnegative coefficients such that every smooth periodic gauge transformation k with $B = A^k$, for smooth connections A and B , satisfies

$$\|k^{\pm 1}\|_{C^{q+1}} \leq P_{q,L}(1 + \|A\|_{C^q} + \|B\|_{C^q}).$$

2. For every integer $r \geq 1$, every finite horizon $T > 0$, and every set $\mathcal{K} \subset \mathcal{A}^\infty$ that is compact in the C^r topology, there exist a C^r -open neighborhood $\mathcal{U}$ of $\mathcal{K}$, a finite open cover $(\mathcal{U}_\lambda)$ of $\mathcal{U}$, and a partition $0 = t_0 < \dots < t_Q = T$ with the following property. For every smooth $b_0 \in \mathcal{U}_\lambda$ there exist smooth solutions $b_{\lambda,i}(s; b_0)$, $0 \leq s \leq t_{i+1} - t_i$, of (24.23), with $b_{\lambda,0}(0; b_0) = b_0$, and smooth periodic gauge transformations $k_{\lambda,i}$ that do not depend on $b_0 \in \mathcal{U}_\lambda$, such that

$$b_{\lambda,i+1}(0; b_0) = b_{\lambda,i}(t_{i+1} - t_i; b_0)^{k_{\lambda,i}}, \quad [b_{\lambda,i}(s; b_0)] = [\Phi_{t_i+s}(b_0)].$$

The numbers of cover members and of intervals, the C^r bounds of the $b_{\lambda,i}$, and their Lipschitz constants as maps from $\mathcal{U}_\lambda$, with the C^r norm, to $C([0, t_{i+1} - t_i]; \mathcal{A}^r)$ are finite and uniform on $\mathcal{U}$. For every q , the C^q norms of the finitely many $k_{\lambda,i}^{\pm 1}$ are bounded. These constants depend on $L, T, r, \mathcal{K}$, and the construction, and on q where indicated; they do not depend on any coupling, lattice mesh, lattice side, M , or J .

3. Fix an integer $r \geq 3$ and a finite horizon $T > 0$, and let b_0, b'_0 be smooth. As $b'_0 \rightarrow b_0$ in C^r , or $b'_0 \rightarrow b_0^h$ in C^r for a fixed smooth time-independent gauge transformation h , every finite vector of values $W_{\ell_i, \chi_i, t}$ of loops and characters along the trajectory of b'_0 , and the energies $E_L(\Phi_t b'_0)$, converge to the corresponding quantities for b_0 uniformly for $t \in [0, T]$, and

$$\sup_{0 \leq t \leq T} \inf_{k \in C^\infty(\mathbb{T}_L^3; \text{SU}(2))} \|\Phi_t b_0 - (\Phi_t b'_0)^k\|_\infty \longrightarrow 0.$$

The map $b_0 \mapsto ([\Phi_t b_0])_{t>0}$ from the smooth connections, with the C^r topology, to $(\mathcal{X}_L, \mathcal{T}_W)$ is continuous, and hence Borel. The energy is nonincreasing along the flow, and

$$E_L(\Phi_t b_0) \leq E_L(b_0) \leq CL^3(K_1 + K_1^2)^2, \quad K_1 = \|b_0\|_{C^1}.$$

4. If $\mathcal{K}$ is a set of smooth connections that is compact in the C^3 topology, then there exist a neighborhood $\mathcal{U}$ as in part (2) with $r = 3$ and $\eta > 0$ such that every smooth connection at C^3 distance less than η from $\mathcal{K}$ lies in $\mathcal{U}$. In particular, smooth connections approaching $\mathcal{K}$ in C^3 eventually lie in $\mathcal{U}$; no assumption on the closure of the approximating family is needed.

In part (2), the members $\mathcal{U}_\lambda$ of the cover are neighborhoods of initial data, while the index i labels time intervals; the construction is not a chart on the space of gauge orbits. On one member of the cover, the representative of the trajectory is obtained by alternating local solutions of the DeTurck equation and fixed gauge transformations:

$$b_0 = c_0 \xrightarrow{D_{h_0}} D_{h_0} c_0 \xrightarrow{(\cdot)^{k_0}} c_1 \xrightarrow{D_{h_1}} D_{h_1} c_1 \xrightarrow{(\cdot)^{k_1}} c_2 \longrightarrow \dots \longrightarrow c_{Q-1} \xrightarrow{D_{h_{Q-1}}} D_{h_{Q-1}} c_{Q-1}, \quad (24.24)$$

where $h_i = t_{i+1} - t_i$ and c_i represents the orbit at time t_i .

Proof. We first prove the local estimates in C^r and identify smooth solutions with the solutions of (C2) and (C5). Next we remove the gauge fixing and prove the bounds on gauge transformations. The neighborhoods of part (2) are then constructed around one reference trajectory, by propagating errors through (24.24), and finally around a compact family. The remaining steps prove parts (3) and (4).

Step 1: the local solution map in C^r . By the product rule and $||[X, Y]|| \leq |X||Y|$, for $r \geq 1$ and $\|b\|_{C^r}, \|c\|_{C^r} \leq S$,

$$\|\mathcal{N}(b)\|_{C^{r-1}} \leq C_r(S^2 + S^3), \quad \|\mathcal{N}(b) - \mathcal{N}(c)\|_{C^{r-1}} \leq C_r(S + S^2)\|b - c\|_{C^r}. \quad (24.25)$$

Indeed, for $|\nu| \leq r - 1$, $\partial^\nu[b_j, \partial_k b_i] = \sum_{\mu \leq \nu} \binom{\nu}{\mu} [\partial^\mu b_j, \partial^{\nu-\mu} \partial_k b_i]$, and both factors involve at most r derivatives. For the cubic brackets, we distribute ν over the three factors. For differences, we replace one factor at a time by the corresponding factor of $b - c$ and keep the other factors at b or c . The constants depend only on r and the conventions for directions and $\mathfrak{su}(2)$; these formulas also give the estimates (C4).

By (C3), after the rescaling to $\mathbb{T}_L^3$, for $0 < h \leq \min(1, L^2)$,

$$\|P_h f\|_{C^r} \leq \|f\|_{C^r}, \quad \|P_h f\|_{C^r} \leq C_{r,L} h^{-1/2} \|f\|_{C^{r-1}}.$$

The heat semigroup is strongly continuous at zero in C^r : each derivative of order at most r is continuous and periodic, and convolution with the positive approximate identity converges uniformly to it. For a continuous path f with values in C^{r-1} and bounded by K , the heat convolution $\int_0^t P_{t-s} f(s) ds$ has C^r norm at most $2C_{r,L} \sqrt{t} K$. It is also continuous with values in C^r , including at $t = 0$. To see this, we split off the upper part of the integral, of length η , whose norm is at most $2C_{r,L} \sqrt{\eta} K$. On the remaining interval the heat times are bounded below, so the kernel and all derivatives needed are continuous and bounded, and dominated convergence applies. We then let $\eta \rightarrow 0$.

For $R \geq 0$ we set $S = 2(R + 1)$ and use the Banach space $C([0, 2d]; \mathcal{A}^r)$ with the supremum norm. On its closed ball of radius S , we consider

$$\mathcal{M}_{b_0}(b)(t) = P_t b_0 + \int_0^t P_{t-s} \mathcal{N}(b(s)) ds.$$

We choose $d > 0$ such that $2d \leq \min(1, L^2)$ and

$$2C_{r,L} C_r \sqrt{2d} (S^2 + S^3) \leq S - R, \quad 2C_{r,L} C_r \sqrt{2d} (S + S^2) \leq \frac{1}{2}. \quad (24.26)$$

These choices depend only on r , R , and L . By (24.25), the map preserves the ball and is a contraction with constant $1/2$. Its iterates converge, since successive differences form a geometric series. Subtracting the fixed-point equations for two initial data, we find

$$\sup_{t \leq 2d} \|D_t b_0 - D_t b'_0\|_{C^r} \leq \|b_0 - b'_0\|_{C^r} + \frac{1}{2} \sup_{t \leq 2d} \|D_t b_0 - D_t b'_0\|_{C^r},$$

which implies the Lipschitz bound with constant two; the ball gives the size bound. Any two mild solutions in C^r are bounded on a common compact interval. On an interval of length h , by (24.25), their difference is at most $C\sqrt{h}$ times its supremum when the initial values agree. Choosing h so that this coefficient is less than one and restarting the mild equation by the semigroup property, finitely many intervals give uniqueness. (If $R < 0$ there are no initial data, and any choice made for $R = 0$ will do.)

Step 2: smooth initial data. Let b_0 be smooth. The solution just constructed is a fixed point in $C([0, 2d]; C^1)$ of the map (C.9) of [42, p. 58], and the time $2d$ corresponds to a unit-torus time at most one. By (C5) it belongs to $C([0, 2d]; C^q)$ for every q and is a smooth solution on $[0, 2d]$; conversely, a smooth solution satisfies the mild equation on every closed subinterval of its interval of definition. The local smooth existence and uniqueness of (C2) describe the same solution. Restriction to $[0, d]$ leaves a smooth solution beyond the right endpoint, which is used at the endpoints of the intervals below.

Step 3: removal of the gauge fixing. Let $v = \sum_i \partial_i b_i = -d^*b$. Expanding the curvature and the covariant derivatives gives

$$\Delta b + \mathcal{N}(b) = -d_b^* F_b + d_b v. \quad (24.27)$$

Indeed, in components $\sum_j D_j F_{ji}$ has linear part $\Delta b_i - \partial_i v$, quadratic part $[v, b_i] + 2\sum_j [b_j, \partial_j b_i] - \sum_j [b_j, \partial_i b_j]$, and cubic part $\sum_j [b_j, [b_j, b_i]]$, and adding $(d_b v)_i = \partial_i v + [b_i, v]$ cancels the two terms $-\partial_i v$ and $[v, b_i]$.

At every point, we solve $\dot{k} = kv$, $k(0) = I$. The ordered integral series converges on each finite interval on which b is smooth. Its spatial derivatives also converge: after any fixed number of derivatives, their absolute majorant is a polynomial in the index of the iterated integral times the same factorially convergent series, with constants from finitely many derivatives of v . It follows that k is smooth in space and time. Its coefficient is traceless and skew-Hermitian, and differentiating kk^* and $\det k$ shows that $k \in \text{SU}(2)$. Uniqueness for the matrix integral equation and periodicity of v give periodicity of k .

For a time-dependent gauge transformation, we set $\lambda = \dot{k}k^{-1}$. Differentiating the gauge action, we find

$$\partial_t(b^k) = \text{Ad}(k)\partial_t b - d_{b^k}\lambda. \quad (24.28)$$

For example, the derivative of $-\partial_i k k^{-1}$ is $-\partial_i \lambda - [\lambda, \partial_i k k^{-1}]$, and combining it with the derivative $[\lambda, k b_i k^{-1}] + \text{Ad}(k)\partial_t b_i$ of $k b_i k^{-1}$ gives the covariant term. Since $\dot{k} = kv$, we have $\lambda = \text{Ad}(k)v$, and the covariance $\text{Ad}(k)d_b v = d_{b^k} \text{Ad}(k)v$ cancels the last term of (24.27). Hence b^k solves the Yang–Mills heat flow with initial value b_0 .

The initial datum is a smooth periodic connection on the trivial $\text{SU}(2)$ bundle, so (C1) applies, and the global solution is smooth up to $t = 0$. The connection b^k belongs to the same solution class on the interval where it is defined: it is smooth on a compact initial time interval, its strong time derivative is given by the equation, and its curvature

is bounded there, which also bounds $t^{3/4}\|F(t)\|_\infty$ near zero. Uniqueness in (C1) implies that $b(t)^{k(t)} = \Phi_t(b_0)$.

Step 4: fixed gauge transformations. For a smooth time-independent gauge transformation h , differentiating the gauge formula, we find $F(A^h) = \text{Ad}(h)F(A)$ and $d_{A^h} \text{Ad}(h)z = \text{Ad}(h)d_A z$. In particular, a fixed gauge transformation maps smooth solutions of the Yang–Mills heat flow to smooth solutions, and by the uniqueness in (C1) we have $\Phi_t(A^h) = \Phi_t(A)^h$. Time translation and uniqueness give the semigroup property. These two consequences identify the orbits on successive intervals below.

If $B = A^k$, rearranging the formula for the gauge action in our convention, we obtain

$$\partial_i k = kA_i - B_i k, \quad \partial_i k^{-1} = k^{-1}B_i - A_i k^{-1}. \quad (24.29)$$

The norms of order zero of k and k^{-1} are fixed. Applying ∂^ν to the first identity, we obtain

$$\partial^\nu \partial_i k = \sum_{\mu \leq \nu} \binom{\nu}{\mu} ((\partial^\mu k)(\partial^{\nu-\mu} A_i) - (\partial^\mu B_i)(\partial^{\nu-\mu} k)).$$

By induction on $|\nu| \leq q$, all derivatives of k of order at most $q+1$ are bounded by a polynomial with nonnegative coefficients in $1 + \|A\|_{C^q} + \|B\|_{C^q}$. The second identity gives the same bound for k^{-1} , after enlarging the polynomial. This defines $P_{q,L}$; no normalization at a point is needed. For a fixed k , the identity $A^k - B^k = k(A - B)k^{-1}$ and the Leibniz rule give a Lipschitz bound in C^r for the action of k , involving at most r derivatives of k . The inhomogeneous term of the action involves at most $r+1$ derivatives of k .

Step 5: a neighborhood of one reference trajectory. Fix a smooth reference $b_* \in \mathcal{K}$. By (C1), $A_*(t) = \Phi_t b_*$ is smooth on $[0, T]$; we set

$$R_* = 1 + \sup_{t \leq T} \|A_*(t)\|_{C^r} < \infty.$$

We choose a partition whose lengths $h_i = t_{i+1} - t_i$ are at most $d(r, R_*, L)$. On the i th interval, let $b_i^*(s)$ be the solution of the DeTurck equation started from $A_*(t_i)$. By Step 3 and the semigroup property, there exists a smooth periodic gauge transformation $k_i^*(s)$ with $(b_i^*(s))^{k_i^*(s)} = A_*(t_i + s)$. For each $i < Q-1$ set $k_i = k_i^*(h_i)$. These fixed gauge transformations depend only on the reference and the partition, and they map the right endpoints exactly: $(b_i^*(h_i))^{k_i} = A_*(t_{i+1})$. They and their inverses have bounded derivatives of every fixed order, by smoothness and compactness, or quantitatively by (24.29).

Let $L_i \geq 1$ be a Lipschitz constant in C^r for the action of k_i , and set $K_0 = 1$ and $K_{i+1} = 2L_i K_i$. We fix $\rho_* > 0$ so small that $K_i \rho_* < 1/2$ for every i . For b_0 in the open ball $\|b_0 - b_*\|_{C^r} < \rho_*$ define successively

$$c_0(b_0) = b_0, \quad b_i(s; b_0) = D_s c_i(b_0), \quad c_{i+1}(b_0) = (D_{h_i} c_i(b_0))^{k_i}, \quad (24.30)$$

as in (24.24). By induction, $\|c_i(b_0) - A_*(t_i)\|_{C^r} \leq K_i \|b_0 - b_*\|_{C^r} < 1/2$. Indeed, $c_i(b_0)$ lies in the ball of radius R_* , the local solution map has Lipschitz constant two by Step 1 and maps $A_*(t_i)$ to b_i^* , and the fixed gauge transformation k_i then contributes the factor L_i . This proves the induction step, and it shows that every map in (24.30) is defined on the whole ball. Arguing in the same way for two data in the ball, we obtain

$$\|c_i(b_0) - c_i(b'_0)\|_{C^r} \leq K_i \|b_0 - b'_0\|_{C^r}, \quad \sup_{s \leq h_i} \|b_i(s; b_0) - b_i(s; b'_0)\|_{C^r} \leq 2K_i \|b_0 - b'_0\|_{C^r}.$$

The solutions on the intervals have C^r norm at most $2(R_* + 1)$. For smooth initial data all the solutions b_i and all the transformed endpoints are smooth.

On the first interval, Step 3 shows that $b_0(s; b_0)$ represents the orbit of $\Phi_s(b_0)$. If the right endpoint of the i th interval represents $[\Phi_{t_{i+1}} b_0]$, then the fixed gauge transformation k_i preserves this orbit, and by Step 3, Step 4, and the semigroup property the next solution represents $[\Phi_{t_{i+1}+s} b_0]$. By induction, the orbit identity holds on every interval. This argument provides, for each interval, a smooth periodic gauge transformation relating the representative to the Yang–Mills heat flow; no continuous choice of such gauge transformations in b_0 is needed.

Step 6: compact families. The open balls of Step 5 around the points of $\mathcal{K}$ cover $\mathcal{K}$. By compactness in C^r finitely many of them suffice; let $\mathcal{U}$ be their union and let these balls be the $\mathcal{U}_\lambda$. We take maxima of the finitely many bounds and Lipschitz constants; for each q , the finitely many fixed gauge transformations and their inverses have a finite maximal C^q norm. To obtain one partition for all members of the cover, we take the union of the finitely many partitions. When a new time subdivides an interval, we restrict the solution and translate its time origin; it still solves the autonomous DeTurck equation. We use the identity gauge transformation at each new endpoint and keep the original fixed gauge transformation at each old endpoint. The recursion, the orbit identities, and all bounds persist, and the number of intervals remains finite. This proves part (2). Only the finitely many references, their smooth trajectories, the fixed gauge transformations, and the local constants in C^r enter the dependence of the constants. Compactness in C^r is not used to bound higher derivatives of the whole family.

Step 7: observables and continuity. Let $r \geq 3$ and let smooth $b'_0 \rightarrow b_0$ in C^r . We use one member of the cover containing b_0 . All sufficiently close data lie in it, and by the bounds of Step 5 the representatives on each interval converge in C^r uniformly in time. For two representatives b, c and a piecewise C^1 loop ℓ , we subtract their transport equations in a fixed unitary representation ϱ . By the variation-of-constants argument of Lemma 24.7(1), we have

$$\|\text{Hol}_\ell(b) - \text{Hol}_\ell(c)\|_{\text{op}} \leq C_\varrho \text{length}(\ell) \|b - c\|_\infty,$$

where C_ϱ is the norm of the differential of ϱ , a linear map on the finite-dimensional Lie algebra. Taking the trace costs at most the dimension of ϱ . By gauge invariance, these

characters coincide at the endpoints of the intervals with the characters of the Yang–Mills heat flow. We conclude that every finite vector of loop and character values converges uniformly on $[0, T]$.

On a bounded set in C^1 , by the formula for the curvature, we have

$$\|F_b - F_c\|_\infty \leq C(1 + \|b\|_{C^1} + \|c\|_{C^1})\|b - c\|_{C^1}, \quad \|F_b\|_\infty \leq C(\|b\|_{C^1} + \|b\|_{C^1}^2).$$

Using $\|X\|^2 - \|Y\|^2 \leq (\|X\| + \|Y\|)\|X - Y\|$ and integrating over the volume L^3 , the energies of the representatives converge uniformly. By gauge invariance they are the energies of the Yang–Mills heat flow, also at the endpoints of the intervals.

To check the convergence of the orbit infimum without using any later result on metrics, we write the two representatives at time t as $b = (\Phi_t b_0)^h$ and $c = (\Phi_t b'_0)^{h'}$ for smooth periodic gauge transformations h, h' . We apply h^{-1} to both. The inhomogeneous terms cancel in the difference, and the linear part is a pointwise isometry, so

$$\|\Phi_t b_0 - (\Phi_t b'_0)^{h^{-1}h'}\|_\infty = \|b - c\|_\infty.$$

The infimum in part (3) is at most this quantity, and the bounds are uniform in t , which implies the uniform limit. If instead $b'_0 \rightarrow b_0^h$ for a fixed smooth time-independent h , first use $\Phi_t(b_0^h) = \Phi_t(b_0)^h$ from Step 4; characters, energies, and the infimum take the same values.

By the global smooth semigroup of (C1), $t \mapsto [\Phi_t b_0]$ is a compatible trajectory at positive times, that is, a point of $\mathcal{X}_L$. Each function $W_{\ell, \mathcal{X}, t}$ that generates $\mathcal{T}_W$ is a continuous function of b_0 by what we have just proved. Since $\mathcal{T}_W$ is the weakest topology for which these functions are continuous, the trajectory map is continuous; preimages of open sets are open, so it is Borel. No completeness or separability of a metric on orbits enters this argument.

Step 8: the energy. For a smooth solution of the Yang–Mills heat flow, differentiation and periodic integration by parts give

$$\begin{aligned} \frac{d}{dt} E_L(A) &= 2 \sum_{i < j} \int \langle F_{ij}, D_i \dot{A}_j - D_j \dot{A}_i \rangle dx \\ &= -2 \sum_i \int \left\langle \sum_j D_j F_{ji}, \dot{A}_i \right\rangle dx = -2 \sum_i \int |\dot{A}_i|^2 dx \leq 0. \end{aligned}$$

The integration by parts for the brackets uses the invariance of the inner product, and periodicity eliminates boundary terms. The bound on $\|F_b\|_\infty$ in Step 7 at $t = 0$ then implies $E_L(b_0) \leq CL^3(K_1 + K_1^2)^2$.

Step 9: approximation of a compact family. Let $\mathcal{K}$ be compact in C^3 and consist of smooth connections, and apply part (2) with $r = 3$. For each point of $\mathcal{K}$, we choose a

ball around it contained in $\mathcal{U}$, cover $\mathcal{K}$ by finitely many of the balls with half these radii, and let η be smaller than half the least chosen radius. Every point within η of $\mathcal{K}$ then lies in one of the original balls, hence in $\mathcal{U}$. Smooth connections approaching $\mathcal{K}$ in C^3 eventually lie in $\mathcal{U}$. $\square$

24.7 Modified plaquette actions

In this subsection $c \geq 0$ is a parameter of the action. For a fixed upper bound $C_0 \geq 0$ and $0 \leq c \leq C_0$ set

$$\mathcal{A}_c(U) = \sum_{p \in \mathcal{P}_n} \{q(U_p) + c q(U_p)^2\}, \quad \Psi_t^{(n,c)} = \Phi_{4t/a^2}^{\mathcal{A}_c}, \quad a = L/n,$$

where $\Phi^{\mathcal{A}_c}$ is the flow of $-\nabla \mathcal{A}_c$ for the product metric of Section 24.1. In particular $\Psi^{(n,0)} = \Psi^{(n)}$, and these are the flows of Theorem 2.6. The vector field is smooth on the compact configuration space, so its flow exists for all times, as for the Wilson flow. Since the action is gauge invariant and the metric is invariant under gauge transformations, the flow commutes with lattice gauge transformations.

The Taylor expansion of $q(U_p)^2$ in link logarithms starts in degree four, so its force begins in degree three. The linear operator and the bilinear map of the damped equation are unchanged, and the modified flows have the same flow data. On every fixed bounded C_a^1 ball, the plaquette logarithms are $O(a^2)$ and the added physical force is $O(a^3)$, by (24.34) below. In Theorem 24.12 we use this factor near rough initial data by measuring the difference of the solutions in a norm with a slightly weaker time weight, which leaves a positive power of the mesh.

Lemma 24.11. *Let $C_0 \geq 0$, and use the stencil, $\mathcal{J}$, the gauge velocity, $\mathcal{N}_a$, and R_3 of Lemma 24.2. Decreasing r_* if necessary, depending only on the stencil, the following holds. The damped equation for $\mathcal{A}_c$, obtained as in Lemma 24.2(3) with $\sum_p q(U_p)$ replaced by $\mathcal{A}_c$ and the same gauge velocity, is, for $a\|A\|_\infty \leq r_*$,*

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + F_a(A) + E_{a,c}(A), \quad F_a(A) = a^{-3} R_3(aA). \quad (24.31)$$

For $0 \leq c \leq C_0$ and $0 < a \leq 1$, the following statements hold uniformly in the mesh and the volume.

1. $E_{a,c}(A) = a^{-3} \mathcal{E}_c(aA)$, where $\mathcal{E}_c$ is a local analytic map whose Taylor coefficients vanish through degree two; after rescaling, its term of degree m carries the factor a^{m-3} , $m \geq 3$. If $M = \|A\|_\infty$ and aM lies in a fixed smaller ball, then for every $j \geq 0$

$$\|D^j E_{a,c}(A)[V_1, \dots, V_j]\|_\infty \leq K_{j,C_0} \begin{cases} M^{3-j}, & 0 \leq j \leq 3, \\ a^{j-3}, & j > 3, \end{cases} \cdot \prod_{r=1}^j \|V_r\|_\infty, \quad (24.32)$$

with the usual conventions for $j = 0$ and M^0 . In particular,

$$\|E_{a,c}(A)\|_\infty \leq K_{C_0} M^3, \quad \|E_{a,c}(A) - E_{a,c}(B)\|_\infty \leq K_{C_0} (\|A\|_\infty^2 + \|B\|_\infty^2) \|A - B\|_\infty.$$

2. All forward and backward differences obey the composition rule of Lemma 24.1, with the exact lattice translations: for $m \geq 1$,

$$\|D_a^m E_{a,c}(A)\|_\infty \leq K_{m,C_0} \sum_{j=1}^m b_j(M, a) \sum_{\substack{\nu_1 + \dots + \nu_j = m \\ \nu_r \geq 1}} \prod_{r=1}^j \|D_a^{\nu_r} A\|_\infty, \quad (24.33)$$

where $b_j(M, a) = M^{3-j}$ for $j \leq 3$ and $b_j(M, a) = a^{j-3}$ for $j > 3$. For two fields in the same smaller ball, with $M = \max(\|A\|_\infty, \|B\|_\infty)$,

$$\|D_a(E_{a,c}(A) - E_{a,c}(B))\|_\infty \leq K_{C_0} \{M^2 \|D_a(A - B)\|_\infty + M(\|D_a A\|_\infty + \|D_a B\|_\infty) \|A - B\|_\infty\}.$$

3. For A in the smaller ball,

$$\|E_{a,c}(A)\|_\infty \leq K_{C_0} a^3 (\|D_a A\|_\infty + \|A\|_\infty^2)^3. \quad (24.34)$$

Consequently $E_{a,c}(A) = O_{C_0, \rho}(a^3)$ on $\{\|A\|_{C_a^1} \leq \rho\}$.

The constants depend only on the stencil, the margin of the smaller ball, the orders, and C_0 , and not on L , n , the coupling, or any probability law. Part (6) of Lemma 24.2 holds for (24.31) with $\Psi^{(n,c)}$ in place of $\Psi^{(n)}$, with the same damping equation, started at the identity.

Proof. We first compute the added force exactly on the group, then bound it with Lemma 24.1, prove the bound (24.34) from the size of the plaquette logarithms, and identify the flow.

Let e be a positive link and p a plaquette containing it. We vary U_e to $e^{\varepsilon X} U_e$. If P is the product preceding the entry of e in the oriented plaquette word, then

$$\left. \frac{d}{d\varepsilon} U_p(\varepsilon) \right|_{\varepsilon=0} U_p^{-1} = B_{p,e} X, \quad B_{p,e} = \begin{cases} \text{Ad}(P), & \text{if } e \text{ enters positively,} \\ -\text{Ad}(PU_e^{-1}), & \text{if } e \text{ enters negatively.} \end{cases}$$

Each $B_{p,e}$ is an isometry. Direct differentiation, with $\langle X, Y \rangle = -2 \text{tr}(XY)$, gives

$$h(V) = \nabla q(V) = \frac{V - V^{-1}}{8}, \quad h(e^X) = \frac{\sin(|X|/2)}{2|X|} X, \quad \nabla_e q(U_p) = B_{p,e}^* h(U_p),$$

the quotient being continuous at $X = 0$. Since $\nabla(c q_p^2) = 2c q_p \nabla q_p$, the added velocity in logarithmic coordinates, in the variable $s = t/a^2$ and with the factor four, is exactly

$$\mathcal{E}_{c,e}(\theta) = -8c \mathcal{J}(\theta_e)^{-1} \sum_{p \ni e} q(U_p(\theta)) B_{p,e}(\theta)^* h(U_p(\theta)). \quad (24.35)$$

This is the additional negative-gradient velocity expressed in logarithmic coordinates: the factor $\mathcal{J}(\theta_e)^{-1}$ converts the velocity on the group into the logarithmic velocity, and the Euclidean gradient in the coordinates θ is not used. The gauge velocity is unchanged. Substituting $\theta = aA$ and $s = t/a^2$ and multiplying by a^{-3} , we obtain (24.31).

On a fixed small ball all functions of the finite plaquette words in (24.35) are analytic, and $\text{Log } U_p = O(\|\theta\|_{\infty,p})$, $q(U_p) = O(\|\theta\|_{\infty,p}^2)$, $h(U_p) = O(\|\theta\|_{\infty,p})$. The inverse $\mathcal{J}^{-1}$ is analytic on a smaller fixed ball. We conclude that the Taylor coefficients of (24.35) vanish through degree two. Its Taylor series at an output link is a series in a fixed finite stencil, and, extending the finite exponential, inverse, and trace formulas to a smaller complex polydisc and using the Cauchy estimates there, its homogeneous part of degree m has coefficient sum at most $K_1 K_2^m C_0$. Lemma 24.1, with $K_0 = K_1 C_0$ and the smaller ball chosen so that $K_2 a M \leq \eta < 1$, implies (24.32), the bound on the values and their differences, (24.33), and the estimate for the difference of two fields. In these estimates every intermediate stencil is a convex combination of translates of the fields, hence lies in the same ball, and no factor a^{-1} remains outside a difference of A .

For (24.34), note that the four entries of the plaquette $p_{ij}(x)$ for $\theta = aA$ have sum $a^2(D_{a,i}^+ A_j - D_{a,j}^+ A_i)(x)$. The Baker–Campbell–Hausdorff formula on the fixed ball, with quadratic error bounded by a constant times the square of the largest entry, gives

$$|\text{Log } U_p| \leq K a^2 (\|D_a A\|_\infty + \|A\|_\infty^2).$$

On the plaquette chart, $|q(e^X)| \leq K|X|^2$ and $|h(e^X)| \leq K|X|$. Inserting these bounds into (24.35), and using bounded incidence and the boundedness of $\mathcal{J}^{-1}$, we obtain $|\mathcal{E}_{c,e}(aA)| \leq K C_0 a^6 (\|D_a A\|_\infty + \|A\|_\infty^2)^3$; multiplication by a^{-3} gives (24.34).

Finally, we remove the damping by solving the same equation for the damping gauge transformation, started at the identity. Gauge equivariance of $\nabla \mathcal{A}_c$ cancels the gauge velocity exactly, as in Step 7 of the proof of Lemma 24.2. The remaining ordinary differential equation is the one that defines $\Psi^{(n,c)}$, and uniqueness identifies the flows. This argument uses the force (24.35), not a truncation of it. All formulas involve boundedly many links, which gives the stated dependence of the constants. $\square$

Theorem 24.12. *Fix $L > 0$, $1/2 < \alpha < 2/3$, $0 \leq R < \infty$, and $0 \leq C_0 < \infty$, and let $\kappa = 1 - 3\alpha/2$. There exist $T_{R,C_0} \in (0, 1/2]$, $a_{R,C_0} > 0$, and $K_{R,C_0} < \infty$ such that the following holds for every mesh $a \leq a_{R,C_0}$, every $c \in [0, C_0]$, and every field A_0 whose flow data, as in Definition 24.3, satisfy $\|(x_a, y_a)\|_{\mathfrak{A},a} \leq R$ on $[0, T_{R,C_0}]$. The equation (24.31) with $A(0) = A_0$ has a unique solution of the form*

$$A_a^c = x_a + y_a + r_a^c, \quad r_a^c(0) = 0, \quad \|r_a^c\|_{R_a} \leq K_{R,C_0},$$

where the remainder norm (24.13) is taken on $[0, T_{R,C_0}]$; uniqueness holds in the class of Theorem 24.5. The Lipschitz bounds of Theorem 24.5 for two fields, at the same c , hold uniformly in

$c \in [0, C_0]$. These solutions represent the flows $\Psi^{(n,c)}$ in their damping gauges, started at the identity.

For every fixed $0 < \varepsilon < 2 - 3\alpha$ put $\kappa_\varepsilon = \kappa - \varepsilon/2 > 0$ and

$$\|w\|_{R_a^{\kappa_\varepsilon}} = \sup_{0 \leq t \leq T_{R,C_0}} \{ (t + a^2)^{-\kappa_\varepsilon} \|w(t)\|_\infty + (t + a^2)^{1/2-\kappa_\varepsilon} \|D_a w(t)\|_\infty \}.$$

The time T_{R,C_0} can be chosen independently of ε such that, for the same field A_0 and uniformly in $c \in [0, C_0]$,

$$\|A_a^c - A_a^0\|_{R_a^{\kappa_\varepsilon}} \leq K_{R,C_0,\varepsilon} a^\varepsilon. \quad (24.36)$$

In particular, for every fixed $t_0 \in (0, T_{R,C_0}]$, $\|A_a^c(t_0) - A_a^0(t_0)\|_{C_a^1} \rightarrow 0$ as $a \rightarrow 0$, uniformly in c . The damping gauge transformations of A_a^c and A_a^0 , both equal to the identity at $t = 0$, differ by $O_{R,C_0,\varepsilon}(a^\varepsilon)$ at every site, uniformly on $[0, T_{R,C_0}]$ and in c . The constants depend only on $L, \alpha, R, C_0, \varepsilon$, and the stencil and chart, and not on any coupling or probability law.

Proof. The existence part repeats the contraction of Theorem 24.5 with the cubic term F_a replaced by $F_a + E_{a,c}$. The new information is the comparison: the difference of the two solutions satisfies a difference equation with the inhomogeneous source $E_{a,c}(A_a^c)$, which (24.34) bounds by a^ε times a power of $t + a^2$ that is still integrable.

Step 1: construction. We write $p = 3\alpha/2 < 1$, $\delta = (1 - \alpha)/2$, and $v = t + a^2$, and let r_* be as in Lemma 24.11. By Lemma 24.2(5), Lemma 24.1, and Lemma 24.11(1), the map $F_a + E_{a,c}$ satisfies the bounds (24.15) on $\{B : a\|B\|_\infty \leq r_*/2\}$, with a constant depending also on C_0 and uniform in $c \in [0, C_0]$. Subtracting the equations for x_a and y_a from (24.31) leaves the source

$$\begin{aligned} & \mathcal{N}_a(x, y) + \mathcal{N}_a(y, x) + \mathcal{N}_a(y, y) \\ & + \mathcal{N}_a(x, r) + \mathcal{N}_a(r, x) + \mathcal{N}_a(y, r) + \mathcal{N}_a(r, y) + \mathcal{N}_a(r, r) + (F_a + E_{a,c})(x + y + r). \end{aligned}$$

The linear operator and the map $\mathcal{N}_a$ are the same as in Theorem 24.5. The five types of quadratic sources therefore have the exponents $-p, -p + \delta, -p + \delta, -p + 2\delta$, and $-p + 3\delta$, with the amplitudes KR^2, KR^2, KRS, KRS , and KS^2 , respectively, as in the table in Step 3 of that proof, with the arguments in the displayed order. The cubic source at $r = 0$ is at most $K_{R,C_0} v^{-p}$, and its Lipschitz constant in r is at most $K_{R,S,C_0} v^{-p+2\delta} \|r - r'\|_{R_a}$. For the latter, the possible term $S^2 v^{3\kappa}$ is bounded by this power, since $3\kappa - (-p + 2\delta) = 4\delta > 0$ and $v \leq 1$. Steps 3–7 of the proof of Theorem 24.5 now apply with constants depending also on C_0 : the heat integral of the source at $r = 0$ has norm at most K_{R,C_0} , and on a ball of fixed radius S the contraction constant is at most $K_{R,S,C_0} (2T)^\delta$. We choose S larger than twice the first bound, then $T \leq 1/2$ such that the second is at most $1/2$, and then $a_{R,C_0} \leq \min(1, \sqrt{T})$ such that

$$Ra^{1-\alpha} + Ra^{2-2\alpha} + Sa(2T)^\kappa < r_*/2.$$

Every candidate then stays in the ball where the estimates hold. The complete space of remainders on a fixed lattice and the Picard iteration give the fixed point. Subtracting the sources for two fields with distance d_a gives an additional source bounded by $K_{R,S,C_0}d_av^{-p}$, and the contraction yields the Lipschitz bound for the same c . Uniqueness in the class of Theorem 24.5, outside the constructed ball, also uses local Lipschitz continuity of the finite-dimensional vector field on the full chart and follows on successive compact subintervals as in Step 5 of that proof. After reducing a_{R,C_0} as in Step 7, the solution stays in the chart, and Lemma 24.11 identifies it with $\Psi^{(n,c)}$ in its damping gauge. The construction applies on every shorter time interval, so T_{R,C_0} may be decreased further below.

Step 2: bounds on the solutions and on the added force. For $v \leq 1$ the constructed solutions satisfy

$$\|A_a^c(t)\|_\infty \leq Kv^{-\alpha/2}, \quad \|D_a A_a^c(t)\|_\infty \leq Kv^{-(\alpha+1)/2}.$$

Indeed, the value bounds of x , y , and r^c are $Rv^{-\alpha/2}$, $Rv^{-\beta}$, and Sv^κ , and $-\beta > -\alpha/2$, $\kappa > 0$. The difference bounds are $Rv^{-(\alpha+1)/2}$, $Rv^{-\alpha}$, and $Sv^{\kappa-1/2}$; here $\alpha < (\alpha+1)/2$ since $\alpha < 1$, and $\kappa - 1/2 + (\alpha+1)/2 = 1 - \alpha > 0$. By (24.34) and $v^{-\alpha} \leq v^{-(\alpha+1)/2}$, we obtain

$$\|E_{a,c}(A_a^c(t))\|_\infty \leq Ka^3v^{-p-3/2} \leq Ka^\varepsilon v^{-p-\varepsilon/2},$$

where in the last inequality we used $a/\sqrt{v} \leq 1$ and $\varepsilon < 2 - 3\alpha < 3$. Let $p_\varepsilon = p + \varepsilon/2$. Then $p_\varepsilon < 1$ exactly because $\varepsilon < 2 - 3\alpha$, and $\kappa_\varepsilon = 1 - p_\varepsilon$.

Step 3: the comparison. For $0 < \kappa' < 1$ define $\|\cdot\|_{\mathbb{R}_a^{\kappa'}}$ as in the statement with κ' in place of κ_ε ; thus $\mathbb{R}_a^{\kappa} = \mathbb{R}_a$. By Lemma 24.4, for $0 < q < 1$,

$$\|f(t)\|_\infty \leq H(t+a^2)^{-q} \implies \left\| \int_0^\cdot P_{-u}^a f(u) du \right\|_{\mathbb{R}_a^{1-q}} \leq K_{L,q} H, \quad (24.37)$$

including $t = 0$, where $K_{L,q} = 2C_{q,L}$ is bounded uniformly for q in every interval $(0, 1 - \delta']$, $\delta' > 0$.

The flow data cancel in $w = A_a^c - A_a^0 = r_a^c - r_a^0$, and $w(0) = 0$. Subtracting the two equations and treating $E_{a,c}(A_a^c)$ as an inhomogeneous source, we write the remaining source for this difference w as $\mathcal{L}(w)$, so that

$$\partial_t w = \Delta_a w + \mathcal{L}(w) + E_{a,c}(A_a^c), \quad \mathcal{L}(w) = \mathcal{N}_a(w, A_a^c) + \mathcal{N}_a(A_a^0, w) + F_a(A_a^c) - F_a(A_a^0).$$

Expanding $A_a^c = x + y + r^c$ and $A_a^0 = x + y + r^0$, the terms of $\mathcal{L}(w)$ are $\mathcal{N}_a(w, x)$, $\mathcal{N}_a(x, w)$, $\mathcal{N}_a(w, y)$, $\mathcal{N}_a(y, w)$, $\mathcal{N}_a(w, r^c)$, $\mathcal{N}_a(r^0, w)$, and the difference of F_a .

Let $W = \|w\|_{\mathbb{R}_a^{\kappa_\varepsilon}}$, so that $\|w(t)\|_\infty \leq Wv^{\kappa_\varepsilon}$ and $\|D_a w(t)\|_\infty \leq Wv^{\kappa_\varepsilon-1/2}$. The quadratic terms containing x and w have exponent $-p_\varepsilon + \delta$: for instance $-\alpha/2 + \kappa_\varepsilon - 1/2 = -p_\varepsilon + \delta$

and $\kappa_\varepsilon - (\alpha + 1)/2 = -p_\varepsilon + \delta$. Those containing y and w have exponent $-p_\varepsilon + 2\delta$, and those containing r^c and w , or w and r^0 , have exponent $-p_\varepsilon + 3\delta$. By (24.15) and the value bound of Step 2 for both solutions, the difference of F_a is at most $Kv^{-\alpha}Wv^{\kappa_\varepsilon}$, with exponent $-p_\varepsilon + 2\delta$. Since $v \leq 1$, we obtain $\|\mathcal{L}(w)(t)\|_\infty \leq K_{R,S,C_0}Wv^{-(p_\varepsilon-\delta)}$.

We apply (24.37) with $q = p_\varepsilon - \delta$; the output is bounded in $R_a^{\kappa_\varepsilon+\delta}$, and multiplying by $(2T)^\delta$ bounds it in $R_a^{\kappa_\varepsilon}$, since $v^{-\kappa_\varepsilon} = v^\delta v^{-\kappa_\varepsilon-\delta}$. The constants $K_{L,q}$ are bounded uniformly, because $q = p_\varepsilon - \delta \in [p - \delta, 1 - \delta)$ and $p - \delta = 2\alpha - 1/2 > 1/2$. Hence, decreasing T_{R,C_0} once, independently of ε , we may assume that the heat integral of $\mathcal{L}(w)$ has $R_a^{\kappa_\varepsilon}$ norm at most $W/2$. We then reduce the mesh threshold so that $a_{R,C_0} \leq \sqrt{T_{R,C_0}}$ for this final time. This preserves $v \leq 2T_{R,C_0}$ and the hypothesis of Lemma 24.4; both reductions depend only on R, C_0, L , and α .

For the inhomogeneous term, (24.37) with $q = p_\varepsilon < 1$ and Step 2 give the bound $K_\varepsilon a^\varepsilon$ in $R_a^{\kappa_\varepsilon}$. On a fixed lattice W is finite, since w is continuous, $w(0) = 0$, and $v \geq a^2$; so the inequality $W \leq W/2 + K_\varepsilon a^\varepsilon$ implies $W \leq 2K_\varepsilon a^\varepsilon$, which is (24.36). At a fixed $t_0 > 0$, this implies $\|w(t_0)\|_{C_a^1} \leq 2K_\varepsilon a^\varepsilon(1 + t_0^{\min(\kappa_\varepsilon-1/2, 0)})$, which tends to zero uniformly in c .

Step 4: the damping gauge transformations. The gauge velocity is the same for all c , so by (24.8) both damping generators are the same linear combination of differences of the respective solutions. Their difference is bounded by $3\|D_a w(t)\|_\infty \leq Ka^\varepsilon v^{\kappa_\varepsilon-1/2}$. Since $\kappa_\varepsilon - 1/2 > -1/2$ and $T_{R,C_0} \leq 1/2$, the integral of $v^{\kappa_\varepsilon-1/2}$ over $[0, T_{R,C_0}]$ is bounded uniformly in a , so the integral of the difference of the generators is $O(a^\varepsilon)$. For skew-Hermitian generators, the variation-of-constants argument in the proof of Lemma 24.7(2) bounds the difference of the two gauge transformations, started at the identity, by this integral. Together with Step 1, this proves the statements on the gauges and the identification of the flows. The choices of S, T_{R,C_0} , and a_{R,C_0} were made in this order and have the stated dependence. $\square$

Consistency and continuation of the Wilson flow

This section completes the deterministic theory of the Wilson flow used in the rest of the paper. It turns bounds on the flow data of a lattice initial potential, which the stochastic estimates of Part VIII provide, into statements about flowed Wilson loops at fixed positive times. Section 24 solved the damped Wilson equation on a short time interval for initial potentials with bounded flow data (Theorem 24.5). For a compact set of smooth initial data, it also constructed a continuation of the Yang–Mills heat flow using a finite reference family: finitely many smooth DeTurck solutions on consecutive time intervals, joined by fixed gauge transformations (Theorem 24.10). At a fixed mesh the Wilson flow exists for all times, since it is the gradient flow of a smooth function on a compact manifold. The problem is to obtain estimates in physical units that are uniform as $a \rightarrow 0$, together with logarithmic representatives of the flowed configuration in which these estimates hold.

We prove that every initial potential with bounded flow data can be continued to any fixed time interval $[\tau, T]$ with $\tau > 0$, uniformly in the mesh. We also prove that the based holonomy matrices and the characters of flowed Wilson loops are Lipschitz functions of the flow data (Lemma 25.9, Theorem 25.10, , and Corollary 25.11). For initial potentials depending on finitely many real parameters, we bound two parameter derivatives of the based holonomy matrices of the coordinate cycles uniformly in the mesh. For fixed smooth families of initial data, we identify the limits of these matrices and their derivatives (Theorem 25.13). Finally, we show that the flows of the plaquette potentials $q + cq^2$, $0 \leq c \leq C_0$, give asymptotically the same flowed loops as the Wilson flow (Lemma 25.15). The main tools are the consistency of the lattice vector field on smooth fields (Lemma 25.3 and Theorem 25.4), stability in C_a^1 near smooth solutions (Lemma 25.5), and smoothing after a positive time together with compactness (Theorem 25.7 and Lemma 25.6).

The section contains three kinds of comparison, with different kinds of error.

Comparison	Hypothesis	Conclusion
Wilson flow from the exponentials of a sampled smooth solution, versus that solution	initial link-logarithm error $O(a^2)$	link-logarithm error $O(a^2)$; error $O(a)$ for loops of bounded length
Lattice solution versus a selected smooth DeTurck solution	initial error η in C_a^1	error $C(\eta + a)$ in C_a^1
Two lattice solutions on the same lattice	difference d_a of the flow data	difference Cd_a in C_a^1 after a waiting time, and in the based holonomy matrices

Only the last comparison has no additive error in the mesh; this makes the flowed loops Lipschitz functions of the flow data. For matrices, as opposed to characters, we must track every gauge transformation applied at the basepoint.

Two observations organize the proofs. First, we expand the lattice vector field completely before taking the continuum limit. The contributions of the two plaquettes on either side of a link cancel at order a^2 , and their common part also cancels at order a^3 ; what remains at order a^3 is the DeTurck vector field. Second, after smoothing for a positive time, all solutions whose flow data obey a fixed bound are uniformly close to a compact set of smooth connections. Finitely many selected smooth DeTurck solutions then give continuation and stability uniformly over these solutions, without assuming that the initial potentials converge as the mesh varies. We work both in dimensionless units, with link logarithms $\theta = aA$ and time $s = t/a^2$, where the lattice structure is simplest, and in physical units. Since $\partial_s \theta = a^3 \partial_t A$, a dimensionless error $O(a^4)$ is a physical error $O(a)$, which explains the factors a^{-3} below.

We use the local formulas for the Wilson vector field, the local solution, the sampled gauge transformations, and the continuation of the smooth flow from Lemmas 24.2 and 24.7, Definition 24.3, , and Theorems 24.5 and 24.10; the estimates for the modified plaquette potential in Lemma 24.11 and Theorem 24.12; and the heat estimates Lemmas 16.3 and 16.9. All existence statements for the continuum flow used here are contained in Theorem 24.10, which rests on the deterministic smooth-data results of Cao and Chatterjee [42, 43]; no result about random initial data is used. The Wilson flow was introduced by Lüscher [128]. The damping gauge transformation that turns it into a parabolic equation is in the spirit of the method of Zwanziger [157], DeTurck [66], and Donaldson [76], and for the Wilson flow it is described in [128, Section 2]. The arguments are standard parabolic ones (weakly singular Volterra estimates, a bootstrap for higher differences, and compactness by the Arzelà–Ascoli theorem), carried out on the exact lattice equation with constants independent of the mesh. The results are used in Sec-

tions 28 and 29 for the convergence of flowed loops and energies and for the construction of the limiting measure, in Section 32 for macroscopic loops in the short-distance analysis, and in Sections 34 to 36 for holonomy matrices and their parameter derivatives at weak coupling.

25.1 Conventions and heat estimates

Throughout this section $a = L/n$, with $n \geq 4$, is a lattice mesh, and $\Lambda_{L/a}$ is the corresponding periodic lattice. For a continuous field f on $\mathbb{T}_L^3$ we write $S_a f = f|_{\Lambda_{L/a}}$ for its sample on this lattice. A link field A has components $A_i(x) \in \mathfrak{su}(2)$ on the positively oriented links $(x, x + ae_i)$. We locate each component at the starting vertex x of its link, and we write T_i for translation by ae_i . We write D_a for any physical forward or backward difference $D_{a,j}^\pm$, and D_a^ν for an ordered composition of such differences. In a norm, D_a^m stands for the maximum over all ordered compositions of order m and over all link and color components. For a composition ν we write ∂^ν for the corresponding composition of partial derivatives. As in Theorem 24.5, we use the norm

$$\|A\|_{C_a^1} = \|A\|_\infty + \max_{j \in \{+, -\}} \|D_{a,j}^\epsilon A\|_\infty.$$

Translations do not change these norms, since they permute the sites of the periodic lattice. In dimensionless coordinates we write $\nabla_i = T_i - I$ and $H_0 = \sum_i \nabla_i^* \nabla_i$, so that $\Delta_a = -a^{-2}H_0$, and $S_s^a = e^{-sH_0} = P_{a^2s}^a$ for the heat semigroup in dimensionless time. This semigroup differs from the sampling S_a .

Flow data. As in Definition 24.3, the flow data of a lattice initial potential A_0 on mesh a are the linear heat evolution and the quadratic Picard term

$$x_a(t) = P_t^a A_0, \quad y_a(t) = \mathcal{Q}_a(x_a, x_a)(t) = \int_0^t P_{t-r}^a \mathcal{N}_a(x_a(r), x_a(r)) dr.$$

Here $\mathcal{N}_a$ is the ordered bilinear map fixed in Lemma 24.2; it is not symmetric, and we always take its arguments in the displayed order. On a time interval $[0, T_0]$ we write

$$\|(x_a, y_a)\|_{\mathfrak{f},a} = X_\alpha^a(x_a) + Y_\alpha^a(y_a),$$

with the norms of Definition 4.8 taken on $[0, T_0]$. These norms contain the weighted first differences, with all translated components. Differences of flow data on the same lattice are measured by the same norms applied to the difference. Throughout, $1/2 < \alpha < 2/3$, and we write

$$\beta = \alpha - \frac{1}{2}, \quad \kappa = 1 - \frac{3}{2}\alpha, \quad v_a(t) = t + a^2.$$

The remainder norm of Theorem 24.5 is

$$\|r\|_{R_a} = \sup_{0 \leq t \leq T_R} \{v_a(t)^{-\kappa} \|r(t)\|_\infty + v_a(t)^{1/2-\kappa} \|D_a r(t)\|_\infty\}.$$

The chart and damped representatives. Let r_* be the radius of the link-logarithm chart in Lemma 24.2. A potential A on mesh a lies in the chart if $a\|A\|_\infty < r_*$. All local analytic bounds below are used on a fixed smaller chart $\{a\|A\|_\infty \leq r'\}$ with $r' < r_*$, and the corresponding constants may depend on $r_* - r'$. In the chart, the damped Wilson equation in physical units reads

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + a^{-3} R_3(aA),$$

with the analytic remainder R_3 of Lemma 24.2. By that lemma, a solution in the chart is related to the Wilson flow by

$$e^{aA_i(x,t)} = k(x,t) (\Psi_t^{(n)} U)_i(x) k(x + ae_i, t)^{-1}, \quad U = e^{aA(0)},$$

where the damping gauge transformation k solves $\partial_t k k^{-1} = a^{-2} \phi(aA(t))$ with $k(0) = I$, and ϕ is the damping parameter of Lemma 24.2. As in Section 24, we call A the damped representative of the Wilson flow. The Wilson flow is used at physical time t , that is, at time $4t/a^2$ for the vector field $-\nabla \sum_p q(U_p)$; there is no coupling factor in this conversion. When the flow is restarted at a time t_0 from a gauge-transformed configuration, the new damping gauge transformation has value I at t_0 .

Constants. A constant that comes only from the finite local formulas (the finitely many terms of $\mathcal{N}_a$, the analytic remainder R_3 , and their derivatives) depends only on the chart radius r' and on the order of differentiation, as in item (1) of Definition 4.14. Other constants may in addition depend on L , on the displayed bounds, waiting times, and horizons, and on finitely many smooth DeTurck solutions selected in the proofs; this is the dependence described in item (7) of Definition 4.14. No constant in this section depends on the coupling, on a probability measure, on the number of averaging steps, or on M and J . When applying the local results of Section 24, we may decrease the chart radius and the local existence time, and increase the minimal lattice side; we do this once, before choosing the constants below. Every restriction $a \leq a_*$ on the mesh is chosen after the displayed data and before the particular initial potentials to which it is applied.

Lemma 25.1. *Fix $L, H_1 > 0$ and an integer $q \geq 0$. The lattice heat semigroup P_u^a is positivity preserving, preserves constants, is a contraction for the supremum norm, and commutes with every translation and every difference. Further,*

$$\|D_a^q P_u^a f\|_\infty \leq C_{q,H_1/L^2} (u + a^2)^{-q/2} \|f\|_\infty, \quad 0 \leq u \leq H_1.$$

In particular, for $a \leq 1$ and $0 \leq s \leq H_1/a^2$,

$$\|S_s^a\|_{\infty \rightarrow \infty} \leq 1, \quad \|\nabla S_s^a\|_{\infty \rightarrow \infty} \leq C_{H_1/L^2} (1 + s)^{-1/2}.$$

For $p < 1$, $0 \leq q < 1$, and $\epsilon > 0$,

$$\int_0^t (r + \epsilon)^{-p} (t - r + \epsilon)^{-q} dr \leq C_{p,q} (t + \epsilon)^{1-p-q}, \quad (25.1)$$

and the same bound holds for $\epsilon = 0$ and $t > 0$.

Proof. We have $\Delta_a = a^{-2}(\sum_i (T_i + T_i^{-1}) - 6I)$. Factoring out e^{-6u/a^2} and expanding the exponential of the remaining sum of translations shows that P_u^a is positivity preserving and preserves constants; in particular it is a contraction for the supremum norm. It commutes with translations and differences, since all translations commute. For the difference bound we apply Lemma 16.3 with equal input and output meshes, physical horizon $T = H_1$, and refinement ratio one. The cardinal interpolation is then the identity, and the heat estimate of that lemma is the displayed bound for every lattice array. (Where the lemma is used below for fixed smooth interpolants, the refinement ratio is the integer H/a , and the horizon is the displayed finite T .) Multiplying by a and setting $u = a^2 s$, we obtain the dimensionless bounds.

We now prove (25.1). Suppose first that $p < 0$. Then $(r + \epsilon)^{-p} \leq (t + \epsilon)^{-p}$, and the integral of the remaining factor is at most $(t + \epsilon)^{1-q}/(1 - q)$, which gives the bound. Suppose now that $p \geq 0$. If $t \geq \epsilon$, we bound the integrand by $r^{-p}(t - r)^{-q}$ and substitute $r = tv$. The integral of $v^{-p}(1 - v)^{-q}$ over $[0, 1]$ is finite, as we see by bounding $(1 - v)^{-q}$ by 2^q on $[0, 1/2]$ and v^{-p} by 2^p on $[1/2, 1]$. Since $t \leq t + \epsilon \leq 2t$, we obtain the bound, and the same argument proves the case $\epsilon = 0$. If $t < \epsilon$, the integral is at most $t\epsilon^{-p-q} \leq \epsilon^{1-p-q}$, which is at most a constant times $(t + \epsilon)^{1-p-q}$ because $\epsilon \leq t + \epsilon \leq 2\epsilon$. These arguments apply for either sign of $1 - p - q$. $\square$

We use the ordered expansion of $\mathcal{N}_a$ fixed in Lemma 24.2: a fixed finite sum of brackets of translated components, with one physical difference in the second argument of each bracket. Differences of brackets obey the exact product rules

$$D_{a,k}^+[u, v] = [D_{a,k}^+ u, T_k v] + [u, D_{a,k}^+ v], \quad D_{a,k}^-[u, v] = [D_{a,k}^- u, T_k^{-1} v] + [u, D_{a,k}^- v]. \quad (25.2)$$

Repeated application distributes a fixed number of differences among the factors and produces finitely many terms with bounded translations. Whenever we take differences of a discrete nonlinearity, we use these identities and never differentiate an interpolated product.

25.2 Consistency for fixed smooth interpolants

The first consistency statement concerns the flow data of a fixed finite family of smooth cardinal interpolants. Such families arise in Section 28 from finite-scale reconstructions, the fields reconstructed from finitely many RG steps. The point is that the constants do not depend on the finer mesh on which the flow is computed.

Lemma 25.2. Fix $L, T, H > 0$ with $L/H \in \mathbb{N}$, and $M_H < \infty$. Let $f = I_H v$ and $g = I_H w$ be the smooth cardinal interpolants, as in Lemma 16.3, of link fields v, w on the lattice of mesh H with $\|v\|_\infty + \|w\|_\infty \leq M_H$. For every mesh $a = H/m$ with $m \in \mathbb{N}$, set

$$x_a = P_a^{\cdot}(S_a f), \quad z_a = P_a^{\cdot}(S_a g), \quad x = P.f, \quad z = P.g.$$

Let $\mathcal{N}_0$ be the continuum bilinear map obtained from the fixed expansion of $\mathcal{N}_a$ by replacing every translation by the identity and every physical difference by the corresponding partial derivative. Then

$$(\mathcal{N}_0(b, b))_i = \sum_j \{2[b_j, \partial_j b_i] - [b_j, \partial_i b_j]\}.$$

Uniformly over this family of inputs and over $0 \leq t \leq T$, the values and all first differences of $\mathcal{N}_a(x_a, z_a)$ differ from the samples of $\mathcal{N}_0(x, z)$ and of its corresponding first derivatives by at most $C_{H, M_H, L, T} a$, and the same holds for $\mathcal{N}_a(z_a, x_a)$ and $\mathcal{N}_0(z, x)$. The same estimate holds for

$$\mathcal{Q}_a(x_a, z_a)(t) - S_a \int_0^t P_{t-s} \mathcal{N}_0(x(s), z(s)) ds$$

and its first differences, and for the reversed order of the arguments. Consequently, for every bounded finite sum f of such interpolants, the flow data of $S_a f$ converge, after separate interpolation of value and difference arrays, to the continuum pair $(x, \int_0^t P_{t-s} \mathcal{N}_0(x(s), x(s)) ds)$ in the four weighted uniform norms of Definition 24.3, whose weights are powers of t , and $\|(x_a, y_a)\|_{\mathfrak{H}, a}$ is bounded uniformly in a . The constants are deterministic, do not depend on a , and may grow arbitrarily as $H \downarrow 0$. The lemma compares two discretizations of the same smooth input on the mesh a ; it does not compare the quadratic term $\mathcal{Q}_H$ on the mesh H with $\mathcal{Q}_a$.

The diagonal formula for $\mathcal{N}_0$ is the quadratic part of the DeTurck form of the Yang–Mills heat flow, written in Theorem 24.10; compare the gauge-fixed flow of Lüscher [128, Section 2.2]. We verify it from the lattice formula, since its sign and the factor two must match the lattice map exactly.

Proof. Step 1: bounds on the inputs. For each integer $q \geq 0$, by the bounds for cardinal interpolants in Lemma 16.3, we have

$$\|f\|_{C^q} + \|g\|_{C^q} \leq C_q(1 + H^{-q})M_H.$$

By heat contraction on differentiated smooth data, and its discrete counterpart, every fixed spatial difference of the input fields is bounded uniformly in a and $t \leq T$. Applying Lemma 16.9 through order two, and not only through order one, we obtain

$$\max_{0 \leq q \leq 2} \sup_{t \leq T} (\|D_a^q x_a - S_a \partial^q x\|_\infty + \|D_a^q z_a - S_a \partial^q z\|_\infty) \leq C_{H, M_H, L, T} a. \quad (25.3)$$

The maximum includes every ordered choice of differences; for $q = 0$ the stronger error $O(a^2)$ is available. A translation by any vector of the fixed stencil changes a sampled smooth q th derivative by at most $Ca\|\partial^{q+1}x\|_\infty$, and the same holds for z .

Step 2: the diagonal of $\mathcal{N}_0$. Recall that $\mathcal{N}_0$ is defined by the termwise replacement in the statement. To identify its diagonal, let $(Cb)_{ij} = \partial_i b_j - \partial_j b_i$ and $Q(b)_{ij} = [b_i, b_j]$, extended antisymmetrically in i, j . At zero translation the logarithms of the four factors of the plaquette in Lemma 24.2 are $b_i, b_j, -b_i, -b_j$, and half the sum of their pairwise brackets is $Q(b)_{ij}$. By the adjoint formula in Lemma 24.2, we have

$$(C^*Q(b))_i = \sum_j \partial_j [b_i, b_j], \quad (DQ(b)^*Cb)_i = \sum_j [b_j, \partial_i b_j - \partial_j b_i].$$

For the second identity, note that a variation h_i of b_i changes $[b_i, b_j]$ by $[h_i, b_j]$, and invariance of the inner product moves this bracket to $[b_j, (Cb)_{ij}]$. Combining the two orientations of each plaquette, we obtain the displayed formula. Further, since $d_0^*(ab)/a^2$ tends to $\sum_j \partial_j b_j$, the quadratic gauge term G_2 of Lemma 24.2, divided by a^3 , tends to $[b_i, \sum_j \partial_j b_j]$. The diagonal coefficient is therefore

$$-C^*Q(b) - DQ(b)^*Cb + [b, \operatorname{div} b] = \left(\sum_j \{2[b_j, \partial_j b_i] - [b_j, \partial_i b_j]\} \right)_i,$$

where we used $[\partial_j b_i, b_j] = -[b_j, \partial_j b_i]$, and the terms $[b_i, \partial_j b_j]$ cancel. This verifies the sign and the factor two without symmetrizing the ordered map.

Step 3: the nonlinearity. Every summand of $\mathcal{N}_a(x_a, z_a)$ is a bracket of a factor of order zero and a factor of order one. Applying (25.2) once, we obtain factors of orders $(1, 1)$ or $(0, 2)$, with the corresponding translations. By (25.3) and Step 1, each factor differs from its continuum counterpart by $O(a)$, and every other factor is bounded by a constant depending on the family. Using the inequality $||[u, v]|| \leq |u||v|$ and subtracting the two products, we obtain the asserted $O(a)$ errors for the values and the first differences. Reversing the inputs replaces the factors by z_a, x_a in the same list; their bounds through order two and their errors are again those in (25.3), which gives the estimate for the reversed order.

Step 4: the Duhamel integrals. Let $h(t) = \mathcal{N}_0(x(t), z(t))$, a uniformly bounded smooth family at every fixed spatial order. In the value of the Duhamel integral we first replace $\mathcal{N}_a(x_a, z_a)$ by $S_a h$; by contraction this costs at most CTa . Then we replace $P_{t-r}^a S_a h(r)$ by $S_a P_{t-r} h(r)$. By the general smooth-input bound of Lemma 16.9, applied separately at each r , this costs at most $CT^2 a^2 \sup_r \|h(r)\|_{C^4}$. For a first difference of the integral, we commute the difference through P_{t-r}^a onto the source. The replacement of the source difference by $S_a \partial h$ costs CTa , and the heat comparison on $\partial h(r)$ together with the comparison of $D_a S_a$ with $S_a \partial$ cost at most $C_{H, M_H, L, T} a$ after integration. This order of comparisons avoids dividing a value error by a . Replacing h by $\mathcal{N}_0(z, x)$ gives the assertion for the reversed order.

Step 5: weighted norms and convergence. We first bound the flow data near time zero, without using any convergence. Let $B = \|f\|_\infty + \|g\|_\infty$. By Lemma 25.1 and the bound $\|\mathcal{N}_a(u, v)\|_\infty \leq C\|u\|_\infty\|D_a v\|_\infty$, we have

$$\begin{aligned} \|x_a(t)\|_\infty + \|z_a(t)\|_\infty &\leq B, & \|D_a x_a(t)\|_\infty + \|D_a z_a(t)\|_\infty &\leq CB(t + a^2)^{-1/2}, \\ \|\mathcal{Q}_a(x_a, z_a)(t)\|_\infty &\leq CB^2\sqrt{t}, & \|D_a \mathcal{Q}_a(x_a, z_a)(t)\|_\infty &\leq CB^2. \end{aligned}$$

In the last inequality we used $\int_0^t (t-r+a^2)^{-1/2}(r+a^2)^{-1/2} dr \leq \pi$. For a fixed family one can instead bound all input and source differences by its smooth bounds, which also covers $t = 0$. Since $a \leq H$ and all four weights $v_a^{\alpha/2}$, $v_a^{(\alpha+1)/2}$, v_a^β , v_a^α have positive exponents, multiplying by them we obtain a bound on $\|(x_a, y_a)\|_{\mathfrak{H}, a}$ depending only on the data of the family. Interpolation of each array separately is a contraction, since the cardinal weights are nonnegative and sum to one. Because of its support radius, it has error at most $Ca\|\partial F\|_\infty$ for a smooth F . Then the uniform errors already proved imply convergence in all four weighted uniform norms. For the quadratic Picard term we take $g = f$, and for a finite sum of interpolants we use the corresponding bounded smooth family. All constants have the asserted dependence, and $\mathcal{Q}_H$ is never compared with $\mathcal{Q}_a$. $\square$

25.3 Consistency and stability for smooth initial data

We now expand the exact lattice vector field on sampled smooth potentials. This vector field consists of the Riemannian gradient of the plaquette action for the bi-invariant product metric and the damping term, written in logarithmic coordinates by means of the inverse differential of the exponential. All three parts enter the term of order a^3 .

Lemma 25.3. *Let $\mathcal{F}$ be the dimensionless damped logarithmic vector field of Lemma 24.2, so that the damped Wilson equation reads*

$$\partial_s \theta = \mathcal{F}(\theta) = -H_0 \theta + N_2(\theta) + R_3(\theta)$$

in dimensionless time $s = t/a^2$. Let $K_0 > 0$, and let b be a smooth periodic connection with $\|b\|_{C^3} \leq K_0$. If a is so small that $K_0 a$ lies in a fixed smaller chart, then

$$a^{-3} \mathcal{F}(aS_a b) = S_a \{ \Delta b + \mathcal{N}_0(b, b) + \mathcal{N}_3(b) \} + O_{K_0}(a), \quad (\mathcal{N}_3(b))_i = \sum_j [b_j, [b_j, b_i]],$$

and

$$a^{-3} R_3(aS_a b) = S_a \mathcal{N}_3(b) + O(a).$$

More generally, for every integer $q \geq 0$ the errors in both identities remain $O(a)$ after any q physical differences, compared with the samples of the corresponding derivatives, uniformly over families of b bounded in C^{q+4} . The constants depend only on q , on the smooth bounds, and on the chart.

Proof. Step 1: the exact vector field. For $i \neq j$, let

$$P_{ij}(x) = U_i(x)U_j(x + ae_i)U_i(x + ae_j)^{-1}U_j(x)^{-1}, \quad p(V)^c = -2\operatorname{tr}(\tau_c V),$$

and identify the color vector $p(V)$ with its Lie-algebra element. For $V = e^f$ in the plaquette chart, $p(e^f) = 2\sin(|f|/2)f/|f| = f + O(|f|^3)$. When $j < i$ we have $P_{ij}(x) = P_{ji}(x)^{-1}$ and $q(V^{-1}) = q(V)$, so each unoriented plaquette is counted once. Varying $U_i(x)$ by $e^{\varepsilon w}U_i(x)$ in $4\sum_p q(U_p)$ gives $\langle w, p(P_{ij}(x)) \rangle$ from the plaquette $P_{ij}(x)$ and

$$-\langle w, \operatorname{Ad}(U_j(x - ae_j)^{-1}) p(P_{ij}(x - ae_j)) \rangle$$

from the plaquette $P_{ij}(x - ae_j)$; the second formula follows by moving the varied inverse factor cyclically inside the trace. The negative gradient for the bi-invariant metric, in right trivialization, is therefore

$$V_{W,i}(x) = \sum_{j \neq i} \{ \operatorname{Ad}(U_j(x - ae_j)^{-1}) p(P_{ij}(x - ae_j)) - p(P_{ij}(x)) \}. \quad (25.4)$$

This is an identity at every configuration. By Lemma 24.2, with $U_i(x) = e^{\theta_i(x)}$, the damping parameter, the damping velocity, and the logarithmic vector field are

$$\phi(x) = \sum_j (\theta_j(x - ae_j) - \theta_j(x)), \quad (25.5)$$

$$V_{g,i} = \phi(x) - \operatorname{Ad}(e^{\theta_i(x)})\phi(x + ae_i), \quad \mathcal{F}_i(\theta) = \mathcal{J}(\theta_i)^{-1}(V_{W,i} + V_{g,i}),$$

where $\mathcal{J}(\theta) = (e^{\operatorname{ad} \theta} - I) / \operatorname{ad} \theta$ is the differential of the exponential in right trivialization.

Step 2: expansion of one plaquette. Let $\theta = aS_a b$, that is, $U_i(x) = e^{ab_i(x)}$, and write

$$F_{ij} = \partial_i b_j - \partial_j b_i + [b_i, b_j], \quad v = \sum_j \partial_j b_j.$$

By Taylor expansion at x through order a^3 , the logarithms of the four factors of $P_{ij}(x)$ are

$$\begin{aligned} & ab_i, \quad ab_j + a^2 \partial_i b_j + \frac{1}{2} a^3 \partial_i^2 b_j + O_{K_0}(a^4), \\ & -ab_i - a^2 \partial_j b_i - \frac{1}{2} a^3 \partial_j^2 b_i + O_{K_0}(a^4), \quad -ab_j. \end{aligned}$$

Taking the finite analytic product and the logarithm, we obtain

$$p(P_{ij}(x)) = a^2 F_{ij}(x) + a^3 B_{ij}(x) + O_{K_0}(a^4), \quad \|B_{ij}\|_{C^1} \leq C_{K_0}. \quad (25.6)$$

Here B_{ij} is the homogeneous coefficient of a^3 in the product of the four factors, with their logarithms replaced by the displayed jets. It is a fixed polynomial in the two-jets of b , and its first derivative involves only three-jets. Taylor's integral remainder for each translated b and the analytic remainder of fourth order of the finite product are bounded by $C_{K_0} a^4$.

The coefficient of a^2 is F_{ij} : these logarithms add up to $a^2(\partial_i b_j - \partial_j b_i)$ modulo $O(a^3)$, and half the sum of their pairwise brackets is $a^2[b_i, b_j]$ modulo $O(a^3)$, as in Step 2 of the proof of Lemma 25.2. Replacing the logarithm by p changes the result only by $O_{K_0}(a^6)$. This proves (25.6) without computing the polynomial B_{ij} .

Step 3: cancellation between the two plaquettes. At the neighboring basepoint, $F_{ij}(x - ae_j) = F_{ij}(x) - a\partial_j F_{ij}(x) + O_{K_0}(a^2)$ and $B_{ij}(x - ae_j) = B_{ij}(x) + O_{K_0}(a)$, and the adjoint in (25.4) is $I - a \operatorname{ad}(b_j(x)) + O_{K_0}(a^2)$. Subtracting the contribution of $P_{ij}(x)$, we obtain

$$V_{W,i}(aS_a b) = -a^3 \sum_j (\partial_j F_{ij} + [b_j, F_{ij}]) + O_{K_0}(a^4).$$

The terms of order a^2 and the common terms $a^3 B_{ij}$ cancel before the inverse differential $\mathcal{J}^{-1}$ is applied. For the damping parameter we have

$$\phi = -a^2 v + a^3 w + O_{K_0}(a^4), \quad w = \frac{1}{2} \sum_j \partial_j^2 b_j, \quad \|w\|_{C^1} \leq C_{K_0}.$$

The translated terms $a^3 w$ cancel in V_g , and expanding the adjoint gives $V_{g,i} = a^3(\partial_i v + [b_i, v]) + O_{K_0}(a^4)$. Then the total group velocity is $O_{K_0}(a^3)$. Since $\mathcal{J}(ab_i)^{-1} = I + O_{K_0}(a)$, applying this inverse changes it only by $O_{K_0}(a^4)$. Finally,

$$- \sum_j (\partial_j F_{ij} + [b_j, F_{ij}]) + \partial_i v + [b_i, v] = \Delta b_i + \sum_j \{2[b_j, \partial_j b_i] - [b_j, \partial_i b_j] + [b_j, [b_j, b_i]]\},$$

and by Lemma 25.2 the right side is $(\Delta b + \mathcal{N}_0(b, b) + \mathcal{N}_3(b))_i$. This proves the first identity. The only condition on a is that $K_0 a$ lie in a fixed smaller chart, and all spatial remainders used at most three derivatives of b .

Step 4: the analytic remainder. By Lemma 24.2,

$$a^{-3} \mathcal{F}(aS_a b) = \Delta_a S_a b + \mathcal{N}_a(S_a b, S_a b) + a^{-3} R_3(aS_a b).$$

Central-difference consistency gives $\Delta_a S_a b = S_a \Delta b + O(a)$, and the calculation of Step 3 of the proof of Lemma 25.2, applied to the smooth field b , implies that $\mathcal{N}_a(S_a b, S_a b) = S_a \mathcal{N}_0(b, b) + O(a)$. Subtracting these from the first identity, we obtain the second.

Step 5: differences. For a first difference, we regard the finite sampled formula for $a^{-3} \mathcal{F}(aS_a b)$ as a smooth function $B_a(u)$ of a continuous basepoint $u \in \mathbb{T}_L^3$, evaluating every stencil entry at $u + a\tilde{\xi}$ instead of $x + a\tilde{\xi}$ for the corresponding stencil vector $\tilde{\xi}$. Differentiating the Taylor calculation of Steps 2 and 3 in u , we find that $\partial_k B_a$ differs from $\partial_k \{\Delta b + \mathcal{N}_0(b, b) + \mathcal{N}_3(b)\}$ by $O(a)$, since this calculation uses one additional bounded spatial derivative of b . Using the exact identity

$$D_{a,k}^+ B_a(x) = \int_0^1 \partial_k B_a(x + rae_k) dr$$

we compare the difference with the sampled continuum derivative, at error $O(a)$; for a backward difference we integrate over $x - rae_k$ instead. Repeating this argument at a fixed order q , we obtain the assertion through order q on families bounded in C^{q+4} . The same argument applies to $a^{-3}R_3(aS_ab)$. There is no negative power of a in the remainder: its term of degree d in the analytic series has coefficient a^{d-3} , $d \geq 3$, by Lemma 24.2. On a smaller chart, the derivatives of this series converge absolutely after multiplication by polynomial factors in d of fixed order. Hence the product-rule argument applies to the whole analytic remainder, not only to its cubic truncation. $\square$

Theorem 25.4. Fix $L, T, K_0, C_{\text{init}} > 0$. Let b be a smooth periodic solution on $[0, T]$ of the DeTurck equation of Theorem 24.10, continuously differentiable in time, with $\sup_{t \leq T} \|b(t)\|_{C^3} \leq K_0$. For $a = L/n$ write $s = t/a^2$, $S = T/a^2$, and $\theta^{\text{ref}}(s, x) = ab(a^2s, x)$. There is $a_* > 0$ such that the following statements hold for all $a \leq a_*$.

1. The residual of θ^{ref} for the exact vector field $\mathcal{F}$ of Lemma 24.2 satisfies

$$\sup_{s \leq S} \|\partial_s \theta^{\text{ref}}(s) - \mathcal{F}(\theta^{\text{ref}}(s))\|_{\infty} \leq C_{K_0} a^4.$$

Here $\mathcal{F}$ contains the Riemannian gradient of the plaquette action, the damping term, and the inverse differential of the exponential.

2. Let $\vartheta(s)$ be a path on $[0, S]$, not necessarily a solution, with $\|\vartheta\|_{\infty} \leq K_0 a$ and $\|\nabla \vartheta\|_{\infty} \leq K_0 a^2$. Its linearized lower-order operator $A_{\vartheta} = D\mathcal{F}(\vartheta) + H_0$ satisfies

$$\|A_{\vartheta} z\|_{\infty} \leq C_{K_0} (a \|\nabla z\|_{\infty} + a^2 \|z\|_{\infty}).$$

Define

$$\|z\|_{R_{\text{sm}}} = \sup_{s \leq S} \|z(s)\|_{\infty} + \sup_{s \leq S} \sqrt{1+s} \|\nabla z(s)\|_{\infty}$$

and

$$L_{\vartheta} z(s) = \int_0^s e^{-(s-r)H_0} A_{\vartheta}(r) z(r) dr.$$

Then $\|(I - L_{\vartheta})^{-1}\|_{R_{\text{sm}} \rightarrow R_{\text{sm}}} \leq C_{L,T,K_0}$, uniformly in a . A forcing F enters through its whole Volterra integral, and if $\sup_s \|F(s)\|_{\infty} \leq c$, then

$$\left\| \int_0^{\cdot} e^{-(\cdot-r)H_0} F(r) dr \right\|_{R_{\text{sm}}} \leq C_{L,T} c (1 + S).$$

3. If $\|\theta(0) - \theta^{\text{ref}}(0)\|_{\infty} \leq C_{\text{init}} a^2$, then the damped lattice solution θ with this initial value exists in the chart on $[0, S]$, is a damped representative of the Wilson flow, and satisfies

$$\|\theta - \theta^{\text{ref}}\|_{R_{\text{sm}}} \leq C_{L,T,K_0,C_{\text{init}}} a^2.$$

4. Let b_0 be smooth, and let the partition $0 = t_0 < \dots < t_Q = T$, the DeTurck solutions b_i on $[0, t_{i+1} - t_i]$, and the fixed smooth gauge transformations k_i be those given by Theorem 24.10 for the initial value b_0 . Start the Wilson flow from $\exp(aS_a b_0)$, and at each time t_{i+1} apply the gauge transformation k_i , sampled at the lattice sites, to the links before restarting the damped equation. Let $\theta_{a,i}(s)$ be the link logarithms of the damped representative restarted on the i th interval, with lattice time $s = (t - t_i)/a^2$. Then, uniformly for $t \in [t_i, t_{i+1}]$, $\|\theta_{a,i}((t - t_i)/a^2) - aS_a b_i(t - t_i)\|_\infty = O(a^2)$, and hence the links of this representative differ from $\exp(aS_a b_i(t - t_i))$ by $O(a^2)$.

The constants and a_* depend only on $L, T, K_0, C_{\text{init}}$, and in (4) also on the smooth bounds and margins of the continuation. The hypothesis of (3) is an initial error of order a^2 ; an initial link error that is only $o(a)$ is not covered.

The proof follows the usual pattern for semilinear parabolic equations: a consistency estimate for the residual, a bound on the linearized Volterra operator in a weighted norm, and a contraction argument for the error. The only delicate point is the scaling. In dimensionless units the coefficient of the first difference in A_ϑ is $O(a)$, the time interval has length T/a^2 , and the weight $e^{-\lambda s}$ must be taken with λ of order a^2 .

Proof. Step 1: the residual. By Lemma 25.3,

$$\mathcal{F}(\theta^{\text{ref}}(s)) = a^3 S_a \{ \Delta b + \mathcal{N}_0(b, b) + \mathcal{N}_3(b) \} (a^2 s) + O_{K_0}(a^4).$$

By the DeTurck equation the bracket is $\partial_t b$, and $\partial_s \theta^{\text{ref}} = a^3 S_a \partial_t b$. This proves (1). The threshold only requires that $K_0 a$ lie in a fixed smaller chart.

Step 2: the linearized operator. Let ϑ satisfy the two bounds in (2). We have $A_\vartheta = DN_2(\vartheta) + DR_3(\vartheta)$. By Lemma 24.2, the quadratic part N_2 is a fixed sum of translated products of the form $[u, \nabla v]$. In $DN_2(\vartheta)z$, the terms with the difference on z are bounded by $CK_0 a \|\nabla z\|_\infty$, and the terms with the difference on ϑ by $CK_0 a^2 \|z\|_\infty$. The derivative of the remainder satisfies $\|DR_3(\vartheta)z\|_\infty \leq C \|\vartheta\|_\infty^2 \|z\|_\infty$. This proves the bound for A_ϑ .

Step 3: the Volterra inverse. Let $k(u) = (1 + u)^{-1/2}$, and for $\lambda > 0$ consider the equivalent weighted norm

$$\|z\|_\lambda = \sup_{s \leq S} e^{-\lambda s} \|z(s)\|_\infty + \sup_{s \leq S} e^{-\lambda s} \sqrt{1 + s} \|\nabla z(s)\|_\infty.$$

For every $\lambda > 0$,

$$\int_0^\infty e^{-\lambda u} k(u) du \leq C \lambda^{-1/2}, \quad \sup_s \int_0^s e^{-\lambda(s-r)} k(r) dr \leq C \lambda^{-1/2}. \quad (25.7)$$

The first bound follows from $k(u) \leq u^{-1/2}$ and rescaling. For the second, if $s \leq \lambda^{-1}$ we discard the exponential and integrate $r^{-1/2}$. If $s > \lambda^{-1}$, the intervals $r \leq s/2$ and $r \geq s/2$

contribute at most $C\sqrt{s}e^{-\lambda s/2}$ and $C/(\lambda\sqrt{s})$ respectively, and each of these is at most $C\lambda^{-1/2}$. We also use the elementary inequality

$$\sqrt{1+s}k(s-r)k(r) \leq k(s-r) + k(r), \quad 0 \leq r \leq s,$$

which follows from $\sqrt{1+s} \leq \sqrt{1+r} + \sqrt{1+s-r}$. We insert the bound for A_θ into the Volterra integral. For the value we use that $e^{-(s-r)H_0}$ is a contraction, and for the first difference the bound $\|\nabla e^{-(s-r)H_0}\|_{\infty \rightarrow \infty} \leq Ck(s-r)$ of Lemma 25.1 with $H_1 = T$. Using (25.7) and the elementary inequality for the term with $a\|\nabla z\|_\infty$ in the differentiated estimate, we obtain

$$\|L_\theta\|_{\lambda \rightarrow \lambda} \leq C_{L,T,K_0} \left\{ \frac{a}{\sqrt{\lambda}} + \frac{a^2}{\lambda} + \frac{a^2\sqrt{1+S}}{\sqrt{\lambda}} \right\}. \quad (25.8)$$

The last term comes from the coefficient of order zero in the differentiated estimate. We take $\lambda = \kappa_0 a^2$ with $\kappa_0 = \kappa_0(L, T, K_0)$ so large that the right side is at most $1/2$; here we used $a\sqrt{1+S} = \sqrt{a^2+T} \leq \sqrt{1+T}$ for $a \leq 1$. The series $\sum_{j \geq 0} L_\theta^j$ then converges, with weighted norm at most two, and removing the weight costs at most $e^{\lambda S} = e^{\kappa_0 T}$. This proves the bound on $(I - L_\theta)^{-1}$ in R_{sm} .

Step 4: the forcing. An initial heat term $e^{-sH_0}z_0$ has R_{sm} norm at most $C_{L,T}\|z_0\|_\infty$. A forcing of uniform norm c has a value integral at most cs , and its weighted first difference is at most

$$Cc\sqrt{1+s} \int_0^s (1+s-r)^{-1/2} dr \leq 2Cc(1+s).$$

This proves the forcing bound in (2), including the constant Fourier mode, on which the heat semigroup does not decay.

Step 5: the error. Let θ be the damped solution with the initial value in (3), set $z = \theta - \theta^{\text{ref}}$, and let $\mathcal{R} = \partial_s \theta^{\text{ref}} - \mathcal{F}(\theta^{\text{ref}})$ be the residual of (1). Since N_2 is quadratic, the difference satisfies the exact equation

$$z' = -H_0 z + A_{\theta^{\text{ref}}} z + N_2(z) + Q_{\theta^{\text{ref}}}(z) - \mathcal{R}, \quad Q_\theta(z) = R_3(\theta + z) - R_3(\theta) - DR_3(\theta)z. \quad (25.9)$$

By analytic Taylor expansion on a smaller chart, we have

$$\|Q_\theta(z)\|_\infty \leq C(K_0 a \|z\|_\infty^2 + \|z\|_\infty^3),$$

and the difference of Q_θ at two points of a ball of radius δ has the corresponding factor $K_0 a \delta + \delta^2$. If $\|z\|_{R_{\text{sm}}} \leq \delta$, the quadratic source $N_2(z)$ has size at most $C\delta^2 k(s)$. Its Volterra integral has R_{sm} norm at most $C\delta^2 \sqrt{1+S}$; for the differentiated component we use $\int_0^s k(s-r)k(r) dr \leq \pi$ and multiply by $\sqrt{1+s}$. By Step 4, the integral of $Q_{\theta^{\text{ref}}}(z)$ costs at most $C(K_0 a \delta^2 + \delta^3)(1+S)$. After the Volterra inverse, the Lipschitz constant of these nonlinear terms on the ball of radius δ is at most

$$C_{L,T,K_0} \{ \delta \sqrt{1+S} + K_0 a \delta (1+S) + \delta^2 (1+S) \}. \quad (25.10)$$

The initial heat term and the residual in (25.9) have R_{sm} norm at most $C_{L,T,K_0,C_{\text{init}}}(a^2 + a^4(1+S)) \leq Ca^2$, by Step 4 and (1). Let D be twice this constant multiplied by the bound for the Volterra inverse, and set $\delta = Da^2$. Every term in (25.10) then tends to zero: the first two are $O_{L,T,K_0,D}(a)$ and the last is $O_{T,D}(a^2)$. For a small enough, the mild form of (25.9) defines a map of the ball of radius δ into itself which is a contraction with factor at most $1/2$. Iteration from zero converges in the complete space defined by the norm R_{sm} ; this proves existence of the mild solution and the error bound in (3). The mild solution solves the finite-dimensional differential equation. Since its amplitude is at most $K_0a + Da^2 < r_*$, it does not leave the chart. Local uniqueness for the differential equation and the gauge identification of Lemma 24.2 show that it is the damped representative of the Wilson flow.

Step 6: continuation by restarts. We use the continuation of Theorem 24.10 with $r = 3$ and its fixed margins. On one interval, Step 5 gives an error $O(a^2)$ for the link logarithms, and thus $O(a^2)$ for the links. At a restart time we apply the fixed sampled gauge transformation to the links. Left and right multiplication are isometries, and the logarithm is Lipschitz on the next smaller chart. By Lemma 24.7, applying the gauge transformation to the exponentials of a sample differs from sampling the transformed smooth connection by $O(a^2)$ per link. Then the initial error on the next interval is again of the form $C_{\text{init},i}a^2$, which is the hypothesis of (3). Iterating over the finite partition preserves this order, with the maximum of finitely many constants and the minimum of finitely many restrictions on the mesh. This proves (4). $\square$

25.4 Continuation near smooth solutions

We now pass to physical units and to the norm C_a^1 , and we write $S_ab = b|_a$. In physical time, the damped equation of Lemma 24.2 reads

$$\partial_t B = \Delta_a B + N_a(B), \quad N_a(B) = \mathcal{N}_a(B, B) + \mathcal{C}_a(B), \quad \mathcal{C}_a(B) = a^{-3}R_3(aB).$$

We write

$$\mathcal{B}_a(B)V = \mathcal{N}_a(B, V) + \mathcal{N}_a(V, B) + D\mathcal{C}_a(B)V$$

for the linearization of N_a at B . Here D on the cubic-and-higher remainder $\mathcal{C}_a$ differentiates in the field, and D_a is a physical spatial difference. On the smaller chart,

$$\|D^j \mathcal{C}_a(B)[V_1, \dots, V_j]\|_\infty \leq C \|B\|_\infty^{3-j} \prod_{i=1}^j \|V_i\|_\infty, \quad 0 \leq j \leq 3, \quad (25.11)$$

where for $j = 0$ the empty product is one. Indeed, the derivative is $a^{j-3}D^j R_3(aB)$, and since the Taylor coefficients of R_3 vanish through order two, $D^j R_3(aB)$ has a factor $(a\|B\|_\infty)^{3-j}$; for $j = 3$ we use the bounded analytic third derivative on the smaller chart.

All translated components are included. We use these bounds to show that, on a bounded set of the norm C_a^1 , say $\|B\|_{C_a^1}, \|B'\|_{C_a^1} \leq K$ in the chart,

$$\|N_a(B) - N_a(B')\|_\infty \leq C_K \|B - B'\|_{C_a^1}. \quad (25.12)$$

For the quadratic term, we write the difference as $\mathcal{N}_a(B, B - B') + \mathcal{N}_a(B - B', B')$ and use $\|\mathcal{N}_a(u, v)\|_\infty \leq C \|u\|_\infty \|D_a v\|_\infty$ from Lemma 24.2. For the remainder we use the Lipschitz bound for R_3 in that lemma, rescaled to physical units. No interpolation enters (25.12).

Lemma 25.5. *The following statements hold.*

1. Fix $L, h, K > 0$. Let $B_a(t)$ be a damped lattice solution on $[0, h]$ in the smaller chart with $\sup_{t \leq h} \|B_a(t)\|_{C_a^1} \leq K$. Then the solution of

$$V(t) = P_t^a V_0 + \int_0^t P_{t-r}^a \{ \mathcal{B}_a(B_a(r)) V(r) + f(r) \} dr$$

satisfies

$$\sup_{t \leq h} \|V(t)\|_{C_a^1} \leq C_{L,h,K} (\|V_0\|_{C_a^1} + \sup_{r \leq h} \|f(r)\|_\infty). \quad (25.13)$$

The same bound holds if the right side is replaced by $C_{L,h,K}$ times the norm in $C([0, h]; C_a^1)$ of the whole inhomogeneous term $P_t^a V_0 + \int_0^t P_{t-r}^a f(r) dr$. The constant is uniform in small meshes, and no derivative of f is needed.

2. Let b be a fixed smooth DeTurck solution on $[0, h]$ whose spatial derivatives through order five are bounded by K_b . There are $\eta_b, a_b > 0$ and $C_b < \infty$, depending only on L, h, K_b , and the chart, such that every lattice initial field with $\|B_a(0) - S_a b(0)\|_{C_a^1} \leq \eta \leq \eta_b$, on a mesh $a \leq a_b$, has a damped solution through time h with

$$\sup_{t \leq h} \|B_a(t) - S_a b(t)\|_{C_a^1} \leq C_b(\eta + a), \quad \sup_{t \leq h} \|B_a(t)\|_{C_a^1} \leq C_b. \quad (25.14)$$

The same holds on each time interval of a fixed continuation as in Theorem 24.10, when the flow is restarted with its fixed sampled gauge transformations. In particular, this comparison supplies the C_a^1 bound that is the hypothesis of the smoothing on later time intervals in Theorem 25.7.

3. If two initial fields satisfy the hypothesis of (2) with respect to the same solution b , and their solutions are restarted with the same fixed sampled gauge transformations, then

$$\max_i \sup_{t \in I_i} \|B_a^{(i)}(t) - B_a'^{(i)}(t)\|_{C_a^1} \leq C_b \|B_a(0) - B_a'(0)\|_{C_a^1}. \quad (25.15)$$

Here I_i are the finitely many time intervals of the continuation (or the single interval $[0, h]$), $B_a^{(i)}$ and $B_a'^{(i)}$ are the representatives on I_i , and the constant may also depend on the continuation.

4. Fix $L, h > 0$ and a set $\mathcal{K} \subset C^\infty(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$ that is compact in the C^3 topology. There are $\eta_{\mathcal{K}}, a_{\mathcal{K}} > 0$, $C_{\mathcal{K}} < \infty$, and finitely many continuations as in Theorem 24.10 with the following property. For every $b_0 \in \mathcal{K}$, every $a \leq a_{\mathcal{K}}$, and any two lattice initial fields with

$$\max(\|B_a(0) - S_a b_0\|_{C_a^1}, \|B'_a(0) - S_a b_0\|_{C_a^1}) \leq \eta_{\mathcal{K}},$$

one of these continuations gives the bound $\sup \|B_a(t)\|_{C_a^1} \leq C_{\mathcal{K}}$ through time h for both solutions, and (25.15) with constant $C_{\mathcal{K}}$. The constants depend on $L, h, \mathcal{K}$, on the bounds of the finitely many selected smooth DeTurck solutions, and on the margins of the chart and of the continuations. Higher derivatives are only bounded for these finitely many selected solutions, not on a C^3 neighborhood of $\mathcal{K}$.

In (2)–(4), every representative is gauge equivalent to the Wilson flow from its initial links.

Proof. Part (1). By the bound on $\mathcal{N}_a$ and (25.11), we have $\|\mathcal{B}_a(B)V\|_\infty \leq C_K \|V\|_{C_a^1}$. We use Lemma 25.1 with physical horizon h . The heat semigroup is a contraction for C_a^1 , since it commutes with differences, and for $t > 0$

$$\|P_t^a f\|_{C_a^1} \leq C_{L,h}(1 + (t + a^2)^{-1/2})\|f\|_\infty \leq C_{L,h}(1 + t^{-1/2})\|f\|_\infty.$$

In the weighted norm $\|V\|_\lambda = \sup_{t \leq h} e^{-\lambda t} \|V(t)\|_{C_a^1}$, the integral operator $V \mapsto \int_0^t P_{t-r}^a \mathcal{B}_a(B_a(r))V(r) dr$ has norm at most

$$C_{L,h,K} \int_0^\infty e^{-\lambda v} (1 + v^{-1/2}) dv = C_{L,h,K} (\lambda^{-1} + \sqrt{\pi} \lambda^{-1/2}).$$

We choose λ , independently of the mesh, so that this is at most $1/2$. The Neumann series then gives an inverse of weighted norm at most two, and removing the weight costs $e^{\lambda h}$. The initial heat term has norm at most $\|V_0\|_{C_a^1}$, and the integral of the forcing has unweighted norm at most $C_{L,h}(h + 2\sqrt{h}) \sup \|f\|_\infty$. This proves both linear assertions.

Part (2). Set $b_a = S_a b$ and $e_a = \partial_t b_a - \Delta_a b_a - N_a(b_a)$. The residual in Theorem 25.4(1) is $O(a^4)$ in dimensionless link-logarithm units. Since $\theta = ab$ and physical time is a^2 times dimensionless time, we divide by a^3 and obtain $\sup_t \|e_a(t)\|_\infty \leq C_{K_b} a$. For $Z = B_a - b_a$ the mild difference equation is

$$Z(t) = P_t^a Z(0) + \int_0^t P_{t-u}^a \{N_a(b_a(u) + Z(u)) - N_a(b_a(u)) - e_a(u)\} du.$$

Up to the first time at which $\|Z\|_{C_a^1} = 1$, the difference of the nonlinear terms is at most $C_{K_b} \|Z\|_{C_a^1}$ by (25.12). Using the weighted estimate of Part (1) on every such interval, we obtain

$$\sup_t \|Z(t)\|_{C_a^1} \leq C_b(\eta + a).$$

We choose the constants in the following order: first the reference solution and the horizon; then λ and C_b ; then η_b and a_b so small that $C_b(\eta_b + a_b) \leq 1/2$ and $a_b(\sup \|b\|_\infty + 1)$ is strictly below the radius of the smaller chart. No first time with $\|Z\|_{C_a^1} = 1$ exists, since it would contradict the bound $1/2$. At fixed mesh, the finite-dimensional analytic differential equation can be continued as long as the solution stays in a compact part of the chart, which gives existence through time h . The gauge identification of Lemma 24.2 applies through time h . This proves (25.14).

Part (3) and continuations. For two solutions we subtract their exact equations. The source is $N_a(B_a) - N_a(B'_a)$, and there is no residual. Using the common bound in C_a^1 already proved and the estimate of Part (1), we obtain (25.15) on one interval. By Lemma 24.7, at a restart with the fixed smooth gauge transformation k of the continuation, we have

$$\begin{aligned}\|\mathcal{G}_{a,k}(B_a) - S_a(b^k)\|_{C_a^1} &\leq C_k \|B_a - S_a b\|_{C_a^1} + C_{k,b} a, \\ \|\mathcal{G}_{a,k}(B_a) - \mathcal{G}_{a,k}(B'_a)\|_{C_a^1} &\leq C_k \|B_a - B'_a\|_{C_a^1},\end{aligned}$$

where $\mathcal{G}_{a,k}(A)_i(x) = a^{-1} \text{Log}(k(x)e^{aA_i(x)}k(x+ae_i)^{-1})$. Let L_i be the constant of (25.14) or (25.15) on the i th interval, let C_i, C'_i be the Lipschitz constants of the restart, and let $d_i = C_{k_i,b_i}$ be the coefficient of a in the first restart estimate. For the errors e_i relative to the selected DeTurck solution at the restart times, we have the finite affine recurrence $e_{i+1} \leq L_i C_i (e_i + a) + d_i a$. For the differences δ_i of a pair at the restart times, we have the homogeneous recurrence $\delta_{i+1} \leq L_i C'_i \delta_i$. Their finite products and sums give the asserted constants. We choose the initial error and the mesh only afterwards, below the threshold of every interval divided by the resulting finite multiplier. This proves the assertions for continuations. It also provides the C_a^1 bound needed for the later smoothing, without using that smoothing.

Part (4). For each $c \in \mathcal{K}$ we take one continuation through time h from Theorem 24.10. Its finitely many DeTurck solutions are smooth on their closed time intervals and have finite C^5 bounds, and its gauge transformations are fixed and smooth. Parts (2) and (3) give an initial radius $\varepsilon_c > 0$, a threshold $a_c > 0$, and finite constants for this continuation. The sets $\{b_0 : \|b_0 - c\|_{C^1} < \varepsilon_c/4\}$ are open in the C^3 topology and cover $\mathcal{K}$; we select a finite subcover. By the fundamental theorem of calculus along coordinate segments, sampling has operator norm at most two from the continuum C^1 maximum norm into the sum norm C_a^1 . Thus, if $\eta_{\mathcal{K}} \leq \min_c \varepsilon_c/4$ and $\|b_0 - c\|_{C^1} < \varepsilon_c/4$ for a selected c , both lattice initial fields are within $3\varepsilon_c/4 < \varepsilon_c$ of $S_a c$. We apply Parts (2) and (3) with this c and take the maximum of the finitely many constants and the minimum of the finitely many thresholds. This proves Part (4). $\square$

25.5 Smoothing after a positive waiting time

In the statements about loops below, a closed lattice path ℓ_a approximating a continuum loop ℓ is required to have bounded physical length. It must also be joined to ℓ by a based piecewise smooth homotopy whose swept area, counted with multiplicity, tends to zero, as in Definition 2.1. An area bound $O(a)$ suffices, and coordinate polygons whose vertices lie on the lattice can be taken exactly. The following estimate for a smooth connection b explains this condition.

Let $\Gamma(h, r)$, $h, r \in [0, 1]$, be a smooth homotopy with $\Gamma(h, 0)$ and $\Gamma(h, 1)$ equal to a fixed basepoint, and let $V(h, r)$ be the parallel transport along $\Gamma(h, \cdot)$ up to r , so that $\partial_r V = V b(\partial_r \Gamma)$ and $V(h, 0) = I$, where $b(\xi) = \sum_i b_i \xi^i$. We define $X = \partial_h V - V b(\partial_h \Gamma)$. Differentiating the transport equation in h and subtracting the r -derivative of $V b(\partial_h \Gamma)$, we find $\partial_r X = X b(\partial_r \Gamma) + V F_b(\partial_h \Gamma, \partial_r \Gamma)$, where $F_b(\xi, \zeta) = \sum_{i,j} F_{ij} \xi^i \zeta^j$ and $F_{ij} = \partial_i b_j - \partial_j b_i + [b_i, b_j]$. Since the basepoint is fixed, $X(h, 0) = 0$ and $\partial_h V(h, 1) = X(h, 1)$. Solving for XV^{-1} , we obtain that $\partial_h V(h, 1)V(h, 1)^{-1}$ is the integral over $r \in [0, 1]$ of the conjugate $V F_b(\partial_h \Gamma, \partial_r \Gamma) V^{-1}$. Since conjugation by unitary matrices has norm one and $\|X\|_{\text{op}} = |X|/2$ for $X \in \mathfrak{su}(2)$, this proves

$$\|\text{Hol}_b(\Gamma(1, \cdot)) - \text{Hol}_b(\Gamma(0, \cdot))\|_{\text{op}} \leq C \|F_b\|_{\infty} \iint |\partial_h \Gamma \wedge \partial_r \Gamma| dr dh, \quad (25.16)$$

where C only converts the componentwise curvature norm into the norm of a two-form. Subdivision gives the same estimate for piecewise smooth homotopies. A moving basepoint only adds a conjugation, which does not change traces. The approximation error therefore tends to zero uniformly on each bounded set of smooth connections.

The next lemma is a compactness statement for lattice arrays with bounded differences. In its proof, limits of difference arrays are identified with derivatives by discrete summation by parts, not by differentiating an interpolation of the arrays.

Lemma 25.6. *Let $a_k \rightarrow 0$ be meshes, and let A_k be $\mathfrak{su}(2)$ -valued link fields on Λ_{L/a_k} . Suppose that for every integer $m \geq 0$ there are $K_m < \infty$ and k_m such that $\max_{|v| \leq m} \|D_{a_k}^v A_k\|_{\infty} \leq K_m$ for all $k \geq k_m$, where v ranges over ordered compositions of forward and backward differences.*

1. *There are a subsequence (k_j) and a smooth $b \in C^{\infty}(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$ with $\|\partial^v b\|_{\infty} \leq K_{|v|}$ for all v such that, for every v ,*

$$\|D_{a_{k_j}}^v A_{k_j} - S_{a_{k_j}} \partial^v b\|_{\infty} \rightarrow 0.$$

Equivalently, the cardinal interpolants of Lemma 16.3, applied separately to each difference array $D_{a_{k_j}}^v A_{k_j}$, converge to $\partial^v b$ uniformly on $\mathbb{T}_L^3$.

2. *Let $\mathcal{K}$ be the set of all $b \in C^{\infty}(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$ with $\|\partial^v b\|_{\infty} \leq K_{|v|}$ for every v . Then $\mathcal{K}$ is compact in the C^3 topology, and it contains 0.*

3. Suppose that for each mesh a a set $\mathcal{F}_a$ of link fields on $\Lambda_{L/a}$ is given such that, for every m , the bound $\max_{|\nu| \leq m} \|D_a^\nu A\|_\infty \leq K_m$ holds for all $A \in \mathcal{F}_a$ whenever $a \leq a_m$. Then

$$\sup_{A \in \mathcal{F}_a} \inf_{b \in \mathcal{K}} \|A - S_a b\|_{C_a^1} \longrightarrow 0 \quad (a \rightarrow 0).$$

Here the outer supremum is defined to be zero when $\mathcal{F}_a = \emptyset$.

