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The continuum limit of
three-dimensional $SU(2)$
lattice Yang–Mills theory
on a torus III:
Regional renormalization and
large-field control

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WARNING!

We have extensively audited this manuscript using large language models, including adversarial reviews by multiple models. However, the proof is too complex for Lean formalization at the present time, and it has not yet undergone verification by a human expert.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).

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<https://www.loc.gov/resource/g3200m.gct00126/?sp=106>

AUTHORS' NOTE

With the exception of this note, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited.

The manuscript began as an attempt to pursue the program of Magnen, Rivasseau, and Sénéor [132] using large language models. Our hope was that understanding the problem on the torus might offer some insight into the general case. The construction that resulted from this starting point is three-dimensional and lattice-based and draws extensively on Bałaban's renormalization group methods.

As we were verifying these results, we learned that Alpöge and Sussman announced that they appear to have a YM4 existence argument, following Bałaban's program (<https://github.com/ethanwsussman/Yang--Mills-LLM-experiment>). We are thrilled and look forward to learning more about these exciting developments. We hope that this manuscript nonetheless contributes to understanding constructive approaches to three-dimensional Yang–Mills theory. The first version of these results was posted to the first author's GitHub on October 17, 2026 (<https://github.com/pmaronow/3d-ym-torus>). Prior to posting the first version of this work, we did not inspect the ongoing work of Alpöge and Sussman. Further, to the best of our knowledge, the LLMs we used did not access their work during the mathematical development and drafting of this manuscript. This version contains additional clarifications relative to the first version; it has also been partitioned into five volumes.

We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We consider the Wilson lattice gauge measure for $SU(2)$ on a periodic three-dimensional torus of fixed size, with bare coupling $g^2 = g_3^2 a$ proportional to the lattice spacing a , as dictated by the superrenormalizability of three-dimensional Yang–Mills theory. Observables are Wilson loops of the lattice configuration evolved by the Wilson flow, the lattice gradient flow of the Wilson action, at positive flow time. We prove that, at every fixed value of the coupling, the joint laws of finitely many flowed Wilson loops converge as the lattice spacing tends to zero along the sequences of lattice sides nM^k , for fixed odd $M \geq 3$ and fixed positive n divisible by four. For each such refinement class, the limits are realized by a unique probability measure on the space of gauge orbits of Yang–Mills heat-flow trajectories at positive times, a version of the state space introduced by Cao and Chatterjee. At each fixed positive time, the energies of the selected trajectories $\pi_k(U)$, with U distributed under the lattice Wilson measures, are tight uniformly in the lattice spacing. The limiting measure is invariant under the lattice symmetries and the center symmetries. We also show that, at short distances, the rescaled deficits $(1 - W)/(g_3^2 \varepsilon)$ of the normalized fundamental Wilson loops W of loops $x + \varepsilon \ell$, flowed for time $\sigma \varepsilon^2$ with fixed $\sigma > 0$, converge to explicit quadratic functions of the heat-smoothed Gaussian free field. As the coupling tends to zero, the measure concentrates on flat connections, with Gaussian fluctuations of explicit covariance near regular commuting holonomies and fluctuations governed by the constant-mode quartic form near the central holonomies. The proof is a multiscale renormalization group analysis in the tradition of Balaban, carried out on a small-field event whose complement we show to have small probability, uniformly in the lattice spacing, using reflection positivity and chessboard estimates. It is combined with stochastic estimates for the lattice heat flow and a deterministic analysis of the Wilson flow for rough initial data.

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Part VI: Large-field control and the small-field event (continued)

Comparison of reconstructions on block regions

This section compares one renormalization group step on a block region with the same step on the whole torus. Fix a composition T_M^s of $s \leq J$ averaging steps and a union Ω of blocks of its output lattice. In Section 18 we reconstructed the fine configuration on Ω from the coarse field on Ω and free logarithmic coordinates, with the coordinates outside Ω held fixed. We also constructed the stationary point of the effective classical action in these coordinates and proved a gap for its free Hessian. Here we prove that the regional construction differs from the torus construction with the coarse field extended by zero outside Ω only by terms which decay exponentially with the distance to the boundary of Ω . We also prove that the regional construction is stable under small analytic perturbations of the incoming action. All estimates are bounds on absolute Taylor coefficients. Their constants do not depend on the shape of Ω , the number of its components, the depth, the number of preceding renormalization group steps, or the values of previously frozen coordinates.

We use these estimates in the localized renormalization group analysis behind the exponential-moment bound for blocked actions (Step 1 in Section 3.4). There the configurations are decomposed according to the regions where some averaged plaquette has large curvature, and the renormalization group is run outside these regions with the link variables inside them held fixed. Section 20 integrates the fluctuations on block regions using the reconstructions and perturbation estimates proved here. Section 21 uses the gauge transformation of Lemma 19.5 and the perturbation theorem Theorem 19.9 to construct large-field decompositions, and Section 23 uses the domain conditions of Lemmas 19.3 and 19.7 and Theorem 19.9 to choose the parameters.

The results, in order, are as follows: Lemma 19.3 treats successive minimization; Theorem 19.4 compares the regional and torus stationary points, together with the minimized actions, the Hessians, and the Haar disintegration densities; Lemma 19.5 constructs a gauge transformation which is the identity at prescribed vertices; Theorem 19.6 compares the gauge-transformed reconstructions and bounds their input columns; Lemma 19.7 collects the conditions on the radii; and Theorem 19.9 proves stability under perturbations of the incoming action.

The inputs are Theorem 18.12 (regional coordinates, the gap of the free Hessian, and the regional stationary point), Lemma 18.15 (coefficient bounds for connected networks which reach the fixed complement), Lemma 18.8 (derivatives of constrained stationary

values), Lemma 18.9 (freezing of coordinates), the coefficient algebra of Definition 18.1, the target-dependent gauge transformation of Theorem 13.2, the classical coefficient bounds of Theorem 10.11, and the equality of effective classical actions on compact gauge overlaps, Theorem 11.6.

We compare constrained minimizers through the implicit equation for their response, and we control the response by an exponentially decaying inverse of the constrained Hessian. This approach follows Balaban's analysis of the variational problem for background fields [16, (182)–(190), pp. 307–308, and Proposition 9, p. 309], although our constraints, coordinate domains, and coefficient norms differ from his. The comparison between the regional and the torus problems and the gauge transformation fixing the prescribed endpoints are constructed here. The exponentially weighted resolvent bound, proved in Lemma 18.3, is the finite-dimensional version of the exponential-conjugation method of Combes and Thomas [61].

The mechanism. The regional problem and the torus problem with coarse field extended by zero have the same constraints away from a fixed neighborhood of the boundary of Ω . If we restrict the torus stationary point to the regional coordinates, the gradient of the regional objective at this point is therefore concentrated near the boundary, up to exponentially decaying tails. This residual gradient has two sources, which are the two terms of the formula for r_R in Theorem 19.4. First, the reconstructed fine field changes. Second, the tangent map of the constraints changes, since the regional problem omits the coarse constraints on links leaving Ω . Writing p for the restricted torus stationary point and z_R for the regional stationary point, we have $z_R - p = -\mathcal{A}_R^{-1}r_R$, where r_R is the residual gradient and $\mathcal{A}_R$, the regional Hessian averaged along the segment from p to z_R , has a positive gap. Since this inverse decays exponentially in the distance, the two stationary points differ by an amount which decays exponentially away from the boundary. The remaining work is to make this mechanism stable under differentiation, composition, and successive minimization. This requires bounds on all Taylor coefficients, with any one external index fixed. A bound on the values of a holomorphic map does not give the spatial weights and the fixed-index sums which are needed when these functions are composed later.

Figure 19.1 shows the sets which appear below. Two of them play different roles: the prescribed vertices, where the gauge transformation must be the identity, and the complement of the incoming list, which controls the spatial decay of the perturbation in Theorem 19.9.

19.1 Setting and conventions

Geometry. Throughout this section $M \geq 3$ is odd, $J \geq 1$ is fixed, $1 \leq s \leq J$, and $B_s = M^s$. The composition T_M^s of s averaging steps maps configurations of side $M^s m$ to

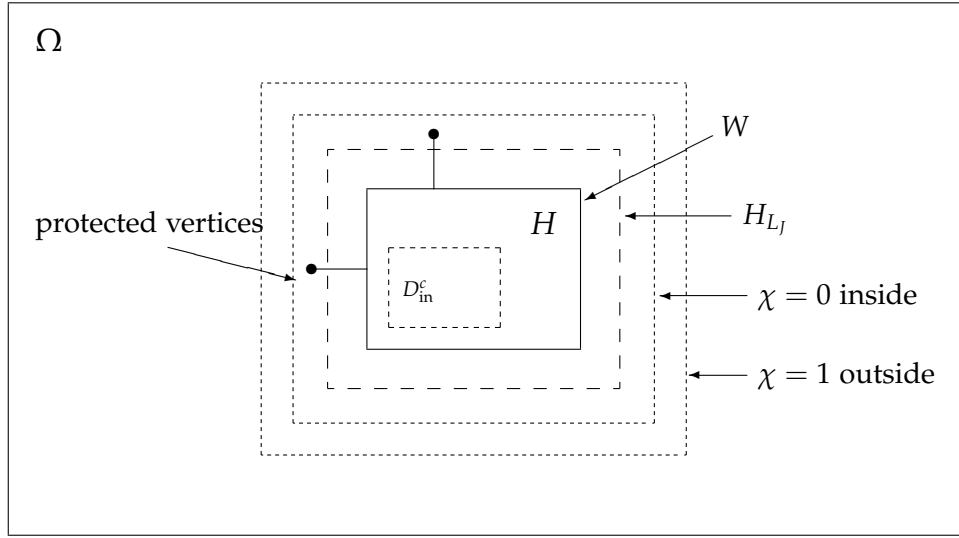


Figure 19.1. The geometry in the output lattice (schematic). The solid inner rectangle is the fixed complement $H = \Omega^c$, and its boundary is the interface W . The dashed rectangle bounds the enlargement H_{L_J} of H by the reconstruction stencil. The cutoff χ of Lemma 19.5 vanishes inside the inner dotted rectangle and equals one outside the outer dotted rectangle. The black dots are protected vertices, here the endpoints of the paths of two based transports of frozen coordinates. The small dotted rectangle inside H is the complement of the incoming list D_{in} of Theorem 19.9, which lies within a fixed distance of H .

configurations of side $m \geq 4$, and the prior depth $d_0 \geq 0$ is an arbitrary integer. When $s = J$ the composition is one renormalization group step, with blocking factor $B = M^J$. Positions are measured in units of the output lattice, of side m . Let Ω be a union of blocks of the output lattice, let $H = \Omega^c$ be its fixed complement, let W be the interface between Ω and H , and let H_{L_J} be the enlargement of H by the fixed reconstruction stencil L_J of Theorem 18.12. A coordinate is a label carrying a position, a direction, a color, and a scale index; its position is taken at the lattice where the coordinate was introduced and embedded in output units. In particular the newly frozen coordinates are counted with the density of the lattice on which they were introduced, not as coordinates of the output lattice. The density constant v_J of Definition 18.1 includes the colors, the directions, and the at most J scale indices. For a label i in Ω we write $\delta(i) = \text{dist}(i, W)$. Here dist and t are the graph distance and the ℓ^1 network length of the refined position graph fixed before Definition 18.1; coefficient bounds imported at a Euclidean rate b are used at rate $b/\sqrt{3}$. If Ω is empty, every statement below with a fixed index in Ω is void. If H is empty, every boundary difference considered below is zero; we never use an expression of the form $0 \cdot e^\infty$.