Proof. Part (1). An array g on mesh a whose first differences are bounded by K has oscillation at most $CK(|x - y| + a)$ between two lattice sites x, y , as we see by following a shortest coordinate path on the torus. Since cardinal interpolation is a convex combination of samples within distance Ca , the interpolated array has the same modulus of continuity, up to a fixed constant and an additional CKa . It also takes the given values at the lattice sites. Applying this to $D^\nu A_k$ with the bound $K_{|\nu|+1}$, we find that the interpolants of the arrays of order $|\nu|$ are bounded by $K_{|\nu|}$ and have modulus $CK_{|\nu|+1}(|x - y| + a_k)$. We choose finite nets of $\mathbb{T}_L^3$ with spacing tending to zero. The bounded values at the points of one net have a convergent subsequence, and the modulus of continuity extends the convergence uniformly over the torus up to an error that vanishes as the spacing and a_k tend to zero. By successive extractions and a diagonal subsequence, we obtain, for every one of the countably many ν , a continuous limit b_ν of the interpolants of $D^\nu A_{k_j}$, with $\|b_\nu\|_\infty \leq K_{|\nu|}$. Since the interpolants take the given values at the lattice sites, $\|D^\nu A_{k_j} - S_{a_{k_j}} b_\nu\|_\infty \rightarrow 0$.

Let $b = b_\emptyset$ be the limit of the arrays themselves. For a smooth periodic test function φ , discrete summation by parts gives

$$a^3 \sum_x D_a^\nu A(x) \varphi(x) = (-1)^{|\nu|} a^3 \sum_x A(x) \tilde{D}_a^\nu \varphi(x),$$

where $\tilde{D}_a^\nu$ reverses the order of the differences and exchanges forward and backward differences. By Taylor expansion of φ and Riemann sums, along the subsequence the left side tends to $\int b_\nu \varphi$ and the right side to $(-1)^{|\nu|} \int b \partial^\nu \varphi$. We conclude that $b_\nu = \partial^\nu b$ in the sense of distributions, for every forward or backward choice in ν . Further, the left side is bounded by $K_{|\nu|} a^3 \sum_x |\varphi(x)|$, which gives

$$|\langle \partial^\nu b, \varphi \rangle| \leq K_{|\nu|} \|\varphi\|_{L^1}, \quad (25.17)$$

with all direction and color components included. To see that these derivatives are classical, we convolve b with a nonnegative smooth periodic approximate identity χ_ϵ of mass one. Then $\partial^\nu (b * \chi_\epsilon) = b_\nu * \chi_\epsilon$ is bounded by $K_{|\nu|}$ by (25.17), and it converges uniformly to b_ν as $\epsilon \rightarrow 0$, because b_ν is continuous; also $b * \chi_\epsilon \rightarrow b$ uniformly. By the fundamental theorem of calculus along coordinate segments, when a sequence of functions and their first derivatives converge uniformly, the limit is continuously differentiable with the limiting derivatives. Applying this order by order, we conclude that b is smooth,

with $\partial^\nu b = b_\nu$ and $\|\partial^\nu b\|_\infty \leq K_{|\nu|}$. The equivalence with uniform convergence of the interpolants follows from the modulus of continuity and the fact that the interpolants take the sampled values at the lattice sites.

Part (2). The bounds on the derivatives of order four imply that the derivatives of order at most three of the elements of $\mathcal{K}$ are bounded and equicontinuous. By the Arzelà–Ascoli theorem, $\mathcal{K}$ is relatively compact in C^3 . Let $b_n \in \mathcal{K}$ converge in C^3 to b . The uniform bounds at every higher order allow a diagonal subsequence along which every derivative of b_n converges uniformly. The fundamental theorem of calculus along coordinate segments identifies the limits with the derivatives of b . Hence b is smooth and satisfies the same bounds, and $\mathcal{K}$ is closed in C^3 . This uses the bounds at every order; a family controlled at only one finite order need not have a closure consisting of smooth fields.

Part (3). If the assertion fails, there are $\epsilon > 0$, meshes $a_k \rightarrow 0$, and $A_k \in \mathcal{F}_{a_k}$ with $\inf_{b \in \mathcal{K}} \|A_k - S_{a_k} b\|_{C_{a_k}^1} \geq \epsilon$. Part (1) implies that a subsequence satisfies $\|A_k - S_{a_k} b\|_\infty \rightarrow 0$ and $\|D_{a_k} A_k - S_{a_k} \partial b\|_\infty \rightarrow 0$ for some $b \in \mathcal{K}$. Since $\|D_a S_a b - S_a \partial b\|_\infty \leq C a \|b\|_{C^2} \leq C a K_2$, it follows that $\|A_k - S_{a_k} b\|_{C_{a_k}^1} \rightarrow 0$, a contradiction. $\square$

Theorem 25.7. Fix L , $1/2 < \alpha < 2/3$, and R , and let T_R be the local existence time of Theorem 24.5. For every integer $m \geq 0$ and every $0 < s < T_R$ there are $a_{m,R,s} > 0$ and $C_{m,L,R,s,T_R} < \infty$ such that

$$\sup_{a \leq a_{m,R,s}} \sup_{s \leq t \leq T_R} \max_{|\nu| \leq m} \|D_a^\nu A_a(t)\|_\infty \leq C_{m,L,R,s,T_R}$$

for all solutions A_a given by Theorem 24.5 whose flow data satisfy $\|(x_a, y_a)\|_{\mathbb{A},a} \leq R$ on $[0, T_R]$. All ordered compositions of forward and backward differences and all translated components are included. More generally, the same conclusion holds on any smaller interval that begins after a positive waiting time inside a time interval on which a restarted damped representative is used, provided that this representative lies in the smaller chart and a uniform bound in C_a^1 on the preceding interval is given. The constant then depends on that bound, on the waiting time, on the length of the interval, and on m .

Suppose that b is a continuous periodic field and that such solutions, along meshes $a_k \rightarrow 0$, satisfy $\|A_{a_k}(s) - S_{a_k} b\|_{C_{a_k}^1} \rightarrow 0$ at a fixed time $s \in (0, T_R)$. Then b is smooth, with bounds on its derivatives of every order depending only on the order and on L, R, s, T_R . The closure in C^3 of the set of all such limits is compact and consists of smooth connections, so Theorem 24.10 applies to it for every finite horizon. For each such limit b , the Wilson flow started from $\exp(a S_a b)$ has link error $O(a^2)$ on every time interval of a continuation as in Theorem 24.10, including the restarts with the sampled fixed gauge transformations. The normalized traces of its holonomies around closed lattice paths of bounded length that approximate continuum loops, in the sense described before Lemma 25.6, converge to the corresponding continuum traces.

On each later time interval of a continuation by restarts with fixed smooth gauge transformations, the first paragraph applies as soon as a uniform C_a^1 bound for the restarted representative is known, for example from Lemma 25.5. Then every fixed difference order is bounded at every fixed positive later time. The C_a^1 bound is a hypothesis of this statement; it is not deduced from the smoothing. The constants are deterministic and do not depend on the mesh below the stated thresholds. No bound is claimed as the waiting time tends to zero, and no bound on high norms at time zero follows from the compactness in C^3 .

The proof of the first assertion is the standard parabolic bootstrap, carried out on the lattice equation. In a difference of order $m - 1$ of the nonlinearity, the only difference of order m of the solution appears linearly, with a bounded factor in front. The heat semigroup gains one difference at the integrable cost $(t - r)^{-1/2}$, and the resulting term can be absorbed on a short interval.

Proof. Step 1: the base case. By Theorem 24.5, on $[s/2, T_R]$ we have

$$\|A_a(t)\|_\infty + \|D_a A_a(t)\|_\infty \leq K_1(L, R, s, T_R). \quad (25.18)$$

This proves the cases $m = 0, 1$. In the smaller chart the equation is

$$\partial_t A_a = \Delta_a A_a + \mathcal{N}_a(A_a, A_a) + a^{-3} R_3(a A_a). \quad (25.19)$$

Step 2: the highest difference. Let $m \geq 2$, and suppose that all differences through order $m - 1$ are bounded on a preceding interval, by $K_1, \dots, K_{m-1}$. We show that

$$\|D_a^{m-1} \{\mathcal{N}_a(A_a, A_a) + a^{-3} R_3(a A_a)\}\|_\infty \leq B_m(K_1, \dots, K_{m-1}) + C_m K_1 \|D_a^m A_a\|_\infty. \quad (25.20)$$

Each summand of $\mathcal{N}_a$ contains exactly one physical difference. When we apply (25.2) $m - 1$ times, a factor of order m occurs only when all these differences fall on the factor that already carries a difference; its companion is then a value of A_a , bounded by K_1 . If at least one difference falls on the companion, both factors have orders at most $m - 1$ and are bounded by the K_j . The number of terms and all translations depend only on m and on the fixed stencil. In the analytic remainder, the term of degree d in physical units has the factor a^{d-3} , $d \geq 3$, and a difference of order $m - 1$ of a product involves differences of A_a of order at most $m - 1$ only. To see that the series converges uniformly, we shrink the threshold so that $a K_1$ lies in a fixed smaller chart. A differentiated term of degree d is bounded by a fixed polynomial in d times its coefficient in the series times products of the finitely many K_j . The remaining undifferentiated factors keep a geometric factor from the smaller chart. The resulting series, even multiplied by the polynomial, is summable uniformly in a . This proves that the remainder contributes only to B_m , with no negative power of the mesh. These observations prove (25.20) for every ordered choice of forward and backward differences and all translated components.

Step 3: absorption. We start the mild equation at a time u_0 in the preceding interval, and define

$$Z_m(d) = \sup_{0 < v \leq d} \sqrt{v} \|D_a^m A_a(u_0 + v)\|_\infty.$$

At each fixed mesh this number is finite, since the arrays are finite dimensional. We commute $m - 1$ differences onto the initial value and onto the source, and place one difference on the heat semigroup, whose bound is $C(v + a^2)^{-1/2} \leq Cv^{-1/2}$. Multiplying the resulting mild estimate by $\sqrt{v}$ and using (25.20), we obtain

$$Z_m(d) \leq C_m K_{m-1} + C_m B_m d + C_m K_1 \sqrt{d} Z_m(d). \quad (25.21)$$

For the last term we used $\int_0^v (v - r)^{-1/2} r^{-1/2} dr = \pi$, together with the additional factor $\sqrt{v} \leq \sqrt{d}$; for the bounded source the corresponding factor is $2v \leq 2d$. We choose $d \leq 1$ such that $C_m K_1 \sqrt{d} \leq 1/2$ and such that $[u_0, u_0 + d]$ lies in the preceding interval. Then (25.21) implies that $Z_m(d) \leq 2C_m(K_{m-1} + B_m d)$, and hence a uniform bound on $D_a^m A_a$ at time $u_0 + d$.

Step 4: the whole interval. For a given finite $m \geq 2$ we choose times $s/2 = s_1 < s_2 < \dots < s_m = s$. At stage $j \geq 2$ we take $d_j > 0$ less than $s_j - s_{j-1}$ and satisfying the absorption condition of Step 3 for order j . For any $t \in [s_j, T_R]$ we apply Step 3 with $u_0 = t - d_j \geq s_{j-1}$ and evaluate at $v = d_j$. This gives one bound K_j for all such t . Since there are finitely many orders, the restrictions $a \leq a_{j,R,s}$ have a positive minimum, and the constants are finite. This proves the first assertion. On a restarted time interval, the same proof starts from the given chart and the given C_a^1 bound on the preceding interval instead of (25.18). Inspection of the choices shows the stated dependence of the constant.

Step 5: limits at a positive time. Let $A_{a_k}(s)$ satisfy $\|A_{a_k}(s) - S_{a_k} b\|_{C_{a_k}^1} \rightarrow 0$. The first assertion implies that the arrays $A_{a_k}(s)$ satisfy the hypotheses of Lemma 25.6, with bounds $K_m = C_{m,L,R,s,T_R}$. By part (1) of that lemma, a subsequence converges, together with all its difference arrays, to a smooth field with these bounds. This field is b , since the arrays converge to $S_{a_k} b$. Hence every limit belongs to the set $\mathcal{K}$ of part (2) of that lemma, which is compact in C^3 and consists of smooth connections, and so does the C^3 closure of the limits.

Step 6: continuation and loops. We apply Theorem 24.10 with $r = 3$ to this compact set and a fixed horizon. Its open neighborhood contains every smooth connection sufficiently close to the compact set in C^3 . Indeed, for each point of the compact set we choose a C^3 ball whose doubled radius lies in the neighborhood, take a finite subcover, and use the minimum of the finitely many radii. Theorem 25.4(4), applied on the finitely many time intervals, yields link error $O(a^2)$ for the Wilson flow started from $\exp(a S_a b)$, including the restarts with the sampled gauge transformations. The Wilson flow and its damped representatives have the same traces of holonomies around closed paths. Since a closed lattice path of length at most ℓ has at most ℓ/a links, unitary telescoping turns the link

error $O(a^2)$ into an error $O(\ell a)$ for the holonomy. By Lemma 24.7, the product of the exponentials e^{ab_i} along the path differs from the smooth parallel transport along the same path by an error of the same order. Finally, (25.16) bounds the difference between the smooth transport along the lattice path and along the continuum loop. This proves the assertion on loops.

Step 7: later restarts. Suppose that lattice differences of A of order up to m are bounded and that k is a fixed smooth periodic gauge transformation. Then

$$Y_i(x) = k(x)e^{aA_i(x)}k(x + ae_i)^{-1} = I + O(a),$$

and every difference of fixed order of $Y - I$ keeps this factor a . To verify this, we split $Y - I$ into $k(x)(e^{aA_i(x)} - I)k(x + ae_i)^{-1}$ and $k(x)k(x + ae_i)^{-1} - I$. We handle the first part with the convergent exponential series and the exact product rule, and the second with $k(x + ae_i) - k(x) = \int_0^a \partial_i k(x + re_i) dr$. Both give a times a polynomial in the given bounds and in smooth norms of k . In the analytic composition $a^{-1} \text{Log } Y$, a term with $j \geq 1$ differentiated factors carries a^{j-1} , and the value is $a^{-1}O(a)$. The bounds for the composition are therefore uniform in the mesh, and such a gauge transformation preserves the bounds on lower differences. On each restarted interval, a given uniform C_a^1 bound in the common gauge restarts the induction of Step 2, and waiting intervals before each given later time give every given finite order. At a restart time we use a preceding interval and the bound for the composition just proved to transfer the bounds on the differences. This proves the last assertion. $\square$

The next lemma is the space-time version of Lemma 25.6, together with the passage to the limit in the Wilson equation.

Lemma 25.8. *Let $a_k \downarrow 0$, and let $J \subset (0, \infty)$ be a compact interval. Let B_{a_k} be time-dependent link fields on J such that, for every m , all differences of order at most m of B_{a_k} and their time derivatives are bounded on J uniformly in k . We interpolate each difference array separately by the cardinal interpolation of Lemma 16.3. Then every subsequence has a further subsequence along which the interpolated spatial difference arrays converge, uniformly on $J \times \mathbb{T}_L^3$ at every fixed order, to the corresponding derivatives of one field B that is smooth in space for each time; by a diagonal argument the same holds for countably many intervals simultaneously. If, in addition, each B_{a_k} is a damped representative of the Wilson flow on J , that is, solves $\partial_t B = \Delta_a B + \mathcal{N}_a(B, B) + \mathcal{C}_a(B)$ in the smaller chart, then the limit B solves the DeTurck equation $\partial_t B = \Delta B + \mathcal{N}_0(B, B) + \mathcal{N}_3(B)$ on J . The same holds for the mild form of the equation with a fixed lower integration endpoint in J .*

The passage to the limit in the equation uses three properties of the Wilson equation. Its quadratic part is a fixed finite sum of translated brackets containing one difference (Lemma 24.2). The remainder $\mathcal{C}_a$ has local Lipschitz bounds that are uniform on bounded

sets ((25.11)). Finally, $\mathcal{C}_a$ is consistent, uniformly on bounded families of smooth fields (Lemma 25.3). The same proof applies to any equation with these three properties.

Proof. We obtain the spatial modulus of continuity of each interpolated difference array from the bound on the next difference, as in the proof of Lemma 25.6(1). The bounds on the time derivatives give a uniform Lipschitz modulus in time. Using finite nets in space and time, successive subsequences of the bounded finite-dimensional values at their points, and a diagonal argument, we obtain uniform convergence on $J \times \mathbb{T}_L^3$, first for one array and then for the countable list of orders and positive intervals. As in the proof of Lemma 25.6(1), summation by parts against smooth test functions, mollification, and the fundamental theorem of calculus identify the limits with the corresponding classical derivatives.

For the nonlinear terms, every summand of $\mathcal{N}_a$ involves finitely many translations and one difference, and these converge to the corresponding untranslated value and derivative. For the analytic remainder, the uniform bound on B_a on J implies that the analytic composition converges uniformly on the smaller chart. Its local Lipschitz constants are bounded by (25.11), and its values on the sample of the smooth limit converge to $S_a \mathcal{N}_3(B)$ by Lemma 25.3. Subtracting the value at the sample, we obtain the convergence at B_a itself. The same reasoning applies after any fixed difference for which bounds are given. The lattice Laplacian is a sum of second differences and converges to ΔB . Integrating the differential equations in time, and using uniform convergence to pass every term through the integral, we obtain the limiting equation.

For the mild form, the heat semigroup applied to a sampled smooth family converges by the explicit bounds for general smooth inputs in Lemma 16.9. For a convergent array, we first replace it by the sample of its smooth limit; the replacement error is controlled by contraction. For a first difference of the output, we either commute it onto a convergent difference of the source, or use the integrable heat bound of Lemma 25.1. These arguments pass each term of the mild equation with a positive lower endpoint to the limit. $\square$

25.6 Continuation and stability for bounded flow data

We now apply smoothing and compactness to bounded flow data. After a short positive time u , Theorem 25.7 bounds all lattice differences of the solution, and Lemma 25.6 shows that the solutions at time u are uniformly close to samples of a compact set $\mathcal{K}$ of smooth connections. This compact set depends only on the bound R on the flow data, not on any convergence of the initial potentials. Part (4) of Lemma 25.5 then provides finitely many continuations, chosen before the initial potential. Along them we compare every solution with a selected smooth DeTurck solution, and two solutions with each other.

Lemma 25.9. Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, and $0 < \tau \leq T < \infty$. Let T_R be the local existence time of Theorem 24.5 for flow data of norm at most R , replaced by $\min(T_R, T_0)$, and choose $0 < u < \min(T_R, T_0, \tau)/4$. There are $a_*, K, C, \delta > 0$ and finitely many continuations as in Theorem 24.10, all chosen after these data, with the following properties. For every mesh $a = L/n \leq a_*$ and every lattice initial potential whose flow data satisfy $\|(x_a, y_a)\|_{\mathfrak{A}, a} \leq R$ on $[0, T_0]$, the Wilson flow on $[u, T]$ has damped representatives, restarted at the times of one of these continuations with its sampled gauge transformations, whose C_a^1 norms are at most K . If two such initial potentials on the same lattice, compared without any gauge transformation, satisfy

$$d_a = X_\alpha^a(x_a - x'_a) + Y_\alpha^a(y_a - y'_a) \leq \delta,$$

with both norms taken on $[0, T_0]$ and including all first differences and translated components, then the two flows can use the same continuation with the same sampled gauge transformations, and on each of its time intervals

$$\sup_t \|B_a(t) - B'_a(t)\|_{C_a^1} \leq C d_a.$$

Each representative is gauge equivalent to the Wilson flow $\Psi_t^{(n)}$. The constants and a_* depend only on $L, T_0, R, \alpha, \tau, T, u$, and the chart. In particular they do not depend on higher derivatives of the initial potentials, nor on any finite approximation from which these potentials are constructed. Only the finitely many selected smooth DeTurck solutions need bounds on higher derivatives.

Proof. Step 1: bounds at time u . By the individual bounds and the bound on differences in Theorem 24.5, we have

$$\|A_a(u) - A'_a(u)\|_{C_a^1} \leq L_u d_a, \quad \sup_{u/2 \leq t \leq 2u} \|A_a(t)\|_{C_a^1} \leq K_1,$$

and the same individual bound holds for the second initial potential. Theorem 25.7, applied with a preceding interval after a positive waiting time, yields for every fixed m a bound K_m on all differences of order m at time u , for all sufficiently small meshes. The restriction on the mesh may depend on m ; along every sequence $a_j \rightarrow 0$ it is eventually satisfied at each fixed order.

Step 2: compactness. Let $\mathcal{K}$ be the set of smooth connections b with $\|\partial^\nu b\|_\infty \leq K_{|\nu|}$ for all ν . By Lemma 25.6(2), $\mathcal{K}$ is compact in C^3 and contains 0; by part (1) of that lemma, it contains every possible limit of the endpoints $A_a(u)$. Applying part (3) of that lemma to the set $\mathcal{F}_a$ of all endpoints $A_a(u)$ of initial potentials with flow data bounded by R , we obtain

$$\sup \inf_{b \in \mathcal{K}} \|A_a(u) - S_a b\|_{C_a^1} \longrightarrow 0 \quad (a \rightarrow 0),$$

where the supremum is over all such initial potentials.

Step 3: choice of the continuations. We apply part (4) of Lemma 25.5 to $\mathcal{K}$ with horizon $T - u$. Its neighborhood radius $\eta_{\mathcal{K}}$, its restriction $a \leq a_{\mathcal{K}}$, its uniform bound, and its finitely many continuations are fixed before we choose either initial potential. We decrease a_* until every endpoint $A_a(u)$ lies within $\eta_{\mathcal{K}}/4$ of $S_a b_0$ for some $b_0 \in \mathcal{K}$, and we choose $\delta \leq \eta_{\mathcal{K}} / (4 \max(1, L_u))$. If $d_a \leq \delta$, then $A'_a(u)$ lies within $\eta_{\mathcal{K}}/4 + L_u \delta \leq \eta_{\mathcal{K}}/2$ of the same $S_a b_0$. Thus both endpoints lie in the neighborhood of one common element of $\mathcal{K}$.

Step 4: conclusion. The paired part of Lemma 25.5(4) propagates the initial difference $L_u d_a$ at time u through the finite continuation, using the same sampled gauge transformations for both flows. Its individual part bounds the representative of every initial potential, whether or not it is paired with another. All representatives are gauge equivalent to the Wilson flow by Lemma 24.2. Since the two compared equations are exact lattice equations, their difference has no consistency residual. This proves the asserted bounds and their dependence. $\square$

For matrices, we must also compare the gauge transformations used to define the representatives. On the early interval the damping gauge transformation solves $\partial_t k = \lambda_a k$ with an anti-Hermitian generator. Since such equations have unitary propagators, the difference of two solutions is bounded by the integral of the difference of the generators, without any exponential of the size of the generators. By the estimates of Lemma 24.7, the difference of the generators has an integrable singularity at time zero.

Theorem 25.10. Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, $0 < t_- \leq t_+ \leq T < \infty$, and $\mathfrak{l}_1, \dots, \mathfrak{l}_b > 0$. For each mesh $a = L/N$, let $\ell_{1,a}, \dots, \ell_{b,a}$ be closed paths in Λ_N with given basepoints and with physical lengths $a N_{\ell_{i,a}} \leq \mathfrak{l}_i$, where N_{ℓ} is the number of links of ℓ . Let A_0, A'_0 be lattice initial potentials on Λ_N whose flow data $(x, y), (x', y')$ satisfy $\|(x, y)\|_{\mathfrak{f},a}, \|(x', y')\|_{\mathfrak{f},a} \leq R$ on $[0, T_0]$, put $U = e^{aA_0}$ and $U' = e^{aA'_0}$, and let $d_a = X_{\alpha}^a(x - x') + Y_{\alpha}^a(y - y')$, with both norms on $[0, T_0]$. Then

$$\sup_{t_- \leq t \leq t_+} \max_i \|\text{Hol}_{\ell_{i,a}} \Psi_t^{(N)} U - \text{Hol}_{\ell_{i,a}} \Psi_t^{(N)} U'\|_{\text{op}} \leq C d_a.$$

The holonomies are compared as matrices: no gauge transformation is applied to either configuration, and no conjugation at the basepoints is optimized. The constant depends only on $L, T_0, R, \alpha, t_{\pm}, T$, the lengths $\mathfrak{l}_i$, the chart, and the finitely many smooth DeTurck solutions selected in the proof. It holds for every mesh $a = L/N$ with $N \geq 4$, and it does not depend on the position or shape of the paths beyond their lengths, nor on higher derivatives of the initial potentials.

Let $h_R = \min(T_R, T_0)$, where T_R is the local existence time of Theorem 24.5. After decreasing the mesh bound a_R of that theorem if necessary, the following holds for $a \leq a_R$ and $0 \leq s \leq h_R$. The damped representatives A, A' and their damping gauge transformations k, k' with initial value I satisfy, at every vertex,

$$\|D_a(A - A')(s)\|_{\infty} \leq C_R d_a \{v_a(s)^{-(\alpha+1)/2} + v_a(s)^{-\alpha} + v_a(s)^{\kappa-1/2}\},$$

$$\|k(s) - k'(s)\|_\infty \leq C \int_0^s \|D_a(A - A')(v)\|_\infty dv \leq C_R d_a.$$

For $a \leq a_*$ and $d_a \leq \delta$, where a_* and δ are the constants of Lemma 25.9 with $\tau = t_-$, both flows can use the same continuation and the same sampled gauge transformations on the later intervals. The matrices are always expressed in the gauges of the initial configurations. For other pairs and meshes, the matrix bound follows from the diameter bound for unitary matrices or from the finite-dimensional Wilson flow, as in the proof.

Proof. Step 1: the early interval. Fix $u < \min(T_R, T_0, t_-)/4$. Since the flow data are bounded by R only on $[0, T_0]$, we apply Theorem 24.5 on $[0, h_R]$. Adding the bound on the difference of the remainders in Theorem 24.5 to the differences of the linear heat evolutions and of the quadratic Picard terms, we obtain exactly the three powers in the bound for $D_a(A - A')$. This bound is also stated in Lemma 24.7. By the formula for the generator in that lemma, the damping generator in physical units is the signed lattice divergence of the potential, which gives $\|\lambda_a - \lambda'_a\|_\infty \leq C \|D_a(A - A')\|_\infty$. The damping gauge transformations solve $\dot{k} = \lambda_a k$ and $\dot{k}' = \lambda'_a k'$ with $k(0) = k'(0) = I$. Since the generators are anti-Hermitian, the propagators are unitary, and variation of constants gives, at every vertex,

$$\|k(s) - k'(s)\|_\infty \leq \int_0^s \|\lambda_a(v) - \lambda'_a(v)\|_\infty dv. \quad (25.22)$$

Indeed, $\partial_s(k^{-1}k') = k^{-1}(\lambda'_a - \lambda_a)k'$, and $\|k' - k\| = \|k^{-1}k' - I\|$. All three powers in the bound for $D_a(A - A')$ are integrable uniformly in the mesh, since $(\alpha + 1)/2 < 1$, $\alpha < 1$, and $1/2 - \kappa < 1$; for $0 < p < 1$ we have $\int_0^{h_R} (v + a^2)^{-p} dv \leq h_R^{1-p}/(1-p)$. This proves the estimate for the gauge transformations on the early interval $[0, h_R]$. The representatives remain in the chart, since Theorem 24.5, applied on $[0, h_R]$, bounds the contribution of the remainder to the link angles by $C_R a(h_R + a^2)^\kappa$.

Step 2: later intervals and the composed gauge transformations. We apply Lemma 25.9 at time u . If $d_a \leq \delta$, the paired estimate implies that $\sup \|B_a - B'_a\|_{C_a^1} \leq C d_a$ on every later time interval. Let g, g' be the composed gauge transformations, from the initial configurations to the current damped representatives. At time u their difference is $O(d_a)$ by Step 1. On a time interval I_i of the continuation, $g = k_i g_i$, where g_i is the value at the start of I_i and k_i solves the damping equation with initial value I . The generators of k_i and k'_i differ by at most a constant times the C_a^1 difference of the representatives. The inhomogeneous version of (25.22) therefore carries the incoming difference $\|g_i - g'_i\|$ with coefficient one and adds at most $C|I_i|d_a$. At a restart both composed gauge transformations are multiplied on the left by the same fixed unitary gauge transformation, which does not change the norm of their difference. By induction over the finitely many intervals, we obtain $\sup \|g - g'\|_\infty \leq C d_a$ through time T , including the possibly different damping gauge transformations of the two flows.

Step 3: the matrices. On a common time interval, the exponential $e^{aB_{a,e}}$ of each link e changes by at most $C a \|B_a - B'_a\|_\infty$, and unitary telescoping along a path with at most $\mathfrak{l}_i/a$ links gives

$$\|H_i(B_a) - H_i(B'_a)\|_{\text{op}} \leq C \mathfrak{l}_i \|B_a - B'_a\|_\infty,$$

where $H_i(B_a)$ is the holonomy of e^{aB_a} around $\ell_{i,a}$. The holonomy of the Wilson flow is $g(x_0)^{-1} H_i(B_a) g(x_0)$, where x_0 is the basepoint, and undoing the composed gauge transformation at x_0 adds at most $2 \|g(x_0) - g'(x_0)\|_{\text{op}}$. This proves the estimate for $d_a \leq \delta$, in the gauge of the initial configurations. For $d_a > \delta$, the distance between the two matrices is at most two, and it suffices to take $C \geq 2/\delta$.

Step 4: coarse meshes. Only finitely many lattice sides have $a > a_*$. On each of them the Wilson vector field is smooth on the compact manifold $\text{SU}(2)^{\mathcal{E}_N}$, and its solution map over a finite time is Lipschitz. To see this, we subtract the two integral equations, apply the integral form of Gronwall's inequality, and concatenate up to time $4T/a^2$. At $t = 0$ the first weighted norm implies that $\|A_0 - A'_0\|_\infty \leq a^{-\alpha} d_a$. Then the initial links differ by at most $C a^{1-\alpha} d_a$. Telescoping after the finite-dimensional flow, with $N_{\ell_{i,a}} \leq \mathfrak{l}_i/a$, gives a finite constant on each such side, and we take the maximum over these finitely many sides. This proves the estimate for all meshes. The assertions about the early damped representatives concern the interval $[0, h_R]$ and the mesh threshold of Theorem 24.5, as stated, and the common continuation is used only when $a \leq a_*$ and $d_a \leq \delta$. $\square$

Corollary 25.11. Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, $0 < \tau \leq T < \infty$, lengths $\mathfrak{l}_1, \dots, \mathfrak{l}_b > 0$, and characters $\chi_1, \dots, \chi_b$ of finite-dimensional representations of $\text{SU}(2)$. Let T_R be the local existence time of Theorem 24.5, replaced by $\min(T_R, T_0)$, and choose

$$0 < u < \frac{1}{4} \min\{T_R, T_0, \tau\}.$$

There are $a_* > 0$ and $C < \infty$, chosen after these data, such that the following holds for every mesh $a = L/n \leq a_*$. Let (x_a, y_a) and (x'_a, y'_a) be the flow data of two lattice initial potentials $A_a(0), A'_a(0)$ on the same lattice, with $\|(x_a, y_a)\|_{\mathfrak{f},a}, \|(x'_a, y'_a)\|_{\mathfrak{f},a} \leq R$ on $[0, T_0]$. Write

$$d_a = X_\alpha^a(x_a - x'_a) + Y_\alpha^a(y_a - y'_a),$$

with both norms on $[0, T_0]$ and including all first differences, and put $U_a = e^{aA_a(0)}$, $U'_a = e^{aA'_a(0)}$. Then for every choice of closed lattice paths $\ell_{r,a}$ with $aN_{\ell_{r,a}} \leq \mathfrak{l}_r$ and every choice of times $t_r \in [\tau, T]$,

$$\left(\sum_{r=1}^b |\chi_r(\text{Hol}_{\ell_{r,a}} \Psi_{t_r}^{(n)} U_a) - \chi_r(\text{Hol}_{\ell_{r,a}} \Psi_{t_r}^{(n)} U'_a)|^2 \right)^{1/2} \leq C d_a. \quad (25.23)$$

The threshold and the constant depend only on the displayed data, the chart, and the finitely many smooth DeTurck solutions selected in the proof. They do not depend on the position or shape of the paths beyond their lengths, nor on higher derivatives of the initial potentials or on any finite

approximation from which they are constructed. No convergence of the flow data as the mesh varies is assumed, and there is no additive error in the mesh. For normalized fundamental traces, take $\chi_r(V) = (1/2) \operatorname{tr} V$.

Proof. We apply Theorem 25.10 with the given lengths and with $t_- = \tau$, $t_+ = T$. Every character χ of a finite-dimensional representation of $\mathrm{SU}(2)$ is a polynomial p_χ in $(1/2) \operatorname{tr} V$. Indeed, on the diagonal matrices $\operatorname{diag}(e^{i\theta}, e^{-i\theta})$ the character is $\sum_k m_k e^{ik\theta}$ with integer weights k and multiplicities $m_k = m_{-k}$, which is a combination of the functions $\cos(k\theta)$. The recurrence for $\cos(k\theta)$ expresses these as polynomials in $\cos \theta = (1/2) \operatorname{tr} V$. Then $\sup_{[-1,1]} |p'_\chi|$ bounds the Lipschitz constant of χ as a function of the normalized trace, and the normalized trace is Lipschitz in the matrix norm. Summing the finitely many bounds in Euclidean norm, we obtain (25.23). $\square$

25.7 Parameter derivatives in a fixed initial gauge

In the weak-coupling analysis of Sections 34 to 36, the initial potential depends on finitely many real parameters, and we need two parameter derivatives of the flowed holonomy matrices, uniformly in the mesh. We first treat a single time interval on which the representative is bounded in C_a^1 , and then the singular initial interval and the continuation.

Lemma 25.12. *Assume the hypothesis of Lemma 25.5(1) for every value of a finite-dimensional real parameter, and suppose that the initial field depends twice continuously differentiablely on the parameter. For unit directions v, w write $V_v = \partial_v B_a$ and $W_{vw} = \partial_v \partial_w B_a$. If the initial C_a^1 norms of V_v and W_{vw} are bounded by b_1 and b_2 , then*

$$\sup_{t \leq h} \|V_v(t)\|_{C_a^1} \leq C_{L,h,K} b_1, \quad \sup_{t \leq h} \|W_{vw}(t)\|_{C_a^1} \leq C_{L,h,K} (b_2 + b_1^2). \quad (25.24)$$

These estimates persist through finitely many restarts with fixed sampled smooth gauge transformations. After positive waiting times they also give bounds on every given fixed order of spatial differences of B_a , V_v , and W_{vw} . The corresponding assertions hold for smooth continuum solutions on the time intervals of a continuation as in Theorem 24.10.

Proof. Step 1: differentiability. At each fixed mesh, the initial fields and the finite-dimensional analytic vector field have two continuous derivatives. Difference quotients in the integral equation converge to the solutions of the differentiated linear integral equations. Indeed, Taylor's integral formula for the vector field leaves a remainder that tends to zero uniformly on compact sets of parameters and times, and the linear Volterra inverse of Lemma 25.5(1) bounds the effect of this remainder. Applying this first to the fields and then to their first derivatives, we obtain the differentiability. The uniform bounds below do not depend on the fixed mesh.

Step 2: the equations. The derivatives satisfy

$$\begin{aligned}\partial_t V_v &= \Delta_a V_v + \mathcal{B}_a(B_a) V_v, \\ \partial_t W_{vw} &= \Delta_a W_{vw} + \mathcal{B}_a(B_a) W_{vw} + f_{vw}, \\ f_{vw} &= \mathcal{N}_a(V_v, V_w) + \mathcal{N}_a(V_w, V_v) + D^2 \mathcal{C}_a(B_a)[V_v, V_w].\end{aligned}\tag{25.25}$$

The Volterra estimate (25.13) implies the first bound in (25.24). By the bound on $\mathcal{N}_a$ and (25.11), we have $\|f_{vw}\|_\infty \leq C_K \|V_v\|_{C_a^1} \|V_w\|_{C_a^1}$, and applying the Volterra estimate to the inhomogeneous second equation gives the second bound. Note that the second variation satisfies an inhomogeneous equation, and the forcing f_{vw} must be kept.

Step 3: restarts. For a fixed smooth gauge transformation k , the map $\mathcal{G}_{a,k}(B)_i(x) = a^{-1} \text{Log}(k(x)e^{aB_i(x)}k(x+ae_i)^{-1})$ satisfies, on a bounded set of the norm C_a^1 ,

$$\|D^j \mathcal{G}_{a,k}(B)[Z_1, \dots, Z_j]\|_{C_a^1} \leq C_{k,K} \prod_{i=1}^j \|Z_i\|_{C_a^1}, \quad j = 1, 2, \tag{25.26}$$

where D differentiates in the field. The case $j = 1$ is the Lipschitz bound of Lemma 24.7; we check the case $j = 2$ directly. For $Y = k(x)e^{aB_i(x)}k(x+ae_i)^{-1}$ we have $Y - I = O_K(a)$, and the j th field derivative of Y has a factor a^j for $j = 1, 2$. By the exact product rule, the given C_a^1 norms, and the bounded smooth differences of k , a further spatial difference keeps these factors. Since the derivatives of Log are bounded near I , a differentiated term in $a^{-1} \text{Log} Y$ keeps at least one factor a to compensate the factor a^{-1} , also after one spatial difference. Terms with more differentiated factors carry further nonnegative powers of a . This proves (25.26), using only fixed smooth norms of k and a margin of the chart. Counting factors in the same way, with three field arguments, we bound $D^3 \mathcal{G}_{a,k}$ on this set: it has the factor a^3 before the division by a , and products of derivatives of the logarithm have at least the same total number of field arguments; one spatial difference again keeps these factors. Hence the first two field derivatives in (25.26) are locally Lipschitz on this set, with uniform constants, which we use below to subtract second derivatives at restarts. The chain rule at a restart gives

$$V_v^+ = D \mathcal{G}_{a,k}(B) V_v, \quad W_{vw}^+ = D \mathcal{G}_{a,k}(B) W_{vw} + D^2 \mathcal{G}_{a,k}(B)[V_v, V_w].$$

These bounds and (25.24) can be composed finitely many times. Since the gauge transformations k are fixed, no parameter dependence of a gauge choice is differentiated in these formulas.

Step 4: higher differences. We apply Theorem 25.7 to B_a , whose C_a^1 bound is given. After $m - 1$ differences of the first equation in (25.25), computed with the product rules (25.2), the only unknown term of highest order is bounded by $C_K \|D_a^m V_v\|_\infty$. The term containing $D_a^m B_a$ has the companion V_v , whose value is already bounded, and it belongs to the known forcing. Every other term contains only lower differences of V_v and

bounded differences of B_a . The field derivative of the remainder involves no difference of order larger than $m - 1$ and no negative power of the mesh. This estimate has the form of (25.20), with a known bounded forcing. As in (25.21), the weighted quantity $\sup_v \sqrt{v} \|D_a^m V_v(u_0 + v)\|_\infty$ and the integral $\int_0^v (v - r)^{-1/2} r^{-1/2} dr = \pi$ then bound $D_a^m V_v$ after a positive waiting time. In the second equation the unknown term of highest order is bounded by $C_K \|D_a^m W_{vw}\|_\infty$. Its additional forcing involves differences of V_v and V_w of order at most m , which are already bounded by the argument for the first derivatives, and the same absorption gives order m for W_{vw} . Finitely many successive waiting intervals give any given finite list of orders, and the argument for restarts extends to these orders by the exact product rule.

Step 5: continuum solutions. For smooth continuum solutions on the time intervals of a continuation, the same proof applies, with the ordinary product rule and the continuum heat estimates in Step 1 of the proof of Theorem 24.10, on a fixed physical horizon subdivided if necessary. Starting from a smooth finite-dimensional family of initial data, we take difference quotients of the continuum mild equation. The same Volterra inverse and Taylor's integral remainder give their convergence, first at order one and then at order two. If bounds through a fixed order are needed up to the initial time, they hold for the smooth family at the initial time. We then apply the product rule one order below the highest desired derivative and the same Volterra estimate; the highest term is linear, with a bounded undifferentiated companion. Induction gives the required mixed bounds in space and in the parameters, down to the initial time of each such smooth solution. $\square$

Theorem 25.13. Fix $L > 0$, $1/2 < \alpha < 2/3$, $T_0 > 0$, a finite horizon $T > 0$, and $t_* \in (0, T]$, and consider the holonomies around the three coordinate cycles of $\mathbb{T}_L^3$, based at a fixed vertex x_0 , at times in $[t_*, T]$. Let $U_0 \Subset U_1 \subset \mathbb{R}^d$ be bounded parameter domains, covered by finitely many smooth input charts with fixed positive margins; changes between these charts act on the initial potentials by gauge transformations equal to the identity at the basepoint. For each mesh $a = L/n$, let $u \mapsto \mathcal{E}_a(u) = (x_a(u), y_a(u))$ be the flow data of a lattice initial potential $A_{0,a}(u)$, twice continuously differentiable on U_1 , with

$$\sup_{a, u \in U_1} \sum_{r=0}^2 \|D_u^r \mathcal{E}_a(u)\|_{\mathbb{A}, a} \leq R_2 < \infty,$$

where the norms are taken on $[0, T_0]$ and contain both components and all their first physical differences, and the parameter derivatives are measured in a fixed Euclidean operator norm. For every u , start the damping gauge transformation at I , in the gauge of the initial configuration $e^{aA_{0,a}(u)}$. Let T_{R_2} be the local existence time of Theorem 24.5 for the bound R_2 , replaced by $\min(T_{R_2}, T_0)$.

1. There is $a_* > 0$, chosen after the displayed data and below the restrictions on the mesh in Theorem 24.5, such that the following holds for every mesh $a \leq a_*$. On the early interval

$[0, T_{R_2}]$, the remainder $r_a(u)$ and the damped representative $A_a(u, t)$ satisfy, for $r = 1, 2$,

$$\sup_{a \leq a_*, u \in U_0} \|D_u^r r_a(u)\|_{R_a} \leq C_{R_2},$$

$$\|D_u^r D_a A_a(u, t)\|_\infty \leq C_{R_2} \{(t + a^2)^{-(\alpha+1)/2} + (t + a^2)^{-\alpha} + (t + a^2)^{\kappa-1/2}\}.$$

All three powers of $t + a^2$ are integrable at zero, uniformly in the mesh. The first and second parameter derivatives of the damping gauge transformation are bounded uniformly at every vertex on this interval. The representatives remain in the chart, by the bound

$$a \sup_{t \leq T_{R_2}} \|A_a(t)\|_\infty \leq C_{R_2} \{a^{1-\alpha} + a^{2-2\alpha} + a(T_{R_2} + a^2)^\kappa\} \longrightarrow 0.$$

2. At a positive restart time, the solutions are smoothed by Theorem 25.7, and the continuations of Theorem 24.10 are used on a neighborhood of the compact closure of their possible limits; on each restart, the same fixed sampled gauge transformation is applied to all parameters in a neighborhood. The first and second parameter derivatives of the resulting based holonomy matrices of the three coordinate cycles on $[t_*, T]$, expressed in the gauge of the initial configuration, are bounded by a constant $C_{R_2, U_0, U_1, t_*, T}$ uniformly in the mesh. These matrices therefore have a uniform $C^{1,1}$ bound on smaller convex parameter charts. The conjugation at the basepoint by all damping and restart gauge transformations is included; this is a statement about matrices, not only about their traces.
3. Let $f(u)$, $u \in U_1$, be a fixed compact family in a common fixed finite-dimensional span of smooth cardinal interpolants on fixed input meshes, with coefficients twice continuously differentiable in u , and let $A_{0,a}(u) = S_a f(u)$. Then the matrices of (2) converge, uniformly in $u \in U_0$ and $t \in [t_*, T]$, to the holonomy matrices, based at x_0 , of the solution of $\partial_t A_i = \sum_j D_j F_{ji}$ with initial value $f(u)$; the proof shows that their first two parameter derivatives converge as well. On corresponding intervals of the lattice and continuum continuations, the damping generators converge uniformly away from the initial time and in L^1 down to the left endpoint. The composed damping and restart gauge transformations, with initial value I , converge at the basepoint. The same conclusions hold when the coefficients in this same fixed span depend on a and converge, together with their first two parameter derivatives, uniformly on compact subsets of U_1 . No rate that is uniform over growing families is asserted.

The constants in (1) and (2) depend only on d , U_0 , U_1 , and the margins of the input charts, on R_2 , L , α , T_0 , t_* , T , and on the chart and the margins of the continuations. The bounds for the based holonomy matrices in (2) hold for every mesh $a = L/n$, $n \geq 4$. The assertions about logarithmic representatives and their damping gauge transformations hold for $a \leq a_*$; finitely many additional meshes can be included when the corresponding representatives remain in the fixed smaller chart. The constants in (3) may also depend on the fixed family. The derivatives are real derivatives of the Wilson flow; neither a complex continuation of the flow nor a derivative of a Borel selection is involved.

We outline the proof. On the early interval, where the flow data are singular, we differentiate the fixed-point equation for the remainder $r = A - x - y$ of Theorem 24.5 twice in the parameters. The second derivative of the quadratic term contains products of two differentiated linear heat evolutions, whose singularity $v_a^{-\alpha-1/2}$ is not integrable. These products cancel against the corresponding products in the definition of r , and the remaining terms have the integrable singularity $v_a^{-3\alpha/2}$. We control the parameter derivatives of the damping gauge transformation by unitary variation of constants. After a positive time, the solutions are smooth uniformly in the mesh, and we continue them by comparison with finitely many selected smooth DeTurck solutions, as in Lemma 25.9. Along this continuation, Lemma 25.12 controls the parameter derivatives. Finally we return to the gauge of the initial configuration by conjugating with the composed gauge transformation at the basepoint, and differentiate this conjugation as well. For a fixed smooth family, the consistency results of Sections 25.2 and 25.3, differentiated twice in the parameters, identify the limit.

Proof. Step 0: setup. We work first in one of the finitely many input charts. Since changes between these charts are gauge transformations equal to the identity at the basepoint, they do not change the gauge in which the matrices are expressed. We choose an intermediate parameter domain V with $\overline{U}_0 \subset V \Subset U_1$. We take all difference quotients inside U_1 and first establish all bounds on V ; the minimal parameter margin and the constants of the charts and of the conversion of Euclidean norms enter the final constant. In formulas involving directions, v, w have Euclidean norm at most one. Set

$$h_0 = T_{R_2}, \quad p = \frac{3}{2}\alpha < 1,$$

and recall $\beta = \alpha - 1/2$, $\kappa = 1 - 3\alpha/2 = 1 - p$, and $v_a(t) = t + a^2$. Using the contraction proof of Theorem 24.5, we may decrease the local existence time while restricting the horizon of the flow data. All weighted norms and all the gains of the contraction then only decrease, and none of the bounds on the early remainder becomes larger. We choose h_0 small enough for the contractions below, and take $a^2 \leq h_0$ and $a \leq 1$. All heat integrals in the early argument have physical horizon h_0 .

Step 1: the differentiated equation on the early interval. We write $x = x_a(u)$, $y = y_a(u)$, and $A = x + y + r$, and let

$$(\mathcal{T}_a f)(t) = \int_0^t P_{t-s}^a f(s) ds, \quad F_a((x, y), r) = \mathcal{T}_a \{ \mathcal{N}_a(A, A) - \mathcal{N}_a(x, x) + \mathcal{C}_a(A) \}.$$

The remainder satisfies the fixed-point equation $r = F_a((x, y), r)$ exactly, with no term of the vector field truncated. By Theorem 24.5, r lies in a ball of fixed radius $D = D(R_2)$ in the norm R_a . Using the estimates of the local contraction on a slightly larger fixed ball, we obtain

$$\|A(t)\|_\infty \leq C_{R_2, D} v_a(t)^{-\alpha/2}, \quad \|D_a A(t)\|_\infty \leq C_{R_2, D} v_a(t)^{-(\alpha+1)/2}. \quad (25.27)$$

Indeed, the exponents β, α of y are smaller than the corresponding exponents $\alpha/2, (\alpha + 1)/2$ of x , the remainder has exponents $\kappa, \kappa - 1/2$, and $v_a \leq 2$. The derivative in the remainder variable is

$$\mathcal{L}_a \rho = D_r F_a((x, y), r) \rho = \mathcal{T}_a \{ \mathcal{B}_a(A) \rho \}.$$

If $\|\rho\|_{R_a} \leq 1$, its source is bounded by

$$C_{R_2, D} \{ v_a^{\kappa - (\alpha + 1)/2} + v_a^{\kappa - \alpha} \};$$

the first term comes from the two ordered quadratic products and the second from $D\mathcal{C}_a(A)\rho$. Both exponents exceed -1 . Bounding the value of the convolution and one difference of the heat semigroup, and then applying (25.1), we obtain

$$\|\mathcal{L}_a\|_{R_a \rightarrow R_a} \leq C_{R_2, D} \{ (h_0 + a^2)^{(1-\alpha)/2} + (h_0 + a^2)^{1-\alpha} \} \leq \frac{1}{2}, \quad \|(I - \mathcal{L}_a)^{-1}\| \leq 2. \quad (25.28)$$

We choose h_0 , and then the restriction $a \leq a_*$ on the mesh, so that this holds on the slightly larger ball. The inverse is given by the norm-convergent Neumann series, and it acts on the whole Volterra integral.

For a direction (ξ, η) in the flow data, the derivative of the source with r fixed is

$$\begin{aligned} J_a(\xi, \eta) &= \mathcal{N}_a(x, \eta) + \mathcal{N}_a(\eta, x) \\ &\quad + \mathcal{N}_a(y + r, \xi + \eta) + \mathcal{N}_a(\xi + \eta, y + r) + D\mathcal{C}_a(A)(\xi + \eta); \end{aligned} \quad (25.29)$$

the products $\mathcal{N}_a(x, \xi)$ and $\mathcal{N}_a(\xi, x)$ cancel against the derivative of $\mathcal{N}_a(x, x)$. If $b = \|\xi\|_{X_a^a} + \|\eta\|_{Y_a^a}$, then

$$\|J_a(\xi, \eta)(t)\|_\infty \leq C_{R_2, D} b v_a(t)^{-p}, \quad \|\mathcal{T}_a J_a(\xi, \eta)\|_{R_a} \leq C_{R_2, D} b. \quad (25.30)$$

To verify the time exponents, recall that $\|\mathcal{N}_a(u, v)\|_\infty \leq C\|u\|_\infty\|D_a v\|_\infty$. The products of one factor x or ξ with one factor y or η have worst exponents $-\alpha/2 - \alpha = -p$ and $-\beta - (\alpha + 1)/2 = -p$. Two factors of the type y, η give the exponent $1/2 - 2\alpha$. A remainder factor against a factor of type x, ξ gives the same exponent $1/2 - 2\alpha$, and against a factor of type y, η the exponent $\kappa - \alpha$. All of these are at least $-p$ for $\alpha < 2/3$. Finally, (25.11) bounds the last term by $C\|A\|_\infty^2\|\xi + \eta\|_\infty$, again with exponent $-p$. Since $\kappa = 1 - p$, the convolutions have exactly the powers v_a^κ and $v_a^{\kappa - 1/2}$ of the norm R_a .

For fixed a , the initial value $A_0(u) = x_a(u, 0)$ is twice continuously differentiable in u . The differential equation in the chart, or difference quotients of its contracting integral equation, allow us to differentiate $r = F_a((x, y), r)$ twice. To justify the second route, we subtract two fixed-point identities and integrate the derivative in the field along the segment joining them. The difference of the remainders is then obtained by applying an inverse of norm at most two to the difference in the flow data. By Taylor's integral

remainder and (25.30), the first difference quotient converges. Repeating this with the first differentiated equation, and using the second field derivatives of the analytic vector field, we obtain the convergence of the second difference quotient. This justifies the differentiation before we derive the uniform bounds below.

We write $\xi_v = \partial_v x$, $\eta_v = \partial_v y$, $\rho_v = \partial_v r$, and $Z_v = \partial_v A$. The first derivative satisfies

$$(I - \mathcal{L}_a)\rho_v = \mathcal{T}_a J_a(\xi_v, \eta_v), \quad Z_v = \xi_v + \eta_v + \rho_v. \quad (25.31)$$

By hypothesis, $b \leq R_2$ here, and (25.28) and (25.30) imply that $\|\rho_v\|_{R_a} \leq C_{R_2}$. The second derivative satisfies, with all inhomogeneous terms,

$$\begin{aligned} (I - \mathcal{L}_a)\rho_{vw} &= \mathcal{T}_a \{J_a(\xi_{vw}, \eta_{vw}) + K_{vw}\}, \\ K_{vw} &= \mathcal{N}_a(Z_v, Z_w) + \mathcal{N}_a(Z_w, Z_v) - \mathcal{N}_a(\xi_v, \xi_w) - \mathcal{N}_a(\xi_w, \xi_v) \\ &\quad + D^2\mathcal{C}_a(A)[Z_v, Z_w]. \end{aligned} \quad (25.32)$$

The two subtracted products of differentiated linear heat evolutions must be kept: bounding the two unsubtracted products separately would produce the exponent $-\alpha - 1/2$, which is not integrable for $\alpha > 1/2$. We write $Z_v = \xi_v + \delta_v$ with $\delta_v = \eta_v + \rho_v$. Each ordered quadratic difference in K_{vw} then contains at least one factor δ ; for instance,

$$\mathcal{N}_a(Z_v, Z_w) - \mathcal{N}_a(\xi_v, \xi_w) = \mathcal{N}_a(\xi_v, \delta_w) + \mathcal{N}_a(\delta_v, \xi_w) + \mathcal{N}_a(\delta_v, \delta_w).$$

The part η of δ has the exponents of the quadratic Picard term, and the part ρ has the exponents of the remainder, which were bounded above. By the exponent calculation for (25.30), the sum of these terms is bounded by $C_{R_2} v_a^{-p}$. For the last term we use $\|D^2\mathcal{C}_a(A)[Z_v, Z_w]\|_\infty \leq C\|A\|_\infty\|Z_v\|_\infty\|Z_w\|_\infty$; each factor has the value bound $Cv_a^{-\alpha/2}$, which again gives the exponent $-p$. In total, the second forcing has, before inversion, R_a norm at most C_{R_2} , and by (25.28) we have $\|\rho_{vw}\|_{R_a} \leq C_{R_2}$. These directional estimates give the asserted bounds on both parameter derivatives in Euclidean operator norm, with all directions and all first physical differences included.

Adding the differentiated flow data to these two estimates for the remainder, we find, for $r = 1, 2$,

$$\|D_u^r D_a A(u, t)\|_\infty \leq C_{R_2} \{v_a(t)^{-(\alpha+1)/2} + v_a(t)^{-\alpha} + v_a(t)^{\kappa-1/2}\}. \quad (25.33)$$

The same bound, with a possibly larger C_{R_2} , holds for $r = 0$. All three negative exponents are smaller than one in absolute value. In particular, the integrals of the right side over $[0, \epsilon]$ are at most

$$C_{R_2} \omega(\epsilon), \quad \omega(\epsilon) = \epsilon^{(1-\alpha)/2} + \epsilon^{1-\alpha} + \epsilon^{\kappa+1/2}, \quad (25.34)$$

uniformly in a , for $0 < \epsilon \leq h_0$. Finally, the representatives remain in the chart because

$$a \sup_{t \leq h_0} \|A(t)\|_\infty \leq C_{R_2} \{a^{1-\alpha} + a^{2-2\alpha} + a(h_0 + a^2)^\kappa\} < r_*;$$

In the last term we used $\sup_{0 \leq t \leq h_0} \|r(t)\|_\infty \leq C_{R_2}(h_0 + a^2)^\kappa$, because the upper bound $C_{R_2}(t + a^2)^\kappa$ is increasing in t . The differentiated mild solution is therefore, for every parameter, the damped representative of the Wilson flow.

Step 2: the damping gauge transformation on the early interval. The generator in physical units is linear in the potential:

$$\lambda_a = a^{-2}\phi(aA) = -\sum_i D_{a,i}^- A_i, \quad \partial_t k_a = \lambda_a k_a, \quad k_a(0) = I. \quad (25.35)$$

It is anti-Hermitian. Its parameter derivatives of order zero, one, and two are bounded in $L^1([0, h_0]; \ell^\infty)$ by (25.33), and their integrals over $[0, \epsilon]$ are bounded by (25.34). At any vertex write $l_v = \partial_v k_a$ and $m_{vw} = \partial_v \partial_w k_a$. Direct differentiation gives

$$\begin{aligned} \partial_t l_v &= \lambda_a l_v + (\partial_v \lambda_a) k_a, & l_v(0) &= 0, \\ \partial_t m_{vw} &= \lambda_a m_{vw} + (\partial_v \lambda_a) l_w + (\partial_w \lambda_a) l_v + (\partial_{vw} \lambda_a) k_a, & m_{vw}(0) &= 0. \end{aligned} \quad (25.36)$$

The homogeneous propagator is unitary. If $B_j = \int_0^{h_0} \|D_u^j \lambda_a\|_\infty dt$ for $j = 1, 2$, variation of constants gives

$$\sup_t \|l_v(t)\|_\infty \leq B_1, \quad \sup_t \|m_{vw}(t)\|_\infty \leq B_2 + 2B_1^2.$$

These bounds hold at every vertex, uniformly in the mesh, with no exponential of the size of λ_a , and they include the two products of first derivatives in the second equation. Differentiating $k_a^{-1} k_a = I$, we also bound both derivatives of k_a^{-1} . This completes the proof of (1).

Step 3: bounds on later intervals. Fix $0 < s < \min(h_0/2, t_*/2)$, which leaves room for a positive waiting time before s . By the early estimates we have uniform C_a^1 bounds on A , Z_v , and Z_{vw} on $[s/2, h_0]$. We apply Theorem 25.7 to A , and then Lemma 25.12 on intervals after positive waiting times inside the early interval. At time s these give bounds on every given fixed order of differences of the three fields, with constants depending on that order, on L, R_2, s , and on the parameter data.

Next, we justify the use of finitely many selected smooth DeTurck solutions, although we have not assumed that the flow data converge as the mesh varies. Let $\mathcal{K}$ be the set of smooth connections whose derivatives of every order satisfy the bounds just obtained for $A_a(s, u)$; it depends only on L, R_2, s . By Lemma 25.6, it is compact in C^3 , it contains every limit of a sequence $A_{a_j}(s, u_j)$ with $a_j \rightarrow 0$, and

$$\sup_u \inf_{b \in \mathcal{K}} \|A_a(s, u) - S_a b\|_{C_a^1} \longrightarrow 0 \quad (a \rightarrow 0). \quad (25.37)$$

Only the spatial bounds at time s are needed here.

We apply part (4) of Lemma 25.5 to $\mathcal{K}$ with horizon $T - s$. It provides $\eta_{\mathcal{K}} > 0$ and finitely many continuations, built from the DeTurck solutions of Theorem 24.10 with

$r = 3$, whose needed C^5 norms are finite. It also provides one restriction $a \leq a_{\mathcal{K}}$ and one C_a^1 bound for all comparisons with lattice fields within $\eta_{\mathcal{K}}$ of the sample of an element of $\mathcal{K}$. Then (25.37) implies that for every sufficiently small mesh and every u there exists $b \in \mathcal{K}$ with $\|A_a(s, u) - S_a b\|_{C_a^1} \leq \eta_{\mathcal{K}}/4$. By the early bound on the first parameter derivative, we have

$$\|A_a(s, u') - A_a(s, u)\|_{C_a^1} \leq C_{R_{2,s}}|u' - u|$$

on every segment in V . Hence, for $|u' - u|$ smaller than the input margin and smaller than $\eta_{\mathcal{K}}/(4C_{R_{2,s}})$, the field $A_a(s, u')$ lies within $\eta_{\mathcal{K}}/2$ of the same $S_a b$. Then all parameters in this neighborhood of u can use the same continuation, with the same fixed sampled gauge transformations. Then the individual estimate (25.14) gives the C_a^1 bound for the representatives through time T . This estimate applies to an incoming small discrepancy in C_a^1 ; it does not require a link error of order a^2 .

The two initial parameter derivatives at time s already have uniform C_a^1 bounds. Applying (25.24) and (25.26) along this finite continuation, we obtain one uniform C_a^1 bound for the first and second parameter derivatives on every time interval, including its endpoints. Only after the C_a^1 bound for the representative has been obtained in this way do we apply the last part of Theorem 25.7. Together with the last part of Lemma 25.12, it gives further spatial orders at every given positive waiting time.

Step 4: composed gauge transformations and matrices. On a later time interval I_j , let $g_j(t)$ be the damping gauge transformation with initial value I at the start of I_j . By (25.35) and the C_a^1 bounds just proved, its generator and the first two parameter derivatives of the generator are bounded. Then the equations (25.36) bound the first two parameter derivatives of g_j on I_j . We keep track of the whole gauge transformation at every vertex, and not only of its value modulo a conjugation at the basepoint. If Q_j is the composed gauge transformation at the start of I_j , then

$$Q(t) = g_j(t)Q_j, \quad Q_{j+1} = k_j g_j(h_j)Q_j, \quad Q_0 = k_a(s), \quad (25.38)$$

where h_j is the length of I_j and k_j is the fixed sampled gauge transformation applied at its end. The order of these products follows from $(U^k)^l = U^{lk}$. Since all factors are unitary, the k_j do not depend on the parameter, and there are finitely many of them, the first and second product rules bound $D_u Q$ and $D_u^2 Q$ at every vertex. This includes the early damping gauge transformation with its initial value I .

Let $H_a^{\text{rep}}(t, u)$ be the holonomy of one of the coordinate cycles in the current damped representative. If the cycle has N links and $aN \leq \ell$, differentiating a single exponential gives

$$\|\partial_v e^{aA_e}\|_{\text{op}} \leq Ca\|Z_v\|_{\infty}, \quad \|\partial_{vw} e^{aA_e}\|_{\text{op}} \leq Ca\|Z_{vw}\|_{\infty} + Ca^2\|Z_v\|_{\infty}\|Z_w\|_{\infty};$$

these follow by integrating the derivative of the exponential once and twice along its unitary factors. An inverse factor satisfies bounds of the same type, by differentiating the identity for the inverse. The product rule over the links of the cycle gives

$$\begin{aligned}\|\partial_v H_a^{\text{rep}}\|_{\text{op}} &\leq C\ell\|Z_v\|_{\infty}, \\ \|\partial_{vw} H_a^{\text{rep}}\|_{\text{op}} &\leq C\ell\|Z_{vw}\|_{\infty} + C(\ell a + \ell^2)\|Z_v\|_{\infty}\|Z_w\|_{\infty}.\end{aligned}\tag{25.39}$$

The term ℓa counts the N second derivatives of a single factor, and the term ℓ^2 counts the at most $N(N-1)$ products of two first derivatives. Since every observation time is later than s , the fields in these estimates are uniformly bounded by Step 3. At the basepoint x_0 , the holonomy matrix of the Wilson flow in the gauge of the initial configuration is

$$H_a(t, u) = Q(t, x_0)^{-1} H_a^{\text{rep}}(t, u) Q(t, x_0).\tag{25.40}$$

Its two derivatives include the derivatives of both factors $Q(t, x_0)^{\pm 1}$. By the bounds for Q and Q^{-1} and (25.39), all these terms are bounded uniformly. At a restart time both representations describe the same holonomy matrix of the Wilson flow, so their parameter derivatives agree there. The choice of a reference neighborhood was used only to prove a bound for the map $u \mapsto H_a(t, u)$, which is defined independently of this choice. The choice is constant on the parameter neighborhood where the derivatives were computed, and we take no derivative of it. Taking the maximum over the finitely many possible continuations, we obtain the uniform bounds on the first and second derivatives on U_0 . On a smaller convex chart, the fundamental theorem of calculus along segments and the bound on the second derivative imply that $\|DH_a(u) - DH_a(u')\| \leq C|u - u'|$. Together with $\|H_a\|_{\text{op}} = 1$, this is the uniform $C^{1,1}$ bound in (2).

Step 5: smooth families, consistency of the fields. Let $f(u)$ be a family in the common fixed finite-dimensional span of interpolants as in (3). On $\bar{V} \Subset U_1$, its finitely many coefficients and their first two parameter derivatives are bounded. Consequently its spatial derivatives of every fixed order, and its mixed derivatives with up to two parameter derivatives, have uniform finite bounds there. These bounds also hold uniformly for all sufficiently small a when the coefficients depend on a and converge through order two uniformly on compact subsets of U_1 . Lemma 25.2 and Lemma 16.9 apply to f , to its parameter derivatives, and to their ordered products. On meshes that do not refine the fixed input lattices, we use the general smooth-input bounds of Lemma 16.9 in place of its bounds for interpolants. The proof of Lemma 25.2 uses only these bounds and the fixed smooth norms of the inputs, so its conclusions hold on those meshes as well. In particular, the quadratic Picard term satisfies the exact identities

$$\begin{aligned}\partial_v y_a &= \mathcal{Q}_a(\partial_v x_a, x_a) + \mathcal{Q}_a(x_a, \partial_v x_a), \\ \partial_{vw} y_a &= \mathcal{Q}_a(\partial_{vw} x_a, x_a) + \mathcal{Q}_a(x_a, \partial_{vw} x_a) + \mathcal{Q}_a(\partial_v x_a, \partial_w x_a) + \mathcal{Q}_a(\partial_w x_a, \partial_v x_a).\end{aligned}$$

Hence the bounds on the flow data through two parameter derivatives, and the consistency of both components of the flow data, hold for this family, with a constant R_2 depending on the family. All heat comparisons are on the same fixed physical horizon. This also covers coefficients that converge as described.

Theorem 24.10 with $r = 3$ applies to the compact family of smooth initial data, and on each of its continuations the gauge transformations do not depend on u . For assertion (3), decrease the early time h_0 so that $[0, h_0]$ lies strictly before the first restart of every one of these finitely many continuations, and reduce the mesh threshold to preserve $a^2 \leq h_0$. On this interval the lattice continuation and the early damped representative coincide by uniqueness for their finite-dimensional equation; the continuum continuation is the unrestarted DeTurck solution from $f(u)$. On subsequent intervals we use the damping generators in the corresponding continued gauges on both lattices. By the continuum part of Lemma 25.12, the DeTurck solutions $b_j(u)$ on the time intervals have two parameter derivatives, with all the spatial bounds needed below. The bounds of the smooth initial family and the finitely many fixed gauge transformations give these bounds also at the left endpoints of the intervals. This verifies the regularity needed to differentiate the residual of the sampled smooth solution, without deducing it from the Lipschitz statement of Theorem 24.10 alone.

Let B_a be the lattice representative and $b_a = S_a b$ the sample of the selected DeTurck solution on a common time interval. Expanding the vector field, we obtain for $r = 0, 1, 2$

$$\sup_{u,t} \|D_u^r \{ \partial_t b_a - \Delta_a b_a - \mathcal{N}_a(b_a, b_a) - \mathcal{C}_a(b_a) \} \|_\infty \leq Ca. \quad (25.41)$$

This follows by differentiating the finite analytic expression and its Taylor remainder with respect to u . The integral remainder is bounded by the corresponding mixed spatial and parameter derivatives of the fixed smooth family. The linear part is central-difference consistency, the quadratic part is the calculation for $\mathcal{N}_a$ in Lemma 25.2, and the remaining terms form the analytic remainder treated in Lemma 25.3. The first physical difference of the residual is consistent as well, since the smooth spatial derivatives of one more order are bounded.

We now prove stability of the derivatives. For the value we use (25.14). Recall that $\mathcal{B}_a(B)V$ is the linearization, and let $f_a(B; V, W)$ be defined by the last line of (25.25), with V, W as the two first-derivative arguments. On a common bounded set of the norm C_a^1 , the displayed products and (25.11) imply

$$\|(\mathcal{B}_a(B) - \mathcal{B}_a(C))V\|_\infty \leq C_K \|B - C\|_{C_a^1} \|V\|_{C_a^1},$$

and the difference of f_a at two triples is at most

$$C_K (\|B - C\|_{C_a^1} \|V\|_{C_a^1} \|W\|_{C_a^1} + \|V - V'\|_{C_a^1} \|W\|_{C_a^1} + \|V'\|_{C_a^1} \|W - W'\|_{C_a^1})$$

when the remaining fields are bounded by K . For the remainder, this uses the bound on the third derivative in (25.11). We subtract the equations for the first derivatives, keep the linear operator at B_a , and place the difference of the coefficients and the differentiated residual in the forcing. By the Volterra inverse of Lemma 25.5(1), the C_a^1 difference of the first derivatives is at most C times the initial difference plus $C(a + \sup \|B_a - b_a\|_{C_a^1})$. We subtract the equations for the second derivatives in the same way. By the bound for f_a , the additional forcing is at most a constant times the errors of the fields and of the first derivatives already controlled, plus the second differentiated residual. Applying the Volterra inverse again, with the same fixed multiplicative constant on the incoming second-derivative error, we obtain the convergence of this difference as well. At a fixed sampled gauge transformation, the consistency error and its first two parameter derivatives are $O(a)$ in C_a^1 . To see this, we expand $k(x + ae_i)$ to first order in the formula for $\mathcal{G}_{a,k}$ and differentiate twice in the bounded smooth family. The remainder of second order, before division by a , is $O(a^2)$, and one spatial difference uses one more smooth derivative. Together with (25.26), this propagates the three error bounds across the finite continuation.

For initial values that are exact samples, the three initial errors vanish. For converging finite coefficients they are $O(\delta_a)$ with $\delta_a \rightarrow 0$, in all needed fixed spatial norms. The preceding argument then gives, on every common time interval,

$$\sup_{u,t} \|D_u^r(B_a - S_a b)\|_{C_a^1} \leq C(a + \delta_a), \quad r = 0, 1, 2. \quad (25.42)$$

For $r = 0$, the sharper link consistency of Theorem 25.4 is also available, since its hypothesis of an initial link error $O(a^2)$ holds at every restart from an exact sample. The estimate in C_a^1 above is used for the parameter derivatives and for the generators.

Step 6: smooth families, generators, gauge transformations, and matrices. We verify the convergence of the generators in L^1 on the early interval directly from the weighted estimates, without a uniform unweighted estimate at time zero. On each continuation interval, let λ_a and $\lambda = -\sum_i \partial_i b_i$ be the generators of its matched lattice and continuum representatives, with the sign fixed by (25.35). By the choice of h_0 in Step 5, these are the original early generators on $[0, h_0]$. On every interval $[\epsilon, h_0]$, by the consistency of the values and first differences, we have

$$\sup_u \|D_u^r \lambda_a - S_a D_u^r \lambda\|_\infty \longrightarrow 0, \quad r = 0, 1, 2.$$

By (25.34), the integral of each lattice term over $[0, \epsilon]$ is at most $C_{R_2} \omega(\epsilon)$. The limiting continuum bound has the same property, by the limit of the same estimates at positive times or by the stronger smooth bounds of the fixed family. Taking first $a \rightarrow 0$ at fixed ϵ and then $\epsilon \downarrow 0$, we obtain

$$\sup_u \int_0^{h_0} \|D_u^r \lambda_a - S_a D_u^r \lambda\|_\infty dt \longrightarrow 0, \quad r = 0, 1, 2. \quad (25.43)$$

On later time intervals the generators and their two parameter derivatives are bounded by the C_a^1 estimates of Step 3, and (25.42) implies the analogous convergence in L^1 on each whole interval.

On each continuation interval, reset time to zero at its left endpoint and let k solve $\partial_t k = \lambda k$ with $k(0) = I$ in the continuum; define k_a on the matching lattice interval in the same way. Then $\partial_t(k^{-1}) = k^{-1} \sum_i \partial_i b_i$, and k^{-1} is the gauge transformation that undoes the damping in Theorem 24.10, with the convention used there. At every vertex, unitary variation of constants gives

$$\sup_{t,u} \|k_a(t,u) - S_a k(t,u)\|_{\text{op}} \leq \sup_u \int \|\lambda_a - S_a \lambda\|_{\infty} dt \longrightarrow 0. \quad (25.44)$$

For the first derivative we use its integral formula from (25.36). Its homogeneous propagators $k_a(t)k_a(r)^{-1}$ converge uniformly by (25.44), its source generators converge in L^1 by (25.43), and k_a converges uniformly. Subtracting the two integrals, we bound the difference by the L^1 error of the generators plus a uniform error of the factors times a bounded L^1 norm, which proves the convergence of the first derivative. For the second derivative, the two additional sources are $(\partial_v \lambda_a)l_w$ and $(\partial_w \lambda_a)l_v$. Since the first derivatives just treated converge uniformly and the generators converge in L^1 , the same subtraction proves the convergence of the second derivative; the term $(\partial_{vw} \lambda_a)k_a$ is handled by its own L^1 convergence. Since the fixed sampled gauge transformation at a restart has the same value at the basepoint as its continuum counterpart, the finite products (25.38) converge at the basepoint, together with their first two parameter derivatives.

On each of the three coordinate cycles, the physical path is the same coordinate circle, of length L and with the same basepoint, for every mesh. Using the estimate for smooth edge transport in Lemma 24.7 and unitary telescoping, we compare the product of the exponentials of the sampled selected DeTurck solution with its smooth holonomy. To include derivatives, we regard this product as the parallel transport, along the same coordinate circle, for a piecewise constant coefficient. This coefficient and its two parameter derivatives differ from the corresponding smooth coefficients by $O(a)$, using one smooth spatial derivative. Differentiating the one-dimensional transport equation once and twice, we find the same linear and quadratic forcing structure as in (25.36), now over the fixed interval of arclength L . By unitary variation of constants and subtraction, the products and their two parameter derivatives converge at rate $O(a)$. The errors between the lattice fields and the samples in (25.42) add $O(a + \delta_a)$, by the same transport argument or by (25.39) and its version for differences. Finally we undo the gauge transformations at the basepoint by (25.40). The convergence at the basepoint proved above identifies the holonomy matrices of the Wilson flow with the holonomy matrices of the covariant continuum flow, in the gauge of the initial configuration, uniformly over the compact smooth family. This proves (3).

Step 7: dependence of the constants and coarse meshes. All restrictions on the mesh and all constants in Steps 5 and 6 may depend on the fixed smooth family and on the displayed bounds. All bounds in Steps 1–4 used only R_2 , the parameter domains, the waiting times, the horizon, the torus, and the margins of the finitely many continuations. For the finitely many meshes with $a > a_*$, the initial fields and their two parameter derivatives are bounded by the weighted norm at time zero, since these meshes are bounded below. Since the Wilson vector field and its first two derivatives are bounded on the corresponding compact products of copies of $SU(2)$, the real integral equations for the derivatives give finite uniform bounds on the fixed horizon. This gives the bounds for the based holonomy matrices on these meshes, and we take a maximum over this finite set. For any additional meshes on which the damped representatives remain in the fixed smaller chart, the positive chart margin also gives bounds for their logarithms and for the equations of their gauge transformations. $\square$

25.8 Flows with the modified plaquette potential

We now treat the flows of the plaquette potential $q + cq^2$ with $0 \leq c \leq C_0$, which are used in Section 29 to show that the limit does not depend on the choice of smoothing within this family. As in Theorem 24.12, let

$$\mathcal{A}_c(U) = \sum_{p \in \mathcal{P}_N} \{q(U_p) + cq(U_p)^2\}, \quad \Psi_t^{(N,c)} = \Phi_{4t/a^2}^{\mathcal{A}_c}, \quad a = L/N,$$

where $\Phi^{\mathcal{A}_c}$ is the flow of the vector field $-\nabla \mathcal{A}_c$ for the product metric; thus $\Psi^{(N,0)} = \Psi^{(N)}$. By Lemma 24.11, the damped representatives of this flow solve

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + F_a(A) + E_{a,c}(A)$$

in a smaller chart, where $F_a(A) = a^{-3}R_3(aA) = \mathcal{C}_a(A)$ is the cubic-and-higher remainder and $E_{a,c}$ is the additional analytic term coming from cq^2 . Since the linear and quadratic parts are the same as for the Wilson flow, the flow data are unchanged. The additional term satisfies the bounds on field derivatives (24.32), the bounds on differences (24.33), and the stronger bound (24.34), which makes it $O(a^3)$ on bounded sets of the norm C_a^1 . Only bounded $c \geq 0$ is considered.

Lemma 25.14. *Fix $L > 0$, $0 \leq C_0 < \infty$, and a smaller chart.*

1. *For flow data bounded by R , the solutions of Theorem 24.12 have bounds on every fixed order of physical differences on every interval after a positive waiting time, uniformly in a and $c \in [0, C_0]$. The same holds on any restarted time interval for which a uniform C_a^1 bound is given, with the dependence stated in Theorem 25.7 and an additional dependence on C_0 .*

2. Let b be a fixed smooth periodic DeTurck solution on $[0, h]$ whose spatial derivatives through order five are bounded by K_b . There are $\eta_b, a_b > 0$ and $K'_b < \infty$, depending only on L, h, K_b, C_0 , and the chart, such that, whenever

$$\|B_a(0) - S_a b(0)\|_{C_a^1} \leq \eta \leq \eta_b, \quad a \leq a_b,$$

the damped flow for $\mathcal{A}_c$ exists in the chart through time h and

$$\sup_{t \leq h} \|B_a(t) - S_a b(t)\|_{C_a^1} \leq K'_b(\eta + a), \quad \sup_{t \leq h} \|B_a(t)\|_{C_a^1} \leq K'_b. \quad (25.45)$$

This holds uniformly for $c \in [0, C_0]$, and also on each time interval of a fixed continuation as in Theorem 24.10, with its fixed sampled gauge transformations at the restarts. In particular, every sufficiently small initial discrepancy in C_a^1 is allowed.

The constants are deterministic and do not depend on the mesh below the thresholds.

Proof. Part (1). The local solution of Theorem 24.12 gives the initial C_a^1 bound after a positive waiting time. Suppose that differences through order $m - 1$ have been bounded by $K_1, \dots, K_{m-1}$ on a preceding interval. By the product rules (25.2), we have

$$\|D_a^{m-1} \mathcal{N}_a(B, B)\|_\infty \leq P_m(K_1, \dots, K_{m-1}) + C_m K_1 \|D_a^m B\|_\infty,$$

since the only factor of order m occurs when all additional differences fall on the factor that already carries a difference, and its companion is a bounded value of B . By the composition rule of Lemma 24.2 and by (24.33), both analytic terms F_a and $E_{a,c}$ contribute only a bounded function of $K_1, \dots, K_{m-1}$. Indeed, their arguments carry no difference to begin with, and all their powers of the mesh are nonnegative. We start the mild equation at the left endpoint u_0 of the preceding interval, commute $m - 1$ differences onto the initial value and onto the source, and place one difference on the heat semigroup. For $Z_m(d) = \sup_{0 < v \leq d} \sqrt{v} \|D_a^m B(u_0 + v)\|_\infty$, using the bound for one difference of the heat semigroup, we obtain

$$Z_m(d) \leq C_m K_{m-1} + C_m P_m d + C_m K_1 \sqrt{d} Z_m(d),$$

where for the highest term we used $\int_0^v (v - r)^{-1/2} r^{-1/2} dr = \pi$. At fixed mesh, $Z_m(d)$ is finite before the absorption. We choose d so that the last coefficient is at most $1/2$, and cover the smaller interval by intervals of this length. By induction, every fixed order is bounded, with uniform constants of the stated dependence.

Part (2). Let $b_a = S_a b$, and define

$$e_{a,c} = \partial_t b_a - \Delta_a b_a - \mathcal{N}_a(b_a, b_a) - F_a(b_a) - E_{a,c}(b_a).$$

The residual of Theorem 25.4(1) bounds the residual for the Wilson flow in physical units by $K_{K_b} a$, after dividing the dimensionless residual $O(a^4)$ of the link logarithms by

a^3 . By (24.34), the additional term is at most $K_{K_b, C_0} a^3$. Hence $\sup_t \|e_{a,c}\|_\infty \leq Ka$, and no difference of this residual will be needed. Let $Z = B_a - b_a$, and stop at the first time at which $\|Z\|_{C_a^1} = 1$. Apart from $-e_{a,c}$, the source of order zero in the equation for Z is

$$\mathcal{N}_a(b_a, Z) + \mathcal{N}_a(Z, b_a) + \mathcal{N}_a(Z, Z) + (F_a + E_{a,c})(b_a + Z) - (F_a + E_{a,c})(b_a).$$

By the bound on $\mathcal{N}_a$ and the bounds on the cubic field derivatives, its supremum norm is at most $K_{K_b, C_0} \|Z\|_{C_a^1}$ on this interval. The stopped comparison with the smooth DeTurck solution b and the induction over restarts in the proof of Lemma 25.5 apply with this larger local Lipschitz constant and the residual just bounded. They prove (25.45), remove the stopping condition, and give the assertion for continuations, with the additional dependence on C_0 . $\square$

Lemma 25.15. *Fix $L > 0$, $1/2 < \alpha < 2/3$, $0 \leq R, C_0 < \infty$, $0 < t_* \leq T < \infty$, and $0 \leq \ell < \infty$. There is a deterministic function $\omega(a)$ with $\omega(a) \rightarrow 0$ as $a \downarrow 0$ with the following property. Let $A_{0,a}$ be a lattice initial potential whose flow data on the common early interval $[0, T_{R, C_0}]$ of Theorem 24.12 have norm at most R , and let $U_a^c(t) = \Psi_t^{(N, c)}(e^{aA_{0,a}})$. Then, for every sufficiently small mesh,*

$$\sup_{0 \leq c \leq C_0} \sup_{t_* \leq t \leq T} \sup_{\substack{\ell_a \text{ closed path} \\ aN_{\ell_a} \leq \ell}} \left| \frac{1}{2} \text{tr Hol}_{\ell_a} U_a^c(t) - \frac{1}{2} \text{tr Hol}_{\ell_a} U_a^0(t) \right| \leq \omega(a). \quad (25.46)$$

The same holds for any fixed finite collection of characters of finite-dimensional representations of $SU(2)$. No approximation or area hypothesis on the closed paths is needed, since paths in the same lattice are compared. The function ω may depend on the displayed data, on the representations, and on the chart; no rate is asserted.

Proof. Fix $0 < s < \min(t_*, T_{R, C_0})$ such that there is a positive waiting time before s . The early comparison (24.36), evaluated at time s , gives

$$\sup_c \|A_a^c(s) - A_a^0(s)\|_{C_a^1} \longrightarrow 0$$

uniformly over the initial potentials considered. We prove the remaining uniform assertion by contradiction, which avoids the need for a single smooth continuation that is uniform over arbitrary families with bounds of finite order. Suppose that the assertion fails. Then there exist meshes $a_j \downarrow 0$, initial potentials with flow data bounded by R , parameters c_j , closed paths, and times that exhibit a fixed positive discrepancy. On a fixed interval before s , all fixed differences of the Wilson fields $A_{a_j}^0$ are bounded by Theorem 25.7.

By Lemma 25.6, applied at time s , there exist a subsequence and a smooth periodic b_s such that

$$\|A_{a_j}^0(s) - S_{a_j} b_s\|_{C_{a_j}^1} \longrightarrow 0.$$

Only the spatial bounds at time s are needed here. (Since the time derivatives of all differences are bounded by the equation and the bounds on the next two differences, Lemma 25.8 would give convergence on the whole interval before s as well.)

We use the continuation of Theorem 24.10 for the single initial value b_s , with $r \geq 5$, through the horizon $T - s$. Its finitely many DeTurck solutions have finite smooth bounds, and its gauge transformations are fixed and smooth. The endpoints at time s of the Wilson flow and of the flows for $\mathcal{A}_c$ approach the sample of b_s in C_a^1 , the latter uniformly in c . We apply Lemma 25.14 on each time interval of the continuation (for $c = 0$ this is also Lemma 25.5). In the gauges of the continuation, both flows approach the sample of the same selected DeTurck solution uniformly in C_a^1 on the finite continuation, with error $K(\eta_j + a_j)$, where $\eta_j \rightarrow 0$, uniformly over $c \in [0, C_0]$.

For anti-Hermitian X, Y , the integral formula for the derivative of the exponential implies that $\|e^{aX} - e^{aY}\| \leq Ka|X - Y|$. We telescope over the $N_{\ell_a} \leq \ell/a$ unitary factors of a closed path, including the inverse factors. The matrix error is then at most $K\ell\|B_a - S_ab\|_\infty$, where b is the selected DeTurck solution at the given time. Since a conjugation at the basepoint does not change the character, each of the two character values approaches the character of the product of the exponentials e^{aS_ab} along the same path, uniformly in the path and in the time. For a fixed representation, the same calculation in its unitary matrices involves in addition the norm of the corresponding representation of $\mathfrak{su}(2)$ and its dimension. This contradicts the fixed positive discrepancy. The supremum of the bounded left side of (25.46) over the initial potentials considered defines a function of a that tends to zero, which proves the lemma. $\square$

PART VIII

Fluctuation estimates and Wilson-loop
convergence

Concentration of reconstruction fluctuations

This section and the next control the heat evolution of the reconstructed field and its quadratic Picard term under the conditioned Wilson measure. Recall from Section 3.6 that the field on the finest lattice is reconstructed from the coarse field one renormalization group step at a time, from coarse to fine, by adding the normalized fluctuation variables of each step. At the step from mesh a_{j-1} to mesh a_j , the heat evolution x_j of the reconstructed field changes by $d_j = x_j - x_{j-1}$. In Lemma 26.2, we write d_j exactly as a sum of four terms:

- the linear fluctuation s_j , a linear function of the newly integrated fluctuation variables after subtraction of their conditional mean;
- the heat evolution z_j of the conditionally centered nonlinear reconstruction residual;
- an interpolation defect r_j , which compares the linearized reconstruction of the coarser field with the direct interpolation of that field;
- a predictable term u_j , which contains the nonlinear part of the zero-fluctuation reconstruction and the conditional mean correction, that is, the conditional mean of the difference between the reconstructed field and its zero-fluctuation reconstruction.

The quadratic Picard term $y_j = \mathcal{Q}_\delta(x_j, x_j)$ then changes by the two mixed terms $\mathcal{Q}_\delta(x_{j-1}, d_j)$, $\mathcal{Q}_\delta(d_j, x_{j-1})$, and by the sixteen ordered products of the four terms.

This section proves the estimates that involve the newly integrated fluctuation variables. Theorem 26.10 bounds the linear fluctuation s_j and the mixed Duhamel terms of s_j with a predictable background field, in either order; the backgrounds of interest are x_{j-1} and r_j . Theorem 26.11 bounds z_j and its mixed terms with x_{j-1} . Theorem 26.12 bounds the quadratic Duhamel term $\mathcal{Q}_\delta(s_j, s_j)$ and, separately, its conditional mean, which is not zero. The terms built only from r_j and u_j are treated in Section 27. Section 28 assembles the estimates of both sections into bounds on the linear and quadratic flow data of the reconstructed field, and the coefficient bounds proved here are also used in Sections 29 and 34.

At mesh $a = a_j$, the linear fluctuation has size $g_3 a^{\alpha-1/2} \sqrt{\log(2 + L/a)}$ in the norm X_α^δ , and the heat evolution of the nonlinear reconstruction residual has size $g_3^2 a^\alpha \log(2 + L/a) + g_3^3 a^{1/2} p_j^3$, where p_j is the common fluctuation cutoff radius of the step. The centered part of $\mathcal{Q}_\delta(s_j, s_j)$ has size $g_3^2 a^{2\alpha-1} \log(2 + L/a)$ in the norm Y_α^δ , and its conditional mean is at most $C g_3^2 \|\theta_{j-1}\|_\infty$, where θ_{j-1} is the logarithm of the coarser reconstructed

field. Since $1/2 < \alpha < 2/3$ and p_j grows only logarithmically, the first three rates are summable over the geometric sequence of meshes. The mean is summable because the coarser fields are bounded by a summable deterministic sequence; see (26.8).

Heuristics. At times t of order a^2 , a linear reconstruction fluctuation is a lattice potential of size $g_j/a = g_3 a^{-1/2}$, where $g_j^2 = g_3^2 a$ is the coupling at mesh a ; multiplying by the weight a^α of X_α^δ gives $g_3 a^{\alpha-1/2}$. A quadratic reconstruction fluctuation has the coefficient $g_j^2/a = g_3^2$, and the weight a^η of the corresponding heat norm gives $g_3^2 a^\eta$. The source of the quadratic Duhamel term contains one lattice derivative, and for the linear fluctuation it has size $g_3^2 a^{-2}$. Integration over a time interval of length a^2 and the weight $a^{2\alpha-1}$ of Y_α^δ give $g_3^2 a^{2\alpha-1}$. The maximum over space and time contributes a factor $\sqrt{\log(2 + L/a)}$ for linear forms in the fluctuation variables and $\log(2 + L/a)$ for quadratic forms. Scaling alone does not give the long-time bounds. They come from the spatial averaging of the heat kernel, and their proof keeps track of the physical volume of each lattice cell.

Inputs and two points that need care. The probabilistic input is the conditional fluctuation kernel of Theorem 13.8: given the coarser field, the normalized fluctuation variables of one renormalization group step have the density $Z^{-1} \mathbf{1}_\Omega e^{-V}$ on a product Ω of three-dimensional balls, with $D_w^2 V \geq I/2$. By the logarithmic Sobolev inequality of Theorem 5.4, linear and quadratic functions of these variables concentrate, with constants independent of the dimension. We also use the bounds on conditional means and covariances of Theorem 13.10, and the reconstruction maps $\hat{\mathcal{R}}$ of Theorem 15.13, with their disintegration (Lemma 15.10) and uniform bounds on their absolute Taylor coefficients. The deterministic inputs are the cardinal interpolation and the chaining lemma of Lemmas 16.3 and 16.4, the cancellation estimate of Lemma 17.2, the covariance estimate of Theorem 17.4, and the quadratic coefficient of the damped Wilson flow and its Duhamel estimates from Lemmas 24.2 and 24.6 and Definition 24.3.

Two points need care. First, the normalized fluctuation variables are neither Gaussian nor independent, and their conditional mean is not zero when the coarser field is not the identity. We subtract the conditional mean under the conditional fluctuation kernel itself, and we keep the nonzero mean of every quadratic term. Second, all fields are interpolated to the finest lattice, of mesh δ , while their randomness lives at the much coarser mesh a . To obtain the factor $a^{3/2}$ in the operator bounds, and logarithms of L/a rather than of L/δ in the maximal inequalities, we keep the cell volume of each coefficient on its own lattice and cover space and time at the scale a .

We separate the heat evolution of rough initial data from its quadratic Picard term as Cao and Chatterjee do for the Yang–Mills heat flow with random initial data in [42, Definition 2.8 and Theorem 2.10]; their treatment follows the approach of Bourgain [32] and Da Prato and Debussche [63]. The first nonlinear correction of Cao and Chatterjee also contains a cubic term, and their random-data theorem treats a class of initial data

that includes Gaussian free fields; here the estimates are proved from the conditional fluctuation kernels. The concentration inequalities use the convexity methods of Gross [103] and Bakry and Émery [8], in the form proved in Theorem 5.4. The quadratic tail bound is of Hanson–Wright type [108]; for dependent coordinates satisfying a concentration inequality, compare Adamczak [1, Theorem 2.5]. The maximal inequalities use elementary chaining in the sense of Dudley [79] (see also [74]), in the form of Lemma 16.4.

26.1 Setting and the exact decomposition of an increment

The hierarchy. Throughout this section we fix $L > 0$, $\gamma = g_3^2 L > 0$, an odd integer $M \geq 3$, and an integer $J \geq 1$ for which the constructions of Theorems 13.8 and 15.13 apply, and we set $B = M^J$. We fix a coarse side $n \in 4\mathbb{N}$ large enough that $t_1 = \sqrt{\gamma/(Mn)} \leq g_*$ and the common domain and smallness conditions below hold. For $K \geq 0$ we set $N = nB^K$ and

$$a_j = (L/n)B^{-j}, \quad g_j = g_3\sqrt{a_j} \quad (0 \leq j \leq K), \quad \delta = a_K,$$

so that $g_j^2 = \gamma/(nB^j)$ is the lattice coupling at mesh a_j . The j th renormalization group step from the coarse side maps the lattice of mesh a_j to the lattice of mesh $a_{j-1} = Ba_j$; it consists of J averaging steps, and its intermediate scales have couplings $g_{j,i}$, $0 \leq i \leq J-1$, whose ratios to g_j lie in an interval depending only on M and J . The largest of them is

$$t_j = M^{(J-1)/2}g_j, \quad t_j^2 = \frac{\gamma}{Mm_j}, \quad m_j = nB^{j-1}.$$

Here $m_j \geq 4$ is the side of the coarser lattice of the step. The zero-fluctuation reconstruction at this step is built from a classical background field whose effective classical action has been obtained by $\mathfrak{d}_j = J(K-j)$ preceding averaging steps, counted from the finest lattice.

The measure and the filtration. All reconstruction variables and the fine configuration are coupled as in Remark 15.11. We use the same symbol for the conditioned measure and this joint law; its fine-configuration marginal is the conditioned Wilson measure. We use the conditioned Wilson measure

$$\nu_K = \mu_{N,\gamma}(\cdot \mid S_{N,n,\gamma})$$

and the reconstruction maps $\widehat{\mathcal{R}}$ of Theorem 15.13. Let θ_0 be the logarithm of the coarsest configuration in the gauge chosen there, and let w_j be the normalized fluctuation variables of the j th renormalization group step from the coarse side. We write $\theta_j = a_j A_j$ for the link logarithm of the field reconstructed at mesh a_j , so that A_j is the corresponding lattice potential. Let $\mathcal{R}_j(\theta, w)$ be the logarithmic form of the reconstruction map at step j ; thus $\theta_j = \mathcal{R}_j(\theta_{j-1}, w_j)$, and $\mathcal{R}_j(\theta, w) = \widehat{\Phi}_{\mathfrak{d}_{j,j}}(\theta, (g_{j,i}w_i)_i)$ in the notation of Section 17. Let $\mathcal{F}_j$ be

the σ -algebra generated by the coarsest configuration and by $w_1, \dots, w_j$. All conditional statements below hold for every value of θ_{j-1} that occurs on the small-field event, and *uniformly in θ_{j-1}* refers to these values.

Each scalar fluctuation coordinate λ of the j th step is attached to a lattice location on one of its intermediate scales. We write a_λ for the mesh of that lattice, $v_\lambda = a_\lambda^3$ for its cell volume, and $g_\lambda = g_3 \sqrt{a_\lambda}$ for its coupling. Then $C_J^{-1} a_j^3 \leq v_\lambda \leq C_J a_j^3$, and the ratios g_λ / g_j are bounded above and below by constants depending only on M and J . The fluctuation cutoff domain of the step is

$$\Omega_j = \prod_{i,e} \{w_{i,e} \in \mathbb{R}^3 : |w_{i,e}| \leq p_{j,i}\}, \quad p_{j,i} = p_j = A_w \log(1/t_j) \geq 1 \quad \text{for every } i,$$

as prescribed by Definition 4.4. The same radius is used at every intermediate scale of this RG step; estimates for products with unequal radii are auxiliary bounds, not a change in this conditional law.

Common domains. We assume that all integrated couplings lie below the common upper bound g_* of Theorems 13.8, 13.10, and 17.4 and Lemma 15.10; in particular, $t_j \leq g_*$ for every j . We also assume that the coarser fields lie in the small-link domain B_r , the target set of Theorem 13.8, and that the reconstruction is evaluated in the larger analytic domain of Theorem 15.13, with the fixed positive distances between its nested domains. These choices are made before K , j , or any sample, and they do not change with K or j .

For a coarser field θ and a number $\mathfrak{d} \geq 0$ of preceding averaging steps, we write $\mu_{\mathfrak{d},\theta}$ for the conditional fluctuation kernel of Theorem 13.8 on Ω_j . For coarser fields in the small-link domain, $\mu_{\mathfrak{d},\theta}(dw) = k(e^\theta, dw)$ in the notation of (4.8), for a step preceded by $\mathfrak{d}$ averaging steps. Let $\mathbb{E}_\theta$, $\mathbb{P}_\theta$, and Cov_θ denote expectation, probability, and covariance under this kernel. We write

$$m_\theta = \mathbb{E}_\theta w_j, \quad \zeta_j = w_j - m_\theta,$$

and Σ_θ for the conditional fluctuation covariance of w_j . (This Σ_θ is unrelated to the normal covariance of (4.12).)

Norms and adjoints. We fix a finite horizon $T > 0$ for all heat estimates. Coefficient arrays indexed by fluctuation coordinates carry the counting norm, which sums over every scalar coordinate of every intermediate scale; we write $\|l\|_2$ or $\|l\|_{\ell^2(w)}$ for it, and the transpose $^\top$ is taken with respect to the counting inner product. Fields on a lattice of mesh b carry the physical norms

$$\|f\|_{L_b^2} = \left(b^3 \sum_x |f(x)|^2 \right)^{1/2}, \quad \|f\|_{L_b^1} = b^3 \sum_x |f(x)|,$$

and the adjoint W^* of a map $W : \ell^2(w) \rightarrow L_\delta^2$ is taken with respect to L_δ^2 , so that matrix products with W^* contain the factor δ^3 . The covariance matrix of the linear reconstruction coefficient is the counting matrix

$$Z_\theta = E_\theta \Sigma_\theta E_\theta^\top,$$

where E_θ is defined in Lemma 26.2; the corresponding physical operator is $a^3 Z_\theta$. We always use the counting matrix Z_θ between counting rows. The heat semigroup $P_t^\delta = e^{t\Delta_\delta}$, the cardinal interpolation $I_b = I_{b \rightarrow \delta}$ from mesh b to mesh δ , the differences D_δ , and the norms X_α^δ , Y_α^δ , H_σ^δ , and V_σ^δ are those of Definition 4.8, with $v_\delta(t) = t + \delta^2$; here D_δ includes every componentwise difference and every translation required by the quadratic coefficient of Lemma 24.2. For $0 < \eta < 3/2$ we also write

$$H_\eta^\delta(f) = X_\eta^\delta(P_t^\delta f) = \sup_{0 \leq t \leq T} \{ (t + \delta^2)^{\eta/2} \|P_t^\delta f\|_\infty + (t + \delta^2)^{(\eta+1)/2} \|D_\delta P_t^\delta f\|_\infty \};$$

for $0 < \eta < 1 - \alpha$ this is the norm of Definition 4.8. Finally,

$$\mathcal{Q}_\delta(u, v)(t) = \int_0^t P_{t-s}^\delta \mathcal{N}_\delta(u(s), v(s)) ds, \quad \mathcal{B}_\delta(u, v) = \mathcal{Q}_\delta(u, v) + \mathcal{Q}_\delta(v, u),$$

where $\mathcal{N}_\delta$ is the ordered quadratic coefficient of the damped Wilson flow on the lattice of mesh δ ; it is not symmetric, and we always keep the order of its arguments.

Constants. Unless stated otherwise, constants C , C_J , and $C_{J,q}$ depend only on L , T , M , J , the fixed analytic domains and the distances between them, the bounds on the absolute Taylor coefficients of the reconstruction maps, the interpolation function of the cardinal interpolation, the exponents, and the indicated derivative orders. They do not depend on the lattice side, on K , on the number of preceding averaging steps, on the coarser field, on the coupling below g_* , or on the refinement ratio a/δ . The constants of maximal inequalities proved with the fluctuation cutoff (Lemma 26.9) may in addition depend on the fixed γ , n , and the prescription of the cutoff radii; the covering and chaining bounds of Lemmas 26.7 and 26.8 have no such dependence. Among the constants of this section, only the bound C_{CP} on finitely many derivatives of the zero-fluctuation reconstruction (see the proof of Lemma 26.2) is chosen before J ; all other bounds are fixed- J bounds. All estimates concern the whole torus.

The conditional law and the Taylor coefficients

The first lemma identifies the conditional law of the fluctuation variables of one step and records the coefficient bounds used in all later estimates. Its proof mostly verifies the hypotheses of earlier results; the new ingredient is the passage from absolute coefficient bounds on a larger domain to derivative bounds, without loss in the volume-normalized column sums.

Lemma 26.1. *Assume the setting of Section 26.1. For every $K \geq 0$ and $1 \leq j \leq K$, the conditional law of w_j given $\mathcal{F}_{j-1}$ is the normalized density $Z_{j,\theta}^{-1} e^{-V_{j,\theta}(w)} dw$ on Ω_j , where $\theta = \theta_{j-1}$, the potential $V_{j,\theta}$ is smooth on a neighborhood of the closed product Ω_j , and*

$$D_w^2 V_{j,\theta} \geq \frac{1}{2} I \quad \text{on } \Omega_j.$$

This density and the normalized fluctuation coordinates are the same before and after the change to the reconstruction maps $\widehat{\mathcal{R}}$ at a fixed coarser field. For each base point $w_* = 0$ or $w_* = m_\theta$, the normalized tensors $g_j^{-1} D_w \mathcal{R}_j(\theta, w_*)$ and $(2g_j^2)^{-1} D_w^2 \mathcal{R}_j(\theta, w_*)$ have uniformly bounded absolute output row sums and volume-normalized input column sums for every choice of fixed input, on common nested domains in the unscaled coarser field and the coupling-scaled fluctuation inputs $q_{j,i} = g_{j,i} w_i$. On a smaller domain these bounds extend to every fixed-order mixed derivative in the unscaled coarser field and the q coordinates, with a constant depending on the order. These assertions are uniform in the lattice side and in the number of preceding averaging steps, at fixed M, J , and domains.

Proof. At step j , the coarser lattice has side $m_j = nB^{j-1} \geq 4$, the number of preceding averaging steps is $\mathfrak{d}_j = J(K-j)$, and $t_j^2 = \gamma/(Mm_j)$. By Lemma 15.10, the coarser field lies in the small-link domain B_r . At every intermediate scale the cutoff radius is exactly $p_j = A_w \log(1/t_j)$. The common upper bound g_* includes the smallness conditions of Theorems 13.8 and 13.10. With $p = p_j$ and $t = t_j$, we have in particular

$$Ctp < 1, \quad \frac{C[tp + t^2(1 + E_*)]}{(1 - Ctp)^3} \leq \frac{1}{2}, \quad tp^3 \leq c_*. \quad (26.1)$$

Here E_* is the upper bound on the weighted analytic coefficient norm of the effective interaction in Theorem 13.8, and C and c_* are constants of that theorem and of Theorem 13.10. Since $p(t) = A_w \log(1/t)$ and $t[\log(1/t)]^k \rightarrow 0$ as $t \rightarrow 0$ for each fixed k , we may choose g_* to satisfy (26.1) before K, j , or any sample is chosen. The domain conditions on the product Ω_j , and the positive distances from the analytic images to the boundaries of their domains, are the common-domain hypotheses preserved in Theorem 15.13.

All hypotheses of Theorem 13.8 hold at this coarser side, number of averaging steps, coarser field, and product Ω_j . That theorem gives the potential, its smooth extension to a neighborhood, and the lower bound $I/2$ on its Hessian; the integration domain is Ω_j itself, not a larger enclosing product domain. By the last sentence of part (1) of Theorem 13.8 and by Lemma 15.10, this law is the conditional law of w_j under ν_K , since the factors that depend only on the coarser field cancel. By Theorem 15.14, the reconstruction maps preserve, at this coarser field, the normalized fluctuation coordinates, the balls, the action, the Haar disintegration density, the triangular determinant, the input effective interaction produced by the finer-scale integrations, the normalizer, and the conditional fluctuation kernel. Two equal positive normalized densities have potentials that differ by a constant

depending only on the coarser field. Fixing that constant, we obtain the same $V_{j,\theta}$, and hence the same Hessian, after the change of representatives. In particular, we never transport a Hessian through a nonlinear change of the fluctuation variables. The auxiliary integral of Theorem 14.6 over larger domains does not enter, because the coarser field lies in B_r .

For the analytic assertion, we use the norm of Theorem 15.13 on the absolute joint Taylor coefficients, including its volume-normalized column sums, rather than a bound on the values of a holomorphic map. The products of coarser fields and coupling-scaled inputs occurring here lie in the inner domain $\mathcal{D}_{\text{in}}$ of that theorem, at a positive distance from its boundary, and the absolute coefficient norm is bounded uniformly on $\mathcal{D}_{\text{in}}$. We choose an evaluation polydisc and a larger polydisc,

$$\mathcal{P}_- \Subset \mathcal{P}_+ \Subset \mathcal{D}_{\text{in}}, \quad \eta_\lambda = \text{rad}_\lambda(\mathcal{P}_+) - \text{rad}_\lambda(\mathcal{P}_-) > 0.$$

They may be centered at any point of evaluation, with gaps bounded below by fixed fractions of the distance to the boundary of $\mathcal{D}_{\text{in}}$; inscribing coordinate polydiscs in the product balls needs only a fixed change of color coordinates. The nested domains of Theorem 15.13 give these gaps uniformly, and nothing depends on a modulus of convergence or on the size of the lattice. The coupling-scaled images $(g_{j,i}w_i)_i$ of every point of Ω_j , of the point $w = 0$, and of the conditional mean m_θ lie in such an evaluation polydisc, because the mean of a probability measure on a ball lies in that ball.

Differentiating the absolute joint series term by term and evaluating it on $\mathcal{P}_-$ introduces the factorial of the derivative order and inverse powers of the gaps. These operations preserve the column sum with a fixed input, because all other inputs, including the output, are already summed in that column sum. Equivalently, one may apply the Cauchy integral formula on $\mathcal{P}_+$ to the coefficients, in these finite-dimensional row and column norms. We obtain uniform bounds for the first, second, and every fixed higher derivative. A derivative in a fluctuation variable of intermediate scale i introduces the factor $g_{j,i}$, and the ratios $g_{j,i}/g_j$ and the ratios of cell volumes are bounded by constants depending only on M and J . Dividing the first and second derivatives in w by g_j and $2g_j^2$, respectively, gives the normalized tensors asserted in this lemma, including their volume-normalized column sums. These are the tensors used in the two Taylor expansions of Lemma 26.2 below. All constants in this argument are fixed- J constants; the constant C_{CP} chosen before J is not enlarged. $\square$

The exact decomposition

Table 26.1 lists the four terms of the increment of the heat evolution; their formulas are in the next lemma.

The classical part uses the zero-fluctuation reconstruction $C_j = a_j^{-1} \mathcal{R}_j(\theta_{j-1}, 0)$, which is a minimizing reconstruction and not a conditional mean. Its linearization P_j maps a

Term	Field before heat evolution P^δ	Conditional mean	Treated in
s_j	$I_{a_j}(g_j/a_j)E_\theta\zeta_j$, linear in ζ_j	zero	Theorem 26.10
z_j	$I_{a_j}Z_j$, nonlinear in ζ_j	zero	Theorem 26.11
r_j	$H^{-1}(I_{a_j}P_j - I_H E_H)\theta_{j-1}$	predictable	Lemma 27.1
u_j	$I_{a_j}U_j^0$, nonlinear classical part plus mean	predictable	Theorem 27.7

Table 26.1. The four terms of the increment $d_j = x_j - x_{j-1}$ in Lemma 26.2.

lattice potential on the coarser lattice, of mesh $H = a_{j-1}$, to a lattice potential on the lattice of mesh a_j . The interpolation defect compares this linearization, interpolated from mesh a_j , with the coarser field interpolated directly from its own lattice; the two interpolations are applied on their own lattices before they are subtracted. In the notation of Lemma 28.5, $C_j = a_j^{-1}\widehat{P}_j(\theta_{j-1})$ and $P_j = B\widehat{L}_j$, so the linearized reconstruction is $H^{-1}P_j\theta_{j-1} = a_j^{-1}\widehat{L}_j\theta_{j-1}$.

Lemma 26.2. *Assume the setting of Section 26.1, and fix $K \geq 0$.*

1. *For every $1 \leq j \leq K$, conditional on $\mathcal{F}_{j-1}$, the law of w_j is the conditional fluctuation kernel $\mu_{\mathfrak{d}_j, \theta_{j-1}}$ on the product Ω_j , with $\mathfrak{d}_j = J(K - j)$, including all Haar disintegration densities, determinants, input effective interactions, and indicators. For this fixed coupling and coarse side, $p_j \leq C_\gamma(1 + j)$. Every estimate below for a single kernel holds uniformly for any number $\mathfrak{d} \geq 0$ of preceding averaging steps, on the same domains; the conditioned Wilson measure uses $\mathfrak{d} = \mathfrak{d}_j$.*
2. *Write the Taylor expansion of the reconstruction at the conditional mean as*

$$\mathcal{R}_j(\theta, m_\theta + \zeta) = \mathcal{R}_j(\theta, m_\theta) + g_j E_\theta \zeta + g_j^2 B_\theta(\zeta, \zeta) + R_\theta(\zeta), \quad (26.2)$$

where the ratios of the intermediate couplings to g_j are included in E_θ and B_θ . Let $C_j = a_j^{-1}\mathcal{R}_j(\theta_{j-1}, 0)$ and $m_j^{\text{ph}} = \mathbb{E}(A_j - C_j \mid \mathcal{F}_{j-1})$. With $H = a_{j-1} = Ba_j$, the zero-fluctuation reconstruction has the exact expansion

$$C_j(\theta) = H^{-1}\{P_j\theta + \overline{Q}_j(\theta, \theta) + \overline{R}_j(\theta)\}, \quad (26.3)$$

where P_j is linear, $\overline{Q}_j$ is quadratic, and $\overline{R}_j$ is the remainder. Let E_H be the identity map of arrays on the lattice of mesh H , and set

$$\begin{aligned} x_j &= P^\delta I_{a_j} A_j, & d_j &= x_j - x_{j-1}, \\ s_j &= (g_j/a_j)P^\delta I_{a_j} E_\theta \zeta_j, \\ Z_j &= A_j - \mathbb{E}(A_j \mid \mathcal{F}_{j-1}) - (g_j/a_j)E_\theta \zeta_j, & z_j &= P^\delta I_{a_j} Z_j, \\ r_j &= H^{-1}P^\delta (I_{a_j} P_j - I_H E_H)\theta_{j-1}, \\ U_j^0 &= H^{-1}\{\overline{Q}_j(\theta_{j-1}, \theta_{j-1}) + \overline{R}_j(\theta_{j-1})\} + m_j^{\text{ph}}, & u_j &= P^\delta I_{a_j} U_j^0, \end{aligned} \quad (26.4)$$

with $\theta = \theta_{j-1}$ in E_θ . Then

$$d_j = s_j + z_j + r_j + u_j. \quad (26.5)$$

The terms s_j and z_j have conditional mean zero given $\mathcal{F}_{j-1}$, while r_j and u_j are $\mathcal{F}_{j-1}$ -measurable; neither of the latter is recentered. With $y_j = \mathcal{Q}_\delta(x_j, x_j)$, for every $1 \leq k \leq l \leq K$,

$$y_l - y_{k-1} = \sum_{j=k}^l \{ \mathcal{B}_\delta(x_{j-1}, d_j) + \mathcal{Q}_\delta(d_j, d_j) \}. \quad (26.6)$$

Inserting (26.5) expresses the right side through the four mixed terms $\mathcal{B}_\delta(x_{j-1}, v)$, $v \in \{s_j, z_j, r_j, u_j\}$, and the sixteen ordered products $\mathcal{Q}_\delta(v, v')$, $v, v' \in \{s_j, z_j, r_j, u_j\}$; equivalently, through the nine ordered products of s_j , z_j , and $r_j + u_j$. All these identities are finite and exact, uniformly in K , $\mathfrak{d}_j$, and θ_{j-1} .

3. There is also an expansion at $w = 0$. Set

$$E_\theta^{(0)} = g_j^{-1} D_w \mathcal{R}_j(\theta, 0), \quad B_\theta^{(0)} = (2g_j^2)^{-1} D_w^2 \mathcal{R}_j(\theta, 0),$$

define $s_j^{(0)}$ as s_j with E_θ replaced by $E_\theta^{(0)}$, and define $Z_j^{(0)}$ and $z_j^{(0)}$ by the same centered subtraction. Then (26.5) and (26.6) hold with $s_j^{(0)}$, $z_j^{(0)}$ in place of s_j , z_j , and with r_j , u_j unchanged. Define $\tilde{R}_\theta$ by

$$\mathcal{R}_j(\theta, m_\theta + \zeta) - \mathcal{R}_j(\theta, m_\theta) = g_j E_\theta^{(0)} \zeta + g_j^2 B_\theta^{(0)}(\zeta, \zeta) + \tilde{R}_\theta(\zeta). \quad (26.7)$$

Then $\|\tilde{R}_\theta\|_\infty \leq C_J g_j^3 p_j^3$ on Ω_j . Theorems 26.10 to 26.12 hold for either expansion, with E_θ , B_θ , R_θ replaced consistently by $E_\theta^{(0)}$, $B_\theta^{(0)}$, $\tilde{R}_\theta$ in the second. This is a second decomposition of the same sample, under the same law.

Proof. The proof consists of exact computations, apart from two estimates: the deterministic bound on the coarser fields in Step 2, which is used throughout the section, and the bound on $\tilde{R}_\theta$ in Step 7.

Step 1: the conditional law. The disintegration of Lemma 15.10 applies with coarse mesh L/n , total number K of renormalization group steps, and the reconstruction maps $\hat{\mathcal{R}}$, whose gauge transformations depend only on the coarser field. Every restriction imposed by the small-field event at coarser scales is measurable with respect to the coarser field and therefore cancels in the normalized conditional law of the finer variables. This proves the assertion about the law of w_j given $\mathcal{F}_{j-1}$, which is a statement about the joint law of all variables under ν_K . Every cutoff radius in this step equals $p_j = A_w \log(1/t_j)$. Since $t_j = M^{(J-1)/2} g_3 \sqrt{L/n} B^{-j/2}$, this common radius satisfies $p_j \leq C_{\gamma, n, M, J}(1+j)$.

Step 2: a deterministic bound on the coarser fields. Let $C_{\text{rec},J}$ be the fixed- J bound on the absolute row sums of the derivative of one reconstruction step in the coupling-scaled inputs. Step 1 of the proof of Lemma 17.1 verifies the linear amplitude hypothesis of Lemma 15.10, using the zero-fluctuation value bound and the free-input derivative bound of Theorem 15.13. Hence

$$\|\theta_j\|_\infty \leq a_{\text{cap}} \|\theta_{j-1}\|_\infty + C_{\text{rec},J} t_j p_j, \quad a_{\text{cap}} = C_{\text{CP}}/B, \quad a_{\text{cap}} \sqrt{B} < 1.$$

Here C_{CP} bounds finitely many derivatives of the zero-fluctuation reconstruction and is chosen before J . We iterate this recursion, using $p_i \leq p_j$ for $i \leq j$ and $t_i = M^{(J-1)/2} g_3 \sqrt{a_j} B^{(j-i)/2}$, and obtain

$$\|\theta_j\|_\infty \leq r a_{\text{cap}}^j + M^{(J-1)/2} C_{\text{rec},J} g_3 \sqrt{a_j} p_j \sum_{h=0}^{j-1} (a_{\text{cap}} \sqrt{B})^h \leq r a_{\text{cap}}^j + C_{\text{cap},J} g_3 \sqrt{a_j} p_j, \quad (26.8)$$

where r is the radius of B_r and $C_{\text{cap},J} \geq M^{(J-1)/2} C_{\text{rec},J} / (1 - a_{\text{cap}} \sqrt{B})$ is a separate constant. A constant C_J in a deterministic bound on the coarser fields below may include $C_{\text{cap},J}$; it is never used to enlarge C_{CP} or to change J . The conditional fluctuation kernel and the coefficient bounds used in the Taylor expansions below are verified in Lemma 26.1.

Step 3: the expansion at the conditional mean. By convexity, the product of balls contains the coordinatewise mean m_θ and the segment $m_\theta + v\zeta$, $0 \leq v \leq 1$. We repeat the Taylor argument of Theorem 13.10(5) for the replacement reconstruction $\mathcal{R}_j$, using the derivative bounds on the common domain supplied by Lemma 26.1, including the ratios between intermediate scales. The conditional law is unchanged, but the reconstruction function need not be the one appearing in that theorem. With

$$E_\theta = g_j^{-1} D_w \mathcal{R}_j(\theta, m_\theta), \quad B_\theta = (2g_j^2)^{-1} D_w^2 \mathcal{R}_j(\theta, m_\theta),$$

Taylor's formula gives (26.2) exactly, with

$$R_\theta(\zeta) = \frac{1}{2} \int_0^1 (1-v)^2 D_w^3 \mathcal{R}_j(\theta, m_\theta + v\zeta) [\zeta, \zeta, \zeta] dv, \quad \|R_\theta\|_\infty \leq C_J g_j^3 p_j^3. \quad (26.9)$$

The derivatives here are taken in the normalized fluctuation variables. Writing an increment of the coupling-scaled input $q_{j,i}$ as $g_{j,i} w_i$ contributes its own factor $g_{j,i}$ at each input, and all ratios $g_{j,i}/g_j$ lie in an interval depending only on M and J . This accounts for the normalization of both tensors.

Step 4: the expansion of the zero-fluctuation reconstruction. We define $\bar{A}_j(\theta) = B \mathcal{R}_j(\theta, 0)$, $P_j = D \bar{A}_j(0)$, and $\bar{Q}_j = D^2 \bar{A}_j(0)/2$, and let $\bar{R}_j$ be what remains after subtracting these two terms from $\bar{A}_j$. We have $\bar{A}_j(0) = 0$, because the zero-fluctuation reconstruction of the identity is the identity. Since $B a_j = H$, we have $a_j^{-1} \mathcal{R}_j(\theta, 0) = H^{-1} \bar{A}_j(\theta)$, which is (26.3).

Step 5: the four terms. By the definitions of C_j , m_j^{ph} , and Z_j ,

$$A_j = C_j + m_j^{\text{ph}} + (g_j/a_j)E_\theta \zeta_j + Z_j.$$

We apply I_{a_j} to this identity, subtract $I_H A_{j-1} = H^{-1} I_H E_H \theta_{j-1}$, and apply P_t^δ . By (26.3), the linear part of C_j and the coarser field combine to $H^{-1} P_t^\delta (I_{a_j} P_j - I_H E_H) \theta_{j-1} = r_j(t)$. The remaining predictable terms are u_j , which proves (26.5). Since $\mathbb{E}_\theta \zeta_j = 0$, the definitions imply that $\mathbb{E}_\theta s_j = \mathbb{E}_\theta z_j = 0$, while r_j and u_j depend only on θ_{j-1} . The conditional mean correction m_j^{ph} , which is in general not zero, has not been removed.

Step 6: the quadratic telescope. For any bilinear map $\mathcal{Q}$ and any x, d ,

$$\mathcal{Q}(x + d, x + d) - \mathcal{Q}(x, x) = \mathcal{Q}(x, d) + \mathcal{Q}(d, x) + \mathcal{Q}(d, d).$$

We apply this identity to $\mathcal{Q}_\delta$ on the lattice of mesh δ , with $x = x_{j-1}$ and $d = d_j$, and sum from $j = k$ to $j = l$. This is (26.6). Inserting $d_j = \sum_{v \in \{s_j, z_j, r_j, u_j\}} v$, we obtain $\mathcal{B}_\delta(x_{j-1}, d_j) = \sum_v \mathcal{B}_\delta(x_{j-1}, v)$ and $\mathcal{Q}_\delta(d_j, d_j) = \sum_{v, v'} \mathcal{Q}_\delta(v, v')$, which are the four mixed terms and the sixteen ordered products. Combining $r_j + u_j$ gives the nine ordered products.

Step 7: the expansion at zero. We derive the second expansion from an exact Taylor computation; we do not assume that $m_\theta = 0$. At $w = 0$ we have

$$\mathcal{R}_j(\theta, v) = \mathcal{R}_j(\theta, 0) + g_j E_\theta^{(0)} v + g_j^2 B_\theta^{(0)}(v, v) + R_\theta^{(0)}(v).$$

By Lemma 26.1, the third directional derivative is bounded, and we obtain $\|R_\theta^{(0)}(v)\|_\infty \leq C_J g_j^3 \|v\|_\infty^3$. One may also obtain this bound directly from the one-variable Cauchy formula for the bounded reconstruction along the line through 0 and v , written in the coupling-scaled inputs $(g_{j,i} v_i)_i$; the radius available on that line is the distance to the boundary of the domain of these inputs divided by the supremum norm of the direction. By convexity, the segments from 0 to the points used here lie in the product. We subtract the expansions at $v = m_\theta + \zeta$ and at $v = m_\theta$ and obtain

$$\tilde{R}_\theta(\zeta) = g_j^2 \{B_\theta^{(0)}(m_\theta, \zeta) + B_\theta^{(0)}(\zeta, m_\theta)\} + R_\theta^{(0)}(m_\theta + \zeta) - R_\theta^{(0)}(m_\theta).$$

The bilinear form $B_\theta^{(0)}$ is bounded in the supremum norm, by polarization of the same directional Taylor formula. Together with the bound $\|m_\theta\|_\infty \leq C_J t_j \|\theta\|_\infty$ of Theorem 13.10, this bounds the first term by $C_J g_j^2 t_j \|\theta\|_\infty p_j$. Since θ lies in a fixed ball, t_j/g_j is fixed, and $p_j \geq 1$, this is at most $C_J g_j^3 p_j^3$. The two remainders $R_\theta^{(0)}$ obey the same bound. We define $z_j^{(0)}$ by the centered subtraction; the algebra of Steps 5 and 6 then applies with r_j and u_j unchanged. The row and column bounds and the Cauchy argument below are made on the common coefficient domain of Lemma 26.1, so they apply at either base point $w = m_\theta$ or $w = 0$. This completes the proof. $\square$

Remark 26.3. In this section s_j and z_j always denote the terms of the expansion (26.2) at the conditional mean, whose linear coefficient is $E_\theta = g_j^{-1} D_w \mathcal{R}_j(\theta, m_\theta)$; the terms of the expansion at $w = 0$, with linear coefficient $E_\theta^{(0)} = g_j^{-1} D_w \mathcal{R}_j(\theta, 0)$, are written $s_j^{(0)}$ and $z_j^{(0)}$. Both pairs use the same centered vector $\zeta_j = w_j - m_\theta$, and

$$s_j + z_j = s_j^{(0)} + z_j^{(0)} = P^\delta I_{a_j}(A_j - \mathbb{E}(A_j \mid \mathcal{F}_{j-1})),$$

so the decomposition (26.5) and the terms r_j and u_j are the same in both conventions. Section 28 uses the expansion at $w = 0$: its terms s_j and z_j are the terms $s_j^{(0)}$ and $z_j^{(0)}$ of this section. By part (3) of Lemma 26.2, Theorems 26.10 to 26.12 hold for these terms as stated.

26.2 Deterministic estimates for heat kernels and coefficients

Standing conventions. In this section and the next, $a = a_j$ for some $1 \leq j \leq K$, $H = Ba = a_{j-1}$, and $\delta = a_K$. Then $0 < \delta \leq a \leq L/4$, and $a/\delta = B^{K-j}$ and $H/\delta = B^{K-j+1}$ are integers. We fix $1/2 < \alpha < 2/3$ and set

$$\beta = \alpha - 1/2, \quad p_\alpha = (\alpha + 1)/2, \quad \Lambda_a = \log(2 + L/a),$$

where Λ_a is the logarithmic lattice-size factor, written $\Lambda_{a_j} = \ell_j$ in Section 17; it is unrelated to the normalization constant Λ . For $q > 0$ and $0 \leq t \leq T$, we also set

$$I_q^a(t) = \int_0^t (s + a^2)^{-p_\alpha} (t - s + a^2)^{-q} ds. \quad (26.10)$$

For a heat path $B(t) = P_t^\delta b$ on the lattice of mesh δ we write

$$R_a(B) = \sup_{0 \leq t \leq T} \{ (t + a^2)^{\alpha/2} \|B(t)\|_\infty + (t + a^2)^{(\alpha+1)/2} \|D_\delta B(t)\|_\infty \} \quad (26.11)$$

for its radius at scale a ; it is the norm $X_\alpha^\delta(B)$ with δ^2 replaced by a^2 in the weights.

Let $f(t, x)$ be a field on $[0, T] \times \Lambda_N$ whose components are linear functions of a vector of counting inputs, with deterministic coefficients. Its *value row* at (t, x) is the coefficient vector of a scalar component of $f(t, x)$, and its *first-difference rows* are those of the components of $D_\delta f(t, x)$. For a field whose components are quadratic forms, rows are replaced by the symmetric coefficient matrices, measured in Hilbert–Schmidt and operator norm. Given exponents (κ_0, κ_1) , the *weighted coefficient bound* of f is

$$\sup_{0 \leq t \leq T} \sup_x \max \{ (t + \delta^2)^{\kappa_0} \|\text{value row}\|, (t + \delta^2)^{\kappa_1} \|\text{first-difference row}\| \}.$$

We use $(\kappa_0, \kappa_1) = (\eta/2, (\eta + 1)/2)$ for the norms X_η^δ and H_η^δ , and $(\kappa_0, \kappa_1) = (\beta, \alpha)$ for the norm Y_α^δ . A bound by S for the weighted coefficient bound means that every scalar component of the weighted value and first difference, evaluated at a fixed point (t, x) , is a linear (or quadratic) form in the inputs with coefficient norm at most S .

The periodic heat kernel

We write $P_t^\delta f(x) = \delta^3 \sum_y p_t^\delta(x, y) f(y)$, so that $p_t^\delta(x, \cdot)$ is the heat kernel relative to the physical measure $\delta^3 \sum_y$, and we let $\|\cdot\|_1, \|\cdot\|_2$ be the norms of L_δ^1 and L_δ^2 .

Lemma 26.4. *For every fixed integer $q \geq 0$ and $0 \leq t \leq T$,*

$$\begin{aligned} \|D_\delta^q p_t^\delta(x, \cdot)\|_1 &\leq C_q (t + \delta^2)^{-q/2}, \\ \|D_\delta^q p_t^\delta(x, \cdot)\|_2 &\leq C_q (t + \delta^2)^{-3/4 - q/2}, \\ \|D_\delta^q p_t^\delta(x, \cdot)\|_\infty &\leq C_q (t + \delta^2)^{-(3+q)/2}. \end{aligned}$$

The kernel is nonnegative and has physical mass one. Fixed translations and adjoint differences obey the same estimates, with the same dependence of the constants.

Proof. We write $\Delta_\delta = \delta^{-2} \sum_i (T_i + T_i^{-1} - 2)$, where T_i is the translation by δe_i . The exponential series of the positive part $\delta^{-2} \sum_i (T_i + T_i^{-1})$, multiplied by e^{-6t/δ^2} , is a convolution with nonnegative coefficients whose sum is one. At $t = 0$ its density is $\delta^{-3} \mathbf{1}_{x=y}$. For $t \leq \delta^2$, contraction and the finite difference stencil give the three bounds, since $(t + \delta^2) \asymp \delta^2$ there.

We give the details for the L^1 bound on differences at longer times. On $\mathbb{Z}$, at dimensionless time $v = t/\delta^2 \geq 1$, the Fourier multiplier of the q th unscaled forward difference of the random walk kernel is

$$f_{v,q}(\xi) = (e^{i\xi} - 1)^q e^{-4v \sin^2(\xi/2)}, \quad -\pi \leq \xi \leq \pi.$$

Backward differences change only a factor of modulus one. Since $\sin^2(\xi/2) \geq c\xi^2$ on this interval, bounding each derivative of a factor and substituting $z = \sqrt{v}\xi$, we find

$$\int |f_{v,q}|^2 d\xi \leq C_q v^{-q-1/2}, \quad \int |\partial_\xi f_{v,q}|^2 d\xi \leq C_q v^{-q+1/2}.$$

For $q = 0$ only the derivative of the exponential occurs in the second integral, and its size is $Cv|\xi|e^{-cv\xi^2}$; for $q \geq 1$ the other term is $C_q|\xi|^{q-1}e^{-cv\xi^2}$, and both integrals have the displayed powers. By Parseval's identity, the difference kernel $h_{v,q}(m)$ satisfies

$$\sum_m (1 + m^2/v) |h_{v,q}(m)|^2 \leq C_q v^{-q-1/2}.$$

Using the Cauchy–Schwarz inequality with weights $1 + m^2/v$, together with $\sum_m (1 + m^2/v)^{-1} \leq C\sqrt{v}$, we obtain $\sum_m |h_{v,q}(m)| \leq C_q v^{-q/2}$. Taking products over the three coordinates, we obtain this bound for every difference in several directions on $\mathbb{Z}^3$. Periodization does not increase the absolute sum, and rescaling each difference by δ^{-1} gives the physical L^1 estimate.

The remaining estimates on the finite torus follow from the finite Fourier sum. Frequencies k in the principal cube satisfy $\omega_\delta(k) = 4\delta^{-2} \sum_i \sin^2(\pi \delta k_i / L) \geq c|k|^2 / L^2$, and the multiplier of a difference of order q is bounded by $C_q(|k|/L)^q$. Grouping the integer frequencies in unit shells $m \leq |k| < m+1$, each of which contains at most $C(m+1)^2$ points, and comparing the decreasing Gaussian tails with integrals after splitting at $m = L/\sqrt{t}$, we obtain

$$L^{-3} \sum_k (|k|/L)^q e^{-ct|k|^2/L^2} \leq C_q t^{-(3+q)/2} + \mathbf{1}_{q=0} L^{-3}.$$

This bounds the absolute Fourier sum, and hence the L^∞ norm. Parseval's identity with order $2q$ and time $2t$ gives the square of the L^2 bound. When $q = 0$, the constant Fourier mode contributes L^{-3} to that square; since $t \leq T$, it is absorbed into the stated estimate with a constant depending on T/L^2 . Differences annihilate the constant mode when $q > 0$. $\square$

Coefficient tensors

Lemma 26.5. *Suppose a tensor $b_{y;\lambda\mu}$, with y on the lattice of mesh a and λ, μ fluctuation coordinates of cell volumes v_λ, v_μ , satisfies*

$$\sup_y \sum_{\lambda, \mu} |b_{y;\lambda\mu}| \leq C_J, \quad \sup_\lambda \frac{a^3}{v_\lambda} \sum_{y, \mu} |b_{y;\lambda\mu}| \leq C_J, \quad v_\lambda / a^3 \leq C_J,$$

and the same column estimate with the roles of the two inputs interchanged. Set $F_t = P_t^\delta I_a b$, regarding the ordered pair (λ, μ) as one counting input. Then, for every fixed integer $q \geq 0$,

$$\begin{aligned} \|D_\delta^q F_t\|_{\ell^2(\lambda, \mu) \rightarrow L_\delta^2} &\leq C_{J,q} a^{3/2} (t + a^2)^{-q/2}, \\ \sup_x \|D_\delta^q F_t(x, \cdot, \cdot)\|_2 &\leq C_{J,q} a^{3/2} (t + a^2)^{-3/4 - q/2}. \end{aligned}$$

Symmetrizing a scalar coefficient matrix increases neither its Hilbert–Schmidt norm nor the displayed bound for its operator norm. The same heat argument, for one input, applies to E_θ .

Proof. By the column hypothesis and $v_\lambda / a^3 \leq C_J$, we have

$$\sup_{\lambda, \mu} \sum_y |b_{y;\lambda\mu}| \leq \sup_\lambda \sum_{y, \mu'} |b_{y;\lambda\mu'}| \leq C_J.$$

In other words, the column of a pair is bounded, because the column sum with one input fixed already sums over the other input. For any array h indexed by pairs, the Cauchy–Schwarz inequality with weights $|b_{y;\lambda\mu}|$ implies that

$$\sum_y \left| \sum_{\lambda, \mu} b_{y;\lambda\mu} h_{\lambda\mu} \right|^2 \leq \left(\sup_y \sum_{\lambda, \mu} |b_{y;\lambda\mu}| \right) \left(\sup_{\lambda, \mu} \sum_y |b_{y;\lambda\mu}| \right) \sum_{\lambda, \mu} |h_{\lambda\mu}|^2.$$

Multiplying by a^3 , we obtain the physical operator factor $a^{3/2}$. By Lemma 16.3, the cardinal weights are nonnegative, their row sums are one, and their column sums, weighted by the output cell volume, are a^3 . By Jensen's inequality, $I_a : L_a^2 \rightarrow L_\delta^2$ has norm at most one. The absolute rows and columns of its differences are bounded by $C_q a^{-q}$ times those of I_a , because each column is a smooth bump at scale a .

For $t \leq a^2$, we commute the differences through the heat semigroup. The absolute row is then $C_{J,q} a^{-q}$, the absolute column, weighted by the output cell volume, is $C_{J,q} a^{3-q}$, and the heat semigroup preserves these bounds. By the Schur test, we have the operator bound. The counting ℓ^2 norm of a row is at most its absolute row sum, which gives the pointwise bound in this range. For $t \geq a^2$, we put all differences on the heat kernel. Its L^1 estimate in Lemma 26.4 and the Schur test imply the operator bound, and its physical L^2 row, composed with the undifferentiated operator, gives the pointwise bound. The bounds of the two ranges agree at $t = a^2$ up to fixed constants. This proves the estimates for all orders, with no factor counting output sites.

For scalar coefficients, the counting norm of the vector indexed by ordered pairs is the Hilbert–Schmidt norm of the corresponding matrix. The orthogonal projection $A \mapsto (A + A^T)/2$ does not increase that norm, and the operator norm is at most the Hilbert–Schmidt norm. This proves the last assertion. $\square$

Mixed Duhamel terms with a background field

The next lemma bounds the coefficients of the mixed Duhamel terms $\mathcal{Q}_\delta(B, F)$ and $\mathcal{Q}_\delta(F, B)$, where B is a background heat path, fixed after conditioning, and F is linear in the counting inputs. Both orders are treated, and for each the proof must decide where to put the difference contained in $\mathcal{N}_\delta$. Table 26.2 summarizes the choice, with $u = t - s$. Two differences of the background suffice for these row bounds; the four differences assumed in the lemma are used only for the covering bounds of Lemma 26.7.

Range	Source difference on	Placement
$u \geq a^2$	B	keep it on B ; use the L_δ^2 row of the outer heat kernel
$u \geq a^2, s \leq u$	F_s	move it to the outer heat kernel by summation by parts
$s \geq u \geq a^2$	F_s	keep it on F_s
$u < a^2$	either	put all output differences on the source; contract by the heat semigroup

Table 26.2. Placement of differences in the proof of Lemma 26.6.

Lemma 26.6. *Assume the standing conventions of Section 26.2, and put $S = s + a^2$ and $U = t - s + a^2$. Suppose a coefficient map F_s , from a space of counting inputs to fields on the lattice of mesh δ , satisfies*

$$\|D_\delta^q F_s\|_{2 \rightarrow L_\delta^2} \leq C_q h S^{-q/2}, \quad \sup_x \|D_\delta^q F_s(x, \cdot)\|_2 \leq C_q h S^{-3/4 - q/2}$$

for all fixed orders q required below, and that $B(s) = P_s^\delta b^\delta$ satisfies

$$\|D_\delta^q B(s)\|_\infty \leq C_q R(s + a^2)^{-(\alpha+q)/2}, \quad 0 \leq q \leq 4. \quad (26.12)$$

The constants C_q are fixed, and B is fixed after conditioning. The constants in this lemma may depend on the prescribed constants C_q in the hypotheses on F_s and B , in addition to the standing deterministic data. The hypothesis (26.12) holds for $B(s) = P_s^\delta I_H b$ with $R = R_a(B)$ and C_q depending only on $q, M, J, \alpha, T/L^2$, and the interpolation function. For either order $\mathcal{Q}_\delta(B, F)$ or $\mathcal{Q}_\delta(F, B)$, the coefficient rows of the mixed Duhamel term after $r = 0, 1$ output differences are bounded by

$$C_J R h I_{3/4+r/2}^a(t),$$

and its weighted coefficient bound for Y_α^δ is at most $C_J R h a^{\alpha-3/2}$. These assertions hold with an ordered pair as the counting input, as well as with a single input.

Proof. Steps 3 and 4 are summarized in Table 26.2.

Step 1: backgrounds interpolated from mesh H . We first verify (26.12) for $B(s) = P_s^\delta I_H b$ and $R = R_a(B)$; the radius then controls all higher differences. At $s = 0$, since I_H reproduces the values of b at the sites of the lattice of mesh H , we have $\|b\|_\infty \leq \|I_H b\|_\infty \leq R a^{-\alpha}$. For $s \leq a^2$, by the bound on differences of the cardinal interpolation and heat contraction, we have $\|D_\delta^q B(s)\|_\infty \leq C_q R H^{-q} a^{-\alpha} \leq C_{q,B} R(s + a^2)^{-(\alpha+q)/2}$. For $s \geq a^2$, we write $B(s) = P_{s/2}^\delta B(s/2)$ and apply the L^1 bound of Lemma 26.4 to the differences of $P_{s/2}^\delta$, using $\|B(s/2)\|_\infty \leq R(s/2 + a^2)^{-\alpha/2}$. This proves the same bound for every fixed q . The constant depends on $B = H/a$ and q , but not on b or R . For a general background, the bounds (26.12) are hypotheses. The row estimates below use only $0 \leq q \leq 2$.

Step 2: the quadratic coefficient. We use the expression of the quadratic coefficient in Lemma 24.2, on the periodic lattice of side $L/\delta = nB^K \geq 4$ and with physical difference scale δ . That lemma writes the quadratic coefficient of the damped Wilson flow as a fixed finite sum of terms $c[T_1 u_i, T_2 D_\delta^\pm v_j]$, with fixed translations T_1, T_2 . We keep this order, these coefficients, and these translations in every Duhamel term of this section, including those in the estimates for the residual, for the square, and for the mean. Since this coefficient is a quadratic polynomial, no bound on the amplitude of the fields is needed to evaluate it. Subtracting $[f, g]$ from $[Tf, Tg]$ and dividing by δ gives the product rule

$$D^+[f, g] = [D^+ f, Tg] + [f, D^+ g],$$

and the backward rule follows by replacing T by T^{-1} . By this rule, a difference on F can be moved onto the whole bracket and onto B , with the translated arguments kept. Summation by parts moves a difference of the whole bracket to the outer heat kernel. All translations are isometries of L_δ^2 and of the supremum norm.

Step 3: the range $u = t - s \geq a^2$. A source term whose difference falls on B has coefficient row at most

$$C_J R h S^{-p_\alpha} u^{-3/4-r/2},$$

by the L_δ^2 bound of Lemma 26.4 on the differentiated outer heat kernel and the operator bound on F_s . If the difference of the source falls on F_s and $s \leq u$, summation by parts gives a term of this form and a term bounded by

$$C_J R h S^{-\alpha/2} u^{-5/4-r/2} \leq C_J R h S^{-p_\alpha} U^{-3/4-r/2},$$

since $S \leq U$ and $u \asymp U$. If $s \geq u \geq a^2$, we leave the difference on F_s ; its operator bound gives the factor $S^{-1/2}$, and the same bound follows. For the order $\mathcal{Q}_\delta(F, B)$ the difference of the source falls on B , and the first case applies; for the order $\mathcal{Q}_\delta(B, F)$ the second and third cases apply.

Step 4: the range $u < a^2$. We put every output difference on the source and contract by the heat semigroup. At output order one, the products have the distributions of differences BD^2F , $(DB)DF$, and $(D^2B)F$, with their fixed translations; at output order zero the total order is one. Each coefficient row is bounded by

$$C_J R h S^{-p_\alpha-3/4-r/2} \leq C_J R h S^{-p_\alpha} U^{-3/4-r/2},$$

because $U \leq 2a^2 \leq 2S$. The color brackets contribute a constant factor. Integrating the rows of Steps 3 and 4 in s , we obtain the bounds by $C_J R h I_{3/4+r/2}^a(t)$.

Step 5: the convolution integrals. Set $e = a^2$ and $p = p_\alpha < 1$. For $t \leq e$, direct integration bounds $I_q^a(t)$ by te^{-p-q} . For $t \geq e$ we split at $t/2$. On the early half, $t - s + e \asymp t + e$ and $\int_0^{t/2} (s + e)^{-p} ds \leq C_p(t + e)^{1-p}$. On the late half, $s + e \asymp t + e$, and integrating the other factor yields $C(t + e)^{1/4}$ for $q = 3/4$ and $Ca^{-1/2}$ for $q = 5/4$. In both regimes we obtain

$$I_{3/4}^a(t) \leq C_\alpha(t + a^2)^{1/4-p_\alpha}, \quad I_{5/4}^a(t) \leq C_\alpha a^{-1/2}(t + a^2)^{-p_\alpha}. \quad (26.13)$$

Multiplying by $(t + \delta^2)^\beta$ and $(t + \delta^2)^\alpha$, respectively, gives at most the same expressions with δ^2 replaced by a^2 . The remaining powers of $t + a^2$ are $\alpha/2 - 3/4$ and $(\alpha - 1)/2$, which are both negative. Evaluating at $t + a^2 = a^2$ bounds both weighted rows by $C_\alpha a^{\alpha-3/2}$. This proves the final assertion, with all input factors contained in h . $\square$

Covering numbers of coefficient rows

Lemma 26.7. *Assume the standing conventions of Section 26.2. Let $B(t) = P_t^\delta b^\delta$ be $\mathcal{F}_{j-1}$ -measurable and satisfy*

$$\|D_\delta^q B(t)\|_\infty \leq C_q R(t + a^2)^{-(\alpha+q)/2}, \quad 0 \leq q \leq 4,$$

where R is $\mathcal{F}_{j-1}$ -measurable and the C_q are fixed constants. Put $F_t = (g_j/a)P_t^\delta I_a E_\theta$, for either expansion of Lemma 26.2. For either order of the mixed Duhamel term of B and F , let $l_r(t, x)$ be a scalar coefficient row after r output differences, $r = 0, 1$. Set

$$S = g_3 R a^\beta, \quad A_r(t, x) = S^{-1}(t + \delta^2)^{\kappa_r} l_r(t, x), \quad \kappa_0 = \beta, \quad \kappa_1 = \alpha.$$

If $R = 0$, all these rows vanish, and we define their normalized versions to be zero. Otherwise the distance

$$d((r, t, x), (r', s, y)) = \|A_r(t, x) - A_{r'}(s, y)\|_2,$$

on the index set that includes the finite list of scalar components and differences, has diameter bounded by a fixed constant, and its covering numbers satisfy

$$N(v) \leq C(1 + L/a)^P v^{-Q}, \quad 0 < v \leq 1. \quad (26.14)$$

The same holds for the rows normalized pointwise, $l_r(t, x)/(g_3 R a I_{3/4+r/2}^a(t))$, extended continuously to $t = 0$. The constants depend only on the deterministic data and on $C_0, \dots, C_4$; they do not depend on the fluctuation cutoff, the coupling, the coarser field, the number of preceding averaging steps, or a/δ .

Proof. After conditioning, all rows are deterministic. By Lemmas 26.1 and 16.3, for every fixed order needed,

$$\sup_x \|D_\delta^q F_t(x, \cdot)\|_2 \leq C_{J,q} g_3 a(t + a^2)^{-3/4-q/2} \leq C_{J,q} g_3 a^{-1/2-q}.$$

Each of the two ordered sources, denoted G_t , is the quadratic coefficient applied to $B(t)$ and F_t in that order. By the product rule of Lemma 24.2 and the four assumed differences of the background, we have

$$\sup_x \|D_\delta^q G_t(x, \cdot)\|_2 \leq C_J R g_3 a^{-\alpha-3/2-q}, \quad 0 \leq q \leq 3. \quad (26.15)$$

Order four suffices, because at most $q + 1$ differences fall on either argument. The heat equations for B and F imply that, for $r = 0, 1$,

$$\sup_x \|D_\delta^r \partial_t G_t(x, \cdot)\|_2 \leq C_J R g_3 a^{-\alpha-7/2-r}.$$

Indeed, a time derivative replaces one factor by a sum of its second differences; together with the difference of the source and the r output differences, the largest order on the background is $r + 3 \leq 4$.

For $l_r(t) = \int_0^t P_{t-s}^\delta D_\delta^r G_s ds$, positivity and contraction of the heat semigroup also hold for rows in counting ℓ^2 , by the triangle inequality. Thus

$$\sup_{t,x} \|D_\delta^h l_r(t, x)\|_2 \leq C_J T R g_3 a^{-\alpha-3/2-r-h} \quad (h = 0, 1), \quad \partial_t l_r(t) = D_\delta^r G_t + \Delta_\delta l_r(t).$$

Commuting the Laplacian onto G_s in the last integral bounds the time derivative by $C_J R g_3 [a^{-\alpha-3/2-r} + T a^{-\alpha-7/2-r}]$; the largest order of the source here is $r + 2 \leq 3$, which is covered by (26.15).

Let $\ell_* = 1 + L + \sqrt{T}$ and $\epsilon = a/\ell_* < 1$. Since $a \leq L/4$, every power of a in these polynomial estimates may be replaced by the same power of ϵ , with a constant factor depending only on L and T . After division by $S = R g_3 a^{\alpha-1/2}$, the unweighted row, one spatial difference, and one time derivative are bounded by $C\epsilon^{-4}$, $C\epsilon^{-5}$, and $C\epsilon^{-6}$, respectively. For example, the spatial exponent is $2\alpha + 2 + r < 5$, and the time exponent is $2\alpha + 3 + r < 6$. By Lemma 26.6, the weighted row itself is bounded uniformly.

The weights have powers less than one. For $0 < k < 1$ and $h \geq 0$ we have $(z+h)^k - z^k \leq h^k$ for $z \geq 0$, since the left side decreases in z . This gives $|(t+\delta^2)^k - (s+\delta^2)^k| \leq |t-s|^k$, with no constant depending on δ . Applying this with $k = \beta$ and $k = \alpha$, and using a shortest lattice path on the torus for the spatial difference, we obtain

$$\|A_r(t, x) - A_r(s, y)\|_2 \leq C\epsilon^{-8} \{\text{dist}(x, y) + |t-s|^\beta\}. \quad (26.16)$$

Here $|t-s|$ and $|t-s|^\alpha$ are bounded by fixed multiples, depending on T , of $|t-s|^\beta$, and the exponent eight is a common bound for the preceding exponents. The triangle inequality and $\sup \|A_r\|_2 \leq C$ bound the diameter, also between different members of the finite list of components.

We also check the second normalization, which avoids a division by t at the initial time. We write

$$l_r(t) = t U_r(t), \quad U_r(t) = \int_0^1 P_{(1-v)t}^\delta D_\delta^r G_{vt} dv,$$

and $I_q^a(t) = t J_q(t)$ with

$$J_q(t) = \int_0^1 (vt + a^2)^{-p_\alpha} ((1-v)t + a^2)^{-q} dv.$$

This positive function has a lower bound depending only on L , T , α , and q , because $t + a^2 \leq T + (L/4)^2$. Differentiating its integrand shows that $|J_q'(t)| \leq C_\alpha a^{-2} J_q(t)$. By the formula for U_r and the heat equations, $\partial_t U_r$ is the integral over $v \in [0, 1]$ of

$$(1-v) P_{(1-v)t}^\delta \Delta_\delta D_\delta^r G_{vt} + v P_{(1-v)t}^\delta D_\delta^r \partial_s G_s|_{s=vt}.$$

The bounds on the source control it by $CRg_3a^{-\alpha-7/2-r}$, and one spatial difference of U_r is bounded by $CRg_3a^{-\alpha-5/2-r}$. By the row bounds of Lemma 26.6, $U_r/(g_3RaJ_{3/4+r/2})$ is bounded uniformly. By the quotient rule, its moduli in time and space are bounded by fixed multiples of ϵ^{-8} . For example, the largest factor in the time derivative of the numerator is $a^{-\alpha-9/2-r}$, with exponent below seven, and the derivative of J_q contributes only a^{-2} times the already bounded quotient. The formula defines the continuous value at $t = 0$, and the pointwise normalized rows obey the modulus (26.16) as well.

For $0 < v \leq 1$ we choose the spatial step $cv\epsilon^8$ and the time step $(cv\epsilon^8)^{1/\beta}$, where $c > 0$ depends only on the constant in the modulus. We choose nearest lattice sites when the spatial step is at least δ , and all lattice sites otherwise. Their number is at most $C(1 + L/(v\epsilon^8))^3$, and the time net has at most $C(1 + T/(v\epsilon^8)^{1/\beta})$ points. Both nets are deterministic, independent of the fluctuation variables and of the amplitude of the background. Multiplying these counts by each other and by the number of components and orders proves (26.14), for example with $P = \lceil 24 + 8/\beta \rceil$ and $Q = \lceil 3 + 1/\beta \rceil$. All constants have the stated independence, because only coefficient norms have been used and neither a sample nor the cutoff radius entered. $\square$

26.3 Concentration and maximal inequalities

We now combine the deterministic estimates with concentration under the conditional fluctuation kernel. We control a supremum over space and time in two ways. The first uses the covering bound of Lemma 26.7 and chaining; its constants do not depend on the fluctuation cutoff or on the coupling, and it applies to mixed Duhamel terms with general backgrounds (Lemma 26.8). The second uses the fluctuation cutoff to bound moduli of continuity of the realized fields, followed by a union bound over a finite net with spatial and time steps ca^{30} , whose logarithmic cardinality is bounded by $C \log(2 + L/a)$ (Lemma 26.9); its constants may depend on the fixed coupling and coarse side.

Lemma 26.8. *Assume the standing conventions of Section 26.2, and fix $\theta = \theta_{j-1}$ and $\mathfrak{d} \geq 0$. Let $(X_i)_{i \in \mathcal{T}}$ be a family of linear evaluations $X_i = l_i \zeta$, where $\zeta = w - \mathbb{E}_\theta w$ under $\mu_{\mathfrak{d}, \theta}$, indexed by a finite list of components, the output sites, and the times in $[0, T]$, with coefficients continuous in time at each site. Suppose that $\sup_i \|l_i\|_2 \leq C_0 S$, and that the distance $\|l_i - l_{i'}\|_2/S$ on $\mathcal{T}$ satisfies the covering bound (26.14). Then, for every $u \geq 0$ and $1 \leq p < \infty$,*

$$\mathbb{P}_\theta \left\{ \sup_i |X_i| > CS \sqrt{\Lambda_a + u} \right\} \leq Ce^{-u}, \quad \left\| \sup_i |X_i| \right\|_{L^p(\mu_{\mathfrak{d}, \theta})} \leq CS(\sqrt{\Lambda_a} + \sqrt{p}).$$

The constants depend only on the lower bound $1/2$ for the Hessian and on the constants C_0, C, P, Q in the covering bound; in particular, they do not depend on the fluctuation cutoff.

Proof. By Lemma 26.1 and Theorem 5.4, for each coefficient row l , we have $\log \mathbb{E}_\theta e^{z l \zeta} \leq z^2 \|l\|_2^2$ for $z \in \mathbb{R}$, and the exponential Markov inequality shows that $l \zeta$ satisfies $\mathbb{P}_\theta \{|l \zeta| >$

$2\|l\|_2\sqrt{u}\} \leq 2e^{-u}$; the same holds for every difference of two rows. We apply Lemma 16.4 with the distance $\|l_i - l_{i'}\|_2/S$, $s = CS$, $b = 0$, diameter at most $2C_0$, $A = C(1 + L/a)^P$, and entropy exponent Q . At a fixed reference index, the evaluation has coefficient norm at most C_0S . By the Cauchy–Schwarz inequality in the finite-dimensional space of ζ , limits of rows in this distance correspond to limits of the evaluations. Because the coefficients are continuous in time and the lattice is finite, the supremum is determined by a countable dense set. Since $\log(2A) + Q \leq C\Lambda_a$, the tail and moment bounds of Lemma 16.4 imply the assertions, with the explicit factor $\sqrt{p}$. If $S = 0$, all rows vanish. $\square$

Lemma 26.9. *Assume the standing conventions of Section 26.2. Fix $\theta = \theta_{j-1}$ and the conditional fluctuation kernel of Lemma 26.1, and let $\zeta = w_j - m_\theta$.*

1. *A linear evaluation $l\zeta$ with $\|l\|_2 \leq S$ satisfies $\mathbb{P}_\theta\{|l\zeta| > CS\sqrt{v}\} \leq 2e^{-v}$ for $v > 0$.*
2. *A quadratic evaluation $\zeta^T A \zeta - \mathbb{E}_\theta \zeta^T A \zeta$, with A symmetric and $\|A\|_{\text{HS}} \vee \|A\|_{\text{op}} \leq S$, satisfies*

$$\mathbb{P}_\theta\{|\zeta^T A \zeta - \mathbb{E}_\theta \zeta^T A \zeta| > C(\|A\|_{\text{HS}}\sqrt{v} + \|A\|_{\text{op}}v)\} \leq 2e^{-v}, \quad v > 0;$$

in particular, the threshold $CS(1 + v)$ suffices.

3. *Consider the following fields, all centered under $\mu_{\mathfrak{d}_j, \theta}$: the heat fields $P^\delta I_a f$ of arrays f on the lattice of mesh a whose entries are linear or quadratic in ζ , measured in H_η^δ for some $0 < \eta < 3/2$; the mixed Duhamel terms of s_j , or of the quadratic part of z_j , with a background $P^\delta I_{Hb}$ fixed after conditioning, measured in Y_α^δ ; and the quadratic Duhamel term $\mathcal{Q}_\delta(s_j, s_j) - \mathbb{E}_\theta \mathcal{Q}_\delta(s_j, s_j)$, measured in Y_α^δ . Suppose S bounds the weighted coefficient bound of the field and, for heat fields, also a^η times the coefficient norms of the entries of f . For a mixed Duhamel term let $R \geq R_a(P^\delta I_{Hb})$. In the mixed linear, mixed quadratic, and self-quadratic Duhamel cases, respectively, also require $S \geq Rg_3 a^{\alpha-1/2}$, $S \geq Rg_3^2 a^\alpha$, and $S \geq g_3^2 a^{2\alpha-1}$. These additional lower bounds are not imposed on heat fields. Then the norm of the field exceeds $CS\sqrt{\Lambda_a + u}$ in the linear case, and $CS(\Lambda_a + u)$ in the quadratic case, with probability at most Ce^{-u} , for every $u \geq 0$; the corresponding moment bounds hold for every $1 \leq p < \infty$.*

The constants in (1) and (2) depend only on the lower bound $1/2$ for the Hessian. The constants in (3) may in addition depend on the fixed γ , n , and the prescription of the cutoff radii, as stated in Section 26.1.

Proof. The proof of (3) uses the fluctuation cutoff to bound the moduli of continuity of the realized fields. The short times $t \leq a^2$, for heat fields, and $t \leq \min(a^2, a^8)$, for Duhamel terms, are treated without a time net. On the remaining times, crude bounds by a fixed power of a^{-1} suffice, because the net then has polynomially many points in $1/a$ and only the logarithm of its size enters.

Step 1: pointwise tails. By Lemma 26.1, the potential is smooth on a neighborhood of the closed product Ω_j and $D_w^2 V \geq m_* I$ with $m_* = 1/2$. The domain is a finite product of Euclidean balls, with the reflecting boundary condition. These are the hypotheses of Theorem 5.4, applied to the potential itself, including its determinant and Haar disintegration terms. For a linear row l , that theorem implies that $\log \mathbb{E}_\theta e^{z l \zeta} \leq z^2 \|l\|_2^2 / (2m_*)$, and optimizing Markov's inequality in z , we obtain the two-sided sub-Gaussian tail (1). For a symmetric matrix A , we write $a_2 = \|A\|_{\text{HS}}$ and $a_\infty = \|A\|_{\text{op}}$. By the same theorem, we have $\log \mathbb{E}_\theta e^{z(\zeta^\top A \zeta - \mathbb{E}_\theta \zeta^\top A \zeta)} \leq 8z^2 a_2^2 / m_*^2$ for $|z| \leq m_* / (4a_\infty)$. Chernoff's inequality, applied with both signs and $z = \min(c_1 \sqrt{v} / a_2, c_2 / a_\infty)$, gives (2). Vanishing norms are treated by continuity. Throughout, ζ is centered at the conditional mean m_θ under the same kernel.

Step 2: consequences of the cutoff. We now prove (3) by a finite net, including the endpoints of the time interval; this avoids any entropy at the scale δ . On Ω_j we have $\|\zeta\|_\infty \leq C_J p_j$, and, at fixed γ and n , $p_j \leq C_{\gamma,n} [1 + \log(2 + L/a)]$. In particular, $p_j^k \leq C_{\gamma,n,k} a^{-1}$ for every fixed k when $a \leq \min(1, \sqrt{T})$. The bounded remaining range of a is absorbed by enlarging the constants, which then depend on L , T , γ , and n .

Step 3: continuous extensions. We estimate the deterministic moduli through continuous extensions in space, starting from the smooth periodic cardinal interpolant on the torus. The exponential of the positive shift part defining P_t^δ acts on its translates for every real x , not only at lattice sites; this defines a smooth function whose restriction to the lattice is the discrete heat evolution. Continuous spatial derivatives commute with this convolution. Further,

$$\|\Delta_\delta f\|_{C^q} \leq C \|f\|_{C^{q+2}},$$

by the second-order Taylor formula with integral remainder for each symmetric difference. Thus a time derivative costs two derivatives of the smooth data, never a factor δ^{-2} . The same extension applies to translated brackets and to their Duhamel integrals.

Step 4: unweighted bounds on the cutoff. Let the background $P^\delta I_H b$ have radius at most R , and fix an order q of continuous spatial derivatives and an order k of time derivatives. On Ω_j , the supremum of $D^q \partial_t^k$ of each field or source satisfies the following bounds:

field or source	bound
$(g_j/a) P_t^\delta I_a E_\theta \zeta$	$C_{J,q,k} g_3 p_j a^{-1/2-q-2k}$
$P_t^\delta I_a B_\theta(\zeta, \zeta)$	$C_{J,q,k} p_j^2 a^{-q-2k}$
$\mathcal{N}_\delta(B, s_j), \mathcal{N}_\delta(s_j, B)$	$C_{J,q,k} R g_3 p_j a^{-\alpha-3/2-q-2k}$
$\mathcal{N}_\delta(B, g_3^2 P^\delta I_a B_\theta(\zeta, \zeta)), \text{ and the reversed order}$	$C_{J,q,k} R g_3^2 p_j^2 a^{-\alpha-1-q-2k}$
$\mathcal{N}_\delta(s_j, s_j)$	$C_{J,q,k} g_3^2 p_j^2 a^{-2-q-2k}$

To verify these entries, note that the absolute row bounds give $C_J p_j$ for $E_\theta \zeta$ and $C_J p_j^2$ for $B_\theta(\zeta, \zeta)$. A spatial derivative of their cardinal interpolants costs $C_q a^{-q}$, and the heat semigroup contracts every continuous derivative. By its initial value and the heat equation, the background is bounded by $C_{J,q,k} R a^{-\alpha-q-2k}$. Using the quadratic coefficient of Lemma 24.2, we multiply the two factors in each bracket and distribute its difference; this gives the last three entries. By the product rule for differences, the same bounds hold for differences of these smooth extensions. A time derivative is the sum of the terms with one factor differentiated, and it satisfies the listed bound. Centering at the conditional mean doubles these bounds. For a Duhamel field, spatial derivatives are bounded by T times the corresponding bound on the source, and its first time derivative is the source plus Δ_δ of the Duhamel field.

Step 5: heat fields. For a heat field measured with exponent $0 < \eta < 3/2$, we treat the times $0 \leq t \leq a^2$ without a time net. By heat contraction and the bound on differences of the cardinal interpolation, we have

$$\sup_{t \leq a^2} \{ (t + \delta^2)^{\eta/2} \|P_t^\delta I_a f\|_\infty + (t + \delta^2)^{(\eta+1)/2} \|D_\delta P_t^\delta I_a f\|_\infty \} \leq C_\eta a^\eta \max_y |f(y)|.$$

The maximum on the right is over at most $C_J (L/a)^3$ scalar entries, and we take a union bound over the tails of (1) or (2) at these entries. For $t \geq a^2$, the weights and their first time derivatives are bounded by fixed powers of a^{-1} . The following coefficient argument treats arbitrary admissible S , including for the special heat fields listed in the table. Let d_w be the number of scalar fluctuation coordinates of the step. Since there are J intermediate scales with fixed ratios of meshes, $d_w \leq C_J (L/a)^3$. On the cutoff domain, we have $\|\zeta\|_2 \leq C_J p_j \sqrt{d_w}$, and the assumed bounds on the coefficients of the entries of f give

$$\max_y |f_y| \leq C_J S a^{-\eta} \times \begin{cases} p_j \sqrt{d_w} & \text{for centered linear forms,} \\ p_j^2 d_w & \text{for centered quadratic forms;} \end{cases}$$

in the quadratic case we apply the bound on the operator norm both to the value of the form and to its expectation. By the bounds on the cardinal interpolation, heat contraction, and the continuous extension of Step 3, we have

$$\|D^q \partial_t^k P_t^\delta I_a f\|_\infty \leq C_{J,q,k} a^{-q-2k} \max_y |f_y|.$$

For $t \geq a^2$, the weighted value and first-difference components and their first spatial and time derivatives are bounded, after division by S , by a fixed power of a^{-1} , independently of a/δ . By the bounds of Step 2 on p_j , and since $d_w \leq C a^{-3}$, the exponent 30 suffices. If $S = 0$, the coefficients of the entries of f vanish, and so does the field.

Step 6: Duhamel fields. For a Duhamel term, we set $t_0 = \min(T, a^2, a^8)$. If $t \leq t_0 \leq a^2$, its weighted value and first difference, divided by S , are bounded by $Cp_j^k t/a^2$, with $k = 1$ for a mixed linear term and $k = 2$ for a quadratic term. Indeed, the last three source bounds of the table, multiplied by $ta^{2\alpha-1}$ and $ta^{2\alpha}$ at output orders zero and one, respectively, give this ratio for the natural scales $Rg_3a^{\alpha-1/2}$, $Rg_3^2a^\alpha$, and $g_3^2a^{2\alpha-1}$, respectively. The additional lower bounds on S make the same bounds valid after division by S . If $R = 0$, each mixed field vanishes. Since $p_j^k t_0/a^2 \leq p_j^k \min(1, a^6)$ is bounded at fixed coupling, using Step 2 also on the bounded coarse-mesh range, this is a deterministic contribution CS. If $t_0 < T$, then $t_0^{-1} = \max(a^{-2}, a^{-8}) \leq \max\{1, (L/4)^6\}a^{-8}$. Thus on $t \geq t_0$ a differentiated weight costs at most Ca^{-8} , with the allowed dependence on L . The table and the formula for the time derivative of a Duhamel term again bound the first moduli in space and time by CSa^{-30} . The largest unweighted exponent of a source, after division by its scale, is less than 6; together with the factor a^{-8} from the weight and the growth of the cutoff, the total exponent stays below 30. If $t_0 = T$, the deterministic bound covers the whole interval.

Step 7: the net. We choose spatial and time steps ca^{30} , with c small depending on the constant in the modulus. A net of the torus and of the time interval then has at most

$$C(1 + La^{-30})^3(1 + Ta^{-30})$$

points, and the error in each normalized weighted component is at most one. When a step of the spatial net exceeds δ , we choose nearest lattice sites, and otherwise all lattice sites; the displayed count bounds either choice. The continuous extension is used only to bound the modulus between these sites, and all probability estimates are evaluated on the lattice. The logarithm of this count, and of the count of the short-time entries in Step 5, is at most $C\Lambda_a$, independently of a/δ . We apply the finite-family clause of Lemma 16.4 with $s = CS$ and $b = 0$ for a linear field, and with $s = CS$ and $b = CS$ for a quadratic field, using the tails of Step 1. The deterministic error from interpolation and from short times is $e \leq CS$, and the logarithmic count is $C\Lambda_a$, with the dependence on the fixed γ and n stated above. This proves the maximal tails and all their moments. $\square$

26.4 The linear fluctuation and the mixed Duhamel terms

We now prove the estimates for the linear fluctuation s_j and its mixed Duhamel terms with a predictable background. The background x_{j-1} , the heat evolution of the coarser field interpolated from mesh H , is covered by part (3) of the next theorem. The interpolation defect r_j , a predictable heat path on the finest lattice, is covered by parts (4) and (5). For the general class of part (4), we assume bounds on four differences of the background at scale a . These are needed for the covering bound of Lemma 26.7, and a bound on the value and the first difference alone does not give them uniformly at small times.

Theorem 26.10. Assume the setting of Section 26.1 and the standing conventions of Section 26.2, and put $W_\theta(t) = P_t^\delta I_a E_\theta$, with $\theta = \theta_{j-1}$.

1. For every fixed integer $r \geq 0$, the coefficient rows and the operator of W_θ satisfy

$$\sup_x \|D_\delta^r W_\theta(t, x, \cdot)\|_{\ell^2(w)} \leq C_{J,r} a^{3/2} (t + a^2)^{-3/4-r/2},$$

$$\|D_\delta^r W_\theta(t)\|_{\ell^2(w) \rightarrow L_\delta^2} \leq C_{J,r} a^{3/2} (t + a^2)^{-r/2}.$$

Each fluctuation coordinate enters with its own cell volume v_λ , which is comparable to a^3 up to a factor depending only on M and J . For the same scalar output component, times $s, t \in [0, T]$ with $\tau = \min(s, t)$, and output sites x, y , the difference of the coefficient rows at (t, x) and at (s, y) is bounded in $\ell^2(w)$ by

$$C_{J,r} a^{3/2} (\tau + a^2)^{-3/4-r/2} \min \left\{ 1, \frac{\text{dist}(x, y)}{\sqrt{\tau + a^2}} + \frac{|t - s|}{\tau + a^2} \right\}.$$

2. For every $1 \leq p < \infty$, uniformly in θ_{j-1} ,

$$\|X_\alpha^\delta(s_j)\|_{L^p(\mu_{\mathfrak{d}_{j,\theta}})} \leq C_J g_3 a^\beta (\sqrt{\Lambda_a} + \sqrt{p}),$$

with C_J independent of p .

3. Let $B_j(t) = P_t^\delta I_H b$ be an $\mathcal{F}_{j-1}$ -measurable heat evolution of an array b interpolated from the lattice of mesh H , and let $R_j(B_j) = R_a(B_j)$ be its radius (26.11). For each fixed $r \geq 2$,

$$\|D_\delta^r B_j(t)\|_\infty \leq C_{r,B} R_j(B_j) (t + a^2)^{-(\alpha+r)/2},$$

where $C_{r,B}$ depends only on $r, M, J, \alpha, T/L^2$, and the interpolation function, and not on b or its radius. For each of the two orders $\mathcal{M}_j = \mathcal{Q}_\delta(B_j, s_j)$ and $\mathcal{M}_j = \mathcal{Q}_\delta(s_j, B_j)$, the value rows and first-difference rows of the coefficients are bounded by

$$C_J g_3 R_j(B_j) a I_{3/4}^a(t) \quad \text{and} \quad C_J g_3 R_j(B_j) a I_{5/4}^a(t),$$

respectively. Each order has conditional mean zero given $\mathcal{F}_{j-1}$. For every $u \geq 0$, on the event $\{R_j(B_j) \leq R\}$,

$$\mathbb{P} \left\{ Y_\alpha^\delta(\mathcal{M}_j) > C_J g_3 R a^\beta \sqrt{\Lambda_a + u} \mid \mathcal{F}_{j-1} \right\} \leq C e^{-u}.$$

The sum $\mathcal{B}_\delta(B_j, s_j)$ of the two orders satisfies the same estimate with a larger C_J .

4. The two mixed-term coefficient row bounds, conditional mean-zero assertions, and conditional maximal inequalities of (3) also hold for an $\mathcal{F}_{j-1}$ -measurable heat path $B_j(t) = P_t^\delta b_j^\delta$ on the lattice of mesh δ , provided an $\mathcal{F}_{j-1}$ -measurable radius $R_j^\sharp$ satisfies

$$\|D_\delta^q B_j(t)\|_\infty \leq C_q R_j^\sharp (t + a^2)^{-(\alpha+q)/2} \quad (0 \leq t \leq T, 0 \leq q \leq 4),$$

with fixed constants $C_0, \dots, C_4$; the heat equation then supplies the increments in time. For this class we replace $R_j(B_j)$ by $R_j^\sharp$ in both row bounds and in the conditional tail. The normalized coefficient distances of Lemma 26.7 have diameter bounded by a fixed constant and covering numbers $N(v) \leq C(1 + L/a)^P v^{-Q}$, $0 < v \leq 1$, with P, Q, C independent of a/δ , the lattice side, the number of averaging steps, θ_{j-1} , and the coupling. The covering is at scale a ; no maximum over all sites of the finest lattice is used.

5. The interpolation defect

$$B_j(t) = H^{-1} P_t^\delta (I_a P_j - I_H E_H) \theta_{j-1} = r_j(t)$$

belongs to the class of (4), with $R_j^\sharp = C_J \|\theta_{j-1}\|_\infty H^{\alpha-1}$. If $\|\theta_{j-1}\|_\infty \leq \bar{b}_{j-1}$ on the small-field event, for a deterministic $\bar{b}_{j-1}$, then for every $1 \leq p < \infty$,

$$\|Y_\alpha^\delta(\mathcal{Q}_\delta(s_j, r_j))\|_{L^p(v_K)} + \|Y_\alpha^\delta(\mathcal{Q}_\delta(r_j, s_j))\|_{L^p(v_K)} \leq C_{J,p} g_3 a^\beta (\sqrt{\Lambda_a} + \sqrt{p}) \bar{b}_{j-1} H^{\alpha-1}.$$

All assertions hold for either expansion of Lemma 26.2, with the conditionally centered vector ζ_j in both. The constants in the coefficient bounds depend only on L, T, M, J , the analytic domains, the column bounds, α , and r ; they do not depend on $K, j, \mathfrak{d}_j, a/\delta$, the dimension, the coupling, or θ_{j-1} . The constants in the conditional maximal inequalities may in addition depend on the fixed coupling, the coarse side, and the prescription of the cutoff radii. In part (4), the constants in the coefficient bounds, conditional maximal inequalities, and normalized diameter and covering bounds may additionally depend on the prescribed constants $C_0, \dots, C_4$, held fixed as the lattice parameters and the coupling vary. No bound on the expectation of a background radius is assumed or proved.

Proof. We first prove the deterministic assertions, then the probability estimates, and then the application to r_j .

Step 1: coefficient rows. By Lemma 26.1, the first tensor of the reconstruction maps has bounded absolute rows and volume-normalized column sums. Each fluctuation coordinate has the volume $v_\lambda = a_\lambda^3$ of the lattice of its intermediate scale, and the finite ratios between intermediate scales imply that $C_J^{-1} a^3 \leq v_\lambda \leq C_J a^3$; there is no comparison with δ^3 . The paragraph “Row sums and volume-normalized column sums” in the proof of Lemma 16.3 establishes the numerical row and operator bounds without exponential column decay. That argument therefore applies with input mesh a and output mesh δ and yields the two bounds of (1) for every fixed order r .

Step 2: increments of the rows. Fix r . A shortest coordinate path on the torus from x to y and the bound on rows of order $r+1$ bound the spatial increment by $C_{J,r} \text{dist}(x, y) a^{3/2} (t + a^2)^{-5/4-r/2}$; the triangle inequality gives twice the bound of order r instead. For the time increment, we integrate $\partial_t D_\delta^r W_\theta = \Delta_\delta D_\delta^r W_\theta$ from $\min(s, t)$ to $\max(s, t)$ and use the bound

of order $r + 2$. Combining these linear increment bounds with twice the bound on values, we obtain the stated minimum. The coordinate path has length at most $\sqrt{3}$ times the distance on the torus, which is absorbed into $C_{J,r}$.

Step 3: the linear fluctuation. Since $g_j/a = g_3a^{-1/2}$ and $\delta \leq a$, the weighted coefficient bound of s_j for X_α^δ is at most the supremum over t of

$$(t + \delta^2)^{\alpha/2} g_3a^{-1/2} a^{3/2} (t + a^2)^{-3/4} \quad \text{and} \quad (t + \delta^2)^{(\alpha+1)/2} g_3a^{-1/2} a^{3/2} (t + a^2)^{-5/4}.$$

Both are bounded by $C_J g_3a(t + a^2)^{\alpha/2-3/4} \leq C_J g_3a^{\alpha-1/2}$, because $\alpha < 3/2$. At a fixed coarser field, these evaluations are linear in ζ_j , and their conditional mean is zero. The entries of the initial array $(g_j/a)E_\theta \zeta_j$ have coefficient norms at most $C_J g_3a^{-1/2}$, and a^α times these norms is at most $C_J g_3a^\beta$. The conditional fluctuation kernel and its Hessian bound $1/2$ are verified in Lemma 26.1. By Lemma 26.9, with $S = C_J g_3a^\beta$, we have

$$\mathbb{P}_\theta\{X_\alpha^\delta(s_j) > C_J g_3a^\beta \sqrt{\Lambda_a + u}\} \leq Ce^{-u}.$$

Integrating this tail, we obtain the moment bound of (2). In detail, $\sqrt{\Lambda_a + u} \leq \sqrt{\Lambda_a} + \sqrt{u}$, and the positive part of the normalized norm beyond $\sqrt{\Lambda_a}$ has a tail bounded by Ce^{-cv^2} at level v . Its p th moment is at most $(C\sqrt{p})^p$: in the integral $p \int_0^\infty v^{p-1} e^{-cv^2} dv$, we bound $v^{p-1} e^{-cv^2/2}$ by its maximum and integrate $e^{-cv^2/2}$. The constants of Lemma 26.9 do not depend on p , so C_J does not depend on p .

Step 4: rows of the mixed terms. The bound on higher differences of $B_j = P^\delta I_H b$ was proved in Step 1 of the proof of Lemma 26.6, from the values of the interpolation at the sites of the lattice of mesh H at time zero and from the splitting of the heat semigroup at later times. Its constant $C_{r,B}$ depends only on $r, M, J, \alpha, T/L^2$, and the interpolation function, and not on the array b or its radius. We apply Lemma 26.6 with $F_s = (g_j/a)W_\theta(s)$ and $h = (g_j/a)a^{3/2} = g_3a$; by (1), F_s satisfies its hypotheses. For each order separately, we obtain the value rows $C_J g_3 R_j(B_j) a I_{3/4}^a(t)$ and the first-difference rows $C_J g_3 R_j(B_j) a I_{5/4}^a(t)$. The two orders are linear functions of ζ_j with $\mathcal{F}_{j-1}$ -measurable coefficients. Each therefore has conditional mean zero as a field in space and time, and no covariance needs to be computed.

Step 5: conditional maxima of the mixed terms. By Lemma 26.6, the weighted coefficient bound of each order is $C_J g_3 R_j(B_j) a^\beta$. We fix a coarser field in the measurable event $\{R_j(B_j) \leq R\}$. The four differences of the background were just proved, and Lemmas 26.7 and 26.8 yield the conditional tail of (3) for each order. A union bound for the two tails and the triangle inequality for the sum give the assertion for $\mathcal{B}_\delta(B_j, s_j)$; the factor two is absorbed into C_J and C . If $R = 0$, then on this event $R_j(B_j) = 0$ and $I_H b = B_j(0) = 0$; since the interpolation reproduces b at the sites of the lattice of mesh H , we have $b = 0$, and both terms vanish identically.

Step 6: general backgrounds. For the class of (4), we condition on $\mathcal{F}_{j-1}$ and use the radius $R_j^\sharp$ and the four assumed difference bounds. These are the hypotheses of Lemmas 26.6 and 26.7. The former implies both row bounds and the common weighted bound $C_J g_3 R_j^\sharp a^\beta$; its proof treats the difference in either argument. The latter implies the bounded diameter and the covering bound (26.14), independently of the cutoff and of the coupling; the heat equation gives the increments in time used there, with at most four differences of the background. Each term is linear in the conditionally centered ζ_j and has conditional mean zero. On $\{R_j^\sharp \leq R\}$, Lemma 26.8 implies the conditional maximum and its moments. If the radius vanishes, the bound on the background of order zero shows that both terms vanish. Every conditional event is measurable: the reconstruction and its coefficients are Borel, the lattice is finite, and suprema over continuous time are determined by rational times.

Step 7: the interpolation defect. We verify that r_j belongs to the class of (4), keeping the two lattices on which it is defined. Here $a = a_j$, $H = Ba$, and $\delta = a_K$, so

$$\delta = a/m, \quad m = B^{K-j} \in \mathbb{N}, \quad H/\delta = B^{K-j+1} \in \mathbb{N}.$$

We apply Lemma 17.2 with coarse mesh H , fine mesh a , $\mathfrak{d}_j$ preceding averaging steps, horizon T , and the integer refinements just displayed. By the expansion (26.3), its fine reconstruction of a lattice potential v on the coarser lattice is $P_j v$, and its coarse embedding is $E_H v$. It follows that its difference operator is $I_a P_j - I_H E_H$, and with $v = \theta_{j-1}/H$ its heat evolution is r_j . By that lemma, for every fixed integer q , we have

$$\|D_\delta^q r_j(t)\|_\infty \leq C_{J,q} \frac{\|\theta_{j-1}\|_\infty}{H} (t + H^2)^{-q/2} \frac{H}{\sqrt{t + H^2}}.$$

For $z = t/H^2$, $0 < \alpha < 1$, and $B \geq 1$ we have

$$(z + B^{-2})^{(\alpha+q)/2} (1+z)^{-(q+1)/2} \leq (1+z)^{(\alpha-1)/2} \leq 1.$$

Multiplying the bound on $\|D_\delta^q r_j(t)\|_\infty$ by $(t + a^2)^{(\alpha+q)/2} = H^{\alpha+q} (z + B^{-2})^{(\alpha+q)/2}$, we obtain

$$(t + a^2)^{(\alpha+q)/2} \|D_\delta^q r_j(t)\|_\infty \leq C_{J,q} \|\theta_{j-1}\|_\infty H^{\alpha-1}.$$

Choosing C_J large enough for $q = 0, 1, 2, 3, 4$, we obtain the radius $R_j^\sharp = C_J \|\theta_{j-1}\|_\infty H^{\alpha-1}$.

Step 8: moments under the conditioned Wilson measure. For either order separately, the conditional moment bound of (4) implies that

$$\left(\mathbb{E} \left[Y_\alpha^\delta (\mathcal{Q}_\delta(s_j, r_j))^p \mid \mathcal{F}_{j-1} \right] \right)^{1/p} \leq C_J g_3 a^\beta (\sqrt{\Lambda_a} + \sqrt{p}) \|\theta_{j-1}\|_\infty H^{\alpha-1},$$

and the same inequality holds with the two arguments interchanged, by the separate row computation for that order. We replace $\|\theta_{j-1}\|_\infty$ by $\bar{b}_{j-1}$, raise to the power p , and integrate over the law of θ_{j-1} under ν_K . By the tower property, we obtain both $L^p(\nu_K)$ bounds, and adding them gives (5) after changing $C_{J,p}$.

Step 9: the expansion at zero. By Lemma 26.1, we have the same rows and columns for $E_\theta^{(0)}$. Replacing E_θ by $E_\theta^{(0)}$ in the Schur, mixed-term, covering, and concentration arguments proves all assertions for the second expansion. The centered vector ζ_j , its domain, and its law are the same in both cases. $\square$

26.5 The conditionally centered nonlinear residual

The conditionally centered nonlinear reconstruction residual Z_j is the reconstructed field after subtraction of its conditional mean and of the linear term; it is not a quadratic approximation. Its quadratic part is a centered quadratic form in ζ_j , to which the quadratic concentration of Lemma 26.9 applies, and the rest is bounded deterministically on the cutoff domain.

Theorem 26.11. *Assume the setting of Section 26.1 and the standing conventions of Section 26.2, and put $a = a_j$, $\theta = \theta_{j-1}$, and $\zeta = w_j - m_\theta$. Let $p_j \geq 1$ bound the cutoff radii of this step, so that $\|\zeta\|_\infty \leq 2C_J p_j$.*

1. *The initial residual is*

$$Z_j = g_3^2 \{B_\theta(\zeta, \zeta) - \mathbb{E}_\theta B_\theta(\zeta, \zeta)\} + a^{-1} \{R_\theta(\zeta) - \mathbb{E}_\theta R_\theta(\zeta)\},$$

$$\text{and } \|a^{-1}(R_\theta - \mathbb{E}_\theta R_\theta)\|_\infty \leq C_J g_3^3 a^{1/2} p_j^3.$$

2. *The coefficients $b_{y;\lambda\mu}$ of B_θ satisfy*

$$\sup_y \sum_{\lambda, \mu} |b_{y;\lambda\mu}| \leq C_J, \quad \sup_\lambda \frac{a^3}{v_\lambda} \sum_{y, \mu} |b_{y;\lambda\mu}| \leq C_J$$

for either fixed input, where v_λ is the cell volume of the coordinate λ and $v_\lambda/a^3 \leq C_J$. Consequently

$$\|B_\theta\|_{\ell^2(\text{ordered pairs}) \rightarrow L_a^2} \leq C_J a^{3/2}.$$

For each scalar component and every fixed $r \geq 0$, the symmetric coefficient matrix $A_{t,x}^{(r)}$ of $D_\delta^r P_t^\delta I_a B_\theta(\zeta, \zeta)$ at x satisfies

$$\|A_{t,x}^{(r)}\|_{\text{HS}} \vee \|A_{t,x}^{(r)}\|_{\text{op}} \leq C_{J,r} a^{3/2} (t + a^2)^{-3/4-r/2}.$$

Centering changes only the constant term of the quadratic form, not this matrix.

3. For $0 < \sigma < 1 - \alpha$ and every $u \geq 0$,

$$\mathbb{P}_\theta \left\{ H_\sigma^\delta(I_a[B_\theta(\zeta, \zeta) - \mathbb{E}_\theta B_\theta(\zeta, \zeta)]) > C_J a^\sigma (\Lambda_a + u) \right\} \leq C e^{-u},$$

and the same bound holds for $H_\alpha^\delta = X_\alpha^\delta(P^\delta \cdot)$ with a^α in place of a^σ . In particular, for every fixed $1 \leq p < \infty$, both conditionally at every θ_{j-1} and under ν_K ,

$$\begin{aligned} \|H_\sigma^\delta(I_a Z_j)\|_p &\leq C_{p,J} \{g_3^2 a^\sigma (1+j) + g_3^3 a^{1/2} p_j^3\}, \\ \|X_\alpha^\delta(z_j)\|_p &\leq C_{p,J} \{g_3^2 a^\alpha (1+j) + g_3^3 a^{1/2} p_j^3\}. \end{aligned}$$

4. Let $B_j = x_{j-1}$, with radius $R_j = R_a(x_{j-1})$ as in Theorem 26.10. Each of $\mathcal{Q}_\delta(B_j, z_j)$ and $\mathcal{Q}_\delta(z_j, B_j)$ has conditional mean zero given $\mathcal{F}_{j-1}$. Its part that is quadratic in ζ has weighted coefficient bound, in Hilbert–Schmidt and operator norm for Y_α^δ , at most $C_J R_j g_3^2 a^\alpha$, and its remaining centered part has Y_α^δ norm at most $C_J R_j g_3^3 a^{1/2} p_j^3$. In particular, on $\{R_j \leq R\}$, each order and their sum satisfy

$$\mathbb{P} \left\{ Y_\alpha^\delta(\mathcal{B}_\delta(B_j, z_j)) > C_J R [g_3^2 a^\alpha (\Lambda_a + u) + g_3^3 a^{1/2} p_j^3] \mid \mathcal{F}_{j-1} \right\} \leq C e^{-u}.$$

The constants in the coefficient bounds have the dependence stated in Theorem 26.10; the constants in the maximal and L^p bounds may in addition depend on the fixed γ , n , p , and σ . All constants are uniform in $K \geq j$, in the number of preceding averaging steps, in θ_{j-1} , in the dimension, and in a/δ . Here $p_j \leq C_\gamma(1+j)$, and the factor $1+j$ absorbs Λ_a at the fixed coarse side n ; this does not give uniformity as $n \rightarrow \infty$. All norms include the first-difference component at short times. Both expansions of Lemma 26.2 are allowed.

Proof. The proof has five steps: the exact form of Z_j , the coefficient bounds, the maximal inequality for the quadratic part, the remainder and the moments, and the two mixed terms.

Step 1: the exact residual. We take the conditional expectation of the Taylor expansion (26.2); its linear term has expectation zero. Subtracting this conditional expectation, then the linear term, and dividing by a , we obtain

$$Z_j = \frac{g_j^2}{a} (B_\theta(\zeta, \zeta) - \mathbb{E}_\theta B_\theta(\zeta, \zeta)) + \frac{1}{a} (R_\theta(\zeta) - \mathbb{E}_\theta R_\theta(\zeta)).$$

Since $g_j^2/a = g_3^2$, this is the identity of (1). By (26.9) and $\|\zeta\|_\infty \leq 2C_J p_j$, the centered remainder is at most $2C_J g_j^3 p_j^3/a = C_J g_3^3 a^{1/2} p_j^3$.

Step 2: coefficient bounds. We apply the row and column bounds of Lemma 26.1 to the scalar coefficients of B_θ . A fluctuation coordinate on its intermediate lattice contributes its own volume v_λ , and converting the output volume to a^3 , we obtain the two bounds

of (2); the same holds with μ as the fixed input. The normalization by g_j^2 introduces only the fixed ratios of intermediate couplings, and the coefficients do not depend on the realized value of w_j . Lemma 26.5 now implies the operator bound for pairs and the bounds on the matrices $A_{t,x}^{(r)}$. The symmetric coefficient matrix represents the same quadratic polynomial, and subtracting the conditional mean changes only its constant term.

Step 3: the maximal inequality for the quadratic part. Let $f_y = B_\theta(\zeta, \zeta)_y - \mathbb{E}_\theta B_\theta(\zeta, \zeta)_y$. We first treat short times explicitly, including the first difference. For every $0 < \eta < 3/2$, by heat contraction and the bound on differences of the cardinal interpolation, we have

$$\begin{aligned} \sup_{0 \leq t \leq a^2} (t + \delta^2)^{\eta/2} \|P_t^\delta I_a f\|_\infty &\leq C_\eta a^\eta \max_y |f_y|, \\ \sup_{0 \leq t \leq a^2} (t + \delta^2)^{(\eta+1)/2} \|D_\delta P_t^\delta I_a f\|_\infty &\leq C_\eta a^\eta \max_y |f_y|. \end{aligned}$$

In the second line, we commute D_δ with the heat semigroup and use $\|D_\delta I_a f\|_\infty \leq C a^{-1} \max_y |f_y|$. There are only $C(L/a)^3$ scalar entries on the right, and the matrix of each entry has Hilbert–Schmidt and operator norm at most C_J , by its absolute row. For $t \geq a^2$, by (2), the weighted coefficient matrices have norms at most

$$C_{J,q} a^{3/2} (t + a^2)^{(\eta-3/2)/2} \leq C_{J,q} a^\eta, \quad q = 0, 1,$$

where the weight is $(t + \delta^2)^{\eta/2}$ for the value and $(t + \delta^2)^{(\eta+1)/2}$ for the difference. This proves the bound $C_J a^\eta$ for the weighted coefficient bound, including its first-difference component. By the Hessian bound verified in Lemma 26.1, the quadratic part of Lemma 26.9 applies; taking its union over short-time entries and its net at scale a , we obtain

$$\mathbb{P}_\theta \{H_\eta^\delta(I_a f) > C_J a^\eta (\Lambda_a + u)\} \leq C e^{-u}.$$

We take $\eta = \sigma$ for the first bound of (3), and $\eta = \alpha$ for the second; in the latter case $H_\alpha^\delta(I_a f)$ is the same heat expression $X_\alpha^\delta(P^\delta I_a f)$. No restriction on η coming from products is needed for this heat estimate; only the later products require $\alpha + \sigma < 1$.

Step 4: the remainder and the moments. Let $e = a^{-1}(R_\theta - \mathbb{E}_\theta R_\theta)$ and $\epsilon_j = C_J g_3^3 a^{1/2} p_j^3$. By the bounds on the cardinal interpolation and on the heat kernel, we have

$$\|P_t^\delta I_a e\|_\infty \leq \epsilon_j, \quad \|D_\delta P_t^\delta I_a e\|_\infty \leq C_J \epsilon_j (t + a^2)^{-1/2}.$$

In particular, both $H_\sigma^\delta(I_a e)$ and $H_\alpha^\delta(I_a e)$ are at most $C_{J,T,L,\alpha,\sigma} \epsilon_j$. For example, the difference component is bounded by $C \epsilon_j (t + a^2)^{\eta/2}$ after multiplication by its weight, for either $\eta = \sigma$ or $\eta = \alpha$, and the horizon and $a \leq L/4$ bound the last factor. Integrating the quadratic tail of Step 3, we obtain the conditional moment bound $C_{p,J} a^\eta (\Lambda_a + p + 1)$.

Since $\Lambda_{a_j} \leq C_{n,B}(1+j)$, combining this with the deterministic bound on the remainder, we find the two conditional L^p bounds of (3). For fixed p , we raise these conditional bounds to the power p and integrate over the law of $\mathcal{F}_{j-1}$ under ν_K ; the tower property gives the same bounds under ν_K . This changes neither the law of the coarsest configuration nor the conditional kernels of the finer variables.

Step 5: the mixed terms. We first treat the quadratic part. Let $V_s = P_s^\delta I_a B_\theta(\zeta, \zeta)$, regarded as a coefficient map from ordered pairs of fluctuation coordinates; its rows and operator norm are bounded by Lemma 26.5. We apply Lemma 26.6 with $F_s = g_3^2 V_s$ and $h = g_3^2 a^{3/2}$; the background $x_{j-1} = P^\delta I_H A_{j-1}$ satisfies its hypotheses with $R = R_j$. The unweighted value rows and first-difference rows of either order are

$$C_J R_j g_3^2 a^{3/2} I_{3/4}^a(t) \quad \text{and} \quad C_J R_j g_3^2 a^{3/2} I_{5/4}^a(t),$$

respectively, and after weighting they are at most $C_J R_j g_3^2 a^\alpha$. Rows indexed by pairs have the Hilbert–Schmidt norms of the corresponding matrices, and symmetrization gives the same bound for the operator norm. Centering the quadratic polynomial at its conditional expectation centers the Duhamel term, because, with the background fixed, the term is a fixed linear function of its initial input.

We prove the bound for the remaining part directly. Let $V_s^e = P_s^\delta I_a e$. By the bounds of Step 4 and the radius of the background, we have, in either order,

$$\|\mathcal{N}_\delta(B_j(s), V_s^e)\|_\infty + \|\mathcal{N}_\delta(V_s^e, B_j(s))\|_\infty \leq C_J R_j \epsilon_j (s + a^2)^{-p_\alpha}.$$

For the value of a Duhamel term, contraction and integration give $C R_j \epsilon_j (t + \delta^2)^{1-p_\alpha}$. For its first difference, we put the difference on the outer heat kernel and use Lemma 26.4; this gives the bound

$$C_J R_j \epsilon_j \int_0^t (t - s + \delta^2)^{-1/2} (s + a^2)^{-p_\alpha} ds \leq C_J R_j \epsilon_j (t + \delta^2)^{1/2-p_\alpha}.$$

To check the last inequality, we replace $s + a^2$ by $s + \delta^2$ and split the integral at $t/2$; both exponents are below one, and for $t \leq \delta^2$ the length of the interval suffices. The weights of Y_α^δ for the value and the difference leave the same nonnegative power $(t + \delta^2)^{\alpha/2}$. This proves the bound $C_J R_j \epsilon_j$ for the Y_α^δ norm. The same bound follows from Lemma 24.6 applied on the lattice of mesh $\delta = L/(nB^K)$, with first argument the heat path B_j , bounded in X_α^δ , and second argument the heat evolution of $I_a e$, bounded in H_σ^δ : there $X_\alpha^\delta(B_j) \leq R_j$, $H_\sigma^\delta(I_a e) \leq C \epsilon_j$ by Step 4, $\alpha + \sigma < 1$, and T is fixed. No hypothesis $\delta \leq 1$ is needed there: when $\delta > 1$, the clause of that lemma for coarse meshes applies to the finitely many integers $4 \leq nB^K < L$ at this fixed L , and the maximum of its constants over these meshes is an allowed constant.

Conditional on $\mathcal{F}_{j-1}$, the map from Z_j to either Duhamel term is linear, and $\mathbb{E}(Z_j \mid \mathcal{F}_{j-1}) = 0$. Hence the term, including the remaining part, has conditional mean zero. On

$\{R_j \leq R\}$, the quadratic part of Lemma 26.9 implies the threshold $Rg_3^2 a^\alpha (\Lambda_a + u)$, and the deterministic bound just proved yields the threshold $Rg_3^3 a^{1/2} p_j^3$. A union bound over the two orders gives the estimate for each order and for their sum.

Finally, for the expansion at zero, Lemma 26.1 provides the second tensor $B_\theta^{(0)}$ with the same column bounds, and the remainder $\tilde{R}_\theta$ of Lemma 26.2 has, after centering, the same deterministic bound. The Schur and mixed-term arguments then apply to this second decomposition. $\square$

26.6 The quadratic Duhamel term of the linear fluctuation

The last estimate of this section concerns $\mathcal{Q}_\delta(s_j, s_j)$. Its centered part is a quadratic form in ζ_j , and its conditional mean is in general not zero. The mean vanishes when the coarser field is the identity, by the invariance of the conditional law under global color rotations; at a nonzero coarser field we bound it by comparing the covariance of the linear reconstruction term with its value at the identity.

Theorem 26.12. *Assume the setting of Section 26.1 and the standing conventions of Section 26.2, and put $a = a_j$ and $r_\theta = \|\theta_{j-1}\|_\infty$. Either expansion of Lemma 26.2 may be used.*

1. *For each translated bracket in the quadratic coefficient $\mathcal{N}_\delta$, and for their sum, the matrix $C_{s,x}^{(r)}$ of the quadratic form $D_\delta^r \mathcal{N}_\delta(s_j(s), s_j(s))(x)$ in ζ_j satisfies*

$$\|C_{s,x}^{(r)}\|_{\text{HS}} \vee \|C_{s,x}^{(r)}\|_{\text{op}} \leq C_J g_3^2 a^2 (s + a^2)^{-2-r/2}, \quad r = 0, 1.$$

The corresponding matrices $A_{t,x}^{(r)}$ of $D_\delta^r \mathcal{Q}_\delta(s_j, s_j)(t, x)$ satisfy

$$\|A_{t,x}^{(r)}\|_{\text{HS}} \vee \|A_{t,x}^{(r)}\|_{\text{op}} \leq C_J g_3^2 \frac{t}{t + a^2} a^{3/2} (t + a^2)^{-3/4-r/2}, \quad r = 0, 1,$$

and hence the weighted coefficient bound of $\mathcal{Q}_\delta(s_j, s_j)$ for Υ_α^δ is $C_J g_3^2 a^{2\alpha-1}$. These coefficients keep the physical factor δ^3 of the heat semigroup and the meshes of the input coordinates; no factor $(a/\delta)^{3/2}$ is lost. Every further fixed difference in space or time needed for the covering at scale a is bounded by a fixed power of a^{-1} , with a constant depending on its order, and with the same factor g_3^2 .

2. *Regard the normalized first derivative of the reconstruction as a jointly analytic function of the unscaled coarser field and the coupling-scaled fluctuation inputs $q_{j,i} = g_{j,i} w_i$, before its base point in these inputs is evaluated at the conditional mean. Choose an inner polydisc whose gaps from the fixed larger polydisc are $\eta_\lambda > 0$ in each coordinate, and evaluate the bounds of (1) on the larger polydisc. For every finite multi-index $\mathbf{k}$, the bounds of (1) for the source and Duhamel coefficients hold for the derivatives $\partial^{\mathbf{k}}$, with the additional factor $\mathbf{k}! \prod_\lambda \eta_\lambda^{-k_\lambda}$. If an input coordinate on mesh a_λ is written instead as a lattice potential, or*

as a normalized fluctuation variable of coupling g_λ , it contributes the factor $a_\lambda^{k_\lambda}$, or $g_\lambda^{k_\lambda}$, respectively, and each difference at the location of that input may cost $2/a_\lambda$. Coordinates of the coarser field use the mesh H , the q coordinates use the mesh of their intermediate scale, and none uses δ . These derivatives are taken with the conditional centering held fixed; they are not derivatives of m_θ or of the conditional law.

3. Let Σ_θ be the conditional fluctuation covariance of w_j and $Z_\theta = E_\theta \Sigma_\theta E_\theta^\top$. Then $\|Z_\theta - Z_0\|_{\text{op}} \leq C_J r_\theta$, and Z_0 is proportional to the identity in the color indices. The conditional mean of the Duhamel term satisfies

$$\begin{aligned} \|\mathbb{E}_\theta \mathcal{Q}_\delta(s_j, s_j)(t)\|_\infty &\leq C_J g_3^2 r_\theta \frac{t}{t + a^2}, \\ \|D_\delta \mathbb{E}_\theta \mathcal{Q}_\delta(s_j, s_j)(t)\|_\infty &\leq C_J g_3^2 r_\theta (t + a^2)^{-1/2} \min(1, t/a^2), \\ Y_\alpha^\delta(\mathbb{E}_\theta \mathcal{Q}_\delta(s_j, s_j)) &\leq C_{J,T,\alpha} g_3^2 r_\theta. \end{aligned}$$

These means vanish when the coarser field is the identity, and they are kept when it is not.

4. For every $u \geq 0$,

$$\mathbb{P}_\theta \left\{ Y_\alpha^\delta(\mathcal{Q}_\delta(s_j, s_j) - \mathbb{E}_\theta \mathcal{Q}_\delta(s_j, s_j)) > C_J g_3^2 a^{2\alpha-1} [\log(2 + L/a) + u] \right\} \leq C e^{-u}.$$

5. Let $b_0 = \rho_0$ and $b_i = \rho_0 a_{\text{cap}}^i + C_J g_3 \sqrt{a_i} p_i$ for $i \geq 1$ be the deterministic bounds on $\|\theta_i\|_\infty$ on the small-field event, as in Theorem 17.4, with $a_{\text{cap}} \sqrt{B} < 1$. Then

$$\sup_{K \geq j} \mathbb{E}_{v_K} Y_\alpha^\delta(\mathcal{Q}_\delta(s_j, s_j)) \leq C_J g_3^2 \{a_j^{2\alpha-1} (1 + j) + b_{j-1}\}.$$

The right sides are summable over j , with tails that vanish uniformly in K , and each fixed higher moment satisfies the analogous bound.

The constants in the coefficient and mean bounds depend only on L , T , M , J , α , and the fixed analytic, covariance, and column bounds; they do not depend on the coupling, θ_{j-1} , K , j , the number of preceding averaging steps, or a/δ . The constants in the maximal inequalities may in addition depend on the fixed coupling, the coarse side, and the prescription of the cutoff radii.

The long-time bound in (1) is the main analytic point. Integrating the bound on the source alone would lose the averaging of the outer heat kernel over the cells of the input lattice. Instead, for $u = t - s \geq a^2$, we factor the coefficient matrix through physical space,

$$\ell^2(w) \xrightarrow{D_\delta W_s} L_\delta^2 \xrightarrow{M_{D_\delta^r p_u^\delta(x, \cdot)}} L_\delta^2 \xrightarrow{W_s^*} \ell^2(w),$$

where M_k denotes multiplication by k and W_s^* is the adjoint with respect to L_δ^2 . In the computation of the mean, on the other hand, the covariance enters as the counting matrix Z_θ between counting rows.

Proof. The proof has eight steps: the source matrices, the Duhamel matrices for long and short times, their weighted bounds, higher differences, derivatives in unscaled coordinates, the conditional mean, and the maximal and summability claims.

Step 1: the source matrices. We use the quadratic coefficient of Lemma 24.2 on the same periodic lattice of side $nB^K \geq 4$ and mesh δ as in Lemma 26.6. Set $\lambda = g_j/a = g_3a^{-1/2}$ and $W_s = P_s^\delta I_a E_\theta$. For one scalar color component of one translated bracket of the ordered source, the coefficient matrix is a fixed finite sum of rank-one matrices

$$\lambda^2 l_s \otimes m_s,$$

where l_s is a translated value row of W_s and m_s is a translated first-difference row of W_s . The color coefficients are the structure constants of $\mathfrak{su}(2)$, which correspond to the cross product on $\mathbb{R}^3$. The counting Hilbert–Schmidt norm of $l \otimes m$ is $\|l\|_2 \|m\|_2$. At output order $r = 0, 1$, the product rule distributes a total of $r + 1$ differences over the two rows. By Theorem 26.10(1), at each (s, x) ,

$$\|C_{s,x}^{(r)}\|_{\text{HS}} \leq C_J \lambda^2 a^3 (s + a^2)^{-3/2-(r+1)/2} = C_J g_3^2 a^2 (s + a^2)^{-2-r/2}.$$

Symmetrization gives the same quadratic polynomial and does not increase this bound, and the operator norm is at most the Hilbert–Schmidt norm. This proves the first bound of (1) for each translated bracket and, by the triangle inequality, for the ordered source, which is their finite sum.

Step 2: long times. Let $u = t - s \geq a^2$, and fix one translated scalar summand. After r output differences on the heat kernel, its coefficient for the outer heat semigroup has the form

$$\lambda^2 W_s^* M_{D_\delta^r p_u^\delta(x, \cdot)} D_\delta W_s,$$

with the fixed translations and color projections inserted in their places. For a multiplier k , expanding in an orthonormal basis of $\ell^2(w)$, we find

$$\|M_k D_\delta W_s\|_{\text{HS}}^2 = \delta^3 \sum_y |k(y)|^2 \|D_\delta W_s(y, \cdot)\|_2^2.$$

Translations and color projections change only the constant. The ideal inequality $\|AB\|_{\text{HS}} \leq \|A\|_{\text{op}} \|B\|_{\text{HS}}$, obtained by summing $\|ABe_i\|^2 \leq \|A\|_{\text{op}}^2 \|Be_i\|^2$ over the basis, implies that

$$\begin{aligned} \|\lambda^2 W_s^* M_{D_\delta^r p_u^\delta} D_\delta W_s\|_{\text{HS}} &\leq C_J \lambda^2 a^{3/2} a^{3/2} (s + a^2)^{-5/4} u^{-3/4-r/2} \\ &= C_J g_3^2 a^2 (s + a^2)^{-5/4} u^{-3/4-r/2}. \end{aligned}$$

Here we used the operator norm of W_s from Theorem 26.10(1) and the L_δ^2 bound on the heat kernel from Lemma 26.4. The factor δ^3 is part of the identity, and therefore neither a counting norm over the output lattice nor a factor $(a/\delta)^{3/2}$ occurs.

Step 3: short times. For $0 \leq u < a^2$, we commute the output differences onto the source and use heat contraction on the matrix-valued field. By the bound of Step 1, we have

$$\|A_{u,s,x}^{(r)}\|_{\text{HS}} \vee \|A_{u,s,x}^{(r)}\|_{\text{op}} \leq C_J g_3^2 a^2 (s + a^2)^{-2-r/2},$$

where $A_{u,s,x}^{(r)}$ is the matrix of the integrand at time s . The differences of the smooth cardinal interpolation up to order two justify this bound also at $s = 0$.

Step 4: integration in time and weights. For $t \leq 2a^2$, the source bound and contraction give $C_J g_3^2 t a^{-2-r}$, which is at most a fixed multiple of the claimed bound and retains its factor $t/(t + a^2)$. For $t \geq 2a^2$, we split the integral at $s = t/2$. On the early half, $u \geq t/2 \geq a^2$, and

$$a^2 \int_0^{t/2} (s + a^2)^{-5/4} u^{-3/4-r/2} ds \leq C a^{3/2} t^{-3/4-r/2},$$

because $\int_0^\infty (s + a^2)^{-5/4} ds = 4a^{-1/2}$. On the part of the late half with $u \geq a^2$, the bound is

$$C a^2 t^{-5/4} \int_{a^2}^{t/2} u^{-3/4-r/2} du.$$

For $r = 0$ this is at most $C a^2 t^{-1}$, which is at most $C a^{3/2} t^{-3/4}$ since $a^2 \leq t$; for $r = 1$ the integral is at most $C a^{-1/2}$, and the bound is $C a^{3/2} t^{-5/4}$. The final interval $u < a^2$ contributes at most $C a^4 t^{-2-r/2}$, which is at most $C a^{3/2} t^{-3/4-r/2}$ for the same reason. Multiplying by g_3^2 , and noting that $t/(t + a^2)$ lies between $1/2$ and 1 in this range, we obtain the bound of (1) at all times.

For either component of Y_α^δ , replacing the weight $t + \delta^2$ by the larger $t + a^2$ leaves the power $(t + a^2)^{\alpha-5/4}$ in that bound, and this exponent is negative. The supremum is therefore at most $C_J g_3^2 a^{3/2} (a^2)^{\alpha-5/4} = C_J g_3^2 a^{2\alpha-1}$. The factor $t/(t + a^2)$ remains in the unweighted bound; in particular the Duhamel term vanishes at $t = 0$.

Step 5: higher differences. Every further fixed spatial difference follows by distributing differences in the finite translated products and on the rows of the heat-smoothed interpolation kernel. For a polynomial bound, it suffices to put all outer differences on the smooth source. If the total spatial order is q and the time order is k , the source coefficients are bounded in Hilbert–Schmidt norm by $C_{J,q,k} g_3^2 a^{-2-q-2k}$ on $[0, T]$. The Duhamel coefficients are bounded by T times this bound at time order zero, and their time derivative is the source plus the discrete Laplacian of the Duhamel term. This proves the bounds by fixed powers of a^{-1} , with the factor g_3^2 .