Coordinates. We use the following coordinates.

- The coarse field c is the logarithm of the output links internal to Ω , and $c \oplus 0$ is its extension by zero to all output links of the torus.
- The free logarithmic coordinates $z = (z_i)_{0 \leq i < s}$ of Theorem 18.12 parametrize the solutions of the block averaging constraints in Ω , with one block z_i for each intermediate scale i ; here $i = 0$ is the finest scale. On the torus we write $z = (z_L, z_H)$, where z_L are the coordinates of the regional list and z_H are the remaining torus coordinates.
- $\Psi_R(c, z; \xi)$ is the logarithm of the regional reconstruction of the fine field before gauge transformation, including its prescribed small values outside Ω , and $\Psi_T(c_T, z_T)$ is the logarithm of the torus reconstruction before gauge transformation. We call the links of the input lattice of T_M^s fine links.
- In the successive reconstruction the free coordinates of scale i are $a_i + K_i^{-1/2} q_i$, where a_i is the conditional stationary point and K_i the constrained Hessian of that scale, and q_i is the fluctuation increment. The increments q_i replace $g_i w_i$, where w_i are the normalized fluctuation variables; no coupling is absorbed into K_i .
- In Section 19.7 we also use the jointly normalized coordinates u , defined by $z = z_0 + K_0^{-1/2} u$, where z_0 is the joint stationary point and K_0 the Hessian there.
- The newly frozen coordinates ξ are the small logarithmic coordinates frozen at the present step; they enter Ψ_R through its prescribed values outside Ω . The previously frozen coordinates b_{rec} are fixed real parameters, kept at the lattices on which they were introduced.

The points $z = 0$, $q = 0$, and $u = 0$ are different. Both $q = 0$ and $u = 0$ correspond to the joint stationary point $z = z_0$ (for q this is part of Lemma 19.3). The point $z = 0$ is the origin of the constraint coordinates.

Frozen coordinates and stationary points. No derivative in the previously frozen coordinates b_{rec} is taken or asserted in this section. In Theorems 19.4 and 19.6 the newly frozen coordinates ξ are analytic variables, which may be complexified and differentiated. In Theorem 19.9 the perturbed objects are differentiated only in the coarse field, the free coordinates, and the interpolation parameter, at fixed real (ξ, b_{rec}) . All stationary points below are the holomorphic stationary branches in the coordinate domains of Theorem 18.12. At real data the positive free Hessian makes them local minima, and the conditional stationary points are the conditional constrained minimizers of the successive minimization.

Dependence of constants. All constants and radii in this section have the dependence described in item (3) of Definition 4.14. They may depend on M and J , on the larger classical and reconstruction domains and the ratios of their radii, on the coefficient bounds on these domains, on the gaps of the free Hessians and the bounds on their weighted inverses, on the fixed spatial decay exponents, on v_J , and on any fixed bound

on the size of the perturbation. In Lemma 19.7 and Theorem 19.9 they may also depend on the fixed analytic radii, on K_* , and on the ratios of these radii. They do not depend on L , γ , the lattice side, the depth d_0 , the shape or topology of Ω , the number of components of Ω or H , the values of the previously frozen coordinates, or the number of preceding renormalization group steps. They are evaluated before the later choices of the boundary-layer width w_{bd} , the erosion width, the widths of the protected neighborhoods, the bound β_{rec} on newly frozen coordinates, and the enclosing product domains of Section 20. Each constant includes a finite maximum over $1 \leq s \leq J$ and over the finitely many initial segments. We do not assert uniformity as $J \rightarrow \infty$. Theorem 18.12 provides the gap of the free Hessian at every integer depth d_0 , and when only this classical estimate is used, no condition on an effective interaction is imposed.

The torus construction domain. Every use of Theorem 13.2 below takes place on its construction product. For a composition of s averaging steps, we denote the radius and coefficient constant of that theorem by r_s and $C_s^{\mathbb{T}}$. Before the later profiles and widths are chosen, we fix positive U_T, Q_T in the common larger product of coarse fields and fluctuation increments such that

$$U_T + Q_T < \frac{1}{2} \min_{1 \leq s \leq J} r_s, \quad C_T(U_T + Q_T) < \frac{1}{2}, \quad C_T = \max_{1 \leq s \leq J} C_s^{\mathbb{T}}. \quad (19.1)$$

Here Q_T bounds $\max_i \|q_i\|_\infty$ for the fluctuation increments, not the raw free coordinates $a_i + K_i^{-1/2} q_i$. Decreasing U_T, Q_T if necessary, we may assume that the intermediate classical, inverse, and Haar-density margins of Theorem 13.2 hold. Later radii for coarse fields and fluctuation increments are chosen below U_T and Q_T , so each use of Theorem 13.2 has $u + v < r_s$, a positive denominator, and its intermediate margins. The rational row bound of that theorem alone does not show that a larger product is allowed.

Coefficient bounds. We use the coefficient algebra of Definition 18.1, Theorem 18.12, , and Lemma 18.15 and its coefficient bounds with each external index fixed in turn. A rooted coefficient sum of a tensor fixes one external index, its root, which is either an output or an input, and sums the absolute values of the Taylor coefficients over the remaining indices, with the weight e^{at} , where t is the ℓ^1 length of a shortest connected network spanning the positions involved. For a fixed input index the sum is the volume-normalized column sum of Definition 4.6, with the mesh of that input. A coefficient bound is a bound on all these sums, with any external index fixed. Taylor tensors at complex arguments use bilinear transposes. A majorant series is the series of absolute coefficients after the prescribed rooted sums and weights; the original coefficients need not be positive. In a scalar majorant the variable t is a radius after these rooted sums, never a sum of radii over all lattice sites. We say that a tensor, or a convergent series of tensors, has boundary-localized coefficient bounds if it satisfies the bounds of Lemma 18.15: with

any index i in Ω fixed, its rooted coefficient sums carry the additional weight $e^{c\delta(i)}$ for a fixed $c > 0$.

A product or contraction joins the connecting networks of its factors at the contracted indices. We root the joined network at any external index and sum over the internal indices, using the corresponding rooted bounds of the factors. The boundary-localized estimate applies if one factor is boundary-localized, or if the joined network visits H_{L_J} . An index deep in H is bounded there, and the decay comes from its connection to the fixed index in Ω . In particular, for decay exponents $b > a > 0$, the sum over such indices is bounded by

$$e^{a\delta(e)} \sum_{h \in H_{L_J}} e^{-b \text{dist}(e,h)} \leq e^{aL'_J} S_J(b-a), \quad S_J(u) = Cv_J \sum_{r \geq 0} (r+1)^3 e^{-ur}, \quad (19.2)$$

where L'_J also includes the fixed convention for link endpoints. This is the case of a fixed index in Ω in Lemma 18.15; there is no factor of the volume of H .

Implicit series. The implicit equations below are solved in the absolute coefficient algebra. After the construction constant C_J is increased, the scalar majorant of the solution without boundary weight can be taken to satisfy

$$Y = C_J t + \frac{1}{2} Y + \frac{C_J(t+Y)^2}{1 - C_J(t+Y)}. \quad (19.3)$$

Fix $t_0 > 0$ such that, on $0 \leq t \leq t_0$ and $0 \leq Y \leq 4C_J t_0$, the rational term has Y -derivative at most $1/4$ and denominator at least $1/2$, and its value at $(t, 4C_J t)$ is at most $C_J t$. Such a t_0 exists and depends only on C_J : the derivative and denominator conditions have strict margins at $(t, Y) = (0, 0)$, while the rational term at $(t, 4C_J t)$ equals $C_J(1 + 4C_J)^2 t^2 / [1 - C_J(1 + 4C_J)t] = O(t^2)$ as $t \downarrow 0$. Iteration from zero produces series with nonnegative coefficients which stay below $4C_J t$. The iteration converges uniformly on every smaller disc, since the derivative in Y of the right side of (19.3) is at most $3/4$ on its majorant disc. The degree- l coefficients of the solution are bounded by $4C_J t_0^{1-l}$, and dividing Taylor tensors by $l!$ before this comparison gives factorial tensor bounds. The argument works in the absolute multivariate coefficient algebra because each product satisfies the rooted contraction bound just described. Uniform convergence of the power series implies holomorphy.

A derivative which inserts one difference factor, localized near H , has the majorant

$$Z \preceq H(t, Y(t)) + A(t)Z, \quad A(t) = \frac{1}{2} + \partial_Y \frac{C_J(t+Y)^2}{1 - C_J(t+Y)} \Big|_{Y=Y(t)}, \quad (19.4)$$

where $\preceq$ denotes coefficientwise domination and $H(t, Y)$ is the series with the difference factor inserted. On a smaller fixed radius $A(t) \leq 3/4$, and the solution is majorized

coefficientwise by $H(t, Y(t))/(1 - A(t))$; this is a convergent majorant series with one inserted difference factor (compare (18.55)). Fixing an analytic input differentiates this same series and keeps the index of that input. By Lemma 18.15, the connecting network and the route to the boundary are bounded once for the whole network, and their exponent is not divided by the Taylor order.

Matrix functions. We use matrix functions only on free coordinate spaces where a gap has been proved. The next lemma collects the facts we need.

Lemma 19.1. *Let $A^b \geq \lambda I$ be a real symmetric free matrix whose two-sided weighted Schur norm at exponent b is at most Q , and set*

$$\mu = \frac{b\lambda}{8(Q + \lambda)}, \quad C_R = 2S_J(\mu/2).$$

1. *For every $t \geq 0$, the entries of $(A^b + t)^{-1}$ are bounded by $2e^{-\mu \text{dist}(x,y)} / (\lambda + t)$, and its weighted Schur norm at exponent $\mu/2$ is at most $C_R / (\lambda + t)$; the constant C_R is the same for every subset of the label list.*
2. *If L is a complex coefficient perturbation with $C_R \|L\|_{\mu/2} / \lambda \leq 1/4$, then the series*

$$(A^b + L + t)^{-1} = \sum_{k \geq 0} [-(A^b + t)^{-1} L]^k (A^b + t)^{-1} \quad (19.5)$$

converges normally, and its coefficient Schur norm is at most $4C_R / [3(\lambda + t)]$.

3. *For positive definite real matrices A, B ,*

$$A^{-1/2} = \frac{1}{\pi} \int_0^\infty t^{-1/2} (A + t)^{-1} dt, \quad (19.6)$$

$$A^{-1/2} - B^{-1/2} = -\frac{1}{\pi} \int_0^\infty t^{-1/2} (A + t)^{-1} (A - B) (B + t)^{-1} dt, \quad (19.7)$$

$$\log A - \log B = \int_0^\infty (A + t)^{-1} (A - B) (B + t)^{-1} dt, \quad (19.8)$$

and these identities extend holomorphically to the complex perturbations in (2).

4. *If $T = \begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}$ is invertible with invertible blocks A and $D_0 - CA^{-1}B$, then*

$$(T^{-1})_{LL} - A^{-1} = A^{-1}B(D_0 - CA^{-1}B)^{-1}CA^{-1}. \quad (19.9)$$

Proof. Parts (1) and (2) are the Combes–Thomas bounds proved in Lemma 18.3(3); see (18.5). We recall the choice of constants. For a real one-Lipschitz function f on the label metric, conjugation by $\text{diag}(e^{\mu f})$ changes A^b by a matrix of Schur norm at most $Q\mu / (b - \mu) \leq \lambda/8$; here we used $e^{\mu r} - 1 \leq \mu r e^{\mu r}$ and $\sup_{r \geq 0} r e^{-(b-\mu)r} \leq (b - \mu)^{-1}$.