Step 6: derivatives in the unscaled coordinates. We use the bounds on the absolute joint Taylor coefficients verified in Lemma 26.1, and we choose the evaluation and larger polydiscs there, with the displayed gaps. Before the conditional mean is substituted,

we regard E as a function of the unscaled coarser field and of the base point q in the coupling-scaled fluctuation inputs. The scalar quadratic coefficient matrices above are holomorphic functions of these variables: they use bilinear transposes, and neither complex conjugation nor a conditional covariance. Repeating the absolute row, Schur, and heat arguments with the coefficient bounds valid on the larger polydisc, we find that the bounds of (1) hold there. For a finite multi-index $\mathbf{k}$, we iterate the Cauchy integral formula on circles of radii η_λ about the evaluation point. For a matrix $A(q)$ in finite dimensions, this gives

$$\partial^{\mathbf{k}} A(q) = \frac{\mathbf{k}!}{(2\pi i)^{|\text{supp } \mathbf{k}|}} \int \cdots \int \frac{A(q+z) \prod_{\lambda \in \text{supp } \mathbf{k}} dz_\lambda}{\prod_{\lambda \in \text{supp } \mathbf{k}} z_\lambda^{k_\lambda+1}}.$$

This identity follows by expanding the power series on slightly smaller circles, integrating term by term, and noting that a nonconstant Fourier monomial integrates to zero over a circle; one then increases the circles to the chosen radii. By the triangle inequality in Hilbert–Schmidt or operator norm, the integral is at most the bound on the larger polydisc times $\mathbf{k}! \prod_\lambda \eta_\lambda^{-k_\lambda}$, and the constants of the heat estimates are unaffected. If the input coordinate is $q_\lambda = a_\lambda A_\lambda$ or $q_\lambda = g_\lambda w_\lambda$, the chain rule gives the factor $a_\lambda^{k_\lambda}$ or $g_\lambda^{k_\lambda}$. For an array at an input location on mesh a_λ , the bound $\|D_{a_\lambda} f\| \leq 2a_\lambda^{-1} \|f\|$ gives the possible cost of a difference. Coordinates of the coarser field and the q coordinates keep their own meshes. Only after the differentiation do we evaluate at the real base point given by the conditional mean; this base point and the conditional law are held fixed throughout.

Step 7: the conditional mean. We first treat the expansion at the conditional mean. We repeat the single-kernel calculation in Step 1 of the proof of Theorem 17.4, leading to (17.22), for the replacement reconstruction of Lemma 26.1, tracking uniformity below g_* . The unchanged conditional kernel satisfies

$$\|\Sigma_\theta\|_{\text{op}} \leq 2, \quad \|\Sigma_\theta - \Sigma_0\|_{\text{op}} \leq C_J t_j p_j^3 r_\theta, \quad \|m_\theta\|_\infty \leq C_J t_j r_\theta$$

by Theorem 13.10. The coarse-field segment and the coupling-scaled mean segment lie in the fixed evaluation product. The latter has length at most $C_J g_j t_j r_\theta$, and the mean at zero coarse field is zero. The normalized joint coefficient bounds of Lemma 26.1 and Schur’s test therefore give $\|E_\theta\|_{\text{op}} + \|E_0\|_{\text{op}} \leq C_J$ and $\|E_\theta - E_0\|_{\text{op}} \leq C_J r_\theta$, with constants uniform below g_* . The ratios of intermediate couplings and cell volumes contribute only fixed M, J factors. The identity

$$\begin{aligned} E_\theta \Sigma_\theta E_\theta^\top - E_0 \Sigma_0 E_0^\top &= (E_\theta - E_0) \Sigma_\theta E_\theta^\top + E_0 (\Sigma_\theta - \Sigma_0) E_\theta^\top \\ &\quad + E_0 \Sigma_0 (E_\theta - E_0)^\top \end{aligned}$$

then gives

$$\|Z_\theta - Z_0\|_{\text{op}} = \|E_\theta \Sigma_\theta E_\theta^\top - E_0 \Sigma_0 E_0^\top\|_{\text{op}} \leq C_J r_\theta (1 + t_j p_j^3) \leq C_J r_\theta,$$

because the common upper bound on the couplings bounds $t_j p_j^3$, by (26.1). The conditional covariance Σ_θ itself is that of the unchanged kernel in Theorem 13.10. The reference Z_0 is built from the conditional law at the zero coarser field. The blocks of its covariance in the color indices are scalar, and so is the linear coefficient at the zero coarser field. Hence Z_0 is proportional to the identity in the color indices, with all its spatial correlations kept.

Here Z_θ is the counting matrix of Section 26.1, and its operator norm is the counting operator norm. The physical operator $a^3 Z_\theta$ is not inserted between counting rows, and the adjoint in the factorization of Step 2 is a different object.

Let $V_s = P_s^\delta I_a$ be the heat evolution of the cardinal interpolation with identity input. For one bracket and one color component, the expected source is the contraction of two rows of V_s against $\lambda^2 Z_\theta$. The contraction against Z_0 vanishes: its two color indices enter through δ_{bc} , while the bracket coefficient ϵ_{bcd} is antisymmetric in b and c . This holds for every translation and difference in the quadratic coefficient. Equivalently, the expected vector is fixed by every adjoint color rotation, and such a vector is zero. We subtract this vanishing contraction and use the operator norm of $Z_\theta - Z_0$ between the rows of V_s . By Lemma 16.3 with identity input, or by the Schur argument of Lemma 26.5,

$$\|\mathbb{E}_\theta D_\delta^r \mathcal{N}_\delta(s_j(s), s_j(s))\|_\infty \leq C_J g_3^2 r_\theta a^2 (s + a^2)^{-2-r/2}, \quad r = 0, 1.$$

For the value, heat contraction and the exact integral

$$a^2 \int_0^t (s + a^2)^{-2} ds = \frac{t}{t + a^2}$$

give the first bound of (3). For the first difference at $t \leq a^2$, the differentiated source integrates to $C_J g_3^2 r_\theta t a^{-3}$, which is at most $C_J g_3^2 r_\theta (t + a^2)^{-1/2} (t/a^2)$. For $t \geq a^2$, we split at $s = t/2$. On the early half, we put the difference on the outer heat kernel and use its L^1 bound; the contribution is at most

$$C_J g_3^2 r_\theta t^{-1/2} a^2 \int_0^{t/2} (s + a^2)^{-2} ds \leq C_J g_3^2 r_\theta t^{-1/2}.$$

On the late half, we put the difference on the source and contract by the heat semigroup, which gives $C_J g_3^2 r_\theta a^2 t (t + a^2)^{-5/2} \leq C_J g_3^2 r_\theta t^{-1/2}$. This proves the second bound of (3) in both regimes. The weight of the value has the positive exponent β , and the weight of the difference leaves the bounded power $(t + a^2)^{\alpha-1/2}$; the fixed horizon and $a \leq L/4$ then give the bound on Y_α^δ of the mean.

For the expansion at zero, the same covariance comparison follows from Lemma 26.1: we integrate the derivative of $E_\theta^{(0)}$ in the coarser field along the segment to the zero coarser field, and use the bound on $\Sigma_\theta - \Sigma_0$ of Theorem 13.10. At the zero coarser field, $m_0 = 0$, which implies that the reference covariance and the reference linear tensor are the same for both expansions. The contraction above then implies the same bounds on the mean.

Step 8: maxima and summability. At a fixed coarser field, the Duhamel term is a quadratic polynomial in the centered vector ζ_j . Its matrices are bounded in (1), and its conditional mean in (3). By the Hessian bound verified in Lemma 26.1, Lemma 26.9 applies with $S = C_J g_3^2 a^{2\alpha-1}$; its proof uses the Duhamel factor at very short times and the polynomial bounds of Step 5 afterwards, and it covers both the value and the first-difference components. We obtain the centered tail (4), with $\Lambda_a = \log(2 + L/a)$.

For each fixed $p \geq 1$, integrating this conditional tail and adding the bound on the mean, we obtain

$$\left(\mathbb{E}_\theta Y_\alpha^\delta(\mathcal{Q}_\delta(s_j, s_j))^p \right)^{1/p} \leq C_{p,J} g_3^2 \{ a_j^{2\alpha-1} (\Lambda_{a_j} + p + 1) + r_\theta \}.$$

The deterministic bound (26.8), from Lemma 15.10, with $C_J \geq C_{\text{cap},J}$ and with the bound on the coarsest field denoted by ρ_0 , implies that $r_\theta \leq b_{j-1}$ on the small-field event; this is the bound used in Theorem 17.4. We integrate the conditional bound on the p th moment over the law of θ_{j-1} under ν_K and use $\Lambda_{a_j} \leq C_{n,B}(1+j)$. For $p = 1$, we obtain the bound of (5), and the same computation gives its versions for fixed higher moments.

Finally, if $c > 0$ and $q \geq 0$, then, since $a_{m+h} = a_m B^{-h}$ and $1 + m + h \leq (1 + m)(1 + h)$,

$$\sum_{j \geq m} a_j^c (1+j)^q \leq a_m^c (1+m)^q \sum_{h \geq 0} B^{-ch} (1+h)^q < \infty.$$

The last series converges because geometric decay eventually dominates the polynomial factor, and the right side tends to zero as $m \rightarrow \infty$. We apply this with $c = 2\alpha - 1 > 0$ and $c = 1/2$, and we sum the geometric sequence $\rho_0 a_{\text{cap}}^{j-1}$ separately; the condition $a_{\text{cap}} \sqrt{B} < 1$ implies $a_{\text{cap}} < 1$. Since $p_j \leq C_\gamma(1+j)$, every tail is summable, uniformly in K . The deterministic majorants do not depend on K or on the coarser field, and therefore the convergence over short scales is uniform in the number of steps, at fixed coupling and coarse side. $\square$

Remark 26.13. The mixed terms with one conditionally centered input have conditional mean zero. The centered part of the contribution $\mathcal{Q}_\delta(s_j, s_j)$ to the quadratic Picard term is summable because $2\alpha - 1 > 0$; its mean is summable because $\|\theta_{j-1}\|_\infty \leq b_{j-1}$.

Conditional means and propagated fluctuations

This section completes the estimates of Section 26 on the heat evolution of the reconstructed field and its quadratic Picard term under the conditioned Wilson measure. Recall the decomposition of Lemma 26.2,

$$x_j - x_{j-1} = s_j + z_j + r_j + u_j,$$

where s_j is the centered linear fluctuation term, z_j the centered nonlinear heat term, r_j the interpolation defect, and u_j the predictable term. The latter is $u_j = P^\delta I_{a_j} [U_j^{\text{cl}} + m_j^{\text{ph}}]$, the heat evolution of the sum of the nonlinear part U_j^{cl} of the zero-fluctuation reconstruction and the conditional mean correction m_j^{ph} . Section 26 proved the fluctuation estimates for s_j and z_j ; Section 28 combines them with the estimates proved here to bound all ordered products. Here we treat the terms built only from r_j and u_j . For the quadratic Picard term, these are $\mathcal{B}_\delta(x_{j-1}, r_j)$ and $\mathcal{Q}_\delta(r_j, r_j)$. The products involving u_j are handled in Section 28 by the bilinear estimate of Lemma 24.6, once u_j is bounded in the heat norm H_σ^δ of its initial field; we prove that bound here.

Three ordered terms. Let $H = a_{j-1}$ and $a = a_j$. The coarser field enters both x_{j-1} and r_j linearly: with

$$U = P^\delta I_a P_j / H, \quad V = P^\delta (I_a P_j - I_H E_H) / H,$$

we have $x_{j-1} = (U - V)\theta_{j-1}$ and $r_j = V\theta_{j-1}$. For any bilinear map $\mathcal{N}$,

$$\begin{aligned} \mathcal{N}(U - V, V) + \mathcal{N}(V, U - V) &= \mathcal{N}(U, V) + \mathcal{N}(V, U) - 2\mathcal{N}(V, V), \\ \mathcal{N}(U, U) - \mathcal{N}(U - V, U - V) &= \mathcal{N}(U, V) + \mathcal{N}(V, U) - \mathcal{N}(V, V). \end{aligned}$$

Applying these identities with $U = U\theta_{j-1}$ and $V = V\theta_{j-1}$, we obtain $\mathcal{B}_\delta(x_{j-1}, r_j) = \mathcal{T}^{UV} + \mathcal{T}^{VU} - 2\mathcal{T}^{VV}$ and $\mathcal{Q}_\delta(r_j, r_j) = \mathcal{T}^{VV}$, where $\mathcal{T}^{UV}$, $\mathcal{T}^{VU}$, and $\mathcal{T}^{VV}$ are the quadratic Duhamel terms of the ordered pairs $(U\theta_{j-1}, V\theta_{j-1})$, $(V\theta_{j-1}, U\theta_{j-1})$, and $(V\theta_{j-1}, V\theta_{j-1})$. Since $\mathcal{N}_\delta$ is not symmetric, the three terms are estimated separately; a bound on their sum would not bound the summands.

Propagation through coarser scales. The coarser field θ_{j-1} depends on the fluctuation variables of all coarser steps. We propagate both inputs back through the linearized reconstructions $L_{i+1}, \dots, L_{j-1}$ and telescope over the coarser scales i ; each step of the

telescope contributes a conditionally centered term $Z_{j,i}^\omega$ and a conditional drift $e_{j,i}^\omega$ (see (27.30)). Theorem 27.5 bounds the coefficients and the drifts, Theorem 27.6 bounds and sums the centered terms, and Theorem 27.7 controls the predictable term.

The scaling. The maps between lattices are

$$G \xrightarrow{L_{j-1} \cdots L_{i+1}} H \xrightarrow{P_j} a \xrightarrow{I_a} \delta \quad \text{and} \quad H \xrightarrow{I_H} \delta,$$

where $G = a_i$ and each arrow is labeled by its map and points to the mesh of its target. The interpolation defect is the difference of the two branches out of H . Each column of this difference with a fixed input on the lattice of mesh H has total mass zero (Lemma 27.1); after heat evolution for time s this gives the factor $d_H(s) = H(s + H^2)^{-1/2}$. The cancellation must be used at the scale H , before the earlier maps are composed: a first moment measured at the scale G would give the wrong length. Each linearization L_i has absolute row sums and volume-normalized column sums at most $a_{\text{cap}} = C_{\text{CP}}/B$. Dividing by the mesh H of the coarser lattice, we obtain the contraction

$$\frac{a_{\text{cap}}^{2(j-1-i)}}{H} = \frac{q_{\text{anc}}^{j-1-i}}{G}, \quad q_{\text{anc}} = Ba_{\text{cap}}^2 < 1,$$

for a quadratic term propagated from scale i . Finally, the Schur bound in counting norms gives a factor $(G/\delta)^{3/2}$, which cancels against the factor $\delta^{3/2}$ of the ℓ^2 row of the heat kernel on the finest lattice.

Inputs and constants. We use the setting and notation of Sections 26.1 and 26.2, including the conditioned Wilson measure ν_K , the filtration $\mathcal{F}_j$, the normalized fluctuation variables w_j , the norms X_α^δ , Y_α^δ , H_σ^δ , and V_σ^δ of Definition 4.8, and the four terms of Lemma 26.2. Throughout, $1/2 < \alpha < 2/3$, $\beta = \alpha - 1/2$, $0 < \sigma < 1 - \alpha$, and the horizon T is fixed. All heat kernels keep their constant mode, and all estimates concern the whole torus. The inputs are the conditional fluctuation kernel and the bounds on conditional means and covariances of Theorems 13.8 and 13.10; the reconstruction maps and the bounds on their absolute Taylor coefficients of Theorems 15.13 and 15.14; the derivatives of the zero-fluctuation reconstruction and the bounds on reconstructions of Lemmas 17.1 and 17.2 and Theorem 17.4; the interpolation and heat estimates of Lemmas 16.1, 16.3, and 16.4; the quadratic coefficient of Lemma 24.2; and the concentration inequalities of Theorem 5.4. The results are used in Section 28, which assembles the bounds on the linear and quadratic flow data, and in Sections 29 and 34.

We summarize the dependence of the constants in Table 27.1. The bounds on coefficients and on conditional means hold uniformly in the coupling below the common upper bound g_* . The maximal inequalities and the summations use the bound $b_i/g_i \leq C_\gamma(1+i)$ on the coarser fields, which holds only at fixed positive coupling.

Estimate	Constants may depend on
Unweighted coefficients and derivatives in w (Lemma 27.1; Theorems 27.5 and 27.7, time-dependent rows)	L, T, M, J , the interpolation function, the analytic domains and column bounds, and the orders of differences
Pointwise conditional means and drifts, unweighted (Lemmas 27.3 and 27.4)	the analytic domains and coefficient bounds, M, J , and the common upper bound g_*
Weighted coefficients and drifts in Y_α^δ (Theorem 27.5)	in addition α and the coarse mesh $a_0 = L/n$
Drifts in V_σ^δ (Theorem 27.7)	in addition σ ; not on a_0
Maximal inequalities, moments, and summations (Theorems 27.6 and 27.7)	in addition the fixed γ, n, ρ_0 , the moment, the exponents, and the prescription of the cutoff radii

Table 27.1. Dependence of constants in Section 27. No constant depends on K , on the scale indices i and j , on the number of preceding averaging steps, on a/δ , or on the values of the coarser fields.

The tools are standard: the weighted Schur test, cancellation for convolution with a signed kernel of mass zero, the logarithmic Sobolev inequality (Gross [103]; Bakry and Émery [8]) and the covariance representation of Helffer and Sjöstrand [110] in the forms proved in Theorem 5.4, the derivative formula for normalized expectations, and union bounds over finite nets. As in Section 26, we control a heat evolution together with its quadratic Picard term following Cao and Chatterjee [42, Section 2.1]; here the estimates are proved from the conditional fluctuation kernels.

27.1 Volume-normalized column sums and the interpolation defect

We use the following norms for matrices between arrays on lattices. Each index u of such an array has a location and a cell volume v_u : the cube of the mesh of its lattice. For a matrix A with entries A_{yu} we set

$$R(A) = \sup_y \sum_u |A_{yu}|, \quad C(A) = \sup_u \frac{v_{\text{out}}}{v_{\text{in}}} \sum_y |A_{yu}|,$$

the absolute row sum and the volume-normalized column sum, where v_{in} and v_{out} are the cell volumes of the input and output lattices. All sums include the finitely many direction and color indices.

Lemma 27.1. *Assume the setting of Section 26.1.*

1. *The norms R and C are submultiplicative under composition. For a block with fixed input and output volumes,*

$$\|A\|_{\ell_{\text{count}}^2 \rightarrow \ell_{\text{count}}^2} \leq \sqrt{R(A)C(A)v_{\text{in}}/v_{\text{out}}}.$$

For several types of input or output indices, the column ratio in each summand is replaced by $v_{\text{out}}(y)/v_{\text{in}}(u)$. The operator norm from $\ell^2(v_{\text{in}})$ to $\ell^2(v_{\text{out}})$, the ℓ^2 spaces weighted by the cell volumes, is then at most $\sqrt{RC}$, and counting norms use the corresponding conversions of volumes for each block. The ratios of volumes within one renormalization group step cost only C_J . For a pointwise product $c_{y;u,v} = A_{yu}B_{yv}$ of two matrices, the absolute row sum is at most $R(A)R(B)$, and the two column sums with a fixed input are at most $C(A)R(B)$ and $R(A)C(B)$.

2. Let $a = a_j$, $H = a_{j-1} = Ba$, $\delta = a_K$, and put

$$D_j = I_a P_j - I_H E_H, \quad d_H(s) = H(s + H^2)^{-1/2}.$$

Each column of D_j with a fixed input u on the lattice of mesh H has total physical mass zero, separately in every output direction and color, and first absolute moment at most CH^4 about the location of u :

$$\delta^3 \sum_y (D_j)_{yu} = 0, \quad \delta^3 \sum_y \text{dist}(y, u) |(D_j)_{yu}| \leq CH^4.$$

The matrix D_j and the matrix $I_a |P_j| + I_H |E_H|$ of absolute values have bounded absolute row sums and volume-normalized column sums. For every $s \in [0, T]$ and $r = 0, 1, 2$,

$$\begin{aligned} (R + C)(D_\delta^r P_s^\delta D_j) &\leq C_r (s + H^2)^{-r/2} d_H(s), \\ (R + C)(D_\delta^r P_s^\delta I_a P_j) &\leq C_r (s + H^2)^{-r/2}. \end{aligned}$$

The cancellation is used for each column with input on the lattice of mesh H , before any earlier linearization is composed.

3. Let $L_i = D_\theta \mathcal{R}_i(0, 0)$ be the linearization at the zero field of the zero-fluctuation reconstruction of step i , acting on link logarithms, with $R(L_i), C(L_i) \leq a_{\text{cap}} = C_{\text{CP}}/B$ and $q_{\text{anc}} = Ba_{\text{cap}}^2 < 1$. For $0 \leq i \leq j-1$ set $m = j-1-i$, $G = a_i = B^m H$, $T_{ji} = L_{j-1} \cdots L_{i+1}$ (the identity if $i = j-1$), and

$$U_s = P_s^\delta I_a P_j T_{ji} / H, \quad V_s = P_s^\delta D_j T_{ji} / H, \quad \mathfrak{A}_{ji} = C_{\text{CP}}^m / G.$$

If $\|\theta_i\|_\infty \leq b_i$, then for $r = 0, 1, 2$,

$$\|D_\delta^r U_s \theta_i\|_\infty \leq C_r \mathfrak{A}_{ji} b_i (s + H^2)^{-r/2}, \quad \|D_\delta^r V_s \theta_i\|_\infty \leq C_r \mathfrak{A}_{ji} b_i (s + H^2)^{-r/2} d_H(s).$$

For $i \geq 1$, the derivatives in w_i of $U_s \theta_i$ and $V_s \theta_i$, where $\theta_i = \mathcal{R}_i(\theta_{i-1}, w_i)$ is differentiated at the given point w_i of Ω_i , have absolute row sums and volume-normalized column sums bounded by the same expressions with b_i replaced by g_i . In bounds on derivatives, b_i may be any deterministic bound on $\|\theta_i(w)\|_\infty$ valid for every $w \in \Omega_i$ at the fixed coarser field θ_{i-1} ; a bound sharper than the global one may be used when it holds on that whole set.

A derivative in the dimensionless input θ_i removes a factor b_i ; for an input written as a lattice potential, $\theta_i = Gv_i$, the derivative in v_i acquires the factor G ; and a derivative in a normalized fluctuation coordinate $w_{i,\lambda}$ acquires its own coupling $g_{i,\lambda}$. These conversions use the mesh of the input, never δ .

4. Let $\mathcal{S}_s$ be any one of the three ordered sources

$$\mathcal{N}_\delta(\mathbf{U}_s\theta_i, \mathbf{V}_s\theta_i), \quad \mathcal{N}_\delta(\mathbf{V}_s\theta_i, \mathbf{U}_s\theta_i), \quad \mathcal{N}_\delta(\mathbf{V}_s\theta_i, \mathbf{V}_s\theta_i),$$

or their linear combinations with coefficients $1, 1, -1$ or $1, 1, -2$. Then

$$(\mathbf{R} + \mathbf{C})(D_{w_i}D_{\delta\mathcal{S}_s}) \leq C_J C_{\text{CP}}^{2m} g_i b_i G^{-2} (s + H^2)^{-1} d_H(s).$$

The constants depend on L, T, M, J , the interpolation function, the analytic domains and column bounds, and the order of differences; they do not depend on K, j, i , the number of preceding averaging steps, a/δ , the coupling, or the coarser field. These are estimates on the whole torus.

Proof. Constants C_J may absorb the number of coordinate types and their volume ratios within one renormalization group step, but never a ratio between the mesh G of a coarser scale and the finest mesh. The horizon in every heat estimate of this section is the fixed T . The refinement hypotheses of the cardinal interpolation and of Lemma 17.2 hold because

$$\frac{a_j}{\delta} = B^{K-j} \in \mathbb{N}, \quad \frac{a_{j-1}}{\delta} = B^{K-j+1} \in \mathbb{N}. \quad (27.1)$$

Every order of differences used below is a nonnegative integer. The coarse side $n \geq 4$ and all finer sides nB^l satisfy the hypotheses on lattice sides of the heat and interpolation estimates, including meshes larger than one.

Step 1: the matrix norms. We write v_z for the cell volume of an intermediate index z . By finite summation, we have

$$\sum_u |(\mathbf{AB})_{yu}| \leq \sum_z |\mathbf{A}_{yz}| \sum_u |\mathbf{B}_{zu}|,$$

and

$$\frac{1}{v_u} \sum_y v_y |(\mathbf{AB})_{yu}| \leq \sum_z \frac{v_z}{v_u} |\mathbf{B}_{zu}| \left(\frac{1}{v_z} \sum_y v_y |\mathbf{A}_{yz}| \right).$$

Taking suprema gives both composition inequalities. For an array f , the Cauchy–Schwarz inequality with weights $|\mathbf{A}_{yu}|$ implies that

$$\sum_y v_y |(\mathbf{A}f)_y|^2 \leq \mathbf{R}(\mathbf{A}) \sum_u |f_u|^2 \sum_y v_y |\mathbf{A}_{yu}| \leq \mathbf{R}(\mathbf{A}) \mathbf{C}(\mathbf{A}) \sum_u v_u |f_u|^2.$$

For fixed input and output volumes, replacing both weighted norms by counting norms introduces exactly the factor $(v_{\text{in}}/v_{\text{out}})^{1/2}$. The same computation gives the assertion for

several types of indices. For $c_{y;u,v} = A_{yu}B_{yv}$, summing first over u and v gives the row bound. Fixing u and summing first over v , we obtain

$$\sup_u \frac{v_{\text{out}}}{v_{\text{in}}} \sum_{y,v} |A_{yu}B_{yv}| \leq C(A)R(B),$$

and fixing v gives the other column bound. The same computation applies to two different types of input indices with their volumes. Fixed translations of either output index are permutations of the output indices, so the proof applies to them with the same norms. A color bracket costs only the norm of a fixed bilinear map on $\mathbb{R}^3$.

Step 2: zero mass of the defect. We apply Lemma 17.2 with coarse mesh H and fine mesh $a = H/B$; its linear reconstruction is P_j of Lemma 26.2. The two interpolations are applied on their own lattices. By Lemma 16.3, after interpolation a column with input on the lattice of mesh a has total physical mass a^3 , and a column with input on the lattice of mesh H has mass H^3 . The identity for the total mass of the columns of P_j in Lemma 17.2 implies that

$$\delta^3 \sum_y (I_a P_j - I_H E_H)_{yu} = 0$$

in every output component, without summing over colors or directions. For an index z on the lattice of mesh a , we use the triangle inequality $\text{dist}(y, u) \leq \text{dist}(y, z) + \text{dist}(z, u)$ and the first moment of the column of the cardinal interpolation matrix at z to obtain

$$\delta^3 \sum_y \text{dist}(y, u) |(I_a P_j)_{yu}| \leq C \sum_z a^3 \{a + \text{dist}(z, u)\} |(P_j)_{zu}| \leq CH^4.$$

The term $I_H E_H$ has the same bound, and periodic distances may be used in each triangle inequality. The rows and volume-normalized columns of Lemma 17.2, together with Step 1, also bound $I_a |P_j| + I_H |E_H|$.

Step 3: heat evolution of the defect. The estimate on the finest lattice in Lemma 17.2 gives the bound on the absolute rows of $D_\delta^r P_s^\delta D_j$. We give a separate argument for the columns. For $s \leq H^2$ we commute every difference through the heat semigroup and apply it to the smooth columns of the cardinal interpolation matrix. Their r th differences have volume-normalized column sums at most $C_{J,r} H^{-r}$, since $a = H/B$ with B fixed, and the heat semigroup does not increase absolute column sums. For $s \geq H^2$ we subtract the heat kernel at the location of the input u , which is allowed by the zero mass of the column:

$$(D_\delta^r P_s^\delta D_j)_{yu} = \delta^3 \sum_z \{D_\delta^r p_s^\delta(y, z) - D_\delta^r p_s^\delta(y, u)\} (D_j)_{zu}.$$

Using a shortest coordinate path from z to u , the telescoping identity along it, and Lemma 16.1, we obtain

$$\delta^3 \sum_y |D_\delta^r p_s^\delta(y, z) - D_\delta^r p_s^\delta(y, u)| \leq C_r \text{dist}(z, u) (s + \delta^2)^{-(r+1)/2};$$

the coordinate path has length at most $\sqrt{3} \operatorname{dist}(z, u)$ on the torus. Dividing the resulting column bound by the input volume H^3 and using the first moment of Step 2, we obtain $C_r H s^{-(r+1)/2}$, which is comparable to $(s + H^2)^{-r/2} d_H(s)$ in this range. For $r = 2$ the proof uses at most three differences of the heat kernel.

For $I_a P_j$ we use differences of the cardinal interpolation for $s \leq H^2$, and differences of the heat kernel for $s \geq H^2$, without subtraction; this proves both of its bounds. Thus the cancellation is used at the scale H ; a first moment about a location at the scale G would not suffice.

Step 4: derivatives of the reconstruction at an arbitrary point. The coefficient bounds come from Theorem 15.13, not from the heat estimate of Lemma 16.3. We write $\mathcal{D}_{\text{in}} \subseteq \mathcal{D}_{\text{mid}} \subseteq \mathcal{D}_{\text{out}}$ for its nested analytic domains, with fixed positive gaps between their radii. The products of coarser fields and free coordinates in Lemma 26.2 lie in $\mathcal{D}_{\text{in}}$ at a fixed positive distance from its boundary, uniformly in the side of the coarser lattice and in the number of preceding averaging steps. On that domain the reconstruction has a bounded absolute coefficient norm, including every volume-normalized column sum with a fixed input. Differentiating its majorant series on a smaller domain, we obtain bounds on the first and second derivatives in the free coordinates and on the fixed mixed derivatives used below. Indeed, a derivative of order k of a monomial of degree l multiplies its majorant by at most l^k times a fixed inverse power of the radius. Since the ratio η of the radii is less than one, $\sup_{l \geq k} l^k \eta^{l-k}$ is finite. The same argument with a fixed input gives the column bounds, without summing over a new output volume. These are absolute coefficient bounds, not bounds on values of a holomorphic map.

The coupling-scaled fluctuation inputs are $q_{i,\lambda} = g_{i,\lambda} w_{i,\lambda}$, and the ratios $g_{i,\lambda}/g_i$ lie in a fixed interval determined by M and J . The chain rule supplies one such coupling for every derivative in w . The resulting coefficients satisfy the hypotheses of Lemma 16.3: a fixed number of types of coordinates, bounded rows and volume-normalized columns, and exponential decay in their own units. In particular, for the normalized derivative of the reconstruction,

$$W_i(w) = D_{w_i} \mathcal{R}_i(\theta_{i-1}, w), \quad R(W_i) + C(W_i) \leq C_J g_i. \quad (27.2)$$

Here the coarser field is fixed, and the coefficient is evaluated on the whole cutoff domain Ω_i , within the domain of Lemma 26.2, not only at $w = 0$. Equivalently, Lemma 16.3 applies to $g_i^{-1} W_i$, with the ratios of intermediate couplings included. Its fixed mixed derivatives satisfy the same bounds, with the coupling factors of their inputs. The same conclusion can be reached from the mean. The segment from the conditional mean $\mathbb{E}w$ to a point w of the product lies in the product by convexity. Along it, Taylor's formula for the first derivative involves the second derivative in w , whose normalized absolute rows and columns are bounded by the same coefficient norm. Since $\|w - \mathbb{E}w\|_\infty \leq 2p_i$, its

contribution to either norm is at most $C_J g_i^2 p_i$; the first derivative at the mean has either norm at most $C_J g_i$; and $g_i p_i$ is bounded by the smallness conditions of Lemma 26.1.

Step 5: the zero-fluctuation reconstruction and composition. By the bounds on the derivatives of the zero-fluctuation reconstruction in Lemma 17.2, with the constant C_{CP} chosen before J , we have

$$\begin{aligned} \|\mathcal{R}_i(\phi, 0)\|_\infty &\leq a_{\text{cap}} \|\phi\|_\infty, \\ \|\mathcal{R}_i(\phi, 0) - \mathcal{L}_i \phi\|_\infty &\leq C a_{\text{cap}} \|\phi\|_\infty^2. \end{aligned} \quad (27.3)$$

The first line follows by integrating the first derivative along $s\phi$, $0 \leq s \leq 1$, and the second from Taylor's formula with integral remainder and the second derivative in the coarser field. Both use the constant chosen before J ; it is not enlarged after J is fixed.

By composition, we have $R(\mathcal{T}_{ji}), C(\mathcal{T}_{ji}) \leq a_{\text{cap}}^m$. The division is by the mesh H of the coarser lattice of step j . Thus

$$\frac{a_{\text{cap}}^m}{H} = \frac{C_{CP}^m}{G} = \mathfrak{A}_{ji}, \quad \mathfrak{A}_{ji}^2 H = \frac{q_{\text{anc}}^m}{G} = K_{ji}. \quad (27.4)$$

The bounds of (3) follow from those of Step 3. Replacing the rightmost input by W_i , we obtain the bounds on rows and columns of derivatives in w_i . At a fixed coarser field, a smaller bound b_i is allowed only if it bounds $\theta_i(w)$ at every point w of Ω_i . Differentiating in a dimensionless input removes its bound b_i , the substitution $\theta_i = Gv_i$ multiplies that derivative by G , and for a fluctuation coordinate the chain rule supplies its own $g_{i,\lambda}$.

Step 6: one more difference of the source. By Lemma 24.2, it suffices to consider one translated summand $[T_\xi f, T_\xi D_k^\pm h]$, with (f, h) equal to $(U_s \theta_i, V_s \theta_i)$, $(V_s \theta_i, U_s \theta_i)$, or $(V_s \theta_i, V_s \theta_i)$. For a forward difference D_l , the product rule is

$$D_l[T_\xi f, T_\xi D_k^\pm h] = [T_\xi D_l f, T_{\xi+\delta e_l} D_k^\pm h] + [T_\xi f, T_\xi D_l D_k^\pm h].$$

Its derivative in w_i has four terms, obtained by differentiating either argument of either bracket; their orders of differences are $(1, 1)$, $(1, 1)$, $(0, 2)$, and $(0, 2)$. A backward difference is a translated forward difference up to sign, so the same four bounds apply to it. In every term one factor is the deterministic defect or the derivative in w_i of the defect. Multiplying the corresponding bounds of (3), we find

$$C_J \mathfrak{A}_{ji}^2 g_i b_i (s + H^2)^{-1} d_H(s).$$

For the row we use the row of the differentiated factor and the supremum of the other factor; for the column we use the volume-normalized column of that factor and the same supremum. For $\mathcal{S}_s = \mathcal{N}_\delta(V_s \theta_i, V_s \theta_i)$ there may be two factors $d_H(s)$; since $d_H(s) \leq 1$, we may discard one. The two signed combinations cost at most the sum of the absolute values of their coefficients. Summing over the fixed list of summands proves (4), with the stated dependence of the constants. $\square$

27.2 Time-dependent coefficients and conditional drifts

This section bounds the coefficients of the ordered quadratic terms propagated from scale i , and the conditional drifts in the telescope over scales. We first record an elementary summation, then the properties of the conditional fluctuation kernel that we need, with constants uniform in the coupling, and a bound on conditional means of quadratic functions of the reconstructed field. The main estimate is Theorem 27.5.

An elementary summation

We write $\lambda_q = -\log q_{\text{anc}} / \log B > 0$, so that $q_{\text{anc}} = B^{-\lambda_q}$, and recall that $a_i = a_0 B^{-i}$ with $a_0 = L/n$.

Lemma 27.2. *Let $r > 0$ and $d \geq 0$.*

1. *For every $0 < \zeta < \min(r, \lambda_q)$ there exists C , depending only on r, ζ, λ_q, B , and a_0 , such that for all $j \geq 1$,*

$$\sum_{i=0}^{j-1} q_{\text{anc}}^{j-1-i} a_i^r \leq C a_{j-1}^\zeta.$$

2. *For every $0 < \eta < \min(r, \lambda_q)$ there exists C , depending only on r, d, η, λ_q, B , and a_0 , such that for all $k \geq 1$,*

$$\sum_{j \geq k} (1+j)^d \sum_{i=0}^{j-1} q_{\text{anc}}^{j-1-i} a_i^r \leq C a_k^\eta.$$

In particular, the left side is finite and tends to zero as $k \rightarrow \infty$.

Proof. Let $0 < \zeta < \min(r, \lambda_q)$ and $m = j - 1 - i$. Since $j - 1 = m + i$,

$$q_{\text{anc}}^m a_i^r = a_0^r B^{-\lambda_q m - r i} = a_0^r B^{-\zeta(j-1)} B^{-(\lambda_q - \zeta)m - (r - \zeta)i}.$$

The two remaining exponents are positive, so the sum over $0 \leq i \leq j - 1$ of the last factor is at most $\sum_{m, i \geq 0} B^{-(\lambda_q - \zeta)m - (r - \zeta)i} < \infty$. This gives (1), since $B^{-\zeta(j-1)} = (a_{j-1}/a_0)^\zeta$. For (2) we choose ζ strictly between η and $\min(r, \lambda_q)$ and apply (1). Since $\zeta - \eta > 0$, the polynomial factor is absorbed: $(1+j)^d B^{-\zeta(j-1)} \leq C_{d, \zeta - \eta} B^{-\eta(j-1)}$, and summing over $j \geq k$ gives at most $C B^{-\eta k} = C(a_k/a_0)^\eta$. Polynomial factors in i or m are bounded by $(1+j)^d$, since $i, m \leq j$. $\square$

The conditional fluctuation kernel

The next lemma collects the properties of the conditional law of w_i given $\mathcal{F}_{i-1}$ that we use. In the notation of the lemma, $\mu_{\mathfrak{d}_i, \phi} = k_{\mathfrak{d}_i}(e^\phi, dw)$ is the conditional fluctuation kernel of (4.8) after $\mathfrak{d}_i$ preceding averaging steps, and its mean $\mu_\phi = m_\phi$ is the conditional mean of Section 26.1. For the small-field event of Definition 4.4, all cutoff radii of one

renormalization group step are equal, $p_i = A_w \log(1/t_i) \geq 1$. This is the case treated here; it is a special case of the prescription allowed in Section 26.1. The point of the lemma is that its constants do not depend on the coupling below the common upper bound g_* .

Lemma 27.3. *Assume the setting of Section 26.1, with the equal-radius prescription of the preceding paragraph. Fix $K \geq 1$ and $1 \leq i \leq K$, and let $\phi = \theta_{i-1}$.*

1. *The conditional law of w_i given $\mathcal{F}_{i-1}$ is the conditional fluctuation kernel $\mu_{\mathfrak{d}_i, \phi}$, $\mathfrak{d}_i = J(K - i)$, with density $Z_\phi^{-1} \mathbf{1}_{\Omega_i} e^{-V_{\mathfrak{d}_i, \phi}}$, and, uniformly in $K, i, \mathfrak{d}_i, \phi$, and the coupling below g_* ,*

$$1 - Ct_i p_i > 0, \quad \frac{C[t_i p_i + t_i^2(1 + E_*)]}{(1 - Ct_i p_i)^3} \leq \frac{1}{2}, \quad t_i p_i^3 \leq \varepsilon_{sc}, \quad (27.5)$$

where E_* is the fixed upper bound on the weighted analytic coefficient norm of the effective interaction in Theorem 13.8, and ε_{sc} is fixed by the construction. The same holds at every point $u\phi$, $0 \leq u \leq 1$, of the segment from the zero field to ϕ , with the same product Ω_i .

2. *The potential is C^2 on a neighborhood of the closed product Ω_i , and*

$$D_w^2 V_{\mathfrak{d}_i, \phi} \geq \frac{1}{2} I \quad \text{on } \Omega_i. \quad (27.6)$$

In particular, the conditional fluctuation covariance Σ_ϕ of w_i satisfies $\|\Sigma_\phi\|_{\text{op}} \leq 2$, and every real function F on Ω_i that is Lipschitz for the Euclidean distance satisfies

$$\log \mathbb{E}_\phi e^{z(F - \mathbb{E}_\phi F)} \leq z^2 \text{Lip}(F)^2 \quad (z \in \mathbb{R}). \quad (27.7)$$

3. *Let $\mu_\phi = \mathbb{E}_\phi w_i$. Then*

$$\|\mu_\phi\|_\infty \leq C_J t_i \|\phi\|_\infty, \quad \mu_0 = 0, \quad \|\Sigma_\phi - \Sigma_0\|_{\text{op}} \leq C_J t_i p_i^3 \|\phi\|_\infty, \quad \|\Sigma_\phi\|_{\text{op}} \leq 2. \quad (27.8)$$

4. *Let $E_\phi = g_i^{-1} D_w \mathcal{R}_i(\phi, \mu_\phi)$ and $Z_\phi = E_\phi \Sigma_\phi E_\phi^\top$. Then*

$$\|E_\phi\|_{\text{op}} + \|E_0\|_{\text{op}} \leq C_J, \quad \|E_\phi - E_0\|_{\text{op}} \leq C_J \|\phi\|_\infty, \quad (27.9)$$

and

$$\|Z_\phi - Z_0\|_{\text{op}} \leq C_J (1 + t_i p_i^3) \|\phi\|_\infty \leq C_J \|\phi\|_\infty. \quad (27.10)$$

The constants depend only on the fixed analytic domains and coefficient bounds, M, J , and the common upper bound g_* ; they do not depend on the coupling below g_* , on the lattice side, on $K, i, \mathfrak{d}_i$, or on ϕ .

Proof. For fixed K and i , Lemma 26.2 identifies the conditional law of w_i given $\mathcal{F}_{i-1}$ with the conditional fluctuation kernel at the coarser field ϕ , with $\mathfrak{d}_i = J(K - i)$ preceding averaging steps. The side of the coarser lattice of the step is $nB^{i-1} \geq 4$, the coarser field

lies in B_r , and the coupling-scaled fluctuation inputs $q_{i,\lambda} = g_{i,\lambda} w_{i,\lambda}$ of every point of Ω_i lie in the common analytic domain, as checked in the proof of Lemma 26.1. The domain conditions there include those on every intermediate classical background field, on the block averaging constraint, and on the Haar disintegration density, and the condition that the reconstruction of every point of the product lies in D_r . These are the common domains of Lemma 26.2, and Theorems 15.13 and 15.14 preserve them. In particular, the change to the reconstruction maps preserves each ball of fluctuation coordinates, the free logarithmic coordinates, the Haar disintegration density, the triangular determinant, the inherited normalization constant, and the normalizer, and hence the normalized kernel.

The hypotheses of Theorem 13.8 hold on this product. Let $t_i = M^{(J-1)/2} g_i$; the radii are $p_i = A_w \log(1/t_i) \geq 1$, and the ratios of intermediate couplings are included in the reconstruction map. The common upper bound g_* of Lemma 26.2 is also the reduced bound of Theorem 13.10. Hence (27.5) holds uniformly in $K, i, \mathfrak{d}_i, \phi$, and the coupling below g_* . The last condition is uniform in the coupling: since $t[A_w \log(1/t)]^3 \rightarrow 0$ as $t \downarrow 0$, the reduced bound controls it on the whole interval of couplings. Every point $u\phi$ of the segment lies in B_r and satisfies the same product and domain conditions, so the theorem on means and covariances also applies along the segment.

We write the density as $Z_\phi^{-1} \mathbf{1}_{\Omega_i} e^{-V_{\mathfrak{d}_i, \phi}} dw$. By Theorem 13.8, the potential is smooth on an open neighborhood of the closed product, and (27.6) holds. The restrictions imposed at coarser scales depend only on the coarser field and cancel in this normalized conditional law, by the same theorem. The product is a finite product of convex balls in $\mathbb{R}^3$, the potential is C^2 on its neighborhood, and the Hessian bound holds on the whole product. These are the hypotheses of Theorem 5.4, with $m = 1/2$, and that theorem implies that $\|\Sigma_\phi\|_{\text{op}} \leq 2$ and (27.7). All later uses of conditional concentration in this section are for this kernel, so their constants do not depend on the dimension, the number of averaging steps, or the coupling.

By Theorem 13.10, which applies by the preceding checks and the preservation of the kernel, we have (27.8). To bound the linear coefficient of the reconstruction map, we write the map as a function $\hat{F}_i(\phi, q)$ of the coupling-scaled fluctuation inputs. Let $S_i = \text{diag}(g_{i,\lambda}/g_i)$ and $q_\phi = (g_{i,\lambda} \mu_{\phi,\lambda})_\lambda$. Then

$$E_\phi = g_i^{-1} D_w \mathcal{R}_i(\phi, \mu_\phi) = D_q \hat{F}_i(\phi, q_\phi) S_i, \quad E_0 = D_q \hat{F}_i(0, 0) S_i,$$

and the norm of S_i is bounded by the fixed ratio of intermediate couplings. The inner domain contains these points and the segments joining them: the conditional mean lies in the convex product, and $u\phi$ lies in the ball of coarser fields. The second derivatives in the free coordinates, and the mixed second derivatives in the coarser field and the free coordinates, have bounded absolute row sums and volume-normalized column sums. This follows from Theorem 15.13 and the computation with majorant series in Step 4 of

the proof of Lemma 27.1. Integrating these derivatives along the two segments, we obtain

$$(R + C)(E_\phi - E_0) \leq C_J(\|\phi\|_\infty + \|q_\phi\|_\infty) \leq C_J(1 + t_i^2)\|\phi\|_\infty,$$

where we used $\|q_\phi\|_\infty \leq t_i\|\mu_\phi\|_\infty$. Converting the volumes of the fluctuation coordinates to the output volume uses only fixed ratios within one renormalization group step, so the weighted Schur bound of Lemma 27.1(1) implies (27.9). For $Z_\phi = E_\phi \Sigma_\phi E_\phi^\top$ we expand

$$Z_\phi - Z_0 = (E_\phi - E_0)\Sigma_\phi E_\phi^\top + E_0(\Sigma_\phi - \Sigma_0)E_\phi^\top + E_0\Sigma_0(E_\phi^\top - E_0^\top). \quad (27.11)$$

By (27.8) and (27.9), the three terms are bounded by $C_J\|\phi\|_\infty$, $C_J t_i p_i^3 \|\phi\|_\infty$, and $C_J\|\phi\|_\infty$, respectively, which gives (27.10). Every constant here comes from the common upper bound on the couplings and from the coefficient bounds of the reconstruction maps; none depends on the coupling. The covariance bound of Theorem 17.4, whose constants may depend on the fixed coupling, is not used. $\square$

Conditional means of quadratic functions

The next lemma bounds the conditional mean of a quadratic function of the reconstructed field without using any spatial decay of its coefficients. This allows us to apply it after heat evolution, which spreads the coefficients over the torus. We say that a quadratic form Q on link logarithms, with values in arrays of $\mathfrak{su}(2)$ -valued components, is equivariant if $Q(\text{Ad}_h v, \text{Ad}_h v) = \text{Ad}_h Q(v, v)$ for every constant $h \in \text{SU}(2)$ acting on all color indices.

Lemma 27.4. *Assume the setting of Lemma 27.3. Let Q be an equivariant quadratic form on link logarithms on the lattice of mesh a_i , fix one scalar output component of $Q(v, v)$, and let k_Q be the absolute sum of the coefficients of that component over both inputs. Put $r = \|\phi\|_\infty$ and $c = \mathcal{R}_i(\phi, 0)$. Then*

$$|\mathbb{E}_\phi Q(\theta_i, \theta_i) - Q(c, c)| \leq C_J k_Q (g_i^2 r + g_i^3 p_i^3), \quad (27.12)$$

$$|Q(c, c) - Q(L_i \phi, L_i \phi)| \leq C k_Q a_{\text{cap}}^2 r^3. \quad (27.13)$$

The constants depend only on the fixed coefficient bounds, M , J , and the common upper bound g_* ; they do not depend on the coupling below g_* , the lattice side, K , i , or the number of preceding averaging steps.

Proof. The matrix C_Q of the fixed scalar component has nuclear norm at most k_Q , since it is a sum of coordinate matrix units, each of nuclear norm one. Thus, for every matrix C ,

$$|Q : C| = |\text{tr}(C_Q C)| \leq k_Q \|C\|_{\text{op}}. \quad (27.14)$$

This needs no spatial weight on Q .

Let $\mu = \mathbb{E}_\phi w_i$, $\xi = w_i - \mu$, and $v = \mathcal{R}_i(\phi, \mu)$. Using the bound on the mean in (27.8), the expansion (26.2) of Lemma 26.2, and the bounds (27.2) on the derivatives of the reconstruction, we have

$$\begin{aligned} \theta_i &= v + g_i E_\phi \xi + g_i^2 B_\phi(\xi, \xi) + R_\phi(\xi), & \|R_\phi\|_\infty &\leq C_J g_i^3 p_i^3, \\ \|v - c\|_\infty &\leq C_J g_i \|\mu\|_\infty \leq C_J g_i t_i r \leq C_J g_i^2 r, & \|v\|_\infty &\leq C_J r. \end{aligned} \quad (27.15)$$

The normalized coefficients of E_ϕ and of each output component of B_ϕ have bounded absolute rows. By (27.14) and the bound $\|\Sigma_\phi\|_{\text{op}} \leq 2$, which follows from the Poincaré inequality, we obtain

$$\|\mathbb{E}_\phi B_\phi(\xi, \xi)\|_\infty \leq 2C_J. \quad (27.16)$$

We do not need the quadratic mean to vanish at a nonzero coarser field.

We insert (27.15) into both arguments of Q . The terms linear in ξ have mean zero, since ξ is centered under the same law. The term with $g_i E_\phi \xi$ in both arguments has mean exactly $g_i^2 Q : (E_\phi \Sigma_\phi E_\phi^\top)$. By (27.14) and (27.10), it differs from $g_i^2 Q : Z_0$ by at most $C_J k_Q g_i^2 r$, uniformly in the coupling. The vector-valued contraction $Q : Z_0$ vanishes. Indeed, Z_0 is scalar in the color indices, so by equivariance the contraction is a vector fixed by all adjoint rotations; such a vector is zero, since a rotation through π about an axis perpendicular to a nonzero vector changes its sign.

The two terms that pair v with $g_i^2 \mathbb{E}_\phi B_\phi(\xi, \xi)$ are bounded by $C_J k_Q g_i^2 r$, by (27.16). Replacing $Q(v, v)$ by $Q(c, c)$ costs at most $C_J k_Q g_i^2 r$, since the extra factor r is bounded on the fixed chart. Every remaining term contains R_ϕ , or one argument linear and one quadratic in ξ , or two arguments quadratic in ξ . On the cutoff domain $|\xi| \leq 2p_i$ coordinatewise, so these terms are bounded by $C_J k_Q g_i^3 p_i^3$, $C_J k_Q g_i^3 p_i^3$, and $C_J k_Q g_i^4 p_i^4$, respectively; the last is absorbed into the first because $g_i p_i$ is bounded by (27.5). The factors v and $g_i p_i$ in products with R_ϕ are bounded on the same domain. This proves (27.12). For the classical error, by (27.3), we have

$$|Q(c, c) - Q(L_i \phi, L_i \phi)| \leq k_Q (\|c\|_\infty + \|L_i \phi\|_\infty) \|c - L_i \phi\|_\infty \leq C k_Q a_{\text{cap}}^2 r^3,$$

which is (27.13). The only quantities involving the cutoff that were absorbed into constants are $g_i p_i$, $g_i t_i / g_i^2 = M^{(J-1)/2}$, and $t_i p_i^3$, all bounded by fixed data as checked in Lemma 27.3. $\square$

Coefficients and drifts of the propagated terms

For $\omega \in \{UV, VU, VV\}$ and $0 \leq i \leq j-1$ we define the bilinear sources

$$S_s^{UV}(v, z) = \mathcal{N}_\delta(U_s v, V_s z), \quad S_s^{VU}(v, z) = \mathcal{N}_\delta(V_s v, U_s z), \quad S_s^{VV}(v, z) = \mathcal{N}_\delta(V_s v, V_s z),$$

with U_s, V_s as in Lemma 27.1(3), and the bilinear Duhamel terms

$$\mathcal{T}_{j,i}^\omega(v, z)(t) = \int_0^t P_{t-s}^\delta S_s^\omega(v, z) ds.$$

Taking $z = v$ gives the quadratic terms of the introduction; each argument has a fixed place, and no symmetry of $\mathcal{N}_\delta$ is used. Both arguments are propagated through the same linearizations, so $\mathcal{T}_{j,i-1}^\omega(v, v) = \mathcal{T}_{j,i}^\omega(L_i v, L_i v)$. We write $\mathcal{S}_s$ for any of the sources $S_s^\omega(\theta_i, \theta_i)$.

Theorem 27.5. *Assume the setting of Section 26.1 and the notation of Lemma 27.1, with $m = j - 1 - i$ and $G = a_i$. Set*

$$K_{ji} = C_{\text{CP}}^{2m} H / G^2 = q_{\text{anc}}^m / G, \quad \ell_H(t) = 1 + \log(1 + t / H^2),$$

$$J_H(t) = \int_0^t (t - s)^{-1/2} (s + H^2)^{-1} ds.$$

1. *For $1 \leq i \leq j - 1$, the derivative $F_s = D_{w_i} \mathcal{S}_s$ in w_i , at any point of Ω_i , satisfies*

$$(R + C)(F_s) \leq C_J K_{ji} g_i b_i (s + H^2)^{-1},$$

$$\|F_s\|_{\ell^2(w_i) \rightarrow \ell^2(\delta)} \leq C_J K_{ji} g_i b_i (s + H^2)^{-1} (G / \delta)^{3/2},$$

where both ℓ^2 norms are counting norms. For $s \geq G^2$, the Euclidean norm of its value row improves to $C_J K_{ji} g_i b_i G^{3/2} s^{-7/4}$; no such improvement is asserted for its volume-normalized column sum.

2. *The rows of the derivatives in w_i of the Duhamel terms satisfy, for $0 \leq t \leq G^2$,*

$$\sup_x \|D_{w_i} \mathcal{T}_{j,i}^\omega(t, x)\|_2 \leq C_J K_{ji} g_i b_i \log(1 + t / H^2),$$

$$\sup_x \|D_{w_i} D_\delta \mathcal{T}_{j,i}^\omega(t, x)\|_2 \leq C_J K_{ji} g_i b_i J_H(t),$$

and, for $t \geq G^2$ and $r = 0, 1$,

$$\sup_x \|D_{w_i} D_\delta^r \mathcal{T}_{j,i}^\omega(t, x)\|_2 \leq C_J K_{ji} g_i b_i G^{3/2} t^{-3/4-r/2} \ell_H(T).$$

At small times the displayed integrals, which vanish at $t = 0$, are kept; in particular $J_H(t) \leq C \min\{\sqrt{t} / H^2, (t + H^2)^{-1/2} \ell_H(t)\}$. The bounds on first differences use the outer heat kernel.

3. *For a quadratic coefficient $c_{y;u,v}$, let R_2 be the supremum over outputs of the absolute sum over both inputs, and, for a coefficient with input mesh G and output mesh h ,*

$$C_2(c) = \max \left\{ \sup_u \frac{h^3}{G^3} \sum_{y,v} |c_{y;u,v}|, \sup_v \frac{h^3}{G^3} \sum_{y,u} |c_{y;u,v}| \right\}$$

the larger of its two volume-normalized column sums with a fixed input. Here $h = \delta$; for $Q_{j,i}$ in Theorem 27.7, before interpolation, $h = a_j$. For each ω ,

$$(R_2 + C_2)(\mathcal{T}_{j,i}^\omega(t)) \leq C K_{ji} \log(1 + t / H^2),$$

$$(R_2 + C_2)(D_\delta \mathcal{T}_{j,i}^\omega(t)) \leq C K_{ji} J_H(t).$$

With the weights of Y_α^δ , that is, after multiplying the first bound by $(t + \delta^2)^\beta$ and the second by $(t + \delta^2)^\alpha$, the supremum over $t \leq T$ of these coefficient norms is therefore at most $C_{L,T,\alpha,a_0}(1+j)K_{ji}$; we call this the weighted coefficient bound. On the whole cutoff domain Ω_i we also have, for $r = 0, 1, 2, 3$,

$$\sup_{w_i, x} \|D_{w_i} D_\delta^r \mathcal{T}_{j,i}^\omega(t, x)\|_2 \leq C_{J,r} C_{CP}^{2m} g_i b_i G^{-2} t H^{-1-r},$$

and, for $r = 0, 1$,

$$\sup_{w_i, x} \|D_{w_i} \partial_t D_\delta^r \mathcal{T}_{j,i}^\omega(t, x)\|_2 \leq C_{J,r} C_{CP}^{2m} g_i b_i G^{-2} (H^{-1-r} + t H^{-3-r}).$$

These bounds keep the factor $g_i b_i$, for later use in coverings.

4. For $1 \leq i \leq j-1$, define the conditional drift

$$e_{j,i}^\omega = \mathbb{E}[\mathcal{T}_{j,i}^\omega(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}] - \mathcal{T}_{j,i-1}^\omega(\theta_{i-1}, \theta_{i-1}).$$

With the deterministic bounds $b_0 = \rho_0$ and $b_i = \rho_0 a_{\text{cap}}^i + C_J g_3 \sqrt{a_i} p_i$, $i \geq 1$, on $\|\theta_i\|_\infty$,

$$Y_\alpha^\delta(e_{j,i}^\omega) \leq C_J(1+j) \frac{q_{\text{anc}}^{j-1-i}}{a_i} \{a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3\},$$

and

$$Y_\alpha^\delta(\mathcal{T}_{j,0}^\omega(\theta_0, \theta_0)) \leq C \rho_0^2 (1+j) q_{\text{anc}}^{j-1} / a_0.$$

More precisely, for $\phi = \theta_{i-1}$ and $\theta_i^{\text{cl}} = \mathcal{R}_i(\phi, 0)$, write $e_{j,i}^\omega = e_{j,i}^{\omega, \text{cl}} + e_{j,i}^{\omega, \text{q}}$, where the linearization error and the fluctuation correction are

$$\begin{aligned} e_{j,i}^{\omega, \text{cl}} &= \mathcal{T}_{j,i}^\omega(\theta_i^{\text{cl}}, \theta_i^{\text{cl}}) - \mathcal{T}_{j,i-1}^\omega(\phi, \phi), \\ e_{j,i}^{\omega, \text{q}} &= \mathbb{E}[\mathcal{T}_{j,i}^\omega(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}] - \mathcal{T}_{j,i}^\omega(\theta_i^{\text{cl}}, \theta_i^{\text{cl}}). \end{aligned}$$

Then

$$\begin{aligned} Y_\alpha^\delta(e_{j,i}^{\omega, \text{cl}}) &\leq C_J(1+j) K_{ji} a_{\text{cap}}^2 \|\phi\|_\infty^3, \\ Y_\alpha^\delta(e_{j,i}^{\omega, \text{q}}) &\leq C_J(1+j) K_{ji} (g_i^2 \|\phi\|_\infty + g_i^3 p_i^3). \end{aligned}$$

Both deterministic majorants have summable tails in j , uniformly in K . When the coarser field is the identity, the vector-valued quadratic mean vanishes by the invariance of the conditional law under global color rotations; at a nonzero coarser field, the variation of the covariance, the nonlinear mean, and the classical Taylor error are all kept.

The constants in the coefficient and mean bounds do not depend on the coupling, K , j , i , the number of preceding averaging steps, or a/δ ; they have the dependence of Lemma 27.1, with α added. The weighted coefficient bounds and the weighted bounds on drifts may in addition depend on the fixed coarse mesh $a_0 = L/n$; this does not apply to the unweighted time-dependent rows. The constants in the summability statements may depend on the fixed γ , n , ρ_0 , and the prescription of the cutoff radii.

Proof. We first prove the deterministic estimates on coefficients and on derivatives in w_i , and then the estimates on drifts, which use Lemmas 27.3 and 27.4. Throughout, K_{ji} is the exact value displayed in the statement.

Step 1: sources and late rows. Combining the bound on pointwise products in Lemma 27.1(1) with part (3) of that lemma without a derivative in w_i , and using (27.4), we find that the bilinear coefficient of the source satisfies

$$(R_2 + C_2)(S_s^\omega) \leq C\mathfrak{A}_{ji}^2(s + H^2)^{-1/2}d_H(s) = \frac{CK_{ji}}{s + H^2}, \quad (27.17)$$

One argument uses a volume-normalized column and the other a row; hence no factor of the output volume occurs. Differentiating the two arguments in w_i instead, we obtain

$$(R + C)(F_s) \leq \frac{C_J K_{ji} g_i b_i}{s + H^2}. \quad (27.18)$$

The counting Schur bound of Lemma 27.1(1) then implies the operator bound of (1), with the factor $(G/\delta)^{3/2}$. For several types of fluctuation coordinates, the intermediate volumes are their own cell volumes, and replacing them by G^3 costs only C_J .

For $s \geq G^2$ we split the heat semigroup in either argument into two equal halves. The first half, applied to the derivative in w_i , has row and volume-normalized column at most $C_J \mathfrak{A}_{ji} g_i$; in the argument with the defect, both acquire the factor $d_H(s/2) \leq \sqrt{2}d_H(s)$. The second half has counting row norm at most $C\delta^{3/2}s^{-3/4-r/2}$ after r differences, by Lemma 16.1. The Schur bound for the first half implies that

$$\begin{aligned} \sup_x \|D_{w_i} D_\delta^r U_s \theta_i(x)\|_2 &\leq C_{J,r} \mathfrak{A}_{ji} g_i G^{3/2} s^{-3/4-r/2}, \\ \sup_x \|D_{w_i} D_\delta^r V_s \theta_i(x)\|_2 &\leq C_{J,r} \mathfrak{A}_{ji} g_i G^{3/2} s^{-3/4-r/2} d_H(s) \quad (r = 0, 1), \end{aligned} \quad (27.19)$$

where we used $\delta^{3/2}(G/\delta)^{3/2} = G^{3/2}$. In a translated bracket, the derivative in w_i falls either on the argument without a difference or on the argument with the difference. We use, respectively, the bound of (27.19) with $r = 0$ or $r = 1$, and the complementary deterministic bound of Lemma 27.1(3) for the other argument. Every term keeps one factor d_H . Since $d_H(s) \leq Hs^{-1/2}$ and $\mathfrak{A}_{ji}^2 H = K_{ji}$, we find

$$\sup_x \|F_s(x, \cdot)\|_2 \leq C_J K_{ji} g_i b_i G^{3/2} s^{-7/4} \quad (s \geq G^2). \quad (27.20)$$

For the order VV we use $d_H(s)^2 \leq d_H(s)$. These bounds also hold on $G^2/2 \leq s \leq G^2$ after increasing the constant: on that interval the absolute row in (27.18) is bounded by the right side of (27.20). The heat semigroup conserves the mass of columns, so this argument gives no improvement at late times for the absolute column.

Step 2: the outer heat semigroup at small times. Let $\Lambda = C_j K_{ji} g_i b_i$. By heat contraction and (27.18), we have the value row $\Lambda \int_0^t (s + H^2)^{-1} ds = \Lambda \log(1 + t/H^2)$ at every t . For the first difference we use the bound on the outer heat kernel

$$\|D_\delta P_r^\delta\|_{\infty \rightarrow \infty} \leq C(r + \delta^2)^{-1/2} \leq Cr^{-1/2} \quad (r > 0),$$

which gives $C\Lambda J_H(t)$, without differentiating the source; the singularity at $s = t$ is integrable. The substitution $z = \sqrt{t-s}$ yields

$$J_H(t) = \frac{2}{\sqrt{t+H^2}} \operatorname{artanh} \sqrt{\frac{t}{t+H^2}}. \quad (27.21)$$

Bounding $(s + H^2)^{-1}$ by H^{-2} , we find $J_H(t) \leq 2\sqrt{t}/H^2$ at all t . For $t \geq H^2$, the identity $2 \operatorname{artanh} z = \log((1+z)/(1-z))$ gives $J_H(t) \leq C(t + H^2)^{-1/2} \ell_H(t)$; for $t \leq H^2$ this bound also follows from $\sqrt{t}/H^2 \leq C/H$. This proves the stated minimum, including its vanishing at $t = 0$.

Step 3: late times. For $2G^2 \leq t \leq T$ we split the Duhamel integral at $t/2$. On the early half we use the counting operator bound of (1) for F_s and the counting row of the heat kernel,

$$\|D_\delta^r P_{t-s}^\delta(x, \cdot)\|_2 \leq C_r \delta^{3/2} t^{-3/4-r/2}, \quad r = 0, 1;$$

the integral is at most $C\Lambda G^{3/2} t^{-3/4-r/2} \log(1 + t/H^2)$. On the late half we use (27.20) and the ℓ^1 row of $D_\delta^r P_{t-s}^\delta$; the integral is at most

$$C\Lambda G^{3/2} \int_{t/2}^t (t-s)^{-r/2} s^{-7/4} ds \leq C\Lambda G^{3/2} t^{-3/4-r/2}.$$

If $G^2 \leq t \leq 2G^2$, the bounds of Step 2 give the same result, since $t \asymp G^2$ and $J_H(t) \leq Ct^{-1/2} \ell_H(T)$. Thus the late estimates of (2) hold from $t = G^2$ on. The constants for the fixed horizon may depend on T/L^2 .

Step 4: quadratic coefficients and weights. The outer heat kernel has absolute row and volume-normalized column equal to one, and its first difference has both norms at most $C(t-s)^{-1/2}$. Applying these facts to (27.17), we obtain the two coefficient bounds of (3), with the exact integral profiles. If $t \leq H^2$, the weighted sum is at most

$$CK_{ji} \{H^{2\beta} t/H^2 + H^{2\alpha} \sqrt{t}/H^2\} \leq CK_{ji} H^{2\alpha-1}.$$

If $t \geq H^2$, it is at most $C_{T,\alpha} K_{ji} \ell_H(T)$, because both powers of the weights are positive and $\delta \leq H$. In the first regime we use $H \leq a_0$. Since $H = a_0 B^{-(j-1)}$,

$$\begin{aligned} \ell_H(T) &= 1 + \log(1 + (T/a_0^2) B^{2(j-1)}) \\ &\leq 1 + \log(1 + T/a_0^2) + 2(j-1) \log B \\ &\leq C_{T,a_0,B}(1+j), \end{aligned} \quad (27.22)$$

where we used $1 + xB^{2(j-1)} \leq (1 + x)B^{2(j-1)}$ for $x = T/a_0^2$, since $B^{2(j-1)} \geq 1$. The constant in the weighted coefficient bound may therefore depend on the fixed $a_0 = L/n$. The unweighted bounds on time-dependent coefficients and the bounds on rows of derivatives in w_i keep their stated dependence and their logarithmic profiles.

Step 5: crude bounds for coverings. For any fixed integer $r \geq 0$ we commute the differences through the heat semigroup and apply them to the cardinal interpolations on their own lattices before smoothing. The arguments then satisfy

$$\|D_\delta^r U_s \theta_i\|_\infty + \|D_\delta^r V_s \theta_i\|_\infty \leq C_{J,r} \mathfrak{A}_{ji} b_i H^{-r},$$

and the rows of their derivatives in w_i have the same bound with b_i replaced by g_i . By the finite product rule, we have, for each fixed r ,

$$\begin{aligned} \sup_{w_i, s, x} |D_\delta^r \mathcal{S}_s(x)| &\leq C_{J,r} \mathfrak{A}_{ji}^2 b_i^2 H^{-1-r}, \\ \sup_{w_i, s, x} \|D_{w_i} D_\delta^r \mathcal{S}_s(x)\|_2 &\leq C_{J,r} \mathfrak{A}_{ji}^2 g_i b_i H^{-1-r}. \end{aligned} \quad (27.23)$$

Heat contraction and integration in time give the factor t for the Duhamel term. For the time derivative we use the exact identity $\partial_t \mathcal{T}(t) = \mathcal{S}_t + \Delta_\delta \mathcal{T}(t)$ in finite dimensions. The Laplacian is a sum of three second differences, which are bounded by (27.23) with two more differences. This proves both crude bounds of (3), uniformly on Ω_i , and avoids a bound of size δ^{-2} for the generator.

Step 6: the drifts. We apply Lemma 27.4 to each output component of $\mathcal{T}_{j,i}^\omega(t)$ and of its first difference. Their deterministic coefficients are fixed before conditioning, they are equivariant, and their absolute sums k_Q are bounded by the unweighted bounds of (3). Conditional integration commutes with finite differences of outputs. With $\phi = \theta_{i-1}$ and $\mathcal{T}_{j,i-1}^\omega(\phi, \phi) = \mathcal{T}_{j,i}^\omega(L_i \phi, L_i \phi)$, the weighted coefficient bound of Step 4 implies that

$$\begin{aligned} Y_\alpha^\delta(e_{j,i}^{\omega, \text{cl}}) &\leq C_{J,a_0} (1+j) K_{ji} a_{\text{cap}}^2 \|\phi\|_\infty^3, \\ Y_\alpha^\delta(e_{j,i}^{\omega, \text{q}}) &\leq C_{J,a_0} (1+j) K_{ji} (g_i^2 \|\phi\|_\infty + g_i^3 p_i^3). \end{aligned} \quad (27.24)$$

Here C_{J,a_0} has the deterministic dependence already listed, including a_0 from (27.22); it does not depend on the coupling, K , j , i , the number of averaging steps, or a/δ . In the weighted bounds on drifts in the statement, C_J denotes this constant, while the constants of (27.12) and (27.13) are unchanged. Adding the two bounds and using $\|\phi\|_\infty \leq b_{i-1}$, we obtain the bound on $e_{j,i}^\omega$.

The identity $\mathcal{T}_{j,i-1}^\omega(v, v) = \mathcal{T}_{j,i}^\omega(L_i v, L_i v)$ holds because $\mathbb{T}_{j,i-1} = \mathbb{T}_{ji} L_i$ in both arguments. Substituting θ_0 , whose norm is at most ρ_0 , into the weighted coefficient bound, we obtain the bound on the coarsest term, with the same dependence on a_0 . When the

coarser field is the identity, the conditional law at the zero field and the reconstruction are equivariant under global color rotations, so the vector-valued quadratic mean itself vanishes, and not only its contribution from the linear covariance.

Step 7: bounds on the coarser fields and summability. Integrating (27.2) from $w = 0$ to w_i along the segment in the convex product, and using (27.3), we obtain

$$\|\theta_i\|_\infty \leq a_{\text{cap}} \|\theta_{i-1}\|_\infty + C_J g_i p_i.$$

Iterating, and using $g_{i-l} = B^{l/2} g_i$ and that the cutoff radii do not decrease with i , we obtain

$$\|\theta_i\|_\infty \leq \rho_0 a_{\text{cap}}^i + C_J g_i p_i \sum_{l=0}^{i-1} (a_{\text{cap}} \sqrt{B})^l \leq \rho_0 a_{\text{cap}}^i + C_J g_3 \sqrt{a_i} p_i,$$

where the geometric series is included in the fixed C_J . For fixed γ and n , we obtain

$$b_i \leq C_\gamma (1+i) \sqrt{a_i}, \quad b_i / g_i \leq C_\gamma (1+i), \quad (27.25)$$

because $a_{\text{cap}}^i = (a_i / a_0)^{\kappa_{\text{term}}}$ with $\kappa_{\text{term}} = -\log a_{\text{cap}} / \log B > 1/2$. In particular, the three contributions to the drifts, divided by a_i , satisfy

$$\frac{a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3}{a_i} \leq C_\gamma (1+i)^3 a_i^{1/2}. \quad (27.26)$$

For the first term we use $a_{i-1} = B a_i$; the other two are exactly $g_3^2 b_{i-1}$ and $g_3^3 \sqrt{a_i} p_i^3$. The double series of majorants of the drifts is bounded by

$$C_\gamma \sum_{i \geq 1} \sum_{m \geq 0} (2+i+m) (1+i)^3 q_{\text{anc}}^m a_i^{1/2},$$

and by Lemma 27.2(2), with $r = 1/2$, its tail over $j = i + 1 + m \geq k$ is at most $C a_k^\eta$ for some $\eta > 0$, and tends to zero; truncation at any K only decreases it. The majorant of the coarsest terms is summable because $\sum_j (1+j) q_{\text{anc}}^{j-1} < \infty$. This proves the summability statement. Only this summation uses the bound (27.25), which holds at fixed coupling; the constants in the coefficient bounds and in the pointwise bounds on means are uniform in the coupling below g_* . $\square$

27.3 Maximal estimates and ultraviolet tails of the ordered terms

We now bound the conditionally centered terms $Z_{j,i}^\omega$ of the telescope, sum the telescope, and obtain the bounds on $\mathcal{Q}_\delta(r_j, r_j)$ and $\mathcal{B}_\delta(x_{j-1}, r_j)$. The maximal inequality uses a finite net in space and time at the scale H , whose size is a fixed power of H^{-1} , independent of δ . It also uses the bound $b_i / g_i \leq C_\gamma (1+i)$ of (27.25), so it holds at fixed positive coupling.

Theorem 27.6. Assume the setting of Section 26.1 and the notation of Theorem 27.5, and fix $\gamma > 0$, L , n , T , M , J , and the domains before any ultraviolet limit is taken. Put

$$Z_{j,i}^\omega = \mathcal{T}_{j,i}^\omega(\theta_i, \theta_i) - \mathbb{E}[\mathcal{T}_{j,i}^\omega(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}], \quad A_{ji} = K_{ji} g_i b_i G^{2\alpha-1} \ell_H(T).$$

1. There exist $C_\gamma < \infty$ and a fixed finite exponent c such that, for every $u \geq 0$, every $1 \leq i \leq j-1 \leq K-1$, and each ω separately,

$$\mathbb{P}\left\{Y_\alpha^\delta(Z_{j,i}^\omega) > C_\gamma A_{ji} \sqrt{\log(2 + L/H) + u} \mid \mathcal{F}_{i-1}\right\} \leq C e^{-u}.$$

It follows that, for every fixed $0 < p < \infty$,

$$\|Y_\alpha^\delta(Z_{j,i}^\omega)\|_{L^p(\cdot \mid \mathcal{F}_{i-1})} \leq C_{\gamma,p} (1+j)^c q_{\text{anc}}^{j-1-i} g_i b_i a_i^{2\alpha-2} [\sqrt{\log(2 + L/a_i)} + \sqrt{p}].$$

The proof uses a net in physical space and time at the scale H , with at most a fixed power of H^{-1} points, independently of δ/H ; a short initial interval of times is controlled by the factor t of the Duhamel term. The proof uses $b_i/g_i \leq C_\gamma(1+i)$, and no constant uniform in the coupling is asserted.

2. At $i = j-1$ we have $(U - V)\theta_{j-1} = x_{j-1}$ and $\forall \theta_{j-1} = r_j$, and

$$\mathcal{B}_\delta(x_{j-1}, r_j) = \mathcal{T}_{j,j-1}^{UV} + \mathcal{T}_{j,j-1}^{VU} - 2\mathcal{T}_{j,j-1}^{VV}, \quad \mathcal{Q}_\delta(r_j, r_j) = \mathcal{T}_{j,j-1}^{VV},$$

with θ_{j-1} in both arguments of each term on the right. For the full quadratic difference, $\mathcal{Q}_\delta(x_{j-1} + r_j, x_{j-1} + r_j) - \mathcal{Q}_\delta(x_{j-1}, x_{j-1}) = \mathcal{T}_{j,j-1}^{UV} + \mathcal{T}_{j,j-1}^{VU} - \mathcal{T}_{j,j-1}^{VV}$, so the coefficient of $\mathcal{T}_{j,j-1}^{VV}$ is -1 .

3. For sufficiently large fixed C_γ and c , let

$$W_j = C_\gamma (1+j)^c \left\{ \rho_0^2 q_{\text{anc}}^{j-1} / a_0 + \sum_{i=1}^{j-1} q_{\text{anc}}^{j-1-i} \left[g_i b_i a_i^{2\alpha-2} \sqrt{\log(2 + L/a_i)} + \frac{a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3}{a_i} \right] \right\}.$$

For each ω separately, $\mathbb{E}_{v_K} Y_\alpha^\delta(\mathcal{T}_{j,j-1}^\omega(\theta_{j-1}, \theta_{j-1})) \leq W_j$, including the coarsest term and all drifts. In particular,

$$\mathbb{E}_{v_K} Y_\alpha^\delta(\mathcal{Q}_\delta(r_j, r_j)) \leq W_j, \quad \mathbb{E}_{v_K} Y_\alpha^\delta(\mathcal{B}_\delta(x_{j-1}, r_j)) \leq 4W_j.$$

For every $0 < \eta < \min\{2\alpha - 1, -\log q_{\text{anc}} / \log B\}$ there exists $C_{\gamma,\eta}$ such that $\sum_{j \geq k} W_j \leq C_{\gamma,\eta} a_k^\eta$. By the triangle inequality, the same sum bounds the expected norms of all finite partial sums over $j \geq k$, uniformly in K .

All constants may depend on the fixed coupling, the coarse side, the moment, the exponents, the horizon, and the fixed analytic and column bounds; they do not depend on K , on the number of preceding averaging steps, on i and j , on a/δ , or on the values of the coarser fields. These are estimates on the whole torus.

Proof. Throughout, we use the conditional law and the properties verified in Lemma 27.3; see (27.5), (27.6), and (27.7). The proof has five steps: a pointwise Lipschitz bound, a deterministic net, concentration, the telescope, and the summation.

Step 1: the Lipschitz constant of one evaluation. Let $\Lambda = K_{ji}g_i b_i$. For $t \leq H^2$, by the bounds of Theorem 27.5(2) for small times and $\delta \leq H$, we have

$$(t + \delta^2)^\beta \Lambda \log(1 + t/H^2) + (t + \delta^2)^\alpha \Lambda J_H(t) \leq C \Lambda H^{2\beta}.$$

For $H^2 \leq t \leq G^2$, the two terms are at most $C \Lambda t^\beta \ell_H(T)$. For $t \geq G^2$, the late bounds, multiplied by their weights, are at most

$$C \Lambda G^{3/2} t^{\alpha-5/4} \ell_H(T) \leq C \Lambda G^{2\alpha-1} \ell_H(T),$$

since $\alpha < 2/3 < 5/4$ and $t \geq G^2$. These three ranges show that every scalar weighted evaluation, of the value or of a first difference, is Lipschitz in w_i with constant at most $C_J A_{ji}$ on the whole product Ω_i . The sum of the two maxima bounds the norm Y_{α}^δ , at the cost of a factor two. Translations and the finite number of colors and directions cost a fixed factor.

Step 2: a deterministic net. This step uses the bound b_i on the coarser fields rather than a distance between coefficient rows, so its constants depend on the fixed coupling. It suffices to consider $H \leq \min(1, \sqrt{T})$; the other levels form a finite set, independent of the number of remaining steps. By (27.23) and heat contraction, the uncentered Duhamel term and its first three differences satisfy

$$\|D_\delta^r \mathcal{T}(t)\|_\infty \leq C_{J,r} \mathfrak{A}_{ji}^2 b_i^2 t H^{-1-r}, \quad 0 \leq r \leq 3, \quad (27.27)$$

and the corresponding time derivatives for $r = 0, 1$ are bounded by $C_{J,r} \mathfrak{A}_{ji}^2 b_i^2 (H^{-1-r} + t H^{-3-r})$. These bounds are deterministic on the whole product Ω_i . Conditional expectation preserves them, and centering costs a factor two.

For $0 \leq t \leq H^6$, both weights are bounded by their values at $2H^2$, up to fixed powers of two. The centered weighted term on this interval is at most $C \mathfrak{A}_{ji}^2 b_i^2 H^{6+2\alpha-2}$, and its ratio to A_{ji} is at most

$$C \frac{b_i}{g_i} H^4 (H/G)^{2\alpha-1} \ell_H(T)^{-1} \leq C_\gamma (1+j) H^4, \quad (27.28)$$

by (27.25). The right side tends to zero geometrically in j , uniformly in i and in $j-i$. For all sufficiently fine levels it is less than $1/8$, and enlarging the constant handles the finitely many other levels. This uses the factor t of the Duhamel term, not a bound on a source at time zero.

On $[H^6, T]$ we consider the two scalar processes

$$(t + \delta^2)^\beta (\mathcal{T} - \mathbb{E}[\mathcal{T} \mid \mathcal{F}_{i-1}]) (t, x), \quad (t + \delta^2)^\alpha D_\delta (\mathcal{T} - \mathbb{E}[\mathcal{T} \mid \mathcal{F}_{i-1}]) (t, x).$$

Their Lipschitz constants in physical space follow from one more output difference in (27.27). For their time derivatives we use the bounds following that display and the product rule for the weights. Differentiating a weight costs no power of the mesh: for $\zeta = \beta$ or $\zeta = \alpha$,

$$t(t + \delta^2)^{\zeta-1} \leq (t + \delta^2)^\zeta \leq C_{T,\zeta}.$$

Thus all these Lipschitz constants are bounded by $C_{T,J} \mathfrak{A}_{ji}^2 b_i^2 H^{-4}$. Dividing by A_{ji} , we obtain the bound

$$C_{T,J} \frac{b_i}{g_i} H^{-5} G^{-(2\alpha-1)} \leq C_\gamma (1+j) H^{-5-(2\alpha-1)} \leq C_\gamma H^{-7},$$

after enlarging C_γ , since $2\alpha - 1 < 1/3$ and $j = O(1 + |\log H|)$ for the fixed sequence of lattices. After another fixed adjustment, these inequalities also hold on the finite set of coarse levels.

We choose a spacing $\varepsilon_H = c_\gamma H^{12}$ in physical space and time, with $c_\gamma > 0$ small. If $\delta \geq \varepsilon_H$, we take every lattice site; there are at most $C_L \varepsilon_H^{-3}$ of them. If $\delta < \varepsilon_H$, we partition the torus into cubes of side between $\varepsilon_H/2$ and ε_H , and choose one lattice site in each occupied cube. Every lattice site can then be joined to its chosen site by a coordinate lattice path of length at most $C\varepsilon_H$. The bound on physical differences applies along these paths, although the net points need not form a lattice. We add a time net of spacing at most ε_H , including both endpoints. The total number of points is at most

$$C_{\gamma,L,T} (1 + L/\varepsilon_H)^3 (1 + T/\varepsilon_H) \leq C_{\gamma,L,T,a_0} H^{-48}. \quad (27.29)$$

After the choice of c_γ , the error of interpolation for the centered weighted processes is at most $A_{ji}/8$, and so is the contribution of short times. The finite numbers of components are included in the constant.

Step 3: concentration and conditional moments. Consider an evaluation F at a point of the net. The coarser field is fixed, the domain is the same finite product of balls, and F is a smooth real function that is Lipschitz for the Euclidean distance on a neighborhood of the product. We may apply (27.7) with Lipschitz constant at most $C_J A_{ji}$; Chernoff's inequality with positive and negative z then gives

$$\mathbb{P}\{|F - \mathbb{E}[F \mid \mathcal{F}_{i-1}]| > z A_{ji} \mid \mathcal{F}_{i-1}\} \leq 2e^{-c_J z^2}.$$

We apply the finite-family clause of Lemma 16.4 with $s = C_J A_{ji}$, $b = 0$, and deterministic error $e \leq C A_{ji}$. The logarithm of the number of points is at most $C_\gamma \log(2 + L/H)$,

which implies the tail of (1). The moment clause yields all $p \geq 1$, and for $0 < p < 1$ Jensen's inequality gives $\|M\|_p \leq \mathbb{E}M$. The moments in (1) then follow from deterministic comparisons between H , G , and j . By (27.22), $\ell_H(T) \leq C_{T,a_0,B}(1+j)$, and $\log(2+L/H) \leq C(1+j)\log(2+L/G)$. Inserting $K_{ji} = q_{\text{anc}}^{j-1-i}/G$ and absorbing these polynomial factors, we obtain the moment bound of (1) with a fixed finite c independent of K , j , and i . The constant $C_{\gamma,p}$ may absorb $C_{T,a_0,B}$, because L and n are fixed and $a_0 = L/n$.

Step 4: the three orders and the telescope. At $i = j - 1$ the product of linearizations is empty, and

$$(U - V)\theta_{j-1} = H^{-1}P^\delta I_H \theta_{j-1} = x_{j-1}, \quad \forall \theta_{j-1} = r_j.$$

For any bilinear map $\mathcal{N}$,

$$\begin{aligned} \mathcal{N}(U - V, V) + \mathcal{N}(V, U - V) &= \mathcal{N}(U, V) + \mathcal{N}(V, U) - 2\mathcal{N}(V, V), \\ \mathcal{N}(U, U) - \mathcal{N}(U - V, U - V) &= \mathcal{N}(U, V) + \mathcal{N}(V, U) - \mathcal{N}(V, V). \end{aligned}$$

Integrating against the same outer heat semigroup gives the identities of (2), including the coefficient -1 for the full difference.

For each order separately, we have the telescoping identity

$$\mathcal{T}_{j,j-1}^\omega(\theta_{j-1}, \theta_{j-1}) = \mathcal{T}_{j,0}^\omega(\theta_0, \theta_0) + \sum_{i=1}^{j-1} (Z_{j,i}^\omega + e_{j,i}^\omega), \quad (27.30)$$

since $Z_{j,i}^\omega + e_{j,i}^\omega = \mathcal{T}_{j,i}^\omega(\theta_i, \theta_i) - \mathcal{T}_{j,i-1}^\omega(\theta_{i-1}, \theta_{i-1})$. Each summand uses the conditional expectation $\mathbb{E}_{v_K}[\cdot \mid \mathcal{F}_{i-1}]$. We take the norm and the expectation, and use the moment bound of (1) with $p = 1$ and the bounds on drifts and on the coarsest term of Theorem 27.5(4). For sufficiently large fixed C_γ and c , we obtain W_j for each order. Here C_γ includes the dependence on a_0 of the weighted bounds on drifts and on the coarsest term. By the identities of (2) and the triangle inequality, we then obtain the bounds W_j and $4W_j$ for $\mathcal{Q}_\delta(r_j, r_j)$ and $\mathcal{B}_\delta(x_{j-1}, r_j)$.

Step 5: the tail of the convolution. Let $r_* = 2\alpha - 1 \in (0, 1/3)$ and $\lambda_q = -\log q_{\text{anc}}/\log B > 0$. By (27.25), the centered summand of W_j without the factor q_{anc}^m is at most $C_\gamma(1+i)^d a_i^{r_*}$ for a fixed finite d ; here $\sqrt{\log(2+L/a_i)} \leq C\sqrt{1+i}$. By (27.26), the summand of the drifts is at most $C_\gamma(1+i)^3 a_i^{1/2}$, and hence also satisfies this weaker bound, after adjusting a constant depending on a_0 . The coarsest term is the term $i = 0$, up to a constant depending on a_0 . Lemma 27.2(2), with $r = r_*$ and the polynomial factor $(1+j)^{c+d}$, implies that

$$\sum_{j \geq k} W_j \leq C_{\gamma,\eta} a_k^\eta$$

for every $0 < \eta < \min(r_*, \lambda_q)$. For any finite partial sum over $j \geq k$, the triangle inequality bounds its expectation by the corresponding part of this series, uniformly in K . At each

finite K the fields and their differences are continuous in t and have finitely many spatial components, so their suprema are determined by a countable dense set of times and are measurable. This completes the proof, under the conditioned Wilson measure ν_K . $\square$

27.4 The predictable term on the finest lattice

The predictable term of Lemma 26.2 is $u_j = P^\delta I_a U_j^0$, with

$$U_j^0 = F_j + H^{-1} \bar{R}_j(\theta_{j-1}) + m_j^{\text{ph}}, \quad F_j = H^{-1} \bar{Q}_j(\theta_{j-1}, \theta_{j-1}).$$

In the notation of the introduction, $U_j^{\text{cl}} = F_j + H^{-1} \bar{R}_j(\theta_{j-1})$ and $U_j^0 = U_j^{\text{cl}} + m_j^{\text{ph}}$. We bound u_j in the norm H_σ^δ of its initial field, which controls both the value and the first difference of the heat evolution. The quadratic part F_j is treated by the same propagation and telescope as in Sections 27.2 and 27.3, now on the lattice of mesh a and without the outer Duhamel integral. The cubic part is bounded deterministically, and the conditional mean correction m_j^{ph} is bounded directly, with an extra factor of the coupling, from the derivative formula for normalized expectations.

Theorem 27.7. *Assume the setting of Section 26.1 and the notation of Theorem 27.5, and let $0 < \sigma < 1 - \alpha$. For $0 \leq i \leq j - 1$ let*

$$Q_{j,i}(v, z) = H^{-1} \bar{Q}_j(\mathsf{T}_{ji}v, \mathsf{T}_{ji}z),$$

the quadratic coefficient of F_j propagated from scale i , including the factor $1/H$, on the lattice of mesh $a = a_j$.

1. *The absolute row sum and both volume-normalized column sums with a fixed input of $Q_{j,i}$ satisfy $(R_2 + C_2)(Q_{j,i}) \leq CK_{ji} = Cq_{\text{anc}}^{j-1-i}/a_i$. For $1 \leq i \leq j - 1$ and $G = a_i$, the derivative in w_i after interpolation satisfies*

$$\sup_x \|D_{w_i} P_t^\delta I_a Q_{j,i}(\theta_i, \theta_i)(x)\|_2 \leq C_J K_{ji} g_i b_i \min\{1, G^{3/2}(t + G^2)^{-3/4}\}.$$

After multiplication by $(t + \delta^2)^{\sigma/2}$ its supremum is at most $C_{T,J} K_{ji} g_i b_i G^\sigma$, for every $0 < \sigma < 3/2$. For every fixed pair of nonnegative integers r, v , the crude bounds

$$\sup_{w_i, t, x} \|D_{w_i} \partial_t^v D_\delta^r P_t^\delta I_a Q_{j,i}(\theta_i, \theta_i)(x)\|_2 \leq C_{J,r,v} K_{ji} g_i b_i a^{-r-2v}$$

hold. The constant mode of the heat kernel is kept.

2. *For the conditionally centered term $Z_{j,i}^{\text{st}} = Q_{j,i}(\theta_i, \theta_i) - \mathbb{E}[Q_{j,i}(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}]$, the conditional logarithmic Sobolev inequality and a net at scale a give, for every $1 \leq p < \infty$,*

$$\|V_\sigma^\delta(I_a Z_{j,i}^{\text{st}})\|_{L^p(\cdot | \mathcal{F}_{i-1})} \leq C_{\gamma,p,J} K_{ji} g_i b_i G^\sigma [\sqrt{\log(2 + L/a)} + \sqrt{p}].$$

3. For the conditional drift $e_{j,i}^{\text{st}} = \mathbb{E}[\mathbf{Q}_{j,i}(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}] - \mathbf{Q}_{j,i-1}(\theta_{i-1}, \theta_{i-1})$ and for the coarsest term,

$$\begin{aligned} \mathbf{V}_\sigma^\delta(I_a e_{j,i}^{\text{st}}) &\leq C_{T,J} \frac{q_{\text{anc}}^{j-1-i}}{a_i} (a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3), \\ \mathbf{V}_\sigma^\delta(I_a \mathbf{Q}_{j,0}(\theta_0, \theta_0)) &\leq C_{T,J} \rho_0^2 q_{\text{anc}}^{j-1} / a_0. \end{aligned}$$

All three contributions to the drift are present. The drift has the same exact splitting into linearization error and fluctuation correction as in Theorem 27.5(4), with the factor $(1+j)K_{ji}$ there replaced by $C_{T,J}K_{ji}$ in the bounds in $\mathbf{V}_\sigma^\delta$. In the bounds on derivatives in w_i , b_i may be replaced by any bound valid on the whole set Ω_i at the fixed coarser field.

4. For every array f on the lattice of mesh δ , including the times $0 \leq t \leq \delta^2$,

$$\mathbf{H}_\sigma^\delta(f) \leq C_{L,T,\sigma} \mathbf{V}_\sigma^\delta(f), \quad \mathbf{X}_\alpha^\delta(P^\delta f) \leq C_{L,T,\alpha,\sigma} \mathbf{H}_\sigma^\delta(f).$$

5. The cubic and mean terms of U_j^0 satisfy $\|\bar{\mathbf{R}}_j/H\|_\infty \leq C_J b_{j-1}^3/H$ and $\|m_j^{\text{ph}}\|_\infty \leq C_J g_3^2 b_{j-1}$. For every $p \in [1, \infty)$ and every

$$0 < \tau < \min\{\sigma, \frac{1}{2}, -\log q_{\text{anc}} / \log B\},$$

there exist a finite exponent c and a constant $C_{\gamma,p,J,\tau}$ such that

$$\sup_{K \geq j} \left\{ \|\mathbf{H}_\sigma^\delta(I_a U_j^0)\|_{L^p(v_K)} + \|\mathbf{X}_\alpha^\delta(u_j)\|_{L^p(v_K)} \right\} \leq C_{\gamma,p,J,\tau} (1+j)^c a_{j-1}^\tau.$$

In particular,

$$\sup_K \sum_{j=k}^K \mathbb{E}_{v_K} \mathbf{H}_\sigma^\delta(I_{a_j} U_j^0) \leq C_{\gamma,J,\tau} (1+k)^c a_k^\tau,$$

and $\mathbf{H}_\sigma^\delta(I_a U_j^0) \leq C_{\gamma,J} (1+j)^c$ on the small-field event.

The constants in the bounds on rows and means depend on L , T , M , J , and the fixed analytic and column bounds; they do not depend on the coupling, K , i , j , the number of preceding averaging steps, or a/δ . The constants in the maximal and summed bounds may in addition depend on the fixed γ , n , p , σ , τ , and the prescription of the cutoff radii. No maximal estimate on regions or uniform in the coupling is asserted.

Proof. All computations take place on the common lattice of mesh δ , and all uses of conditional concentration are for the kernel of Lemma 27.3. The conditional mean correction is identified below with the nonlinear mean of the reconstruction maps.

Step 1: the propagated quadratic coefficient. The quadratic Taylor coefficient of the zero-fluctuation reconstruction, in the normalization of (26.3), is

$$\overline{Q}_j(v, z) = \frac{B}{2} D_{\theta}^2 \mathcal{R}_j(0, 0)[v, z],$$

and its coefficient bounds are given by Lemma 17.2. Propagating both arguments through each L_i , we obtain

$$Q_{j,i}(v, z) = H^{-1} \overline{Q}_j(T_{ji}v, T_{ji}z), \quad Q_{j,i-1}(v, z) = Q_{j,i}(L_i v, L_i z).$$

Here $\overline{Q}_j$ is the symmetric bilinear Taylor coefficient just defined, while the sources of Section 27.2 keep their order. A row acquires the factor a_{cap}^2 at each step. For a column with a fixed input, the argument with the fixed input uses the volume-normalized column of L_i , and the other argument uses its row. The cell volume of the intermediate index then cancels, as in Lemma 27.1(1). Thus

$$(R_2 + C_2)(Q_{j,i}) \leq C \frac{a_{\text{cap}}^{2m}}{H} = C \frac{q_{\text{anc}}^m}{G} = CK_{ji}. \quad (27.31)$$

The same bound on the propagated coefficient, including its weighted version before heat evolution, is part of Theorem 17.4. Interpolation from the lattice of mesh a by I_a preserves these column bounds, since each column of I_a has physical mass exactly a^3 .

Step 2: rows of derivatives and crude bounds. By (27.2), at any point of Ω_i , the derivative in w_i of the two arguments has absolute row and volume-normalized column at most $C_J K_{ji} g_i b_i$. For $t \leq G^2$, the absolute row controls the counting ℓ^2 row. For $t \geq G^2$ we split the heat semigroup into two equal halves. For the first half, the Schur bound gives counting operator norm at most $C_J K_{ji} g_i b_i (G/\delta)^{3/2}$, and the second half has counting row norm at most $C \delta^{3/2} t^{-3/4}$. Their product is $C_J K_{ji} g_i b_i G^{3/2} t^{-3/4}$. The bounds for small and for late times give the profile with $t + G^2$ in (1), since this quantity is comparable to G^2 at small times and to t at late times.

For $0 < \sigma < 3/2$, the weighted supremum follows by splitting at G^2 . Below that time the weight is at most CG^σ , because $\delta \leq G$. Above it the profile gives

$$CG^{3/2} t^{\sigma/2-3/4} \leq CG^\sigma.$$

For fixed r and v we commute all r differences and the v Laplacians through the heat semigroup and apply them to I_a before smoothing. Using the bound on differences of order $r + 2v$ of the cardinal interpolation and heat contraction, we obtain $C_{J,r,v} K_{ji} g_i b_i a^{-r-2v}$, which proves the crude bounds uniformly on Ω_i .

Step 3: the centered term. We write $Z = Z_{j,i}^{\text{st}}$ and $S_i = K_{ji}g_i b_i G^\sigma$. Since the interpolation is positive and preserves constants, heat contraction gives, for $0 \leq t \leq a^2$,

$$\sup_{t \leq a^2} (t + \delta^2)^{\sigma/2} \|P_t^\delta I_a Z\|_\infty \leq C a^\sigma \max_y |Z(y)|, \quad (27.32)$$

where the maximum is over the sites y of the lattice of mesh a . Each scalar output $Z(y)$ is Lipschitz in w_i with constant at most $C_J K_{ji} g_i b_i$. By the conditional logarithmic Sobolev inequality for these finitely many outputs and a union bound over at most $C(L/a)^3$ of them, we control (27.32) at the scale $C_J K_{ji} g_i b_i a^\sigma [\sqrt{\log(2 + L/a)} + \sqrt{u}]$, which is at most the claimed scale since $a \leq G$.

On $[a^2, T]$ the uncentered term has deterministic amplitude at most $CK_{ji}b_i^2$, and its interpolation and heat evolution have fixed derivatives in space and time bounded by $C_{J,r,v}K_{ji}b_i^2a^{-r-2v}$. For $a \leq 1$, the weighted process of values has Lipschitz constants in physical space and time at most $C_{T,J}K_{ji}b_i^2a^{-2}$. For the time derivative of the weight we use $(t + \delta^2)^{\sigma/2-1} \leq C_{T,\sigma}a^{-2}$ on this interval. Centering costs a factor two. Dividing these constants by S_i , we find

$$C_{T,J} \frac{b_i}{g_i} a^{-2} G^{-\sigma} \leq C_\gamma (1+i) a^{-2-\sigma} \leq C_\gamma a^{-4},$$

where we used (27.25), $i \leq j$, the fixed sequence of lattices, and $\sigma < 1 - \alpha < 1/2$. A net in space and time of spacing $c_\gamma a^8$ therefore approximates every centered weighted value within $S_i/8$, and it has at most $C_{\gamma,L,T} a^{-32}$ points. To choose lattice sites when this spacing is smaller or larger than δ , we use the two constructions of Step 2 of the proof of Theorem 27.6, with H replaced by a and the exponent 12 by 8. The lengths of coordinate paths and the count follow from the same partition into cubes. If $a^2 > T$, the interval of small times covers all times, and the finitely many levels with $a > 1$ are included in the constant.

By the bounds of Step 2, at each point of the net, the weighted evaluation is Lipschitz in w_i with constant $C_J S_i$. We use the counts of entries on the lattice of mesh a and of the net, and the deterministic errors of interpolation above. The finite-family clause of Lemma 16.4 then implies the tail and the moments of (2). This net uses b_i/g_i , so it holds at fixed coupling. All bounds on rows remain valid with a smaller bound b_i that holds on the whole set Ω_i .

Step 4: the drifts and the coarsest term. The telescoping identity is

$$F_j = Q_{j,0}(\theta_0, \theta_0) + \sum_{i=1}^{j-1} (Z_{j,i}^{\text{st}} + e_{j,i}^{\text{st}}). \quad (27.33)$$

Before interpolation, each propagated coefficient belongs to the weighted class of Theorem 17.4. By (27.31), its coefficient norm, including $1/H$, is bounded by CK_{ji} ; we use

that theorem for its coefficient bound and its telescope. To obtain a constant uniform in the coupling, we instead apply Lemma 27.4 to each scalar output of Q_{ji} on the lattice of mesh a , with $k_Q \leq CK_{ji}$ and $\phi = \theta_{i-1}$. The propagated coefficients are equivariant under global color rotations, since the zero-fluctuation reconstruction and its linearizations are. The conditional law, the coefficient bounds, and the covariance are those of Lemma 27.3, see (27.5) and (27.10). We obtain

$$\|e_{ji}^{\text{st}}\|_\infty \leq C_J K_{ji} (a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3). \quad (27.34)$$

This bounds the whole drift with a constant uniform in the coupling, as in the computation of Theorem 27.5. Applying (27.12) and (27.13) separately gives the splitting into linearization error and fluctuation correction, with all three contributions kept. The output is still on the lattice of mesh a at this step. Using the contraction of the interpolation and of the heat semigroup, together with $\sup_{t \leq T} (t + \delta^2)^{\sigma/2} \leq C_{L,T,\sigma}$, we obtain the bounds in V_σ^δ of (3). The coarsest term is bounded by substituting $\|\theta_0\|_\infty \leq \rho_0$ into (27.31) and applying the same contractions. These bounds do not contain the factor $1 + j$ of Theorem 27.5.

Step 5: the first difference, including the smallest times. For every array f on the lattice of mesh δ and $t \geq \delta^2$, we split the heat semigroup at $t/2$ and use Lemma 16.1:

$$\|D_\delta P_t^\delta f\|_\infty \leq C(t/2 + \delta^2)^{-1/2} \|P_{t/2}^\delta f\|_\infty \leq C(t + \delta^2)^{-(\sigma+1)/2} V_\sigma^\delta(f).$$

For $0 \leq t \leq \delta^2$, the value term at time zero gives $\|f\|_\infty \leq \delta^{-\sigma} V_\sigma^\delta(f)$, and writing a difference as the difference of two contractions gives

$$\|D_\delta P_t^\delta f\|_\infty \leq 2\delta^{-1} \|f\|_\infty \leq 2\delta^{-\sigma-1} V_\sigma^\delta(f).$$

Multiplication by $(t + \delta^2)^{(\sigma+1)/2}$ makes this bound uniform also at $t = 0$. Including the value term, we obtain $H_\sigma^\delta \leq C V_\sigma^\delta$. Since $\alpha > \sigma$, the two components of $X_\alpha^\delta(P^\delta f)$ are those of $H_\sigma^\delta(f)$ multiplied by $(t + \delta^2)^{(\alpha-\sigma)/2}$, which is at most $[T + (L/4)^2]^{(\alpha-\sigma)/2}$ on the lattices considered. This proves the second comparison of (4), with the stated dependence on L , T , α , and σ . In particular, for a bounded array,

$$H_\sigma^\delta(f) \leq C_{L,T,\sigma} \|f\|_\infty. \quad (27.35)$$

Step 6: the rate of the quadratic part. Applying the conditional and then the unconditional Minkowski inequality to (27.33) for fixed $p \geq 1$, we obtain, with

$$v_i = \frac{a_{\text{cap}}^2 b_{i-1}^3 + g_i^2 b_{i-1} + g_i^3 p_i^3}{a_i},$$

the bound

$$\begin{aligned} \|\mathbf{V}_\sigma^\delta(I_a F_j)\|_{L^p(v_K)} &\leq C_{\gamma,p,J} \left\{ \rho_0^2 q_{\text{anc}}^{j-1} / a_0 + \sum_{i=1}^{j-1} q_{\text{anc}}^{j-1-i} v_i \right. \\ &\quad \left. + \left\{ \sqrt{\log(2 + L/a)} + \sqrt{p} \right\} \sum_{i=1}^{j-1} q_{\text{anc}}^{j-1-i} g_i b_i a_i^{\sigma-1} \right\}. \end{aligned} \quad (27.36)$$

The factor in the last sum satisfies

$$g_i b_i a_i^{\sigma-1} \leq C_J \{ g_3 \rho_0 a_0^{-\kappa_{\text{term}}} a_i^{\kappa_{\text{term}} + \sigma - 1/2} + g_3^2 a_i^\sigma p_i \} \leq C_\gamma (1+i) a_i^\sigma,$$

since $\kappa_{\text{term}} > 1/2$, and v_i is bounded by (27.26), with rate $a_i^{1/2}$. Also $\log(2 + L/a) \leq C(1+j)$. For fixed $\tau < \min(\sigma, 1/2, \lambda_q)$ we choose ζ strictly between τ and this minimum, and apply Lemma 27.2(1) with $r = \sigma$ and with $r = 1/2$. The polynomial factors in i are bounded by powers of $1+j$, and the coarsest term is the term $i = 0$. We obtain

$$\sup_{K \geq j} \|\mathbf{H}_\sigma^\delta(I_a F_j)\|_{L^p(v_K)} \leq C_{\gamma,p,J,\tau} (1+j)^c a_{j-1}^\tau, \quad (27.37)$$

where the comparison of Step 5 has replaced $\mathbf{V}_\sigma^\delta$ by $\mathbf{H}_\sigma^\delta$, so the first difference needs no further probability estimate.

Step 7: the cubic term and the conditional mean. The cubic bound follows from the Taylor remainder of the zero-fluctuation reconstruction in Lemma 17.2, normalized as in (26.3):

$$\|\bar{\mathbf{R}}_j / H\|_\infty \leq C_J b_{j-1}^3 / H \leq C_\gamma (1+j)^3 H^{1/2},$$

where the last inequality is (27.25) on the coarser lattice. For the conditional mean correction, we take in Lemma 17.1 the coarse mesh $a_0 = L/n$ and total number K of renormalization group steps. Its numbers of preceding averaging steps are $J(K-j)$, its reconstruction maps are those used here, and its filtration is $\mathcal{F}_j$. Its lattice potentials and zero-fluctuation reconstructions are therefore

$$A_j = a_j^{-1} \theta_j, \quad C_j = a_j^{-1} \mathcal{R}_j(\theta_{j-1}, 0),$$

and its conditional mean correction $\mathbb{E}[A_j \mid \mathcal{F}_{j-1}] - C_j$ is m_j^{ph} of Lemma 26.2. This identifies the conditional mean correction, including its conditional law.

We prove the uniform bound directly from the derivative formula for normalized expectations, so that no stochastic constant depending on the fixed coarse mesh enters this deterministic bound. Fix $\phi = \theta_{j-1}$, write $r_\phi = \|\phi\|_\infty$, and set

$$\mathcal{G}_\phi(w) = \mathcal{R}_j(\phi, w) - \mathcal{R}_j(\phi, 0).$$

For $0 \leq u \leq 1$ let $\mathbb{E}_u$, Cov_u denote expectation and covariance under the conditional fluctuation kernel at $u\phi$. The segment of coarser fields and every point of the free coordinates remain in the same domain, as checked in Lemma 27.3, and the product of balls does not change. The bound on the mixed derivative of the potential in Theorem 13.10 and the covariance inequality of Theorem 13.8 apply. The majorant series of the joint Taylor coefficients used in (27.2) gives deterministic envelopes $a_{e,l} \geq 0$ such that, on the whole product and the whole segment, $|\partial_{w_l} \mathcal{G}_{u\phi,e}| \leq a_{e,l}$ and $\sum_l a_{e,l} \leq C_J t_j$. In particular $\sum_l \|\partial_{w_l} \mathcal{G}_{u\phi,e}\|_{L^2} \leq C_J t_j$; this uses the absolute coefficient majorant, rather than an exchange of a supremum with a sum of L^2 norms. The bound on the mixed derivative of the potential is $\sup_l \|\partial_{w_l} \dot{V}_u\|_{L^2} \leq C_J t_j r_\phi$, where $\dot{V}_u = \partial_u V_{u\phi}$. The covariance kernel decays exponentially and has a uniformly bounded sum at each index, in the units of the renormalization group step, because the number of fluctuation coordinates in each cell is bounded in terms of J . The covariance inequality then implies that

$$|\text{Cov}_u(\mathcal{G}_{u\phi,e}, \dot{V}_u)| \leq C_J t_j^2 r_\phi. \quad (27.38)$$

To see that no volume factor appears, we sum the covariance kernel over its second index first, using the uniform bound on the mixed derivative there, and then sum the absolute row of derivatives at the fixed output e .

The expectation of $\partial_u \mathcal{G}_{u\phi,e}$ has the same bound. Indeed, its series of absolute joint Taylor coefficients has positive degree in the free coordinates, and one insertion of the coarser field costs $C_J r_\phi$. We fix a radius $R_{\text{free}} > 0$ in the free coordinates from the nested domains, with the segment of coarser fields in its inner domain and a fixed positive margin above every cutoff domain. The differentiated coefficient sum at this radius is bounded by $C_J r_\phi$, and $t_j p_j / R_{\text{free}} \leq \eta_* < 1$, with a fixed margin. For the term of degree one we use the bound $\|\mathbb{E}_u w\|_\infty \leq C_J t_j r_\phi$ of Theorem 13.10; its expectation costs $C_J t_j^2 r_\phi^2$. For degree $k \geq 2$ we put two factors in L^2 and bound the other $k - 2$ by the cutoff p_j . The L^2 factors are bounded uniformly: the Poincaré inequality and the bound on the mean give $\mathbb{E}_u |w_l|^2 \leq 2 + |\mathbb{E}_u w_l|^2 \leq C_J$ for scalar coordinates. After division by the radius of its coefficients, the term of degree k costs at most $C_J t_j^2 R_{\text{free}}^{-2} \eta_*^{k-2}$ times its absolute coefficient mass at R_{free} . Summing this mass, and using the fixed bound on r_ϕ , we obtain

$$|\mathbb{E}_u \partial_u \mathcal{G}_{u\phi,e}| \leq C_J t_j^2 r_\phi. \quad (27.39)$$

Repeated coordinates are handled by the same Cauchy–Schwarz inequality for two factors; no independence is used.

The derivative formula for normalized expectations of Theorem 13.10 implies that

$$\frac{d}{du} \mathbb{E}_u \mathcal{G}_{u\phi,e} = \mathbb{E}_u \partial_u \mathcal{G}_{u\phi,e} - \text{Cov}_u(\mathcal{G}_{u\phi,e}, \dot{V}_u).$$

Its hypotheses hold because these maps and their derivatives are continuous on a neighborhood of the fixed compact product, so all difference quotients in finite dimensions

are dominated there. At $u = 0$, the conditional law at the zero coarser field and the reconstruction map are equivariant under global color rotations, so $\mathbb{E}_0 \mathcal{G}_{0,e} = 0$. Integrating (27.38) and (27.39), we obtain the bound on the nonlinear mean of the reconstruction,

$$\|\mathbb{E}_\phi \mathcal{R}_j(\phi, w) - \mathcal{R}_j(\phi, 0)\|_\infty \leq C_J t_j^2 \|\phi\|_\infty. \quad (27.40)$$

This is the bound of Theorem 13.10 on the mean of the reconstruction, proved here for the reconstruction maps of this section.

Dividing by the mesh a_j of the reconstructed field keeps the factor of the intermediate scales:

$$\|m_j^{\text{ph}}\|_\infty \leq C_J \frac{t_j^2}{a_j} \|\theta_{j-1}\|_\infty = C_J M^{J-1} g_3^2 \|\theta_{j-1}\|_\infty \leq C_J g_3^2 b_{j-1}, \quad (27.41)$$

where the last C_J absorbs only the fixed factor M^{J-1} . All constants in this pointwise bound on the mean are uniform in the lattice side, in the number of preceding averaging steps, and in the coupling below g_* . By this bound and (27.25), we have $\|m_j^{\text{ph}}\|_\infty \leq C_\gamma(1+j)H^{1/2}$.

Step 8: combination and summation. Contraction of the interpolation and (27.35) bound the contributions of the cubic and mean terms to $H_\sigma^\delta(I_a U_j^0)$ by $C_\gamma(1+j)^3 H^{1/2}$, pathwise and hence in every L^p . Combining this with (27.37), and using $\tau < 1/2$, we obtain the bound in L^p of $H_\sigma^\delta(I_a U_j^0)$ in (5). The comparison $X_\alpha^\delta(P^\delta f) \leq CH_\sigma^\delta(f)$ of (4) then implies the bound for u_j under the same law. For the sum we use the bound with $p = 1$ and

$$\sum_{j \geq k} (1+j)^c a_{j-1}^\tau \leq B^\tau a_k^\tau (1+k)^c \sum_{l \geq 0} (1+l)^c B^{-\tau l} \leq C_{B,\tau,c} (1+k)^c a_k^\tau.$$

The finite sum under ν_K is bounded by this deterministic infinite series, uniformly in K . Finally, by the deterministic bounds on the small-field event, we have

$$\|U_j^0\|_\infty \leq C_J \{b_{j-1}^2/H + b_{j-1}^3/H + g_3^2 b_{j-1}\} \leq C_\gamma(1+j)^3,$$

and (27.35) gives the pathwise bound of (5). Only the maximal inequalities and the summations use constants depending on the fixed γ and n . $\square$

Remark 27.8. These are the norms needed for the bilinear estimate of Lemma 24.6. If $x = P^\delta f$, then x and $u_j = P^\delta I_a U_j^0$ are heat evolutions on the same lattice, $\alpha \in (1/2, 2/3)$, and $\sigma \in (0, 1 - \alpha)$, so that lemma applies in both orders with the norm H_σ^δ bounded in Theorem 27.7. Its mesh is $\delta = L/(nB^K)$, with integer side $nB^K \geq 4$. If $\delta \leq 1$ its clause for the heat semigroup applies, and if $\delta > 1$ its clause for finitely many coarse sides applies, with a constant depending on the fixed L and T .

Linear and quadratic flow data and convergence of Wilson loops

In this section we prove that the joint law of finitely many flowed Wilson loops under the Wilson measure $\mu_{N_K, \gamma}$ converges along the lattice sides $N_K = n_* B^K$, $B = M^J$, at fixed coupling γ , first for approximations with swept comparison area $O(a_K)$ (Theorem 28.2). The extension to the full admissible approximation class of Definition 2.1, with swept area tending to zero, is proved in Theorem 29.23. This is part (1) of Theorem 2.3 along the subsequences of Remark 2.4. The passage to all lattice sides nM^k and the construction of the continuum measure on the trajectory space are carried out in Section 29.

The proof has three steps.

1. *Truncation.* Fix a coarse side n and condition $\mu_{nB^K, \gamma}$ on the small-field event. Under the conditioned Wilson measure the fine field is obtained from the coarse field by a coarse-to-fine reconstruction, one RG step at a time. We compare the linear and quadratic flow data (x, y) of the fine field with those of the finite-scale reconstruction from the first q RG steps, interpolated to the fine lattice. The difference tends to zero in probability as $q \rightarrow \infty$, uniformly in K , and the norms of the flow data are tight (Theorem 28.8).
2. *Comparison with smooth flows.* For fixed q the finite-scale reconstruction is a random field on a fixed lattice, with compact support, and its law converges as $K \rightarrow \infty$ (Theorem 15.14). Its smooth cardinal interpolant is a smooth initial datum, and the lattice Wilson flow started from it converges to the Yang–Mills heat flow (Lemma 28.4). On bounded sets of flow data, flowed Wilson loops are Lipschitz functions of the flow data, with constants that do not depend on q (Corollary 25.11). We choose first a bound R on the norm of the flow data, then a tolerance η , then q , and finally let $K \rightarrow \infty$. This gives convergence of the laws of flowed Wilson loops under the conditioned measure (Theorem 28.1).
3. *Removal of the conditioning.* At a fixed lattice side, for a bounded function F and an event S with $\mu(S) > 0$, $|\mathbb{E}_{\mu(\cdot|S)} F - \mathbb{E}_\mu F| \leq 2\|F\|_\infty \mu(S^c)$ (Lemma 5.5). By Theorem 23.9, the small-field failure probability is bounded uniformly in K by a quantity that tends to zero as the coarse side grows. We let first $K \rightarrow \infty$ at a fixed coarse side $n_r = n_* B^r$, and then $r \rightarrow \infty$.

The main estimate behind the first step is Lemma 28.5. Each increment d_j of the linear heat evolution is split into four terms (Lemma 26.2): the centered linear fluctuation term s_j , the centered nonlinear heat term z_j , the interpolation defect r_j , and the predictable term u_j . The increment of the quadratic Picard term consists of a cross term with the previous linear heat evolution and of the sixteen ordered products of these four terms. We bound the expected norm of each ordered product separately, keeping the conditional means of these products, which do not vanish. Since the bilinear Duhamel term is not symmetric, the two orders of each product are treated separately. The stochastic inputs are the estimates of Sections 26 and 27 for the conditional fluctuation kernel. The deterministic inputs are the ordered product inequality Lemma 24.6 and the consistency and stability results of Sections 24 and 25.

The strategy of separating a rough linear evolution from a smoother nonlinear part goes back to Bourgain [32] and to Da Prato and Debussche [63]. Cao and Chatterjee used probabilistic smoothing in this form for the Yang–Mills heat flow with random initial data [42, Section 1.1]. Here the quadratic Picard term is the ordered bilinear Duhamel term of the damped Wilson flow of Section 24, and the stochastic estimates concern the conditioned Wilson measure; no Gaussian hypothesis on the initial data is used. The lattice flow is the Wilson flow of Lüscher [128], in the physical time normalization of Section 2.1.

Section 29 uses Theorems 28.1, 28.2, and 28.8 and the approximation Lemma 28.9 to construct the continuum measure, and Lemma 28.11 to work with the parameters chosen below an arbitrary finite bound on the coupling. Sections 29 and 32 use Lemma 28.4 and the estimates of Lemma 28.10, which are uniform for couplings in a compact subinterval of $(0, \infty)$. Section 34 uses the bounds on the ordered products in Lemma 28.5.

The stochastic truncation estimates are stated at a fixed coarse side and, except in Lemma 28.10, at a fixed coupling. The convergence results for observables concern finitely many Wilson loops. We do not construct a limit of the flow data themselves as random fields, nor a solution map of the flow for rough initial data, and we do not claim a rate of convergence.

Section 28.1 fixes the notation and states the two main theorems. Section 28.2 collects deterministic estimates, including the comparison with smooth flows. Section 28.3 contains the four-term decomposition and the bounds on the ordered products, and Section 28.4 the truncation estimates. The two main theorems are proved in Sections 28.5 and 28.6. Section 28.7 treats compact intervals of couplings and couplings below a finite bound.

28.1 Setting and main results

Standing assumptions. Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \gamma_0 \leq 1$, and let J , $B = M^J$, the tuple Θ of remaining parameters, g_* , and $n_{\min}$ be as in Theorem 23.2; these are the parameters used in Theorem 23.9. Fix $0 < \gamma \leq \gamma_0$, set $g_3^2 = \gamma/L$, and fix a coarse side $n \in 4\mathbb{N}$ with $n \geq n_{\min}$. For an integer $K \geq 0$ we set

$$H = L/n, \quad N = nB^K, \quad a_j = HB^{-j} \quad (0 \leq j \leq K), \quad a = a_K, \quad g_j = g_3 \sqrt{a_j}.$$

Thus a_j is the mesh of the lattice of side nB^j , and $g_j^2 = \gamma/(nB^j)$ is its lattice coupling. We work under the conditioned Wilson measure $\nu_{N,n,\gamma} = \mu_{N,\gamma}(\cdot \mid S_{N,n,\gamma})$ of Definition 4.4. All fields are defined on the whole torus. Tail indices k, l, m are positive integers, and a sum whose upper limit is smaller than its lower limit is zero.

The coarse-to-fine reconstruction. Let $\widehat{\mathcal{R}}$ be the family of reconstruction maps of Theorem 15.13, and consider its coarse-to-fine disintegration of $\nu_{N,n,\gamma}$ (Lemma 15.10) at the classical depths $J(K-j)$:

$$\widehat{\theta}_j = \widehat{\Phi}_j(\widehat{\theta}_{j-1}, g_j, w_j), \quad A_j = \widehat{\theta}_j/a_j, \quad \mathcal{F}_j = \sigma(\widehat{\theta}_0, w_1, \dots, w_j), \quad 1 \leq j \leq K.$$

Here $\widehat{\theta}_0$ is the link logarithm of the coarse field on Λ_n , in the Borel gauge of Lemma 15.10, so that $\|\widehat{\theta}_0\|_\infty < r$, where r is the radius of the small-link domain B_r . Further, w_j is the vector of normalized fluctuation variables of the j th RG step counted from the coarse side, g_j are the couplings of the J intermediate scales of this step, $\widehat{\Phi}_j$ is the logarithmic reconstruction map of the family $\widehat{\mathcal{R}}$ for this step, $\widehat{\theta}_j$ is the link logarithm of the reconstructed field on the lattice of side nB^j , and A_j is the corresponding lattice potential on the mesh a_j . In the notation of Section 26.1, $\widehat{\Phi}_j(\theta, g_j, w) = \mathcal{R}_j(\theta, w)$, at the depth $\mathfrak{d}_j = J(K-j)$. Conditionally on $\mathcal{F}_{j-1}$, the law of w_j is the conditional fluctuation kernel (4.8) of the j th RG step at the coarser field $\widehat{\theta}_{j-1}$, including all action, Jacobian, determinant, and effective-interaction factors and the indicator of the fluctuation cutoff domain. This kernel is not Gaussian. All conditional expectations below refer to it, and we write $\bar{w}_j = \mathbb{E}(w_j \mid \mathcal{F}_{j-1})$; this is the conditional mean $m_{\widehat{\theta}_{j-1}}$ of Section 26.1.

Let $t_j = M^{(J-1)/2} g_j$ be the largest coupling of the intermediate scales of the j th RG step, and $p_j = A_w \log(1/t_j) \geq 1$ the corresponding fluctuation cutoff. Let a_{cap} be the contraction factor of the reconstruction amplitude bound of Lemma 15.10, which satisfies $a_{\text{cap}} \sqrt{B} < 1$, and set

$$q_{\text{anc}} = Ba_{\text{cap}}^2 < 1, \quad \lambda_{\text{anc}} = -\log(q_{\text{anc}})/\log B > 0.$$

Interpolation, heat semigroup, and norms. For meshes $a \leq h$ with h/a an integer, $I_{h \rightarrow a}$ is the cardinal interpolation from the lattice of mesh h to the lattice of mesh a , recalled in

Section 28.2; $I_{a \rightarrow a}$ is the identity. The lattice heat semigroup $P_t^a = e^{t\Delta_a}$ acts componentwise and includes the constant mode. For an array f on the lattice of mesh a we write $P_t^a f$ for the heat path $t \mapsto P_t^a f$. Let $\mathcal{N}_a$ be the ordered quadratic coefficient of the damped Wilson flow (Lemma 24.2), and recall the bilinear Duhamel term and its polarization,

$$\mathcal{Q}_a(u, v)(t) = \int_0^t P_{t-s}^a \mathcal{N}_a(u(s), v(s)) ds, \quad \mathcal{B}_a(u, v) = \mathcal{Q}_a(u, v) + \mathcal{Q}_a(v, u).$$

The term $\mathcal{Q}_a$ is not symmetric. We fix a horizon $T < \infty$ and exponents

$$1/2 < \alpha = \alpha_{\text{fl}} < 2/3, \quad \beta = \beta_{\text{fl}} = \alpha - 1/2, \quad 0 < \sigma < 1 - \alpha, \quad 0 < \tau < \min\{\sigma, 1/2, \lambda_{\text{anc}}\}.$$

We use the norms of Definition 4.8 on $[0, T]$: with $v_a(t) = t + a^2$,

$$\begin{aligned} \mathsf{X}_a^\alpha(x) &= \sup_{0 \leq t \leq T} \{v_a(t)^{\alpha/2} \|x(t)\|_\infty + v_a(t)^{(\alpha+1)/2} \|D_a x(t)\|_\infty\}, \\ \mathsf{Y}_a^\alpha(y) &= \sup_{0 \leq t \leq T} \{v_a(t)^\beta \|y(t)\|_\infty + v_a(t)^\alpha \|D_a y(t)\|_\infty\}, \\ \mathsf{H}_\sigma^a(f) &= \sup_{0 \leq t \leq T} \{v_a(t)^{\sigma/2} \|P_t^a f\|_\infty + v_a(t)^{(\sigma+1)/2} \|D_a P_t^a f\|_\infty\}, \end{aligned}$$

where D_a runs over all physical forward and backward first differences, applied also to the translated components that occur in $\mathcal{N}_a$; $\mathsf{V}_\sigma^a(f)$ is the first term of $\mathsf{H}_\sigma^a(f)$. The linear and quadratic flow data of the reconstructed field at level j , on the fine mesh a , are

$$x_j = P^a I_{a_j \rightarrow a} A_j, \quad y_j = \mathcal{Q}_a(x_j, x_j), \quad \text{and we set} \quad d_j = x_j - x_{j-1}.$$

Thus x_K is the linear heat evolution of the fine field A_K , and y_K is its quadratic Picard term. For $j < K$, the pair (x_j, y_j) is the flow data of the finite-scale reconstruction A_j , interpolated to the fine mesh. The norm of a pair is $\|(x, y)\|_{\text{fl}, a} = \mathsf{X}_a^\alpha(x) + \mathsf{Y}_a^\alpha(y)$, as in Definition 24.3.

By Theorem 15.14, for each fixed q the law of the array A_q under $\nu_{nB^K, n, \gamma}$ converges weakly as $K \rightarrow \infty$, and these arrays take values in a fixed compact set. We denote the limit law by π_q .

Dependence of constants. Unless stated otherwise, the constants in the stochastic estimates of this section depend only on $L, n, \gamma, M, J, \Theta, T$, the exponents, and the moment orders. They do not depend on K , on the level j , on the classical depth $J(K - j)$, or on the ratio a/a_j . The letter $b < \infty$ denotes an exponent with the same dependence, which may increase finitely many times; in a displayed estimate we may replace finitely many such exponents by their maximum. This is the dependence described in item (7) of Definition 4.14. None of the stochastic estimates of this section is claimed to be uniform as $\gamma \rightarrow 0$ or as $n \rightarrow \infty$.

Wilson loops. Let $\ell_1, \dots, \ell_m$ be piecewise C^1 loops in $\mathbb{T}_L^3$, let $\chi_1, \dots, \chi_m$ be characters of finite-dimensional representations of $SU(2)$, and let $t_1, \dots, t_m > 0$. For each i let $(\ell_{i,K})_{K \geq 0}$ be an admissible approximation of ℓ_i along the lattices Λ_{nB^K} (Definition 2.1) whose based homotopies to ℓ_i can be chosen with swept area, counted with multiplicity, of order $O(a_K)$. This stronger area bound is the hypothesis of the convergence theorems in this section; Theorem 29.23 later extends their conclusions to the full admissible class. Under $\nu_{nB^K, n, \gamma}$ we consider the vector $W_K = (W_{K,1}, \dots, W_{K,m})$ with

$$W_{K,i} = \chi_i(\text{Hol}_{\ell_{i,K}} \Psi_{t_i}^{(N)} U), \quad N = nB^K.$$

Each $W_{K,i}$ takes values in $\{w \in \mathbb{C} : |w| \leq \chi_i(I)\}$, where $\chi_i(I)$ is the dimension of the representation.

Main results. The first theorem gives convergence under the conditioned measure at a fixed coarse side; it is proved in Section 28.5.

Theorem 28.1. *Assume the standing assumptions, and let W_K be as above. Then the law of W_K under $\nu_K = \nu_{nB^K, n, \gamma}$ converges weakly as $K \rightarrow \infty$. The limit laws are consistent under finite unions of lists of loops and under coordinate projections. For each q and each array ϑ on the lattice of side nB^q , let $G_q(\vartheta) = (\chi_i(\text{Hol}_{\ell_i} \Phi_{t_i}[f]))_{i=1}^m$, where f is the smooth cardinal interpolant of ϑ , viewed as a smooth connection, and $\Phi_t[f]$ is the gauge orbit at time t of the Yang–Mills heat flow started from f . Then the limit law of W_K is the weak limit, as $q \rightarrow \infty$, of the laws $(G_q)_{\#} \pi_q$. The moment generating functions $\mathbb{E}_{\nu_K} \exp(\sum_i z_i W_{K,i})$ converge uniformly on every bounded subset of $\mathbb{C}^m$, and their cumulants of every finite order converge, the logarithms being taken on a fixed neighborhood of the origin on which none of these functions vanishes.*

The second theorem removes the conditioning; it is proved in Section 28.6. Here the coarse side varies along the sequence $n_r = n_* B^r$, while the fine lattice Λ_{N_K} , the configuration, and the observable stay the same.

Theorem 28.2. *Assume the standing assumptions, except that the coarse side is not fixed. Fix $n_* \in 4\mathbb{N}$, put $n_r = n_* B^r$, $N_K = n_* B^K$, and $\nu_{K|r} = \nu_{N_K, n_r, \gamma}$ for $K \geq r$, and let W_K be the vector of flowed Wilson loops on Λ_{N_K} defined above, with $D = \max_i \chi_i(I)$. Let C' , $\bar{c}$, C_w , and a_w be the constants of Theorem 23.9, and set*

$$F_\gamma(u) = C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}.$$

Then there exists r_0 such that, for all $r \geq r_0$ and all $K \geq r$,

$$e_{K|r} = \mu_{N_K, \gamma}(S_{N_K, n_r, \gamma}^c) \leq F_\gamma(n_r),$$

and $F_\gamma(n_r) \rightarrow 0$ as $r \rightarrow \infty$. For $r \geq r_0$ let $Z_r^{\text{sel}}(z)$ be the limit as $K \rightarrow \infty$ of $\mathbb{E}_{\nu_{K|r}} \exp(\sum_i z_i W_{K,i})$, which exists by Theorem 28.1. For every $R < \infty$ and all $r, r' \geq r_0$,

$$\sup_{\sum_i |z_i| \leq R} |Z_r^{\text{sel}}(z) - Z_{r'}^{\text{sel}}(z)| \leq 2e^{DR} [F_\gamma(n_r) + F_\gamma(n_{r'})].$$

There exists a probability measure ρ on $\prod_i \{w \in \mathbb{C} : |w| \leq \chi_i(I)\}$ such that, as $K \rightarrow \infty$,

$$\text{Law}_{\mu_{N_K, \gamma}}(W_{K,1}, \dots, W_{K,m}) \Longrightarrow \rho, \quad \mathbb{E}_{\mu_{N_K, \gamma}} e^{\sum_i z_i W_{K,i}} \longrightarrow \int e^{\sum_i z_i w_i} d\rho(w),$$

and the second convergence is uniform on $\{\sum_i |z_i| \leq R\}$ for every $R < \infty$. These limits are consistent under taking subfamilies of the loops. The events $S_{N_K, n_r, \gamma}$ for different r need not be nested.

28.2 Deterministic estimates

This section collects the deterministic facts used below: elementary properties of the cardinal interpolation and of the lattice heat semigroup, a geometric summation, and the comparison of lattice and continuum flows for a fixed family of smooth initial data.

Cardinal interpolation. We recall the cardinal interpolation of Lemma 16.3, explicitly enough to justify the elementary facts needed here. Let b_* be the even nonnegative smooth function of that lemma, denoted b there, which is supported in $[-1, 1]$, positive on $(-1, 1)$, and flat at ± 1 , and set

$$\varphi(u) = \frac{b_*(u)}{\sum_{q \in \mathbb{Z}} b_*(u - q)}, \quad \varphi_3(u) = \prod_{i=1}^3 \varphi(u_i).$$

The denominator is positive, smooth, and one-periodic. Hence $\varphi(q) = \mathbf{1}_{q=0}$ for $q \in \mathbb{Z}$ and $\sum_q \varphi(u - q) = 1$. In the notation of that lemma, $\varphi = \zeta$ and $\varphi_3 = \zeta_3$. For an array f on the lattice of mesh h , its periodized smooth interpolant is

$$f_h(x) = \sum_{v \in \Lambda_{L/h}} \varphi_3((x - v)/h) f(v),$$

where the periodic images of v are understood. The formula is applied separately to every directed-link and color component, at the location of that component. For $a \leq h$ with h/a an integer, $I_{h \rightarrow a} f$ is the restriction of f_h to the lattice of mesh a . For an array f on the lattice of mesh a , we write $J_a f = f_a$ for its interpolant, which is the continuous interpolant $I_a f$ of that lemma. At every point only a bounded number of the translates $\varphi_3((x - v)/h)$ are nonzero.

Lemma 28.3. *Let $a \leq h$ be meshes with h/a an integer, let f be an array on the lattice of the indicated mesh, and let F be a smooth periodic field. Take $t \geq 0$ and an integer $q \geq 0$. Let D_a^q be any product of q physical forward or backward differences, possibly applied to translated components. Then*

$$\|J_a f\|_\infty \leq \|f\|_\infty, \quad \|D_a^q I_{h \rightarrow a} f\|_\infty \leq C_q h^{-q} \|f\|_\infty, \quad (28.1)$$

$$\|J_a(F|_a) - F\|_\infty \leq Ca\|\partial F\|_\infty, \quad (28.2)$$

$$\|P_t^a I_{h \rightarrow a} f\|_\infty \leq \|f\|_\infty, \quad \|D_a P_t^a I_{h \rightarrow a} f\|_\infty \leq Ch^{-1}\|f\|_\infty, \quad (28.3)$$

where C_q and C depend only on q and b_* . Further, if $a_j = HB^{-j}$, $\delta > \delta' > 0$, and $c \geq 0$, then for every integer $k \geq 0$,

$$\sum_{j \geq k} (1+j)^c a_j^\delta \leq C_{B,c,\delta} (1+k)^c a_k^\delta \leq C_{H,B,c,\delta,\delta'} a_k^{\delta'}. \quad (28.4)$$

Proof. The first bound in (28.1) follows from nonnegativity of φ_3 and the partition of unity. For the second, a product of q physical differences of $I_{h \rightarrow a} f$ is an average of the corresponding q th derivative of f_h over q segments. Each derivative of order q of $\varphi_3((x-v)/h)$ is bounded by $C_q h^{-q}$, and only a bounded number of terms contribute at each point. This proves the bound, also for backward differences and translated components, with the same constant. For (28.2), every lattice site v that contributes to $J_a(F|_a)(x)$ lies within distance Ca of x . Subtracting $F(x)$ inside the partition sum and integrating the derivative of F along the segment from x to v gives the bound.

For (28.3), we write $\Delta_a = a^{-2} \sum_i (T_i + T_i^{-1} - 2I)$, where T_i is the translation by ae_i . Then $e^{t\Delta_a} = e^{-6t/a^2} \exp(ta^{-2} \sum_i (T_i + T_i^{-1}))$ is a series in translations with nonnegative coefficients of total mass one, so P_t^a is a contraction in the supremum norm. Differences commute with P_t^a . The two bounds then follow from (28.1).

For the first inequality in (28.4), set $j = k + i$ and use $1 + k + i \leq (1+k)(1+i)$ and $a_{k+i} = a_k B^{-i}$; the remaining series $\sum_{i \geq 0} (1+i)^c B^{-i\delta}$ converges by the ratio test. The second inequality holds because $a_k^\delta = H^{\delta-\delta'} B^{-k(\delta-\delta')} a_k^{\delta'}$ and $\sup_k (1+k)^c B^{-k(\delta-\delta')} < \infty$. $\square$

We use (28.3) only for a fixed finite-scale reconstruction or at times $t < h^2$; it does not transfer between meshes uniformly. Under the standing assumptions, $t_j = t_1 B^{-(j-1)/2}$, so the cutoff $p_j = A_w \log(1/t_j)$ is an affine function of j with slope $A_w(\log B)/2$. In particular

$$p_j \leq C_{L,n,\gamma,M,J,A_w} (1+j). \quad (28.5)$$

Smooth initial data. The next lemma compares the lattice objects with their continuum counterparts for a fixed compact family of smooth initial data. It is used for the finite-scale reconstructions, whose interpolants form such a family. The continuum counterpart of $\mathcal{Q}_a$ is the Duhamel term $\int_0^t P_{t-s} \mathcal{N}(u(s), v(s)) ds$, where $P_t = e^{t\Delta}$ and $\mathcal{N}$ is obtained from the finite sum of shifted brackets defining $\mathcal{N}_a$ by replacing translations by the identity and physical differences by derivatives.

Lemma 28.4. *Let $h = L/m$ for an integer $m \geq 4$, let $M_h < \infty$ and $0 < T < \infty$, and let $\mathcal{F}$ be the family of cardinal interpolants $f = \vartheta_h$ of arrays ϑ on the lattice of mesh h with $\|\vartheta\|_\infty \leq M_h$. For meshes $a \leq h$ with h/a an integer, put*

$$x_a = P_a^a(f|_a), \quad y_a = \mathcal{Q}_a(x_a, x_a), \quad f \in \mathcal{F},$$

where $f|_a = I_{h \rightarrow a} \vartheta$ is the restriction of f to the lattice of mesh a . Then, as $a \rightarrow 0$, the four components of (x_a, y_a) , embedded as in Definition 24.3, converge uniformly over $\mathcal{F}$ to the continuum heat path $P.f$, the continuum Duhamel term $\int_0^t P_{t-s} \mathcal{N}(P_s f, P_s f) ds$, and their spatial derivatives in the difference components. The norms $\|(x_a, y_a)\|_{\mathfrak{H}, a}$ are bounded by a constant depending only on h, M_h, L, T, α , and b_* , and not on a .

Further, let U_a be the configuration on the lattice of mesh a with link variables $\exp(af|_a)$. Fix finitely many characters χ_i of finite-dimensional representations of $SU(2)$, loops ℓ_i , and closed lattice paths $\ell_{i,a}$, independent of time, whose physical lengths are bounded uniformly in a and which admit based homotopies to the ℓ_i , including connecting paths between base points, with swept areas counted with multiplicity tending to zero as $a \rightarrow 0$. Then for every compact interval $[\tau, T'] \subset (0, \infty)$,

$$\chi_i(\text{Hol}_{\ell_{i,a}} \Psi_t^{(L/a)} U_a) \longrightarrow \chi_i(\text{Hol}_{\ell_i} \Phi_t[f]) \quad (a \rightarrow 0),$$

uniformly in $t \in [\tau, T']$ and $f \in \mathcal{F}$, where $\Phi_t[f]$ is the Yang–Mills heat flow started from f . The map from ϑ to the path $t \mapsto \chi_i(\text{Hol}_{\ell_i} \Phi_t[\vartheta_h])$ is continuous. The constants in these convergences may depend on h, M_h , and the smooth bounds on $\mathcal{F}$; no uniformity as $h \rightarrow 0$ is asserted.

Proof. Since the cardinal basis is fixed, $\|f\|_{C^k} \leq C_{k,h} M_h$ for every k and every $f \in \mathcal{F}$. Let S_a denote sampling on the lattice of mesh a . By Taylor’s formula, we have

$$\|\Delta_a S_a P_t f - S_a \Delta P_t f\|_\infty \leq C a^2 \|f\|_{C^4}, \quad \|D_a^k S_a P_t f - S_a \partial^k P_t f\|_\infty \leq C_k a \|f\|_{C^{k+1}},$$

where we used that P_t commutes with derivatives and contracts every C^k norm. By variation of constants, we have

$$P_t^a S_a f - S_a P_t f = \int_0^t P_{t-s}^a (\Delta_a S_a - S_a \Delta) P_s f ds,$$

so $\|P_t^a S_a f - S_a P_t f\|_\infty \leq C t a^2 \|f\|_{C^4}$ by heat contraction. Commuting finitely many differences with P_t^a and using the second Taylor estimate, we obtain the same convergence, with error $O_{h, M_h}(a)$, for every fixed number of differences, uniformly over $\mathcal{F}$ and $t \in [0, T]$.

The ordered coefficient $\mathcal{N}_a$ of Lemma 24.2 is a fixed finite sum of shifted brackets with one physical difference. Its first difference is a sum of terms with $(1, 1)$ or $(0, 2)$ differences on the two factors. By the heat convergence just proved, the value and the first difference of $\mathcal{N}_a(x_a, x_a)$ agree with those of the continuum source $S_a \mathcal{N}(P.f, P.f)$ up to an error $O_{h, M_h}(a)$, uniformly. The same holds in each order of the two arguments. The error $y_a - S_a y$, where y is the continuum Duhamel term, solves the lattice heat equation with zero initial value and with source equal to this source error plus the heat residual $(\Delta_a S_a - S_a \Delta)y$ of the smooth continuum output. By heat contraction, also for the equation after commuting one difference, the value and the first differences of y_a converge, uniformly down to time zero. The bounds (28.1) and (28.2) then imply

the convergence of the four embedded components. The same bounds on sources and derivatives, multiplied by the weights in the norms, bound $\|(x_a, y_a)\|_{\mathbb{F},a}$ uniformly in a .

For the Wilson flow we use the consistency of the flow, with all its nonlinear terms, for smooth initial data, namely Lemma 25.2 and Theorem 25.4; no quadratic truncation of the equation is used. The family $\mathcal{F}$ is compact in every fixed smooth norm. Theorem 24.10 gives a finite reference family of local smooth continuum solutions that covers the flows started from $\mathcal{F}$ on a fixed time horizon. On each of these solutions, Lemma 25.5 compares the lattice flow and the sampled smooth flow in values and first physical differences, in the same gauge frame. The consistency of smooth samples gives a residual of order a for the potential, or a^2 for the link variables, including each of the finitely many sampled gauge transformations between the local solutions. We choose this finite family and the corresponding thresholds at the fixed h , before letting $a \rightarrow 0$. Thus the lattice flow is uniformly close, on all local solutions, to the sampled smooth flow. Since transports are unitary, the error of the transport along a closed lattice path telescopes into a sum of edge errors; a path of physical length at most l has at most l/a edges, so the error in its character is $O(la)$.

It remains to compare the lattice paths with the continuum loops for the smooth continuum flow; for this only bounds on the smooth curvature are needed. Differentiating parallel transport along a smooth piece of a based homotopy $H(u, v)$ gives an integral of $F_A(H)(\partial_u H, \partial_v H)$, conjugated by unitary transports. Its norm is at most $C\|F_A\|_\infty$ times the swept area of that piece, counted with multiplicity. Subdivision and smooth approximation in C^1 give the same bound for piecewise C^1 homotopies. The terms coming from moving base points are removed by the transport along the connecting paths. The transport along each edge of a lattice path, with the frozen sampled link variable, differs from the continuum parallel transport along that edge by $O(a^2)$, by the transport equation and a C^1 bound. Writing A_a for the swept area of the homotopy at mesh a , counted with multiplicity, the total error in each character is therefore at most $C_{h,M_h}(a + A_a)$, uniformly in $t \in [\tau, T']$ and in $f \in \mathcal{F}$. Only characters are used, so conjugations at the base point cancel. Continuity with respect to ϑ follows, on the same finite family of local solutions, from the smooth dependence of the flow in Theorem 24.10 and the continuous dependence of the transport equation on the connection. $\square$

28.3 The four-term decomposition and the ordered bilinear terms

Since $\mathcal{Q}_a$ is bilinear and $y_j = \mathcal{Q}_a(x_{j-1} + d_j, x_{j-1} + d_j)$, we have

$$y_j - y_{j-1} = \mathcal{B}_a(x_{j-1}, d_j) + \mathcal{Q}_a(d_j, d_j), \quad 1 \leq j \leq K. \quad (28.6)$$

Summing over j expresses the truncation error of the quadratic Picard term through the cross terms $\mathcal{B}_a(x_{j-1}, d_j)$ and the quadratic increments $\mathcal{Q}_a(d_j, d_j)$; both sums are taken on

the same fine mesh and with the same ordered operator $\mathcal{Q}_a$. The next lemma splits d_j into four terms and bounds the expected norm of each of the sixteen ordered products $\mathcal{Q}_a(v_j, w_j)$. The centered linear fluctuation term s_j is linear in the centered new fluctuations $w_j - \bar{w}_j$. The centered nonlinear heat term z_j is the heat evolution of the rest of the conditionally centered part of A_j , which is nonlinear in w_j . The interpolation defect r_j is the difference between the linearized reconstruction of the coarser field and the cardinal interpolation of that field; it is a deterministic function of $\hat{\theta}_{j-1}$. The predictable term u_j is the nonlinear part of the zero-fluctuation reconstruction plus the conditional mean correction. The cross terms are treated in Section 28.4.

Lemma 28.5. *Assume the standing assumptions of Section 28.1. For $1 \leq j \leq K$ define*

$$\hat{P}_j(v) = \hat{\Phi}_j(v, 0), \quad \hat{L}_j = D\hat{P}_j(0), \quad E_{j,v} = g_j^{-1} D_w \hat{\Phi}_j(v, g_j, w)|_{w=0},$$

and the following four heat paths on the mesh a :

$$\begin{aligned} s_j &= \frac{g_j}{a_j} P^a I_{a_j \rightarrow a} E_{j, \hat{\theta}_{j-1}}(w_j - \bar{w}_j), \\ z_j &= P^a I_{a_j \rightarrow a} (A_j - \mathbb{E}[A_j \mid \mathcal{F}_{j-1}]) - s_j, \\ r_j &= P^a \left(I_{a_j \rightarrow a} \frac{\hat{L}_j \hat{\theta}_{j-1}}{a_j} - I_{a_{j-1} \rightarrow a} \frac{\hat{\theta}_{j-1}}{a_{j-1}} \right), \\ u_j &= P^a I_{a_j \rightarrow a} U_j^0, \quad U_j^0 = \mathbb{E}[A_j \mid \mathcal{F}_{j-1}] - \hat{L}_j \hat{\theta}_{j-1} / a_j. \end{aligned}$$

Then the following statements hold.

1. *We have $d_j = s_j + z_j + r_j + u_j$. The paths s_j and z_j have zero conditional mean given $\mathcal{F}_{j-1}$, and r_j and u_j are $\mathcal{F}_{j-1}$ -measurable. The difference defining r_j involves the two input meshes a_j and a_{j-1} .*
2. *For any two heat paths $v = P^a v_0$ and $w = P^a w_0$ on the mesh a , the two orders satisfy separately, without cancellation,*

$$Y_\alpha^a(\mathcal{Q}_a(v, w)) + Y_\alpha^a(\mathcal{Q}_a(w, v)) \leq C_{L, T, \alpha, \sigma, J, \Theta} X_\alpha^a(v) H_\sigma^a(w_0).$$

The constant does not depend on γ , n , or K .

3. *The reconstructed fields satisfy deterministic bounds $\|\hat{\theta}_j\|_\infty \leq \bar{b}_j$, where $\bar{b}_0 = r$ and*

$$\bar{b}_j \leq C [r a_{\text{cap}}^j + g_3 \sqrt{a_j} (p_j + 1)], \quad a_{\text{cap}} \sqrt{B} < 1 \quad (j \geq 1).$$

For every fixed $p \geq 1$ define

$$\begin{aligned} S_{p,j} &= C_p g_3 a_j^\beta [\sqrt{\log(2 + L/a_j)} + \sqrt{p}], \\ Z_{p,j}^- &= C_p [g_3^2 a_j^\sigma (1 + j) + g_3^3 a_j^{1/2} p_j^3], \\ Z_{p,j}^+ &= C_p [g_3^2 a_j^\alpha (1 + j) + g_3^3 a_j^{1/2} p_j^3], \\ R_j &= C \bar{b}_{j-1} a_{j-1}^{\alpha-1}, \quad U_{p,j} = C_{p,\tau} (1 + j)^b a_j^\tau. \end{aligned}$$

Then $\|X_\alpha^a(s_j)\|_p \leq S_{p,j}$, $\|H_\sigma^a(z_j(0))\|_p \leq Z_{p,j}^-$, $\|X_\alpha^a(z_j)\|_p \leq Z_{p,j}^+$, $X_\alpha^a(r_j) \leq R_j$ pathwise, and $\|H_\sigma^a(I_{a_j \rightarrow a} U_j^0)\|_p \leq U_{p,j}$ and $\|X_\alpha^a(u_j)\|_p \leq U_{p,j}$. The first three bounds also hold conditionally on $\mathcal{F}_{j-1}$, for every value of the coarser field. Further, $\mathcal{U}_j = H_\sigma^a(I_{a_j \rightarrow a} U_j^0)$ satisfies $\mathcal{U}_j \leq C(1+j)^b$ pathwise.

4. There is a nonnegative deterministic sequence (W_j) with $\mathbb{E}Y_\alpha^a(\mathcal{Q}_a(r_j, r_j)) \leq W_j$ and such that $W_j \leq C_\zeta(1+j)^b a_j^\zeta$ for every $0 < \zeta < \min\{2\alpha - 1, \lambda_{\text{anc}}\}$.
5. Put $F_j = g_3^2[a_j^{2\alpha-1}(1+j) + \bar{b}_{j-1}]$. For $v, w \in \{s, z, r, u\}$, the expectation $\mathbb{E}Y_\alpha^a(\mathcal{Q}_a(v_j, w_j))$ is bounded by a constant times the entry in row v and column w of the following table:

	s	z	r	u
s	F_j	$S_{2,j}Z_{2,j}^-$	$S_{1,j}R_j$	$S_{1,j}U_{1,j}$
z	$S_{2,j}Z_{2,j}^-$	$Z_{2,j}^+Z_{2,j}^-$	$R_jZ_{1,j}^-$	$U_{1,j}Z_{1,j}^-$
r	$R_jS_{1,j}$	$R_jZ_{1,j}^-$	W_j	$R_jU_{1,j}$
u	$U_{1,j}S_{1,j}$	$U_{1,j}Z_{1,j}^-$	$R_jU_{1,j}$	$(1+j)^b U_{1,j}$

6. For every $0 < \zeta_2 < \min\{2\alpha - 1, \tau\}$,

$$\sup_{K \geq j} \sum_{v, w \in \{s, z, r, u\}} \mathbb{E}_{v_{N,n,\gamma}} Y_\alpha^a(\mathcal{Q}_a(v_j, w_j)) \leq C_{\zeta_2}(1+j)^b a_j^{\zeta_2}.$$

The constants in (3)–(6) have the dependence stated in Section 28.1; the same holds in the remaining results of this section.

The lemma uses the expansion at $w = 0$. By Remark 26.3, s_j and $z_j = P^a I_{a_j \rightarrow a} Z_j^{(0)}$ are the terms $s_j^{(0)}$ and $z_j^{(0)}$ of Lemma 26.2, while $s_j + z_j$, r_j , and u_j are the same as in the expansion at the conditional mean. In the notation of that lemma, $\hat{P}_j(\theta)/a_j = C_j(\theta)$ is the zero-fluctuation reconstruction in physical units; the linearization $\hat{L}_j$ of $\hat{P}_j$ acts on link logarithms, and $P_j = \hat{B}\hat{L}_j$ acts on lattice potentials, so that the linear part of C_j is $a_j^{-1}\hat{L}_j\theta = a_{j-1}^{-1}P_j\theta$.

Three groups of entries in the table require different inputs. The entry F_j for $\mathcal{Q}_a(s_j, s_j)$, which includes the conditional mean of this product at a nonzero coarser field, and the two entries $S_{1,j}R_j$ for $\mathcal{Q}_a(s_j, r_j)$ and $\mathcal{Q}_a(r_j, s_j)$ rely on the conditional concentration estimates of Section 26. The entry W_j for $\mathcal{Q}_a(r_j, r_j)$ is a separate estimate from Theorem 27.6. The remaining twelve entries follow from the product inequality (2) and the bounds in (3), combined either with the Cauchy–Schwarz inequality or with conditioning on $\mathcal{F}_{j-1}$. All entries decay like positive powers of a_j ; by (28.12) below, $\bar{b}_{j-1} \leq C(1+j)^b a_j^{1/2}$, R_j and $S_{p,j}$ are $O((1+j)^b a_j^\beta)$, and $2\beta = 2\alpha - 1 > 0$.

Proof of Lemma 28.5. We first prove the decomposition and the bounds on the four factors, then the product inequality, then the table, and finally the summed bound (6).

Step 1: the decomposition. The arrays A_j are bounded at every finite K by the amplitude bounds in (3), so conditional expectation commutes with the fixed linear heat and cardinal maps. By the definition of z_j , we have

$$s_j + z_j = P^a I_{a_j \rightarrow a}(A_j - \mathbb{E}[A_j \mid \mathcal{F}_{j-1}]).$$

Adding u_j replaces the conditional expectation by $\widehat{L}_j \widehat{\theta}_{j-1}/a_j$, and adding r_j cancels this linearized reconstruction and subtracts $P^a I_{a_{j-1} \rightarrow a} A_{j-1} = x_{j-1}$. This proves $d_j = s_j + z_j + r_j + u_j$. The vector $w_j - \bar{w}_j$ has conditional mean zero under the conditional fluctuation kernel, and the derivative $E_{j, \widehat{\theta}_{j-1}}$ and the conditional mean of A_j are $\mathcal{F}_{j-1}$ -measurable. This gives the four assertions on conditional means and measurability.

Step 2: the linear fluctuation term and the centered nonlinear term. We use the convention of Theorems 26.10 to 26.12 in which the reconstruction is Taylor expanded at $w = 0$. The input mesh, coarser mesh, and common fine mesh of those theorems are a_j , $a_{j-1} = Ba_j$, and a_K , and the refinement ratios are the integers B and B^{K-j} . The classical depth is $J(K-j)$, and the coarser dimensionless field is $\widehat{\theta}_{j-1}$. The parameter choice of Theorem 23.2 gives $t_1 \leq g_*$, where g_* is the bound on the coupling under which the conditional fluctuation kernel is controlled (Theorem 13.8). Hence $t_j = t_1 B^{-(j-1)/2} \leq g_*$ for every $j \geq 1$. The coarser fields that occur lie in the inner domain of the small-field event, and their reconstructions are evaluated on the fixed larger analytic domain. All conditional estimates below use the conditional fluctuation kernel on its fluctuation cutoff domain and its mean $\bar{w}_j$. The larger domain supplies bounds on coefficients only; no conditional law on a larger domain of coarser fields is used. Thus the terms of those theorems in the convention at $w = 0$ are precisely s_j and z_j . By the conditional estimates of the first two theorems, we obtain $S_{p,j}$, $Z_{p,j}^-$, and $Z_{p,j}^+$, and taking expectations of their conditional p th moments gives the same unconditional bounds. The amplitude bounds in Theorem 26.12 give the displayed $\bar{b}_j$, where r is the radius of B_r .

Step 3: the interpolation defect. Let $H_j = a_{j-1}$ and $d_{H_j}(t) = H_j(t + H_j^2)^{-1/2}$. In the notation of Lemma 27.1, the linearized reconstruction in physical units is $B\widehat{L}_j$, since

$$\widehat{L}_j \widehat{\theta}_{j-1}/a_j = (B\widehat{L}_j) \widehat{\theta}_{j-1}/H_j.$$

The difference of the two interpolations in that lemma is therefore the one in r_j . The bounds of that lemma on the rows of the heat-evolved difference and of its first differences imply

$$\|r_j(t)\|_\infty \leq C\bar{b}_{j-1}H_j^{-1}d_{H_j}(t), \quad \|D_a r_j(t)\|_\infty \leq C\bar{b}_{j-1}H_j^{-1}(t + H_j^2)^{-1/2}d_{H_j}(t).$$

Since $a \leq H_j$ and $\alpha < 1$, each of the two weighted terms in $X_\alpha^a(r_j)$ is at most $C\bar{b}_{j-1}H_j^{\alpha-1}$: we replace $t + a^2$ by $t + H_j^2$ in the weights, and the remaining power $(t + H_j^2)^{(\alpha-1)/2}$ is largest at $t = 0$. This proves the pathwise bound R_j .

Step 4: the predictable term. The field U_j^0 is the nonlinear part of the zero-fluctuation reconstruction, in physical units, plus the conditional mean correction under the conditional fluctuation kernel. It is the field of Theorem 27.7. That theorem bounds the moments of $H_\sigma^a(I_{a_j \rightarrow a} U_j^0)$, including the first differences at times $t \leq a^2$, and of $X_\alpha^a(u_j)$, and it gives the pathwise polynomial bound. Its moment bound has $H_j^\tau = a_{j-1}^\tau$ in place of a_j^τ ; the fixed factor B^τ is absorbed into $C_{p,\tau}$. This gives $U_{p,j}$ and the pathwise bound on $\mathcal{U}_j$.

Step 5: the product inequality. Statement (2) is Lemma 24.6, with the two orders kept separately. Its hypotheses hold, since the side $N = nB^K \geq 4$ is an integer, $a = L/N$, $1/2 < \alpha < 2/3$, and $0 < \sigma < 1 - \alpha$. For $a \leq 1$ its heat-kernel proof applies. For $1 < a \leq L/4$ we use its clause for coarse meshes, which we check directly. Lemma 24.2 gives the finite stencil of shifted brackets on every periodic lattice of side $N \geq 4$. Its bilinear polynomial defines $\mathcal{N}_a$ on arbitrary arrays; this does not require the smallness of link logarithms needed for the nonlinear equation. A physical first difference has supremum operator norm at most $2/a$, translations are isometries, and $\mathcal{N}_a$ is a fixed finite sum of brackets with one difference. Together with heat contraction, these facts imply that

$$\|\mathcal{Q}_a(v, w)(t)\|_\infty \leq Cta^{-1}\|v_0\|_\infty\|w_0\|_\infty, \quad \|D_a\mathcal{Q}_a(v, w)(t)\|_\infty \leq Cta^{-2}\|v_0\|_\infty\|w_0\|_\infty;$$

for the second bound we commute the output difference with the outer heat semigroup and use the bound $2/a$ on the source. The reversed order has the same bound. Evaluating the input norms at $t = 0$, we find $\|v_0\|_\infty \leq a^{-\alpha}X_\alpha^a(v)$ and $\|w_0\|_\infty \leq a^{-\sigma}H_\sigma^a(w_0)$. Since $a > 1$, the sum of the two output norms is at most

$$CT[(T + (L/4)^2)^\beta + (T + (L/4)^2)^\alpha]X_\alpha^a(v)H_\sigma^a(w_0). \quad (28.7)$$

If $L \leq 4$ this case does not occur. For $a \leq 1$ the factor $(T + a^2)^{(\alpha-\sigma)/2}$ in Lemma 24.6 is bounded by a constant depending only on L , T , α , and σ , since $\alpha > \sigma$ and $a \leq L/4$. Taking the larger of the two constants proves (2) for every mesh that occurs, independently of the coupling, the coarse side, and K . In particular, if $x = P^a f$ for an array f on the mesh a , then x and u_j are heat paths on the same mesh, and (2) applies to $\mathcal{Q}_a(x, u_j)$ and $\mathcal{Q}_a(u_j, x)$ with the norm $H_\sigma^a(I_{a_j \rightarrow a} U_j^0)$ bounded in Theorem 27.7.

Step 6: the sequence W_j . Let W_j be the nonnegative deterministic bound of Theorem 27.6, with $\rho_0 = r$ and with the amplitude bounds of that theorem replaced by $\bar{b}_i$. After enlarging its constant to account for this choice, both conclusions of that theorem remain valid:

$$\mathbb{E}Y_\alpha^a(\mathcal{Q}_a(r_j, r_j)) \leq W_j, \quad \mathbb{E}Y_\alpha^a(\mathcal{B}_a(x_{j-1}, r_j)) \leq 4W_j. \quad (28.8)$$

The tail estimate of that theorem, for any $\zeta < \min(2\alpha - 1, \lambda_{\text{anc}})$, implies that $W_j \leq \sum_{i \geq j} W_i \leq C_\zeta a_j^\zeta$, which is stronger than (4). Both quantities in (28.8) include all contributions of the coarse field and all propagated conditional means.

Step 7: all products except those of s_j and r_j . The bound on $\mathcal{Q}_a(s_j, s_j)$, including its conditional mean at a nonzero coarser field, is Theorem 26.12; the expectation it gives is at most CF_j .

For $\mathcal{Q}_a(s_j, z_j)$ and $\mathcal{Q}_a(z_j, s_j)$ we put s_j in the argument measured by X_α^a and $z_j(0)$ in the argument measured by H_σ^a , in (2). By the Cauchy–Schwarz inequality under the same measure, we have

$$\mathbb{E}[X_\alpha^a(s_j)H_\sigma^a(z_j(0))] \leq S_{2,j}Z_{2,j}^-.$$

For $\mathcal{Q}_a(z_j, z_j)$ the same argument with z_j in the X_α^a argument gives $Z_{2,j}^+Z_{2,j}^-$.

For $\mathcal{Q}_a(z_j, r_j)$ and $\mathcal{Q}_a(r_j, z_j)$ we use the pathwise bound R_j on $X_\alpha^a(r_j)$ and the bound on $H_\sigma^a(z_j(0))$; each expectation is at most $CR_jZ_{1,j}^-$, in either order, without any symmetry of $\mathcal{Q}_a$.

For $\mathcal{Q}_a(s_j, u_j)$ and $\mathcal{Q}_a(u_j, s_j)$, statement (2) gives the bound $CX_\alpha^a(s_j)\mathcal{U}_j$. Here $\mathcal{U}_j$ is $\mathcal{F}_{j-1}$ -measurable, and the conditional first moment of $X_\alpha^a(s_j)$ is at most $S_{1,j}$ for every value of the coarser field. Taking first the conditional expectation and then the expectation, we obtain $CS_{1,j}U_{1,j}$. For $\mathcal{Q}_a(z_j, u_j)$ and $\mathcal{Q}_a(u_j, z_j)$ we put u_j in the X_α^a argument and $z_j(0)$ in the H_σ^a argument; conditioning on $\mathcal{F}_{j-1}$ and using the conditional bound $Z_{1,j}^-$ gives $CU_{1,j}Z_{1,j}^-$. In these four products one factor is $\mathcal{F}_{j-1}$ -measurable and the other has a conditional moment bound that is uniform in the coarser field; this is why their bounds are products of first moments.

For $\mathcal{Q}_a(r_j, u_j)$ and $\mathcal{Q}_a(u_j, r_j)$, statement (2) gives $CR_j\mathcal{U}_j$, hence $CR_jU_{1,j}$ after taking expectations. For $\mathcal{Q}_a(u_j, u_j)$, the splitting of the heat semigroup in Theorem 27.7 gives $X_\alpha^a(u_j) \leq C\mathcal{U}_j$, so the product is at most $C\mathcal{U}_j^2$. By the pathwise bound on $\mathcal{U}_j$, we have

$$\mathbb{E}\mathcal{U}_j^2 \leq C(1+j)^b \mathbb{E}\mathcal{U}_j \leq C(1+j)^b U_{1,j}.$$

Finally, $\mathcal{Q}_a(r_j, r_j)$ is bounded by the first estimate in (28.8).

Step 8: the products of s_j and r_j . For $\mathcal{Q}_a(s_j, r_j)$ and $\mathcal{Q}_a(r_j, s_j)$ we use the clause of Theorem 26.10 on predictable heat backgrounds on the common fine mesh. To check its hypotheses, let $h = a_j$, $H_j = a_{j-1} = Bh$, $\delta = a_K$, $\theta = \hat{\theta}_{j-1}$, and $P_j = B\hat{L}_j$, and let E_{H_j} be the identity map of arrays on the lattice of mesh H_j . Then the background in that clause is exactly

$$r_j(t) = H_j^{-1}P_t^\delta(I_{h \rightarrow \delta}P_j - I_{H_j \rightarrow \delta}E_{H_j})\theta, \quad R_j^\sharp = C_J\|\theta\|_\infty H_j^{\alpha-1}. \quad (28.9)$$

It is a predictable heat path on the whole torus, and both input meshes are kept. That clause gives, for each integer $0 \leq q \leq 4$ and every choice of differences and shifts,

$$\|D_\delta^q r_j(t)\|_\infty \leq C_q R_j^\sharp (t + h^2)^{-(\alpha+q)/2}, \quad (28.10)$$

which are the hypotheses on higher differences in its class of backgrounds, including times $t < h^2$. The finitely many constants C_q depend on the ratio B , the cardinal basis,

the stencil, L , T , and the exponents, but not on the coarser field or on δ/h . Since r_j is a heat path, $\partial_t D_\delta^q r_j = \Delta_\delta D_\delta^q r_j$; the time increments of its values and of its first and second differences involve differences of order at most four, which are controlled by (28.10). The same clause gives a bounded diameter for the normalized coefficient metric and the covering bound $N(v) \leq C(1 + L/h)^P v^{-Q}$, $0 < v \leq 1$, independently of $\delta/h = B^{-(K-j)}$. Thus the conditional maximum is taken over a cover at the input mesh h .

In the expansion at $w = 0$ used in that theorem, the linear coefficient is $E_{j,\theta}$ and the centered vector is $w_j - \bar{w}_j$, so the theorem concerns the same s_j and the same conditional fluctuation kernel as here. For each order separately let V_j be the norm Y_α^δ of the product. For each value of the coarser field, the theorem implies that

$$\mathbb{P}\left\{V_j > Cg_3 h^\beta R_j^\sharp \sqrt{\log(2 + L/h) + u} \mid \mathcal{F}_{j-1}\right\} \leq Ce^{-u}, \quad u \geq 0.$$

If $R_j^\sharp = 0$, then $r_j = 0$ and the product vanishes. Otherwise we integrate this tail. With $\Lambda_h = \log(2 + L/h)$ and $A_j^\sharp = Cg_3 h^\beta R_j^\sharp$, the layer-cake formula and the substitution $v = A_j^\sharp \sqrt{\Lambda_h + u}$ give

$$\mathbb{E}[V_j \mid \mathcal{F}_{j-1}] \leq A_j^\sharp \sqrt{\Lambda_h} + \frac{CA_j^\sharp}{2} \int_0^\infty \frac{e^{-u}}{\sqrt{\Lambda_h + u}} du \leq Cg_3 h^\beta (\sqrt{\Lambda_h} + 1) R_j^\sharp,$$

where we used $\Lambda_h \geq \log 2$. The amplitude bound gives $R_j^\sharp \leq C_J \bar{b}_{j-1} H_j^{\alpha-1} \leq CR_j$. By the tower property we obtain, separately for the two orders,

$$\mathbb{E}Y_\alpha^a(Q_a(s_j, r_j)) \leq CS_{1,j} R_j, \quad \mathbb{E}Y_\alpha^a(Q_a(r_j, s_j)) \leq CS_{1,j} R_j. \quad (28.11)$$

No cancellation between the two orders is used. This completes the proof of (5).

Step 9: summation. Let $\kappa_{\text{cap}} = -\log a_{\text{cap}} / \log B$, so that $\kappa_{\text{cap}} > 1/2$. For $j \geq 2$, by the amplitude bounds, (28.5), and (28.4), we have

$$\begin{aligned} \bar{b}_{j-1} &\leq C(1+j)^b a_j^{1/2}, \\ R_j &\leq C(1+j)^b a_j^\beta, \quad S_{p,j} \leq C_p(1+j)^b a_j^\beta, \\ Z_{p,j}^- &\leq C_p(1+j)^b a_j^\sigma, \quad Z_{p,j}^+ \leq C_p(1+j)^b a_j^{1/2}, \quad U_{p,j} \leq C_{p,\tau}(1+j)^b a_j^\tau. \end{aligned} \quad (28.12)$$

In the first line, the contribution of the coarse field is $ra_{\text{cap}}^{j-1} = r(a_{j-1}/H)^{\kappa_{\text{cap}}}$, whose exponent is strictly larger than $1/2$. The first level $j = 1$ is absorbed into a constant. In

the bounds on $Z_{p,j}^\pm$ we used $\sigma < 1 - \alpha < 1/2$ and $\alpha > 1/2$. By (28.12), the slowest powers of a_j in the sixteen entries of the table are

	s	z	r	u
s	2β	$\beta + \sigma$	2β	$\beta + \tau$
z	$\beta + \sigma$	$1/2 + \sigma$	$\beta + \sigma$	$\tau + \sigma$
r	2β	$\beta + \sigma$	ζ	$\beta + \tau$
u	$\beta + \tau$	$\tau + \sigma$	$\beta + \tau$	τ

where we choose ζ strictly between ζ_2 and $\min(2\beta, \lambda_{\text{anc}})$; such a choice exists since $\zeta_2 < 2\beta$ and $\zeta_2 < \tau < \lambda_{\text{anc}}$. The term of F_j that contains the conditional mean has power $1/2 > 2\beta$. Every displayed power exceeds ζ_2 . Absorbing the fixed powers of H into the constant, we obtain (6). $\square$

28.4 Truncation estimates

We now bound the truncation error $(x_K - x_{K-1}, y_K - y_{K-1})$. Its first component is the sum of the increments d_j , $k \leq j \leq K$, which we control by the factor bounds of Lemma 28.5. By (28.6), its second component is the sum of the cross terms $\mathcal{M}_j = \mathcal{B}_a(x_{j-1}, d_j)$ and of the quadratic increments $\mathcal{Q}_a(d_j, d_j)$. The quadratic increments are controlled by Lemma 28.5(6). For the cross terms we use the splitting $\mathcal{M}_j = \mathcal{M}_j^s + \mathcal{M}_j^z + \mathcal{M}_j^r + \mathcal{M}_j^u$ with $\mathcal{M}_j^v = \mathcal{B}_a(x_{j-1}, v_j)$. The terms $\mathcal{M}_j^s$ and $\mathcal{M}_j^z$ are conditionally centered fields with the predictable background x_{j-1} ; we bound them by the conditional estimates of Section 26, in terms of a radius of the background at the input scale a_j . The bound on $\mathcal{M}_j^r$ is part of (28.8), and $\mathcal{M}_j^u$ is bounded by the product inequality and the Cauchy–Schwarz inequality. In every case we bound the first moments of the norms and sum them over j ; the norms themselves are not centered.

Theorem 28.6. *Assume the standing assumptions, and use the notation of Lemma 28.5. For every fixed $p \geq 1$ and every $0 < \zeta_1 < \min\{\beta, \tau\}$,*

$$\sup_{K \geq m \geq l \geq k} \left\| X_\alpha^{a_K} \left(\sum_{j=l}^m d_j \right) \right\|_p \leq C_{p, \zeta_1} a_k^{\zeta_1}, \quad \sup_{K \geq 0} \sup_{0 \leq j \leq K} \|X_\alpha^{a_K}(x_j)\|_p \leq C_p.$$

For $1 \leq j \leq K$ the radius of the background x_{j-1} at the input scale,

$$\mathcal{R}_{j-1} = \sup_{0 \leq t \leq T} \left\{ (t + a_j^2)^{\alpha/2} \|x_{j-1}(t)\|_\infty + (t + a_j^2)^{(\alpha+1)/2} \|D_{a_K} x_{j-1}(t)\|_\infty \right\},$$

satisfies $\sup_{K,j} \|\mathcal{R}_{j-1}\|_p \leq C_p$. With $\mathcal{M}_j^v = \mathcal{B}_{a_K}(x_{j-1}, v_j)$ we have

$$\begin{aligned} \sup_{K \geq m \geq l \geq k} \mathbb{E} Y_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{M}_j^s \right) &\leq C g_3 \sum_{j \geq k} a_j^\beta \sqrt{1+j}, \\ \sup_{K \geq m \geq l \geq k} \mathbb{E} Y_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{M}_j^z \right) &\leq C \sum_{j \geq k} [g_3^2 a_j^\alpha (1+j) + g_3^3 a_j^{1/2} p_j^3], \\ \sup_{K \geq m \geq l \geq k} \mathbb{E} Y_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{M}_j^r \right) &\leq 4 \sum_{j \geq k} W_j. \end{aligned}$$

All three right sides tend to zero as $k \rightarrow \infty$. The fields $\mathcal{M}_j^s$ and $\mathcal{M}_j^z$ have zero conditional mean given $\mathcal{F}_{j-1}$, while $\mathcal{M}_j^r$ includes all propagated conditional means. These statements include the level $j = K$, for which $I_{a_K \rightarrow a_K}$ is the identity.

Proof. Let $a = a_K$. By the triangle inequality for the path norm and Minkowski's inequality,

$$\|X_\alpha^a(d_j)\|_p \leq S_{p,j} + Z_{p,j}^+ + R_j + U_{p,j}.$$

By (28.12) and (28.4), the sum of the right side over $j \geq k$ is at most $C_{p,\zeta_1} a_k^{\zeta_1}$ for every $\zeta_1 < \min(\beta, \tau)$. This proves the first bound, uniformly over all partial sums and all K .

On the coarse lattice, of mesh $H = L/n$, we have $\|A_0\|_\infty \leq r/H$. By (28.3) and the bounded horizon,

$$X_\alpha^a(x_0) \leq (T + H^2)^{\alpha/2} r/H + C(T + H^2)^{(\alpha+1)/2} r/H^2.$$

This bound is deterministic and independent of K . Adding the sum of the increments from level 1 to level j proves the uniform moment bound on x_j , including the case $K = j = 0$.

For the radius $\mathcal{R}_{j-1}$, let $h = a_j$ and $H_j = Bh$. For $t \geq h^2$ we have $t + h^2 \leq 2(t + a^2)$, so the two weighted terms in $\mathcal{R}_{j-1}$ are bounded by a fixed multiple of $X_\alpha^a(x_{j-1})$. For $t < h^2$, (28.3) applied to A_{j-1} , an array on the mesh H_j with $\|A_{j-1}\|_\infty \leq \bar{b}_{j-1}/H_j$, bounds the sum of the two terms by

$$C(h^\alpha + h^{\alpha+1}/H_j) \bar{b}_{j-1}/H_j \leq C_B \bar{b}_{j-1} H_j^{\alpha-1}.$$

This bound is deterministic, bounded uniformly in j , and decays at fine levels by (28.12); at the first level it uses only $\bar{b}_0 = r$ and the fixed mesh H . Taking p th moments proves the bound on $\mathcal{R}_{j-1}$. At short times the weight is the one at the input scale a_j , not the one at the output mesh a .

For $\mathcal{M}_j^s$ note that the background x_{j-1} is the heat path of a cardinal interpolation from the coarser mesh, so it satisfies the hypothesis on backgrounds of Theorem 26.10. The coarser mesh is $a_{j-1} = Ba_j$, the common fine mesh is a_K , an integer refinement of both

input meshes, the input is A_{j-1} , the radius of the background is $\mathcal{R}_{j-1}$, and the horizon is T . The clause of that theorem on cardinal backgrounds from the coarser mesh gives the bounds on their higher differences with fixed geometric constants; the larger class of backgrounds is needed only for (28.9). The conditional kernel and the convention of Taylor expansion at $w = 0$ are those matched in Step 2 of the proof of Lemma 28.5. For each value of the coarser field we use $\mathcal{R}_{j-1}$ as the number R in the conditional tail bound of that theorem. Integrating the tail in u , which is justified since $\int_0^\infty e^{-u}(1 + \sqrt{u}) du < \infty$, we obtain

$$\mathbb{E}[\Upsilon_\alpha^a(\mathcal{M}_j^s) \mid \mathcal{F}_{j-1}] \leq C g_3 \mathcal{R}_{j-1} a_j^\beta [\sqrt{\log(2 + L/a_j)} + 1].$$

Using the bound on the first moment of $\mathcal{R}_{j-1}$ and $\log(2 + L/a_j) \leq C(1 + j)$, we obtain the bound on each summand. The triangle inequality and expectation then sum it over $l \leq j \leq m$, and the sum is at most the sum over $j \geq k$.

The same background satisfies the hypotheses of Theorem 26.11. Integrating its conditional tail, which has a quadratic part and a remainder, we obtain

$$\mathbb{E}[\Upsilon_\alpha^a(\mathcal{M}_j^z) \mid \mathcal{F}_{j-1}] \leq C \mathcal{R}_{j-1} [g_3^2 a_j^\alpha (1 + j) + g_3^3 a_j^{1/2} p_j^3].$$

Taking expectations and using the moment bound on $\mathcal{R}_{j-1}$, we obtain the second bound. The fields $\mathcal{M}_j^s$ and $\mathcal{M}_j^z$ have zero conditional mean, since their backgrounds are predictable and $s_j(0)$ and $z_j(0)$ have zero conditional mean.

For $\mathcal{M}_j^r$ we use the second estimate in (28.8), which is a conclusion of Theorem 27.6; the triangle inequality gives $4 \sum_{j \geq k} W_j$. All three right sides tend to zero as $k \rightarrow \infty$, by (28.4), (28.5), and the summability of W_j in Theorem 27.6. At the last level $I_{a_K \rightarrow a_K}$ is the identity, and none of these estimates omits that level. $\square$

Theorem 28.7. *Assume the standing assumptions, use the notation of Lemma 28.5, and write $\mathcal{M}_j = \mathcal{B}_{a_K}(x_{j-1}, d_j)$ and $\mathcal{M}_j^v = \mathcal{B}_{a_K}(x_{j-1}, v_j)$ for $v \in \{s, z, r, u\}$. There exist $b \geq 0$ and $C < \infty$ such that, for every $k \geq 1$,*

$$\sup_{K \geq m \geq l \geq k} \mathbb{E}_{v_{nBK, n, \gamma}} \Upsilon_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{M}_j^u \right) \leq C(1 + k)^b a_k^\tau.$$

Further, both

$$\sup_{K \geq m \geq l \geq k} \mathbb{E} \Upsilon_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{M}_j \right) \quad \text{and} \quad \sup_{K \geq m \geq l \geq k} \mathbb{E} \Upsilon_\alpha^{a_K} \left(\sum_{j=l}^m \mathcal{Q}_{a_K}(d_j, d_j) \right)$$

tend to zero as $k \rightarrow \infty$. In particular, the sums $\sum_{j=l}^m \mathcal{M}_j$ and $\sum_{j=l}^m \mathcal{Q}_{a_K}(d_j, d_j)$ tend to zero in probability in the norm $\Upsilon_\alpha^{a_K}$, at every fixed threshold, uniformly over $K \geq m \geq l \geq k$. The constants have the dependence stated in Section 28.1, with the exponents added.

Proof. Let $a = a_K$, $X_j = X_\alpha^a(x_{j-1})$, and $\mathcal{U}_j = H_\sigma^a(I_{a_j \rightarrow a} U_j^0)$. The case $p = 2$ of Theorem 28.6 gives $\|X_j\|_2 \leq C_2$, uniformly in K and j . The case $p = 2$ of Theorem 27.7, applied under the same conditioned measure to the predictable term, which includes the contributions of the coarse field, the cubic part of the zero-fluctuation reconstruction, and the conditional mean correction, gives

$$\|\mathcal{U}_j\|_2 \leq C(1+j)^b a_{j-1}^\tau \leq C'(1+j)^b a_j^\tau,$$

since $a_{j-1} = Ba_j$ with B fixed. Neither input uses an estimate of Section 34.

We apply Lemma 24.6 separately to $\mathcal{Q}_a(x_{j-1}, u_j)$ and $\mathcal{Q}_a(u_j, x_{j-1})$; both arguments are heat paths on the mesh a , and the argument measured by H_σ^a has initial datum $I_{a_j \rightarrow a} U_j^0$. The factor $(T + a^2)^{(\alpha-\sigma)/2}$ in that lemma is bounded uniformly, since $a \leq L/n$. By this inequality and the Cauchy–Schwarz inequality under the same measure, we have

$$\mathbb{E} Y_\alpha^a(\mathcal{M}_j^u) \leq C \mathbb{E}(X_j \mathcal{U}_j) \leq C \|X_j\|_2 \|\mathcal{U}_j\|_2 \leq C(1+j)^b a_j^\tau.$$

By the triangle inequality for the path norm and (28.4), for every $K \geq m \geq l \geq k$,

$$\mathbb{E} Y_\alpha^a\left(\sum_{j=l}^m \mathcal{M}_j^u\right) \leq C \sum_{j \geq k} (1+j)^b a_j^\tau \leq C'(1+k)^b a_k^\tau.$$

This proves the first assertion.

By Lemma 28.5(1), $\mathcal{M}_j = \mathcal{M}_j^s + \mathcal{M}_j^z + \mathcal{M}_j^r + \mathcal{M}_j^u$. Combining the three bounds on first moments in Theorem 28.6 with the bound just proved, we obtain a bound on the first moment of the sum of the $\mathcal{M}_j$ that tends to zero as $k \rightarrow \infty$. The means of $\mathcal{M}_j^r$ are included in W_j by Theorem 27.6. At fixed coupling $p_j \leq C(1+j)$ by (28.5), so all the sums of powers converge and their tails tend to zero, and the same holds for $\sum_j W_j$ by Theorem 27.6.

For the quadratic increments, let $0 < \zeta_2 < \min(2\alpha - 1, \tau)$. By Lemma 28.5(6), at every level the sum of the expected norms of the sixteen ordered products is at most $C_{\zeta_2}(1+j)^b a_j^{\zeta_2}$. We expand $\mathcal{Q}_a(d_j, d_j)$ by bilinearity, using $d_j = s_j + z_j + r_j + u_j$, apply the triangle inequality, and sum the geometric series (28.4). This proves the last limit, with the bound $C_{\zeta_2}(1+k)^b a_k^{\zeta_2}$. Markov's inequality at a fixed positive threshold implies the two statements in probability. All these operations take place under the same conditioned measure and on the same fine mesh, including the level $j = K$. $\square$

The next theorem combines the two previous results. Here the finite-scale reconstruction A_j is extended to the fine mesh by cardinal interpolation, not by further reconstruction with zero fluctuations, and the quadratic operator is the same ordered $\mathcal{Q}_{a_K}$ for every j .

Theorem 28.8. Assume the standing assumptions, with the norms taken on a fixed horizon $[0, T_0]$, $0 < T_0 < \infty$, and let $\nu_K = \nu_{nBK, n, \gamma}$. Use the joint realization of Remark 15.11, with fine-configuration marginal ν_K . Let the arrays A_j , $0 \leq j \leq K$, be those of its coarse-to-fine reconstruction, at the classical depths $J(K - j)$, and put

$$\mathcal{E}_{K,j} = (x_j, y_j), \quad x_j = P^{a_K} I_{a_j \rightarrow a_K} A_j, \quad y_j = \mathcal{Q}_{a_K}(x_j, x_j),$$

$d_j = x_j - x_{j-1}$ for $1 \leq j \leq K$, and $R_K = \|\mathcal{E}_{K,K}\|_{\mathfrak{fl}, a_K} = X_{\alpha}^{a_K}(x_K) + Y_{\alpha}^{a_K}(y_K)$, as in Definition 24.3. Then, for $1 \leq k \leq K$,

$$\mathcal{E}_{K,K} - \mathcal{E}_{K,k-1} = \left(\sum_{j=k}^K d_j, \sum_{j=k}^K \{ \mathcal{B}_{a_K}(x_{j-1}, d_j) + \mathcal{Q}_{a_K}(d_j, d_j) \} \right). \quad (28.13)$$

For every $\eta > 0$,

$$\limsup_{k \rightarrow \infty} \sup_{K \geq k} \nu_K \{ X_{\alpha}^{a_K}(x_K - x_{k-1}) + Y_{\alpha}^{a_K}(y_K - y_{k-1}) > \eta \} = 0, \quad (28.14)$$

and $\rho(R) = \sup_{K \geq 0} \nu_K \{ R_K > R \}$ tends to zero as $R \rightarrow \infty$. All norms include the physical first differences and shifts. For each fixed q , the laws of the arrays A_q converge, on a fixed compact set, to the law π_q (Theorem 15.14), and their smooth cardinal interpolants satisfy the uniform consistency and the deterministic norm bound of Lemma 28.4. In this joint realization, the configuration $e^{a_K A_K}$ on Λ_{nBK} is gauge equivalent to U on every sample. The constants and the rates of convergence in these statements are taken at fixed coarse side and fixed coupling, uniformly in K .

Proof. Summing (28.6) over $k \leq j \leq K$ gives (28.13). We apply Theorems 28.6 and 28.7 with $T = T_0$. The first component in (28.14) tends to zero by Markov's inequality and Theorem 28.6. For the second component we apply Theorem 28.7 separately to the two sums in (28.13), with threshold $\eta/3$ for each of the three contributions, and use a union bound. This proves (28.14).

For the tightness, let $\varepsilon > 0$ and choose k such that the probability in (28.14) with $\eta = 1$ is less than ε for all $K \geq k$. The array A_{k-1} lies in a fixed compact set, and $x_{k-1} = P^{a_K}(f|_{a_K})$, where f is its cardinal interpolant. By Lemma 28.4, $\|\mathcal{E}_{K,k-1}\|_{\mathfrak{fl}, a_K} \leq C_k$ for a deterministic C_k that does not depend on K . By the triangle inequality, $R_K \leq C_k + 1$ outside the exceptional event. The finitely many $K < k$ have bounded arrays, so their linear heat evolutions and quadratic Picard terms have a common finite bound. This proves that $\rho(R) \rightarrow 0$. The fixed-scale convergence assertions are Theorem 15.14 applied to the family $\widehat{\mathcal{R}}$ at the classical depths $J(K - j)$. For the samplewise comparison, use the joint realization of Remark 15.11: take the terminal reconstructed configuration $U_K = e^{a_K A_K}$, an independent Haar fine-lattice gauge transformation h , and set $U = U_K^h$. Gauge invariance of ν_K and equality of the terminal orbit laws give $\text{Law}(U) = \nu_K$. The

block maps commute with gauge transformations, so all intermediate reconstructed fields are gauge equivalent to the corresponding block fields of this same U , through restrictions of h . Thus all joint gauge-invariant observables agree sample by sample in this realization; the reconstruction variables need not be functions of U . $\square$

28.5 Convergence under the conditioned measure

We first state an approximation lemma for laws, a version of the standard approximation theorem for weak convergence (compare [28, Theorem 3.2]). In that theorem the limit of the approximating laws is assumed to exist; here its existence is part of the conclusion, so we include the proof. For a metric space E let $\mathcal{L}$ be the set of functions $f : E \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq 1$ and Lipschitz constant at most one, and let $d_{\text{BL}}(\mu, \nu) = \sup_{f \in \mathcal{L}} |\int f d\mu - \int f d\nu|$ be the bounded-Lipschitz distance between probability measures on E .

Lemma 28.9. *Let (E, d) be a complete separable metric space. For each $K \geq 0$ let Z_K be an E -valued random variable with law μ_K , and for $0 \leq k \leq K$ let $Z_{K,k}$ be E -valued random variables defined on the same probability space as Z_K . Suppose that for each fixed k the laws of $Z_{K,k}$ converge weakly as $K \rightarrow \infty$, that the union over $K \geq k$ of the supports of these laws has compact closure, that each μ_K has compact support, and that for every $\eta > 0$*

$$\lim_{k \rightarrow \infty} \sup_{K \geq k} \mathbb{P}\{d(Z_K, Z_{K,k}) > \eta\} = 0.$$

Then μ_K converges weakly to a probability measure on E .

Proof. We show that (μ_K) is a Cauchy sequence for d_{BL} and is tight, and then identify the limit.

Step 1: the Cauchy property. For $f \in \mathcal{L}$ and $\eta > 0$ we have

$$|\mathbb{E}f(Z_K) - \mathbb{E}f(Z_{K,k})| \leq \eta + 2\mathbb{P}\{d(Z_K, Z_{K,k}) > \eta\}.$$

Fix k , and let C_k be the compact closure of the union of the supports of the laws of $Z_{K,k}$, $K \geq k$. Given $\varepsilon > 0$, let $x_1, \dots, x_N$ be a finite ε -net of C_k . We group the functions $f \in \mathcal{L}$ according to the values $f(x_1), \dots, f(x_N)$ rounded to multiples of ε . There are finitely many groups, and two functions in the same group differ by at most 3ε on C_k . Choosing one representative in each group and applying weak convergence to the finitely many representatives, we find that $\sup_{f \in \mathcal{L}} |\mathbb{E}f(Z_{K,k}) - \mathbb{E}f(Z_{K',k})| \leq 6\varepsilon + o(1)$ as $K, K' \rightarrow \infty$. Combining the two bounds and letting first $K, K' \rightarrow \infty$ with k fixed, then $k \rightarrow \infty$, and then $\eta, \varepsilon \rightarrow 0$, we obtain that (μ_K) is a Cauchy sequence for d_{BL} .

Step 2: tightness. Fix $\varepsilon > 0$, and choose increasing k_h such that $\mathbb{P}\{d(Z_K, Z_{K,k_h}) > 2^{-h}\} \leq \varepsilon 2^{-h}$ for all $K \geq k_h$. Let C_h be the compact closure of the union of the supports of the laws of Z_{K,k_h} , $K \geq k_h$, together with the supports of the finitely many μ_K with $K < k_h$. For $K \geq k_h$ we have $d(Z_K, C_h) \leq 2^{-h}$ unless $d(Z_K, Z_{K,k_h}) > 2^{-h}$, and for $K < k_h$ we have $Z_K \in C_h$ almost surely. By the union bound, the closed set

$$C_\varepsilon = \bigcap_{h \geq 1} \{z : d(z, C_h) \leq 2^{-h}\}$$

has μ_K -mass at least $1 - \varepsilon$ for every K . It is totally bounded: for a given tolerance, choose h large and a finite net of the compact set C_h . Since C_ε is closed in a complete space, it is compact. Hence (μ_K) is tight.

Step 3: the limit. By Prokhorov's theorem [138], in the direction that holds on every metric space [28, Theorem 5.1], every subsequence of (μ_K) has a further subsequence that converges weakly to a probability measure. By Step 1, $\int f d\mu_K$ converges for every $f \in \mathcal{L}$, so any two subsequential limits μ and μ' satisfy $\int f d\mu = \int f d\mu'$ for all bounded Lipschitz f . For a closed set F , the functions $\max(0, 1 - m d(\cdot, F))$ decrease to $\mathbf{1}_F$ as $m \rightarrow \infty$, so $\mu(F) = \mu'(F)$. Closed sets form a π -system generating the Borel σ -algebra, so $\mu = \mu'$. Since every subsequence has a further subsequence converging weakly to the same limit, the whole sequence converges weakly to it. $\square$

Proof of Theorem 28.1. We compare W_K with the Wilson loops of the flow started from the finite-scale reconstruction from the first q RG steps. By Theorem 28.8 and the stability of flowed Wilson loops, this approximation is uniform in K . We then identify the limit law of the approximation at fixed q and apply Lemma 28.9. We fix a horizon $T_0 > 0$ and use Theorem 28.8 with it.

Step 1: comparison with finite-scale reconstructions. On the same probability space, let $W_K^{[q]}$ be the vector of the same characters of holonomies for the Wilson flow on Λ_{nBK} started from

$$U_K^{[q]} = \exp(a_K I_{a_q \rightarrow a_K} A_q),$$

and let $D_{K,q} = \|\mathcal{E}_{K,K} - \mathcal{E}_{K,q}\|_{\mathfrak{f}, a_K}$. The Wilson flow commutes with lattice gauge transformations and characters of holonomies are gauge invariant. Since $e^{a_K A_K}$ is gauge equivalent to U (Theorem 28.8), W_K is also the vector of Wilson loops of the flow started from $e^{a_K A_K} = e^{a_K x_K(0)}$, while $U_K^{[q]} = e^{a_K x_q(0)}$. On the event $\{R_K \leq R, D_{K,q} \leq \eta \leq 1\}$ both flow data $\mathcal{E}_{K,K}$ and $\mathcal{E}_{K,q}$ have norm at most $R + 1$. By Corollary 25.11 there exist $a_* > 0$ and C_R such that $|W_K - W_K^{[q]}| \leq C_R \eta$ on this event whenever $a_K \leq a_*$. These constants depend on R , the loops, the characters, the times, and the deterministic data of

the flow, but not on q , on the higher derivatives of the finite-scale reconstruction, or on the coupling. Thus for $f \in \mathcal{L}$ the replacement error is at most

$$C_R \eta + 2\rho(R) + 2\nu_K \{D_{K,q} > \eta\}. \quad (28.15)$$

We choose R first, then η , and then q , increasing q until $a_q \leq a_*$; then $a_K \leq a_*$ for all $K \geq q$. By Theorem 28.8, the last term in (28.15) tends to zero as $q \rightarrow \infty$, uniformly in $K \geq q$. In particular, for every $\eta' > 0$, $\lim_{q \rightarrow \infty} \sup_{K \geq q} \nu_K \{|W_K - W_K^{[q]}| > \eta'\} = 0$.

Step 2: the limit at fixed q . Fix q . The vector $W_K^{[q]}$ is a function of the array A_q , which lies in a fixed compact set. By Lemma 28.4, applied with $h = a_q$, these functions converge as $K \rightarrow \infty$, uniformly on that compact set, to the continuous map G_q of the statement. By Theorem 15.14 the law of A_q converges weakly to π_q . Together, these two facts imply that $\text{Law}(W_K^{[q]}) \Rightarrow (G_q)_\# \pi_q$. All the vectors W_K and $W_K^{[q]}$ take values in the compact box $\prod_i \{w \in \mathbb{C} : |w| \leq \chi_i(I)\}$. We apply Lemma 28.9 in this box, with $Z_K = W_K$ and $Z_{K,q} = W_K^{[q]}$; its hypotheses were verified in Step 1 and above. Hence the law of W_K converges weakly. Applying the same argument to the union of two lists of loops proves the consistency under finite unions and projections. Taking $K \rightarrow \infty$ at fixed q in the bound (28.15), we find that the d_{BL} -distance between the limit law of W_K and $(G_q)_\# \pi_q$ is at most $C_R \eta + 2\rho(R) + 2 \sup_{K \geq q} \nu_K \{D_{K,q} > \eta\}$. This tends to zero in the order R, η, q chosen above, which identifies the limit law of W_K as the limit of $(G_q)_\# \pi_q$.

Step 3: moment generating functions. Let $D \geq \max_i \chi_i(I)$. On the box, $|\exp(z \cdot w)| \leq e^{D\|z\|_1}$, and on $\{\|z\|_1 \leq Z\}$ the function $z \mapsto \exp(z \cdot w)$ is Lipschitz with constant at most De^{DZ} , uniformly in w . Weak convergence of the laws of W_K implies convergence of the moment generating functions at each z . With the common Lipschitz constant and a finite net of z , we obtain uniform convergence on $\{\|z\|_1 \leq Z\}$. On the ball where $e^{D\|z\|_1} - 1 < 1/2$, we have $|\mathbb{E} \exp(z \cdot W_K) - 1| \leq e^{D\|z\|_1} - 1 < 1/2$, so the logarithm with value zero at the origin is defined and holomorphic. Cauchy's formula on a smaller ball gives the convergence of every finite logarithmic derivative, that is, of the cumulants. $\square$

28.6 Removal of the small-field conditioning

Proof of Theorem 28.2. We first bound the small-field failure probability, then compare conditioned and unconditioned expectations at the same fine lattice, and finally pass to the limit, first in K and then in r .

Step 1: the failure probability. Theorem 23.9 implies that, for every $u \in 4\mathbb{N}$ with $u \geq n_{\min}$ and every $K' \geq 0$, we have $\mu_{uBK', \gamma}(S_{uBK', u, \gamma}^c) \leq F_\gamma(u)$, uniformly in K' . For fixed $\gamma > 0$ both terms of $F_\gamma(u)$ tend to zero as $u \rightarrow \infty$. For the second term, with $v = \log u$,

the exponent is $3v - (a_w/4)(v + \log(M/\gamma))^2$, which tends to $-\infty$. Let r_0 be such that $n_r \geq n_{\min}$ and $F_\gamma(n_r) < 1$ for $r \geq r_0$. For $r \geq r_0$ and $K \geq r$ the conditioned hierarchy with coarse side n_r has $K - r$ RG steps and fine side $n_r B^{K-r} = n_* B^K = N_K$. Applying the bound with $u = n_r$ and $K' = K - r$, we obtain $e_{K|r} \leq F_\gamma(n_r)$. This bounds the probability of the complement of the event that defines $\nu_{K|r}$, including its initial test and the tests at the end of each RG step.

Step 2: conditioning at a fixed fine lattice. For $K \geq r$, the observable $W_K = (W_{K,1}, \dots, W_{K,m})$ under $\nu_{K|r}$ is the same measurable function of the configuration on Λ_{N_K} as under $\mu_{N_K, \gamma}$. Lemma 5.5, applied to the measure $\mu_{N_K, \gamma}$, the event $S_{N_K, n_r, \gamma}$, and this map, gives, for every bounded complex measurable f on the range of W_K ,

$$|\mathbb{E}_{\mu_{N_K, \gamma}} f(W_K) - \mathbb{E}_{\nu_{K|r}} f(W_K)| \leq 2\|f\|_\infty e_{K|r}. \quad (28.16)$$

The events for different r need not be nested.

Step 3: the Cauchy property of the limits. We take $f(w) = \exp(\sum_i z_i w_i)$, so that $\|f\|_\infty \leq e^{DR}$ on $\{\sum_i |z_i| \leq R\}$. For $K \geq \max(r, r')$, we compare both conditioned expectations with the same unconditioned expectation in (28.16) and obtain

$$\sup_{\sum_i |z_i| \leq R} |\mathbb{E}_{\nu_{K|r}} e^{\sum_i z_i W_{K,i}} - \mathbb{E}_{\nu_{K|r'}} e^{\sum_i z_i W_{K,i}}| \leq 2e^{DR} (e_{K|r} + e_{K|r'}).$$

At fixed r and r' the conditioned moment generating functions converge uniformly by Theorem 28.1, applied with coarse side n_r , respectively $n_{r'}$. Letting $K \rightarrow \infty$ and using Step 1, we obtain the estimate of the theorem for Z_r^{sel} and $Z_{r'}^{\text{sel}}$. In particular these functions form a uniform Cauchy family on every ball as $r, r' \rightarrow \infty$.

Step 4: the limit law. We show that the laws of W_K under $\mu_{N_K, \gamma}$ converge. For $r \geq r_0$ let ρ_r be the weak limit of the laws of W_K under $\nu_{K|r}$, which exists by Theorem 28.1. For $f \in \mathcal{L}$, by (28.16) and Step 1, we have $|\mathbb{E}_{\mu_{N_K, \gamma}} f(W_K) - \mathbb{E}_{\nu_{K|r}} f(W_K)| \leq 2F_\gamma(n_r)$ for all $K \geq r$. Letting $K \rightarrow \infty$, we obtain

$$\limsup_{K, K' \rightarrow \infty} |\mathbb{E}_{\mu_{N_K, \gamma}} f(W_K) - \mathbb{E}_{\mu_{N_{K'}, \gamma}} f(W_{K'})| \leq 4F_\gamma(n_r),$$

and the right side tends to zero as $r \rightarrow \infty$. Hence $\mathbb{E}_{\mu_{N_K, \gamma}} f(W_K)$ converges for every bounded Lipschitz f . The vectors W_K take values in the compact box $\prod_i \{w \in \mathbb{C} : |w| \leq \chi_i(I)\}$, so every subsequence of their laws has a weakly convergent further subsequence. All these subsequential limits agree on bounded Lipschitz functions, and therefore coincide, as in Step 3 of the proof of Lemma 28.9. Hence the laws of W_K under $\mu_{N_K, \gamma}$ converge weakly to a unique probability measure ρ on the box. Passing to the limit $K \rightarrow \infty$ in the bound above also gives $d_{\text{BL}}(\rho, \rho_r) \leq 2F_\gamma(n_r)$.

Step 5: moment generating functions and consistency. Weak convergence to ρ implies pointwise convergence of the unconditioned moment generating functions. Their common Lipschitz constant on $\{\sum_i |z_i| \leq R\}$ is at most De^{DR} , so the finite-net argument of the proof of Theorem 28.1 gives uniform convergence on this set. Further, letting $K \rightarrow \infty$ in (28.16) at fixed r , we find

$$\sup_{\sum_i |z_i| \leq R} \left| \int e^{\sum_i z_i w_i} d\rho(w) - Z_r^{\text{sel}}(z) \right| \leq 2e^{DR} F_\gamma(n_r),$$

which shows how the error at a fixed coarse side disappears as $r \rightarrow \infty$. For a subfamily of the loops, the laws at each fixed K are coordinate projections of the laws of the larger family. Coordinate projection is continuous on the compact box, so the weak limits have the same relation. This proves the consistency under taking subfamilies. Along the limit only the coarse side n_r changes, within the single sequence $n_* B^r$. $\square$

28.7 Compact intervals of couplings and couplings below a finite bound

The results above use the parameters of Theorem 23.2, chosen for couplings $0 < \gamma \leq \gamma_0 \leq 1$, and their constants depend on the fixed coupling. This section gives two extensions. The first concerns the parameters of Theorem 23.5, chosen for couplings up to an arbitrary finite bound Γ , and shows that the truncation and tightness estimates hold uniformly for γ in a compact interval $[\gamma_-, \gamma_+] \subset (0, \Gamma]$. This uniformity requires checking how each constant depends on the coupling; taking the supremum of constants obtained at fixed coupling would not suffice. The second extension states that all fixed-coupling results of this section hold for these parameters at every fixed $\gamma \leq \Gamma$.

Lemma 28.10. *Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$, and let $J, B = M^J, \Theta, g_*$, and n_Γ be as in Theorem 23.5. Fix a horizon $T_0 > 0$ and an interval $I = [\gamma_-, \gamma_+] \subset (0, \Gamma]$ with $\gamma_- > 0$, and fix exponents $1/2 < \alpha < 2/3$, $\beta = \alpha - 1/2$, $0 < \sigma < 1 - \alpha$, and $0 < \tau < \min\{\sigma, 1/2, \lambda_{\text{anc}}\}$. There is a threshold $n_I \geq n_\Gamma$, which does not depend on $\gamma \in I$, such that the following holds for every coarse side $n \in 4\mathbb{N}$ with $n \geq n_I$. Put $H = L/n$ and $a_j = HB^{-j}$. Under $\nu_{nB^k, n, \gamma}$ use the arrays $A_j = \hat{\theta}_j / a_j$ of the coarse-to-fine reconstruction, and on the fine mesh $a = a_k$ set*

$$x_j = P^a I_{a_j \rightarrow a} A_j, \quad y_j = \mathcal{Q}_a(x_j, x_j), \quad d_{k,q} = X_\alpha^a(x_k - x_q) + Y_\alpha^a(y_k - y_q), \quad 0 \leq q \leq k,$$

with norms on $[0, T_0]$ that include all first differences and shifts. Then every estimate of Lemma 28.5 and Theorems 28.6 and 28.7 holds with constants that are uniform for $\gamma \in I$, and with the coupling factors displayed there. In particular, for each $\eta > 0$ there exist $b < \infty$, $C_{I,\eta} < \infty$, $0 < \xi < \tau$, and $\delta > 0$ such that

$$\sup_{\gamma \in I} \sup_{k \geq q} \nu_{nB^k, n, \gamma} \{d_{k,q} > \eta\} \leq C_{I,\eta} [(1+q)^b (a_q^\delta + a_q^{\tau-\xi}) + a_q^\xi], \quad (28.17)$$

and the right side tends to zero as $q \rightarrow \infty$. Further, the norms of the flow data are tight uniformly in $\gamma \in I$:

$$\lim_{R \rightarrow \infty} \sup_{\gamma \in I, k \geq 0} \nu_{nB^k, n, \gamma} \{X_\alpha^{a_k}(x_k) + Y_\alpha^{a_k}(y_k) > R\} = 0. \quad (28.18)$$

All constants may depend on $I, L, n, M, J, \Theta, T_0$, the exponents, and the moment orders. They do not depend on $\gamma \in I, k$, the classical depth, or the ratio of the fine mesh to a_j . For a one-point interval $I = \{\gamma\}$ this gives the assertions at every fixed $\gamma \in (0, \Gamma]$.

Proof. We go through the inputs of Lemma 28.5 and Theorems 28.6 and 28.7 and check that their constants, including the constants in the conditional maximal estimates, are uniform on I . Let $g_\pm = \sqrt{\gamma_\pm/L}$ and $\kappa = a_{\text{cap}}\sqrt{B} < 1$.

Step 1: cutoffs and amplitude bounds. The cutoffs in the definition of the small-field event differ from $p(t_j) = A_w \log(1/t_j)$ by at most a fixed c_J . Let $p_{\max, j}$ be the largest of them in the j th RG step. We have

$$\begin{aligned} t_j^2 &= \frac{\gamma}{MnB^{j-1}}, & p(t_j) &= \frac{A_w}{2} [\log(Mn/\gamma) + (j-1) \log B], & p_{\max, j} &\leq P_I(1+j), \\ \frac{\bar{b}_j}{\sqrt{a_j}} &\leq C[rH^{-1/2}\kappa^j + g_+P_I(1+j)], & \frac{\bar{b}_j}{g_j} &\leq C[r(g_-\sqrt{H})^{-1}\kappa^j + P_I(1+j)]. \end{aligned} \quad (28.19)$$

Here the amplitude bounds are those of Lemmas 17.1 and 15.10; we used $ra_{\text{cap}}^j/\sqrt{a_j} = rH^{-1/2}\kappa^j$, and the first level is included by enlarging C . For every fixed integer v we have $p_{\max, j}^v \leq C_{I, v} a_j^{-1}$, since $\sup_j HB^{-j}(1+j)^v < \infty$. None of these bounds requires $\gamma_+ \leq 1$.

Step 2: hypotheses of the conditional estimates. The conditional fluctuation potential, the lower bound $I/2$ on its Hessian in w , the conditional means, the local bounds on derivatives of the conditional log-density, and the volume-normalized column sums are given by Theorems 13.8, 13.10, and 15.13 and Lemma 26.1. Their hypotheses are the fixed domains and finitely many smallness conditions on $t[p(t) + c_J]^v$, including (26.1). We choose a small $t_* > 0$ such that these conditions hold for $0 < t \leq t_*$. We require $n \geq \gamma_+/(Mt_*^2)$, so that $t_1^2 = \gamma/(Mn) \leq t_*^2$ for all $\gamma \in I$, together with the thresholds on the coarse side and on the levels in Theorem 23.5; this defines n_I . Then every conditional fluctuation kernel at a finer level satisfies these hypotheses, for every value of the coarser field. Factors that depend only on coarser fields cancel in a normalized conditional kernel; no factor that depends on w is dropped.

Step 3: uniform maximal constants. In Lemma 26.9 the deterministic moduli contain only the fixed coefficients, powers of g_3 , and powers of $p_{\max, j}$. By Step 1 they give the same normalized modulus $C_I a_j^{-30}$ in space and time as in the proof of that lemma. A physical net of spacing $c_I a_j^{30}$ has at most $C_I(1 + La_j^{-30})^3(1 + T_0 a_j^{-30})$ points, and the logarithm of this number is at most $C_I \log(2 + L/a_j)$. The part with short times is estimated at

the outputs on the mesh a_j before this net is used. The conditional linear and quadratic concentration of Theorem 5.4, with the Hessian lower bound $1/2$, followed by a union bound and the deterministic modulus, gives maximal constants that are uniform on I . For the mixed linear Duhamel terms, Lemmas 26.7 and 26.8 also give an alternative proof with constants independent of the coupling.