By the geometric resolvent series, the conjugated resolvent has operator norm at most $2/(\lambda + t)$ for $t \geq 0$. Taking $f(x) = \text{dist}(x, y)$, we obtain the entry bound, and summing at exponent $\mu/2$ gives $C_R/(\lambda + t)$, uniformly over every subset of the label list. Under the condition in (2) each term of (19.5) after the first is smaller than the preceding one by the factor $1/4$, which gives the bound $4C_R/[3(\lambda + t)]$.

For (3), we diagonalize the positive matrices and substitute $t = \lambda u^2$ in each eigenvalue integral, which gives (19.6). The resolvent identity $(A + t)^{-1} - (B + t)^{-1} = -(A + t)^{-1}(A - B)(B + t)^{-1}$ fixes the order of the three factors and gives (19.7) after integration against $\pi^{-1}t^{-1/2} dt$. For (19.8), we integrate $(1 + t)^{-1}I - (A + t)^{-1}$ to obtain $\log A$ and subtract the corresponding formula for B . The series (19.5) extends these identities holomorphically to the complex perturbations in (2). Their majorants are integrable: they are bounded by $C_J t^{-1/2}(\lambda + t)^{-2}$ and $C_J(\lambda + t)^{-2}$ for the two differences, whose integrals are $\pi/(2\lambda^{3/2})$ and $1/\lambda$. Thus a boundary-localized middle factor $A - B$ gives boundary-localized differences of these functions, rooted at every index, and differentiation under the integral is justified by the same normally convergent majorants. Restricted diagonal sums are taken after this calculation; only a complete scalar trace permits a cyclic permutation of the factors. Part (4) follows by block multiplication. $\square$

Additive constants. The coefficient seminorm with boundary decay $\mathcal{X}$ of Definition 18.1 vanishes on constants, so it never estimates additive constants. The constant term of a difference of minimized actions remains in the full action. Scalar determinants, Haar disintegration densities, and traces over H , which have no index in Ω , are kept as their exact finite values. The difference between two exterior extensions of the classical action is kept with all its constants (Theorem 19.9). After freezing at c , replacing an effective interaction $R(v, c)$ by $R(v, c) - R(0, c)$ changes the normalization constant from Λ to $\Lambda - R(0, c)$ and preserves the density, as in Lemma 18.9.

19.2 Conditional minimization and successive substitution

The reconstruction of the fine field proceeds from the coarsest intermediate scale to the finest. At scale i the free coordinates are $a_i(c, z_{>i}) + K_i^{-1/2}q_i$, where a_i is a conditional stationary point which depends on the coarser coordinates $z_{>i}$ already reconstructed. We must check two things. First, for every fixed value of the coarser coordinates in some domain, the conditional stationary point exists. Second, the coarser coordinates produced by the earlier substitutions lie in that domain. The first does not imply the second, as the following example shows.

Example 19.2. Let $F_0(x, y) = (x^2 + y^2)/2$ and $F_1(x, y) = F_0(x, y) - \delta y$ with $\delta > 0$, and eliminate x first, so that y is the coarser coordinate. For every fixed y the conditional stationary point of F_1 in x is $x = 0$, and the joint Hessian is the identity. The joint

stationary point of F_1 is $(0, \delta)$. If the conditional stationary points are constructed only for $|y| < \eta$ with $\eta < \delta$, the successive substitution never reaches the joint stationary point, however small the fluctuation radius in x is chosen. The quantity δ_V of Lemma 19.3 detects this: for the initial segment consisting of both coordinates it equals δ , so the condition in part (3) of the lemma fails.

In the next lemma, I is an initial segment of the finer scale coordinates eliminated in the successive minimization, and $y = z_{I^c}$ denotes the coarser coordinates held as parameters. The segment containing all free coordinates is included.

Lemma 19.3. *Fix the larger construction domains of Theorem 18.12, within (19.1), and the margins of their coefficient, inverse, and image bounds and of the lower bound for the real constrained Hessian. Let $F_0(c, z; \xi)$ be the unperturbed regional objective and $z^0(c; \xi)$ its joint stationary point. For each initial segment I write $y = z_{I^c}$, and let $a_I(c, y; \xi)$ be the conditional stationary point of F_0 in the variables z_I .*

1. *One can first choose a solution radius $r_I > 0$, and then a strictly smaller radius η_I for the deviation $y - z_{I^c}^0(c; \xi)$, uniformly over the coarse fields in the domain, such that*

$$a_I(c, z_{I^c}^0; \xi) = z_I^0, \quad D_y a_I = -F_{0,II}^{-1} F_{0,II^c}, \quad \|a_I - z_I^0\|_{\text{coeff}} \leq C_J \|y - z_{I^c}^0\|_{\text{coeff}}.$$

The derivative identity is evaluated at the conditional stationary point. We write $\mathcal{T}_I$ for the resulting domain of y .

2. *Let $F_\lambda = F_0 + \lambda V$, $0 \leq \lambda \leq 1$, be holomorphic and real at real arguments, with the same coefficient bounds on the larger domains. Set $A_I = F_{0,II}(c, a_I, y; \xi)$ and assume that, throughout the domains of part (1), in the operator norm of the coefficientwise majorant,*

$$\|A_I^{-1} F_{\lambda,I}(c, a_I, y)\| \leq r_I/4, \quad \|A_I^{-1}(F_{\lambda,II} - A_I)\| \leq 1/2.$$

Assume also that the real Hessian is perturbed by less than half of the lower bound for the constrained Hessian, and that the complex resolvent condition of Lemma 19.1(2) holds. Then for each I and each $y \in \mathcal{T}_I$ there is a unique holomorphic stationary point $a_{\lambda,I}(c, y; \xi)$ of F_λ in z_I within distance $r_I/2$ of a_I . It is real at real data and a local real minimum there, and it is unique in the convex domain on which the second inequality holds.

3. *Put $V = \partial_\lambda F_\lambda$ and*

$$\delta_V = \max_I \sup_{y \in \mathcal{T}_I} \|A_I^{-1} V_I(c, a_I(c, y; \xi), y)\|_{\text{coeff}},$$

where the coefficient norm includes the common product of coarse fields, uniformly at fixed real frozen coordinates, and let $\varepsilon_{\text{tube}}$ be the minimum over I of the solution and parameter margins at I . Then, for every I and every $y \in \mathcal{T}_I$,

$$\|a_{\lambda,I}(c, y; \xi) - a_I(c, y; \xi)\|_{\text{coeff}} \leq 2\lambda\delta_V.$$

Assume further that $\delta_V < \infty$, that the mixed coefficient bounds hold uniformly on the larger domains, and that $r_q > 0$ satisfies

$$C_{\text{seq},J}(2\delta_V + r_q) < \frac{1}{2}\varepsilon_{\text{tube}},$$

where $C_{\text{seq},J}$ is the constant given in the proof. For I_i consisting of scales $0, \dots, i$, let $a_{\lambda,i}$ be the scale- i component of the conditional stationary point a_{λ,I_i} of F_λ , and let $K_{\lambda,i}$ be M^i times the Schur block of F_λ at scale i in input units. Then, whenever $\max_i \|q_i\|_\infty \leq r_q$, the successive substitutions in the original free coordinates

$$z_i = a_{\lambda,i}(c, z_{>i}; \xi) + K_{\lambda,i}^{-1/2}(c, z_{>i}; \xi)q_i$$

stay in every parameter domain and every solution domain, with half of each margin unused. At $q = 0$ they give the joint stationary point.

The radii are chosen in the following order: first the unperturbed product, for which $\delta_V = 0$; then a bound on the size of the perturbation on the fixed larger unperturbed products, and a smaller perturbed product; only afterwards the smaller enclosing and fluctuation radii. Reducing r_q does not reduce the radius of the coarse field, since the chain at $q = 0$ is the joint stationary point. All radii and constants have the dependence of Section 19.1, independently of the later widths, the depth, and the geometry of Ω .

Proof. Part (1) is a quantitative implicit function argument about the joint stationary point, and part (2) is the same argument about a_I . The new point is part (3), which first bounds the displacement of the conditional stationary points at every fixed y and then propagates these bounds through the successive substitution by a finite recurrence.

Step 1: the unperturbed conditional stationary points. Fix I and write $y = z_{I^c}^0(c; \xi) + v$. This shift is an analytic change of coordinates with the coefficient bounds of Theorem 18.12. In these coordinates we set $u = z_I - z_I^0(c; \xi)$ and

$$B_I(c; \xi) = F_{0,II}(c, z^0(c; \xi); \xi).$$

By the bounds on the inverse and on the joint third derivative, there exist constants I_0, C_0 on the fixed larger products such that

$$\|B_I^{-1}F_{0,I}(c, z_I^0, z_{I^c}^0 + v; \xi)\|_{\text{coeff}} \leq I_0 C_0 \|v\|_{\text{coeff}}, \quad (19.10)$$

$$\|B_I^{-1}(F_{0,II}(c, z_I^0 + u, z_{I^c}^0 + v; \xi) - B_I)\|_{\text{coeff}} \leq I_0 C_0 (\|u\|_{\text{coeff}} + \|v\|_{\text{coeff}}). \quad (19.11)$$

For the first inequality we integrate F_{0,II^c} in v , starting from the joint stationarity $F_{0,I}(c, z^0; \xi) = 0$; for the second we integrate the third derivative tensor in (u, v) . These are coefficient inequalities, since integration along a segment from the origin multiplies

each term of positive degree by a number at most one, and the rooted bounds on the larger domains control the products.

Fix $r_I > 0$ within a fixed fraction of the solution margin of the chart, with $I_0 C_0 r_I < 1/8$. Then fix $0 < \eta_I < r_I$ within the parameter margin, with $I_0 C_0 \eta_I < r_I/8$ and $I_0 C_0 (r_I + \eta_I) < 1/4$. For $\|v\| \leq \eta_I$, the map

$$u \mapsto u - B_I^{-1} F_{0,I}(c, z_I^0 + u, z_{I^c}^0 + v; \xi)$$

has derivative $-B_I^{-1}(F_{0,II} - B_I)$, whose norm is less than $1/4$ on $\|u\| \leq r_I/2$ by (19.11), and its value at $u = 0$ has norm less than $r_I/8$ by (19.10). It is therefore a coefficient contraction of the ball $\|u\| \leq r_I/2$ into itself. Its Picard iterates converge in the coefficient norm, and their power series converge locally uniformly on the parameter product, which implies that the limit $a_I - z_I^0$ is holomorphic. At $v = 0$ the stationary point is the joint one, which proves the first identity of part (1). Differentiating the gradient equation at the conditional stationary point and using Lemma 18.8, we obtain

$$D_y a_I = -F_{0,II}^{-1} F_{0,II^c}.$$

The inverse and the mixed matrices have convergent coefficient bounds on a product larger than the chosen one. Integrating this identity from $y = z_{I^c}^0$ to y , we obtain the Lipschitz bound of part (1), with C_J a common bound for the product of these matrices. The choices are uniform in the coarse field, because the estimates were made on its fixed larger domain before v was introduced.