Let $\tilde{x}_j = x_{j-1} + r_j$. The three ordered terms $\mathcal{Q}_a(\tilde{x}_j, r_j)$, $\mathcal{Q}_a(r_j, \tilde{x}_j)$, and $\mathcal{Q}_a(r_j, r_j)$ of Theorem 27.6, with their propagation through the coarser levels as in that theorem, are estimated separately there. The cross term satisfies $\mathcal{B}_a(x_{j-1}, r_j) = \mathcal{Q}_a(\tilde{x}_j, r_j) + \mathcal{Q}_a(r_j, \tilde{x}_j) - 2\mathcal{Q}_a(r_j, r_j)$, which explains the factor 4 in (28.8). For each of these three terms the ratio of the contribution of short times to the scale of the estimate is at most $C_I(1+j)a_{j-1}^4$ (see (27.28)). A single threshold on the level makes it small, and the finitely many coarser levels are handled by increasing C_I . The normalized modulus is $C_I a_{j-1}^{-7}$, and a net of spacing $c_I a_{j-1}^{12}$ has at most $C_I a_{j-1}^{-48}$ points (see (27.29)). Conditional concentration, the union bound, and the modulus give the same conditional moments as in that theorem, now uniformly on I . In Theorem 27.7 the normalized modulus is $C_I a_j^{-4}$, and a net of spacing $c_I a_j^8$ has at most $C_I a_j^{-32}$ points (see the estimates following (27.32)); this covers its centered terms. The estimates of Lemma 27.1 and the coefficient estimates of Theorem 27.5 are deterministic under the same amplitude bounds. Their conditional drifts keep the explicit terms

$$a_{\text{cap}}^2 \bar{b}_{i-1}^3 + g_i^2 \bar{b}_{i-1} + g_i^3 p_{\text{max},i}^3.$$

These arguments use $g_- > 0$ only to bound the inverse coupling in $\bar{b}_j/g_j$, and $g_+ < \infty$ only in upper bounds.

Step 4: the factor bounds. By Steps 1–3, the four-term decomposition of Lemma 26.2 and Theorems 26.10 to 26.12 give, in the notation of Lemma 28.5,

$$\begin{aligned} S_{p,j} &\leq C_{I,p}(1+j)^{1/2} a_j^\beta, \\ Z_{p,j}^+ &\leq C_{I,p}(1+j)^3 (a_j^\alpha + a_j^{1/2}), \\ Z_{p,j}^- &\leq C_{I,p}(1+j)^3 (a_j^\sigma + a_j^{1/2}), \\ R_j &\leq C_I(1+j) a_j^\beta, \\ U_{p,j} &\leq C_{I,p}(1+j)^b a_j^\tau, \\ \sum_{j \geq q} W_j &\leq C_{I,\zeta}(1+q)^b a_q^\zeta \quad (0 < \zeta < \min(2\beta, \lambda_{\text{anc}})). \end{aligned}$$

For the last line we insert (28.19) into the sum over earlier levels in Theorem 27.6, with factor q_{anc}^{j-1-i} , keeping the term from the coarse field. A convolution of two geometric

sequences multiplied by polynomials decays with every exponent strictly below the smaller of the two exponents. Explicitly, if $\lambda = -\log q_{\text{anc}} / \log B$, $\varrho > \zeta$, and $\lambda > \zeta$, then

$$\frac{q_{\text{anc}}^{j-1-i} a_i^{\varrho}}{a_j^{\zeta}} = H^{\varrho-\zeta} B^{\zeta} B^{-(\lambda-\zeta)(j-1-i)} B^{-(\varrho-\zeta)i}.$$

We bound the sum over i by dropping the first geometric factor, which is at most one, and summing the second. The polynomial factors contribute at most $(1+j)^b$. The term from the coarse field is geometric by itself.

Step 5: the ordered products. With these factors, each of the sixteen ordered products is bounded by a constant, uniform on I , times the entry of the table in Lemma 28.5(5). Indeed, the proof of that table uses only the product inequality, whose constant does not depend on the coupling, and the following inputs. For (s, s) it uses Theorem 26.12, including the conditional mean; for (s, r) and (r, s) , the two ordered estimates with a predictable background from Theorem 26.10; for (r, r) , the separate estimate for $\mathcal{Q}_a(r_j, r_j)$ in Theorem 27.6; for (u, u) , the pathwise bound on $\mathcal{U}_j$ and one moment; and for the remaining products, conditional estimates, the deterministic bound on r_j , or the Cauchy–Schwarz inequality under the same measure. No product of first moments of unrelated quantities occurs. The hypotheses of all these inputs were verified above, without an upper bound $\gamma \leq 1$. Every entry is summable with a positive power of a_j , after an arbitrarily small loss in the exponent.

Step 6: the truncation error. By Minkowski’s inequality applied to $d_j = s_j + z_j + r_j + u_j$, we obtain the bounds on partial sums of the increments of the linear heat evolution, and all fixed moments of $X_\alpha^a(x_j)$, uniformly on I . The term of the coarse field is bounded deterministically by $C_{H, T_0, \alpha} r / H$. At the fixed order $p = 2$ the factor bounds have constants uniform on I , and the constant of the product inequality does not depend on the coupling. Hence the proof of Theorem 28.7 gives its bounds on first moments uniformly on I : the cross terms with s_j and z_j are bounded by their conditional estimates times the moments of the background radius, the cross term with r_j is bounded by $4W_j$, and the cross term with u_j is bounded, by the Cauchy–Schwarz inequality, by $C_I(1+j)^b a_j^\tau$. By (28.4), for every $v > 0$,

$$\sum_{j \geq q} (1+j)^b a_j^v \leq (1+q)^b a_q^v \sum_{h \geq 0} (1+h)^b B^{-hv} < \infty.$$

We apply this to every cross term and every ordered product, with $\delta > 0$ smaller than all the positive exponents that occur, and with $\delta \leq \tau$.

The identity

$$y_k - y_q = \sum_{j=q+1}^k \{ \mathcal{B}_a(x_{j-1}, d_j) + \mathcal{Q}_a(d_j, d_j) \}$$

follows from (28.6). Together with $x_k - x_q = \sum_{j=q+1}^k d_j$ and Markov's inequality, it implies that $\sup_{\gamma \in I} \sup_{k \geq q} \nu_{nB^k, n, \gamma} \{d_{k,q} > \eta\} \leq C_{I, \eta} (1+q)^b a_q^\delta$. Since $a_q \leq H$, the right side of (28.17) is at least a constant multiple of $(1+q)^b a_q^\delta$, for every choice of $0 < \xi < \tau$. This proves (28.17).

Step 7: tightness. For fixed q the array A_q lies in a deterministic compact ball of a finite-dimensional space, uniformly for $\gamma \in I$, by (28.19). By Lemma 28.4, its flow data on every finer lattice have a deterministic norm bound C_q . We choose q so that the probability in (28.17) with $\eta = 1$ is as small as desired, uniformly for $\gamma \in I$ and $k \geq q$. Outside this event the norm $X_\alpha^{a_k}(x_k) + Y_\alpha^{a_k}(y_k)$ is at most $C_q + 1$. The finitely many $k < q$ have bounded arrays and bounded norms, uniformly on I . This proves (28.18). $\square$

Lemma 28.11. Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$, let $J, B = M^J, \Theta, g_*$, and n_Γ be as in Theorem 23.5, and fix $0 < \gamma \leq \Gamma$. Then Lemmas 28.4 and 28.5 and Theorems 28.1, 28.2, and 28.6 to 28.8 hold with these parameters in place of those of Theorem 23.2, for coarse sides $n \in 4\mathbb{N}$ with $n \geq n_\Gamma$ above finitely many further thresholds, which may depend on γ . They hold for the same conditional fluctuation kernels, conditional means, laws of finite-scale reconstructions, reconstruction maps, and configurations of the Wilson measure, and their constants have the dependence on the fixed coupling stated in Section 28.1. In Theorem 28.2 the constants of F_γ are those of Theorem 23.10. No bound on the norms of the flow data, and no rate of convergence, that is uniform over $\gamma \in (0, \Gamma]$ is asserted.

Proof. From Theorems 23.5 and 23.10 we use only the common smallness conditions and the bound on the small-field failure probability. At this fixed coupling, Lemma 28.10 with $I = \{\gamma\}$ supplies the truncation, factor, and tightness estimates once $n \geq n_I$; this is one of the permitted further thresholds. The convergence of the laws of finite-scale reconstructions is Theorem 15.14. The smallness condition at the first RG step is $t_1^2 = \gamma/(Mn) \leq g_*^2$, and $t_j = t_1 B^{-(j-1)/2}$ for $j \geq 1$. The local estimates on coefficients and means in Lemmas 27.1 and 26.2 and Theorems 26.10 to 26.12, 27.6, and 27.7 use these conditions and the same product domains. Their summations over levels use only $p_j \leq C_{\gamma, n, J}(1+j)$, which holds at fixed positive coupling. Thus the factor bounds of Lemma 28.5 and Theorems 28.6 and 28.7 keep their proofs and constants. Lemma 28.4 is deterministic and does not involve the coupling. The proofs of Theorems 28.1 and 28.8 use only these results, the convergence of the laws of finite-scale reconstructions, and deterministic flow estimates, which do not require a bound on the coupling. The proof of Theorem 28.2 uses Theorem 23.10 in place of Theorem 23.9, at coarse sides n_r tending to infinity; the bound (23.13) has the same form as the bound used there. $\square$

PART IX

The continuum measure

The continuum measure on gauge orbits

In this section we construct the continuum measure $\lambda_{L,\gamma,[n]_M}$ on the trajectory space $\mathcal{X}_L$ of Definition 2.2 and prove Theorem 2.3. The main result is Theorem 29.28. Fix $L > 0$, $\gamma > 0$, an odd integer $M \geq 3$, and $n \in 4\mathbb{N}$. Along the full sequence of lattice sides $N_k = nM^k$, every finite vector of flowed Wilson loops, evaluated on admissible approximations of loops in the sense of Definition 2.1, converges in law under the Wilson measure $\mu_{N_k,\gamma}$, jointly with the lattice energies at positive times. The limits are the joint laws of the corresponding continuum observables under one Borel probability measure on $\mathcal{X}_L$. This measure is the limit of the images of $\mu_{N_k,\gamma}$ under Borel maps π_k that are invariant under all lattice gauge transformations. The energies of the trajectories $\pi_k(U)$ are tight at every positive time, and the measure depends only on L , γ , and the class $[n]_M$ of n under the relation $nM^a = n'M^b$.

There are three inputs. The first is the convergence of the joint laws of flowed Wilson loops under the conditioned Wilson measures at a fixed coarse side, proved in Section 28, together with the description of these limits through continuum flows started from finite-scale reconstructions (Theorems 28.1 and 28.8). Combined with the bounds of Section 23 on the small-field failure probability, these results give convergence of finite families of flowed loops under the Wilson measures (Theorem 28.2). The second is the deterministic theory of the Wilson flow and its comparison with the Yang–Mills heat flow, from Sections 24 and 25. The third is the trajectory-space framework of Cao and Chatterjee [43]: compactness modulo gauge, smoothing by the heat flow at positive times, and the Borel structure of the space of trajectories.

Convergence of finite families of flowed loops does not by itself produce a measure on $\mathcal{X}_L$. The determining coordinates w_j of Theorem 29.3 define a metric d_c that determines Borel probability measures on $\mathcal{X}_L$, but the energy $X \mapsto E_L(X(t))$ is not known to be continuous for d_c or for the Wilson topology. We therefore also use the topology of Cao and Chatterjee on trajectories of Sobolev connections modulo gauge. Compactness of trajectories comes from simultaneous bounds on their energies at the times $s_j = L^2 2^{-j}$. Energy at each positive time is continuous in the Sobolev trajectory topology because the heat flow regularizes weakly convergent data (Lemma 29.7), and this identifies the limits of the lattice energies. To realize the limit as a limit of the lattice measures themselves, we construct the maps π_k explicitly, by a search over a countable family of smooth trajectories subject to one-sided energy bounds (Theorem 29.19).

Section 29.1 collects the properties of $\mathcal{X}_L$ that we use, and Section 29.2 the deterministic lattice estimates. Sections 29.3 to 29.5 prove joint convergence of energies and loops under the conditioned Wilson measures, construct the maps π_K , and remove the conditioning, for the parameters of Theorem 23.2; Section 29.6 extends this to couplings below an arbitrary fixed bound and treats general admissible approximations. Section 29.7 states the resulting limit theorems along the sequences $N_K = n_* M^{J^K}$, and Section 29.8 passes to the full sequence nM^k and completes the proof of Theorem 2.3. The remaining parts of Theorem 2.5 are proved in Section 30. The results of this section are used in Section 32 and in the weak-coupling analysis of Sections 34 to 36, mainly through Theorems 29.23 to 29.25 and Corollary 29.26.

Conventions. Constants in deterministic flow estimates depend only on the data listed in each statement, with the dependence described in item (7) of Definition 4.14. Stochastic convergence moduli and the number of constraints $m(K)$ used to define π_K may in addition depend on the fixed coupling, on the parameters of the renormalization group hierarchy, on the base side, and on the observables. We claim no rate of convergence, no convergence of moments of the energy, and no uniformity as $\gamma \rightarrow 0$. We use the Lie-algebra norm $|\cdot|$ of Section 4 on connections, and for a lattice potential A at mesh a we write

$$\|A\|_{C_a^1} = \|A\|_\infty + \max_{j \in \{1,2,3\}, \epsilon \in \{+,-\}} \|D_{a,j}^\epsilon A\|_\infty,$$

where $D_{a,j}^\pm$ are the forward and backward physical differences. The finite-difference norms of Definition 4.8 likewise include both directions and all translates.

29.1 The trajectory space and its Borel structure

Recall from Definition 2.2 the space $\mathcal{A}_L = C^\infty(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$ of smooth connections, the group $\mathcal{G}_L = C^\infty(\mathbb{T}_L^3; \mathrm{SU}(2))$ of smooth periodic gauge transformations (including all its connected components), acting by $A^k = kAk^{-1} - (dk)k^{-1}$, and the Yang–Mills heat semigroup $(\Phi_t)_{t \geq 0}$ on $\mathcal{A}_L/\mathcal{G}_L$. Global existence of smooth solutions for smooth initial data is due to Råde [139]; we use it in the form of [43, Theorem 3.14, p. 25], which also relies on Charalambous and Gross [47]. The trajectory space is

$$\mathcal{X}_L = \{X : (0, \infty) \rightarrow \mathcal{A}_L/\mathcal{G}_L : X(t) = \Phi_{t-s}X(s) \text{ for all } 0 < s < t\},$$

the Wilson topology $\mathcal{T}_W$ is the weakest topology for which all the functions $W_{\ell, \chi, t}(X) = \chi(\mathrm{Hol}_\ell X(t))$ are continuous, where ℓ ranges over piecewise C^1 loops, χ over characters of finite-dimensional representations of $\mathrm{SU}(2)$, and t over $(0, \infty)$. The energy E_L is defined in (2.3). Our parallel transport solves $H'(s) = H(s)A(\ell(s))\dot{\ell}(s)$ with $H(0) = I$, as in Definition 4.10, and our gauge variable is the inverse of the one in [43, (1.1), p. 2].

In [43], Cao and Chatterjee introduced trajectories of the heat flow at positive times as a state space for three-dimensional Yang–Mills theory, following the proposal of Charalambous and Gross [47] to regularize Wilson loops by the heat flow. We work in their framework. The separation of smooth gauge orbits by holonomies is due to Sengupta [144]; we use it in the formulation of [43, Lemma 1.2, p. 4]. The compactness of energy sublevel sets modulo gauge is the form of gauge compactness proved in [43, Theorem 3.8 and Theorem 3.51], which rests on Uhlenbeck’s theorem [151]. We add a countable determining metric, a comparison of its Borel structure with that of the Wilson topology, and the continuity of the energy after a positive time in the form needed below.

Sobolev connections modulo gauge

Definition 29.1. Let $\mathcal{G}_L^{2,2}$ be the group of all periodic maps $\mathbb{T}_L^3 \rightarrow \mathrm{SU}(2)$ of class $W^{2,2}$, including all connected components, and set

$$\mathcal{H}_L = H^1(\mathbb{T}_L^3; \mathfrak{su}(2)^3) / \mathcal{G}_L^{2,2}.$$

For $[A_n], [A] \in \mathcal{H}_L$ we write $[A_n] \rightarrow_q [A]$ if there exist representatives B_n of $[A_n]$ and B of $[A]$ such that $B_n \rightharpoonup B$ weakly in H^1 . The *sequential weak quotient topology* $\mathcal{T}_q$ on $\mathcal{H}_L$ is defined by declaring a set $\mathcal{U}$ open if every sequence $[A_n] \rightarrow_q [A]$ with $[A] \in \mathcal{U}$ eventually lies in $\mathcal{U}$. We write $\Phi_t^{1,2}$, $t \geq 0$, for the Yang–Mills heat semigroup on $\mathcal{H}_L$ of Cao and Chatterjee [43]; the properties of it that we use are recalled, with locators, in Lemma 29.2. Let $\mathcal{X}_L^{1,2}$ be the set of trajectories $Z : (0, \infty) \rightarrow \mathcal{H}_L$ with $Z(t) = \Phi_{t-s}^{1,2} Z(s)$ for all $0 < s < t$, with the weakest topology $\mathcal{T}_{1,2}$ for which every evaluation $Z \mapsto Z(t)$, $t > 0$, is continuous into $(\mathcal{H}_L, \mathcal{T}_q)$. This is the point-evaluation topology of [43, Definition 3.70, p. 41]; it is not a topology of uniform weak H^1 convergence on time intervals. Finally, let $i^{1,2} : \mathcal{X}_L \rightarrow \mathcal{X}_L^{1,2}$ send a trajectory of smooth orbits to the trajectory of the Sobolev orbits containing them.

For a finite energy bound R , we write $\Omega_R = \{[A] \in \mathcal{H}_L : E_L(A) \leq R\}$. We also need a countable family of regularized Wilson loop functions on $\mathcal{H}_L$: let $\Psi_0 : \mathcal{H}_L \rightarrow \mathbb{C}^{\mathbb{N}}$ be the map whose components are the functions $[A] \mapsto \chi(\mathrm{Hol}_\ell \Phi_v^{1,2}[A])$, where ℓ ranges over the countable family of loops of Theorem 29.3 below, v over $L^2\mathbb{Q}_{>0}$, and χ over the characters of the irreducible representations of $\mathrm{SU}(2)$. The following facts are taken from [43].

Lemma 29.2. *Let $L > 0$. Then the following statements hold.*

1. *The group $\mathcal{G}_L^{2,2}$ coincides with the set of continuous maps $k : \mathbb{T}_L^3 \rightarrow \mathrm{SU}(2)$ with a weak derivative such that $k^{-1}dk \in H^1$, and with the set of maps $s \exp \xi$, where $s : \mathbb{T}_L^3 \rightarrow \mathrm{SU}(2)$ is smooth and $\xi \in H^2(\mathbb{T}_L^3; \mathfrak{su}(2))$.*
2. *Every $A \in H^1(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$ has finite energy, and $\|F_A\|_2 \leq C_L(\|A\|_{H^1} + \|A\|_{H^1}^2)$ with C_L depending only on L .*

3. The space $(\mathcal{H}_L, \mathcal{T}_q)$ is Hausdorff, each Ω_R is closed and compact metrizable, and the map Ψ_0 is continuous and injective, with $\Psi_0(F \cap \Omega_R)$ compact for every closed set $F \subset \mathcal{H}_L$ and every R .
4. For every $v \geq 0$ the map $\Phi_v^{1,2}$ is continuous on $(\mathcal{H}_L, \mathcal{T}_q)$.
5. Restriction to times in $L^2\mathbb{Q}_{>0}$ is a homeomorphism of $\mathcal{X}_L^{1,2}$ onto the corresponding space of compatible trajectories at these times, and the Borel σ -algebra of $(\mathcal{X}_L^{1,2}, \mathcal{T}_{1,2})$ is generated by the evaluations at these times.
6. For every $t > 0$, every orbit $Z(t)$ of a trajectory $Z \in \mathcal{X}_L^{1,2}$ has a smooth representative, and a $W^{2,2}$ gauge transformation relating two smooth connections is smooth, so that the map $j^{1,2} : \mathcal{X}_L^{1,2} \rightarrow \mathcal{X}_L$ that replaces each orbit by its smooth representative is well defined. This map satisfies $j^{1,2}i^{1,2} = \text{Id}$ and $i^{1,2}j^{1,2} = \text{Id}$, it is continuous from $\mathcal{T}_{1,2}$ to $\mathcal{T}_W$, and it preserves every function $W_{\ell, \chi, t}$.
7. Every function $W_{\ell, \chi, t} \circ j^{1,2}$ is continuous on $(\mathcal{X}_L^{1,2}, \mathcal{T}_{1,2})$.

Proof. On the unit torus, (1) is [43, Definition 3.5 and (3.2), p. 22] (we verify the identification below), (2) is [43, Lemma 3.7, p. 23], (3) is [43, Lemma 3.49, p. 35; Definition 3.50 and Theorem 3.51, pp. 35–36], (4) is [43, Lemma 3.68, p. 40], (5) is [43, Definitions 3.70 and 3.74, pp. 41–42; Lemmas 3.75 and 3.78, pp. 42–44], (6) is [43, Lemma 3.10, p. 24; Lemmas 3.84–3.85, pp. 47–48; Definition 3.86 and Lemmas 3.87–3.89, pp. 48–49], and (7) is [43, Definition 3.82, p. 47]. In (3) the countable family of regularized Wilson loop functions may be chosen as above, from our dense loop family, rational positive delays, and all characters. The identity $i^{1,2}j^{1,2} = \text{Id}$ in (6) holds because at each time the smooth representative lies in the given Sobolev orbit. It remains to check the hypotheses of these results and to transfer them to side L .

The bundle is trivial, the base is the periodic three-torus, and the group $\text{SU}(2)$ belongs to the class allowed in [43, Section 1.1, p. 2]. All connected components of the gauge group are included. For $y = x/L$ let $\tilde{A}(y) = LA(Ly)$. Differentiating, we find

$$F_{ij}(\tilde{A})(y) = L^2 F_{ij}(A)(Ly), \quad E_1(\tilde{A}) = LE_L(A), \quad \tilde{t} = t/L^2,$$

where $\tilde{t}$ is the time of the rescaled flow. Gauge transformations transform by composition with $y \mapsto Ly$, and inversion converts the gauge convention of [43] into ours. All results on the unit torus therefore apply at every fixed L to the spaces defined here. For $f_L(y) = f(Ly)$ we have

$$\|f_L\|_{L^p(\mathbb{T}_L^3)} = L^{-3/p} \|f\|_{L^p(\mathbb{T}_L^3)}, \quad \|f_L\|_{H^1(\mathbb{T}_L^3)}^2 = L^{-3} \|f\|_{L^2(\mathbb{T}_L^3)}^2 + L^{-1} \|\nabla f\|_{L^2(\mathbb{T}_L^3)}^2.$$

Applying the Sobolev inequality $\|f\|_p \leq C_p \|f\|_{H^1}$ on the unit torus for $1 \leq p \leq 6$, with compact embedding for $p < 6$ [43, Theorem 3.3, p. 21], componentwise to Lie-algebra and matrix coefficients, we obtain at each fixed $L > 0$

$$\|f\|_{L^p(\mathbb{T}_L^3)} \leq C_p (L^{3/p-3/2} \|f\|_{L^2(\mathbb{T}_L^3)} + L^{3/p-1/2} \|\nabla f\|_{L^2(\mathbb{T}_L^3)}), \quad 1 \leq p \leq 6, \quad (29.1)$$

and the compactness for $p < 6$ transfers under the same linear rescaling. Below we use $p = 6$ for the gauge group and $p = 4$ for the curvature. The constants depend only on L , p , and the fixed coefficient norms.

We now verify (1), that is, that $\mathcal{G}_L^{2,2}$ is the full group used in [43]. The description there, as the set of maps $s \exp \zeta$ with s smooth (ranging over all components) and $\zeta \in H^2$, and as the set of continuous maps with $k^{-1}dk \in H^1$, transfers to side L by composition with $y \mapsto Ly$. A periodic H^2 matrix function is continuous, since its Fourier series converges absolutely and uniformly by the Cauchy–Schwarz inequality and $\sum_{p \in \mathbb{Z}^3} (1 + |p|^2)^{-2} < \infty$. For a $W^{2,2}$ group-valued map k we have $dk \in H^1 \subset L^6$ by (29.1). The derivative of $k^{-1}dk$ consists of a bounded matrix times $d^2k \in L^2$ and products of two factors dk , which lie in $L^3 \subset L^2$. This shows that $k^{-1}dk \in H^1$. Conversely, let k be continuous with values in the compact group, with $\alpha = k^{-1}dk \in H^1$. Then $dk = k\alpha \in L^6$, the derivative of this identity consists of $dk\alpha \in L^3$ and $k d\alpha \in L^2$, and we conclude that $k \in W^{2,2}$. These product identities hold weakly, by smooth convolution and convergence in the displayed Lebesgue norms. This identifies the two groups, including all periodic components. Finally, for (2) we apply [43, Lemma 3.7] to $\tilde{A}$; its only hypothesis is $A \in H^1$ on the unit torus. We then use the scaling of curvature and energy displayed above. The constant C_L depends only on L and the fixed coefficient norms. $\square$

The determining metric

The trajectory space carries the Wilson topology, which is defined by uncountably many functions. The following theorem replaces it, for measure-theoretic purposes, by a countable family.

Theorem 29.3. *Let $L > 0$. On piecewise C^1 loops $\ell, \ell' : [0, 1] \rightarrow \mathbb{T}_L^3$ consider the metric*

$$d_{\mathcal{L}}(\ell, \ell') = \sup_{u \in [0, 1]} d_{\mathbb{T}_L^3}(\ell(u), \ell'(u)) + \int_0^1 |\dot{\ell}(u) - \dot{\ell}'(u)| du.$$

Fix a countable family of piecewise C^1 loops that is dense for $d_{\mathcal{L}}$ and contains loops of every winding class (such families exist; see the proof), and enumerate its product with $L^2\mathbb{Q}_{>0}$ as $(\ell_j, t_j)_{j \geq 1}$. Set

$$w_j(X) = \frac{1}{2} \operatorname{tr} \operatorname{Hol}_{\ell_j} X(t_j), \quad d_c(X, Y) = \sum_{j \geq 1} 2^{-j} |w_j(X) - w_j(Y)|, \quad X, Y \in \mathcal{X}_L.$$

Then the following statements hold.

1. d_c is a metric on $\mathcal{X}_L$, and $(\mathcal{X}_L, d_c)$ is separable.
2. $(\mathcal{X}_L, \mathcal{T}_W)$ is separable, and the identity map from $(\mathcal{X}_L, \mathcal{T}_W)$ to $(\mathcal{X}_L, d_c)$ is continuous.
3. The Borel σ -algebras satisfy

$$\mathcal{B}(d_c) = \sigma(w_j : j \geq 1) = \mathcal{B}(\mathcal{T}_W).$$

4. Two Borel probability measures on $\mathcal{X}_L$ under which all finite vectors $(w_1, \dots, w_b)$ have the same laws are equal. In particular, a Borel probability measure on $\mathcal{X}_L$ is determined by the joint laws of all finite families of the functions $W_{\ell, \chi, t}$.

Remark 29.4. We do not claim that the two topologies coincide, that d_c is complete, or that $\mathcal{X}_L$ or $\mathcal{H}_L$ is a Polish space. Weak convergence on $\mathcal{X}_L$ is always stated in two forms: against bounded $\mathcal{T}_W$ -continuous functions, and weakly with respect to d_c .

We call the functions w_j the determining coordinates. Each is one half of a regularized Wilson loop for the fundamental character. The proof of Theorem 29.3 uses the reduction of all characters of $SU(2)$ to the fundamental trace. Let P_p be the polynomials defined by

$$P_0(u) = 1, \quad P_1(u) = u, \quad P_{p+1}(u) = uP_p(u) - P_{p-1}(u).$$

Lemma 29.5. *The character of every finite-dimensional representation of $SU(2)$ is a finite sum of the functions $H \mapsto P_p(\text{tr } H)$, $p \geq 0$.*

Proof. This is the classical highest-weight classification; we include the short argument. If H has eigenvalues z, z^{-1} , the recursion shows that $P_p(z + z^{-1}) = z^p + z^{p-2} + \dots + z^{-p}$ for $z \neq \pm 1$, and by continuity the identity also holds for $z = \pm 1$. Averaging a Hermitian inner product over the Haar probability measure makes a finite-dimensional representation ρ unitary. An invariant subspace then has an invariant orthogonal complement, and by induction on the dimension ρ is a direct sum of irreducible representations. If ρ is only continuous, convolving its matrix coefficients with a smooth bump supported near the identity gives a matrix arbitrarily close to I , and in particular an invertible one. Translating the bump on the left shows that ρ multiplied by this matrix is smooth, which implies that ρ is smooth. In the complexified differential of an irreducible ρ we set $H = d\rho(h)$, $E = d\rho(e)$, $F = d\rho(f)$ for

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$, $H^* = H$, and $E^* = F$, the last two because ih , $e - f$, and $i(e + f)$ are anti-Hermitian. Since $\exp(2\pi ih) = I$, the eigenvalues of H are integers. Let n be the largest one and $Hv = nv$ with $v \neq 0$; then $Ev = 0$, and by induction, using the commutation relations, we obtain

$$EF^q v = q(n - q + 1)F^{q-1}v, \quad \|F^{q+1}v\|^2 = (q + 1)(n - q)\|F^q v\|^2.$$

The case $q = 0$ shows that $n \geq 0$. The vectors $v, Fv, \dots, F^n v$ are nonzero with distinct weights $n, n - 2, \dots, -n$, and $F^{n+1}v = 0$. Their span is invariant under H, E, F . It is also invariant under the group, since every element of $SU(2)$ is an exponential after unitary diagonalization. By irreducibility the span is the whole space, and the character on $\text{diag}(z, z^{-1})$ is $P_n(z + z^{-1})$. Unitary diagonalization covers every group element. $\square$

Proof of Theorem 29.3. The proof has six steps.

Step 1: a dense family. The continuity and separability results [43, Lemmas 3.24–3.25 and 3.28, p. 28] apply to every fixed smooth connection and character here. A dense countable family of loops can be chosen among polygonal loops whose lifts have vertices in LQ^3 and rational subdivision parameters, with final vertex equal to the initial vertex plus Lz , for every $z \in \mathbb{Z}^3$. To see that this family is dense for $d_{\mathcal{L}}$, we approximate each C^1 piece of a loop by its polygonal interpolant and then perturb the finitely many vertices and subdivision parameters within these dense sets. The first approximation converges uniformly, with derivatives in L^1 , by uniform continuity of the derivative on each piece; for a fixed interpolant the second does so because the velocities are constant on finitely many intervals. The final vertex is imposed using the same Lz . This family contains every winding class and the constant loops at rational points.

Step 2: separation. Suppose that $w_j(X) = w_j(Y)$ for all j . Fix $t \in L^2\mathbb{Q}_{>0}$. Then $\text{tr Hol}_{\ell} X(t) = \text{tr Hol}_{\ell} Y(t)$ for every loop ℓ of the dense family, and by the continuity of holonomies in $d_{\mathcal{L}}$ [43, Lemmas 3.24–3.25] for every piecewise C^1 loop. By Lemma 29.5, all characters of $\text{Hol}_{\ell} X(t)$ and $\text{Hol}_{\ell} Y(t)$ agree on every piecewise C^1 loop. The holonomy separation theorem [43, Lemma 1.2, p. 4], whose hypotheses are exactly these, implies that $X(t) = Y(t)$. For an arbitrary $t > 0$, let $s \in L^2\mathbb{Q}_{>0}$ with $s < t$; by compatibility we have $X(t) = \Phi_{t-s}X(s) = \Phi_{t-s}Y(s) = Y(t)$. No backward uniqueness is used. Thus d_c separates points.

Step 3: metric properties. Symmetry and the triangle inequality hold termwise. Since $|w_j| \leq 1$, the tail $\sum_{j>b} 2^{-j}|w_j(X) - w_j(Y)|$ is at most 2^{1-b} . This implies that convergence in d_c is equivalent to convergence of every coordinate, and the coordinate map $X \mapsto (w_j(X))_{j \geq 1}$ is a homeomorphism of $(\mathcal{X}_L, d_c)$ onto a subspace of $[-1, 1]^{\mathbb{N}}$. The countable base of cylinder sets of the product restricts to a countable base, and choosing a point in each nonempty member, we obtain a countable dense set. The same observation shows that $\mathcal{B}(d_c) = \sigma(w_j : j \geq 1)$. The partial sums defining d_c are $\mathcal{T}_W$ -continuous and the tails are uniformly small. Then the coordinate map is continuous from $\mathcal{T}_W$ into the product, and the identity from $(\mathcal{X}_L, \mathcal{T}_W)$ to $(\mathcal{X}_L, d_c)$ is continuous.

Step 4: the reverse Borel inclusion. Since every w_j is $\mathcal{T}_W$ -continuous, it remains to show that $\mathcal{B}(\mathcal{T}_W) \subset \sigma(w_j : j \geq 1)$. We first show that the coordinates of Ψ_0 generate the Borel σ -algebra of $(\mathcal{H}_L, \mathcal{T}_q)$. By Lemma 29.2(2), every element of $\mathcal{H}_L$ has finite energy, and the sets Ω_b , $b \in \mathbb{N}$, exhaust $\mathcal{H}_L$. For a closed set $F \subset \mathcal{H}_L$ we therefore have

$$\Psi_0(F) = \bigcup_{b \geq 1} \Psi_0(F \cap \Omega_b). \quad (29.2)$$

By Lemma 29.2(3), the right side is a countable union of compact, hence closed, subsets of the Hausdorff space $\mathbb{C}^{\mathbb{N}}$, and in particular it is Borel. Since Ψ_0 is injective,

$F = \Psi_0^{-1}(\Psi_0(F))$ lies in the σ -algebra generated by the coordinates of Ψ_0 ; the converse inclusion holds because Ψ_0 is continuous. This argument uses the compactness of the images $\Psi_0(F \cap \Omega_b)$; injectivity and measurability of Ψ_0 alone would not give measurability of its inverse.

Now let $u \in L^2\mathbb{Q}_{>0}$ and $X \in \mathcal{X}_L$. Each component of $\Psi_0(i^{1,2}X(u))$ is of the form $\chi(\text{Hol}_\ell X(u+v))$ with ℓ in the dense family and $v \in L^2\mathbb{Q}_{>0}$. By Lemma 29.5 it is a polynomial in $(\text{tr Hol}_\ell X(u+v))/2$, which is one of the functions w_j , since $u+v \in L^2\mathbb{Q}_{>0}$. Since the coordinates of Ψ_0 generate the Borel σ -algebra of $\mathcal{H}_L$, the map $X \mapsto i^{1,2}X(u)$ is measurable from $\sigma(w_j : j \geq 1)$ into the Borel σ -algebra of $\mathcal{H}_L$. By Lemma 29.2(5), the map $i^{1,2}$ is measurable from $\sigma(w_j : j \geq 1)$ into the Borel σ -algebra of $\mathcal{X}_L^{1,2}$. Let $\mathcal{B} \in \mathcal{B}(\mathcal{T}_W)$. Since $j^{1,2}$ is continuous (Lemma 29.2(6)), the set $(j^{1,2})^{-1}(\mathcal{B})$ is Borel in $\mathcal{X}_L^{1,2}$, and since $j^{1,2}i^{1,2} = \text{Id}$,

$$\mathcal{B} = (i^{1,2})^{-1}((j^{1,2})^{-1}(\mathcal{B})) \in \sigma(w_j : j \geq 1).$$

This proves the Borel identity in (3).

Step 5: separability of the Wilson topology. We consider the countable set of trajectories $t \mapsto \Phi_t[b]$, where b ranges over the trigonometric polynomial one-forms with rational coefficients. Let $X \in \mathcal{X}_L$, and fix a basic Wilson neighborhood of X determined by finitely many functions $W_{\ell,\chi,t}$. The shifted trajectories $X_\epsilon(t) = X(t+\epsilon)$ approach X in every finite list of these functions as $\epsilon \downarrow 0$, since X is the smooth solution started at any earlier positive time. Each X_ϵ is the trajectory started from a smooth representative of $X(\epsilon)$. Approximating this representative in C^3 by a rational trigonometric polynomial and using Theorem 24.10 on a finite horizon containing the listed times, we find a trajectory of the countable set in the given neighborhood.

Step 6: determination of probability measures. If two measures have the same finite joint laws of the w_j , then their image measures on $[-1, 1]^{\mathbb{N}}$ agree on the cylinder sets. The class of sets on which the two image measures agree is a Dynkin class containing the class of cylinder sets, which is closed under intersections. By Dynkin's π - λ theorem, it contains the σ -algebra generated by the cylinders. Pulling back and using (3), we find that the two measures agree on $\mathcal{B}(\mathcal{T}_W)$. The joint laws of finite families of the functions $W_{\ell,\chi,t}$ include those of the w_j , which proves the last assertion. This also verifies the hypotheses of the uniqueness statement [43, Lemma 1.4, p. 7]. $\square$

Compactness of energy sublevel sets

Theorem 29.6. *Let $L > 0$. On the Borel space of Theorem 29.3, the function $X \mapsto E_L(X(t))$ is measurable for every $t > 0$, and $t \mapsto E_L(X(t))$ is nonincreasing for every $X \in \mathcal{X}_L$. Let $s_j = L^2 2^{-j}$ for $j \geq 1$. For every sequence $R = (R_j)_{j \geq 1}$ of finite numbers $R_j \geq 0$, the set*

$$\mathcal{K}(R) = \{X \in \mathcal{X}_L : E_L(X(s_j)) \leq R_j \text{ for all } j \geq 1\}$$

is compact for $\mathcal{T}_W$ and for d_c . Let $\{\lambda_i\}$ be a family of Borel probability measures on $\mathcal{X}_L$ such that

$$\lim_{R \rightarrow \infty} \sup_i \lambda_i \{E_L(X(s_j)) > R\} = 0 \quad \text{for every } j \geq 1.$$

Then for every $\eta > 0$ there exists R with $\lambda_i(\mathcal{K}(R)) \geq 1 - \eta$ for all i , and every sequence in the family has a subsequence that converges, against every bounded $\mathcal{T}_W$ -continuous function and weakly with respect to d_c , to a Borel probability measure on $\mathcal{X}_L$. Finally, if A is a smooth connection with $\|A\|_{C^1} \leq R$, then

$$E_L(\Phi_t[A]) \leq E_L(A) \leq CL^3(R + R^2)^2 \quad (t \geq 0),$$

where C depends only on the normalization of the Lie-algebra norm.

No moment condition on the measures λ_i is needed. Subsequential compactness from energy tightness is due to Cao and Chatterjee [43, Theorem 1.6, p. 7]. The theorem adds the explicit compact sets $\mathcal{K}(R)$, which are used in Section 29.4. We say that a family of Borel probability measures on $\mathcal{X}_L$ has *compact containment* if for every $\eta > 0$ some set $\mathcal{K}(R)$ has mass at least $1 - \eta$ under every member of the family. Since the sets $\mathcal{K}(R)$ are compact for both $\mathcal{T}_W$ and d_c , this is compact containment in both topologies.

Proof. For probability measures on the trajectory space with its Wilson Borel σ -algebra, the energy at each positive time is measurable and nonincreasing in time [43, Lemma 1.5, p. 7; Lemma 4.1, p. 50]. The hypotheses on the group, the bundle, the scaling, and the gauge group were checked in Lemma 29.2, and the energy in the statement has exactly this normalization.

We consider the product $\prod_{j \geq 1} \Omega_{R_j}$ of subsets of $\mathcal{H}_L$. It is compact and metrizable, by Lemma 29.2(3) and a diagonal extraction of subsequences in the factors, using a summable bounded product metric. We impose the compatibility conditions

$$x_i = \Phi_{s_i - s_j}^{1,2} x_j \quad (j > i). \quad (29.3)$$

Each of these equations defines a closed set, since $\mathcal{H}_L$ is Hausdorff and $\Phi_v^{1,2}$ is continuous for $\mathcal{T}_q$ (Lemma 29.2(3),(4)). The compatible subset is therefore compact. Given a compatible sequence (x_j) , we define $Z(t) = \Phi_{t - s_j}^{1,2} x_j$ for any j with $s_j < t$. By (29.3) and the semigroup property, $Z(t)$ does not depend on j , and Z is compatible at all times. Fixing one j with $s_j < t$ for each t , we see that $Z(t)$ is a continuous function of x_j , and therefore the map $(x_j) \mapsto Z$ is continuous into $(\mathcal{X}_L^{1,2}, \mathcal{T}_{1,2})$. By Lemma 29.2(6), composing with $j^{1,2}$, we obtain a continuous map into $(\mathcal{X}_L, \mathcal{T}_W)$.

The image of this map is exactly $\mathcal{K}(R)$. To see this, let $k > j$; then $Z(s_j) = \Phi_{s_j - s_k}^{1,2} x_k = x_j$ by (29.3). Every x_j is therefore the orbit at time s_j of the trajectory, it has a smooth representative, and its energy is at most R_j . Conversely, the values $i^{1,2} X(s_j)$ of any

$X \in \mathcal{K}(R)$ form such a compatible sequence. As a continuous image of a compact set, $\mathcal{K}(R)$ is $\mathcal{T}_W$ -compact, and by Theorem 29.3(2) it is also d_c -compact. It is Borel because the energies are measurable. The argument also covers an empty compatible set and thresholds equal to zero.

Let $\eta > 0$, and for each j fix R_j such that $\sup_i \lambda_i \{E_L(X(s_j)) > R_j\} \leq \eta 2^{-j}$. By the union bound, we have $\lambda_i(\mathcal{K}(R)) \geq 1 - \eta$ for every i . For any $t > 0$, choosing $s_j < t$ and using monotonicity, we see that the energy laws at time t are also tight. The subsequence theorem [43, Theorem 1.6, p. 7] applies to probability measures on this Borel space with tight energy laws at each positive time, and it yields a subsequence converging against every bounded $\mathcal{T}_W$ -continuous function. Every bounded d_c -continuous function is $\mathcal{T}_W$ -continuous by Theorem 29.3(2), and the same subsequence converges weakly for d_c .

Finally, $||[X, Y]| \leq |X||Y|$ in our normalization, which implies that $|F_{ij}(A)| \leq C(R + R^2)$ if $\|A\|_{C^1} \leq R$. There are three pairs $i < j$ and the volume is L^3 , which gives the bound on $E_L(A)$; the bound along the flow follows from monotonicity. $\square$

Continuity of energy after a positive time

The determining coordinates w_j do not control the energy, and we do not know whether $X \mapsto E_L(X(t))$ is continuous for d_c or for $\mathcal{T}_W$. The heat flow provides a substitute: on the Sobolev quotient, the energy after any positive time is a continuous function of the initial orbit. This property lets us identify limits of lattice energies with energies of the limiting trajectories in Theorem 29.18.

Lemma 29.7. *Let $L > 0$ and $u > 0$. The function $[A] \mapsto E_L(\Phi_u^{1,2}[A])$ is continuous on $(\mathcal{H}_L, \mathcal{T}_q)$. Moreover, for every $t > 0$ the function $Z \mapsto E_L(Z(t))$ is continuous on $(\mathcal{X}_L^{1,2}, \mathcal{T}_{1,2})$. Here the energy of an orbit at a positive time is the energy of its smooth representatives, which does not depend on the representative.*

The lemma does not assert that $[A] \mapsto E_L(A)$ itself is continuous on $(\mathcal{H}_L, \mathcal{T}_q)$.

Proof. We work on the unit torus first. Let $A_n \rightharpoonup A$ weakly in H^1 , and set $R_0 = \sup_n \|A_n\|_{H^1} + \|A\|_{H^1}$, which is finite. Fix $0 < v < \min(u, \tau_0(R_0))$, where $\tau_0(R_0)$ is the existence time of [43, Lemma 3.23, pp. 27–28]. By that lemma, the local solutions B_n, B of the gauge-fixed heat equation (in the DeTurck gauge) started at A_n, A satisfy $B_n(v) \rightarrow B(v)$ strongly in H^1 . These endpoints are smooth, by [43, Proposition 3.19, pp. 26–27], and they represent the orbits $\Phi_v^{1,2}[A_n]$ and $\Phi_v^{1,2}[A]$, by [43, Lemma 3.21, p. 27]. Applying [43, Lemma 3.15, p. 25] to the smooth, strongly convergent initial data $B_n(v) \rightarrow B(v)$, we obtain strong H^1 convergence of representatives of the solutions of the covariant flow at time $u - v$, that is, of representatives of $\Phi_u^{1,2}[A_n]$ and $\Phi_u^{1,2}[A]$. The smooth-data lemma is applied only after the smoothing step, and never to the weakly convergent data A_n .

Strong H^1 convergence implies convergence of the energy, by the embedding $H^1 \subset L^4$:

$$\|F(C) - F(D)\|_2 \leq C(1 + \|C\|_{H^1} + \|D\|_{H^1})\|C - D\|_{H^1}.$$

Indeed, the derivative part of F is linear, and $[C_i, C_j] - [D_i, D_j] = [C_i - D_i, C_j] + [D_i, C_j - D_j]$ is bounded in L^2 by Hölder's inequality and the embedding. Multiplying by $\|F(C)\|_2 + \|F(D)\|_2$, we obtain convergence of the squared norms. By gauge invariance, $E_1(\Phi_u^{1,2}[A_n]) \rightarrow E_1(\Phi_u^{1,2}[A])$.

If $[A_n] \rightharpoonup_q [A]$, we apply this argument to weakly convergent representatives. Since $\mathcal{T}_q$ is generated by exactly this sequential convergence, [43, Lemma 3.38 and Corollary 3.39, p. 32; Lemma 3.41, p. 33] give continuity into $\mathbb{R}$; we do not replace weakly convergent nets by bounded sequences. For trajectories, we fix $s \in (0, t)$ and factor the energy as $Z \mapsto Z(s) \mapsto E_1(\Phi_{t-s}^{1,2}Z(s))$. Finally, the rescaling $\tilde{A}(y) = LA(Ly)$, $\tilde{t} = t/L^2$, $E_1(\tilde{A}) = LE_L(A)$ of Lemma 29.2 transfers every assertion to side L . $\square$

A countable dense family of trajectories and finite-dimensional approximation

Fix an enumeration $(b_q)_{q \geq 1}$ of the real $\mathfrak{su}(2)$ -valued trigonometric polynomial one-forms on $\mathbb{T}_L^3$ with rational coefficients in a fixed basis, with $b_1 = 0$, and set

$$X_q(t) = \Phi_t[b_q], \quad t > 0.$$

In particular, X_1 is the zero trajectory, $X_1(t) = [0]$ for all t . The trajectories X_q are the candidates among which the maps π_K of Section 29.4 choose.

Lemma 29.8. *Let $X \in \mathcal{X}_L$, let $W_{\ell_1, \chi_1, t_1}, \dots, W_{\ell_b, \chi_b, t_b}$ be finitely many Wilson loop functions, let $v_1, \dots, v_p > 0$, and let $\delta > 0$. Then there exists q such that*

$$|W_{\ell_h, \chi_h, t_h}(X_q) - W_{\ell_h, \chi_h, t_h}(X)| < \delta \quad (h \leq b), \quad |E_L(X_q(v_i)) - E_L(X(v_i))| < \delta \quad (i \leq p).$$

Proof. Rational trigonometric polynomials are dense in C^3 in the space of smooth real one-forms. Indeed, integration by parts bounds the Fourier coefficients of a smooth form by $C_p(1 + |k|)^{-p}$, so for $p > 6$ the Fourier truncations converge in C^3 , and the finitely many real sine and cosine coefficients of a truncation can be approximated by rational numbers. Since the frequencies are $2\pi k/L$, no rationality condition on L is needed.

For $\eta > 0$, let c_η be a smooth representative of $X(\eta)$. By compatibility, $\Phi_t[c_\eta] = X(t + \eta)$ for $t > 0$. Smooth forward evolution from an earlier positive time shows that the energy and the holonomies are continuous in time at each of the finitely many listed times. Then the listed values on $X(\cdot + \eta)$ converge to those on X as $\eta \downarrow 0$. Fix η such that all these errors are less than $\delta/2$. By Theorem 24.10, the listed energies and loops of the flow are continuous functions of the initial datum in C^3 on the finite horizon. Approximating c_η in C^3 by some b_q , we make the remaining errors less than $\delta/2$. The representative c_η is used only to prove the existence of q . $\square$

Lemma 29.9. *Let $\mathcal{C} \subset \mathcal{X}_L$ be $\mathcal{T}_W$ -compact, and let $h : \mathcal{X}_L \rightarrow \mathbb{R}$ be $\mathcal{T}_W$ -continuous with $|h| \leq D$. For every $\delta > 0$ there exist $b < \infty$ and a continuous function $G : [-1, 1]^b \rightarrow [-D, D]$ such that*

$$\sup_{X \in \mathcal{C}} |h(X) - G(w_1(X), \dots, w_b(X))| < \delta.$$

This applies in particular to the compact sets $\mathcal{K}(R)$ of Theorem 29.6 and to $h = f(W_{\ell_1, \chi_1, t_1}, \dots, W_{\ell_m, \chi_m, t_m})$ with f bounded and continuous.

Proof. If $\mathcal{C}$ is empty, we take $G = 0$. Otherwise, by Theorem 29.3(2) the identity from $(\mathcal{C}, \mathcal{T}_W)$ to $(\mathcal{C}, d_c)$ is a continuous bijection from a compact space to a Hausdorff space, and it is a homeomorphism: closed subsets of the compact domain have compact, hence closed, images. Thus h is uniformly continuous on $(\mathcal{C}, d_c)$. Let $\rho > 0$ be such that $d_c(X, Y) < \rho$ with $X, Y \in \mathcal{C}$ implies $|h(X) - h(Y)| < \delta$. Fix b with $2^{1-b} < \rho/4$ and a finite $\rho/8$ -net $X^1, \dots, X^l$ of $(\mathcal{C}, d_c)$. For $u, v \in [-1, 1]^b$ and $X \in \mathcal{X}_L$ write

$$d_b(u, v) = \sum_{j \leq b} 2^{-j} |u_j - v_j|, \quad p_b X = (w_j(X))_{j \leq b}, \quad h_\alpha(u) = (\rho/2 - d_b(u, p_b X^\alpha))_+,$$

and let $h_0(u)$ be the d_b -distance from u to the compact set $p_b \mathcal{C}$. The function $h_0 + \sum_\alpha h_\alpha$ is positive on the cube: off $p_b \mathcal{C}$ we have $h_0 > 0$, and on $p_b \mathcal{C}$ the net property gives some $h_\alpha > 0$. We define

$$G(u) = \frac{\sum_\alpha h_\alpha(u) h(X^\alpha)}{h_0(u) + \sum_\alpha h_\alpha(u)}.$$

Then G is continuous and $|G| \leq D$. For $X \in \mathcal{C}$ we have $h_0(p_b X) = 0$, and $h_\alpha(p_b X) > 0$ implies $d_c(X, X^\alpha) < \rho/2 + 2^{1-b} < \rho$. Thus $G(p_b X)$ is a convex combination of values $h(X^\alpha)$ within δ of $h(X)$. $\square$

29.2 Lattice observables and deterministic estimates

Lattice approximations of loops

Let $\ell, \ell' : [0, 1] \rightarrow \mathbb{T}_L^3$ be piecewise C^1 loops with base points $x = \ell(0)$ and $x' = \ell'(0)$, and let c be a piecewise C^1 path from x to x' (the constant path if $x = x'$). A *comparison homotopy* between ℓ and ℓ' with connector c is a piecewise C^1 map $H : [0, 1]^2 \rightarrow \mathbb{T}_L^3$ such that $H(\cdot, 0) = \ell$, $H(\cdot, 1)$ is a parametrization of the concatenation $c * \ell' * c^{-1}$, and $H(0, v) = H(1, v) = x$ for all $v \in [0, 1]$. Its swept area is

$$\text{Area}(H) = \int_0^1 \int_0^1 |\partial_u H \wedge \partial_v H| du dv,$$

where for $v, w \in \mathbb{R}^3$ we write $|v \wedge w|$ for the area of the parallelogram spanned by v and w . The swept area is counted with multiplicity: parts of the homotopy with opposite

orientations do not cancel. Since transport along a concatenation is the product of the transports, $\text{Hol}_{c*\ell'*c^{-1}} A = (\text{Hol}_c A)(\text{Hol}_{\ell'} A)(\text{Hol}_c A)^{-1}$, and the characters of $\text{Hol}_{\ell'} A$ and $\text{Hol}_{c*\ell'*c^{-1}} A$ coincide. In this language, a sequence of closed lattice paths (ℓ_k) , with ℓ_k in Λ_{N_k} and $N_k \rightarrow \infty$, is an admissible approximation of ℓ in the sense of Definition 2.1 if the physical lengths of the ℓ_k are bounded and, for all sufficiently large k , there exists a comparison homotopy H_k between ℓ and ℓ_k (with some connector) such that $\text{Area}(H_k) \rightarrow 0$.

Lemma 29.10. *Let $L > 0$, let ℓ be a piecewise C^1 loop in $\mathbb{T}_L^3$, and let $a = L/N$ with an integer $N \geq 4$. There exist a closed lattice path ℓ_a in Λ_N and a comparison homotopy H_a between ℓ and ℓ_a , whose connector is a straight segment of length at most $\sqrt{3}a/2$, such that*

$$\text{length}(\ell_a) \leq C(\text{length}(\ell) + L), \quad \text{Area}(H_a) \leq Ca(\text{length}(\ell) + L), \quad (29.4)$$

where C is an absolute constant. The path ℓ_a has the same winding class as ℓ . The homotopy H_a is piecewise C^1 , and piecewise smooth if ℓ is polygonal. If ℓ is constant, then ℓ_a is constant and $\text{Area}(H_a) = 0$. The path ℓ_a and the homotopy H_a depend only on the parametrized loop ℓ and on a .

In particular, for every sequence of sides $N_k \rightarrow \infty$ the paths $\ell_{a_{N_k}}$ form an admissible approximation of ℓ , with swept areas $O(a_{N_k})$.

Proof. Let $x : [0, 1] \rightarrow \mathbb{R}^3$ be the lift of ℓ with $x(0) \in [0, L)^3$; then $x(1) = x(0) + Lz$ for a unique $z \in \mathbb{Z}^3$. Set $\mathfrak{l} = \text{length}(\ell)$. Suppose first that $\mathfrak{l} > 0$, and set $p = \lceil \mathfrak{l}/a \rceil$, so that $p \leq 1 + \mathfrak{l}/a$. We subdivide ℓ by arclength into p pieces of length $\mathfrak{l}/p \leq a$. At an interior subdivision value we take the earliest parameter with that accumulated length, and we keep the two endpoint parameters; this fixes the choice also when x is stationary on some subintervals. We write x_j , $0 \leq j \leq p$, for the subdivision points.

We round each coordinate by the rule

$$q_j^i = a \lfloor x_j^i/a + \frac{1}{2} \rfloor, \quad \text{so that} \quad |q_j - x_j| \leq \frac{\sqrt{3}}{2}a.$$

Since $L/a = N \in \mathbb{N}$, this rule commutes with translations by $L\mathbb{Z}^3$, and therefore $q_p = q_0 + Lz$. Let σ_j be the lattice path from q_j to q_{j+1} that changes first the first coordinate, then the second, and then the third, each monotonically, and let ℓ_a be the projection of the concatenation of the σ_j to the torus. It is closed and has the same winding class as ℓ . Its length satisfies

$$\sum_{j=0}^{p-1} |q_{j+1} - q_j|_1 \leq \sqrt{3} \sum_{j=0}^{p-1} |x_{j+1} - x_j| + 3pa \leq (\sqrt{3} + 3)\mathfrak{l} + 3a,$$

where we used $|q_{j+1} - q_j|_1 \leq \sqrt{3}|q_{j+1} - q_j| \leq \sqrt{3}|x_{j+1} - x_j| + 3a$ and $pa \leq \mathfrak{l} + a$. Since $a \leq L/4$, this proves the first bound in (29.4), including coarse meshes.

Let η_j be the j th subarc of the lift, from x_j to x_{j+1} , let $c_j = [x_j, q_j]$ be the straight segments, and let $\delta_j = c_j * \sigma_j * c_{j+1}^{-1}$, which also runs from x_j to x_{j+1} . The curves η_j , σ_j , c_j , and c_{j+1} lie in a ball of radius Ca about x_j . We parametrize η_j and δ_j on $[0, 1]$, piecewise C^1 , and consider the strip $H_j(u, v) = (1 - v)\eta_j(u) + v\delta_j(u)$, whose endpoints are fixed. Since $|\partial_v H_j| = |\delta_j(u) - \eta_j(u)| \leq Ca$, its swept area satisfies

$$\begin{aligned} \int_0^1 \int_0^1 |\partial_u H_j \wedge \partial_v H_j| du dv &\leq Ca \int_0^1 \int_0^1 ((1 - v)|\eta_j'(u)| + v|\delta_j'(u)|) du dv \\ &\leq Ca(\text{length}(\eta_j) + \text{length}(\delta_j)) \leq Ca^2. \end{aligned}$$

For polygonal loops we may instead triangulate these strips by coning their polygonal boundaries to x_j ; each triangle has area at most Ca times the length of its boundary edge, which gives the same bound and piecewise smooth strips. We concatenate the strips. The interior pairs $c_j^{-1} * c_j$ are removed by retracing homotopies, whose differentials have rank at most one and which therefore have zero swept area. The final connector is the initial one translated by Lz . After projection to the torus we obtain a comparison homotopy H_a between ℓ and ℓ_a with connector c_0 , and since swept areas add,

$$\text{Area}(H_a) \leq Cpa^2 \leq Ca(l + a) \leq Ca(l + L).$$

All constructions take place in the lift before projection, and they are valid across the periodic identifications and at self-intersections. For piecewise C^1 loops the strips are piecewise C^1 , which suffices for the holonomy variation formula of Lemma 29.11. If $l = 0$, we round $x(0)$ by the same rule and take the constant lattice loop at the rounded point; the connector followed by its inverse and the constant loop are related by a retracing homotopy of zero area. These rules involve only ℓ and a . $\square$

Lemma 29.11. *Let $L > 0$ and let A be a smooth connection on $\mathbb{T}_L^3$.*

1. *Let $H : [0, 1]^2 \rightarrow \mathbb{T}_L^3$ be continuous and C^1 on the cells of a finite C^1 subdivision of the parameter square, with $H(0, v)$ and $H(1, v)$ independent of v . Then*

$$\|\text{Hol}_{H(\cdot, 1)} A - \text{Hol}_{H(\cdot, 0)} A\|_{\text{op}} \leq C\|F_A\|_{\infty} \text{Area}(H),$$

where C is an absolute constant. In particular, if H is a comparison homotopy between loops ℓ and ℓ' with connector c , then $\text{Hol}_{\ell} A$ and $(\text{Hol}_c A)(\text{Hol}_{\ell'} A)(\text{Hol}_c A)^{-1}$ differ by at most $C\|F_A\|_{\infty} \text{Area}(H)$ in operator norm, and for every character χ and every A with $\|A\|_{C^1} \leq R$,

$$|\chi(\text{Hol}_{\ell} A) - \chi(\text{Hol}_{\ell'} A)| \leq C_{\chi}(R + R^2) \text{Area}(H).$$

2. *Let $a = L/N$, and let U^A be the lattice configuration $U_{(x, x+ae_i)}^A = \exp(aA_i(x))$. If $\|A\|_{C^1} \leq R$ and ℓ_a is a closed lattice path of physical length at most l , then $\|\text{Hol}_{\ell_a} A - \text{Hol}_{\ell_a} U^A\|_{\text{op}} \leq C_R la$, and $|\chi(\text{Hol}_{\ell_a} A) - \chi(\text{Hol}_{\ell_a} U^A)| \leq C_{\chi, R} la$.*

3. Let $\ell, \ell_K : [0, 1] \rightarrow \mathbb{T}_L^3$ be piecewise C^1 loops with $\sup_K \text{length}(\ell_K) < \infty$ and $\delta_K = \sup_u d_{\mathbb{T}_L^3}(\ell(u), \ell_K(u)) \rightarrow 0$. If $\delta_K < L/2$, then ℓ_K has the same winding class as ℓ , and there exists a comparison homotopy H_K between ℓ and ℓ_K , whose connector is the shortest segment from $\ell(0)$ to $\ell_K(0)$, such that

$$\text{Area}(H_K) \leq \frac{\delta_K}{2} (\text{length}(\ell) + \text{length}(\ell_K)). \quad (29.5)$$

In particular, a sequence of closed lattice paths of bounded physical length that converges uniformly to ℓ , as parametrized loops, is an admissible approximation of ℓ .

The constants C_χ , C_R , and $C_{\chi,R}$ depend only on the indicated quantities.

Proof. (1) We first consider a piece of H on which H is C^2 , and write $B = A(H)\partial_u H$ and $\tilde{C} = A(H)\partial_v H$ for the pulled-back components of the connection. Let P solve $\partial_u P = PB$ with $P(0, v) = I$, so that $P(1, v) = \text{Hol}_{H(\cdot, v)} A$, and set $Q = \partial_v P - P\tilde{C} + \tilde{C}(0, v)P$. Differentiating, we find

$$\partial_u Q = QB - P(\partial_u \tilde{C} - \partial_v B + [B, \tilde{C}]), \quad Q(0, v) = 0.$$

The expression in parentheses is the curvature $F_A(H)(\partial_u H, \partial_v H)$. Since $\partial_u(QP^{-1}) = -PF_A(H)(\partial_u H, \partial_v H)P^{-1}$ and all transports are unitary, we obtain

$$\|Q(1, v)\|_{\text{op}} \leq C\|F_A\|_\infty \int_0^1 |\partial_u H \wedge \partial_v H| du. \quad (29.6)$$

If $H(0, v)$ and $H(1, v)$ do not depend on v , then $\tilde{C}(0, v) = \tilde{C}(1, v) = 0$, so $Q(1, v) = \partial_v P(1, v)$. Integrating (29.6) over $v \in [0, 1]$, we obtain the first assertion. For a piece that is only C^1 , we extend a C^1 lift slightly beyond its parameter rectangle by reflection, $f(-u, v) = 2f(0, v) - f(u, v)$ at the boundary $u = 0$ and analogously at the other sides, and mollify. The mollified maps and their first derivatives converge uniformly. By their integral form, the solutions of the transport equations then converge uniformly, and the integrands in (29.6) converge in L^1 . The integrated identity and the inequality therefore pass to the C^1 piece. For a piecewise C^1 map, use its finite C^1 cell subdivision, triangulating and using C^1 cell parametrizations when necessary. A triangle with vertices a, b, c is parametrized by $\psi(s, t) = a + s(b - a) + (1 - s)t(c - a)$ on $[0, 1]^2$; the side $s = 1$ is collapsed and its area is counted once, since the Jacobian is $(1 - s)\det(b - a, c - a)$. The moving-endpoint identity above bounds the boundary holonomy of each cell minus the identity by $C\|F_A\|_\infty$ times its swept area. Glue the cells along a dual spanning tree, transporting their boundary loops to a common basepoint. In the resulting path-ordered product the internal edge transports cancel; equivalently, cut along the tree and concatenate the rooted face boundaries. Unitarity and $\|UV - I\|_{\text{op}} \leq \|U - I\|_{\text{op}} + \|V - I\|_{\text{op}}$ bound the outer boundary error by the sum of these cell bounds, without a factor

depending on the number of cells. The two constant sides of the outer square contribute identity transports. Self-intersections of the image are harmless because swept area is counted with multiplicity. For a comparison homotopy, $H(0, v) = H(1, v)$ is the base point of ℓ , and $H(\cdot, 1)$ is the concatenation $c * \ell' * c^{-1}$. A fixed character is Lipschitz on the compact group, and $\|F_A\|_\infty \leq C(R + R^2)$ if $\|A\|_{C^1} \leq R$.

(2) Let P be the exact transport along the edge from x to $x + ae_i$, that is, $P'(v) = P(v)A_i(x + ve_i)$, $P(0) = I$. By Duhamel's formula and unitarity,

$$\|P(a) - e^{aA_i(x)}\|_{\text{op}} \leq C \int_0^a |A_i(x + ve_i) - A_i(x)| dv \leq C_R a^2.$$

Reversed edges satisfy the same bound. A closed lattice path of physical length at most l has at most l/a edges, and by telescoping the product of unitary matrices we obtain the bound $C_R la$. The character bound follows since χ is Lipschitz on $\text{SU}(2)$.

(3) Since $\delta_K < L/2$, the shortest displacement from $\ell(u)$ to $\ell_K(u)$ is unique and has a unique continuous lift $d_K(u) \in \mathbb{R}^3$ with $|d_K(u)| \leq \delta_K$. Let q be a lift of ℓ , and set $q_K = q + d_K$, a lift of ℓ_K . Since $\ell(1) = \ell(0)$ and $\ell_K(1) = \ell_K(0)$, we have $d_K(1) = d_K(0)$. The two lifts therefore have the same displacement $q(1) - q(0)$ and the loops have the same winding class. The ruled homotopy $G_K(u, v) = q(u) + vd_K(u)$ descends to the torus. Differentiating piecewise, we find

$$|\partial_v G_K| \leq \delta_K, \quad \partial_u G_K = (1 - v)q'(u) + vq'_K(u),$$

so $\text{Area}(G_K) \leq \delta_K \int_0^1 \int_0^1 ((1 - v)|q'| + v|q'_K|) du dv$, which is the right side of (29.5). The endpoints $G_K(0, v) = G_K(1, v)$ move along the segment $c_K(v) = q(0) + vd_K(0)$ from $\ell(0)$ to $\ell_K(0)$. The homotopy $v \mapsto c_K|_{[0, v]} * G_K(\cdot, v) * (c_K|_{[0, v]})^{-1}$, suitably parametrized, is a comparison homotopy between ℓ and ℓ_K with connector c_K ; the added retracing parts have differentials of rank at most one and contribute no swept area. This proves (29.5). The finitely many paths with $\delta_K \geq L/2$ may have another winding class; this does not affect convergence statements. $\square$

Remark 29.12. Uniform convergence of the paths alone does not suffice for convergence of holonomies; the bound on the lengths in Definition 2.1 is needed. Consider a smooth periodic connection that agrees in a coordinate ball with $A = bx_1\tau_3 dx_2$, where $b \notin 4\pi\mathbb{Z}$, and the loops that traverse a square of side δ based at the origin $m_\delta = \lfloor \delta^{-2} \rfloor$ times. These loops converge uniformly to the constant loop, but their lengths $4m_\delta\delta$ tend to infinity, and their holonomies $\exp(bm_\delta\delta^2\tau_3)$ converge to $\exp(b\tau_3) \neq I$. Taking the sides of the squares to be integer multiples of the mesh gives the same example with lattice loops. The length and area conditions of Definition 2.1 exclude it.

Lattice loop observables

Definition 29.13. Let $N \geq 4$, let ℓ be a piecewise C^1 loop, let χ be a character of a finite-dimensional representation of $\text{SU}(2)$, let $t > 0$, and let ℓ_N be a closed lattice path in

Λ_N . Set

$$W_{N,\ell,\chi,t}^{\text{lat}}(U) = \chi(\text{Hol}_{\ell_N}(\Psi_t^{(N)}U)), \quad U \in \mathcal{U}_N, \quad (29.7)$$

where the dependence on ℓ_N is suppressed. For the loops and times (ℓ_j, t_j) of Theorem 29.3, let ℓ_{j,a_N} be the lattice paths of Lemma 29.10, fixed once and for all j and N , and set

$$W_{N,j}^{\text{lat}}(U) = \frac{1}{2} \text{tr} \text{Hol}_{\ell_{j,a_N}}(\Psi_{t_j}^{(N)}U), \quad U \in \mathcal{U}_N. \quad (29.8)$$

On a sequence of sides $(N_K)_K$ we write $W_{K,\ell,\chi,t}^{\text{lat}}$ and $W_{K,j}^{\text{lat}}$ for these functions on $\mathcal{U}_{N_K}$.

The paths ℓ_{j,a_N} depend only on the loop and the mesh, and not on the configuration, on any parameter of the renormalization group, or on the coarse comparison side used below. By Lemma 29.10, along every sequence $N_K \rightarrow \infty$ they form admissible approximations of the loops ℓ_j , with swept areas $O(a_{N_K})$.

The Wilson flow and the holonomies of closed lattice paths are continuous and covariant under lattice gauge transformations, and a character removes the conjugation at the base point. The functions (29.7) and (29.8) are therefore continuous, in particular Borel, and invariant under all lattice gauge transformations on the whole configuration space. By unitarity, $|W_{N,\ell,\chi,t}^{\text{lat}}| \leq \chi(I)$, and the functions $W_{N,j}^{\text{lat}}$ are real with absolute value at most one. In particular,

$$|w_j(X) - W_{N,j}^{\text{lat}}(U)| \leq 2 \quad \text{for all } X \in \mathcal{X}_L, U \in \mathcal{U}_N. \quad (29.9)$$

The results of Section 28 allow characters of arbitrary finite-dimensional representations, including the unnormalized fundamental character. Whenever we apply Theorems 28.1 and 28.2 to the functions $W_{N,j}^{\text{lat}}$, we apply them to the vector of fundamental characters and multiply its entries by $1/2$. This continuous linear map preserves joint convergence in law, and for complex source functionals it replaces the sources z_i by $z_i/2$. The limits are the corresponding joint laws of the determining coordinates w_j .

The lattice energy

Recall the physical lattice energy $E_a(U) = (8/a) \sum_{p \in \mathcal{P}_N} q(U_p)$ of (2.2), $a = L/N$, in which each positive plaquette is counted once. In terms of the Wilson action S_W of Definition 4.1, which equals E_0 , we have $E_a = 2S_W/a$. For a lattice potential A at mesh a we write e^{aA} for the configuration with $(e^{aA})_{(x,x+ae_i)} = \exp(aA_i(x))$, and for a smooth one-form b on $\mathbb{T}_L^3$ we write $S_a b$ for its restriction to the lattice, $(S_a b)_i(x) = b_i(x)$.

Lemma 29.14. *Let $L > 0$ and $a = L/N$ with $N \geq 4$.*

1. *The function E_a is continuous on $\mathcal{U}_N$, invariant under lattice gauge transformations, and nonincreasing along the Wilson flow $\Psi_t^{(N)} = \Phi_{4t/a^2}^W$.*

2. For every $R > 0$ there exists $C_{L,R} < \infty$, independent of N , such that for lattice potentials A, B at mesh a with $\|A\|_{C_a^1}, \|B\|_{C_a^1} \leq R$,

$$|\mathbb{E}_a(e^{aA}) - \mathbb{E}_a(e^{aB})| \leq C_{L,R} \|A - B\|_{C_a^1}, \quad 0 \leq \mathbb{E}_a(e^{aA}) \leq C_{L,R}.$$

3. For every smooth periodic one-form b we have $\mathbb{E}_a(e^{aS_a b}) \rightarrow E_L(b)$ as $a \rightarrow 0$. More precisely, if $\|b\|_{C^2} \leq R$ and a is sufficiently small, then $|\mathbb{E}_a(e^{aS_a b}) - E_L(b)| \leq C_{L,R} a$.
4. Let $\mathcal{E}_a(U) = (4/a) \sum_{p \in \mathcal{P}_N} q(U_p)$ be the normalization used in Section 24. Then $\mathbb{E}_a = 2\mathcal{E}_a$ on $\mathcal{U}_N$.

The factor $8/a$ is chosen so that the limit in (3) is the energy E_L of (2.3), without a factor $1/2$; the exponent in (2.1) equals $\mathbb{E}_{a_N}/(2g_3^2)$. The point of (2) is that the Lipschitz constant does not depend on the mesh: a plaquette logarithm is $O(a^2)$, so a plaquette energy and its variation are $O(a^4)$, which compensates the factor a^{-1} and the $3(L/a)^3$ plaquettes.

Proof. (1) Continuity follows from the finite products and the trace. A lattice gauge transformation conjugates each plaquette matrix, which leaves $q(U_p)$ unchanged. Since $\mathbb{E}_a$ is a positive multiple of the Wilson action, it is nonincreasing along the negative gradient flow Φ^W , and the positive change of time $s = 4t/a^2$ does not affect this.

(2) We consider one plaquette $p_{ij}(x)$ and set

$$v = (aA_i(x), aA_j(x + ae_i), -aA_i(x + ae_j), -aA_j(x)).$$

For aR small, the expansion of the logarithm of a product of four exponentials in Lemma 24.2 reads

$$f_A = \text{Log } U_{p_{ij}(x)} = \sum_{k=1}^4 v_k + \frac{1}{2} \sum_{k < l} [v_k, v_l] + \mathcal{R}(v), \quad |\mathcal{R}(v)| \leq C|v|^3, \quad \|D\mathcal{R}(v)\| \leq C|v|^2.$$

The bound on $D\mathcal{R}$ follows from Taylor's integral formula for the analytic logarithm of a word of fixed length: the remainder has vanishing Taylor coefficients up to order two, and its third derivative is bounded on a smaller fixed chart. The linear sum equals $a^2(D_i^+ A_j - D_j^+ A_i)(x)$. If we freeze the shifted entries at x , the quadratic sum equals $a^2[A_i(x), A_j(x)]$, and changing one entry back costs at most $C_R a^3$. We conclude that

$$a^{-2} f_A = D_i^+ A_j - D_j^+ A_i + [A_i, A_j] + O_R(a), \quad |f_A| \leq C_R a^2. \quad (29.10)$$

Let $\delta = \|A - B\|_{C_a^1}$. By the exact cancellation in the linear term, we have $|\sum v(A) - \sum v(B)| \leq C a^2 \delta$, the difference of the quadratic terms is at most $C_R a^2 \delta$, and the difference of the remainders is at most $C_R a^3 \delta$. Thus $|f_A - f_B| \leq C_R a^2 \delta$. By (4.2), $q(e^f) = |f|^2/8 + O(|f|^4)$, and the differential of $f \mapsto q(e^f)$ has norm at most $C|f|$ on this chart. Hence

the plaquette energies differ by at most $C_R a^4 \delta$. There are $3(L/a)^3$ positive plaquettes, and multiplication by $8/a$ gives the bound $24L^3 C_R \delta$, with no remaining power of the mesh. This proves the Lipschitz bound for $a \leq a_0(L, R)$. Meshes $a > a_0$ correspond to finitely many sides $N < L/a_0$; for them, the differential of the unitary exponential and telescoping give the plaquette bound $Ca\|A - B\|_\infty$, which yields a Lipschitz constant $24CN^3 \leq 24C(L/a_0)^3$. We enlarge $C_{L,R}$ to cover these meshes. Since $E_a(I) = 0$, taking $B = 0$, we obtain $E_a(e^{aA}) \leq C_{L,R}R$, and we absorb R into the constant. Nonnegativity follows from $q \geq 0$ on $SU(2)$.

(3) For $A = S_a b$, forward differences differ from derivatives by $O_R(a)$ on the ball $\|b\|_{C^2} \leq R$. By (29.10) and the expansion of q ,

$$E_a(e^{aS_a b}) = a^3 \sum_{x \in \Lambda_N} \sum_{i < j} |F_{ij}(b)(x)|^2 + O_{L,R}(a).$$

Since the summand has bounded first derivatives on this ball, replacing the Riemann sum by the integral costs a further $O_{L,R}(a)$. Finally, since $|X|^2 = -2 \operatorname{tr} X^2$ and $F_{ji} = -F_{ij}$, we have $\sum_{i,j} -\operatorname{tr} F_{ij}^2 = \sum_{i < j} |F_{ij}|^2$, and the limit is exactly $E_L(b)$.

(4) This is immediate from the definitions. $\square$

Stability along the Wilson flow

We recall the setting of Sections 24 and 25. Fix $1/2 < \alpha < 2/3$ and a horizon $T_0 > 0$. For a lattice potential A_0 at mesh a , its linear and quadratic flow data are $x(t) = P_t^a A_0$ and $y = Q_a(x, x)$. We measure them by the flow-data norm $X_\alpha^a(x) + Y_\alpha^a(y)$, with the norms of Definition 4.8 on $[0, T_0]$, including all physical first differences and shifts. For two potentials A_0, A'_0 in the same gauge, with flow data (x, y) and (x', y') , we write

$$d_a = X_\alpha^a(x - x') + Y_\alpha^a(y - y').$$

On the smaller link coordinate domain used in Lemma 24.2, the Wilson flow has damped representatives: they are obtained by a time-dependent lattice gauge transformation and solve the gauge-fixed lattice equation of that lemma, whose linear part is the lattice heat equation. A finite reference family for a smooth solution of the Yang–Mills heat flow on a time interval $[s, T]$ consists of a finite partition of $[s, T]$, smooth solutions of the DeTurck equation on the subintervals, and fixed smooth gauge transformations at the partition times, as provided by Theorem 24.10. Each of these DeTurck solutions is a reference solution.

The first lemma transfers a small error at a positive time s to all later times up to a fixed horizon. It follows directly from the comparison with a fixed smooth reference in Lemma 25.5.

Lemma 29.15. *Let $0 < s < T$, and let b be a smooth solution of the Yang–Mills heat flow on $[s, T]$ with a finite reference family, consisting of DeTurck solutions $b^{(i)}$ on intervals I_i . For lattice meshes $a \rightarrow 0$, let V_a be configurations with representatives $V_a = e^{aA_a(s)}$ such that*

$$e_a = \|A_a(s) - S_a b(s)\|_\infty + \|D_a A_a(s) - S_a \partial b(s)\|_\infty \longrightarrow 0,$$

where D_a and ∂ denote all first differences and the corresponding derivatives. Then for all sufficiently small a , the Wilson flow $t \mapsto \Psi_{t-s}^{(N)} V_a$, $t \in [s, T]$, has representatives $A_a^{(i)}$ solving that gauge-fixed lattice equation on the intervals I_i , in the gauges of the reference family, such that

$$\max_i \sup_{t \in I_i} \|A_a^{(i)}(t) - S_a b^{(i)}(t)\|_{C_a^1} \leq C(e_a + a).$$

The constant C and the smallness threshold depend only on L , the horizon $T - s$, the smooth bounds of the reference family, its gauge transformations, and its chart margins. They do not depend on the mesh, the coupling, or any probability measure, and the convergence $e_a \rightarrow 0$ may be arbitrarily slow.

Proof. We translate the initial time to zero, so that the horizon is $T - s$. By Taylor's formula in our difference conventions, $\|D_a S_a b(s) - S_a \partial b(s)\|_\infty \leq Ca \|b(s)\|_{C^2}$, so

$$\eta_a = \|A_a(s) - S_a b(s)\|_{C_a^1} \leq e_a + Ca \|b(s)\|_{C^2}.$$

Every reference solution of the family is smooth on its closed space-time domain, and in particular its spatial derivatives up to order five are bounded. The finitely many reference solutions and fixed smooth gauge transformations have common bounds, which are part of the data of the reference family and not bounds on higher derivatives of nearby lattice fields. The fixed-reference and finite-family parts of Lemma 25.5 choose the constants and the thresholds for the mesh and the initial error from these bounds, before the incoming error is made small. At the i th partition time, using (25.14) together with the change of gauge, we obtain a recursion

$$\eta_{i+1} \leq G_i C_i (\eta_i + a) + d_i a$$

for the error η_i at the start of the i th interval, with finitely many constants G_i, C_i, d_i depending only on the reference family. Iterating it finitely many times, and taking the minimum of the resulting thresholds, we obtain

$$\max_i \sup_{t \in I_i} \|A_a^{(i)}(t) - S_a b^{(i)}(t)\|_{C_a^1} \leq C(\eta_a + a) \leq C'(e_a + a).$$

The same lemma keeps the representatives inside the coordinate domain of Lemma 24.2, with the required positive distance from its boundary. It also identifies the solution of the gauge-fixed lattice equation on each interval with the Wilson flow, whose time normalization $4t/a^2$ is that of Lemma 24.2. Since $e_a \rightarrow 0$ and $a \rightarrow 0$, the thresholds are eventually satisfied. $\square$

Theorem 29.16. Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, $0 < \tau \leq T < \infty$, and u as in Lemma 25.9, together with finitely many characters $\chi_1, \dots, \chi_b$ and lengths $\ell_1, \dots, \ell_b > 0$. There exist $a_* > 0$ and $C < \infty$, depending only on these data, with the following property. Let $a = L/N \leq a_*$, and let U, U' be the configurations of two lattice potentials at mesh a in the same gauge, whose flow data have norms at most R on $[0, T_0]$ and discrepancy d_a . Then

$$\sup_{\tau \leq t \leq T} \left(|E_a(\Psi_t^{(N)} U) - E_a(\Psi_t^{(N)} U')| + \sum_{i=1}^b |\chi_i(\text{Hol}_{\ell_{i,a}} \Psi_t^{(N)} U) - \chi_i(\text{Hol}_{\ell_{i,a}} \Psi_t^{(N)} U')| \right) \leq C d_a \quad (29.11)$$

for all closed lattice paths $\ell_{i,a}$ in Λ_N of physical length at most ℓ_i , and

$$\sup_{\tau \leq t \leq T} E_a(\Psi_t^{(N)} U) \leq C.$$

The constants do not depend on the coupling, on any probability measure, or on the number of averaging steps. The estimate requires no convergence of the configurations as $a \rightarrow 0$, no condition on swept areas, and no bound on higher derivatives, and it has no additive error term in the mesh. This allows us to apply it to finite-scale reconstructions with a growing number of scales.

Proof. Let δ, K be the constants of Lemma 25.9. Suppose first that $d_a \leq \delta$. By that lemma, the two Wilson flows have damped representatives B, B' , solving the gauge-fixed lattice equation on the intervals of a common reference family. They satisfy the individual bounds $\|B(t)\|_{C_a^1}, \|B'(t)\|_{C_a^1} \leq K$ and the difference bound $\|B(t) - B'(t)\|_{C_a^1} \leq C d_a$. By gauge invariance and Lemma 29.14(2), the energies differ by at most $C_{L,K} C d_a$, and each energy is bounded on every interval. Explicitly, a plaquette logarithm is $O_K(a^2)$, the difference of two of them is $O_K(a^2 \|B - B'\|_{C_a^1})$, and, since the differential of $f \mapsto q(e^f)$ is $O(|f|)$, the difference of two plaquette energies is $O_K(a^4 \|B - B'\|_{C_a^1})$. The normalization cancels as

$$\frac{8}{a} \cdot 3 \left(\frac{L}{a} \right)^3 C_K a^4 \|B - B'\|_{C_a^1} = 24 L^3 C_K \|B - B'\|_{C_a^1}.$$

This is the energy E_a , not $\mathcal{E}_a$. For the loops we apply Corollary 25.11 at each $t \in [\tau, T]$; its constant is uniform over the times in $[\tau, T]$ and over all paths within the length bounds. The observables are continuous in t , which joins the finitely many intervals. If $d_a > \delta$, we use the individual energy bound and $|\chi_i| \leq \chi_i(I)$: the left side of (29.11) is at most $2C + 2 \sum_i \chi_i(I) \leq (2C + 2 \sum_i \chi_i(I)) d_a / \delta$, and we enlarge C accordingly. All constants were fixed before the two configurations were chosen. $\square$

The last deterministic statement concerns a fixed compact family of smooth fields: the cardinal interpolants, in the sense of Lemma 16.3, of arrays on a fixed coarse mesh. Here the constants may depend on all derivatives of the family, unlike those of Theorem 29.16.

Lemma 29.17. *Fix a coarse mesh $h > 0$, a bound $M_h < \infty$, a finite horizon, and the cardinal basis, and let f range over the cardinal interpolants of arrays on the lattice of mesh h with supremum at most M_h , as in Lemma 28.4. Let $0 < \tau < T < \infty$, and let $\ell_1, \dots, \ell_b$ be loops with characters $\chi_1, \dots, \chi_b$ and time-independent admissible approximations $(\ell_{i,a})$, with comparison homotopies of swept area $\mathcal{A}_{i,a} \rightarrow 0$. Then, as $a \rightarrow 0$ along meshes compatible with h , the paths*

$$t \mapsto \left(E_a(\Psi_t^{(N)} e^{af|_a}), (\chi_i(\text{Hol}_{\ell_{i,a}} \Psi_t^{(N)} e^{af|_a}))_{i \leq b} \right), \quad t \in [\tau, T],$$

converge uniformly in $t \in [\tau, T]$, and uniformly over the family, to the paths $t \mapsto (E_L(\Phi_t[f]), (\chi_i(\text{Hol}_{\ell_i} \Phi_t[f]))_{i \leq b})$. More precisely, the uniform distance between these paths is at most

$$C_{h, M_h, \tau, T, \mathcal{W}} \left(a + \max_i \mathcal{A}_{i,a} \right), \quad (29.12)$$

where $\mathcal{W}$ denotes the list of loops, characters, and lengths. The limit map from arrays to $C([\tau, T]; \mathbb{R} \times \mathbb{C}^b)$ is continuous.

Proof. We use the finite reference families and the consistency of the lattice flow with the sampled smooth flow in the proof of Lemma 28.4. On each reference interval the C_a^1 distance between the damped representative and the sampled reference solution tends to zero uniformly over the family. The Lipschitz bound and the consistency for smooth samples in Lemma 29.14 bound the error in the energy by $C\|A_a - S_a b\|_{C_a^1} + Ca$. The reference bounds are uniform over this compact family and its finitely many reference families. The error in the loops is at most $C(a + \mathcal{A}_{i,a})$, by Lemma 29.11(1),(2) (as in the proof of Lemma 28.4), plus the propagated error in the potential. All these bounds are uniform in time, which gives (29.12). The continuity of the limit map follows from the smooth dependence of the flow on its initial datum and the continuity of the energy in C^1 . $\square$

29.3 Joint convergence under the conditioned measures

Standing assumptions. In Sections 29.3 to 29.5 we fix $L > 0$ and an odd integer $M \geq 3$, together with the integer J , the remaining parameters Θ of the small-field event, and the coupling bound $\gamma_0 \in (0, 1]$ of Theorem 23.2, for which Theorem 23.9 holds. We fix $\gamma \in (0, \gamma_0]$ and $n_* \in 4\mathbb{N}$, and set $B = M^J$, $N_K = n_* B^K$, $a_K = L/N_K$, and $n_r = n_* B^r$. For $0 \leq r \leq K$ let

$$v_{K|r} = v_{N_K, n_r, \gamma} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, n_r, \gamma}), \quad e_{K|r} = \mu_{N_K, \gamma}(S_{N_K, n_r, \gamma}^c)$$

be the conditioned Wilson measure with coarse side n_r and the corresponding small-field failure probability. Let $\mathcal{R}$ be the set of indices r for which Theorem 28.8, and hence

Theorems 28.1 and 28.2, apply at the coarse side n_r ; this set is infinite. By Theorem 23.9, as recorded in Theorem 28.2, for $r \in \mathcal{R}$ we have

$$e_{K|r} \leq \epsilon_K + F_\gamma(n_r), \quad \epsilon_K \rightarrow 0 \ (K \rightarrow \infty), \quad F_\gamma(n_r) \rightarrow 0 \ (r \rightarrow \infty), \quad (29.13)$$

and one may take $\epsilon_K = 0$ and $F_\gamma(u) = C'u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}$.

The conditioning is removed by the following elementary identity. If $e_{K|r} < 1$, then $\mu_{N_K, \gamma} = (1 - e_{K|r})\nu_{K|r} + e_{K|r}\rho_{K|r}$ with $\rho_{K|r} = \mu_{N_K, \gamma}(\cdot | S_{N_K, n_r, \gamma}^c)$ (any probability measure if $e_{K|r} = 0$). In particular,

$$\mu_{N_K, \gamma}(\mathcal{B}) \leq \nu_{K|r}(\mathcal{B}) + e_{K|r} \quad \text{for every Borel event } \mathcal{B}, \quad (29.14)$$

and $|\mathbb{E}_{\mu_{N_K, \gamma}} F - \mathbb{E}_{\nu_{K|r}} F| \leq D e_{K|r}$ for every measurable F with $0 \leq F \leq D$, and $\leq 2e_{K|r} \|F\|_\infty$ for every bounded complex measurable F (Lemma 5.5). We always use these bounds in the following order: first $K \rightarrow \infty$ at a fixed $r \in \mathcal{R}$, and then $r \rightarrow \infty$ in $\mathcal{R}$.

Finite-scale reconstructions. Fix $r \in \mathcal{R}$ and write $n = n_r$ and $\nu_K = \nu_{nBK, n, \gamma}$, so that $\nu_K = \nu_{K+r|r}$. We recall the following objects from Theorem 28.8, with input horizon $T_0 = L^2$; they are all defined on the joint realization of Remark 15.11 and Theorem 28.8, whose fine-configuration marginal is ν_K . The reconstruction coupled to U with the reconstruction maps of Theorem 15.13 gives arrays A_j , $0 \leq j \leq K$, on the lattices of mesh $a_j = (L/n)B^{-j}$, and the configuration $e^{a_K A_K}$ is a gauge representative of U . For $0 \leq q \leq K$, the finite-scale reconstruction from the first q renormalization group steps from the coarse side is the fine configuration $U_K^{[q]} = \exp(a_K I_{a_q \rightarrow a_K} A_q)$, where $I_{a_q \rightarrow a_K}$ is the cardinal interpolation of Lemma 16.3. Its flow data at mesh a_K are $x_q = P^{a_K} I_{a_q \rightarrow a_K} A_q$ and $y_q = \mathcal{Q}_{a_K}(x_q, x_q)$, and those of U are x_K, y_K . We set

$$R_K = X_\alpha^{a_K}(x_K) + Y_\alpha^{a_K}(y_K), \quad \rho(R) = \sup_{K \geq 0} \nu_K \{R_K > R\},$$

$$D_{K,q} = X_\alpha^{a_K}(x_K - x_q) + Y_\alpha^{a_K}(y_K - y_q).$$

By Theorem 28.8, we have $\rho(R) \rightarrow 0$ as $R \rightarrow \infty$ and

$$\lim_{q \rightarrow \infty} \sup_{K \geq q} \nu_K \{D_{K,q} > \eta\} = 0 \quad \text{for every } \eta > 0. \quad (29.15)$$

For each fixed q , there exists a compact set $\mathcal{C}_q$ of arrays, independent of $K \geq q$, that supports the law of A_q under every ν_K . These laws converge weakly as $K \rightarrow \infty$ to a law supported on $\mathcal{C}_q$ (Theorem 15.14). Let f_q be the cardinal interpolant of A_q on the torus, a one-form, and let $\lambda_q = \lambda_{r,q}$ be the law of the trajectory $t \mapsto \Phi_t[f_q]$ when A_q has this limiting law.

Theorem 29.18. *Assume the standing assumptions, and let $r \in \mathcal{R}$. There exists a Borel probability measure λ_r on $\mathcal{X}_L$, written $\lambda_{r,\gamma}$ when the coupling is displayed, with the following properties.*

1. As $q \rightarrow \infty$, every finite vector of energies $E_L(X(v))$, $v > 0$, and functions $W_{\ell, \chi, t}$ under $X \sim \lambda_q$ converges in law to the same vector under $X \sim \lambda_r$, and $(i^{1,2})_{\#} \lambda_q \rightarrow (i^{1,2})_{\#} \lambda_r$ against bounded $\mathcal{T}_{1,2}$ -continuous functions.
2. Let $v_1, \dots, v_p > 0$. For $h \leq b$ let ℓ_h be loops with admissible approximations $(\ell_{h,K})_K$ along $(N_K)_K$, let χ_h be characters, let $t_h > 0$, and write $W_{K,h}^{\text{lat}} = \chi_h(\text{Hol}_{\ell_{h,K}} \Psi_{t_h}^{(N_K)} U)$. Then, under $\nu_{K|r}$, as $K \rightarrow \infty$,

$$\left((E_{a_K}(\Psi_{v_i}^{(N_K)} U))_{i \leq p}, (W_{K,h}^{\text{lat}})_{h \leq b} \right) \Longrightarrow \left((E_L(X(v_i)))_{i \leq p}, (W_{\ell_h, \chi_h, t_h}(X))_{h \leq b} \right),$$

where $X \sim \lambda_r$.

3. Suppose that the paths $\ell_{h,K}$ do not depend on the time, and let $0 < \tau < T < \infty$. Then, under $\nu_{K|r}$, the processes

$$t \mapsto \left(E_{a_K}(\Psi_t^{(N_K)} U), (\chi_h(\text{Hol}_{\ell_{h,K}} \Psi_t^{(N_K)} U))_{h \leq b} \right), \quad t \in [\tau, T],$$

converge in law in $C([\tau, T]; \mathbb{R} \times \mathbb{C}^b)$ to $t \mapsto (E_L(X(t)), (W_{\ell_h, \chi_h, t}(X))_{h \leq b})$ with $X \sim \lambda_r$. If $(\ell'_{h,K})_K$ are other time-independent admissible approximations of the same loops, then

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\nu_{K|r}} \sup_{\tau \leq t \leq T} \max_{h \leq b} |\chi_h(\text{Hol}_{\ell_{h,K}} \Psi_t^{(N_K)} U) - \chi_h(\text{Hol}_{\ell'_{h,K}} \Psi_t^{(N_K)} U)| = 0.$$

4. For admissible approximations with swept areas $O(a_K)$, in particular those of Definition 29.13, the limit laws of the loop vectors in (2) are those of Theorem 28.1.
5. For every $t > 0$,

$$\lim_{A \rightarrow \infty} \sup_{K \geq r} \nu_{K|r} \{E_{a_K}(\Psi_t^{(N_K)} U) > A\} = 0, \quad \lim_{A \rightarrow \infty} \sup_{K \geq 0} \mu_{N_K, \gamma} \{E_{a_K}(\Psi_t^{(N_K)} U) > A\} = 0.$$

All convergence moduli are at the fixed coupling γ and, for the conditioned measures, at the fixed index r .

Proof. We fix $r \in \mathcal{R}$ and use the reindexed measures ν_K and the notation introduced above; the statements for $\nu_{K|r}$ follow by the shift $K \mapsto K + r$. We first construct the laws λ_q and show that the lattice observables of U are close in law to those of the finite-scale reconstructions, whose limits are continuum observables under λ_q ; this produces joint limits of the lattice observables. We then prove that the energies under λ_q are tight, extract a limit λ_r using the compactness theorem of Cao and Chatterjee for Sobolev trajectories, identify λ_r by its Wilson loops, and use the continuity of energies after a positive time (Lemma 29.7) to identify the joint limits of energies and loops.

Step 1: the laws λ_q . The map from arrays A_q to the initial datum f_q is linear and continuous in every smooth norm, and by Theorem 24.10 the smooth flow depends continuously on its initial datum on every finite horizon. Hence every determining coordinate w_j of the trajectory $t \mapsto \Phi_t[f_q]$ is a continuous function of the array. Since the Borel σ -algebra of $\mathcal{X}_L$ is generated by the w_j (Theorem 29.3), the map from arrays to trajectories is measurable, and λ_q is a Borel probability measure on $\mathcal{X}_L$. Its image $(i^{1,2})_\# \lambda_q$ is a Borel probability measure on $\mathcal{X}_L^{1,2}$, by Step 4 of the proof of Theorem 29.3.

Step 2: approximation by finite-scale reconstructions. Fix a finite list as in (2), respectively a time-independent list and an interval $[\tau, T]$ as in (3). Let Z_K be the corresponding finite vector, respectively path, of observables of U , with values in the complete separable metric space $E = \mathbb{R}^p \times \mathbb{C}^b$, respectively $E = C([\tau, T]; \mathbb{R} \times \mathbb{C}^b)$, which we equip with the bounded metric $d = \min(2, \|\cdot\|_\infty)$. Let $Z_K^{[q]}$ be the same observables of the finite-scale reconstruction $U_K^{[q]}$, on that same joint realization. Since $e^{a_K A_K}$ is gauge equivalent to U there, their joint gauge-invariant observables agree sample by sample, and their law is the required law under the U -marginal ν_K .

On the event $\{R_K \leq R, D_{K,q} \leq \eta\}$ with $\eta \leq 1$, both flow data have norm at most $R + 1$, and the two potentials are in the same gauge. By Theorem 29.16 at radius $R + 1$, applied on an interval containing all times of the list, we have $d(Z_K, Z_K^{[q]}) \leq C_R \eta$ whenever $a_K \leq a_*(R + 1)$. We conclude that, for every test function ϕ on E bounded by one and with Lipschitz constant at most one,

$$|\mathbb{E}_{\nu_K} \phi(Z_K) - \mathbb{E}_{\nu_K} \phi(Z_K^{[q]})| \leq C_R \eta + 2\rho(R) + 2\nu_K\{D_{K,q} > \eta\} \quad (29.16)$$

for all K with $a_K \leq a_*(R + 1)$. This threshold does not depend on q . By (29.15), the right side of (29.16) tends to zero, uniformly in $K \geq q$, if we choose first R large, then η small, and then q large, as in the proof of Theorem 28.1.

Fix q , and let G be the map that sends a trajectory to the continuum energies and Wilson loops of the list. By Lemma 29.17, applied with the coarse mesh a_q and the compact set $\mathcal{C}_q$, the observables of $U_K^{[q]}$ converge as $K \rightarrow \infty$, uniformly for $A_q \in \mathcal{C}_q$, to the continuous function $A_q \mapsto G(t \mapsto \Phi_t[f_q])$. Combining this with the weak convergence of the laws of A_q , which are all supported on $\mathcal{C}_q$, we obtain $Z_K^{[q]} \Rightarrow G_\# \lambda_q$. The observable maps converge uniformly on $\mathcal{C}_q$ to a continuous map, and each fixed- K map is continuous on $\mathcal{C}_q$. By uniform convergence, compactness of $\mathcal{C}_q$, and completeness of E , the union of their images has compact closure, and so does the union of the supports of the laws of $Z_K^{[q]}$, $K \geq q$. Each Z_K has compact support, since the compact configuration space is mapped continuously into E by the finite-dimensional Wilson flow. By Lemma 28.9, a version of the converging-together principle [28, Theorem 3.2] that also constructs the limit, the laws of Z_K converge to a probability measure ρ_G on E . Letting $K \rightarrow \infty$ at fixed q in (29.16), we find that $G_\# \lambda_q \rightarrow \rho_G$ as $q \rightarrow \infty$. The argument applies to every finite list,

and in particular to the union of any two lists. The limits are therefore consistent under unions and projections, and we never combine separately obtained marginal limits.

Step 3: tightness of the energies under λ_q . Let $t > 0$ and $\varepsilon > 0$. Fix R with $\rho(R) < \varepsilon$. By (29.15), for all sufficiently large q we have $\sup_{K \geq q} \nu_K\{D_{K,q} > 1\} < \varepsilon$, and we may also assume $a_q \leq a_*(R+1)$, so that $a_K \leq a_*(R+1)$ for all $K \geq q$. On the event $\{R_K \leq R, D_{K,q} \leq 1\}$ the flow data of $U_K^{[q]}$ have norm at most $R+1$, and by the individual bound in Theorem 29.16 we have $E_{a_K}(\Psi_t^{(nB^K)} U_K^{[q]}) \leq C_{R,t}$ for a deterministic $C_{R,t}$. Let ϕ be continuous, with $\phi = 0$ on $(-\infty, C_{R,t}]$, $\phi = 1$ on $[C_{R,t} + 1, \infty)$, and $0 \leq \phi \leq 1$. Then $\mathbb{E}_{\nu_K} \phi(E_{a_K}(\Psi_t^{(nB^K)} U_K^{[q]})) \leq 2\varepsilon$ for $K \geq q$. Letting $K \rightarrow \infty$, by Lemma 29.17 and the convergence of the law of A_q , we obtain

$$\lambda_q\{E_L(X(t)) > C_{R,t} + 1\} \leq \mathbb{E}_{\lambda_q} \phi(E_L(X(t))) \leq 2\varepsilon.$$

The finitely many smaller q have laws supported on compact sets of smooth initial data, and their energies at time t are bounded. We conclude that the energies at time t are tight under $\{\lambda_q\}_{q \geq 0}$, for every $t > 0$.

Step 4: the limit λ_r . The compactness theorem for Sobolev trajectories [43, Theorem 3.73, pp. 41–42] applies to Borel $\mathcal{X}_L^{1,2}$ -valued random variables whose energy laws at each positive time are tight, which we have just verified for $(i^{1,2})_{\#} \lambda_q$. For fixed L we use the rescaling of Lemma 29.2. We obtain a subsequence along which $(i^{1,2})_{\#} \lambda_q$ converges, against bounded $\mathcal{T}_{1,2}$ -continuous functions, to a Borel probability measure η on $\mathcal{X}_L^{1,2}$. We set $\lambda_r = (j^{1,2})_{\#} \eta$, a Borel probability measure on $\mathcal{X}_L$. By Lemma 29.2(6),(7), $j^{1,2}$ preserves the functions $W_{\ell, \chi, t}$, and these are continuous on $\mathcal{X}_L^{1,2}$. Hence the finite joint laws of the functions $W_{\ell, \chi, t}$ under λ_r are the limits of those under λ_q along the subsequence. By Step 2, these are the marginals of the limits ρ_G , so they do not depend on the subsequence. By Theorem 29.3(4), λ_r does not depend on the subsequence. Since $i^{1,2} j^{1,2} = \text{Id}$, we have $\eta = (i^{1,2})_{\#} \lambda_r$, so the limit η is unique. The whole sequence $(i^{1,2})_{\#} \lambda_q$ therefore converges to $(i^{1,2})_{\#} \lambda_r$: otherwise a subsequence would stay away from it, and a further subsequence would converge to $(i^{1,2})_{\#} \lambda_r$, a contradiction.

Step 5: identification of the energies. By Lemma 29.7, for every $v > 0$ the function $Z \mapsto E_L(Z(v))$ is continuous on $(\mathcal{X}_L^{1,2}, \mathcal{T}_{1,2})$; one factors the evaluation at time v through an earlier time $s \in (0, v)$. The functions $W_{\ell, \chi, t} \circ j^{1,2}$ are continuous as well. Applying bounded continuous functions to the whole vector, we see from Step 4 that every finite vector of energies and Wilson loops under λ_q converges in law to the same vector under λ_r . This proves (1). By Step 2 applied to finite vectors, the same vectors under λ_q converge to the limit law ρ_G of the lattice observables. We conclude that ρ_G is the law of the continuum vector under λ_r , which proves (2). This step uses the continuity of the

energy after a positive time; measurability of the energy, or convergence of the energy marginals separately, would not suffice.

Step 6: paths and comparison of approximations. In the setting of (3), the limit ρ_G is a probability measure on continuous paths, and by Step 5 its finite-dimensional distributions at rational times coincide with those of $G_\# \lambda_r$. Evaluations at rational times generate the Borel σ -algebra of $C([\tau, T]; \mathbb{R} \times \mathbb{C}^b)$ and determine probability measures on it, and the continuum path map G is Borel for the same reason. This implies that $\rho_G = G_\# \lambda_r$. For two time-independent admissible approximations, we apply this to the union of the two lists. In the limit, the corresponding loop paths coincide. The supremum over $[\tau, T]$ of their maximal difference is a continuous function of the joint path, bounded by $2 \max_h \chi_h(I)$. It converges in law to zero, and since it is bounded, its expectation tends to zero. This proves (3).

For approximations with swept areas $O(a_K)$ and lists consisting only of loops, Step 2 is the argument of Theorem 28.1, and both limits are the limits as $q \rightarrow \infty$ of the laws $G_\# \lambda_q$; this proves (4).

Step 7: tails of the lattice energies. The tails under the conditioned measures follow from the tightness of R_K and the individual energy bound of Theorem 29.16, applied with $U' = U$. The finitely many meshes above the threshold a_* are covered by the deterministic bound $E_a(U) \leq (8/a) \cdot 3N^3 \cdot 2 = 48N^3/a$, since $0 \leq q \leq 2$. For the Wilson measures we use (29.14) and (29.13):

$$\mu_{N_K, \gamma} \{E_{a_K}(\Psi_t^{(N_K)} U) > A\} \leq \nu_{K|r} \{E_{a_K}(\Psi_t^{(N_K)} U) > A\} + \epsilon_K + F_\gamma(n_r).$$

We take the limit superior as $K \rightarrow \infty$, then let $A \rightarrow \infty$ at fixed r , and finally let $r \rightarrow \infty$ in $\mathcal{R}$. This shows that $\lim_{A \rightarrow \infty} \limsup_{K \rightarrow \infty} \mu_{N_K, \gamma} \{E_{a_K}(\Psi_t^{(N_K)} U) > A\} = 0$. The finitely many remaining sides N_K have the deterministic bound $48N_K^3/a_K$, which proves (5). $\square$

29.4 Measurable approximation by smooth trajectories

We now construct Borel maps π_K that approximate the lattice Wilson loop coordinates by smooth trajectories, subject to one-sided energy bounds. The maps choose from the countable family of trajectories X_q of Lemma 29.8, so no measurable choice of gauges is needed. The energy bounds transfer the tightness of the lattice energies to the trajectories $\pi_K(U)$. The map π_K is an auxiliary measurable approximation, not a canonical continuum interpolation of a lattice configuration.

Theorem 29.19. *Assume the standing assumptions of Section 29.3. Let $(X_q)_{q \geq 1}$ be the trajectories of Lemma 29.8, let w_j and $W_{K,j}^{\text{lat}}$ be as in Theorem 29.3 and Definition 29.13, and set*

$s_i = L^2 2^{-i}$ and $\varepsilon_m = 2^{-m}$. For $m \geq 1$, $K \geq 0$, and $U \in \mathcal{U}_{N_K}$, we say that q satisfies the level- m conditions at U if

$$\begin{aligned} |w_j(X_q) - W_{K,j}^{\text{lat}}(U)| &< \varepsilon_m & (j \leq m), \\ E_L(X_q(s_i)) &< E_{a_K}(\Psi_{s_i}^{(N_K)} U) + \varepsilon_m & (i \leq m). \end{aligned} \quad (29.17)$$

Let $\pi_{K,m}(U) = X_q$ for the least q satisfying the level- m conditions at U , let $\pi_{K,m}(U) = X_1$, the zero trajectory, if there is no such q , and let $\pi_{K,0} = X_1$. Then the following statements hold.

1. Each map $\pi_{K,m} : \mathcal{U}_{N_K} \rightarrow \mathcal{X}_L$ is Borel and invariant under all lattice gauge transformations.
2. For each fixed $m \geq 1$, the probability that no q satisfies the level- m conditions tends to zero as $K \rightarrow \infty$, both under $\nu_{K|r}$ for every $r \in \mathcal{R}$ and under $\mu_{N_K, \gamma}$.
3. There exists a nondecreasing sequence of integers $m(K) \rightarrow \infty$, depending only on the data fixed in the standing assumptions and in this theorem, and not on U , such that the maps $\pi_K = \pi_{K,m(K)}$ satisfy, for every $r \in \mathcal{R}$, every $j \geq 1$, and every $t > 0$,

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\nu_{K|r}} |w_j(\pi_K(U)) - W_{K,j}^{\text{lat}}(U)| = 0, \quad \lim_{R \rightarrow \infty} \sup_{K \geq R} \nu_{K|r} \{E_L(\pi_K(U)(t)) > R\} = 0,$$

and such that the image measures $(\pi_K)_\# \nu_{K|r}$ converge to λ_r against every bounded $\mathcal{T}_W$ -continuous function and weakly with respect to d_c .

4. Let $\mathcal{W}$ be a finite list of loops with admissible approximations, characters, and positive times as in Theorem 29.18(2), with lattice vector $\mathcal{W}_K^{\text{lat}}(U)$ and continuum vector $\mathcal{W}(X)$. Then for every bounded continuous function f of this vector and every $r \in \mathcal{R}$,

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\nu_{K|r}} |f(\mathcal{W}(\pi_K(U))) - f(\mathcal{W}_K^{\text{lat}}(U))| = 0.$$

The sequence $m(K)$ may depend on $L, \gamma, M, J, \Theta, n_*$, the enumeration (b_q) , the loops and times (ℓ_j, t_j) , and the convergence moduli in (2) for the countably many indices $r \in \mathcal{R}$. The maps π_K do not depend on the index r that is later used to study their laws.

Proof. We prove (1) directly from the definition, (2) from the joint convergence of Theorem 29.18 and an open-set argument, and then choose $m(K)$ by a diagonal argument. The energy constraint gives the tails in (3), the compactness of energy sublevel sets and the determining property of the w_j identify the limits, and an approximation by functions of finitely many coordinates gives (4).

Step 1: measurability and invariance. Each condition in (29.17) is a strict inequality between continuous gauge-invariant functions of U . Then the set $F_{K,m,q}$ of configurations at which q satisfies the level- m conditions is open, in particular Borel, and invariant under lattice gauge transformations. The event that q is the least such index is $F_{K,m,q} \setminus \bigcup_{q' < q} F_{K,m,q'}$, and the event that no index satisfies the conditions is the complement of

$\mathcal{S}_{K,m} = \bigcup_q F_{K,m,q}$. For a Borel set $\mathcal{B} \subset \mathcal{X}_L$, the preimage $\pi_{K,m}^{-1}(\mathcal{B})$ is the countable union of the events that q is the least index, over those q with $X_q \in \mathcal{B}$, together with the failure event if $X_1 \in \mathcal{B}$. This proves that $\pi_{K,m}$ is Borel. Gauge invariance holds event by event.

Step 2: success at a fixed level. Fix $m \geq 1$. Let $\mathcal{O}_m \subset \mathbb{R}^{2m}$ be the union over q of the sets

$$\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m : |w_j(X_q) - u_j| < \varepsilon_m, E_L(X_q(s_i)) < v_i + \varepsilon_m \ (i, j \leq m)\}.$$

It is open, and by Lemma 29.8 it contains the vector $((w_j(X))_{j \leq m}, (E_L(X(s_i)))_{i \leq m})$ for every $X \in \mathcal{X}_L$. By Theorem 29.18(2), applied to the joint vector of the first m lattice coordinates and the lattice energies at the times $s_1, \dots, s_m$, and by the portmanteau theorem for open sets [28, Theorem 2.1],

$$\liminf_{K \rightarrow \infty} \nu_{K|r}(\mathcal{S}_{K,m}) \geq \lambda_r \{((w_j(X))_{j \leq m}, (E_L(X(s_i)))_{i \leq m}) \in \mathcal{O}_m\} = 1.$$

This uses the joint convergence of energies and loops on the same configuration. By (29.14) and (29.13),

$$\limsup_{K \rightarrow \infty} \mu_{N_K, \gamma}(\mathcal{S}_{K,m}^c) \leq F_\gamma(n_r),$$

and letting $r \rightarrow \infty$ in $\mathcal{R}$, we obtain (2) for the Wilson measures.

Step 3: the choice of $m(K)$ and the coordinates. We enumerate $\mathcal{R}$ as $r(1), r(2), \dots$. We choose strictly increasing integers $K_m \geq \max\{m, r(1), \dots, r(m)\}$ such that for every $K \geq K_m$ the probability of $\mathcal{S}_{K,m}^c$ is at most 2^{-m} , under $\mu_{N_K, \gamma}$ and under each of $\nu_{K|r(1)}, \dots, \nu_{K|r(m)}$. This is possible by Step 2, since each K_m involves only finitely many requirements. We set $m(K) = \max\{m : K_m \leq K\}$, and $m(K) = 0$ if this set is empty. This is a single deterministic definition along the sequence N_K . Fix $j \geq 1$ and $r = r(l) \in \mathcal{R}$. For all sufficiently large K we have $m(K) \geq \max(j, l)$, and then, both under $\nu_{K|r}$ and under $\mu_{N_K, \gamma}$,

$$\mathbb{E}|w_j(\pi_K(U)) - W_{K,j}^{\text{lat}}(U)| \leq 2^{-m(K)} + 2\mathbb{P}(\mathcal{S}_{K,m(K)}^c) \leq 3 \cdot 2^{-m(K)}. \quad (29.18)$$

Here we used the conditions (29.17) with $j \leq m(K)$ on the event $\mathcal{S}_{K,m(K)}$, and (29.9) on its complement.

Step 4: energies. Let $i \leq m(K)$. On the event $\mathcal{S}_{K,m(K)}$ the energy condition in (29.17) holds, and on its complement the zero trajectory has energy zero. We conclude that for every configuration $U \in \mathcal{U}_{N_K}$,

$$E_L(\pi_K(U)(s_i)) \leq E_{a_K}(\Psi_{s_i}^{(N_K)} U) + 2^{-m(K)} \quad (i \leq m(K)). \quad (29.19)$$

Fix i . By Theorem 29.18(5), the lattice energies at time s_i have tails that are uniform in $K \geq r$ under $\nu_{K|r}$ and uniform in $K \geq 0$ under $\mu_{N_K, \gamma}$. By (29.19) the same holds for

$E_L(\pi_K(U)(s_i))$ for all K with $m(K) \geq i$. For each of the finitely many other K , every output $\pi_K(U)$ is one of the smooth trajectories X_q . Then $E_L(\pi_K(U)(s_i))$ is a finite measurable function of U , and its tail probabilities tend to zero by continuity from above of the measure. Combining the two cases, we find that for every i the energies $E_L(\pi_K(U)(s_i))$ have tails that are uniform in $K \geq r$ under $\nu_{K|r}$ and uniform in $K \geq 0$ under $\mu_{N_K, \gamma}$. For an arbitrary $t > 0$ we choose $s_i \leq t$ and use the monotonicity of the continuum energy. This proves the tail bound in (3).

Step 5: convergence of the image measures. Fix $r \in \mathcal{R}$. By Step 4 and Theorem 29.6, the image measures $(\pi_K)_{\#} \nu_{K|r}$ have compact containment, and every subsequence has a further subsequence converging against bounded $\mathcal{T}_W$ -continuous functions and weakly for d_c . By Theorem 29.18(2), the finite vectors $(W_{K,j}^{\text{lat}})_{j \leq b}$ converge in law under $\nu_{K|r}$ to $(w_j)_{j \leq b}$ under λ_r . By (29.18), Markov's inequality, and uniform continuity of bounded continuous functions on the cube $[-1, 1]^b$, the vectors $(w_j(\pi_K(U)))_{j \leq b}$ have the same limit laws. Then every subsequential limit of the image measures has the same finite joint laws of the w_j as λ_r , and by Theorem 29.3(4) it equals λ_r . Since every subsequence has a further subsequence with this limit, every bounded $\mathcal{T}_W$ -continuous function converges along the whole sequence. A bounded d_c -continuous function is $\mathcal{T}_W$ -continuous, which gives the weak convergence for d_c . This completes the proof of (3).

Step 6: additional lists. We now prove (4); the argument uses only compact sets and does not require the functions $W_{\ell, \chi, t}$ of the list to be d_c -continuous. It suffices to treat real f ; for complex f we apply the result to the real and imaginary parts. Let $h(X) = f(\mathcal{W}(X))$ and $D = \|f\|_\infty$; only the values of f on the compact range of the character vector matter. Let $\eta > 0$. By Step 4, Theorem 29.6, and the finiteness of the energies under λ_r , there exists a compact set $\mathcal{C} = \mathcal{K}(R)$ with mass at least $1 - \eta$ under every image measure $(\pi_K)_{\#} \nu_{K|r}$ and under λ_r . Let $\delta > 0$. By Lemma 29.9 there exist b and a continuous $G : [-1, 1]^b \rightarrow [-D, D]$ with $|h - G \circ p_b| < \delta$ on $\mathcal{C}$, where $p_b X = (w_j(X))_{j \leq b}$. Then

$$\mathbb{E}_{\nu_{K|r}} |h(\pi_K(U)) - G(p_b \pi_K(U))| \leq \delta + 2D\eta.$$

By (3) and the uniform continuity of G , $\mathbb{E}_{\nu_{K|r}} |G(p_b \pi_K(U)) - G((W_{K,j}^{\text{lat}}(U))_{j \leq b})| \rightarrow 0$. Next we apply Theorem 29.18(2) to the single joint list consisting of the first b coordinates and the whole list $\mathcal{W}$. Its limit is the corresponding vector under λ_r , and applying the bounded continuous function $|G(u) - f(v)|$ to the joint vector, we obtain

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\nu_{K|r}} |G((W_{K,j}^{\text{lat}}(U))_{j \leq b}) - f(\mathcal{W}_K^{\text{lat}}(U))| = \mathbb{E}_{\lambda_r} |G(p_b X) - h(X)| \leq \delta + 2D\eta.$$

By the triangle inequality, the limit superior in (4) is at most $2\delta + 4D\eta$, and we let $\delta, \eta \rightarrow 0$. $\square$

29.5 Removal of the conditioning

Theorem 29.20. *Assume the standing assumptions of Section 29.3, and let π_K be the maps of Theorem 29.19. Then the following statements hold.*

1. For every $t > 0$,

$$\lim_{R \rightarrow \infty} \sup_{K \geq 0} \mu_{N_K, \gamma} \{E_L(\pi_K(U)(t)) > R\} = 0.$$

2. The image measures $(\pi_K)_\# \mu_{N_K, \gamma}$ have compact containment, and they converge, against every bounded $\mathcal{T}_W$ -continuous function and weakly with respect to d_c , to a Borel probability measure $\lambda_{L, \gamma, [n_*]}$ on $\mathcal{X}_L$.

3. For every $j \geq 1$, $\lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} |w_j(\pi_K(U)) - W_{K,j}^{\text{lat}}(U)| = 0$.

4. For every finite list $\mathcal{W}$ as in Theorem 29.19(4) and every bounded continuous function f of its vector,

$$\begin{aligned} \lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} f(\mathcal{W}_K^{\text{lat}}(U)) &= \int f(\mathcal{W}(X)) d\lambda_{L, \gamma, [n_*]}(X), \\ \lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} |f(\mathcal{W}(\pi_K(U))) - f(\mathcal{W}_K^{\text{lat}}(U))| &= 0. \end{aligned}$$

All statements concern the whole sequence $N_K = n_* B^K$ at the fixed coupling γ , and the convergence moduli and $m(K)$ may depend on γ and on this sequence.

Proof. Statement (1) and the bound (29.18) under $\mu_{N_K, \gamma}$, which gives (3), were proved in Steps 3 and 4 of the proof of Theorem 29.19. By (1) and Theorem 29.6, the image measures have compact containment, and every subsequence has a further subsequence converging against bounded $\mathcal{T}_W$ -continuous functions and weakly for d_c . By Theorem 28.2, applied to the fundamental characters and multiplied by 1/2, every finite vector $(W_{K,j}^{\text{lat}})_{j \leq b}$ converges in law under $\mu_{N_K, \gamma}$ along the whole sequence. By (3), every subsequential limit of the image measures has these limits as the finite joint laws of the w_j , and by Theorem 29.3(4) all subsequential limits coincide. Applying the compactness to a subsequence that stays away from this common limit gives a contradiction, which proves (2).

Fix a finite list $\mathcal{W}$ and a bounded continuous f . By Theorem 29.19(4), the nonnegative discrepancy

$$D_K = |f(\mathcal{W}(\pi_K(U))) - f(\mathcal{W}_K^{\text{lat}}(U))| \leq 2\|f\|_\infty$$

satisfies $\mathbb{E}_{\nu_{K|r}} D_K \rightarrow 0$ for every $r \in \mathcal{R}$. By Lemma 5.5 and (29.13), for every $r \in \mathcal{R}$,

$$\limsup_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} D_K \leq 2\|f\|_\infty F_\gamma(n_r),$$

and letting $r \rightarrow \infty$ shows that the expected absolute discrepancy tends to zero. Since $f \circ \mathcal{W}$ is bounded and $\mathcal{T}_W$ -continuous, by (2) we have $\mathbb{E}_{\mu_{N_K, \gamma}} f(\mathcal{W}(\pi_K(U))) \rightarrow \int f(\mathcal{W}(X)) d\lambda_{L, \gamma, [n_*]}(X)$, and the first assertion of (4) follows. This argument covers

every finite list of loops with admissible approximations, since Theorem 29.18 and Theorem 29.19(4) apply to such lists; no comparison of paths under the Wilson measures is needed. $\square$

29.6 Couplings below a fixed bound

Sections 29.3 to 29.5 use the parameters of Theorem 23.2, which are adapted to small couplings $\gamma \leq \gamma_0$. For an arbitrary fixed coupling we use instead the parameters of Theorem 23.5, which are chosen for all couplings below a given bound Γ . In this subsection we check that the construction carries over, and we collect the statements about general admissible approximations of loops.

The setting below a coupling bound. Fix $0 < \Gamma < \infty$, $L > 0$, and an odd integer $M \geq 3$. Let J , the parameters Θ , and the minimum coarse side $n_\Gamma \in 4\mathbb{N}$ be as in Theorem 23.5; they are chosen before the coupling, and Theorem 23.10 holds for all $\gamma \in (0, \Gamma]$, $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$, and $K \geq 0$. Fix $\gamma \in (0, \Gamma]$ and $n_* \in 4\mathbb{N}$, and define B , N_K , a_K , n_r , $v_{K|r}$, and $e_{K|r}$ as in the standing assumptions of Section 29.3. Let $\mathcal{R}$ be an infinite set of indices r for which Lemma 28.11 applies at the coarse side n_r . By Theorem 23.10, the bound (29.13) holds for $r \in \mathcal{R}$, with $\epsilon_K = 0$ and $F_\gamma(u)$ equal to the right side of (23.13). We refer to these assumptions as the setting below the coupling bound Γ .

Theorem 29.21. *Assume the setting below the coupling bound Γ . Then all conclusions of Theorems 29.18 to 29.20 hold in this setting, with the same lattice energy E_a , the same trajectories X_q , the same level- m conditions, and the zero trajectory when no index satisfies them. In particular, there exist Borel maps $\pi_K : \mathcal{U}_{N_K} \rightarrow \mathcal{X}_L$, invariant under all lattice gauge transformations and independent of r , and Borel probability measures λ_r , $r \in \mathcal{R}$, and $\lambda_{L,\gamma,[n_*]}$ on $\mathcal{X}_L$, with the following properties.*

1. *For every $r \in \mathcal{R}$, the joint limits of lattice energies and loops under $v_{K|r}$ in Theorem 29.18(2),(3) are the laws of the corresponding continuum vectors under λ_r , and the image measures $(\pi_K)_\# v_{K|r}$ converge to λ_r , with tight energies at positive times and L^1 convergence of the coordinates $w_j(\pi_K(U)) - W_{K,j}^{\text{lat}}(U) \rightarrow 0$.*
2. *Under $\mu_{N_K,\gamma}$, the image measures $(\pi_K)_\# \mu_{N_K,\gamma}$ converge to $\lambda_{L,\gamma,[n_*]}$, the energies of $\pi_K(U)$ at every positive time have tails that are uniform in $K \geq 0$, the coordinates converge in L^1 , and every finite list of loops with admissible approximations, characters, and positive times converges in law to its continuum vector under $\lambda_{L,\gamma,[n_*]}$.*

These are statements at each fixed coupling; the success probabilities at a fixed level and the sequence $m(K)$ may depend on γ .

Proof. We fix γ after J , Θ , and n_Γ , and we go through the proofs of the three theorems. The results of Sections 29.1 and 29.2 do not involve the coupling.

The proof of Theorem 29.18 used the following stochastic inputs: the reconstruction arrays, the gauge representative property, the tightness $\rho(R) \rightarrow 0$, the tail bound (29.15), and the convergence of the laws of the arrays A_q , all from Theorem 28.8; the argument of Theorem 28.1; and Lemma 28.4. In the present setting, Lemma 28.11 provides all of them, for the same conditional fluctuation kernels, reconstruction maps, and configurations, with constants depending on the fixed coupling. Steps 1–6 of that proof therefore apply without change and construct the measures λ_r with properties (1)–(4) of Theorem 29.18. Step 7 used Theorem 23.9 only through (29.13), which holds here by Theorem 23.10; property (5) therefore holds as well.

In the proof of Theorem 29.19, Step 1 involves neither the coupling nor a measure. In Step 2, the set $\mathcal{O}_m$ is defined by the trajectories X_q , which do not depend on γ . It contains the vector of every trajectory, and in particular of λ_r -almost every one. By the joint convergence just established, we have $\nu_{K|r}(\mathcal{S}_{K,m}^c) \rightarrow 0$ for every fixed m and $r \in \mathcal{R}$, and (29.14) with (29.13) implies that $\limsup_K \mu_{N_K, \gamma}(\mathcal{S}_{K,m}^c) \leq F_\gamma(n_r)$, which tends to zero as $r \rightarrow \infty$. The choice of K_m and $m(K)$ in Step 3 is the same; it may now depend on the fixed coupling, and (29.18) holds under $\nu_{K|r}$ and under $\mu_{N_K, \gamma}$. The bound (29.19) holds on every configuration. Step 4 then gives the energy tails under both measures, using property (5) for the lattice energies and the finiteness of the energies of the finitely many other outputs. Steps 5 and 6 use only these facts and the measures λ_r constructed above.

In the proof of Theorem 29.20, compact containment follows from the energy tails. The whole-sequence limits of the finite vectors $(W_{K,j}^{\text{lat}})_{j \leq b}$ under $\mu_{N_K, \gamma}$ are given by Theorem 28.2, which holds in this setting by Lemma 28.11. The identification of all subsequential limits of the image measures with one measure $\lambda_{L, \gamma, [n_*]}$ is then the same. So is the treatment of additional lists: the discrepancy $D_K \leq 2\|f\|_\infty$ has conditioned expectations tending to zero, which gives $\limsup_K \mathbb{E}_{\mu_{N_K, \gamma}} D_K \leq 2\|f\|_\infty F_\gamma(n_r)$ for every $r \in \mathcal{R}$, and we let $r \rightarrow \infty$. $\square$

The next lemma records the role of the deterministic estimates of Lemma 29.11 for general admissible approximations of loops, below a coupling bound.

Lemma 29.22. *The following statements hold.*

1. *The estimates of Lemma 29.11 hold with constants that do not depend on the coupling or on any probability measure.*
2. *On every compact family of cardinal interpolants as in Lemma 29.17, they give the bound (29.12) for every finite list of loops with time-independent admissible approximations.*
3. *Assume the setting below a coupling bound Γ , and let $r \in \mathcal{R}$. For every finite list of positive energy times and of loops with admissible approximations, characters, and positive times, the lattice vector converges in law under $\nu_{K|r}$ to the corresponding continuum vector under the measure λ_r of Theorem 29.21. For time-independent admissible approximations, the corresponding processes converge in law in $C([\tau, T]; \mathbb{R} \times \mathbb{C}^b)$ for every $0 < \tau < T < \infty$.*

Proof. Statement (1) is Lemma 29.11, whose proof is purely geometric. For (2), at a fixed coarse mesh the family of cardinal interpolants is compact with bounds on all derivatives. The comparison of the lattice flow with the sampled smooth flow, together with (1), then gives Lemma 29.17, including general admissible approximations. Statement (3) is Theorem 29.18(2),(3) in the setting below the coupling bound, which holds by Theorem 29.21. There is no circularity: the proof of Theorem 29.18 uses (2) through Lemma 29.17, which rests on (1) and on Lemma 28.4, and neither of these uses (3). $\square$

Theorem 29.23. *Let ℓ be a piecewise C^1 loop in $\mathbb{T}_L^3$, let $a_K \rightarrow 0$, and let (ℓ_K) be an admissible approximation of ℓ along these meshes: $\sup_K \text{length}(\ell_K) \leq \mathfrak{l} < \infty$, and for all sufficiently large K there exist comparison homotopies H_K between ℓ and ℓ_K , with connectors c_K , such that $A_K = \text{Area}(H_K) \rightarrow 0$.*

1. *For every smooth connection A we have $\|\text{Hol}_\ell A - (\text{Hol}_{c_K} A)(\text{Hol}_{\ell_K} A)(\text{Hol}_{c_K} A)^{-1}\|_{\text{op}} \leq C\|F_A\|_\infty A_K$. If $\|A\|_{C^1} \leq R$, then $|\chi(\text{Hol}_\ell A) - \chi(\text{Hol}_{\ell_K} A)| \leq C_\chi(R + R^2)A_K$, and replacing the exact transports along the edges of ℓ_K by $\exp(a_K A_i(x))$ changes $\chi(\text{Hol}_{\ell_K} A)$ by at most $C_{\chi,R} a_K$. These constants do not depend on the coupling or on K .*
2. *Assume the setting below a coupling bound Γ , with $a_K = L/N_K$. Every finite list of loops with admissible approximations, characters, and positive times converges in law under $\nu_{K|r}$, $r \in \mathcal{R}$, to its continuum vector under λ_r , and under $\mu_{N_K,\gamma}$ to its continuum vector under the measure $\lambda_{L,\gamma,[n_*]}$ of Theorem 29.21.*
3. *In the setting of (2), let $(\ell_{h,K})$ and $(\ell'_{h,K})$, $h \leq b$, be two admissible approximations of the same loops, with lattice vectors $\mathcal{W}_K^{\text{lat}}$ and $\mathcal{W}_K'^{\text{lat}}$ for fixed characters and times. Then $\mathbb{E}_{\mu_{N_K,\gamma}} |\mathcal{W}_K^{\text{lat}}(U) - \mathcal{W}_K'^{\text{lat}}(U)| \rightarrow 0$, and $\mathbb{E}_{\mu_{N_K,\gamma}} |f(\mathcal{W}_K^{\text{lat}}(U)) - f(\mathcal{W}_K'^{\text{lat}}(U))| \rightarrow 0$ for every bounded continuous f .*
4. *In the setting of (2), for every $R < \infty$,*

$$\lim_{K \rightarrow \infty} \sup_{\sum_h |z_h| \leq R} \left| \mathbb{E}_{\mu_{N_K,\gamma}} \exp\left(\sum_h z_h W_{K,h}^{\text{lat}}\right) - \int \exp\left(\sum_h z_h W_{\ell_h, \chi_h, t_h}(X)\right) d\lambda_{L,\gamma,[n_*]}(X) \right| = 0.$$

5. *In the setting of (2), for time-independent admissible approximations the path convergence of Theorem 29.18(3) under $\nu_{K|r}$ holds on every interval $[\tau, T] \subset (0, \infty)$, and for two such approximations*

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K,\gamma}} \sup_{\tau \leq t \leq T} \max_{h \leq b} |\chi_h(\text{Hol}_{\ell_{h,K}} \Psi_t^{(N_K)} U) - \chi_h(\text{Hol}_{\ell'_{h,K}} \Psi_t^{(N_K)} U)| = 0.$$

6. *A sequence of closed lattice paths of bounded physical length that converges uniformly to ℓ , as parametrized loops, is an admissible approximation of ℓ .*

Proof. Statements (1) and (6) are Lemma 29.11, and the convergence under $\nu_{K|r}$ in (2) and (5) is Lemma 29.22(3). The convergence under $\mu_{N_K,\gamma}$ in (2) is Theorem 29.21(2). We give

the comparison argument, which also proves (3)–(5). Let $D = \max_h \chi_h(I)$ for a nonempty list, and let $G_K = |\mathcal{W}_K^{\text{lat}}(U) - \mathcal{W}'^{\text{lat}}_K(U)|_\infty$, so that $0 \leq G_K \leq 2D$. By Lemma 29.22(3), applied to the union of the two lists, the two vectors converge jointly under $\nu_{K|r}$ to the same continuum vector, and $\mathbb{E}_{\nu_{K|r}} G_K \rightarrow 0$. By Lemma 5.5 and (29.13),

$$\limsup_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} G_K \leq 2DF_\gamma(n_r).$$

We take the limit in K first at this fixed r , and then let $r \rightarrow \infty$ in $\mathcal{R}$; the right side tends to zero. This proves the first assertion of (3) for the maximum norm, hence for any norm. Taking for $(\ell'_{h,K})$ the lattice paths of Lemma 29.10, whose limit under $\mu_{N_K, \gamma}$ is identified by Theorem 29.20 in the setting of Theorem 29.21, we obtain the limit in (2) for every admissible approximation. The second assertion of (3) follows from uniform continuity of f on the compact range of the character vectors and Markov's inequality. For (4), the vectors are bounded by D , and for $\sum_h |z_h| \leq R$,

$$|e^{z \cdot v} - e^{z \cdot v'}| \leq Re^{DR} \max_h |v_h - v'_h|.$$

Then (3) transfers the uniform convergence of source functionals for the paths of Lemma 29.10, which holds by Theorem 28.2 (in this setting by Lemma 28.11), to every admissible approximation, and (2) identifies the limit. For the supremum in (5), we apply Theorem 29.18(3), in the setting below the coupling bound, to the two time-independent approximations simultaneously. Their joint limit has identical loop paths, and the supremum of the difference is bounded by $2D$, so its conditioned expectation tends to zero. The bound of Lemma 5.5 at the same lattice side, followed by the limit $r \rightarrow \infty$, gives the conclusion under $\mu_{N_K, \gamma}$. Two arbitrary time-independent approximations are compared through the paths of Lemma 29.10, which do not depend on time. $\square$

29.7 The continuum limit along the sequences $n_* M^{JK}$

Theorem 29.24. *Let $0 < \Gamma < \infty$, $L > 0$, and $M \geq 3$ odd. Fix any tuple (J, Θ, n_Γ) satisfying the conclusions of Theorems 23.5 and 23.10, chosen before the coupling; in particular one may use the tuples of Corollary 23.15 at any prescribed admissible depth. The following holds for every $\gamma \in (0, \Gamma]$ and every $n_* \in 4\mathbb{N}$ with $n_* \geq n_\Gamma$. Let $B = M^J$ and $N_K = n_* B^K$, and write $\mu_K = \mu_{N_K, \gamma}$. Then, as $K \rightarrow \infty$:*

1. *For loops $\ell_1, \dots, \ell_m$ with admissible approximations $(\ell_{i,K})_K$, characters $\chi_1, \dots, \chi_m$ of finite-dimensional representations of $\text{SU}(2)$, and times $t_1, \dots, t_m > 0$, the vector $(\chi_i(\text{Hol}_{\ell_{i,K}} \Psi_{t_i}^{(N_K)} U))_{i \leq m}$ converges in law under μ_K , and for every $R < \infty$ the expectations $\mathbb{E}_{\mu_K} \exp(\sum_i z_i \chi_i(\text{Hol}_{\ell_{i,K}} \Psi_{t_i}^{(N_K)} U))$ converge uniformly in $z \in \mathbb{C}^m$ with $\sum_i |z_i| \leq R$.*
2. *There exist Borel maps $\pi_K : \mathcal{U}_{N_K} \rightarrow \mathcal{X}_L$, invariant under all lattice gauge transformations, such that the image measures $(\pi_K)_\# \mu_K$ have compact containment and converge, against*

every bounded $\mathcal{T}_W$ -continuous function and weakly with respect to d_c , to a Borel probability measure $\lambda_{L,\gamma,[n_*]}$ on $\mathcal{X}_L$.

3. For every $t > 0$ and every $j \geq 1$,

$$\lim_{R \rightarrow \infty} \sup_{K \geq 0} \mu_K \{E_L(\pi_K(U)(t)) > R\} = 0, \quad \lim_{K \rightarrow \infty} \mathbb{E}_{\mu_K} |w_j(\pi_K(U)) - W_{K,j}^{\text{lat}}(U)| = 0,$$

and the limit law in (1) is the law of $(W_{\ell_i, \chi_i, t_i}(X))_{i \leq m}$ under $X \sim \lambda_{L,\gamma,[n_*]}$.

4. The limits in (1) do not depend on the choice of the admissible approximations, and for two admissible approximations of the same loops the difference of the vectors in (1) tends to zero in $L^1(\mu_K)$.

Here the Wilson flow runs for the lattice time $4t/a_{N_K}^2$, and the characters need not be normalized; they are bounded by the dimensions of the representations. The convergence moduli, the maps π_K , and the measure may depend on γ , on the sequence N_K , and on the observables. Every fixed $\gamma > 0$ lies below some bound Γ , so the theorem applies to every $\gamma > 0$. By Corollary 29.26 below, its conclusions hold for every $n_* \in 4\mathbb{N}$, not only for $n_* \geq n_\Gamma$.

Proof. Fix any tuple J, Θ , and n_Γ satisfying the stated finite-ceiling parameter and payment conclusions; these choices precede the coupling. Fix $\gamma \in (0, \Gamma]$ and $n_* \geq n_\Gamma$. By Lemma 28.11, the conditioned estimates of Section 28 hold at all coarse sides n_r with r in an infinite set $\mathcal{R}$. The minimum coarse sides there, the bounds on the flow data, and the convergence moduli may depend on the fixed coupling. We are therefore in the setting below the coupling bound Γ of Section 29.6. Theorem 29.21 gives (2) and (3), including the energy tails with supremum over all $K \geq 0$ and the identification of every finite list of loops with admissible approximations under the same measure $\lambda_{L,\gamma,[n_*]}$. By Theorem 29.23(2)–(4), we obtain (1) and (4), including the uniform convergence of the source functionals. The maps π_K are defined on the configuration spaces of all sides N_K , $K \geq 0$. For the finitely many K with $m(K) = 0$ their value is the zero trajectory, which adds a single compact point and does not affect compact containment, the energy tails, or any limit. If the conditioned estimates require coarse sides larger than a bound depending on γ , we use only indices $r \in \mathcal{R}$ above it. No parameter of the hierarchy is chosen again. $\square$

The case $\Gamma = 1$ is used in the weak-coupling analysis, where the parameters of Theorem 23.2 enter. We state it separately, with the quantifiers in the form in which it is used.

Theorem 29.25. *Let $L > 0$ and $M \geq 3$ odd. There exist $J \in \mathbb{N}$ and parameters Θ , chosen before the coupling, the lattice sides, and the number of steps, such that for every $\gamma \in (0, 1]$ there exists $n_0 < \infty$, which may depend on L, γ, M, J , and Θ , with the following property. For every*

$n_* \in 4\mathbb{N}$ with $n_* \geq n_0$, with $B = M^J$ and $N_K = n_* B^K$, the conclusions (1)–(4) of Theorem 29.24 hold along the whole sequence $K \rightarrow \infty$, for the Wilson measures $\mu_{N_K, \gamma}$ of Definition 4.1.

All bounds that are uniform in the lattice side range over every $K \geq 0$.

Proof. Fix L and M , and choose the bound $\Gamma = 1$ in Theorems 23.5 and 23.10; the construction there keeps the parameters of Theorem 23.2 and provides the corresponding minimum side. The proof of Theorem 29.24 establishes conclusions (1)–(4) directly from Lemma 28.11 and Theorems 29.21 and 29.23, without using the present theorem. Specializing it to $\Gamma = 1$, with $n_0 = n_1$, we obtain the theorem; in fact n_0 can be chosen independently of γ , and larger minimum sides depending on γ are also allowed. No choice of $m(K)$ uniform in γ is claimed. $\square$

The next corollary removes the lower bound on the base side and records how the limit is used when a comparison side is fixed before the coupling, as in Sections 34 to 36.

Corollary 29.26. *Let $0 < \Gamma < \infty$, $L > 0$, and $M \geq 3$ odd, let (J, Θ, n_Γ) be any tuple satisfying the hypotheses of Theorem 29.24, and set $B = M^J$. Fix $n_* \in 4\mathbb{N}$ before the coupling.*

1. *Let $\gamma \in (0, \Gamma]$, and let $a_\gamma \geq 0$ be an integer with $n_* B^{a_\gamma} \geq n_\Gamma$. The measure given by Theorem 29.24 for the base side $n_* B^{a_\gamma}$ does not depend on the choice of a_γ ; we denote it by $\lambda_{L, \gamma, [n_*]}$. Let π'_K be the maps of that theorem for the base side $n_* B^{a_\gamma}$, and set $\pi_K = \pi'_{K-a_\gamma}$ for $K \geq a_\gamma$ and $\pi_K = X_1$, the zero trajectory, for $K < a_\gamma$. Then the conclusions (1)–(4) of Theorem 29.24 hold along the sequence $N_K = n_* B^K$, $K \geq 0$, for these maps and this measure, including the energy tails with supremum over all $K \geq 0$.*
2. *Let $n = n_* B^a$ be a comparison side, fixed before the coupling, with $n \geq n_\Gamma$, and possibly subject to further side conditions used in later sections, such as the margin (23.52) for the flat connections and the conditions on the analytic radius. Then the bound of Theorem 23.10 on $\mu_{nB^k, \gamma}(S_{nB^k, n, \gamma}^c)$ holds uniformly in $\gamma \in (0, \Gamma]$ and $k \geq 0$. Moreover, for each fixed $\gamma \in (0, \Gamma]$, the limits along $N = nB^k$ of the expectations of bounded continuous functions of finite loop vectors as in Theorem 29.24(1) are the corresponding expectations under $\lambda_{L, \gamma, [n_*]}$, also when $a_\gamma > a$.*

In (2), every further estimate at the side n keeps its own range of couplings, and neither the side n nor a fixed number of scales used in a later weak-coupling estimate is replaced by a side depending on γ . The coupling bound γ_0 of Theorem 23.2 is not changed. We claim no choice of $m(K)$ independent of γ , no interchange of the limits $K \rightarrow \infty$ and $\gamma \rightarrow 0$, and no comparison between different classes $[n]_M$.

Proof. We fix Γ , the parameters, and n_Γ first, and then n_* , before the coupling. Since $B > 1$, there exists an integer $a_\gamma \geq 0$ with $n_* B^{a_\gamma} \geq n_\Gamma$. For fixed $\gamma \in (0, \Gamma]$ we apply Theorem 29.24 with the base side $u = n_* B^{a_\gamma}$. That theorem allows further minimum sides

depending on γ internally, and it gives its conclusions along the sequence $uB^K = N_{K+a_\gamma}$, $K \geq 0$. This sequence is a tail of the sequence $(N_K)_K$, with the same configuration spaces, Wilson measures, Wilson flow, and lattice paths. Hence every finite loop vector, and every expectation of a bounded continuous function of it, has the same limit along $(N_K)_K$. Two choices of a_γ give sequences that share a common tail. The finite joint laws of the limits of the coordinates $W_{K,j}^{\text{lat}}$ therefore coincide, and by Theorem 29.3(4) the two measures coincide.

The maps π_K are Borel and gauge invariant. Changing finitely many terms of a sequence does not affect convergence or the L^1 convergence of the coordinates. Adding the zero trajectory to a compact set does not affect compactness in either topology, so compact containment is preserved. The added image measures have zero energy at every positive time, and the energy tails with supremum over all $K \geq 0$ are therefore unchanged. This proves (1).

For (2), the uniform bound on the small-field failure probability follows from Theorem 23.10, since $n \geq n_\Gamma$. The reindexing in (1) does not change any measure, event, or observable at the side n . For fixed γ , all sufficiently large sides nB^k belong to the tail $(uB^K)_K$ used above, so the expectations of bounded continuous functions of finite loop vectors converge to the expectations under $\lambda_{L,\gamma,[n_*]}$, also when $a_\gamma > a$. Every further estimate at the side n keeps its own range of couplings, and no parameter of the comparison side or bound on the coupling was changed. $\square$

29.8 The sequence nM^k

The results of Section 29.7 give convergence along the subsequences n_*M^{JK} for one fixed J . To obtain convergence along the full sequence nM^k , we use two consecutive values J and $J+1$, both of which are available by Corollary 23.15. Each residue class of k modulo J shares infinitely many lattice sides with the sequence of depth $J+1$, and at a common lattice side the Wilson measures and the lattice observables are identical. This identifies the limits along all residue classes.

Lemma 29.27. *Let $J \geq 1$, $n \in \mathbb{N}$, and $r \in \{0, \dots, J-1\}$. For every integer $s \geq 0$ let $K = r + (J+1)s$ and $K' = r + Js$. Then $r + JK = (J+1)K'$, so that*

$$nM^{r+JK} = nM^{(J+1)K'}, \quad (29.20)$$

and $K, K' \rightarrow \infty$ as $s \rightarrow \infty$.

Proof. We have $r + J(r + (J+1)s) = (J+1)r + J(J+1)s = (J+1)(r + Js)$. $\square$

Theorem 29.28. *Let $0 < \Gamma < \infty$, $L > 0$, $M \geq 3$ odd, and $n \in 4\mathbb{N}$. There exist two choices of the parameters of Theorem 23.5, with consecutive numbers J and $J+1$ of averaging steps per*

renormalization group step, made before the coupling, such that for every $\gamma \in (0, \Gamma]$ the following statements hold, with $N_k = nM^k$ and $a_k = L/N_k$.

1. For loops $\ell_1, \dots, \ell_m$ with admissible approximations $(\ell_{i,k})_k$ along $(N_k)_k$, characters $\chi_1, \dots, \chi_m$, and times $t_1, \dots, t_m > 0$, the vector $(\chi_i(\text{Hol}_{\ell_{i,k}} \Psi_{t_i}^{(N_k)} U))_{i \leq m}$ converges in law under $\mu_{N_k, \gamma}$ as $k \rightarrow \infty$, and for every $R < \infty$ its source functionals $\mathbb{E}_{\mu_{N_k, \gamma}} \exp(\sum_i z_i \chi_i(\text{Hol}_{\ell_{i,k}} \Psi_{t_i}^{(N_k)} U))$ converge uniformly in $z \in \mathbb{C}^m$ with $\sum_i |z_i| \leq R$.
2. There exist Borel maps $\pi_k : \mathcal{U}_{N_k} \rightarrow \mathcal{X}_L$, invariant under all lattice gauge transformations, such that the image measures $(\pi_k)_\# \mu_{N_k, \gamma}$ have compact containment and converge, against every bounded $\mathcal{T}_W$ -continuous function and weakly with respect to d_c , to a Borel probability measure $\lambda_{L, \gamma, [n]_M}$ on $\mathcal{X}_L$. Here $[n]_M$ is the class of n for the relation

$$n \sim_M n' \iff nM^a = n'M^b \text{ for some integers } a, b \geq 0.$$

3. For every $t > 0$ and every $j \geq 1$,

$$\lim_{R \rightarrow \infty} \sup_{k \geq 0} \mu_{N_k, \gamma} \{E_L(\pi_k(U)(t)) > R\} = 0, \quad \lim_{k \rightarrow \infty} \mathbb{E}_{\mu_{N_k, \gamma}} |w_j(\pi_k(U)) - W_{N_k, j}^{\text{lat}}(U)| = 0,$$

and the limit law in (1) is the law of $(W_{\ell_i, \chi_i, t_i}(X))_{i \leq m}$ under $X \sim \lambda_{L, \gamma, [n]_M}$.

4. The measure $\lambda_{L, \gamma, [n]_M}$ depends only on L , γ , and $[n]_M$: for every $n' \in [n]_M$, every bound $\Gamma' \geq \gamma$, and every choice of the parameters of Theorem 23.5 for Γ' , the measure obtained from n' and these parameters in Corollary 29.26, and every measure obtained in (2) from n' using an admissible pair of consecutive depths, coincide with $\lambda_{L, \gamma, [n]_M}$. For two admissible approximations of the same loops, the difference of the vectors in (1) tends to zero in $L^1(\mu_{N_k, \gamma})$.
5. The conclusions of Theorems 30.4 to 30.6 (convergence of the processes of lattice energies and loops on compact intervals of positive times, agreement of time-independent admissible approximations in expected supremum, convergence of the images in the Sobolev trajectory space, and invariance under spatial isometries and center symmetries) hold along the full sequence $N_k = nM^k$, for the maps π_k and the measure $\lambda_{L, \gamma, [n]_M}$.

Since M is odd and $4 \mid n$, every $n' \in [n]_M$ is also divisible by four. The maps and convergence moduli may depend on the fixed coupling, on n , and on the two parameter choices. The proof of (5), which applies the arguments of Section 30 on each residue class and then the assembly below, is given in Section 30.

Remark 29.29. The theorem identifies the limit within one class $[n]_M$. It does not compare the measures for different classes or for different odd integers M . It gives no rate of convergence, no convergence of moments of the energy, and no uniformity as $\gamma \rightarrow 0$, as $\Gamma \rightarrow \infty$, or in J . The statements about conditioned measures remain statements along the sequences $n_* B^K$ at fixed coarse sides; we do not identify the limits of the conditioned measures along different residue classes. Likewise, the bounds on the small-field failure probability refer to the sequences $u B^K$ and the coarse sides u for which they are stated.

Proof of Theorem 29.28(1)–(4). We prove the convergence along each residue class modulo J by Corollary 29.26, identify the limits through the common lattice sides of Lemma 29.27, and then interlace the maps.

Step 1: the two parameter choices. By Corollary 23.15 there exists J_0 such that for every $J \geq J_0$ there exist parameters Θ_J and a minimum side, fixed before the coupling, with all conclusions of Theorems 23.5 and 23.10 for $0 < \gamma \leq \Gamma$. We fix such choices for two consecutive values J and $J + 1$. This use of the corollary is not circular. Its proof first applies Lemma 23.14 with the bound one and then the construction of Theorem 23.5, which changes only the constants in the failure bound and the minimum sides and keeps J and the structural parameters. Now fix $\gamma \in (0, \Gamma]$. For each of the two depths, Theorem 29.24, with Lemma 28.11 and Theorem 29.21, provides maps with all the stated properties along the corresponding sequences. These maps choose among the trajectories X_q , with value X_1 when no index satisfies the conditions and with a sequence $m(K)$ depending on the fixed coupling, and the energy constraint uses the lattice energy of Lemma 29.14. The additional minimum side depending on γ is finite and does not change J or the parameters. No modulus of convergence and no sequence $m(K)$ is claimed to be uniform in the coupling or in J .

Step 2: the residue classes. For each $r \in \{0, \dots, J - 1\}$ we apply Corollary 29.26(1) with depth J and base side nM^r . This is allowed even if nM^r is below the minimum side, since the corollary provides a finite shift and the zero trajectory for the finitely many omitted sides. Let $\lambda^{(r)}$ be the resulting measure and $\pi_K^{(r)}$ the resulting maps along the sequence $nM^r B^K = nM^{r+JK}$. We also apply the corollary with depth $J + 1$ and base side n , and let λ' be the resulting measure along the sequence $nM^{(J+1)K'}$.

Step 3: identification of the limits. For the comparison we use the lattice paths of Lemma 29.10, which depend only on the loop and the mesh and not on the parameters of the hierarchy. By Lemma 29.27, for every r the sequences nM^{r+JK} and $nM^{(J+1)K'}$ have infinitely many common terms. At each common side N , the Wilson measure $\mu_{N,\gamma}$, the Wilson flow, the times, and the lattice loop vectors are identical. The two whole-sequence limits of every finite vector $(W_{N,j}^{\text{lat}})_{j \leq b}$ therefore agree. By Theorem 29.3(4), $\lambda^{(r)} = \lambda'$ for every r ; we write $\lambda_{L,\gamma,[n]_M}$ for this common measure. No conditioned measure, density, or reconstruction has been compared.

Step 4: interlacing. For $k = r + JK$ with $r \in \{0, \dots, J - 1\}$ and $K \geq 0$, set $\pi_k = \pi_K^{(r)}$. Since each side N_k belongs to exactly one residue class, these maps are Borel and gauge invariant. For every bounded $\mathcal{T}_W$ -continuous function, the expectations along each of the J residue sequences converge to its expectation under $\lambda_{L,\gamma,[n]_M}$. Taking the largest of the finitely many thresholds, we obtain convergence along the whole sequence. The same argument gives the weak convergence for d_c and the L^1 convergence of the coordinates.

If $\mathcal{K}(R^{(r)})$ has mass at least $1 - \eta$ under all image measures of residue r , then $\mathcal{K}(R)$ with $R_j = \max_r R_j^{(r)}$ contains all these sets and has mass at least $1 - \eta$ for all k . Moreover,

$$\sup_{k \geq 0} \mu_{N_k, \gamma} \{E_L(\pi_k(U)(t)) > R\} \leq \max_{0 \leq r < J} \sup_{K \geq 0} \mu_{nM^{r+JK}, \gamma} \{E_L(\pi_K^{(r)}(U)(t)) > R\},$$

which tends to zero as $R \rightarrow \infty$. The convergence of finite loop vectors, the uniform convergence of the source functionals, and the agreement of admissible approximations follow by the same finite argument from Corollary 29.26(1) on each residue class, with the limit identified by Step 3. This assembly uses different sequences $m(K)$ on the finitely many residue classes; it never identifies their values, the indices q chosen, or the conditioned measures.

Step 5: dependence only on the class. Let $n' \in [n]_M$ with $nM^a = n'M^b$, and consider a construction as in Corollary 29.26 with base side n' , depth J' , and parameters for a bound $\Gamma' \geq \gamma$. Its sides satisfy $n'M^{J'K} = nM^{a-b+J'K}$ for all sufficiently large K , so they form a subsequence of a tail of $(nM^k)_k$. Its loop limits therefore coincide with those along the full sequence, and Theorem 29.3(4) identifies its measure with $\lambda_{L, \gamma, [n]_M}$. The full sequence $n'M^k = nM^{a-b+k}$, $k \geq b - a$, is a shifted tail of the full sequence nM^k , and the measure obtained in (2) from n' is the same. In particular, constructions with two different bounds $\Gamma, \Gamma' \geq \gamma$ give the same measure. This proves (4) and completes the proof of (1)–(4). $\square$

Proof of Theorem 2.3. Let $L > 0$, $\gamma > 0$, $M \geq 3$ odd, and $n \in 4\mathbb{N}$, and apply Theorem 29.28 with any bound $\Gamma \geq \gamma$. Part (1) of Theorem 2.3 is Theorem 29.28(1); the independence of the limit from the admissible approximations follows from the L^1 agreement in Theorem 29.28(4). For part (2), Theorem 29.28(3) shows that every limit law in part (1) is the law of the corresponding continuum vector under $\lambda = \lambda_{L, \gamma, [n]_M}$, and uniqueness follows from Theorem 29.3(4). Part (3) is Theorem 29.28(2),(3) for the paths of Definition 29.13; for any other fixed admissible approximation $(\ell_{j,k})_k$ of ℓ_j it follows from the L^1 agreement in Theorem 29.28(4) and the triangle inequality. Part (4) is the energy bound in Theorem 29.28(3), and part (5) is Theorem 29.28(4).

Further properties of the continuum measure

In this section, we prove the properties of the continuum measure stated in Theorems 2.5 and 2.6. In Section 29, we constructed the measure $\lambda = \lambda_{L,\gamma,[n]_M}$ on the trajectory space $\mathcal{X}_L$ at each fixed $\gamma > 0$, together with Borel, gauge-invariant maps π_k from lattice configurations to $\mathcal{X}_L$, and showed that every finite family of flowed lattice Wilson loops converges in law to the corresponding regularized Wilson loops under λ (Theorem 29.28). We prove the following further properties.

1. The measures on tori of different side lengths are related by an exact dilation (Theorem 30.1).
2. The joint processes of lattice energy and flowed Wilson loops converge in distribution on every compact time interval in $(0, \infty)$, in the uniform topology (Theorem 30.4). A deterministic input is a uniform Lipschitz bound in time for these processes (Lemma 30.3).
3. The image laws also converge in the finer Sobolev topology of Cao and Chatterjee (Theorem 30.5).
4. The measure is invariant under the isometries $x \mapsto Qx + y$ of $\mathbb{T}_L^3$, with Q a signed permutation matrix, and under the center symmetries; in particular, products of Wilson loops with odd total winding have mean zero (Theorem 30.6).
5. These conclusions hold along the full sequence of lattice sides nM^k (Section 30.6).
6. The map $\gamma \mapsto \lambda_{L,\gamma,[n]_M}$ is weakly continuous. On every compact interval $I \subset (0, \infty)$, each fixed finite vector of flowed lattice Wilson loops converges uniformly in the bounded-Lipschitz distance. The approximating maps can be chosen independently of $\gamma \in I$, with uniform convergence against each bounded Wilson-continuous test and in the bounded-Lipschitz distance for d_c , and with uniform tails of the energies of the image trajectories (Theorem 30.10). Uniformity in γ is not asserted for the path laws or for the laws in the Sobolev topology.
7. At each fixed $0 < \gamma \leq 1$, along the sequence $N_K = n_* B^K$ of Theorem 29.25, replacing the Wilson flow by the gradient flow of $\sum_p [q(U_p) + cq(U_p)^2]$, with $0 \leq c \leq C_0 < \infty$, preserves the limits of finite vectors of Wilson loops and the limits of the joint processes of energies and Wilson loops at positive times (Theorems 30.13 and 30.15).

The inputs are the results of Section 29 on the trajectory space, the energy, the joint limits under conditioned measures, and the approximating maps; the deterministic

estimates for the Wilson flow and its modifications in Sections 24 and 25; the finite-mesh control of the linear and quadratic flow data in Section 28; and the bound on the small-field failure probability in Section 23. The convergence of the energy and loop processes is used in Sections 34 and 36, and the convergence in the Sobolev topology in Sections 32 and 34.

Most proofs below have the same structure. Fix a coarse side $n_r = n_* B^r$. Under the conditioned Wilson measure $\nu_{N_K, n_r, \gamma}$, the configuration is, up to a gauge transformation, the field built by the reconstruction maps of Section 15, and the norm of its linear and quadratic flow data is tight in K (Theorem 28.8). On the event that this norm is at most R , the deterministic flow estimates of Sections 24 and 25 apply. We then remove the conditioning using the elementary inequality (30.2) below, taking first the limit $K \rightarrow \infty$ at fixed r and then $r \rightarrow \infty$. Since the lattice energy is unbounded, only bounded functions of it, such as indicators of tail events or bounded continuous path tests, are transferred in this way. No convergence of moments of the energy is claimed in this section.

30.1 Setting and notation

Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$. Before the coupling is chosen, we fix the integer J , the parameter tuple Θ , and the minimum coarse side n_Γ of Theorem 29.24, that is, of Theorem 23.5. Then fix $0 < \gamma \leq \Gamma$ and $n_* \in 4\mathbb{N}$, and set

$$B = M^J, \quad N_K = n_* B^K, \quad a_K = L/N_K.$$

If $n_* < n_\Gamma$, we use the convention of Corollary 29.26: the results are applied to the shifted base $n_* B^{a_\gamma} \geq n_\Gamma$, and at the finitely many cutoffs $K < a_\gamma$ the maps are the stationary zero trajectory. We write $\lambda = \lambda_{L, \gamma, [n_*]}$ for the continuum measure along $(N_K)_{K \geq 0}$, and π_K for the maps of Theorem 29.21. For the full sequence $N_k = n M^k$ of Theorem 29.28, the maps are instead those of that theorem, which are obtained by interlacing finitely many sequences of the previous form, and the measure is $\lambda_{L, \gamma, [n]_M}$. By part (4) of that theorem, $\lambda = \lambda_{L, \gamma, [n_*]_M}$. The two consecutive values J and $J+1$ and their parameter tuples are given by Corollary 23.15, and no constant uniform in J is used. Every assertion in Sections 30.2 to 30.6 is made at one fixed coupling; Section 30.7 treats compact intervals of couplings.

All gauge transformations are periodic, and all homotopy classes are included. Deterministic constants have the dependence described in item (7) of Definition 4.14. They may depend on the stated bounds on the flow data, the waiting times and horizons, the lengths of the lattice paths, the representations, and the margins of the smooth reference solutions, but not on the coupling, the cutoff, the depth of the hierarchy, or the law. Stochastic constants and convergence moduli may in addition depend on γ , on the coarse side, and on the parameters. No estimate is uniform as the waiting time or the coupling tends to zero.

By Lemma 28.11, the results of Section 28 that we use hold at each fixed $0 < \gamma \leq \Gamma$, with Theorem 23.10 as the bound on the small-field failure probability. These are the finite-mesh control of the flow data (Theorem 28.8), the consistency estimate for a fixed finite-scale reconstruction (Lemma 28.4), and the removal of conditioning (Theorem 28.2). By Theorem 29.21, the conclusions of Theorems 29.18 to 29.20 hold in the present setting, with their fixed-coupling dependence.

Conditioned measures. For an integer $r \geq 0$ and $K \geq r$ write

$$n_r = n_* B^r, \quad \nu_{K|r} = \nu_{N_K, n_r, \gamma} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, n_r, \gamma}), \quad e_{K|r} = \mu_{N_K, \gamma}(S_{N_K, n_r, \gamma}^c).$$

For all sufficiently large r , by Theorems 23.10 and 28.2, we have

$$e_{K|r} \leq \epsilon_K^{\text{sf}} + F_\gamma(n_r), \quad \epsilon_K^{\text{sf}} \longrightarrow 0, \quad F_\gamma(u) = C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}, \quad (30.1)$$

and $F_\gamma(n_r) \rightarrow 0$ as $r \rightarrow \infty$ at fixed $\gamma > 0$; here ϵ_K^{sf} is the remainder denoted by ϵ_K in (29.13). Let r_0 be so large that, for every $r \geq r_0$, we have $n_r \geq n_\Gamma$, all additional thresholds on the coarse side in Lemma 28.11 are satisfied for the fixed coupling and input horizon, and (30.1) holds. All conditioned trajectory laws and conditioned estimates below are used only for these indices. We write λ_r for the trajectory law of Theorem 29.18, which is the limit of the flowed observables under $\nu_{K|r}$ as $K \rightarrow \infty$, and Y^r for the identity map of $\mathcal{X}_L$ under λ_r . Since $\nu_{K|r}$ is $\mu_{N_K, \gamma}$ conditioned on an event of probability $1 - e_{K|r}$, there exists a probability measure $\rho_{K|r}$ (arbitrary if $e_{K|r} = 0$) such that $\mu_{N_K, \gamma} = (1 - e_{K|r})\nu_{K|r} + e_{K|r}\rho_{K|r}$. This implies that, for every bounded complex Borel function F , every Borel function $0 \leq G \leq D$, and every Borel event A ,

$$\begin{aligned} |\mathbb{E}_{\mu_{N_K, \gamma}} F - \mathbb{E}_{\nu_{K|r}} F| &\leq 2e_{K|r} \|F\|_\infty, \\ \mathbb{E}_{\mu_{N_K, \gamma}} G &\leq \mathbb{E}_{\nu_{K|r}} G + De_{K|r}, \quad \mu_{N_K, \gamma}(A) \leq \nu_{K|r}(A) + e_{K|r}. \end{aligned} \quad (30.2)$$

This is Lemma 5.5; the last inequality is (29.14).

Observables. For a mesh $a = L/N$ we use the lattice energy

$$E_a(U) = \frac{8}{a} \sum_{p \in \mathcal{P}_N} q(U_p)$$

of (2.2) and Lemma 29.14, with each positive plaquette counted once. Its smooth limit is the energy E_L of (2.3). For a closed lattice path ℓ_a and a character χ , the flowed lattice Wilson loop is $\chi(\text{Hol}_{\ell_a} \Psi_t^{(N)} U)$, where $\Psi_t^{(N)}$ is the Wilson flow run for the lattice time $4t/a^2$. We use the determining coordinates $w_j(X) = (1/2) \text{tr} \text{Hol}_{\ell_j} X(t_j)$ and the metric d_c of Theorem 29.3, and their lattice counterparts $W_{K,j}^{\text{lat}}$ of Definition 29.13. On lattice one-forms, $\|A\|_{C_a^1} = \|A\|_\infty + \max_{j,\epsilon} \|D_{a,j}^\epsilon A\|_\infty$ is the supremum norm plus the maximum of the forward and backward differences divided by a , and $S_a b$ denotes the sampling of a smooth one-form b at the lattice vertices.

Admissible approximations. Throughout this section, the lattice paths approximating a loop ℓ form an admissible approximation in the sense of Definition 2.1. That is, they are closed lattice paths ℓ_K in Λ_{N_K} , fixed in time and of uniformly bounded physical length, such that for all sufficiently large K there exists a based piecewise C^1 homotopy between ℓ_K and ℓ , together with a connecting path between the basepoints when these differ, whose swept area A_K , counted with multiplicity, tends to zero. No homotopy is required at finitely many initial indices, and all comparisons below are made on the tail. This is the class of Theorem 29.23, and it contains the paths of Lemma 29.10.

Flow data and reference solutions. For an initial potential A_0 on mesh a , its linear and quadratic flow data are the linear heat evolution $x_a(t) = P_t^a A_0$ and the quadratic Picard term $y_a = \mathcal{Q}_a(x_a, x_a)$. We measure them on a horizon $[0, T_0]$ by the norm $X_a^a(x_a) + Y_a^a(y_a)$ of Definitions 24.3 and 4.8, which includes the difference components. Under $\nu_{K|r}$, we write $R_{K|r}$ for this norm of the flow data of the reconstructed field coupled to U as in Theorem 28.8; its gauge-invariant observables have their joint law under $\nu_{K|r}$. By Theorem 28.8, we have

$$\lim_{R \rightarrow \infty} \sup_{K \geq r} \nu_{K|r} \{R_{K|r} > R\} = 0 \quad (30.3)$$

for every fixed $r \geq r_0$. By Theorem 24.10, every smooth trajectory has, on a finite horizon, a finite reference family: smooth solutions of the Yang–Mills heat equation in DeTurck gauge, that is, of the gauge-fixed equation of that theorem, on consecutive time intervals, together with fixed smooth gauge transformations at the transition times. We call each of these solutions a reference solution. On the lattice, after a time-dependent gauge transformation that adds a damping term of the same kind, the Wilson flow is represented by the solution of a damped lattice equation whose linear part is the lattice heat equation (Lemma 24.2). As in Section 25, we call these solutions damped representatives and say that they are in the damped gauge. We compare them with the reference solutions using Lemma 25.5.

30.2 Exact dilation

Once the lattice sites are indexed by $(\mathbb{Z}/N\mathbb{Z})^3$, the Wilson measure (2.1) does not depend on L : the side length enters only through the physical units of time and length. The following theorem transfers this to the continuum measures. We write $\mathcal{X}_1$ for the trajectory space of the unit torus.

Theorem 30.1. Fix an odd integer $M \geq 3$, an equivalence class $[n]_M$, and $\gamma > 0$, and let $\lambda_{L,\gamma,[n]_M}$ be the measure of Theorem 29.28, constructed with any $\Gamma \geq \gamma$. For $L > 0$ define $D_L : \mathcal{X}_1 \rightarrow \mathcal{X}_L$ on representatives by

$$(D_L A)(t, x) = L^{-1} A(t/L^2, x/L).$$

Then D_L is a bijection, a homeomorphism for the Wilson topologies, and a Borel isomorphism. The measures satisfy

$$\lambda_{L,\gamma,[n]_M} = (D_L)_\# \lambda_{1,\gamma,[n]_M}. \quad (30.4)$$

For every $X \in \mathcal{X}_1$ and $t > 0$,

$$LE_L((D_L X)(t)) = E_1(X(t/L^2)). \quad (30.5)$$

In particular, for loops ℓ_i on the unit torus, characters χ_i , and times $\theta_i > 0$, the joint law under $\lambda_{L,\gamma,[n]_M}$ of the Wilson loops $W_{L\ell_i,\chi_i,L^2\theta_i}$ and the energies $LE_L(X(L^2\theta_i))$ does not depend on L .

This independence holds exactly at every finite cutoff for the flowed lattice Wilson loops on the dilated lattice paths. Let $\pi_{k,1}$ be the maps of Theorem 29.28 on the unit torus, and set $\pi_{k,L} = D_L \circ \pi_{k,1}$. These maps are Borel and invariant under lattice gauge transformations on the full configuration spaces of side L , and at every cutoff they have the same dimensionless energy tails and the same coordinate errors $\mathbb{E}|w_j(\pi_{k,L}) - W_{k,j}^{\text{lat}}|$ as on the unit torus, when the determining family on side L is the dilation of that on the unit torus. Consequently, the convergence errors are uniform in $L > 0$ for each fixed bounded Wilson-continuous or bounded d_c -continuous test transported from the unit torus, and for each fixed bounded continuous test of a finite vector of flowed lattice Wilson loops. For other bounded Borel tests the transported errors equal the unit-side errors, and they tend to zero only if the unit-side errors do. No convergence of the energy laws of the maps $\pi_{k,L}$ is asserted, and no uniformity in γ , or for parameters chosen independently at different L , is asserted.

Proof. We first prove the identities at a finite cutoff, then check that D_L is a well-defined map of trajectory spaces that transforms energies and Wilson loops as stated, and then identify the limiting measures. Finally, we construct the maps $\pi_{k,L}$.

Step 1: identities at a finite cutoff. We index the sites by $(\mathbb{Z}/N\mathbb{Z})^3$ and the links by a site and a direction. Under this identification, the finite measures, including their normalizing constants, are identical for all L , since the product Haar measure and the exponent $-4N\gamma^{-1} \sum_p q(U_p)$ in (2.1) do not involve L . The Wilson flow is the gradient flow of $\sum_p q(U_p)$ for the product bi-invariant metric, which does not involve L either. By the definition $\Psi_t^{(N)} = \Phi_{4t/a_N^2}^W$, the physical time t on side L corresponds to the lattice time $4tN^2/L^2$; at $t = L^2\theta$, this is the lattice time $4\theta N^2$ of the physical time θ on the unit torus. Dilation by L sends each edge of the lattice in the unit torus to the corresponding edge of the lattice in $\mathbb{T}_L^3$ and preserves the order of the edges in a path. We conclude that every joint identity for the flowed lattice Wilson loops stated in the theorem holds configuration by configuration, before any expectation is taken.

Step 2: the continuum dilation. The map is well defined on gauge orbits. If k is a periodic gauge transformation of the unit torus, then $k_L(x) = k(x/L)$ is periodic on side

L , and direct differentiation gives $(D_L A)^{k_L} = D_L(A^k)$ at each time. Further,

$$F_{ij}(D_L A)(t, x) = L^{-2} F_{ij}(A)(t/L^2, x/L), \quad \partial_t(D_L A)(t, x) = L^{-3}(\partial_t A)(t/L^2, x/L),$$

and the covariant divergence $\sum_j D_j F_{ji}$ of the curvature scales in the same way as $\partial_t A$, with the factor L^{-3} . Thus D_L maps solutions of the Yang–Mills heat flow to solutions and preserves the compatibility relation $X(t) = \Phi_{t-s} X(s)$. By the uniqueness of smooth covariant solutions recalled in input (C1) of Section 24.6, D_L is a map of trajectory spaces; its inverse is $X \mapsto ((t, y) \mapsto LX(L^2 t, Ly))$. Parallel transport along $u \mapsto L\ell(u)$ has coefficient $L^{-1} A(\ell(u)) \cdot L\dot{\ell}(u) = A(\ell(u)) \cdot \dot{\ell}(u)$, so

$$W_{L\ell, \chi, L^2\theta}(D_L X) = W_{\ell, \chi, \theta}(X). \quad (30.6)$$

Composition with the dilation therefore permutes bijectively the regularized Wilson loops, which generate the Wilson topologies, so D_L is a homeomorphism for these topologies. By Theorem 29.3, it is a Borel isomorphism. Squaring the curvature and changing variables $x = Ly$, we obtain $E_L(D_L A) = L^{-1} E_1(A)$, which is (30.5).

Step 3: identification of the measures. We use the same sequence $N_k = nM^k$ and the dilated lattice paths in the identities of Step 1. Since dilation multiplies lengths by L and swept areas by L^2 , for each fixed L the dilation of an admissible approximation is an admissible approximation; the same holds for the paths of Lemma 29.10. Theorem 29.28 identifies the limiting joint laws of finitely many Wilson loops on both tori. By (30.6) and the determination of probabilities by these laws (Theorem 29.3), we obtain (30.4).

Step 4: a simultaneous choice of maps. We identify the finite link spaces as in Step 1 and set $\pi_{k,L} = D_L \circ \pi_{k,1}$. These maps are Borel and gauge invariant on the full configuration spaces. Since continuous images of compact sets are compact, the image laws have compact containment for each L , and they converge to $(D_L)_\# \lambda_{1,\gamma,[n]_M}$. We choose the determining family on side L by dilating the loops of the unit family by L and its rational times by L^2 . Then $d_{c,L}(D_L X, D_L Y) = d_{c,1}(X, Y)$, and the coordinate errors at each cutoff agree exactly. Further, for every $\theta > 0$ and R ,

$$\mu_{N_k, \gamma} \{LE_L(\pi_{k,L}(L^2\theta)) > R\} = \mu_{N_k, \gamma} \{E_1(\pi_{k,1}(\theta)) > R\}.$$

For a fixed bounded test F on $\mathcal{X}_1$, let $F_L = F \circ D_L^{-1}$ on $\mathcal{X}_L$. Its error at each cutoff, against the limiting measure, is exactly the unit-side error of F . By Theorem 29.28, this error tends to zero for bounded Wilson-continuous F and for bounded d_c -continuous F . The same identity and the same convergence hold for a fixed bounded continuous test of a finite vector of flowed lattice Wilson loops. Finally, the joint identities for energies and Wilson loops under the limiting measures follow from (30.4), (30.5), and (30.6), which do not involve the convergence of lattice energies. $\square$

Remark 30.2. At fixed γ , the continuum coupling $g_3^2 = \gamma/L$ changes with L . At fixed $g_3^2 > 0$, the dimensionless coupling $\gamma = g_3^2 L$ eventually exceeds any fixed Γ as $L \rightarrow \infty$, and Theorem 29.24 gives no uniformity as $\Gamma \rightarrow \infty$. Thus Theorem 30.1 is not a statement about large volumes: it yields no infinite-volume limit, no mass gap, no uniform bound on the unscaled energies, and no comparison between different equivalence classes $[n]_M$. The continuity in L at fixed g_3^2 is Corollary 30.12.

30.3 Energy and loop processes at positive times

In this section, we prove that the processes $t \mapsto E_{a_K}(\Psi_t^{(N_K)} U)$ and $t \mapsto \chi_i(\text{Hol}_{\ell_{i,K}} \Psi_t^{(N_K)} U)$ converge jointly, on every compact interval $[\tau, T] \subset (0, \infty)$ and in the uniform topology, to the corresponding processes under λ . The main difficulty is the energy. The function $X \mapsto E_L(X(t))$ is Borel, but it is not known to be continuous for d_c or for the Wilson topology. The convergence of the image laws $(\pi_K)_\# \mu_{N_K, \gamma}$ therefore does not by itself identify the limit of the lattice energies. We use instead the joint convergence under the conditioned measures in Theorem 29.18, whose proof identifies the energies through the Sobolev topology (Lemma 29.7), together with a comparison of the conditioned and unconditioned limits that holds for all bounded Borel tests.

Lemma 30.3. *Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, $0 < \tau < T < \infty$, finitely many characters χ_i , and lengths $\ell_i^* > 0$. For a mesh $a = L/N$ let $\ell_{i,a}$ be closed lattice paths of physical length at most ℓ_i^* , fixed in time. Let $U \in \mathcal{U}_N$ have a gauge representative whose flow data on $[0, T_0]$ have norm at most R , and set $U(t) = \Psi_t^{(N)} U$. There are $a_0 > 0$ and $C < \infty$ such that, for all $a \leq a_0$ and $t, u \in [\tau, T]$,*

$$\sup_{\tau \leq t \leq T} E_a(U(t)) \leq C, \quad (30.7)$$

$$|E_a(U(t)) - E_a(U(u))| + \sum_i |\chi_i(\text{Hol}_{\ell_{i,a}} U(t)) - \chi_i(\text{Hol}_{\ell_{i,a}} U(u))| \leq C|t - u|.$$

The constants may depend on $L, T_0, R, \alpha, \tau, T$, the characters, the lengths ℓ_i^* , the fixed difference stencil and interpolations, and the margins of the reference solutions used in the proof, but not on the coupling, the cutoff, the depth of the hierarchy, or the particular configuration.

Suppose in addition that, along a sequence $a \rightarrow 0$, the paths $\ell_{i,a}$ form admissible approximations of loops ℓ_i , and that at a fixed time $s < \tau$ a smooth one-form b and the reference solutions of its trajectory are given, such that a representative $A_a(s)$ of $U(s)$, in the gauge of the first reference solution, satisfies $e_a = \|A_a(s) - S_a b\|_{C_a^1} \rightarrow 0$. Then $E_a(U(t))$ and $\chi_i(\text{Hol}_{\ell_{i,a}} U(t))$ converge, uniformly for $t \in [\tau, T]$, to the energy and the Wilson loops at time $t - s$ of the continuum trajectory started at b .

Proof. At a small positive time s , the lattice solutions have bounded differences of every order, so they lie close to samples of a compact family of smooth fields. Finitely many

reference solutions for this compact family control the lattice flow up to time T with uniform bounds on three differences. The equation of the flow then bounds the time derivative in C_a^1 , and the energy and the loops inherit a Lipschitz bound.

Step 1: smoothing at a small positive time. Let T_R be the existence time of Theorem 24.5 for flow data of norm at most R , and fix $0 < s < \min(T_R, T_0, \tau/4)$, with a positive waiting interval before s . The local solution in the damped gauge and its identification with the Wilson flow follow from Theorem 24.5 and Lemma 24.2. By the first part of Theorem 25.7, every fixed order of differences at time s is bounded, uniformly over the configurations in the statement.

These bounds produce a compact family of smooth limiting fields, uniformly over the configurations. We interpolate each value array and each difference array by piecewise multilinear interpolation. A bound on the next difference bounds the modulus of continuity, also across the cells of the lattice. Then the Arzelà–Ascoli theorem and a diagonal extraction give uniform limits of all these arrays along every sequence of meshes tending to zero. Discrete summation by parts identifies the limit of $D_a^\nu A_a$ with the corresponding distributional derivative of the limit of the value arrays, so the limits are smooth. Their bounds at every order imply that the set of all such limiting fields is compact in each fixed C^q norm and consists of smooth fields; closedness follows from a diagonal sequence of approximating lattice fields. Further, as $a \rightarrow 0$, the lattice fields at time s are uniformly close, in value and first differences, to samples of fields in this compact family. Otherwise, an offending sequence and the preceding extraction would give a contradiction.

Step 2: continuation to the time T . We apply Theorem 24.10, with a regularity index at least six, to this compact family of smooth fields. We use the open neighborhood given there, not a claim that families with finitely many bounded derivatives have smooth closure. By the part of Lemma 25.5 on compact families of reference solutions, the restarted lattice solutions in the damped gauge satisfy a uniform C_a^1 bound. The compact set in that lemma is the family just constructed, which is compact in C^3 and in every higher norm, and its thresholds and its finitely many reference solutions are fixed before the configuration or the mesh is chosen. For a quantitative comparison with a given smooth limiting trajectory, we apply the fixed-reference estimate of the same lemma on each time interval; the regularity index six provides the five derivatives that it requires. The incoming error in Lemma 29.15 is measured with sampled first derivatives. By Taylor’s formula, we have $\|D_a S_a b - S_a \partial b\|_\infty \leq Ca \|b\|_{C^2}$, so this error differs from the C_a^1 error by $O(a)$. Thus Lemma 29.15 is an instance of Lemma 25.5. Only after the C_a^1 bound is available do we apply the second part of Theorem 25.7, which gives uniform bounds on three differences.

The waiting times leave no gap at the transition times. By the local existence statement in C^6 of Theorem 24.10, each reference solution extends a fixed positive amount beyond

the right endpoint of its interval. Since the finitely many families of endpoints are bounded, we can choose a common extension time. The comparison in a common gauge extends the lattice representative over this overlap. We choose the waiting time on the next interval shorter than the overlap, so that the bounds of the second part of Theorem 25.7 on the next interval overlap those on the preceding one, and the first waiting interval ends before τ . We obtain finitely many overlapping representations in the damped gauge that cover $[\tau, T]$, each with uniform bounds on three differences. The fixed gauge transformations at the transition times preserve each of these difference bounds. Indeed, for a smooth gauge transformation k , the function $a^{-1} \text{Log}(k(x)e^{aZ}k(x + ae_i)^{-1})$ extends smoothly to $a = 0$, and the exact product rule gives the bound at each order, using finitely many derivatives of k . These derivatives are part of the data of the reference family.

Step 3: the time derivative. In each of these representations, the equation of the flow is exactly

$$\partial_t A_a = \Delta_a A_a + \mathcal{N}_a(A_a, A_a) + a^{-3} R_3(a A_a),$$

by Lemma 24.2. One difference of the Laplacian uses at most three differences of A_a , and one difference of the quadratic term uses at most two. A term of degree $d \geq 3$ in the last summand carries the factor a^{d-3} . Using the absolute convergence of the analytic series and the shifted product rule, we have

$$\|\partial_t A_a\|_{C_a^1} \leq C \quad (30.8)$$

on the bounds of Step 2, with no inverse power of the mesh. In particular, the time derivative of a link variable $e^{aA_{a,i}}$ has norm at most $Ca\|\partial_t A_{a,i}\|_\infty$. Differentiating a path with n_i links and using unitarity, we find that the time derivative of its character is at most $C_{\chi_i} a n_i \|\partial_t A_a\|_\infty \leq C_{\chi_i} \ell_i^* C$, since $a n_i \leq \ell_i^*$. By the Lipschitz estimate of Lemma 29.14 and (30.8), the energy difference is at most $C|t - u|$, and the same lemma bounds the energy itself. Both observables are invariant under the damping gauge transformations and under the fixed gauge transformations at the transition times. Since the lattice flow is continuous in time, the finitely many estimates on the intervals concatenate with the largest constant. This proves (30.7).

Step 4: comparison with a given smooth trajectory. The last assertion uses the given incoming error e_a rather than recovering a limiting field from the flow data. Applying the fixed-reference estimate of Lemma 25.5 on the consecutive intervals of the reference solutions, we find the error $C(e_a + a)$ in C_a^1 up to time T , as in Lemma 29.15. By Lemma 29.14, the energy error is bounded in the same way. For the loops, comparing exact edge transports with sampled link variables and using the curvature-area estimate of Theorem 29.23, we obtain the error $C(e_a + a + A_{i,a})$, where $A_{i,a}$ is the swept area of the comparison homotopy for $\ell_{i,a}$. These errors tend to zero uniformly in $t \in [\tau, T]$. $\square$

Theorem 30.4. *Work in the setting of Section 30.1: fix $0 < \gamma \leq \Gamma$, the sequence $N_K = n_* B^K$, the maps π_K of Theorem 29.21, and the measure λ . Let $0 < \tau < T < \infty$, let $\ell_1, \dots, \ell_m$ be loops, $\chi_1, \dots, \chi_m$ characters of finite-dimensional representations of $\mathrm{SU}(2)$, and $(\ell_{i,K})_K$ admissible approximations of ℓ_i . Then, under $\mu_{N_K, \gamma}$,*

$$\begin{aligned} & \left(t \mapsto (\mathbb{E}_{a_K}(\Psi_t^{(N_K)} U), (\chi_i(\mathrm{Hol}_{\ell_{i,K}} \Psi_t^{(N_K)} U))_{i=1}^m) \right) \\ & \implies \left(t \mapsto (E_L(X(t)), (W_{\ell_i, \chi_i, t}(X))_{i=1}^m) \right), \quad X \sim \lambda, \end{aligned}$$

in distribution in $C([\tau, T]; \mathbb{R} \times \mathbb{C}^m)$ with the uniform norm. If $(\ell_{i,K}^{(1)})_K$ and $(\ell_{i,K}^{(2)})_K$ are two admissible approximations of the same loops, with flowed lattice Wilson loops $W_{K,i}^{(1)}(t)$ and $W_{K,i}^{(2)}(t)$, then

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} \sup_{\tau \leq t \leq T} \max_i |W_{K,i}^{(1)}(t) - W_{K,i}^{(2)}(t)| = 0.$$

In particular, every finite joint vector of lattice energies and flowed lattice Wilson loops at times in $(0, \infty)$ converges in law to the corresponding vector under λ . The same path convergence and the same comparison of two admissible approximations hold along the full sequence $N_k = nM^k$, for the maps and the measure $\lambda_{L, \gamma, [n]_M}$ of Theorem 29.28; this is proved in Section 30.6.

The convergence is tested only by bounded continuous functions on the path space. In particular, the theorem gives no moment bound for time derivatives under the Wilson measure.

Proof. We first recall that the joint path convergence holds under the conditioned measures, with limit λ_r . Then we show that λ_r and λ are close for all bounded Borel tests, which lets us pass to the Wilson measure. The comparison of two approximations is a consequence.

Step 1: conditioned measures. Let $r \geq r_0$, so that (30.1) holds. At fixed r , by Theorem 29.18, the energy and loop paths under $\nu_{K|r}$ converge jointly to the image of λ_r under the continuum path map. The proof of that theorem compares the lattice flows of the full field and of a finite-scale reconstruction, identifies the energies through the Sobolev topology, and uses Theorem 28.8 for the bounds on the flow data; no limiting initial potential is constructed. That theorem applies to every finite union of admissible approximations, so two approximations of the same loops have identical limiting loop paths.

Step 2: comparison of the limits for Borel tests. We now identify the limit of the lattice energies under the Wilson measure with the same λ ; this does not follow from the agreement of the loop marginals alone. By Theorems 29.19 and 29.20, the same maps π_K

have image limit λ_r under $\nu_{K|r}$ and image limit λ under $\mu_{N_K, \gamma}$, for bounded d_c -continuous tests. We claim that

$$\left| \int f d\lambda_r - \int f d\lambda \right| \leq 2\|f\|_\infty F_\gamma(n_r) \quad (30.9)$$

for every bounded Borel function f on $\mathcal{X}_L$. For bounded d_c -continuous f , this follows by applying (30.2) to $f(\pi_K)$ at the same cutoff, using (30.1), and passing to the two image limits. By Theorems 29.3 and 29.6, both limits are tight probability measures on the separable metric space $(\mathcal{X}_L, d_c)$, and in particular inner regular. A bounded Borel function on a metric space can then be approximated in $L^1(\lambda + \lambda_r)$ by bounded continuous functions with the same bound on the supremum. Indeed, we first approximate it uniformly by a simple function $\sum_k c_k \mathbf{1}_{A_k}$ with disjoint A_k and $|c_k| \leq \|f\|_\infty$. We then choose disjoint compact sets $K_k \subset A_k$ that carry all but an arbitrarily small part of the mass of $\lambda + \lambda_r$, then disjoint open neighborhoods $O_k \supset K_k$ of arbitrarily small additional mass, and continuous functions $0 \leq \phi_k \leq 1$ equal to one on K_k and supported in O_k . The function $\sum_k c_k \phi_k$ has the required bound and approximation property. This proves (30.9) for every bounded Borel f , without any continuity of the energy for d_c .

Step 3: the Wilson measure. Let $G(X)$ be the continuum path $t \mapsto (E_L(X(t)), (W_{\ell_i, \chi_i, t}(X))_i)$ on $[\tau, T]$, and $G_K(U)$ the corresponding lattice path. The map G is Borel into $C([\tau, T]; \mathbb{R} \times \mathbb{C}^m)$. Indeed, its Wilson-loop coordinates at rational times are Borel by Theorem 29.3, its energy coordinates are Borel by Theorem 29.6, each trajectory gives continuous paths because the flow is smooth, and evaluations at rational times generate the Borel σ -algebra of the path space. Then (30.9) applies to $f \circ G$ for every bounded continuous path test f . Using (30.2) at the cutoff N_K and then Step 1, we obtain

$$\limsup_{K \rightarrow \infty} \left| \mathbb{E}_{\mu_{N_K, \gamma}} f(G_K(U)) - \int f(G(X)) d\lambda(X) \right| \leq 4\|f\|_\infty F_\gamma(n_r).$$

Letting $r \rightarrow \infty$ proves the convergence along the whole sequence. The tightness of the lattice paths under the Wilson measure can also be seen directly. By Lemma 30.3, on the event $\{R_{K|r} \leq R\}$, the paths lie, for small meshes, in a compact set of the path space given by the Arzelà–Ascoli theorem. We first make $F_\gamma(n_r)$ small, then the probability in (30.3) small, and then take K large. By (30.2), the complement of the compact set has Wilson probability at most its conditioned probability plus $e_{K|r}$. The finitely many remaining cutoffs have compact path images.

The source functionals of admissible approximations also converge locally uniformly. Indeed, path convergence implies the convergence of every finite joint vector of Wilson loops, and the vector is bounded by the dimensions of the representations. On each ball $\{\sum_i |z_i| \leq R'\}$, the functions $e^{\sum_i z_i w_i}$ are bounded and have a common Lipschitz constant in z , so a finite net of the ball gives uniform convergence.

Step 4: two approximations. Applying Theorem 29.18 to the union of the two approximations, we find that the limiting joint path has identical loop paths. Let Δ be the supremum over $t \in [\tau, T]$ of the maximal loop discrepancy; it is continuous in the uniform topology. If the list of loops is empty, there is nothing to prove; otherwise, let $D = \max_i \chi_i(I)$. The bounded continuous test $\min(2D, \Delta)$ equals Δ on all lattice and limiting paths of characters, and it vanishes under the limiting law. Its expectation under $\nu_{K|r}$ then tends to zero. Applying (30.2) at the same cutoff and then letting $r \rightarrow \infty$, we obtain the conclusion under the Wilson measure. $\square$

30.4 Convergence in the Sobolev topology

In [43], Cao and Chatterjee also consider trajectories of Sobolev gauge orbits. Recall from Definition 29.1 that $\mathcal{H}_L = H^1(\mathbb{T}_L^3; \mathfrak{su}(2)^3) / \mathcal{G}_L^{2,2}$ carries the sequential weak quotient topology $\mathcal{T}_q$, that $\mathcal{X}_L^{1,2}$ is the space of trajectories $Z : (0, \infty) \rightarrow \mathcal{H}_L$ with $Z(t) = \Phi_{t-s}^{1,2} Z(s)$ for $0 < s < t$, with the topology $\mathcal{T}_{1,2}$ of pointwise evaluation [43, Definition 3.70, p. 41], and that $i^{1,2} : \mathcal{X}_L \rightarrow \mathcal{X}_L^{1,2}$ sends a smooth orbit at each time to its Sobolev orbit. The compactness behind this topology is the weak compactness of gauge orbits with bounded energy, which goes back to Uhlenbeck [151]; we use it only through the results of Cao and Chatterjee cited below. We consider $\mathcal{T}_{1,2}$ because, for each $t > 0$, the energy $Z \mapsto E_L(Z(t))$ is continuous for it (Lemma 29.7); the energy is not known to be continuous for the Wilson topology. Convergence in $\mathcal{T}_{1,2}$ therefore carries the convergence of the energies of the approximating trajectories.

Theorem 30.5. *Work in the setting of Section 30.1. The maps $i^{1,2} \circ \pi_K$ are Borel on the full configuration spaces, and*

$$(i^{1,2} \circ \pi_K)_\# \mu_{N_K, \gamma} \implies (i^{1,2})_\# \lambda$$

against every bounded $\mathcal{T}_{1,2}$ -continuous real function. The topology on $\mathcal{X}_L$ pulled back by $i^{1,2}$ is finer than $\mathcal{T}_W$ and has the same Borel σ -algebra. For each fixed $r \geq r_0$, the same conclusion holds under the conditioned measures $\nu_{K|r}$, with limit $(i^{1,2})_\# \lambda_r$. The conclusion under the Wilson measures also holds for the maps of Theorem 29.28 along $N_k = nM^k$, with limit $(i^{1,2})_\# \lambda_{L, \gamma, [n]_M}$. The conclusion under conditioned measures is asserted only along the sequences $(n_* B^K)_K$; no sequence of conditioned measures along nM^k is considered, and no conditioned limits along different residue classes are compared. In each case, every finite joint vector of energies at positive times and Wilson loops of the image trajectories converges in distribution.

More generally, let X_K , $K \geq 0$, be Borel $\mathcal{X}_L$ -valued random variables such that, for every $t > 0$, the laws of $E_L(X_K(t))$ are tight, and such that the finite joint laws of the Wilson loops of X_K converge to those of a Borel probability λ' on $\mathcal{X}_L$. Then the laws of $i^{1,2}(X_K)$ converge to $(i^{1,2})_\# \lambda'$ against every bounded $\mathcal{T}_{1,2}$ -continuous real function.

Each assertion has exactly the uniformity of its input convergence: the theorem adds no uniformity in the coupling and no rate.

Proof. We first show that $i^{1,2}$ is a Borel bijection with a Borel inverse that preserves Wilson loops, then prove the general assertion by compactness, and finally check its hypotheses for the maps π_K .

By [43, Lemma 3.84, pp. 47–48], $i^{1,2}$ is Borel. Every trajectory in $\mathcal{X}_L^{1,2}$ has a smooth representative at every positive time [43, Lemma 3.85, p. 48], and choosing its smooth orbit defines a map $j^{1,2} : \mathcal{X}_L^{1,2} \rightarrow \mathcal{X}_L$ [43, Definition 3.86, p. 48]; a Sobolev gauge transformation between two smooth connections is smooth [43, Lemma 3.10, p. 24], so this choice is unambiguous. We have $j^{1,2} \circ i^{1,2} = \text{Id}$ [43, Lemma 3.87, p. 48], and $i^{1,2} \circ j^{1,2} = \text{Id}$ directly at each time. By [43, Lemmas 3.88–3.89, pp. 48–49], $j^{1,2}$ preserves all Wilson loops and is continuous into $\mathcal{T}_W$. Thus $i^{1,2}$ and $j^{1,2}$ are inverse Borel bijections. Since $j^{1,2}$ is continuous, every $\mathcal{T}_W$ -open set $V = (j^{1,2} \circ i^{1,2})^{-1}(V)$ is open for the pulled-back topology, which is therefore finer than $\mathcal{T}_W$. The measurability of $i^{1,2}$ gives the reverse inclusion of Borel σ -algebras; the continuity of $i^{1,2}$ is not used.

We now consider the general sequence. The variables $i^{1,2}(X_K)$ are measurable $\mathcal{X}_L^{1,2}$ -valued, and their energies at every positive time have tight laws, since passing to the Sobolev orbit does not change the energy. These are the hypotheses of [43, Theorem 3.73, pp. 41–42]. Applying this theorem to an arbitrary subsequence, we obtain a further subsequence converging against all bounded $\mathcal{T}_{1,2}$ -continuous functions to a Borel probability measure η on $\mathcal{X}_L^{1,2}$. The regularized Wilson loops on $\mathcal{X}_L^{1,2}$ are bounded and continuous [43, Definition 3.82, p. 47], and they become the usual Wilson loops at positive times after applying $j^{1,2}$ [43, Lemma 3.88, p. 48]. Hence every finite joint law of the Wilson loops under $j_{\#}^{1,2}\eta$ equals that under λ' . Since these laws determine probability measures (Theorem 29.3), we have $j_{\#}^{1,2}\eta = \lambda'$. Applying the inverse Borel map $i^{1,2}$, we find $\eta = (i^{1,2})_{\#}\lambda'$, independently of the subsequence. If the expectations of a bounded continuous test did not converge to the claimed limit, some subsequence would stay at a positive distance from it, which contradicts what we just proved. This proves the general assertion.

For the image laws under the Wilson measures, the hypotheses are the energy tails and the convergence of the finite joint laws of the Wilson loops in Theorems 29.20 and 29.21, which hold in the present setting. For the conditioned measures, they are the corresponding conditioned statements in Theorem 29.19. For the maps along $N_k = nM^k$, they are the convergence of the image laws and the energy tails at every positive time asserted in Theorem 29.28. No other property of the construction of π_K is used. The convergence of finite joint vectors of energies of the image trajectories follows from Lemma 29.7, and the Wilson loops are continuous for $\mathcal{T}_{1,2}$ by the preceding paragraph. The rescaling to side L and all measurability statements are those of Theorem 29.3. $\square$

30.5 Spatial isometries and center symmetry

Two groups act on the trajectory space. The first consists of the isometries $f(x) = Qx + y$ of $\mathbb{T}_L^3$ with $Q \in O(3, \mathbb{Z})$, the signed permutation matrices, and $y \in \mathbb{T}_L^3$; they act at every time by pullback of one-forms, and then on gauge orbits. The second is the group $\{\pm I\}^3$ of center symmetries. On the lattice, a center symmetry multiplies the link variables crossing one coordinate plane by a central element; this is the familiar center symmetry of pure gauge theories on a torus (compare 't Hooft [149]). Both groups preserve the Wilson measure and commute with the Wilson flow, and we pass these exact identities to the limit. For a loop ℓ whose lift to $\mathbb{R}^3$ runs from x to $x + Lw(\ell)$, the vector $w(\ell) \in \mathbb{Z}^3$ is its winding vector; the winding vector of a closed lattice path is defined in the same way. This is the vector denoted by $v(\ell)$ in Theorem 2.5; it is distinct from the determining coordinates w_j .

Theorem 30.6. *Work in the setting of Section 30.1. The measure λ is invariant under the pullback f^* by every isometry $f(x) = Qx + y$ with $Q \in O(3, \mathbb{Z})$ and $y \in \mathbb{T}_L^3$. It is also invariant under the following action of $\{\pm I\}^3$. For $z = (z_1, z_2, z_3) \in \{\pm I\}^3$ choose a smooth map $h_z : \mathbb{R}^3 \rightarrow \text{SU}(2)$ with $h_z(x + Le_i) = z_i h_z(x)$, and set*

$$C_z[A] = [A^{h_z}], \quad A^{h_z} = h_z A h_z^{-1} - (dh_z) h_z^{-1}.$$

The connection A^{h_z} is periodic, the class $C_z[A]$ does not depend on the choice of h_z or of the representative A modulo periodic smooth gauge transformations, and C_z defines a group action on $\mathcal{A}_L/\mathcal{G}_L$ that commutes with the Yang–Mills heat flow, hence an action on $\mathcal{X}_L$. The spatial and center maps are homeomorphisms for the Wilson topology and Borel automorphisms, and they preserve the energy.

For normalized fundamental Wilson loops $W_{\ell,t}(X) = (1/2) \text{tr Hol}_\ell X(t)$, integers $b_1, \dots, b_m \geq 0$, and times $t_1, \dots, t_m > 0$,

$$\mathbb{E}_\lambda \prod_{j=1}^m W_{\ell_j, t_j}^{b_j} = 0 \quad \text{if} \quad \sum_{j=1}^m b_j w(\ell_j) \neq 0 \text{ in } (\mathbb{Z}/2\mathbb{Z})^3.$$

For characters of the irreducible representations of highest weights n_j , the same holds with b_j replaced by $b_j n_j$ in the parity condition. The corresponding selection rule holds at every finite cutoff, with the winding vectors of the lattice paths. All these conclusions also hold for the measure $\lambda_{L, \gamma, [n]_M}$ of Theorem 29.28.

The center symmetry acts on periodic gauge orbits; it is not an additional identification in the quotient by periodic gauge transformations, since h_z is not periodic when $z \neq (I, I, I)$.

Proof. We first prove exact identities on the lattice, then construct the continuum actions and compute their effect on Wilson loops, and finally pass to the limit.

Step 1: lattice symmetries. A lattice translation or a signed permutation of the coordinates relabels the directed edges, inverting a link variable when the orientation of its edge is reversed. It preserves the product Haar measure and the product bi-invariant Riemannian metric. A plaquette path is mapped to a plaquette path, up to cyclic permutation and inversion. Since $q(V) = q(V^{-1})$ and q is a class function, the Wilson action, and hence the Wilson measure (2.1), are preserved. The invariance of Haar measure under inversion also follows from Lemma 5.1 and the uniqueness of the Haar probability measure.

Let $c_e \in \{\pm I\}$, with $c_{\bar{e}} = c_e^{-1}$, be a center-valued function on links such that $\prod_{e \in \partial p} c_e = I$ for every plaquette p . The map $U_e \mapsto c_e U_e$ preserves the product Haar measure, the product metric, and the trace of every plaquette variable, so it preserves the Wilson measure. An explicit choice with monodromy z is $c_e = z_i$ on the positive direction- i links from the plane $x_i = L - a$ to the plane $x_i = 0$, and $c_e = I$ on all other links. Its product around every plaquette is I , and for a closed lattice path ℓ_a ,

$$c(\ell_a) = \prod_{i=1}^3 z_i^{w_i(\ell_a)}, \quad \text{Hol}_{\ell_a}(cU) = c(\ell_a) \text{Hol}_{\ell_a}(U). \quad (30.10)$$

Every such flat center-valued function acts, modulo lattice gauge transformations, as one of these explicit choices. Indeed, we lift the lattice to $a\mathbb{Z}^3$. On the lift, the product of a flat function along a path with fixed endpoints does not depend on the path, since any closed lattice path can be reduced to the trivial path by cancellations and by moves across elementary squares. Thus the lifted function is the lattice differential of a center-valued function on the vertices of $a\mathbb{Z}^3$, whose three monodromies under the periods Le_i are the z_i . For two flat functions with the same monodromies, the ratio of these vertex functions is periodic, so their actions on periodic lattice gauge orbits agree.

An isometry that preserves a smooth function maps its Riemannian gradient to its gradient. All the maps above are isometries of $\mathcal{U}_N$ that preserve the Wilson action, so they commute with the Wilson flow, which exists and is unique globally on the compact configuration space; the rescaling of time by $4/a^2$ preserves this commutation. We conclude that the exact identities of joint laws, and (30.10), hold simultaneously for any finite list of lattice paths, characters, and positive times, without conditioning and without choosing representatives.

Step 2: the continuum actions. Write $z_i = (-I)^{\varepsilon_i}$ with $\varepsilon_i \in \{0, 1\}$, and let

$$h_z(x) = \exp\left(\frac{2\pi}{L}\tau_3 \sum_i \varepsilon_i x_i\right),$$

where $\tau_3 = -i\sigma_3/2$. Since $e^{2\pi\tau_3} = -I$, this map has the required monodromy. Since the z_i are central, A^{h_z} is periodic and smooth whenever A is. If A is replaced by A^k with k

periodic, then $(A^k)^{h_z} = (A^{h_z})^{h_z k h_z^{-1}}$, and $h_z k h_z^{-1}$ is periodic. If h'_z has the same monodromy as h_z , then $h'_z h_z^{-1}$ is periodic and $A^{h'_z} = (A^{h_z})^{h'_z h_z^{-1}}$. This proves both independence claims, for gauge transformations in all homotopy classes. The product $h_z h_{z'}$ has monodromy zz' , so the induced maps on orbits form a group action.

The curvature transforms as $F(A^{h_z}) = h_z F(A) h_z^{-1}$, so the energy is preserved. The Yang–Mills heat equation is covariant under the time-independent smooth gauge transformation h_z on $\mathbb{R}^3$, and both the transformed initial datum and the transformed solution are periodic. By the uniqueness of periodic smooth solutions (Theorem 24.10), C_z commutes with the flow. The same facts for the pullback f^* follow from the chain rule, the orthogonality of Q , the invariance of volume, and the same uniqueness. Hence both kinds of maps act on trajectories.

For a loop ℓ whose lift runs from x to $x + Lw(\ell)$, the transformation rule $H^k(s) = k(\ell(0))H(s)k(\ell(s))^{-1}$ for parallel transport gives

$$\text{Hol}_\ell(A^{h_z}) = h_z(x) \text{Hol}_\ell(A) h_z(x + Lw(\ell))^{-1} = \left(\prod_i z_i^{w_i(\ell)} \right) h_z(x) \text{Hol}_\ell(A) h_z(x)^{-1}. \quad (30.11)$$

For $f(x) = Qx + y$, parallel transport gives $W_{\ell, \chi, t}(f^*X) = W_{f\ell, \chi, t}(X)$. The character of the irreducible representation of highest weight n satisfies $\chi_n(-H) = (-1)^n \chi_n(H)$, since $-I$ acts as $(-1)^n$ on the n th symmetric power of the fundamental representation. For a general finite-dimensional representation ρ , the two eigenspaces of the involution $\rho(-I)$ are invariant subrepresentations, and the transformed character is the difference of their characters. It follows that every Wilson loop composed with either kind of map is a finite linear combination of Wilson loops. Since $\mathcal{T}_W$ is the initial topology of the Wilson loops, both kinds of maps and their inverses are continuous, and by Theorem 29.3 they are Borel.

Step 3: spatial invariance. Fix $y \in \mathbb{T}_L^3$ and $Q \in O(3, \mathbb{Z})$. Let y_K be lattice translations with $d(y_K, y) \leq \sqrt{3}a_K/2$, and set $f_K(x) = Qx + y_K$, which maps the lattice to itself. The exact identity of Step 1 for f_K applies to a finite list of admissible approximations $(\ell_{i,K})_K$. The transformed paths have the same lengths, and the comparison homotopies transformed by f_K connect them to $f_K \ell_i$ with the same swept areas, which tend to zero. We append the translation strip between $f_K \ell_i$ and $f \ell_i$, together with a connecting path between their basepoints. On the universal cover, its swept area is at most $|y_K - y| \text{length}(\ell_i)$, and the same bound holds after projection to the torus, for the swept area counted with multiplicity. Thus the transformed paths form an admissible approximation of $f \ell_i$, with swept areas $A_{i,K} + O(a_K) \rightarrow 0$. Using the identification of the limiting finite joint laws of the Wilson loops in Theorem 29.20 (along nM^k , in Theorem 29.28) together with Theorem 29.23, we pass the exact identities to every finite joint law of the Wilson loops of f^*X and of X under λ . By Theorem 29.3, we have $(f^*)_{\#}\lambda = \lambda$. This proves invariance under continuous as well as lattice translations.

Step 4: center invariance and the selection rule. We now use the exact center identity of Step 1. Since a based homotopy preserves the integer winding vector, for large K the central sign in (30.10) for $\ell_{i,K}$ is the sign in (30.11) for ℓ_i . By the convergence of joint loop laws, the identity passes to every finite joint vector under λ . Since these laws determine probability measures, we have $(C_z)_\# \lambda = \lambda$. For a product of normalized fundamental Wilson loops raised to the powers b_j , this invariance multiplies the expectation by $\prod_i z_i^{\sum_j b_j w_i(\ell_j)}$. If the vector $\sum_j b_j w(\ell_j)$ has a nonzero component modulo two, we choose $z_i = -I$ at such a component and $z_i = I$ elsewhere. Then the multiplier is -1 , so the expectation equals its negative and vanishes; the variables are bounded, so no integrability issue arises. The rule for irreducible characters follows in the same way, with the additional parities n_j , and the rule at a finite cutoff follows directly from the exact identities of Step 1. For the measure $\lambda_{L,\gamma,[n]_M}$ along the full sequence, see Section 30.6. $\square$

30.6 Collected conclusions and the full sequence

We collect the conclusions of Sections 30.3 to 30.5 in the setting of Theorem 29.21, and then prove their extension to the full sequence $N_k = nM^k$.