Step 2: the perturbed conditional stationary points. We write $z_I = a_I + Y$ and consider on the solution domain the map

$$\mathcal{T}_\lambda(Y) = Y - A_I^{-1} F_{\lambda,I}(c, a_I + Y, y). \quad (19.12)$$

Since $D_Y \mathcal{T}_\lambda = -A_I^{-1}(F_{\lambda,II} - A_I)$, the two inequalities of part (2) imply

$$\|\mathcal{T}_\lambda(0)\| \leq r_I/4, \quad \|D_Y \mathcal{T}_\lambda\| \leq 1/2.$$

Then $\mathcal{T}_\lambda$ maps the coefficient ball of radius $r_I/2$ into itself and contracts differences by the factor $1/2$. Iterating from zero proves existence and convergence. The sum of the successive differences is a geometric series in the coefficient norm, so the limit is holomorphic. The differentiated series converge on a fixed smaller domain: they satisfy an equation $Z = H + TZ$ in which the majorant of T is at most $1/2$, and the solution is $\sum_{j \geq 0} T^j H$. This gives the coefficient bounds with any parameter index fixed.

If two stationary points lie in the convex domain on which the derivative bound holds, their difference is unchanged by (19.12) and has norm at most half of itself, so the two points coincide. Since F_λ is real at real arguments, every Picard iterate is real at real data,

and so is the limit. The margin on the real Hessian keeps at least half of the lower bound for the constrained Hessian on a neighborhood of the stationary point. Integrating the real Hessian along segments shows a positive quadratic increase away from it, so it is a local real minimum. The complex resolvent condition is a hypothesis, which provides the inverse square roots used below.

Step 3: the displacement at fixed coarser coordinates. For part (3), note that $F_{0,I}(c, a_I, y) = 0$, so the map (19.12) satisfies $\mathcal{T}_\lambda(0) = -\lambda A_I^{-1} V_I(c, a_I, y)$. Writing $Y_{\lambda,I} = a_{\lambda,I} - a_I$ in the fixed point equation, we obtain for every $y \in \mathcal{T}_I$

$$\|Y_{\lambda,I}\|_{\text{coeff}} \leq \lambda \delta_V + \frac{1}{2} \|Y_{\lambda,I}\|_{\text{coeff}}, \quad \text{hence} \quad \|Y_{\lambda,I}\|_{\text{coeff}} \leq 2\lambda \delta_V. \quad (19.13)$$

This estimate is uniform at each fixed higher argument in $\mathcal{T}_I$, with the common product of coarse fields. A supremum of these fixed-argument coefficient norms does not itself bound substitution of a moving coefficient series. We use it for the joint initial segment, which has no higher argument; the moving substitutions below use the joint mixed-derivative majorants instead. No derivative in a frozen coordinate is involved.

Step 4: the Lipschitz bound in the coarser coordinates. By the coefficient bounds on the larger domains, the mixed matrices F_{λ,II^c} are bounded by a common constant Q_{mix} and the reference inverse by I_0 . The second inequality of part (2) and the Neumann series imply that $\|F_{\lambda,II}^{-1}\| \leq 2I_0$. Implicit differentiation at the perturbed conditional stationary point yields

$$D_y a_{\lambda,I} = -F_{\lambda,II}^{-1} F_{\lambda,II^c}, \quad \|D_y a_{\lambda,I}\|_{\text{coeff}} \leq 2I_0 Q_{\text{mix}}. \quad (19.14)$$

The rooted versions follow from the same ordered products on a fixed smaller domain. From now on let C_0 denote a bound for these maps and their unperturbed versions, and let C_S bound the rows of the inverse square roots. These constants depend on the common bound for the mixed tensors, not on δ_V or on the later free radius.

Step 5: successive substitution. Let $\varepsilon_{\text{tube}}$ be as in part (3). We use the successive maps in the original free coordinates, with $K_{\lambda,i}$ equal to M^i times the Schur blocks in input units. Let $z_\lambda(c)$ be the perturbed joint stationary point. For the initial segment of all free coordinates there is no higher argument, so (19.13) gives

$$\|z_\lambda - z^0\|_{\text{coeff}} \leq 2\delta_V.$$

The assumed tube inequality and the unperturbed Lipschitz bound of part (1) place every projection of this joint point strictly in the corresponding parameter and solution domains. Joint stationarity and uniqueness of each conditional branch therefore give

$$a_{\lambda,I}(c, z_{\lambda,I^c}(c); \xi) = z_{\lambda,I}(c).$$

Let D_i bound the deviations of the already reconstructed coarser coordinates from z_λ , in the joint coefficient algebra of the coarse field and the remaining increments. Integrating $D_y a_{\lambda,I}$ from z_{λ,I^c} to the moving higher argument uses its joint mixed-derivative majorant on the larger product. The small coefficient displacement of z_λ permits this recentering on a fixed smaller product; enlarge C_0 once to include the composition. The increment adds at most $C_S r_q$. Thus

$$D_{i-1} \leq (1 + C_0)D_i + C_S r_q, \quad D_s = 0, \quad (19.15)$$

as an inequality of majorants, rather than a substitution into the fixed- y supremum. After the finite conversion between the original free coordinates and these centered coordinates is included in C_0 and C_S , we may retain

$$C_{\text{seq},J} = 2(1 + C_S)(1 + C_0)^{J+1} \left(1 + \sum_{j=0}^{J-1} (1 + C_0)^j \right).$$

Indeed, iterating (19.15) and then adding the joint displacement, every deviation A_i from z^0 satisfies

$$A_i \leq 2\delta_V + C_S r_q \sum_{j=0}^{J-1} (1 + C_0)^j,$$

which is at most $C_{\text{seq},J}(2\delta_V + r_q)$. The condition of part (3) therefore leaves half of each margin unused. At each admitted higher argument the Picard iterations stay in the fixed solution balls of part (2). This proves admissibility before proceeding to the next scale; it does not assert that every point of those fixed balls has deviation bounded by A_i .

Step 6: zero increments. At $q = 0$, the stationarity at the coarsest scale and the successive stationarities at the finer scales give the joint gradient equation. By uniqueness of its branch, this chain is the joint stationary point; Step 5 shows that the chain stays in the required domains.

Step 7: the order of choices. For $\delta_V = 0$ we first choose r_q by the condition of part (3). On the fixed larger unperturbed products, we then choose the perturbed product and the bound on the perturbation so that the same condition holds, and all later radii are chosen inside that product. Decreasing r_q changes only the terms of positive degree in q , and the image at $q = 0$ is the joint stationary point for every coarse field in the domain; hence no later reduction of the coarse-field radius follows. Finite minima and maxima over the initial segments preserve the dependence of the constants. $\square$

19.3 Comparison of the regional and torus stationary points

We now compare the regional problem with the torus problem at a nonzero coarse field. Recall that $F_R(c, z; \xi) = E_{d_0}(\Psi_R(c, z; \xi))$ is the regional objective of Theorem 18.12, and

that the torus objective is the effective classical action of the torus reconstruction with the coarse field extended by zero. For a target v of the torus we write $z_T(v)$ for the torus stationary point at v , and P for the restriction from the torus free list to the regional list.

Theorem 19.4. *Fix the larger domains of Theorem 18.12, an odd $M \geq 3$, $J \geq 1$, $1 \leq s \leq J$, $m \geq 4$, a depth $d_0 \geq 0$, and a union Ω of output blocks; positions are in output units. Use a smaller fixed product of $(c, z; \xi)$ on which the gap, inverse, and image margins of that theorem hold, with the previously frozen coordinates fixed. Write*

$$F_R = E_{d_0} \circ \Psi_R(c, z; \xi), \quad F_T = E_{d_0} \circ \Psi_T(c \oplus 0, z_L, z_H), \quad (p, h) = z_T(c \oplus 0),$$

put

$$\begin{aligned} a_T^0 &= \Psi_T(c \oplus 0, p, h), & \widehat{a}_R &= \Psi_R(c, p; \xi), & d_a &= \widehat{a}_R - a_T^0, \\ J_R &= \Psi_{R,z}(c, p; \xi), & J_T &= \Psi_{T,z_L}(c \oplus 0, p, h), \end{aligned}$$

and let $r_R = F_{R,z}(c, p; \xi)$ be the regional gradient at the restricted torus stationary point. Then the following statements hold.

1. (Gradient and response.) Let z_R be the regional stationary point. Then

$$\begin{aligned} r_R &= (J_R - J_T)^T E'_{d_0}(a_T^0) + J_R^T \int_0^1 E''_{d_0}(a_T^0 + u d_a) d_a du, \\ \mathcal{A}_R(z_R - p) &= -r_R, \quad \mathcal{A}_R = \int_0^1 F_{R,zz}(c, p + u(z_R - p); \xi) du. \end{aligned}$$

The first term of r_R is the chain-rule contribution of the coarse constraints omitted in the regional problem. With $a_R = \Psi_R(c, z_R; \xi)$, for every free label or fine link i in Ω ,

$$|(r_R)_i| + |(z_R - p)_i| + |(a_R - a_T^0)_i| \leq C_J(\|c\|_\infty + \|\xi\|_\infty) e^{-c_J \text{dist}(i, W)},$$

each summand being read on its own list. These three differences have factorial boundary-localized coefficient bounds, as in Lemma 18.15. In particular, for a fixed input x and a fixed output e in Ω , their differentiated coefficient series are bounded by $C_J e^{-c_J [\text{dist}(e, W) + \text{dist}(e, x)]}$.

2. (Full output.) Let $u_*(c, \xi) = T_M^s \exp a_R$ be the full output of the s averaging steps, including the values of the coarse links which are not constrained in Ω . It lies in the common torus target chart on this product. The bounds of part (1) on Ω , including those with a fixed input, remain true if $c \oplus 0$ is replaced by $\text{Log } u_*$ and a_T^0 by the torus stationary field a_T^* at the target $\text{Log } u_*$. All derivatives through u_* , including those through z_R , are included.
3. (Minimized action.) Let $V_R(c; \xi) = B_s F_R(c, z_R; \xi)$. There are fixed $\rho_0(J), a_W, \alpha_W, A_J^{\text{cl}} > 0$ such that

$$\mathcal{X}_{\Omega, a_W, \alpha_W}^{\rho_0, \text{cl}}(V_R - E_{d_0+s}(c \oplus 0)) \leq A_J^{\text{cl}}.$$

This is the seminorm of Definition 18.1 with the displayed constant classical radius and rates; it includes all degrees $l \geq 1$. The constant term of the difference is kept in the full action.

4. (Newly frozen coordinates.) For the newly frozen coordinates ξ , at their positions and with their density,

$$D_{\xi} z_R = -F_{R,zz}^{-1} F_{R,z\xi}, \quad \sum_f |(D_{\xi_f} z_R)_e| \leq C_J e^{-a_W \text{dist}(e,W)},$$

$$\|F_{R,z\xi}\|_{a_W} + \|F_{R,zc}\|_{a_W} \leq C_J.$$

5. (Hessians.) The constrained Hessians at the two stationary points satisfy

$$| [F_{R,zz}(c, z_R; \xi) - P F_{T,zz}(c \oplus 0, z_T) P^T]_{ef} | \leq C_J e^{-a_W [\text{dist}(e,f) + \text{dist}(e,W) + \text{dist}(f,W)]}.$$

The order- l coefficient tensors of this difference, with every coarse-field and free argument in the connecting network and any root in Ω fixed, are bounded by $(1+2)!C_J^{l+2}$ at fixed exponents.

6. (Successive minima, inverse square roots, and Haar disintegration densities.) Use a common smaller full torus product of higher coordinates $y = (y_L, y_H)$ for each initial segment I . On this product compare the regional conditional center $a_I^R(c, y_L; \xi)$ with the regional components of $a_I^T(c \oplus 0, y_L, y_H)$, and compare the Hessian of the regional partial minimum with the LL block of the torus partial-minimum Hessian at that same full argument. The conditional-center differences have the boundary-localized coefficient bounds of part (1), with every parameter input fixed in turn. The Hessian differences have the bounds of part (5), with the exact normalization M^i at scale i . The common product can be taken as $\|y\|_{\text{coeff}} < \eta_I/2$ after the coarse and frozen radii are reduced as in Step 10 of the proof; it is not a slice at an unspecified y_H . The inverse square roots of these Hessians and the nonconstant coefficients of the logarithms of the Haar disintegration densities satisfy the corresponding boundary-localized comparisons, with any index fixed. Scalar determinants and Haar disintegration densities which are not rooted at a label in Ω are kept as exact quantities.

All constants have the dependence of Section 19.1 and do not depend on the later widths.

The theorem gives no inverse on fixed boundary coordinates and no rank estimate for the boundary terms at a nonzero coarse field.

Proof. We pose both problems on the torus free list by interpolating between the torus objective and the augmented regional objective. The two objectives then differ by one inserted factor localized near H , and all estimates follow from the response of the interpolated stationary point to this factor and from the exponential decay of the inverse Hessian.

Step 1: one coordinate list. We write $z = (z_L, z_H)$ for the torus free list and set

$$\tilde{F}_R(c, z; \xi) = F_R(c, z_L; \xi) + \frac{1}{2}|z_H|^2, \quad F_t = (1-t)F_T + t\tilde{F}_R, \quad 0 \leq t \leq 1, \quad D = \tilde{F}_R - F_T. \quad (19.16)$$

Since its minimum is zero, the added auxiliary quadratic does not change the regional value. Theorem 18.12 gives a positive gap at the identity for the Hessian of this interpolation, for all its principal blocks, and for its Schur complements. We reduce the construction product by a fixed amount so that the coefficient variation of each of these Hessians satisfies the condition of Lemma 19.1(2), uniformly in t . This is possible since the variation is at most $Q_J(\|c\| + \|z\| + \|\xi\|)$, and the gap at the identity of the interpolation is at least the minimum of the regional gap, the torus gap, and 1. All straight segments of fine fields used below then lie strictly inside the larger classical chart.

We retain a split resolvent margin for the later perturbation argument. Choose a common intrinsic exponent $b_{\text{ref}} > 0$ and a constant $C_{0,J}$ such that every flat principal and scale Schur reference K_i^b has resolvent norm at most $C_{0,J}/(\lambda_J + t)$ at this exponent, including the factors M^i at the scale stages. When the larger unperturbed products are fixed, reduce them once so that their conditional matrices $K_{0,i}$ satisfy

$$C_{0,J}\|K_{0,i} - K_i^b\|_{b_{\text{ref}}} \leq \lambda_J/8$$

on those products, uniformly over the finitely many stages. We decrease the unperturbed reserve c_J if necessary so that $c_J \leq b_{\text{ref}}$. These choices use only the unperturbed constructions at fixed J and precede the incoming rates, perturbation threshold, and later widths.

Step 2: localization of the difference of the reconstructions. We use the comparison of the nonlinear words in Theorem 18.12. At common internal coarse values and common regional free values, we set

$$d(c, z; \xi) = \Psi_R(c, z_L; \xi) - \Psi_T(c \oplus 0, z_L, z_H).$$

On fine links in Ω , its defining words vanish outside the fixed interface band. The prescribed values on H can differ, and their difference is a bounded source on H with the separate coefficient bounds of the reconstruction charts. The difference is supported in H together with that band, hence in H_{L_J} ; the two extensions need not agree on H . Its coefficient tensors have the corresponding finite-range bounds, rooted at any index. To check the statement about fine links in Ω at nonzero arguments, consider an internal eliminated-edge equation at the coarsest intermediate scale. On the two sides it has the same endpoint blocks, the same coarse value, and the same free values, so by uniqueness on the common local branch its solution is the same. We repeat this down the s intermediate scales. Only equations meeting a block of H can differ, and the union of their at most J stencils is contained in H_{L_J} . This argument keeps the values of the coarse links across the boundary of the block region as functions of the common free values; it imposes no value on an omitted coarse constraint.

Step 3: the inserted difference. The difference in (19.16) is

$$D = \int_0^1 E'_{d_0}(\Psi_T + vd)[d] dv + \frac{1}{2}|z_H|^2. \quad (19.17)$$

Every derivative of the integral either differentiates d , whose fine position stays in H_{L_j} , or leaves such a factor undifferentiated. In the latter case the classical tensor of Theorem 10.11, which is bounded at every root, connects that position to every differentiated chart argument. A derivative in a coarse field or in a free variable in Ω therefore cannot disconnect the difference factor from the network. The auxiliary quadratic has only indices in H , and any connected contribution from it to an external index in Ω passes through an index in H and is bounded by (19.2). This verifies the hypotheses of Lemma 18.15 for the whole inserted difference, at every order and with any external root.

Step 4: the gradient formula and the bounds on values. Torus stationarity says that $J_T^\top E'_{d_0}(a_T^0) = 0$. Adding and subtracting $J_R^\top E'_{d_0}(a_T^0)$ in $F_{R,z}(c, p; \xi) = J_R^\top E'_{d_0}(\hat{a}_R)$, and applying the fundamental theorem of calculus on the complex segment from a_T^0 to $\hat{a}_R$, we obtain the formula for r_R . The difference $J_R - J_T$ is supported in the construction band, in its regional input indices and its fine output indices. Its product with the gradient of the action on the fine links is kept: torus stationarity makes only the projection of this gradient onto the torus free directions vanish. By the classical bounds, we have

$$\|E'_{d_0}(a_T^0)\|_\infty + \|d_a\|_\infty \leq C_J(\|c\|_\infty + \|\xi\|_\infty);$$

for the first summand we used the bound on the torus stationary point, and for the second the word comparison of Step 2 and the same bound for (p, h) . Contracting the two terms of r_R with their localized sources and using (19.2), we obtain

$$|(r_R)_i| \leq C_J(\|c\|_\infty + \|\xi\|_\infty)e^{-c_J\delta(i)}. \quad (19.18)$$

Subtracting the regional gradient equations at z_R and at p and integrating along the segment between them, we obtain $\mathcal{A}_R(z_R - p) = -r_R$. The segment lies in the convex larger free box, and its averaged Hessian is the positive regional Hessian at the identity plus a coefficient perturbation satisfying the condition of Lemma 19.1(2). By that lemma, the inverse has entries bounded by $C_J e^{-b_J \text{dist}(e, i)}$. Convolving this bound with (19.18) and using $\delta(e) \leq \delta(i) + \text{dist}(e, i) + L'_j$, we obtain the bound on $z_R - p$, at a fixed smaller exponent. For the fine field we have the identity

$$a_R - a_T^0 = d_a + \int_0^1 \Psi_{R,z}(c, p + v(z_R - p); \xi)(z_R - p) dv. \quad (19.19)$$

The first term is supported in H_{L_j} , and the second is the product of a finite-range chart kernel, without boundary weight, with $z_R - p$. This proves the bound on fine links in Ω . No decay is asserted at a fine link deep in H .

Step 5: the coefficient bounds. We now prove the coefficient bounds directly, using the interpolation (19.16). On a common smaller product of parameters, the stationary point z_t of F_t is obtained by the implicit iteration with the inverse of the free Hessian at the identity, which converges coefficientwise. The radius condition and the bound on the inverse times the gradient follow from Theorem 18.12, and the scalar comparison equation is (19.3). Differentiating in t , we obtain

$$\dot{z}_t = -F_{t,zz}^{-1} D_z(c, z_t; \xi). \quad (19.20)$$

This expression contains exactly one inserted difference from (19.17), and its convergent majorant is (19.4). A further derivative in an input may hit the inverse, the inserted difference, or its argument z_t ; in each case the joined network still contains the inserted difference and the index of the new input. We apply Lemma 18.15 and integrate from $t = 0$ to $t = 1$. Since the endpoints are z_T and $(z_R, 0)$, their difference on the regional list has factorial boundary-localized coefficient bounds, rooted at every index. Reading (19.19) as an identity of convergent majorant series, we obtain the corresponding bounds for the fine field, and the same argument applied to the formula for r_R gives its bounds. Keeping one input x fixed and summing over the remaining indices, we obtain the decay with a fixed input stated in part (1).

Step 6: the full output. The composition $T_M^s \exp \Psi_R$ is holomorphic on the common larger chart, by the construction in Theorem 18.12. At the stationary point it satisfies

$$\| \text{Log } u_* \|_\infty \leq C_I (\|c\|_\infty + \|\xi\|_\infty).$$

The internal coordinates of $\text{Log } u_*$ are c , and its other coordinates are the values of the coarse links which are not constrained in Ω . These are analytic functions of $(c, z; \xi)$, whose differentiated series compose the coefficient series of the chart words with the series of the stationary point from Step 5. For instance, for any analytic direction v the first derivative is

$$D_v \text{Log } u_* = D_c \mathcal{T} D_v c + D_z \mathcal{T} D_v z_R + D_\xi \mathcal{T} D_v \xi, \quad \mathcal{T} = \text{Log}(T_M^s \exp \Psi_R).$$

Since higher derivatives are compositions of these factors, all derivatives through the stationary point are included. We reduce the parameter radius so that the segment from $\text{Log } u_*$ to $c \oplus 0$ lies in the common torus target chart. If $\mathcal{A}_T(v)$ denotes the torus stationary fine field at the target v , then

$$a_T^* - a_T^0 = \int_0^1 D \mathcal{A}_T(c \oplus 0 + t(\text{Log } u_* - (c \oplus 0))) [\text{Log } u_* - (c \oplus 0)] dt. \quad (19.21)$$

The difference of targets is supported on coarse links which are not internal to Ω . By the same implicit equation on the torus list, the lower bound for the torus constrained Hessian,

the chart bounds, and the classical tensors give coefficient bounds without boundary weight for the response of $\mathcal{A}_T$. Each term of (19.21) with a root in Ω therefore passes through a coarse link outside Ω , and its composed network, including the derivatives of u_* , visits H_{L_J} . Lemma 18.15 and the finite word bounds then give the same amplitude and fixed-input bounds as before. The same integral with $\mathcal{A}_T$ replaced by the torus free stationary point z_T bounds $p^* - p$ on the regional free labels, where $(p^*, h^*) = z_T(\text{Log } u_*)$. Thus $z_R - p^* = (z_R - p) - (p^* - p)$ satisfies the same bounds. For the new residual gradient, we subtract its value at p and integrate $F_{R,zz}$ along the segment from p to p^* . A Hessian without boundary weight multiplies this boundary-localized displacement, so the residual satisfies the same bounds. Its formula follows again from torus stationarity, now at the fixed target $\text{Log } u_*$. These arguments are identities of convergent series in all variables, so they also give the fixed-input bounds at the full output. A representative in complementary forest coordinates, if needed, can use forest words of fixed depth based at the regional vertices and the coarse roots. These words change only the construction stencil outside Ω , their coefficient series are among the chart bounds, and the comparison in Ω is unchanged.