Corollary 30.7. *At every fixed $0 < \gamma \leq \Gamma$, in the setting of Theorem 29.21, all conclusions of Theorems 30.4 to 30.6 hold, with admissible approximations fixed in time. In particular, the joint processes of lattice energies E_a and flowed lattice Wilson loops converge in distribution in the uniform topology on $[\tau, T]$ to the corresponding processes under the same measure λ , and the expected supremum of the loop discrepancy between two admissible approximations tends to zero. The image laws of the maps π_K in the Sobolev topology converge, together with every finite joint vector of energies and Wilson loops of the image trajectories. The measure λ is invariant under the spatial and center symmetries, and the winding selection rules hold. These conclusions hold at each fixed coupling; no modulus of convergence for the paths or in the Sobolev topology is uniform in the coupling.*

Proof. Theorems 30.4 to 30.6 are formulated and proved at every fixed $0 < \gamma \leq \Gamma$, for the maps and the measure of Theorem 29.21, and their inputs have no further restriction on γ . The deterministic Lemma 30.3 uses Theorems 24.5 and 25.7 and Lemma 25.5, which have no restriction on the coupling, together with the energy estimates of Lemma 29.14; its Lipschitz bound in time uses only the lengths of the paths. The bounds on the flow data are tight by Lemma 28.11. The joint loop limits used in the proof of Theorem 30.6 are given by Theorems 29.21 and 29.23; the strip added when a continuous translation is approximated by lattice translations has vanishing swept area and is covered by Theorem 29.23. The Sobolev assertion requires only the energy tails and the convergence of the finite joint laws of the Wilson loops, which are part of Theorem 29.21.

We also record a second proof of the convergence of the conditioned path laws used in Step 1 of the proof of Theorem 30.4. By Lemma 30.3, on each event $\{R_{K|r} \leq R\}$ and for small meshes, the joint energy and loop paths are bounded and Lipschitz in time with constants depending on R . By (30.3) and the Arzelà–Ascoli theorem, the conditioned path laws are tight. By Theorems 29.21 and 29.23, their joint laws at rational times are those of the paths under λ_r , and these laws determine a probability measure on the continuous path space. Hence the conditioned path laws converge to the image of λ_r under the continuum path map, which is Borel because its coordinates at rational times are Borel and its paths are continuous. This argument does not require the energy to be continuous in the Wilson topology. The passage to the Wilson measure is then Steps 2 and 3 of the proof of Theorem 30.4. $\square$

We now prove the assertions on the full sequence $N_k = nM^k$ in Theorems 30.4 to 30.6, which are also the additional assertions of Theorem 29.28 on paths, the Sobolev topology, and symmetries.

Proof of the assertions on the full sequence. We take the consecutive integers $J, J+1$ of Corollary 23.15 and the maps used in Theorem 29.28. In this paragraph, $r \in \{0, \dots, J-1\}$ denotes a residue index. For each residue r , the subsequence $nM^{r+l}, l \geq 0$, is a sequence of the form n_*B^l with base $nM^r \in 4\mathbb{N}$, and the maps of Theorem 29.28 on it are those of Corollary 29.26.

On every such residue-class subsequence, Theorem 30.4, applied through Theorem 29.21, yields the convergence of the energy and loop paths and the comparison of two admissible approximations. By Theorem 29.28, the limiting measure is the same $\lambda_{L,\gamma,[n]_M}$ for every residue. The continuum path map is Borel: its energy coordinates at fixed times are Borel by Theorem 29.6, its Wilson-loop coordinates are Borel by Theorem 29.3, its paths are continuous, and the Borel σ -algebra of the uniform path space is generated by the evaluations at a countable dense set of times. Hence the limiting path law is the same for every residue. For a bounded continuous path test, the expectations converge along each of the finitely many residue classes, and taking the largest of the finitely many thresholds proves the convergence along the whole sequence. The same finite maximum proves the comparison of two admissible approximations.

For the Sobolev assertion, the general part of Theorem 30.5 applies directly to the interlaced maps, since Theorem 29.28 gives the tightness of the energies of the image trajectories at every positive time and the convergence of all joint laws of determining coordinates. Alternatively, the Sobolev limit along each residue class is the same Borel image $(i^{1,2})_{\#}\lambda_{L,\gamma,[n]_M}$, and the finite maximum argument above applies to every bounded continuous test for $\mathcal{T}_{1,2}$. Lemma 29.7 identifies the finite joint vectors of energies and Wilson loops of the image trajectories.

Finally, the limiting measure equals, for example, the measure along the sequence $n(M^{J+1})^J$ used in the proof of Theorem 29.28. Theorem 30.6 for that sequence therefore gives the spatial and center invariance of this measure, and the selection rule is an identity for this measure. The proof uses only fixed couplings and finitely many values of J , so no uniformity in the coupling, in Γ , or in J is involved. $\square$

30.7 Continuity and uniform convergence in the coupling

In this section, the coupling varies in a compact interval $I = [\gamma_-, \gamma_+] \subset (0, \infty)$, $\gamma_- > 0$. We show that the convergence statements for finite vectors of Wilson loops and for the image laws specified in Theorem 30.10 hold uniformly for $\gamma \in I$, with approximating maps that do not depend on γ , and that $\gamma \mapsto \lambda_{L, \gamma, [n]_M}$ is weakly continuous. The main point is the order of the choices: the parameters, the minimum coarse side, and the sequence $m(K)$ that determines the maps must be fixed before $\gamma \in I$ is chosen. To that end, we first prove, for each fixed number of constraints, that the probability that no element of the countable family of Lemma 29.8 satisfies the constraints tends to zero uniformly in $\gamma \in I$. Only then do we choose $m(K)$.

Constants with a subscript I may depend on both endpoints $\gamma_{\pm}$, on the coarse side, on L, M, J, Θ , on the analytic radii, the exponents, and the moment orders, and on the stated lists of times and observables. They do not depend on $\gamma \in I$, on the depth of the hierarchy, or on the finer mesh; that is, they have the dependence described in item (7) of Definition 4.14, with the additional dependence on I . Lattice paths are admissible approximations as in Section 30.1; they and their swept areas do not depend on γ . The first of two lemmas restates the deterministic comparison of Theorem 29.16 at finitely many observation times.

Lemma 30.8. *Fix $L, T_0, R > 0$, $1/2 < \alpha < 2/3$, $0 < \tau \leq T < \infty$, and a finite list $(\ell_{h,a}, \chi_h, t_h)$ of closed lattice paths, characters, and times $t_h \in [\tau, T]$, where the paths have physical lengths bounded by fixed numbers. For $n = L/a \in \mathbb{N}$ set*

$$W_h^{\text{lat}}(U) = \chi_h(\text{Hol}_{\ell_{h,a}} \Psi_{t_h}^{(n)} U).$$

Consider two flow data (x_a, y_a) and (x'_a, y'_a) on the same lattice and in the same gauge, of initial configurations U and U' , with norms on $[0, T_0]$ at most R , and set $d_a = X_a^a(x_a - x'_a) + Y_a^a(y_a - y'_a)$, where both norms include their difference components. There are $a_* > 0$, $d_* > 0$, and $C_R < \infty$, depending only on the displayed deterministic data, such that $d_a \leq d_*$ and $a \leq a_*$ imply

$$\sup_{\tau \leq t \leq T} |E_a(\Psi_t^{(n)} U) - E_a(\Psi_t^{(n)} U')| + \max_h |W_h^{\text{lat}}(U) - W_h^{\text{lat}}(U')| \leq C_R d_a,$$

and each of these energies is at most C_R on $[\tau, T]$. The constants do not depend on the coupling, the law, the cutoff, the depth of the hierarchy, the index of a finite-scale reconstruction, or the

derivatives of its arrays. For a fixed finite-scale reconstruction, interpolated by the cardinal interpolation of Lemma 16.3, the joint vector of lattice energies and flowed lattice Wilson loops converges, uniformly over the bounded set of its arrays on the coarse mesh, to the corresponding vector of the continuum flow from the interpolated field, for admissible approximations.

Proof. We work in the common gauge in which d_a is measured and apply Theorem 29.16 with the window $[\tau, T]$ and the given length bounds. After enlarging the constant, the supremum over time in that theorem bounds the discrepancy of each loop at its own time t_h , as well as the energy discrepancy, and its bound on the individual energies gives the bound C_R . We take d_* below the threshold for the discrepancy in Lemma 25.9. These are deterministic results with no restriction on the coupling. For the bounded set of coarse arrays, the uniform consistency of the joint energy and loop paths in Lemma 29.17 implies the stated consistency of the finite vector. Only this consistency constant may depend on the derivatives of the arrays of the fixed finite-scale reconstruction. $\square$

For the second lemma, fix a coarse side n and write $a_q = (L/n)B^{-q}$. Under $\nu_{nB^k, n, \gamma}$, let $\hat{\theta}_q$ be the link logarithms of the field reconstructed at scale q by the reconstruction maps (Theorem 15.14), and let $f_q = I_{a_q \rightarrow 0}(\hat{\theta}_q/a_q)$ be its cardinal interpolation to a smooth one-form on $\mathbb{T}_L^3$ (Lemma 16.3). We write $X^{[q]}$ for the continuum trajectory started at f_q , and $d_{k,q}$ for the norm of the difference between the flow data of the fields reconstructed at scales k and q , computed on the lattice of mesh a_k , as in Lemma 28.10.

Lemma 30.9. Fix $L > 0$, $I = [\gamma_-, \gamma_+] \subset (0, \infty)$, $\Gamma \geq \gamma_+$, the parameters of Theorem 23.5, and a coarse side n above the common threshold of Lemma 28.10. Fix a finite list of energies at positive times and of admissible approximations of loops with characters and positive times. Under $\nu_{nB^k, n, \gamma}$, let $Z_{k, \gamma}$ be the joint vector of lattice energies and flowed lattice Wilson loops, and let $Z_{k, \gamma}^{[q]}$ be the joint vector of continuum energies and Wilson loops of $X^{[q]}$, computed on the same sample. Then

$$\lim_{q \rightarrow \infty} \limsup_{k \rightarrow \infty} \sup_{\gamma \in I} \mathbb{E}_{\nu_{nB^k, n, \gamma}} \min(1, |Z_{k, \gamma} - Z_{k, \gamma}^{[q]}|) = 0. \quad (30.12)$$

For each fixed q , the laws of $Z_{k, \gamma}^{[q]}$ converge as $k \rightarrow \infty$, uniformly in $\gamma \in I$, to laws that depend continuously on γ and have a common compact support. For each fixed $t > 0$, the tails of the lattice energy at time t under $\nu_{nB^k, n, \gamma}$ are uniform over $\gamma \in I$ and over all cutoffs.

Proof. The coarse side satisfies $n \geq n_I \geq n_\Gamma$. By Theorem 23.5, all structural, image, product, and coupling conditions of Theorem 15.6 hold for every $\gamma \in I$. For fixed q , the array on the coarse mesh lies in a fixed compact set, and the continuum vector is a continuous function of this array by Theorem 24.10. The assertion about the laws at fixed q , including the continuity in γ and the compact support, then follows from Theorem 15.17.

Let $\rho_I(R)$ be the tail of the norm of the flow data in (28.18), and let $d_*(R+1)$ be the threshold for the discrepancy in Lemma 30.8 at the radius $R+1$. Fix $0 < \delta \leq \min\{1, d_*(R+1)\}$. On the event that the flow data of the full field have norm at most R and $d_{k,q} \leq \delta$, the flow data of the full field and of the field reconstructed at scale q both have norm at most $R+1$, and the hypotheses of Lemma 30.8 hold for all sufficiently small meshes. Applying that lemma, followed by its consistency statement for the fixed scale q , we obtain

$$|Z_{k,\gamma} - Z_{k,\gamma}^{[q]}| \leq C_R \delta + \omega_q(a_k), \quad \omega_q(a_k) \longrightarrow 0,$$

uniformly over the bounded set of coarse arrays and over γ ; the mesh threshold for the first comparison does not depend on q . This implies that the expectation in (30.12) is at most

$$\rho_I(R) + \sup_{\gamma \in I} \nu_{nB^k, n, \gamma} \{d_{k,q} > \delta\} + C_R \delta + \omega_q(a_k).$$

With R and δ fixed, we first take the limit superior in k at fixed q , then let $q \rightarrow \infty$ using (28.17), then let $\delta \rightarrow 0$, and finally let $R \rightarrow \infty$. This proves (30.12). The energy tails follow from the uniform tail $\rho_I(R)$ and the deterministic bound on the energy for flow data of norm at most R in Lemma 30.8. The finitely many coarse cutoffs below the mesh threshold are included by the bound $E_a \leq 48N^3/a$, which holds because $0 \leq q(U_p) \leq 2$ for every plaquette and there are $3N^3$ positive plaquettes. This yields the supremum over all cutoffs. $\square$

Theorem 30.10. Fix $L > 0$, an odd integer $M \geq 3$, and $I = [\gamma_-, \gamma_+] \subset (0, \infty)$ with $\gamma_- > 0$. Choose $\Gamma \geq \gamma_+$ and the parameters of Theorem 23.5. There is a minimum coarse side n_I such that, on every sequence $N_K = n_* B^K$ with $n_* \in 4\mathbb{N}$, after a finite shift of the base, common to all $\gamma \in I$, if $n_* < n_I$, the following statements hold, with $\lambda_\gamma = \lambda_{L, \gamma, [n_*]_M}$.

1. For every fixed finite vector of flowed lattice Wilson loops on admissible approximations, the laws under $\mu_{N_K, \gamma}$ converge to the corresponding law under λ_γ in the bounded-Lipschitz distance, uniformly for $\gamma \in I$. The complex source functionals and each of their fixed derivatives converge uniformly for $\gamma \in I$ and z in each fixed finite ball.
2. There are Borel maps π_K on the full configuration spaces, invariant under periodic lattice gauge transformations and independent of γ and of any coarse side chosen afterwards, such that for each fixed j and $t > 0$,

$$\sup_{\gamma \in I} \mathbb{E}_{\mu_{N_K, \gamma}} |w_j(\pi_K) - W_{K,j}^{\text{lat}}| \longrightarrow 0, \quad \lim_{R \rightarrow \infty} \sup_{K \geq 0, \gamma \in I} \mu_{N_K, \gamma} \{E_L(\pi_K(t)) > R\} = 0.$$

For every further fixed finite vector $\mathscr{W}$ of Wilson loops on admissible approximations, with lattice counterpart $\mathscr{W}_K^{\text{lat}}$, and every bounded continuous test f of that vector,

$$\sup_{\gamma \in I} \mathbb{E}_{\mu_{N_K, \gamma}} |f(\mathscr{W}(\pi_K)) - f(\mathscr{W}_K^{\text{lat}})| \longrightarrow 0.$$

The same statements on determining coordinates and energy tails hold uniformly in $\gamma \in I$ under every fixed conditioned measure $\nu_{N_K, n_r, \gamma}$ with $n_r = n_* B^r \geq n_I$.

3. For every bounded $\mathcal{T}_W$ -continuous function F on $\mathcal{X}_L$,

$$\sup_{\gamma \in I} |\mathbb{E}_{\mu_{N_K, \gamma}} F(\pi_K) - \lambda_\gamma(F)| \longrightarrow 0.$$

The convergence is also uniform in the bounded-Lipschitz distance of $(\mathcal{X}_L, d_c)$. The image laws and the limiting measures have common compact containment in the Wilson topology and in d_c .

4. The map $\gamma \mapsto \lambda_\gamma$ is weakly continuous on I for both topologies, and the law of every fixed finite vector of Wilson loops and its source functional depend continuously on γ . These statements hold on all of $(0, \infty)$, by covering it with compact intervals.

The same conclusions under the Wilson measures hold along the full sequence $N_k = n_* M^k$. The parameters, n_I , and the sequence $m(K)$ defining the maps are fixed before $\gamma \in I$ is chosen; the constants may depend on I and on the sequence.

Remark 30.11. The constants may diverge as $\gamma_- \rightarrow 0$ or $\gamma_+ \rightarrow \infty$, and no rate of convergence is asserted. The uniformity concerns each fixed finite list and each fixed bounded test; it is not asserted for the path laws of Theorem 30.4, for the Sobolev laws of Theorem 30.5, or for the energy laws of the image trajectories, whose convergence at each fixed γ follows from Theorem 30.5. The maps chosen for two overlapping intervals may differ, but their limiting measures agree on the overlap, since these are determined by the joint laws of Wilson loops. Finally, if $I \subset (0, \gamma_0]$ for the coupling bound $\gamma_0 \leq 1$ of Theorem 23.2, one may take $\Gamma = \gamma_0$; by the clause of Theorem 23.5 that retains a prescribed J , the parameter tuple of Theorem 23.2 can then be kept, with one common enlargement of the minimum coarse side.

Proof of Theorem 30.10. The proof has six steps. We fix the parameters and n_I , and then prove uniform convergence of finite vectors of Wilson loops. Next, we prove that the failure probability at each fixed level tends to zero uniformly, choose the maps by a diagonal argument, and treat the early cutoffs. Finally, we deduce the convergence of the image laws, the continuity in γ , and the statement along the full sequence.

Step 1: parameters. Theorem 23.5 gives the common parameter tuple. We fix the horizon $T_0 = L^2$ and the exponents of the norms of the flow data before any list of observables is chosen. All later lists use this horizon; their positive waiting times and later horizons affect the deterministic flow constants and the mesh thresholds, but not the minimum coarse side. Let $t_* > 0$ be so small that the finitely many smallness conditions of Lemma 28.10 on the largest effective coupling $t_1 = (\gamma/(Mn))^{1/2}$ hold whenever $t_1 \leq t_*$, and take $n_I \geq \max(n_\Gamma, \gamma_+/(Mt_*^2))$, rounded up to an element of $4\mathbb{N}$; then $t_1 \leq t_*$ for all

$\gamma \in I$ and $n \geq n_I$. All other conditions are already included in n_Γ . No local constant is chosen using n_I or a later failure-probability constant, and every coarse side $n \geq n_I$ in the sequence is allowed with the same tuple.

Step 2: finite vectors of Wilson loops. At a fixed coarse side, the Wilson loops lie in a fixed compact box, since each is bounded by the dimension of its representation. By Lemma 30.9, the approximation errors are uniformly small, and for fixed q the approximating laws converge uniformly and depend continuously on γ . For a test function f with $\max(\|f\|_\infty, \text{Lip } f) \leq 1$, that is, with bounded-Lipschitz norm at most one, the triangle inequality between two cutoffs shows that the expectations under the conditioned measures form a Cauchy sequence uniformly in $\gamma \in I$; passing to a common finer cutoff, we find that the same holds for the limiting approximating laws at two indices q, q' . The probability measures on the compact box form a compact, hence complete, space for the bounded-Lipschitz distance. To see this, we partition the box into nested finite families of half-open cubes with diameters tending to zero, assigning the outer boundary to the last cells. From any sequence of laws, we extract by a diagonal argument a subsequence along which the mass of every cell converges. The limiting weights are consistent under refinement and sum to one. We subdivide $[0, 1]$ into nested intervals with these lengths, so that the intervals of the children partition that of the parent. Apart from the countably many endpoints of these intervals, each point of $[0, 1]$ selects a nested chain of cells, whose closures determine one point of the box. This map is measurable as a limit of step functions, and its image of Lebesgue measure is a probability measure on the box. At each level of the partition, the integral of a test with bounded-Lipschitz norm at most one against either the laws of the subsequence or this law differs from the corresponding weighted sum of values at fixed representatives of the cells by at most the largest cell diameter. Since the cell masses converge, the subsequence converges to this law in the bounded-Lipschitz distance. Hence for each coarse index r , the laws of the vector under the conditioned measures have a unique limit P_γ^r , and the convergence is uniform on I . The same approximation shows that P_γ^r is continuous in γ , as a uniform limit of the continuous laws at fixed q . The limits for different finite lists are compatible, since the laws at each cutoff are.

We write $n_r = n_* B^r$ and $\nu_{K|r, \gamma} = \nu_{N_K, n_r, \gamma}$. For $u \geq n_\Gamma$, the right side of (23.13) is nondecreasing in $\gamma \in (0, \Gamma]$, since $Mu/\gamma \geq Mn_\Gamma/\Gamma > 1$. Let $F_\Gamma(u)$ denote its value at $\gamma = \Gamma$. By Theorem 23.10, we have

$$\sup_{K \geq r, \gamma \in I} \mu_{N_K, \gamma}(S_{N_K, n_r, \gamma}^c) \leq F_\Gamma(n_r), \quad F_\Gamma(n_r) \longrightarrow 0 \quad (r \rightarrow \infty).$$

By (30.2), at each cutoff, the expectation of a test with bounded-Lipschitz norm at most one under the Wilson measure differs from that under the conditioned measure by at most $2F_\Gamma(n_r)$. Taking the uniform limit in K at fixed r , and then letting $r \rightarrow \infty$, we obtain

uniform convergence of the vector laws under the Wilson measures. By Theorems 29.23 and 29.24, their pointwise limit is the law of the vector under λ_γ . The continuity in γ follows from the continuity of the conditioned limits and the same uniform error. For the source functionals, since the coordinates are bounded, the exponentials and their fixed derivatives in z are uniformly bounded and uniformly continuous on a fixed complex ball; a finite net of the ball gives the stated uniformity. In particular, fixed moments and cumulants converge uniformly and depend continuously on γ . On a common small ball of sources, the logarithm with value zero at the origin is defined, since $|Z(z) - 1| \leq e^{D \sum_h |z_h|} - 1 < 1/2$ there, where D bounds the dimensions of the representations. The uniform convergence of the fixed derivatives then implies the convergence of the derivatives of the logarithms of the source functionals at the origin, with no additional nonvanishing hypothesis.

The tails of the lattice energy under the Wilson measures are also uniform. At fixed r , the conditioned tail is uniform by Lemma 30.9, and by (30.2), the Wilson probability of a tail event is at most its conditioned probability plus $F_\Gamma(n_r)$. We take first the limits in the cutoff and in the size of the tail at fixed r , and then let $r \rightarrow \infty$. The finitely many earlier cutoffs are included by the bound $E_a \leq 48N^3/a$.

Step 3: uniform feasibility at a fixed level. Let $(X_b)_{b \geq 1}$ be the countable family of continuum trajectories of Lemma 29.8, started at the trigonometric polynomials with rational coefficients, with $X_1 = 0$; it does not depend on γ . Set $s_i = L^2 2^{-i}$ and $\epsilon_m = 2^{-m}$, and let $\pi_{K,m}(U)$ be the first X_b that satisfies

$$\begin{aligned} |w_j(X_b) - W_{K,j}^{\text{lat}}(U)| &< \epsilon_m \quad (j \leq m), \\ E_L(X_b(s_i)) &< E_{a_K}(\Psi_{s_i}^{(N_K)} U) + \epsilon_m \quad (i \leq m), \end{aligned} \tag{30.13}$$

and the zero trajectory if there is no such b ; this is the construction of Theorem 29.19. We say that level m fails at U if no b satisfies (30.13).

Fix m , a coarse index r with $n_r \geq n_I$, and $\eta > 0$. By Lemma 30.9, applied to the m determining coordinates and the m energies in (30.13), there exists a fixed q such that, for all sufficiently large K and uniformly in $\gamma \in I$, the lattice vector differs from the continuum vector of $X^{[q]}$ on the same sample by less than $\epsilon_m/4$ with conditioned probability at least $1 - \eta$. To deduce this from the truncated expectation in (30.12), we apply Markov's inequality with the threshold $\min(1, \epsilon_m/4)$ and make that expectation smaller than $\eta \min(1, \epsilon_m/4)$. By Lemma 29.8, for every realization of $X^{[q]}$, there exists an element of the countable family whose joint vector differs from that of $X^{[q]}$ by less than $\epsilon_m/4$. Then the strict inequalities (30.13) hold, since every loop error is less than $\epsilon_m/2$ and every energy of the family element is less than the lattice energy plus $\epsilon_m/2$. Thus the conditioned probability that level m fails is at most η , uniformly in $\gamma \in I$. No measurable choice of the approximating element is needed, since its existence places the

configuration in the countable union of the sets where (30.13) holds. This proves uniform feasibility at each fixed level directly, without a limiting initial potential and without using the compactness of I to make a pointwise statement uniform.

For the failure event under the Wilson measure, (30.2) bounds the limit superior in K by $F_T(n_r)$ for every fixed r with $n_r \geq n_I$, uniformly in $\gamma \in I$. Letting $r \rightarrow \infty$ after this limit, we find that the Wilson probability that level m fails also tends to zero uniformly. For each K and m , the conditions in (30.13) are strict inequalities between continuous gauge-invariant functions of the configuration. Therefore the events that a given b is the first to satisfy them, and the failure event, are Borel, and $\pi_{K,m}$ is Borel and invariant under periodic lattice gauge transformations. The conditions do not involve γ .

Step 4: the diagonal choice. We enumerate the coarse indices r with $n_r \geq n_I$ as $r(1), r(2), \dots$. By Step 3, we can choose increasing integers $K_m \geq \max(m, r(1), \dots, r(m))$ such that for every $K \geq K_m$ the probability that level m fails is at most 2^{-m} , uniformly in $\gamma \in I$, under the Wilson measure and under each of the first m conditioned measures. Set $m(K) = \max\{m : K_m \leq K\}$, or $m(K) = 0$ if this set is empty, and $\pi_K = \pi_{K,m(K)}$. When level $m(K)$ does not fail, the error of each coordinate w_j with $j \leq m(K)$ is less than $2^{-m(K)}$; when it fails, the error is at most two. Thus each fixed coordinate has error in L^1 at most $3 \cdot 2^{-m(K)}$, uniformly in $\gamma \in I$, under the Wilson measure and eventually under each fixed conditioned measure. On every configuration and for $i \leq m(K)$, we have

$$E_L(\pi_K(s_i)) \leq E_{a_K}(\Psi_{s_i}^{(N_K)} U) + 2^{-m(K)}, \quad (30.14)$$

since on failure the output is the zero trajectory, whose energy is zero. Combined with the uniform tails of the lattice energies from Step 2, this yields uniform tails of the energies of the image trajectories for all sufficiently large K . By the monotonicity of the continuum energy along the flow, these tails hold at every $t > 0$.

Step 5: the early cutoffs. It remains to include the finitely many cutoffs below this threshold, uniformly in γ ; the finiteness of the energy of each image trajectory, pointwise in γ , would not suffice. On a fixed finite link space, the densities of the Wilson measures with respect to Haar measure are positive and continuous on the product of the compact link space with I , hence bounded above and below by positive constants. For a fixed conditioned measure, the small-field event contains one open neighborhood of the identity configuration for all $\gamma \in I$. Indeed, its finitely many conditions on charts, constraint Jacobians, and fluctuation variables hold with strict margins at the identity, and the normalized fluctuation variables are continuous near the identity and vanish there. Further, every cutoff radius $A_w \log(1/t_k)$ has a positive minimum over I because $t_k \leq t_* < 1$. We shrink one neighborhood O so that it satisfies all these finitely many conditions. Then $\mu_{N_K, \gamma}(S_{N_K, n_r, \gamma}) \geq c H_{N_K}(O) > 0$ uniformly in $\gamma \in I$, and the densities of the conditioned measures are bounded above by $C_{K,r,I}$ times Haar measure. For each

configuration, $\pi_K(U)$ is an element of the countable family or the zero trajectory, and has finite energy. Its tail under Haar measure tends to zero by continuity from above, and the domination by Haar measure makes this convergence uniform in γ . Hence the finitely many early cutoffs can be included, and the energy tails hold with the supremum over all cutoffs, without assuming a deterministic bound on the energies of all elements of the countable family.

Step 6: convergence of the image laws, continuity, and the full sequence. By Theorem 29.6, the image laws under the Wilson measures have common compact containment. At each fixed γ , the coordinate errors and the convergence of finite vectors identify every subsequential limit of the image laws with λ_γ , by the determination of probabilities in Theorems 29.3 and 29.20. The image laws therefore converge to λ_γ . Fix $\eta > 0$ and choose a common compact set $\mathcal{K}(R)$ of Theorem 29.6 with $(\pi_K)_\# \mu_{N_K, \gamma}(\mathcal{K}(R)) \geq 1 - \eta$ for all $K \geq 0$ and $\gamma \in I$. This set is d_c -compact and hence closed. The closed-set inequality in the portmanteau theorem therefore gives

$$\lambda_\gamma(\mathcal{K}(R)) \geq \limsup_{K \rightarrow \infty} (\pi_K)_\# \mu_{N_K, \gamma}(\mathcal{K}(R)) \geq 1 - \eta, \quad \gamma \in I.$$

Thus the family of limits has the same compact containment.

Let C be a common compact set whose complement has probability at most η under all image laws and all limits. On C , the Wilson topology and the d_c topology agree. By Lemma 29.9, a bounded Wilson-continuous F with $|F| \leq D$ is approximated within δ on C by $G(w_1, \dots, w_b)$, with G continuous on $[-1, 1]^b$ and $|G| \leq D$. Using the coordinate errors and the uniform continuity of G , we compare the expectation of G on the image with its expectation on the lattice coordinates, uniformly in γ , and the latter converges uniformly by Step 2. The two complements of C contribute at most $4D\eta$. Therefore

$$\limsup_{K \rightarrow \infty} \sup_{\gamma \in I} |\mathbb{E}_{\mu_{N_K, \gamma}} F(\pi_K) - \lambda_\gamma(F)| \leq 2\delta + 4D\eta,$$

and we let $\delta, \eta \rightarrow 0$. For the class of functions with bounded-Lipschitz norm at most one for d_c , the restrictions to C have a finite uniform net. To construct it, we take a finite fine net of C , round the values of each test at its points, and choose one test realizing each rounded list of values. The Lipschitz bound controls the error between net points. The uniform convergence for these finitely many tests and the common bound on the complements give uniform convergence in the bounded-Lipschitz distance.

For a further fixed finite vector $\mathscr{W}$ and a bounded continuous test f , the function $h = f \circ \mathscr{W}$ is bounded and Wilson-continuous; let G be its approximation on C as above, and write $p_b X = (w_j(X))_{j \leq b}$ and $\mathbf{W}_{K,b}^{\text{lat}} = (W_{K,j}^{\text{lat}})_{j \leq b}$. Then

$$\begin{aligned} |f(\mathscr{W}(\pi_K)) - f(\mathscr{W}_K^{\text{lat}})| &\leq |h(\pi_K) - G(p_b \pi_K)| + |G(p_b \pi_K) - G(\mathbf{W}_{K,b}^{\text{lat}})| \\ &\quad + |G(\mathbf{W}_{K,b}^{\text{lat}}) - f(\mathscr{W}_K^{\text{lat}})|. \end{aligned}$$

The expectation of the first term is at most $\delta + 2D\eta$, and that of the second tends to zero uniformly by the coordinate errors. The last term is a bounded continuous test of the single joint lattice vector consisting of the determining coordinates and $\mathcal{W}_K^{\text{lat}}$. By Step 2, its expectation converges uniformly to $\mathbb{E}_{\lambda_\gamma} |G \circ p_b - h| \leq \delta + 2D\eta$. This proves the statement for further vectors without combining separately given marginals. For complex tests, we apply the argument to the real and imaginary parts.

At each fixed N , the density $Z_{N,\gamma}^{-1} \exp(-4N\gamma^{-1} \sum_p q(U_p))$ of the Wilson measure is continuous in $L^1(H_N)$ as a function of $\gamma \in I$, since the numerator is bounded and jointly continuous and the normalizing constant is positive and continuous. Hence the finite measures are continuous in total variation. Since the maps π_K do not depend on γ , the expectation of every bounded Borel function of π_K is continuous in γ . Combined with the uniform convergence for bounded Wilson-continuous F , this proves the weak continuity of $\gamma \mapsto \lambda_\gamma$ for the Wilson topology. Since the identity map from the Wilson topology to d_c is continuous, we also obtain weak continuity for d_c . Similarly, the uniform convergence of the finite vectors implies the continuity of their laws and source functionals. The measures λ_γ do not depend on I , so continuity on every compact interval gives continuity on $(0, \infty)$.

In this paragraph, $r \in \{0, \dots, J-1\}$ denotes a residue index. For the full sequence $n_* M^k$, the statements above apply to the finitely many residue bases $n_* M^r$, $0 \leq r < J$, with one common shift of the base. By Theorem 29.28, their pointwise limits are the same λ_γ , and the maximum of finitely many uniform errors tends to zero. Let $m_r(l)$ be the diagonal sequences of the finitely many residue classes. On the full sequence, the maps are defined by (30.13) at the level $m(k) = \min_{0 \leq r < J} m_r(\lfloor k/J \rfloor)$, which is a nondecreasing sequence tending to infinity. A configuration that does not fail at some level does not fail at any lower level, since there are fewer conditions with larger tolerances. Hence the failure probabilities at level $m(k)$ are bounded by those of the residue classes, and all estimates on the coordinate errors and the energies persist. The finitely many initial zero outputs affect no limit and no tail. $\square$

Corollary 30.12. *Fix a physical coupling $c = g_3^2 > 0$, an odd integer $M \geq 3$, and an equivalence class $[n]_M$, and identify $\mathcal{X}_L$ with $\mathcal{X}_1$ by D_L^{-1} . Then the map*

$$L \longmapsto (D_L^{-1})_{\#} \lambda_{L,cL,[n]_M} = \lambda_{1,cL,[n]_M}$$

is weakly continuous on $(0, \infty)$ for both topologies on $\mathcal{X}_1$. On each compact interval of side lengths in $(0, \infty)$, the limits of the dilated finite vectors of flowed lattice Wilson loops, and the convergence of the laws of the common maps, are uniform in L . The same holds jointly in (c, L) on compact rectangles in $(0, \infty)^2$, after transporting tests and times by dilation.

Proof. The notation c is local to this corollary. The displayed identity is (30.4), and the identities of Theorem 30.1 at a finite cutoff include the measure, the lattice time, and

the ordered lattice paths. A compact rectangle of pairs (c, L) is mapped by $(c, L) \mapsto cL$ onto a compact interval of couplings on the unit torus. We apply Theorem 30.10 on that interval and compose the continuous functions $\gamma \mapsto \lambda_{1,\gamma,[n]_M}(F)$ with $(c, L) \mapsto cL$. We then transport the maps on the unit torus, which do not depend on the coupling, by D_L ; their dimensionless errors and energy tails are exactly those on the unit torus. No uniformity as $L \rightarrow \infty$, or for physical observation windows that are not transported by dilation, is obtained. $\square$

30.8 Modified smoothing flows

The flowed Wilson loops are defined with the Wilson flow, the gradient flow of $\sum_p q(U_p)$ introduced by Lüscher [128]. We now show that the limits do not change when the flow is replaced by the gradient flow of a modified plaquette function, within a bounded family. For $c \geq 0$ and $a = L/N$, let

$$\mathcal{A}_c(U) = \sum_{p \in \mathcal{P}_N} \{q(U_p) + c q(U_p)^2\}, \quad \Psi_t^{(N,c)} = \Phi_{4t/a^2}^{\mathcal{A}_c},$$

where $\Phi^{\mathcal{A}_c}$ is the flow of the vector field $-\nabla \mathcal{A}_c$ for the product bi-invariant metric, as in Section 24 (see Lemma 24.11). In particular, $\Psi_t^{(N,0)} = \Psi_t^{(N)}$. Since the vector field is smooth on the compact configuration space, the flow exists for all times, and it commutes with lattice gauge transformations because $\mathcal{A}_c$ is gauge invariant and the metric is invariant. Only the smoothing changes: the measure $\mu_{N,\gamma}$, the block averaging map, and all conditional kernels are unchanged. Near the identity, $q(U_p)^2$ is quartic in the plaquette logarithm, and in the damped gauge the modification adds to the equation of the flow only terms of degree at least three in the field (Lemma 24.11).

In this section and the next we use the setting of Theorem 29.25: the parameters, the order of the quantifiers, and the sequence $N_K = n_* B^K$ are those of that theorem, so the coupling is fixed in $(0, 1]$, and $\lambda_{L,\gamma,[n_*]}$ denotes the corresponding continuum measure. We define the integer r_0 of Section 30.1 for these parameters, enlarged if necessary so that the conditioned estimates used below hold for all $r \geq r_0$.

Theorem 30.13. *Work in the setting of Theorem 29.25, and fix $0 \leq C_0 < \infty$. Under the Wilson measure $\mu_{N_K,\gamma}$ let $U_K^c(t) = \Psi_t^{(N_K,c)} U$. For loops ℓ_i with admissible approximations $(\ell_{i,K})_K$, characters χ_i , and times $t_i > 0$, $1 \leq i \leq m$, set*

$$W_{K,i}^c = \chi_i(\text{Hol}_{\ell_{i,K}} U_K^c(t_i)).$$

Then

$$\lim_{K \rightarrow \infty} \mathbb{E}^{\mu_{N_K,\gamma}} \sup_{0 \leq c \leq C_0} \max_i |W_{K,i}^c - W_{K,i}^0| = 0. \quad (30.15)$$

Consequently, the laws of $(W_{K,i}^c)_i$ converge to the joint law of $(W_{\ell_i, \chi_i, t_i})_i$ under $\lambda_{L, \gamma, [n_*]}$, uniformly in $c \in [0, C_0]$ in the bounded-Lipschitz distance. For every finite Z , the complex source functionals $\mathbb{E}_{\mu_{N_K, \gamma}} \exp(\sum_i z_i W_{K,i}^c)$ converge uniformly on $[0, C_0] \times \{z : \sum_i |z_i| \leq Z\}$ to the limit of those of the Wilson flow. In particular, these conclusions hold for every bounded sequence $c = c_K \geq 0$.

The maps π_K of Theorem 29.25 can be used for every c . Their compact containment, their convergence in both topologies, their limit, and the tightness of their energies over all K are unchanged, and for the normalized fundamental coordinates $W_{K,j}^c = (1/2) \operatorname{tr} \operatorname{Hol}_{\ell_{j,K}} U_K^c(t_j)$ of the determining family,

$$\lim_{K \rightarrow \infty} \mathbb{E}_{\mu_{N_K, \gamma}} \sup_{0 \leq c \leq C_0} |w_j(\pi_K) - W_{K,j}^c| = 0 \quad (j \geq 1). \quad (30.16)$$

Every finite vector of Wilson loops for the modified flows converges jointly to the corresponding vector under $\lambda_{L, \gamma, [n_*]}$, for every choice of admissible approximations.

The number C_0 is chosen after the parameters and the coupling, and neither the parameter tuple nor the minimum side is changed. Deterministic mesh thresholds may depend on C_0 and on the bounds on the flow data; convergence moduli may in addition depend on γ , on the coarse side, and on the observables, as in item (7) of Definition 4.14. No uniformity as $\gamma \rightarrow 0$ or $C_0 \rightarrow \infty$ is asserted.

Proof. If the list of loops is empty, there is nothing to prove, so we assume it is not. We first prove (30.15) under the conditioned measures, using the deterministic comparison of Lemma 25.15, then remove the conditioning, and finally deduce the remaining assertions.

Step 1: conditioned measures. Fix $r \geq r_0$, write $u_r = n_* B^r$ and $\nu_{K|r} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, u_r, \gamma})$. In the joint realization of Theorem 28.8, with U -marginal $\nu_{K|r}$, the reconstructed field is gauge equivalent to U , and the norm $R_{K|r}$ of its flow data is tight by Theorem 28.8. Since each flow $\Psi^{(N_K, c)}$ commutes with gauge transformations, the Wilson loops of the flow of U equal those of the flow of this representative, for every c , and the supremum of the discrepancies is gauge invariant.

Let $D = \max_i \chi_i(I)$ and

$$\Delta_K = \sup_{0 \leq c \leq C_0} \max_i |W_{K,i}^c - W_{K,i}^0| \leq 2D.$$

On each finite lattice, the smooth flow depends continuously on c and on the initial configuration. This follows by subtracting the integral equations for two values of the parameters and applying the Lipschitz bound of the vector field on the compact configuration space. Hence the supremum over c equals the supremum over a countable dense subset of $[0, C_0]$, and Δ_K is Borel. For fixed R , the early time interval of Theorem 24.12 for flow data of norm at most R is deterministic, and the norm of the flow data on it is at most $R_{K|r}$. The admissible approximations have a common bound on their lengths. Then

Lemma 25.15, which applies to any fixed finite set of characters, yields a deterministic $\omega_R(a) \rightarrow 0$ such that

$$\mathbb{E}_{\nu_{K|r}} \Delta_K \leq \omega_R(a_K) + 2D \nu_{K|r} \{R_{K|r} > R\}.$$

We let $K \rightarrow \infty$ at fixed r , R , and C_0 , and then $R \rightarrow \infty$. By (30.3), we conclude that $\mathbb{E}_{\nu_{K|r}} \Delta_K \rightarrow 0$.

Step 2: the Wilson measure. At the same cutoff, by Theorem 23.9, applied with the coupling bound one as in Theorem 29.25, we have $\mu_{N_K, \gamma}(S_{N_K, u_r, \gamma}^c) \leq F_\gamma(u_r)$ for all $K \geq r$, with $F_\gamma(u_r) \rightarrow 0$ as $r \rightarrow \infty$ at fixed $\gamma > 0$. The second inequality in (30.2) then implies that

$$\mathbb{E}_{\mu_{N_K, \gamma}} \Delta_K \leq \mathbb{E}_{\nu_{K|r}} \Delta_K + 2DF_\gamma(u_r).$$

Letting first $K \rightarrow \infty$ and then $r \rightarrow \infty$ proves (30.15), with the same order of limits as in Theorem 28.2.

Step 3: consequences. For a test with bounded-Lipschitz norm, the difference of its expectations for the modified flow and for the Wilson flow is at most $\sqrt{m}$ times its Euclidean Lipschitz constant times $\mathbb{E} \Delta_K$, uniformly in c . By Theorem 29.25, the vector for the Wilson flow converges to its law under $\lambda_{L, \gamma, [n_*]}$, which proves the uniform convergence in the bounded-Lipschitz distance. For a bounded continuous test f with modulus of continuity ω_f for the maximum norm on the compact range of the vector, Markov's inequality implies that the error is at most $\omega_f(\eta) + 2\|f\|_\infty \mathbb{E} \Delta_K / \eta$ for every $\eta > 0$. For the source functionals, we have

$$\sup_{\sum_i |z_i| \leq Z} \left| e^{\sum_i z_i W_{K,i}^c} - e^{\sum_i z_i W_{K,i}^0} \right| \leq Z e^{DZ} \Delta_K,$$

since both exponents are bounded by DZ in absolute value and differ by at most $Z \Delta_K$. Combined with the convergence of the source functionals of the Wilson flow in Theorem 29.25, this implies the uniform convergence. The uniformity in c covers every bounded deterministic sequence c_K .

The maps π_K are those of Theorems 29.20 and 29.25; their definition does not involve c , so their law, gauge invariance, Borel measurability, energy tails, and compact containment are unchanged. For each determining coordinate,

$$\mathbb{E} \sup_c |w_j(\pi_K) - W_{K,j}^c| \leq \mathbb{E} |w_j(\pi_K) - W_{K,j}^0| + \mathbb{E} \sup_c |W_{K,j}^c - W_{K,j}^0| \rightarrow 0,$$

which is (30.16). For every other finite list, (30.15) and the identification of the limit of the Wilson-flow vector in Theorem 29.20 identify its limit under the same $\lambda_{L, \gamma, [n_*]}$; this identifies the law of the joint vector, not only its marginals. By Theorems 29.23 and 29.25, all admissible approximations have the same limits for the Wilson flow, and applying the comparison to each of them proves the independence of the choice of approximation. $\square$

30.9 Energy and loop processes for the modified flows

We now extend Theorem 30.4 to the modified flows, jointly in $c \in [0, C_0]$ and $t \in [\tau, T]$, including the lattice energy. The loops are controlled by Lemma 25.15; the new deterministic input is the comparison of the energies.

Lemma 30.14. *Fix $L > 0$, $1/2 < \alpha < 2/3$, $0 \leq R, C_0 < \infty$, $0 < \tau < T < \infty$, finitely many characters χ_i , and lengths $\ell_i^* > 0$. Consider any initial configuration on the lattice of mesh $a = L/N$ whose flow data, on the common early time interval of Theorem 24.12, have norm at most R , and let $U_a^c(t) = \Psi_t^{(N,c)} U$ for $0 \leq c \leq C_0$. Let $\ell_{i,a}$ be closed lattice paths of physical length at most ℓ_i^* . There is a deterministic function $\omega_{R,C_0,\tau,T}(a) \rightarrow 0$ as $a \rightarrow 0$ such that, uniformly over all such configurations and paths,*

$$\sup_{\substack{0 \leq c \leq C_0 \\ \tau \leq t \leq T}} \left\{ |E_a(U_a^c(t)) - E_a(U_a^0(t))| \right. \\ \left. + \max_i |\chi_i(\text{Hol}_{\ell_{i,a}} U_a^c(t)) - \chi_i(\text{Hol}_{\ell_{i,a}} U_a^0(t))| \right\} \leq \omega_{R,C_0,\tau,T}(a).$$

The maximum over an empty family of loops is zero. The function ω and the mesh thresholds depend only on the displayed data, the representations, and the margins of the charts and of the difference stencil, and not on the coupling or the depth of the hierarchy. No rate is asserted.

Proof. The assertion for the loops is Lemma 25.15; we prove the assertion for the energy on the same configurations, uniformly in both c and t . Fix $s > 0$ smaller than τ and than the length of the common early time interval, with a positive waiting interval before s . By Theorem 24.12, we have

$$\sup_{0 \leq c \leq C_0} \|A_a^c(s) - A_a^0(s)\|_{C_a^1} \rightarrow 0$$

uniformly over the configurations, where A_a^c is the solution in the damped gauge for the flow with parameter c . The bootstrap of Theorem 25.7 bounds every fixed order of differences of the Wilson-flow solution on the waiting interval before s , and the equation of Lemma 24.2 bounds its time derivatives at each fixed order of differences by the next two orders.

Suppose that the assertion for the energy fails. Then there exist meshes $a_j \rightarrow 0$, configurations, parameters $c_j \in [0, C_0]$, and times $t_j \in [\tau, T]$ with a fixed positive discrepancy of the energies. We interpolate the value and difference arrays of the Wilson-flow solution at time s by piecewise multilinear interpolation. The bounds on the next differences give equicontinuity across the cells. Then the Arzelà–Ascoli theorem with a diagonal extraction gives uniform limits of the arrays of all orders, and discrete summation by parts identifies them as the derivatives of one smooth periodic one-form b_s . In particular,

$\|A_{a_j}^0(s) - S_{a_j}b_s\|_{C_{a_j}^1} \rightarrow 0$, and by the early comparison, the same convergence holds for all $c \in [0, C_0]$, uniformly.

We apply Theorem 24.10 to the single field b_s up to time $T - s$, with at least five derivatives and strict chart margins. This gives a finite reference family, whose fixed gauge transformations we sample at the transition times. Next, Lemma 25.14 applies on the consecutive intervals, for $c = 0$ and for every $c \leq C_0$. If the initial errors are $\eta_j \rightarrow 0$, this lemma gives errors at most $C(\eta_j + a_j) \rightarrow 0$ in $C_{a_j}^1$ throughout the reference solutions. At a transition time, the fixed smooth gauge transformation k changes the C_a^1 norm by a bounded factor on the chart, since $a^{-1} \text{Log}(k(x)e^{aA_i(x)}k(x + ae_i)^{-1})$ extends smoothly to $a = 0$. Thus the errors propagate across every transition, with constants uniform in c .

On each interval, both lattice representatives have a common bound on their C_a^1 norms, and their distance tends to zero. By the Lipschitz estimate of Lemma 29.14, which is uniform in the mesh, the difference of their energies is at most this distance times a fixed constant. By the gauge invariance of E_a , these are the energies of the two flows, also at the transition times. This contradicts the assumed discrepancy at t_j . Therefore the supremum over the configurations, c , and t defines a function of a that tends to zero, and combining it with the function for the loops proves the lemma. The only reference solutions used for a hypothetical offending sequence are those of its extracted smooth limit b_s , so no derivatives of finite-scale reconstructions of growing index enter the constants. $\square$

Theorem 30.15. *Work in the setting of Theorem 30.13; in particular $\gamma \in (0, 1]$ is fixed. Fix $0 \leq C_0 < \infty$, $0 < \tau < T < \infty$, loops $\ell_1, \dots, \ell_m$ with admissible approximations $(\ell_{i,K})_K$ fixed in time, and characters $\chi_1, \dots, \chi_m$. Under the Wilson measure $\mu_{N_K, \gamma}$ let*

$$U_K^c(t) = \Psi_t^{(N_K, c)} U, \quad W_{K,i}^c(t) = \chi_i(\text{Hol}_{\ell_{i,K}} U_K^c(t)).$$

Then

$$D_K = \sup_{\substack{c \in [0, C_0] \\ t \in [\tau, T]}} \left\{ |E_{a_K}(U_K^c(t)) - E_{a_K}(U_K^0(t))| + \max_i |W_{K,i}^c(t) - W_{K,i}^0(t)| \right\} \rightarrow 0 \quad (30.17)$$

in $\mu_{N_K, \gamma}$ -probability. The random continuous fields $(c, t) \mapsto (E_{a_K}(U_K^c(t)), (W_{K,i}^c(t))_i)$ converge in distribution in $C([0, C_0] \times [\tau, T]; \mathbb{R} \times \mathbb{C}^m)$, with the uniform norm, to the field that does not depend on c and has the time path

$$t \mapsto (E_L(X(t)), (W_{\ell_i, \chi_i, t}(X))_i), \quad X \sim \lambda_{L, \gamma, [n_*]}.$$

Moreover $\mathbb{E}_{\mu_{N_K, \gamma}} \sup_{c, t} \max_i |W_{K,i}^c(t) - W_{K,i}^0(t)| \rightarrow 0$. The number C_0 , the time interval, and the observables are fixed before $K \rightarrow \infty$, and the convergence moduli have the dependence described in item (7) of Definition 4.14, with these additional data.

The theorem asserts convergence in probability of the energy discrepancy, not convergence of its expectation: only the indicator of the event $\{D_K > \eta\}$ is transferred from the conditioned measures.

Proof. We first recall the tightness of the norm of the flow data under the conditioned measures, then apply Lemma 30.14 on the event where this norm is bounded and transfer the probability of the complementary event, and finally deduce the convergence of the fields.

Step 1: tightness of the flow data. Fix $r \geq r_0$, write $u_r = n_* B^r$ and $\nu_{K|r} = \mu_{N_K, \gamma}(\cdot \mid S_{N_K, u_r, \gamma})$. In the joint realization of Theorem 28.8, the field reconstructed at the finest scale is gauge equivalent to U , whose marginal is $\nu_{K|r}$. Let x_j be the linear heat evolutions, on the finest lattice, of the cardinal interpolations of the fields reconstructed at the scales j , and $y_j = \mathcal{Q}_{a_K}(x_j, x_j)$, as in Theorem 28.8. The tightness (30.3) of $R_{K|r} = X_\alpha^{a_K}(x_{K-r}) + Y_\alpha^{a_K}(y_{K-r})$ is part of Theorem 28.8; we recall the argument, since it uses only finite-scale reconstructions. Lemma 26.2 gives the exact identity

$$(x_k, y_k) - (x_{h-1}, y_{h-1}) = \left(\sum_{j=h}^k d_j, \sum_{j=h}^k \{ \mathcal{B}_{a_K}(x_{j-1}, d_j) + \mathcal{Q}_{a_K}(d_j, d_j) \} \right), \quad d_j = x_j - x_{j-1},$$

where $\mathcal{B}_a(u, v) = \mathcal{Q}_a(u, v) + \mathcal{Q}_a(v, u)$. By the estimate for the linear heat evolution in Theorem 28.6 and the two tail estimates in Theorem 28.7, this difference is small in probability as $h \rightarrow \infty$, uniformly in the remaining depth. For each fixed h , the arrays of the field reconstructed at scale $h-1$ range over a bounded finite-dimensional set, by the construction of the reconstruction maps and the cutoffs on the normalized fluctuation variables, and their cardinal interpolations have bounded derivatives of every fixed order. Using the estimates for cardinal interpolation and the heat semigroup in Lemmas 16.1 and 16.3, we bound x_{h-1} and its fixed differences on every finer lattice. The exact shifted quadratic term of Lemma 24.2, the contraction of the heat semigroup, and the Duhamel integral then bound y_{h-1} and its first difference by $C_h t$. Hence the norm of the flow data of the field reconstructed at scale $h-1$ has a deterministic bound C_h , independent of the finest mesh. We choose h so that the probability that the difference above exceeds one is less than a given $\varepsilon > 0$, and then a bound $R > C_h + 1$. The finitely many remaining depths have bounded arrays and are included by enlarging R . This proves (30.3) at the fixed r and γ . Restricting the horizon to the common early time interval does not increase the norm.

Step 2: the discrepancy. The flows are gauge equivariant and the energies and Wilson loops are gauge invariant. On the event $\{R_{K|r} \leq R\}$, Lemma 30.14 implies that $D_K \leq \omega_R(a_K)$. The random variable D_K is Borel, since the finite flow and its dependence on c and on the initial configuration are continuous, so the supremum over (c, t) equals the

supremum over a countable dense set. For every fixed $\eta > 0$ and all sufficiently large K , we have $\omega_R(a_K) < \eta$ and hence

$$\nu_{K|r}\{D_K > \eta\} \leq \nu_{K|r}\{R_{K|r} > R\}.$$

Applying the last inequality in (30.2) to the event $\{D_K > \eta\}$, and using Theorem 23.9 with the coupling bound one, we obtain

$$\limsup_{K \rightarrow \infty} \mu_{N_K, \gamma}\{D_K > \eta\} \leq \sup_{K \geq r} \nu_{K|r}\{R_{K|r} > R\} + F_\gamma(u_r), \quad F_\gamma(u_r) \rightarrow 0.$$

Letting first $R \rightarrow \infty$ at the fixed r and then $r \rightarrow \infty$ proves (30.17).

Step 3: convergence of the fields. The finite flow is smooth in (c, t) on its compact time interval, so each field is continuous. By Theorem 30.4, the path of the Wilson flow, $c = 0$, converges in distribution. Its image under the continuous map that sends a time path to the field that is constant in c therefore converges to the stated field. With the maximum norm on $\mathbb{R} \times \mathbb{C}^m$, the uniform distance between the field for the modified flows and this image is at most D_K . For a test f with bounded-Lipschitz norm, the difference of the two expectations is at most $\text{Lip}(f)\eta + 2\|f\|_\infty \mu_{N_K, \gamma}\{D_K > \eta\}$, which proves convergence for such tests. For bounded continuous tests, we also need the compact containment of the paths of the Wilson flow. It follows from the bounds on the values and the Lipschitz bound in time of Lemma 30.3, with the same localization to $\{R_{K|r} \leq R\}$ and the same removal of the conditioning as in Step 2. By a finite covering argument, a bounded continuous test is uniformly continuous between the points of a compact set and all sufficiently nearby paths, and outside the compact set we use the bound on the test. This proves the convergence in distribution of the fields. Finally, the supremum of the loop discrepancy is bounded by twice the largest dimension of the representations, so its convergence to zero in probability implies the convergence of its expectation. $\square$

Remark 30.16. The results concern parameters c in a bounded interval and fixed positive observation times. No assertion is made for unbounded c_K , for other modifications of the flow, for observation times tending to zero with the lattice spacing, or for different equivalence classes $[n]_M$, and no rate and no uniformity as $\gamma \rightarrow 0$ is asserted. Statements formulated only in terms of $\lambda_{L, \gamma, [n]_*}$, such as those in Sections 32 and 34 to 36, concern the same measure; the theorems above do not give new proofs of the estimates at finite cutoff on which those statements rest.

Proof of Theorems 2.5 and 2.6. The measure $\lambda = \lambda_{L, \gamma, [n]_M}$ and the maps π_k are those of Theorem 29.28. Part (1) of Theorem 2.5 is the assertion along the full sequence nM^k in Theorem 30.4, proved in Section 30.6. Part (2) is the assertion under the Wilson measures along nM^k in Theorem 30.5. Part (3) is Theorem 30.6, applied to $\lambda_{L, \gamma, [n]_M}$ as

shown in Section 30.6; the winding vectors $w(\ell_j)$ of that theorem are the vectors $v(\ell_j)$ of Theorem 2.5. Part (4) is (30.4) in Theorem 30.1. Part (5) is Theorem 30.10. Items (1)–(3) of that theorem, together with its assertion along the full sequence, give the uniform convergence of finite vectors in the bounded-Lipschitz distance and the choice of maps independent of $\gamma \in I$ for which the convergence in Theorem 2.3(3) and the tightness in Theorem 2.3(4) hold uniformly. Item (4) gives the weak continuity of $\gamma \mapsto \lambda_{L,\gamma,[n]_M}$ on $(0, \infty)$ for both topologies. Theorem 2.6 is Theorems 30.13 and 30.15, along the sequence $N_K = n_* B^K$ of Theorem 29.25. $\square$

References

- [1] Radosław Adamczak. A Note on the Hanson–Wright Inequality for Random Vectors with Dependencies. *Electronic Communications in Probability*, 20(72):1–13, 2015. doi: 10.1214/ECP.v20-3829. URL <https://arxiv.org/abs/1409.8457>.
- [2] Arka Adhikari and Sky Cao. Correlation decay for finite lattice gauge theories at weak coupling. *The Annals of Probability*, 53(1):140–174, 2025. doi: 10.1214/24-AOP1702. URL <https://doi.org/10.1214/24-AOP1702>.
- [3] Robert J. Adler and Jonathan E. Taylor. *Random Fields and Geometry*. Springer Monographs in Mathematics. Springer, New York, 2007. ISBN 978-0-387-48112-8. doi: 10.1007/978-0-387-48116-6. URL <https://doi.org/10.1007/978-0-387-48116-6>. Theorem 2.1.1, printed p. 50, (2.1.5); Lemma 2.1.5, printed p. 54, (2.1.13), proof pp. 54–55. Used with the authors’ correction list of 7 October 2015, entries 54-6, 55-11, and 55-4: <https://robert.net.technion.ac.il/files/2016/08/corrections.pdf>. Gaussian reference only.
- [4] David J. Aldous and Geoffrey K. Eagleson. On mixing and stability of limit theorems. *Annals of Probability*, 6(2):325–331, 1978. doi: 10.1214/aop/1176995577.
- [5] Thomas Appelquist and Robert D. Pisarski. High-temperature Yang–Mills theories and three-dimensional quantum chromodynamics. *Physical Review D*, 23(10):2305–2317, 1981. doi: 10.1103/PhysRevD.23.2305.
- [6] Michael F. Atiyah and Raoul Bott. The Yang–Mills equations over Riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615, 1983. doi: 10.1098/rsta.1983.0017. URL <https://doi.org/10.1098/rsta.1983.0017>.
- [7] Peter Austing and John F. Wheeler. The convergence of Yang–Mills integrals. *Journal of High Energy Physics*, 2001(02):028, 2001. doi: 10.1088/1126-6708/2001/02/028. URL <https://arxiv.org/abs/hep-th/0101071>.
- [8] Dominique Bakry and Michel Émery. Diffusions hypercontractives. In *Séminaire de Probabilités XIX 1983/84*, volume 1123 of *Lecture Notes in Mathematics*, pages 177–206. Springer, 1985. doi: 10.1007/BFb0075847. URL https://www.numdam.org/item/SPS_1985__19__177_0/.
- [9] Tadeusz Bałaban. Regularity and decay of lattice Green’s functions. *Communications in Mathematical Physics*, 89(4):571–597, 1983. doi: 10.1007/BF01214744. URL <https://doi.org/10.1007/BF01214744>.

- [10] Tadeusz Bałaban. Propagators and renormalization transformations for lattice gauge theories. I. *Communications in Mathematical Physics*, 95(1):17–40, 1984. doi: 10.1007/BF01215753. URL <https://doi.org/10.1007/BF01215753>. Proposition 1.1, p. 33; formulas (1.11), (1.18), (1.39)–(1.44), (1.60)–(1.69), (1.103).
- [11] Tadeusz Bałaban. Propagators and renormalization transformations for lattice gauge theories. II. *Communications in Mathematical Physics*, 96(2):223–250, 1984. doi: 10.1007/BF01240221. URL <https://doi.org/10.1007/BF01240221>. Proposition 2.7 and Corollary 2.8, p. 249.
- [12] Tadeusz Bałaban. Averaging operations for lattice gauge theories. *Communications in Mathematical Physics*, 98(1):17–51, 1985. doi: 10.1007/BF01211042. URL <https://doi.org/10.1007/BF01211042>. The reference-transport logarithmic averaging construction is equation (15), p. 19.
- [13] Tadeusz Bałaban. Propagators for lattice gauge theories in a background field. *Communications in Mathematical Physics*, 99(3):389–434, 1985. doi: 10.1007/BF01240355. URL <https://doi.org/10.1007/BF01240355>. Theorems 3.1–3.3, pp. 397–399; Hölder norms (3.40), p. 397, and nonendpoint first-gradient Hölder bounds (3.43), p. 398; extension (3.133), p. 422, and Theorem 3.12, p. 423; constrained covariance (3.148), (3.153), pp. 425–426; Theorem 3.13, p. 426. Only the stated identity-background portions are used.
- [14] Tadeusz Bałaban. Spaces of regular gauge field configurations on a lattice and gauge fixing conditions. *Communications in Mathematical Physics*, 99(1):75–102, 1985. doi: 10.1007/BF01466594. URL <https://doi.org/10.1007/BF01466594>.
- [15] Tadeusz Bałaban. Ultraviolet stability of three-dimensional lattice pure gauge field theories. *Communications in Mathematical Physics*, 102(2):255–275, 1985. doi: 10.1007/BF01229380. URL <https://doi.org/10.1007/BF01229380>.
- [16] Tadeusz Bałaban. The variational problem and background fields in renormalization group method for lattice gauge theories. *Communications in Mathematical Physics*, 102(2):277–309, 1985. doi: 10.1007/BF01229381. URL <https://doi.org/10.1007/BF01229381>.
- [17] Tadeusz Bałaban. Renormalization group approach to lattice gauge field theories. I. Generation of effective actions in a small field approximation and a coupling constant renormalization in four dimensions. *Communications in Mathematical Physics*, 109(2):249–301, 1987. doi: 10.1007/BF01215223. URL <https://doi.org/10.1007/BF01215223>.
- [18] Tadeusz Bałaban. Convergent renormalization expansions for lattice gauge theories. *Communications in Mathematical Physics*, 119(2):243–285, 1988. doi: 10.1007/BF01217741. URL <https://doi.org/10.1007/BF01217741>.

- [19] Tadeusz Bałaban. Renormalization group approach to lattice gauge field theories. II. Cluster expansions. *Communications in Mathematical Physics*, 116(1):1–22, 1988. doi: 10.1007/BF01239022. URL <https://doi.org/10.1007/BF01239022>.
- [20] Tadeusz Bałaban. Large field renormalization. I. The basic step of the R operation. *Communications in Mathematical Physics*, 122(2):175–202, 1989. doi: 10.1007/BF01257412. URL <https://doi.org/10.1007/BF01257412>.
- [21] Tadeusz Bałaban. Large field renormalization. II. Localization, exponentiation, and bounds for the R operation. *Communications in Mathematical Physics*, 122(3):355–392, 1989. doi: 10.1007/BF01238433. URL <https://doi.org/10.1007/BF01238433>.
- [22] Tadeusz Bałaban and Arthur Jaffe. Constructive gauge theory. In *Fundamental Problems of Gauge Field Theory*, volume 141 of *NATO ASI Series B*, pages 207–263. Plenum Press, 1986. doi: 10.1007/978-1-4757-0363-4_5. URL https://link.springer.com/chapter/10.1007/978-1-4757-0363-4_5.
- [23] Tadeusz Bałaban, John Imbrie, and Arthur Jaffe. Exact renormalization group for gauge theories. In *Progress in Gauge Field Theory*, volume 115 of *NATO ASI Series B*, pages 79–103. Plenum Press, 1984. doi: 10.1007/978-1-4757-0280-4_4. URL https://link.springer.com/chapter/10.1007/978-1-4757-0280-4_4.
- [24] Tadeusz Bałaban, John Imbrie, and Arthur Jaffe. Renormalization of the Higgs model: Minimizers, propagators and the stability of mean field theory. *Communications in Mathematical Physics*, 97(1–2):299–329, 1985. doi: 10.1007/BF01206191. URL <https://link.springer.com/article/10.1007/BF01206191>.
- [25] Tadeusz Bałaban, John Z. Imbrie, and Arthur Jaffe. Effective action and cluster properties of the abelian Higgs model. *Communications in Mathematical Physics*, 114(2):257–315, 1988. doi: 10.1007/BF01225038. URL <https://link.springer.com/article/10.1007/BF01225038>.
- [26] Riddhipratim Basu and Shirshendu Ganguly. $SO(N)$ lattice gauge theory, planar and beyond. *Communications on Pure and Applied Mathematics*, 71(10):2016–2064, 2018. doi: 10.1002/cpa.21788. URL <https://doi.org/10.1002/cpa.21788>.
- [27] Rajendra Bhatia. *Matrix Analysis*, volume 169 of *Graduate Texts in Mathematics*. Springer, New York, 1997. doi: 10.1007/978-1-4612-0653-8. URL <https://doi.org/10.1007/978-1-4612-0653-8>.
- [28] Patrick Billingsley. *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. John Wiley & Sons, New York, second edition, 1999. doi: 10.1002/9780470316962.
- [29] Marek Biskup. Reflection positivity and phase transitions in lattice spin models. In Roman Kotecký, editor, *Methods of Contemporary Mathematical Statistical Physics*, volume 1970 of *Lecture Notes in Mathematics*, pages 1–86. Springer, Berlin, Heidelberg, 2009. doi: 10.1007/978-3-540-92796-9_1. URL https://doi.org/10.1007/978-3-540-92796-9_1.

- 978-3-540-92796-9_1. arXiv version 2, 27 April 2009. Theorem 5.8, (5.34)–(5.35), printed p. 44; proof pp. 45–46.
- [30] Christer Borell. The Brunn-Minkowski inequality in Gauss space. *Inventiones Mathematicae*, 30(2):207–216, 1975. doi: 10.1007/BF01425510. URL <https://doi.org/10.1007/BF01425510>.
- [31] Raoul Bott. Nondegenerate critical manifolds. *Annals of Mathematics*, 60(2):248–261, 1954. doi: 10.2307/1969631. URL <https://doi.org/10.2307/1969631>.
- [32] Jean Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Communications in Mathematical Physics*, 176(2):421–445, 1996. doi: 10.1007/BF02099556. URL <https://doi.org/10.1007/BF02099556>.
- [33] Herm Jan Brascamp and Elliott H. Lieb. On Extensions of the Brunn–Minkowski and Prékopa–Leindler Theorems, Including Inequalities for Log Concave Functions, and with an Application to the Diffusion Equation. *Journal of Functional Analysis*, 22(4):366–389, 1976. doi: 10.1016/0022-1236(76)90004-5. URL [https://doi.org/10.1016/0022-1236\(76\)90004-5](https://doi.org/10.1016/0022-1236(76)90004-5).
- [34] Christian Brennecke. On the Leading Order Term of the Lattice Yang–Mills Free Energy. *Journal of Statistical Physics*, 193(7):83, 2026. doi: 10.1007/s10955-026-03649-4. URL <https://doi.org/10.1007/s10955-026-03649-4>.
- [35] Bjoern Bringmann and Sky Cao. A para-controlled approach to the stochastic Yang–Mills equation in two dimensions. *Memoirs of the American Mathematical Society*, 319(1624):v+125, 2026. doi: 10.1090/memo/1624. URL <https://doi.org/10.1090/memo/1624>. Publication metadata corroborated by the zbMATH-linked record DE 8212947 and author publication lists; direct AMS page inaccessible during verification.
- [36] David Brydges, Jürg Fröhlich, and Erhard Seiler. Diamagnetic and critical properties of Higgs lattice gauge theories. *Nuclear Physics B*, 152(3–4):521–532, 1979. doi: 10.1016/0550-3213(79)90095-6. URL <https://www.sciencedirect.com/science/article/pii/0550321379900956>.
- [37] David Brydges, Jürg Fröhlich, and Erhard Seiler. On the construction of quantized gauge fields. I. general results. *Annals of Physics*, 121(1–2):227–284, 1979. doi: 10.1016/0003-4916(79)90098-8. URL <https://www.sciencedirect.com/science/article/abs/pii/0003491679900988>.
- [38] David C. Brydges. A short course on cluster expansions. In Konrad Osterwalder and Raymond Stora, editors, *Critical Phenomena, Random Systems, Gauge Theories, Part I*, Les Houches, Session XLIII, 1984, pages 129–183. North-Holland, Amsterdam, 1986. URL <https://inspirehep.net/literature/211093>. School held in 1984; proceedings published in 1986. No DOI or arXiv identifier found.

- [39] David C. Brydges. Lectures on the renormalisation group. In Scott Sheffield and Thomas Spencer, editors, *Statistical Mechanics*, volume 16 of *IAS/Park City Mathematics Series*, pages 7–93. American Mathematical Society, 2009. doi: 10.1090/pcms/016/02. URL <https://doi.org/10.1090/pcms/016/02>.
- [40] David C. Brydges, Jürg Fröhlich, and Erhard Seiler. Construction of quantised gauge fields. II. convergence of the lattice approximation. *Communications in Mathematical Physics*, 71(2):159–205, 1980. doi: 10.1007/BF01197918. URL <https://link.springer.com/article/10.1007/BF01197918>.
- [41] David C. Brydges, Jürg Fröhlich, and Erhard Seiler. On the construction of quantized gauge fields. III. the two-dimensional abelian Higgs model without cut-offs. *Communications in Mathematical Physics*, 79(3):353–399, 1981. doi: 10.1007/BF01208500. URL <https://link.springer.com/article/10.1007/BF01208500>.
- [42] Sky Cao and Sourav Chatterjee. The Yang–Mills heat flow with random distributional initial data. *Communications in Partial Differential Equations*, 48(2):209–251, 2023. doi: 10.1080/03605302.2023.2169937. URL <https://arxiv.org/abs/2111.10652v4>. All numbered and printed-page locators refer to arXiv:2111.10652v4, 25 August 2022, not journal pagination. Only deterministic smooth-data results are used: Theorem 1.2 and Lemma 1.3, p. 3; Lemmas C.2–C.3, pp. 55–56; Corollary C.6, p. 58; Lemmas C.12–C.13, p. 61.
- [43] Sky Cao and Sourav Chatterjee. A state space for 3D euclidean Yang–Mills theories. *Communications in Mathematical Physics*, 405(1):3, 2024. doi: 10.1007/s00220-023-04870-y. URL <https://arxiv.org/abs/2111.12813v4>. Article 3, 69 pp. All numbered and printed-page locators refer to arXiv:2111.12813v4, 20 November 2023, not journal pagination, including Lemmas 1.4–1.5 and Theorem 1.6, p. 7; Theorem 3.14, p. 25; Theorem 3.51, pp. 35–36; Lemma 3.68, p. 40.
- [44] Sky Cao, Ron Nissim, and Scott Sheffield. Expanded regimes of area law for lattice Yang–Mills theories, 2025. URL <https://arxiv.org/abs/2505.16585>. Preprint; no published version verified.
- [45] Ajay Chandra, Ilya Chevyrev, Martin Hairer, and Hao Shen. Langevin Dynamic for the 2D Yang–Mills Measure. *Publications mathématiques de l’IHÉS*, 136:1–147, 2022. doi: 10.1007/s10240-022-00132-0. URL <https://arxiv.org/abs/2006.04987>.
- [46] Ajay Chandra, Ilya Chevyrev, Martin Hairer, and Hao Shen. Stochastic Quantisation of Yang–Mills–Higgs in 3D. *Inventiones mathematicae*, 237(2):541–696, 2024. doi: 10.1007/s00222-024-01264-2. URL <https://arxiv.org/abs/2201.03487>.
- [47] Nelia Charalambous and Leonard Gross. The Yang–Mills Heat Semigroup on Three-Manifolds with Boundary. *Communications in Mathematical Physics*, 317(3):727–785, 2013. doi: 10.1007/s00220-012-1558-0. URL <https://arxiv.org/abs/1004.1639>.

- [48] Nelia Charalambous and Leonard Gross. Neumann domination for the Yang–Mills heat equation. *Journal of Mathematical Physics*, 56(7):073505, 2015. doi: 10.1063/1.4927250. URL <https://doi.org/10.1063/1.4927250>.
- [49] Nelia Charalambous and Leonard Gross. Initial behavior of solutions to the Yang–Mills heat equation. *Journal of Mathematical Analysis and Applications*, 451(2):873–905, 2017. doi: 10.1016/j.jmaa.2017.02.027. URL <https://doi.org/10.1016/j.jmaa.2017.02.027>.
- [50] Sourav Chatterjee. The Leading Term of the Yang–Mills Free Energy. *Journal of Functional Analysis*, 271(10):2944–3005, 2016. doi: 10.1016/j.jfa.2016.04.032. URL <https://arxiv.org/abs/1602.01222v6>.
- [51] Sourav Chatterjee. Yang–Mills for Probabilists. In Peter Friz, Wolfgang König, Chiranjib Mukherjee, and Stefano Olla, editors, *Probability and Analysis in Interacting Physical Systems*, volume 283 of *Springer Proceedings in Mathematics & Statistics*, pages 1–16. Springer, 2019. doi: 10.1007/978-3-030-15338-0_1. URL <https://arxiv.org/abs/1803.01950>.
- [52] Sourav Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Communications in Mathematical Physics*, 366(1):203–268, 2019. doi: 10.1007/s00220-019-03353-3. URL <https://doi.org/10.1007/s00220-019-03353-3>.
- [53] Sourav Chatterjee. A scaling limit of $SU(2)$ lattice Yang–Mills–Higgs theory. *Probability and Mathematical Physics*, 7(2):339–381, 2026. doi: 10.2140/pmp.2026.7.339. URL <https://arxiv.org/abs/2401.10507v4>.
- [54] Sourav Chatterjee and Jafar Jafarov. The $1/N$ expansion for $SO(N)$ lattice gauge theory at strong coupling, 2016. URL <https://inspirehep.net/literature/1448303>. Preprint; no published version verified.
- [55] Ilya Chevyrev. Yang–Mills Measure on the Two-Dimensional Torus as a Random Distribution. *Communications in Mathematical Physics*, 372(3):1027–1058, 2019. doi: 10.1007/s00220-019-03567-5. URL <https://arxiv.org/abs/1808.09196>.
- [56] Ilya Chevyrev. Stochastic Quantization of Yang–Mills. *Journal of Mathematical Physics*, 63(9):091101, 2022. doi: 10.1063/5.0089431. URL <https://arxiv.org/abs/2202.13359>.
- [57] Ilya Chevyrev and Hao Shen. Invariant Measure and Universality of the 2D Yang–Mills Langevin Dynamic. *Communications on Pure and Applied Mathematics*, 79(8):1973–2102, 2026. doi: 10.1002/cpa.70043. URL <https://arxiv.org/abs/2302.12160v3>.
- [58] Ilya Chevyrev and Hao Shen. Uniqueness of Gauge Covariant Renormalisation of Stochastic 3D Yang–Mills–Higgs. *Archive for Rational Mechanics and Analysis*, 250

- (1):11, 2026. doi: 10.1007/s00205-025-02163-3. URL <https://arxiv.org/abs/2503.03060v2>.
- [59] Ilya Chevyrev, Yahui Qu, and Hao Shen. Scaling limit of the 3D abelian Yang–Mills langevin dynamics, 2026. URL <https://arxiv.org/abs/2608.27828v2>. Preprint, version 2, 17 September 2026.
- [60] B. S. Cirel’son, I. A. Ibragimov, and V. N. Sudakov. Norms of Gaussian sample functions. In Gisiro Maruyama and Jurii V. Prokhorov, editors, *Proceedings of the Third Japan–USSR Symposium on Probability Theory*, volume 550 of *Lecture Notes in Mathematics*, pages 20–41. Springer, 1976. doi: 10.1007/BFb0077482. URL <https://doi.org/10.1007/BFb0077482>.
- [61] Jean-Michel Combes and Lawrence Thomas. Asymptotic Behaviour of Eigenfunctions for Multiparticle Schrödinger Operators. *Communications in Mathematical Physics*, 34(4):251–270, 1973. doi: 10.1007/BF01646473. URL <https://doi.org/10.1007/BF01646473>.
- [62] A. Coste, Antonio González-Arroyo, J. Jurkiewicz, and C. P. Korthals Altes. Zero-momentum contribution to Wilson loops in periodic boxes. *Nuclear Physics B*, 262(1): 67–94, 1985. doi: 10.1016/0550-3213(85)90064-1. URL [https://doi.org/10.1016/0550-3213\(85\)90064-1](https://doi.org/10.1016/0550-3213(85)90064-1).
- [63] Giuseppe Da Prato and Arnaud Debussche. Strong Solutions to the Stochastic Quantization Equations. *The Annals of Probability*, 31(4):1900–1916, 2003. doi: 10.1214/aop/1068646370. URL <https://doi.org/10.1214/aop/1068646370>.
- [64] Paulo A. Faria da Veiga and Michael O’Carroll. On Thermodynamic and Ultra-violet Stability of Yang–Mills, 2019. URL <https://arxiv.org/abs/1903.09829v2>. Preprint, version 2, 7 October 2019. No journal publication verified as of 16 October 2026.
- [65] Paulo A. Faria da Veiga and Michael O’Carroll. On Yang–Mills Stability Bounds and Plaquette Field Generating Function. *Reports on Mathematical Physics*, 95(3): 303–380, 2025. doi: 10.1016/S0034-4877(25)00035-7. URL <https://arxiv.org/abs/2205.07376v2>.
- [66] Dennis M. DeTurck. Deforming Metrics in the Direction of Their Ricci Tensors. *Journal of Differential Geometry*, 18(1):157–162, 1983. doi: 10.4310/jdg/1214509286. URL <https://doi.org/10.4310/jdg/1214509286>.
- [67] Jonathan Dimock. The renormalization group according to Balaban. II. Large fields. *Journal of Mathematical Physics*, 54(9):092301, 2013. doi: 10.1063/1.4821275. URL <https://doi.org/10.1063/1.4821275>.
- [68] Jonathan Dimock. The renormalization group according to Balaban—I. small fields. *Reviews in Mathematical Physics*, 25(7):1330010, 2013. doi: 10.1142/S0129055X13300100. URL <https://arxiv.org/abs/1108.1335v2>. All numbered

- and printed-page locators refer to arXiv:1108.1335v2, 21 December 2012, not journal pagination: Appendix B, Theorem 27, p. 39, and connected hard-core graph estimate (323), p. 42. Only the finite graph bound is imported.
- [69] Jonathan Dimock. The Renormalization Group According to Balaban III. Convergence. *Annales Henri Poincaré*, 15(11):2133–2175, 2014. doi: 10.1007/s00023-013-0303-3. URL <https://doi.org/10.1007/s00023-013-0303-3>.
 - [70] Jonathan Dimock. Covariant Axial Gauge. *Letters in Mathematical Physics*, 105(7): 959–987, 2015. doi: 10.1007/s11005-015-0763-0. URL <https://doi.org/10.1007/s11005-015-0763-0>.
 - [71] Jonathan Dimock. Nonperturbative renormalization of scalar quantum electrodynamics in $d = 3$. *Journal of Mathematical Physics*, 56(10):102304, 2015. doi: 10.1063/1.4933224. URL <https://pubs.aip.org/aip/jmp/issue/56/10>.
 - [72] Jonathan Dimock. Ultraviolet regularity for QED in $d = 3$. *Journal of Mathematical Physics*, 59(1):012301, 2018. doi: 10.1063/1.5009458. URL <https://arxiv.org/abs/1512.04373>.
 - [73] Jonathan Dimock. Multiscale block averaging for QED in $d = 3$. *Journal of Mathematical Physics*, 61(3):032302, 2020. doi: 10.1063/1.5134439. URL <https://arxiv.org/abs/1712.10029>.
 - [74] Sjoerd Dirksen. Tail Bounds via Generic Chaining. *Electronic Journal of Probability*, 20(53):1–29, 2015. doi: 10.1214/EJP.v20-3760. URL <https://arxiv.org/abs/1309.3522>.
 - [75] Simon K. Donaldson. A new proof of a theorem of Narasimhan and Seshadri. *Journal of Differential Geometry*, 18(2):269–277, 1983. doi: 10.4310/jdg/1214437664. URL <https://doi.org/10.4310/jdg/1214437664>.
 - [76] Simon K. Donaldson. Anti Self-Dual Yang–Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles. *Proceedings of the London Mathematical Society*, 50(1):1–26, 1985. doi: 10.1112/plms/s3-50.1.1. URL <https://doi.org/10.1112/plms/s3-50.1.1>.
 - [77] Michael R. Douglas. The Yang–Mills millennium problem. *Nature Reviews Physics*, 8(2):86–97, 2026. doi: 10.1038/s42254-025-00909-2. URL <https://www.nature.com/articles/s42254-025-00909-2>.
 - [78] Bruce K. Driver. YM_2 : Continuum expectations, lattice convergence, and lassos. *Communications in Mathematical Physics*, 123(4):575–616, 1989. doi: 10.1007/BF01218586. URL <https://doi.org/10.1007/BF01218586>.
 - [79] Richard M. Dudley. The Sizes of Compact Subsets of Hilbert Space and Continuity of Gaussian Processes. *Journal of Functional Analysis*, 1(3):290–330, 1967. doi: 10.1016/0022-1236(67)90017-1. URL [https://doi.org/10.1016/0022-1236\(67\)90017-1](https://doi.org/10.1016/0022-1236(67)90017-1).

- [80] Carl Eckart and Gale Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1(3):211–218, 1936. doi: 10.1007/BF02288367.
- [81] Paul Federbush. A phase cell approach to Yang–Mills theory. I. modes, lattice-continuum duality. *Communications in Mathematical Physics*, 107(2):319–329, 1986. doi: 10.1007/BF01209397. URL <https://link.springer.com/article/10.1007/BF01209397>.
- [82] Paul Federbush. A phase cell approach to Yang–Mills theory. III. local stability, modified renormalization group transformation. *Communications in Mathematical Physics*, 110(2):293–309, 1987. doi: 10.1007/BF01207369. URL <https://link.springer.com/article/10.1007/BF01207369>.
- [83] Paul Federbush. A phase cell approach to Yang–Mills theory. VI. non-abelian lattice-continuum duality. *Annales de l’Institut Henri Poincaré, Physique théorique*, 47(1):17–23, 1987. URL https://www.numdam.org/item/AIHPA_1987__47_1_17_0/. Bibliographic record verified in Numdam; no DOI or arXiv identifier located.
- [84] Paul Federbush. A phase cell approach to Yang–Mills theory. IV. the choice of variables. *Communications in Mathematical Physics*, 114(2):317–343, 1988. doi: 10.1007/BF01225039. URL <https://link.springer.com/article/10.1007/BF01225039>.
- [85] Paul Federbush. A phase cell approach to Yang–Mills theory. V. analysis of a chunk. *Communications in Mathematical Physics*, 127(3):433–457, 1990. doi: 10.1007/BF02104497. URL <https://link.springer.com/article/10.1007/BF02104497>.
- [86] Paul Federbush and Calvin Williamson. A phase cell approach to Yang–Mills theory. II. analysis of a mode. *Journal of Mathematical Physics*, 28(6):1416–1419, 1987. doi: 10.1063/1.527495. URL <https://pubs.aip.org/aip/jmp/article/28/6>.
- [87] Roberto Fernández and Aldo Procacci. Cluster Expansion for Abstract Polymer Models. New Bounds from an Old Approach. *Communications in Mathematical Physics*, 274(1):123–140, 2007. doi: 10.1007/s00220-007-0279-2. URL <https://doi.org/10.1007/s00220-007-0279-2>.
- [88] Richard P. Feynman. The qualitative behavior of Yang–Mills theory in 2+1 dimensions. *Nuclear Physics B*, 188(3):479–512, 1981. doi: 10.1016/0550-3213(81)90005-5.
- [89] Dana S. Fine. Quantum Yang–Mills on the two-sphere. *Communications in Mathematical Physics*, 134(2):273–292, 1990. doi: 10.1007/BF02097703. URL <https://doi.org/10.1007/BF02097703>.
- [90] Dana S. Fine. Quantum Yang–Mills on a Riemann surface. *Communications in Mathematical Physics*, 140(2):321–338, 1991. doi: 10.1007/BF02099502. URL <https://doi.org/10.1007/BF02099502>.
- [91] Zoltan Fodor, Kieran Holland, Julius Kuti, Daniel Negradi, and Chik Him Wong. The Yang–Mills gradient flow in finite volume. *Journal of High Energy Physics*, 2012.

- (11):007, 2012. doi: 10.1007/JHEP11(2012)007. URL <https://arxiv.org/abs/1208.1051>.
- [92] Jürg Fröhlich, Robert Israel, Elliott H. Lieb, and Barry Simon. Phase Transitions and Reflection Positivity. I. General Theory and Long Range Lattice Models. *Communications in Mathematical Physics*, 62(1):1–34, 1978. doi: 10.1007/BF01940327. URL <https://doi.org/10.1007/BF01940327>.
- [93] Jürg Fröhlich, Robert B. Israel, Elliott H. Lieb, and Barry Simon. Phase Transitions and Reflection Positivity. II. Lattice Systems with Short-Range and Coulomb Interactions. *Journal of Statistical Physics*, 22(3):297–347, 1980. doi: 10.1007/BF01014646. URL <https://doi.org/10.1007/BF01014646>.
- [94] Krzysztof Gawędzki and Antti Kupiainen. A rigorous block spin approach to massless lattice theories. *Communications in Mathematical Physics*, 77(1):31–64, 1980. doi: 10.1007/BF01205038. URL <https://link.springer.com/article/10.1007/BF01205038>.
- [95] Krzysztof Gawędzki and Antti Kupiainen. Block spin renormalization group for dipole gas and $(\nabla\varphi)^4$. *Annals of Physics*, 147(1):198–243, 1983. doi: 10.1016/0003-4916(83)90071-4. URL <https://www.sciencedirect.com/science/article/pii/0003491683900714>.
- [96] Krzysztof Gawędzki and Antti Kupiainen. Rigorous renormalization group and asymptotic freedom. In *Scaling and Self-Similarity in Physics*, volume 7 of *Progress in Physics*, pages 227–262. Birkhäuser, 1983. doi: 10.1007/978-1-4899-6762-6_7. URL https://link.springer.com/chapter/10.1007/978-1-4899-6762-6_7.
- [97] Krzysztof Gawędzki and Antti Kupiainen. Continuum limit of the hierarchical $O(N)$ non-linear σ -model. *Communications in Mathematical Physics*, 106(4):533–550, 1986. doi: 10.1007/BF01463394. URL <https://link.springer.com/article/10.1007/BF01463394>.
- [98] Paul H. Ginsparg. First order and second order phase transitions in gauge theories at finite temperature. *Nuclear Physics B*, 170:388–408, 1980. doi: 10.1016/0550-3213(80)90418-6.
- [99] James Glimm and Arthur Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer, 2 edition, 1987. doi: 10.1007/978-1-4612-4728-9. URL <https://link.springer.com/book/10.1007/978-1-4612-4728-9>.
- [100] Antonio González-Arroyo. Yang–Mills fields on the 4-dimensional torus. part I: Classical theory, 1998. URL <https://arxiv.org/abs/hep-th/9807108>. First part of lectures given at the 1997 Advanced School on Non-perturbative Quantum Physics, Peñíscola; cited version: arXiv:hep-th/9807108v1.
- [101] Antonio González-Arroyo, J. Jurkiewicz, and C. P. Korthals-Altes. Ground state metamorphosis for Yang–Mills fields on a finite periodic lattice. In J. Honerkamp,

- K. Pohlmeier, and H. Römer, editors, *Structural Elements in Particle Physics and Statistical Mechanics*, pages 339–358. Plenum Press, New York, 1983. doi: 10.1007/978-1-4613-3509-2_25. Freiburg 1981 lectures. Year follows the publisher’s original-volume record (1983); the chapter landing page inconsistently displays 1987.
- [102] David J. Gross and Frank Wilczek. Ultraviolet behavior of non-abelian gauge theories. *Physical Review Letters*, 30(26):1343–1346, 1973. doi: 10.1103/PhysRevLett.30.1343.
- [103] Leonard Gross. Logarithmic Sobolev Inequalities. *American Journal of Mathematics*, 97(4):1061–1083, 1975. doi: 10.2307/2373688. URL <https://doi.org/10.2307/2373688>.
- [104] Leonard Gross. Convergence of $U(1)_3$ Lattice Gauge Theory to Its Continuum Limit. *Communications in Mathematical Physics*, 92(2):137–162, 1983. doi: 10.1007/BF01210842. URL <https://doi.org/10.1007/BF01210842>.
- [105] Leonard Gross. The Yang–Mills heat equation with finite action in three dimensions. *Memoirs of the American Mathematical Society*, 275(1349):v+111, 2022. doi: 10.1090/memo/1349. URL <https://doi.org/10.1090/memo/1349>.
- [106] Leonard Gross, Christopher King, and Ambar Sengupta. Two dimensional Yang–Mills theory via stochastic differential equations. *Annals of Physics*, 194(1):65–112, 1989. doi: 10.1016/0003-4916(89)90032-8. URL [https://doi.org/10.1016/0003-4916\(89\)90032-8](https://doi.org/10.1016/0003-4916(89)90032-8).
- [107] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014. doi: 10.1007/s00222-014-0505-4. URL <https://doi.org/10.1007/s00222-014-0505-4>.
- [108] D. L. Hanson and F. T. Wright. A Bound on Tail Probabilities for Quadratic Forms in Independent Random Variables. *The Annals of Mathematical Statistics*, 42(3):1079–1083, 1971. doi: 10.1214/aoms/1177693335. URL <https://doi.org/10.1214/aoms/1177693335>.
- [109] Bernard Helffer. Remarks on Decay of Correlations and Witten Laplacians: Brascamp–Lieb Inequalities and Semiclassical Limit. *Journal of Functional Analysis*, 155(2):571–586, 1998. doi: 10.1006/jfan.1997.3239. URL <https://doi.org/10.1006/jfan.1997.3239>.
- [110] Bernard Helffer and Johannes Sjöstrand. On the Correlation for Kac-like Models in the Convex Case. *Journal of Statistical Physics*, 74(1–2):349–409, 1994. doi: 10.1007/BF02186817. URL <https://doi.org/10.1007/BF02186817>.
- [111] Robert D. Horowitz. Characters of free groups represented in the two-dimensional special linear group. *Communications on Pure and Applied Mathematics*, 25(6):635–649, 1972. doi: 10.1002/cpa.3160250602. URL <https://doi.org/10.1002/cpa.3160250602>.

- [112] Leon Isserlis. On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, 12(1-2): 134–139, 1918. doi: 10.1093/biomet/12.1-2.134. URL <https://doi.org/10.1093/biomet/12.1-2.134>.
- [113] R. Jackiw and S. Templeton. How super-renormalizable interactions cure their infrared divergences. *Physical Review D*, 23(10):2291–2304, 1981. doi: 10.1103/PhysRevD.23.2291.
- [114] Arthur Jaffe and Edward Witten. Quantum Yang–Mills theory. In James Carlson, Arthur Jaffe, and Andrew Wiles, editors, *The Millennium Prize Problems*, pages 129–152. Clay Mathematics Institute and American Mathematical Society, 2006. URL <https://www.claymath.org/wp-content/uploads/2022/06/yangmills.pdf>. Published version, pp. 129–152; official Clay PDF also available. No chapter DOI or arXiv identifier verified.
- [115] Dimitra Karabali, Chanju Kim, and V. P. Nair. Planar Yang–Mills theory: Hamiltonian, regulators and mass gap. *Nuclear Physics B*, 524:661–694, 1998. doi: 10.1016/S0550-3213(98)00309-5.
- [116] Dimitra Karabali, Chanju Kim, and V. P. Nair. On the vacuum wavefunction and string tension of Yang–Mills theories in (2+1) dimensions. *Physics Letters B*, 434(1-2):103–109, 1998. doi: 10.1016/S0370-2693(98)00751-5.
- [117] Christopher King. The U(1) Higgs model. I. the continuum limit. *Communications in Mathematical Physics*, 102(4):649–677, 1986. doi: 10.1007/BF01221651. URL <https://link.springer.com/article/10.1007/BF01221651>.
- [118] Christopher King. The U(1) Higgs model. II. the infinite volume limit. *Communications in Mathematical Physics*, 103(2):323–349, 1986. doi: 10.1007/BF01206942. URL <https://link.springer.com/article/10.1007/BF01206942>.
- [119] Jeffrey Koller and Pierre van Baal. A non-perturbative analysis in finite volume gauge theory. *Nuclear Physics B*, 302(1):1–64, 1988. doi: 10.1016/0550-3213(88)90665-7.
- [120] Roman Kotecký and David Preiss. Cluster expansion for abstract polymer models. *Communications in Mathematical Physics*, 103(3):491–498, 1986. doi: 10.1007/BF01211762. URL <https://doi.org/10.1007/BF01211762>.
- [121] Werner Krauth and Matthias Staudacher. Finite Yang–Mills integrals. *Physics Letters B*, 435(3-4):350–355, 1998. doi: 10.1016/S0370-2693(98)00814-4. URL <https://arxiv.org/abs/hep-th/9804199>.
- [122] Werner Krauth, Hermann Nicolai, and Matthias Staudacher. Monte Carlo approach to M-theory. *Physics Letters B*, 431(1-2):31–41, 1998. doi: 10.1016/S0370-2693(98)00557-7. URL <https://arxiv.org/abs/hep-th/9803117>.

- [123] Michel Ledoux. Concentration of Measure and Logarithmic Sobolev Inequalities. In *Séminaire de Probabilités XXXIII*, volume 1709 of *Lecture Notes in Mathematics*, pages 120–216. Springer, 1999. doi: 10.1007/BFb0096511. URL https://www.numdam.org/item/SPS_1999__33__120_0/.
- [124] Thierry Lévy. Yang–Mills measure on compact surfaces. *Memoirs of the American Mathematical Society*, 166(790):xiv+122, 2003. doi: 10.1090/memo/0790. URL <https://doi.org/10.1090/memo/0790>.
- [125] Thierry Lévy. Two-dimensional Markovian holonomy fields. *Astérisque*, 329:vi+172, 2010. URL https://www.numdam.org/item/AST_2010__329__R1_0/.
- [126] A. D. Linde. Infrared problem in the thermodynamics of the Yang–Mills gas. *Physics Letters B*, 96(3–4):289–292, 1980. doi: 10.1016/0370-2693(80)90769-8.
- [127] Martin Lüscher. Some analytic results concerning the mass spectrum of Yang–Mills gauge theories on a torus. *Nuclear Physics B*, 219(1):233–261, 1983. doi: 10.1016/0550-3213(83)90436-4. URL [https://doi.org/10.1016/0550-3213\(83\)90436-4](https://doi.org/10.1016/0550-3213(83)90436-4).
- [128] Martin Lüscher. Properties and Uses of the Wilson Flow in Lattice QCD. *Journal of High Energy Physics*, 2010(08):071, 2010. doi: 10.1007/JHEP08(2010)071. URL <https://arxiv.org/abs/1006.4518>. Erratum: *Journal of High Energy Physics* 2014, no. 03, article 092, doi:10.1007/JHEP03(2014)092.
- [129] Martin Lüscher. Trivializing maps, the Wilson flow and the HMC algorithm. *Communications in Mathematical Physics*, 293(3):899–919, 2010. doi: 10.1007/s00220-009-0953-7. URL <https://doi.org/10.1007/s00220-009-0953-7>.
- [130] Martin Lüscher and Gernot Münster. Weak-coupling expansion of the low-lying energy values in the SU(2) gauge theory on a torus. *Nuclear Physics B*, 232(3):445–472, 1984. doi: 10.1016/0550-3213(84)90038-5.
- [131] Martin Lüscher and Peter Weisz. Perturbative analysis of the gradient flow in non-abelian gauge theories. *Journal of High Energy Physics*, 2011(02):051, 2011. doi: 10.1007/JHEP02(2011)051. URL [https://doi.org/10.1007/JHEP02\(2011\)051](https://doi.org/10.1007/JHEP02(2011)051).
- [132] Jacques Magnen, Vincent Rivasseau, and Roland Sénéor. Construction of YM₄ with an Infrared Cutoff. *Communications in Mathematical Physics*, 155(2):325–383, 1993. doi: 10.1007/BF02097397. URL <https://doi.org/10.1007/BF02097397>.
- [133] Rajamani Narayanan and Herbert Neuberger. Infinite N phase transitions in continuum Wilson loop operators. *Journal of High Energy Physics*, 2006(03):064, 2006. doi: 10.1088/1126-6708/2006/03/064. URL <https://doi.org/10.1088/1126-6708/2006/03/064>.
- [134] Konrad Osterwalder and Erhard Seiler. Gauge Field Theories on a Lattice. *Annals of Physics*, 110(2):440–471, 1978. doi: 10.1016/0003-4916(78)90039-8. URL [https://doi.org/10.1016/0003-4916\(78\)90039-8](https://doi.org/10.1016/0003-4916(78)90039-8).

- [135] Giorgio Parisi and Yong-shi Wu. Perturbation theory without gauge fixing. *Scientia Sinica*, 24(4):483–496, 1981. URL <https://chimera.roma1.infn.it/GIORGIO/papers.html>.
- [136] Oliver Penrose. Convergence of fugacity expansions for classical systems. In Thor A. Bak, editor, *Statistical Mechanics: Foundations and Applications*, pages 101–109. W. A. Benjamin, New York, 1967.
- [137] H. David Politzer. Reliable perturbative results for strong interactions? *Physical Review Letters*, 30(26):1346–1349, 1973. doi: 10.1103/PhysRevLett.30.1346.
- [138] Yu. V. Prokhorov. Convergence of random processes and limit theorems in probability theory. *Theory of Probability and Its Applications*, 1(2):157–214, 1956. doi: 10.1137/1101016. URL <https://epubs.siam.org/doi/10.1137/1101016>. English translation; Russian original: *Teoriya Veroyatnostei i ee Primeneniya* 1(2), 177–238 (1956).
- [139] Johan Råde. On the Yang–Mills Heat Equation in Two and Three Dimensions. *Journal für die reine und angewandte Mathematik*, 431:123–163, 1992. doi: 10.1515/crll.1992.431.123. URL <https://doi.org/10.1515/crll.1992.431.123>. Publisher metadata gives pages 123–164, including a blank final verso; substantive text ends at page 163.
- [140] Vincent Rivasseau. *From Perturbative to Constructive Renormalization*. Princeton Series in Physics. Princeton University Press, 1991. doi: 10.1515/9781400862085. URL <https://www.degruyterbrill.com/document/doi/10.1515/9781400862085/html>. Originally published in 1991; DOI refers to the electronic reissue (2014).
- [141] Erhard Seiler. *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, volume 159 of *Lecture Notes in Physics*. Springer, 1982. doi: 10.1007/3-540-11559-5. URL <https://link.springer.com/book/10.1007/3-540-11559-5>.
- [142] Ambar Sengupta. The Yang–Mills measure for S^2 . *Journal of Functional Analysis*, 108(2):231–273, 1992. doi: 10.1016/0022-1236(92)90025-E. URL [https://doi.org/10.1016/0022-1236\(92\)90025-E](https://doi.org/10.1016/0022-1236(92)90025-E).
- [143] Ambar Sengupta. Quantum gauge theory on compact surfaces. *Annals of Physics*, 221(1):17–52, 1993. doi: 10.1006/aphy.1993.1002. URL <https://doi.org/10.1006/aphy.1993.1002>.
- [144] Ambar Sengupta. Gauge Invariant Functions of Connections. *Proceedings of the American Mathematical Society*, 121(3):897–905, 1994. doi: 10.1090/S0002-9939-1994-1215205-7. URL <https://doi.org/10.1090/S0002-9939-1994-1215205-7>.
- [145] Ambar Sengupta. Gauge theory on compact surfaces. *Memoirs of the American Mathematical Society*, 126(600):viii+85, 1997. doi: 10.1090/memo/0600. URL <https://doi.org/10.1090/memo/0600>.

- [146] Hao Shen, Rongchan Zhu, and Xiangchan Zhu. A stochastic analysis approach to lattice Yang–Mills at strong coupling. *Communications in Mathematical Physics*, 400(2):805–851, 2023. doi: 10.1007/s00220-022-04609-1. URL <https://doi.org/10.1007/s00220-022-04609-1>.
- [147] Charles Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In Lucien M. Le Cam, Jerzy Neyman, and Elizabeth L. Scott, editors, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume II: Probability Theory*, pages 583–602. University of California Press, 1972. URL <https://digicoll.lib.berkeley.edu/record/113245>.
- [148] V. N. Sudakov and B. S. Tsirel’son. Extremal properties of half-spaces for spherically invariant measures. *Journal of Soviet Mathematics*, 9(1):9–18, 1978. doi: 10.1007/BF01086099. URL <https://doi.org/10.1007/BF01086099>. English translation. Russian original: Zapiski Nauchnykh Seminarov LOMI 41 (1974), 14–24, 165.
- [149] Gerard ’t Hooft. A property of electric and magnetic flux in non-abelian gauge theories. *Nuclear Physics B*, 153:141–160, 1979. doi: 10.1016/0550-3213(79)90595-9.
- [150] Michael J. Teper. SU(N) gauge theories in 2+1 dimensions. *Physical Review D*, 59(1):014512, 1999. doi: 10.1103/PhysRevD.59.014512. Issue dated 1 January 1999; first published online 9 December 1998.
- [151] Karen K. Uhlenbeck. Connections with L^p Bounds on Curvature. *Communications in Mathematical Physics*, 83(1):31–42, 1982. doi: 10.1007/BF01947069. URL <https://doi.org/10.1007/BF01947069>.
- [152] Pierre van Baal. QCD in a finite volume. In M. Shifman, editor, *At the Frontier of Particle Physics: Handbook of QCD*, volume 2, pages 683–760. World Scientific, 2001. doi: 10.1142/9789812810458_0021.
- [153] Kenneth G. Wilson. Confinement of Quarks. *Physical Review D*, 10(8):2445–2459, 1974. doi: 10.1103/PhysRevD.10.2445. URL <https://doi.org/10.1103/PhysRevD.10.2445>.
- [154] Edward Witten. On quantum gauge theories in two dimensions. *Communications in Mathematical Physics*, 141(1):153–209, 1991. doi: 10.1007/BF02100009. URL <https://doi.org/10.1007/BF02100009>.
- [155] Roderick Wong. *Asymptotic Approximations of Integrals*, volume 34 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics, 2001. doi: 10.1137/1.9780898719260. URL <https://doi.org/10.1137/1.9780898719260>. SIAM reprint of the 1989 Academic Press edition.
- [156] C. N. Yang and R. L. Mills. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Physical Review*, 96(1):191–195, 1954. doi: 10.1103/PhysRev.96.191. URL

<https://doi.org/10.1103/PhysRev.96.191>.

- [157] Daniel Zwanziger. Covariant Quantization of Gauge Fields without Gribov Ambiguity. *Nuclear Physics B*, 192(1):259–269, 1981. doi: 10.1016/0550-3213(81)90202-9. URL [https://doi.org/10.1016/0550-3213\(81\)90202-9](https://doi.org/10.1016/0550-3213(81)90202-9).