Step 7: the minimized action. Let $f_t(c; \xi) = F_t(c, z_t; \xi)$. By stationarity and Lemma 18.8, we have

$$\partial_t f_t = D(c, z_t; \xi). \quad (19.22)$$

At the endpoints, $B_s f_1 = V_R$ and $B_s f_0 = E_{d_0+s}(c \oplus 0)$. The second identity follows by iterating the recurrence $E_{d+1}(c) = M \min_z E_d(b(c, z))$ which defines the effective classical action: by Lemma 18.8 the successive minima equal the joint minimum, and the product of the s factors M is B_s . A nonconstant derivative of the right side of (19.22) has the network structure checked in Step 3. Therefore, for $k \geq 1$ and a root i in Ω ,

$$\sum_{i_2, \dots, i_k} e^{c_J t(i, i_2, \dots, i_k)} |D^k [B_s(f_1 - f_0)](0; \xi)_{i, i_2, \dots, i_k}| \leq k! C_J^{k+1} e^{-c_J \delta(i)}, \quad (19.23)$$

where the factor $B_s \leq M^J$ is included in C_J . For a scalar series $f(c) = \sum_\nu f_\nu c^\nu$ we have, as in (18.2),

$$\sum_{|\nu|=k} \frac{\nu_i}{k} |f_\nu| e^{at(\nu)} = \frac{1}{k!} \sum_{i_2, \dots, i_k} |D^k f(0)_{i, i_2, \dots, i_k}| e^{at(i, i_2, \dots, i_k)}. \quad (19.24)$$

Indeed, the multiset obtained from ν by removing one occurrence of i has $(k-1)! / [(\nu_i - 1)! \prod_{j \neq i} \nu_j!]$ ordered listings, and $D^k f(0)$ evaluated on any of them is $\nu! f_\nu$. Fix $a_W, \alpha_W < c_J/4$, and then $\rho_0 > 0$ in the smaller construction domain with $C_J \rho_0 < 1/4$. Inserting (19.23) into (19.24), we bound the regional seminorm by

$$\sum_{k \geq 1} \rho_0^{k-2} C_J^{k+1} < \infty.$$

This proves part (3), including degree one. The value $B_s(f_1 - f_0)(0; \xi)$ is never subtracted in (19.22); it remains in the full action.

Step 8: derivatives in the newly frozen coordinates. Differentiating the regional gradient equation in ξ and using invertibility of the Hessian, we obtain $D_{\xi}z_R = -F_{R,zz}^{-1}F_{R,z\xi}$. The mixed derivative has exactly two terms,

$$F_{R,uv} = \Psi_{R,u}^T E''_{d_0}(\Psi_R) \Psi_{R,v} + E'_{d_0}(\Psi_R) \Psi_{R,uv}, \quad (19.25)$$

and the bounds of Theorem 18.12 on the larger products control both terms for $(u, v) = (z, \xi)$ and $(u, v) = (z, c)$. An index of ξ is at a fine position outside Ω or in its fixed interface stencil. Contracting it with the free inverse and summing over the labels of ξ , we obtain the row sum $C_J e^{-a_W \delta(e)}$ by (19.2). This proves both bounds on the mixed matrices and the bound on $D_{\xi}z_R$, with no change of the density of the inputs.

Step 9: the Hessians. We compare the endpoints of (19.16) at their stationary points. Before any elimination, the total derivative is

$$\frac{d}{dt} F_{t,zz}(c, z_t; \xi) = D_{zz}(c, z_t; \xi) + F_t^{(3)}(c, z_t; \xi)[\dot{z}_t]. \quad (19.26)$$

Both terms contain the inserted difference, by (19.17) and (19.20), and all terms of (19.25), including the one with the gradient of the action, are part of these tensors. For two matrix roots in Ω , Lemma 18.15 gives the three distances in the Hessian bound of part (5), and at degree l in additional coarse-field and free arguments it gives the bound $(l+2)!C_J^{l+2}$ after C_J is increased. For example, the corresponding coefficient series of the matrix is dominated on radius R by

$$\sum_{l \geq 0} \frac{(l+2)!}{l!} C_J^{l+2} R^l = \frac{2C_J^2}{(1 - C_J R)^3}, \quad C_J R < 1.$$

The same majorant with a fixed input gives every version rooted at an input.

Step 10: successive minima. We restrict to smaller products for the initial segments before this comparison is used, following the construction of Lemma 19.3(1) uniformly in t . For an initial segment I , we take a solution radius r_I about the joint stationary point and a smaller radius η_I for the deviation of the coarser coordinates. The principal inverse has bound I_J and the mixed tensors have bound C_J , so at the joint stationary point the inverse times the conditional gradient is at most $I_J C_J \eta_I$. We require $I_J C_J r_I < 1/8$ and $I_J C_J \eta_I < r_I/8$ within the margin of the chart, uniformly in t . Then the conditional Newton map is a contraction on the solution domain. We reduce the radii of the coarse field and of ξ so that $\sup_t \|z_t(c; \xi)\|_{\text{coeff}} < \min_I \eta_I/8$. Then a common product $\|y\|_{\text{coeff}} < \eta_I/2$ of coarser coordinates lies in the allowed domain for every t . On this product the conditional response is $\dot{z}_I = -F_{t,II}^{-1} D_I$ at fixed y , so the argument of Step 5 applies before a partial minimum is taken. Since there are finitely many initial segments, a common choice

is possible, and this is the smaller product in part (6). At such a partial minimum, Lemma 18.8 gives

$$f_{yy} = F_{yy} - F_{yI}F_{II}^{-1}F_{Iy}, \quad D^3f[\eta_1, \eta_2, \eta_3] = D^3F[V_{\eta_1}, V_{\eta_2}, V_{\eta_3}].$$

Differentiating the Schur formula in t replaces one factor by its derivative, which contains the inserted difference; an inverse contributes $-F_{II}^{-1}\dot{F}_{II}F_{II}^{-1}$, and evaluation at a moving stationary point contributes its response, which also contains the inserted difference. Each resulting network keeps all external roots and a visit to H_{L_J} . At scale i the formula is multiplied by the normalization M^i of the action, and a finite composition over $i < J$ changes only C_J .

Step 11: inverse square roots, and fixing versus eliminating. We apply (19.7) to the Hessians of Step 10. By the common gap and the resolvent condition, the differences of the inverse square roots have integrable boundary-localized coefficient bounds, rooted at every index. To compare $(K_{LL}^T)^{-1/2}$ with $((K^T)^{-1/2})_{LL}$, we set $C = K_{LL}^T \oplus K_{HH}^T$ and apply the resolvent identity to $K^T - C$ before integrating the resolvents. Each off-diagonal factor connects the two sides of the boundary, and hence gives the same boundary-localized estimate rooted in Ω . This also treats the difference between eliminating and fixing the torus free coordinates outside the regional list. For a free matrix T written in blocks with respect to the regional and the remaining torus coordinates, (19.9) expresses $(T^{-1})_{LL} - A^{-1}$ through the Schur complement $D_0 - CA^{-1}B$ on the remaining coordinates, which has the gap of Theorem 18.12. The factors B and C pass through the remaining coordinates and keep the visit to H_{L_J} , also after T is replaced by $T + t$. The same identity inside a larger Schur formula for one scale treats elimination instead of fixing.

Step 12: logarithms of the Haar disintegration densities. In the fixed coordinates, the Haar disintegration density of one intermediate scale is that of Lemma 18.7:

$$c_{\text{Haar}}^f \frac{\prod_{e \text{ nonforest}} j_H(\Gamma_e)}{\prod_c \det D_{X_{e_c^*}} C_{b,c}(\Gamma)}, \quad j_H(X) = \text{sinc}^2(|X|/2), \quad (19.27)$$

where e_c^* is the eliminated edge of the constraint at the coarse link c ; the Haar factors of the solved eliminated edges are included in the numerator. At the common zero the local factors of the numerator equal 1 and the local constraint determinants equal M^{-6} . By the word coefficients of fixed depth and the bounds on the implicit chart, there exists a uniform smaller radius on which each factor, divided by its nonzero value at zero, has nonconstant part with absolute coefficient row sum less than $1/2$. The series $\log(1+x) = \sum_{k \geq 1} (-1)^{k+1} x^k / k$ therefore converges absolutely, with the attached local networks. We compose these logarithms with the conditional stationary points and the inverse square roots. The words for fine links in Ω agree off the interface band, and the prescribed values on H remain bounded sources. Thus every difference has an output in

H_{L_J} , and the derivatives of the other factors stay joined to it. Logarithmic matrix factors are treated by (19.8). A nonconstant scalar logarithmic coefficient has a differentiated index in Ω at which its trace cycle can be rooted, so the rooted sums above bound it. Pure scalar determinants, Haar disintegration densities, and traces over H have no such index and are kept as their exact finite values. All choices involved only the fixed construction data of Section 19.1, which proves the stated dependence of the constants. $\square$

19.4 A gauge transformation fixing prescribed vertices

On the torus, Theorem 13.2 brings the reconstructed field into a small neighborhood of the identity by a gauge transformation which depends only on the coarse field. On a block region this gauge transformation must not change the field at the vertices to which frozen coordinates are attached. A frozen coordinate is stored as its based transport $U(\sigma_e)a_e$: the transport along a path σ_e from its basepoint to a link e , times the frozen value a_e of that link (Lemma 18.7). Under a gauge transformation G the frozen value transforms as $a_e^G = G_v a_e G_w^{-1}$, where v, w are the endpoints of e . We therefore multiply the logarithm of the torus gauge transformation by a cutoff which vanishes near H . To avoid a clash with the block region Ω , we write $\Omega_{d_0, s}^{\mathbb{T}}$ for the gauge transformation of Theorem 13.2, applied to a composition of s averaging steps at depth d_0 .

Lemma 19.5. *Fix M, J , and the geometry of a composition of $s \leq J$ averaging steps as above. Let the prescribed protected set contain all vertices of H , every fine endpoint and every basepoint of each based transport of a newly or previously frozen coordinate, and all endpoints required in finer-lattice representations of these transports. Assume that its protected vertices outside H lie in $\mathcal{B}$, the two-block neighborhood of H in output units, so they lie in the zero region of the cutoff below. Let $0 \leq \chi \leq 1$ be fixed by the geometry, zero on H and its neighborhood of width two blocks, one outside its neighborhood of width three blocks, and zero at every prescribed protected vertex. Define*

$$k(c) = \text{Log } \Omega_{d_0, s}^{\mathbb{T}}(c \oplus 0), \quad G^{\Omega}(c) = \exp(\chi k(c)).$$

There are $u_J, A_J, a_G > 0$ such that the following statements hold on $\|c\|_{\infty} < u_J$.

1. The maps k and G^{Ω} are holomorphic, periodic, and equivariant under constant global color rotations. At real data they take values in $\mathfrak{su}(2)$ and $\text{SU}(2)$, respectively, and $\|k(c)\|_{\infty} \leq A_J \|c\|_{\infty}$. Their order- l absolute coefficients are bounded by $l! C_J^{l+1}$, with the connecting-network weight at exponent a_G and with the volume-normalized column of every fixed input. Replacing the volume normalization of a column for a fine output by counting measure costs at most the factor $M^{3s} \leq M^{3J}$.
2. At fixed geometry G^{Ω} depends only on c , not on the free coordinates or the frozen coordinates. It is exactly I at $c = 0$, at the coarse roots, and at all protected vertices, also for complex c .

3. Consequently, for every reconstruction U in the chart before gauge transformation,

$$(T_M^s U^{G^\Omega(c)})_{\text{internal}} = (T_M^s U)_{\text{internal}}, \quad (U^{G^\Omega(c)})(\sigma_e) a_e^{G^\Omega(c)} = U(\sigma_e) a_e$$

for every based transport of a frozen coordinate. A frozen link across the boundary of the block region with both fine endpoints protected is also unchanged.

4. The gauge transformation is applied after the reconstruction in the rooted spanning-forest coordinates. It does not change the free coordinates, the support restrictions, the Haar disintegration density, or the determinant of the normalization of the fluctuation variables.

The constants have the dependence of Section 19.1 and do not depend on the later widths or on the size of the protected set. The existence of a large-field decomposition whose protected vertices satisfy these hypotheses is a geometric hypothesis, not a conclusion of the lemma.

Proof. We take the torus gauge transformation of Theorem 13.2 for each $s \leq J$, with a common smaller domain of targets. Extending the target by zero on the coarse links outside Ω deletes coefficients from its majorant. After a construction constant A is increased, the gauge transformation minus the identity has absolute row series at most

$$U(t) = \frac{At}{1 - At};$$

here and below this includes the connecting network and the versions with a fixed input. Theorem 13.2 also provides the logarithm of the gauge transformation. To check the domain condition, we fix $u_J \leq U_T$ inside the larger target domain with $Au_J < 1/4$. This is the case $Q_T = 0$ of (19.1), and it gives $u_J < r_s$, a positive denominator, and the fixed intermediate chart margins for every $s \leq J$. The convergent matrix logarithm is bounded by

$$\sum_{r \geq 1} \frac{U(t)^r}{r} = \log(1 - At) - \log(1 - 2At) \leq \frac{At}{1 - 2At} \leq 2At \quad (0 \leq t \leq u_J). \quad (19.28)$$

In the first inequality, we wrote the difference of logarithms as $\log(1 + At/(1 - 2At))$ and used $\log(1 + x) \leq x$ for $x \geq 0$. Products join coefficient networks. Fixing one input differentiates this majorant series, and on a further fixed smaller radius we obtain order- l tensors bounded by $l!C_J^{l+1}$, for an output row or for the volume-normalized column of any input. The same bounds hold for $k(c)$, and in particular $\|k(c)\|_\infty \leq A_J \|c\|_\infty$. The fine mesh in output units is M^{-s} , so the volume-normalized column of a fine output carries the factor M^{-3s} . Replacing it by counting measure multiplies the bound by the reciprocal, at most M^{3J} ; no factor depending on the depth or the volume occurs.

At fixed geometry, multiplication by $\chi(v) \in [0, 1]$ can only decrease an absolute coefficient row. With $L(t) = \sum_{r \geq 1} U(t)^r / r$, exponentiating χk gives the coefficientwise majorant $\exp L(t) = (1 - At)/(1 - 2At)$. This series and its fixed-input derivative series

converge on a fixed smaller radius and give the asserted factorial coefficient bounds for G^Ω . The numerical bound $L(t) \leq 2At$ in (19.28) bounds values only and is not used in this coefficient comparison. Periodicity and global color equivariance hold for the torus gauge transformation and its logarithm by Theorem 13.2, and multiplication by the fixed scalar cutoff and exponentiation preserve them. For real targets the logarithm of the compact gauge transformation lies in $\mathfrak{su}(2)$, and its exponential lies in $SU(2)$; the complex functions are their holomorphic continuations with values in $\mathfrak{sl}_2(\mathbb{C})$ and $SL(2, \mathbb{C})$.

The torus gauge transformation is I at $c = 0$ and at each coarse root, so its logarithm vanishes at these points. At every other protected vertex $\chi = 0$ by hypothesis. Hence $G^\Omega = I$ at all these points, identically as a holomorphic function and not only at real arguments.

The transport along a path σ from v to w transforms as

$$U^G(\sigma) = G_v U(\sigma) G_w^{-1},$$

since all intermediate vertex factors cancel. A based transport $U(\sigma_e)_{a_e}$ therefore transforms only at its basepoint and at the far fine endpoint of e . Both values of the gauge transformation there are I , which implies the second identity in part (3). If both fine endpoints of a frozen link across the boundary of the block region are protected, the gauge transformation also leaves its value unchanged.

For the coarse output, every based loop holonomy in the centered average (4.3) transforms by conjugation at its initial coarse root, and its reference path transforms at its two coarse endpoints. On the branch of the logarithm used here, the logarithm commutes with conjugation, term by term in its power series. Thus one averaging step is covariant, with the gauge transformation acting only at the coarse endpoints. Iterating s steps cancels the intermediate gauge factors and leaves only the gauge transformation at the two coarse roots of the output link, where it is I . This proves the first identity in part (3), also at complex data. No exceptional value of the Borel extension of T_M occurs inside these charts.

Finally, the gauge transformation acts on a field already reconstructed from its free coordinates in the rooted spanning-forest coordinates, and it changes neither these coordinates nor the integration variables. Hence the support restrictions, the Haar disintegration density, and the determinant of the normalization of the fluctuation variables remain the same functions of the original variables. The cutoff enters only as a bounded fixed multiplier; neither the size of its support nor a derivative in its geometric parameters enters any bound. This proves that the constants do not depend on the size of the protected set or on the later widths, under the stated hypothesis on the protected vertices. $\square$

19.5 The gauge-transformed reconstruction and its input columns

We now apply $G^\Omega(c)$ to the regional reconstruction and compare the result with the gauge-transformed torus reconstruction of Theorem 13.2. Let $a_R^{\text{raw}}(c, q; \xi)$ be the logarithm of the regional reconstruction before gauge transformation, built from the successive regional conditional stationary points and constrained inverse square roots, and let a_T^{raw} be the corresponding torus reconstruction. For a fine link $e = (v, w)$ and a logarithmic gauge field k , we write

$$H_e^\chi(a, k) = \text{Log}(e^{\chi(v)k(v)} e^{a_e} e^{-\chi(w)k(w)}).$$

Then $\Phi_R = H^\chi(a_R^{\text{raw}}, k(c))$ is the logarithm of the regional reconstruction transformed by $G^\Omega(c)$, and $\Phi_\mathbb{T} = H^1(a_T^{\text{raw}}, k(c))$ is the gauge-transformed torus reconstruction of Theorem 13.2.

Theorem 19.6. *On the fixed products of Theorem 19.4 and Lemma 19.5, fix a common positive coefficient decay rate b_{com} below the rates of both reconstructions, their conditional matrix functions, and the gauge logarithm of rate a_G . Reduce the common boundary rate so that $8a_W < b_{\text{com}}$, and use network exponents $a > 0$ with $a + 2a_W < b_{\text{com}}/2$, leaving a positive reserve for density sums. Enlarge L_J once to include the three-block cutoff band and the fine-link endpoint buffer. On these products build the reconstruction from the free coordinates with the successive regional conditional stationary points and constrained inverse square roots, with small newly frozen coordinates ξ . There are fixed larger construction radii and positive rates such that, on every smaller product of coarse fields and fluctuation increments satisfying the domain conditions,*

$$\Phi_R(c, q; \xi) = \Phi_\mathbb{T}(c \oplus 0, q \oplus 0)|_\Omega + \Delta(c, q; \xi),$$

where $q \oplus 0$ uses the same scale indices on the regional list and sets the remaining torus increments to zero. The following statements also hold.

1. All terms have convergent majorant series in all variables. The correction Δ has factorial coefficient bounds with a root in Ω , a connecting network through every input, and a visit to H_{L_J} . In particular, let $\mathbf{R}$ be a radius profile on a smaller construction product, let $a > 0$ be a network exponent in the range just fixed, and let $\Delta_{e,v}$ be the Taylor coefficients of Δ_e in its analytic variables, namely the coarse field, the increments q , and the newly frozen coordinates ξ . For a fixed input x , the differentiated series

$$\mathcal{D}_{e,x}(\mathbf{R}) = \sum_{v: v_x \geq 1} v_x |\Delta_{e,v}| \mathbf{R}^{v-1_x} e^{at(\{e\} \cup \text{supp } v)},$$

with the measure conventions of Section 19.1, satisfies

$$\mathcal{D}_{e,x}(\mathbf{R}) \leq C_j e^{-a_W[\text{dist}(e, W) + \text{dist}(e, x)]}.$$

Unlike $C_{e,x}$ in part (4), this series does not contain the factor R_x .

2. At $q = 0$, the dependence of Δ on the newly frozen coordinates satisfies, in the coefficient norm in c ,

$$\|\Delta_e(\cdot, 0; \xi) - \Delta_e(\cdot, 0; 0)\|_{\text{coeff}} \leq C_F \|\xi\|_{\infty} e^{-a_W \text{dist}(e, W)}, \quad \Delta(0, 0; 0) = 0.$$

3. At radius s_q for the fluctuation increments, the absolute coefficient row sum of the terms of Δ of positive degree in q is at most $C_{\Delta, J} s_q$. The rows of the coarse-field terms near the boundary and all rows in the free variables are only bounded by finite constants depending on J ; no contraction factor M^{-J} is asserted for them.
4. (Input columns.) Let $R_y \leq K_* \rho$ be a construction profile, and choose $a_1, b_1 > 0$ with $b_1 > \alpha_{\text{bd}}$ and with $a_1 + b_1$ smaller than a decay exponent for which the full reconstruction has coefficient bounds without boundary weight. Define

$$C_{e,x}(\mathbf{R}) = \sum_{\nu} \nu_x |\Phi_{R,e,\nu}(\xi)| \mathbf{R}^{\nu} e^{a_1 \mathbf{t}(\{e\} \cup \text{supp } \nu)},$$

where $\Phi_{R,e,\nu}(\xi)$ are the Taylor coefficients of Φ_R at the fine output e . Then $C_{e,x}(\mathbf{R}) \leq D_J R_x e^{-b_1 \text{dist}(e,x)}$, and

$$\begin{aligned} \sum_e e^{-\alpha_{\text{bd}} d_{\text{old}}(e)} C_{e,x}(\mathbf{R}) &\leq K_J \rho e^{-\alpha_{\text{bd}} d_{\text{out}}(x)}, \\ K_J &\leq C_{M,J} D_J K_* e^{\alpha_{\text{bd}} C_J} \sum_{l \geq 0} (l+1)^3 e^{-(b_1 - \alpha_{\text{bd}})l}. \end{aligned}$$

Here d_{old} and d_{out} are the distances to the complements of the specified old and output position lists, respectively, both expressed in output units, and we assume that, for every summation label e and every input x ,

$$d_{\text{out}}(x) \leq d_{\text{old}}(e) + \text{dist}(e, x) + C_J,$$

with a buffer C_J depending only on the fixed construction geometry. The sum runs over the full list of old output labels (the fine labels), at their creation-scale density. For an input among the free coordinates the same statement holds with its own radius R_x in place of $K_* \rho$ in the last bound. For a kernel from inputs to outputs, the column of a fixed input carries the volume of the output cells divided by the volume of the cell of that input. The same calculation applies to the profile $R_y = 2\rho$ within the auxiliary construction ball and to the smaller spatially varying radius profiles of Definition 18.1. These column bounds are different from the bounds on the image under the reconstruction majorant in Lemma 19.7.

The constants have the dependence of Section 19.1, are uniform in the depth and in the shape of Ω , and do not depend on the later widths. No equality with a perturbed incoming constrained problem is asserted.

Proof. We compare the successive regional and torus reconstructions before applying the gauge transformation with cutoff, and then estimate the differentiated coefficient series.

Step 1: the domain of the successive substitutions. We take a common larger construction product as in Theorem 19.4, with radii for coarse fields and fluctuation increments at most U_T and Q_T from (19.1). Then the torus reconstruction, its gauge transformation, and all column series below are evaluated with $u + v < r_s$ and denominator at least $1/2$, and the smaller product keeps the intermediate classical and Haar-density margins. For the finitely many conditional problems at the intermediate scales, we use the smaller parameter domains of Step 10 of the proof of Theorem 19.4. We choose the radius of the fluctuation increments so that every successive substitution lies strictly inside these domains. Such a radius exists: by Lemma 18.8, at $q = 0$ the successive conditional stationary points are the joint stationary point. Each conditional stationary point and each inverse square root has a convergent majorant series without boundary weight on a larger product. If A_i bounds the deviation of the coarser coordinates from the joint stationary point after the scales above i have been reconstructed, then by these series we have

$$A_{i-1} \leq (1 + C_0)A_i + C_S s_q, \quad A_s = 0, \quad (19.29)$$

on a fixed smaller free ball, where the constants are maxima of the rows of the conditional derivatives and of the inverse square roots on the larger products. The same recurrence holds in the coefficient norms. Iterating it at most J times, we obtain $A_i \leq C_S s_q \sum_{j=0}^{J-1} (1 + C_0)^j$. Taking s_q smaller than a fixed fraction of every domain margin divided by this finite sum places the successive product inside the domains. This is the case $\delta_V = 0$ of Lemma 19.3(3), and this choice does not reduce the radius of the coarse field.

Step 2: one scale of the reconstruction. On the original list of free coordinates, the reconstruction at scale i is

$$\Gamma_i(c_{i+1}, a_i(c_{i+1}; \xi) + K_i(c_{i+1}; \xi)^{-1/2} q_i; \xi). \quad (19.30)$$

Here a_i is the conditional stationary point of the partial minimum after the finer scales have been eliminated, K_i is its constrained Hessian at scale i , and Γ_i is the local inverse chart of the block averaging constraint. The action at this scale is M^i times the partial minimum of the objective in input units, so K_i includes exactly the factor M^i . The variable q_i is the fluctuation increment, and no coupling is absorbed into K_i .

Step 3: the reconstructions before gauge transformation. On each common list the comparison of the stationary points is given by Theorem 19.4. To compare inverse square roots, we extend the regional matrix to the full list by its direct sum with a principal free block of the torus matrix on the complementary coordinates. Both summands have the gap. The internal difference is the difference of the regional and torus Hessians at nonzero coarse field, estimated for the conditional partial minimum in Theorem 19.4(6). The removed off-diagonal blocks of the torus matrix have their two roots on opposite sides of W , so they are boundary-localized. The identity (19.7) then gives the comparison

on the common list, with every coarse-field and free argument. When a list of finer coordinates is eliminated on one side and fixed on the other, the identity (19.9) inserts a chain through that list, as in Step 11 of the proof of Theorem 19.4, and that chain visits H_{L_j} . Thus the statement concerns the regional matrix itself, not the torus problem with coordinates deleted.

We subtract the regional and the torus versions of (19.30), with the torus free inputs outside the regional list set to zero. First we change the chart words at the same arguments; their difference lies in the fixed interface stencil. For each conditional object O (a center or an inverse square root), let y_R and y_T be the higher arguments in the two chains, and put $y_{\text{mix}} = (y_{R,L}, y_{T,H})$. The augmented regional objective at the endpoint of (19.16) is independent of y_H on its regional components. Thus, on those components,

$$O_R(y_{R,L}) - O_T(y_T) = [O_R - O_T](y_{\text{mix}}) + [O_T(y_{\text{mix}}) - O_T(y_T)].$$

The first difference is the matched-argument comparison in part (6); the second is the integral of the torus derivative along the segment from y_T to y_{mix} . Step 1 restricts both chains to a smaller common product, so this mixed argument and its segment lie in the full product fixed in Step 10. This first changes the conditional object at the same argument and then changes the coarser argument. The difference of an analytic function at two arguments is its derivative integrated along the common segment, that is, a tensor without boundary weight applied to a boundary-localized difference of arguments. At the coarsest scale these are all the terms. Each descent by one scale adds the boundary-localized difference from the scale above as one more difference of arguments. By induction over at most J scales, the difference $a_R^{\text{raw}} - a_T^{\text{raw}}$ is a convergent boundary-localized series. The values of the torus reconstruction on the complementary outputs, which can be nonzero at zero complementary free input, are joined by networks without boundary weight from their output in H to their inputs in Ω . Such networks cross W , and they provide the visit to H_{L_j} at the next substitution. No sum of sources over the volume of H without decay occurs.

Step 4: the gauge transformation. We apply the same torus gauge transformation $k(c)$ to both fields and cut it off only on the regional side. After the construction radius is chosen, all arguments of H_e^χ lie in one fixed larger identity chart, and the difference is

$$\begin{aligned} \Delta_e &= H_e^\chi(a_R^{\text{raw}}, k) - H_e^\chi(a_T^{\text{raw}}, k) \\ &\quad + H_e^\chi(a_T^{\text{raw}}, k) - H_e^1(a_T^{\text{raw}}, k). \end{aligned} \tag{19.31}$$

The first line contains $a_R^{\text{raw}} - a_T^{\text{raw}}$ as a factor, after the derivative in a is integrated. The second line vanishes when both endpoint values of the cutoff equal 1, so its output root lies in a fixed interface band of width three blocks, including the buffer for the endpoints of fine links. Its other factors have the torus and gauge coefficient bounds

of Theorem 13.2 and Lemma 19.5. These facts imply the boundary-localized estimate for both lines of (19.31). A coarse-field derivative which hits a gauge factor appends the network of that factor to the existing network and does not remove the visit to H_{L_J} . The torus term without the cutoff is $\Phi_{\mathbb{T}}$. This proves the decomposition and its factorial coefficient bounds.

Step 5: the cutoff at first order. The contribution of the cutoff can be read off at first order. Let $k_1 = Dk(0)c$. The cutoff part of the linearized correction on the link (v, w) is the difference of $(1 - \chi)k_1$ between its endpoints, which equals

$$(1 - \chi(v))(k_1(w) - k_1(v)) + (\chi(v) - \chi(w))k_1(w).$$

Both terms are contained in (19.31). In particular the second term, which contains the gradient of the cutoff, is not removed by treating the gauge transformation as constant in the target.

Step 6: the local logarithms. The local logarithms in this argument have uniform complex coefficient margins. Indeed, in a submultiplicative matrix norm the exponential series are dominated by e^t , and the logarithm of $I + X$ is dominated by $\sum_{l \geq 1} t^l / l$ on $\|X\| \leq t < 1$. We take the rows of the reconstruction before gauge transformation and of the gauge transformation small enough that the product lies in such a disc with a fixed margin. Composition of majorants and the attached networks then bound every derivative of H_c^χ , with constants independent of $\chi \in [0, 1]$.

Step 7: fixed inputs and the newly frozen coordinates. Fix an input x in the boundary-localized series of the reconstruction and keep its position while summing over the other indices. The connected network contains e , x , and a visit to H_{L_J} , so by the larger decay exponent and Lemma 18.15, we have $\mathcal{D}_{e,x}(\mathbf{R}) \leq C_J e^{-a_W[\delta(e) + \text{dist}(e,x)]}$. At $q = 0$ the dependence on ξ satisfies the coefficient identity

$$\Delta_e(c, 0; \xi) - \Delta_e(c, 0; 0) = \int_0^1 \sum_f \xi_f D_{\xi_f} \Delta_e(c, 0; t\xi) dt. \quad (19.32)$$

The derivatives in ξ come from its analytic input slots. The coefficient bound on their sum follows from the fixed-input bounds in parts (1) and (6) of Theorem 19.4, with the analytic input chosen among the labels of ξ , and from Lemma 18.15(3). Summing these fixed-input envelopes by (19.2) preserves the boundary decay, including in the coefficient norm in c . The finite chart compositions preserve these envelopes, and the gauge transformation does not depend on ξ . Integrating their coefficient series in c in (19.32), we obtain the bound $C_F \|\xi\|_\infty e^{-a_W \delta(e)}$ of part (2). At $c = q = \xi = 0$ the unperturbed stationary points vanish and both gauge transformations are the identity, so $\Delta(0, 0; 0) = 0$.

Step 8: terms of positive degree in q . If a convergent majorant series in all variables is bounded on a larger fixed free radius r_Q , then, for $s_q \leq r_Q/2$, its terms of positive total degree in q are at most s_q times a fixed bound for its derivative majorant on the larger product. Applying this to (19.31), we obtain the bound $C_{\Delta,J}s_q$ of part (3). The rows near the boundary and the rows in the free variables are bounded by the corresponding convergent majorants without boundary weight, with no factor M^{-J} .

Step 9: the input columns. Fix the input x in the differentiated coefficient series. Removing one power of its radius R_x and using the column bound with any input fixed, we see that the connecting network contains a path from e to x . Assigning b_1 of the larger decay exponent to this path gives

$$C_{e,x}(\mathbf{R}) \leq D_J R_x e^{-b_1 \text{dist}(e,x)} \quad (19.33)$$

on every smaller construction profile in the statement; since the majorant has nonnegative coefficients, we may restrict to that profile before summing. A free input is included, since it is an input of (19.30) and of every later composition. By hypothesis the distances satisfy

$$d_{\text{out}}(x) \leq d_{\text{old}}(e) + \text{dist}(e, x) + C_J,$$

where the additive constant is a fixed buffer. For an incoming list D_{in} whose complement lies in H up to the fixed stencil, with $d_{\text{old}}(e) = d_{D_{\text{in}}}(e)/B_s$, this condition follows from Lemma 19.8 after the incoming distance and the buffer are divided by B_s ; we do not identify the incoming and the current fixed complements. Since $B_s \geq 1$, the condition also holds when d_{old} is measured in input units. Hence

$$\begin{aligned} \sum_e e^{-\alpha_{\text{bd}} d_{\text{old}}(e)} C_{e,x}(\mathbf{R}) &\leq D_J R_x e^{\alpha_{\text{bd}} C_J} e^{-\alpha_{\text{bd}} d_{\text{out}}(x)} \sum_e e^{-(b_1 - \alpha_{\text{bd}}) \text{dist}(e,x)} \\ &\leq C_{M,J} D_J R_x e^{\alpha_{\text{bd}} C_J} e^{-\alpha_{\text{bd}} d_{\text{out}}(x)} \sum_{l \geq 0} (l+1)^3 e^{-(b_1 - \alpha_{\text{bd}})l}. \end{aligned}$$

Using $R_x \leq K_* \rho$, we obtain the bound on K_J ; keeping R_x gives the version for a free input. The sum includes every old output label at its density; restricting it to a sublist only decreases it. For a kernel from inputs to outputs it is multiplied by the volume of the fine output cells divided by the volume of the cell of the fixed input, as in Definition 4.6. The colors and the finitely many scale indices enter $C_{M,J}$, not a factor of the lattice volume. Setting $R_y = 2\rho$ is allowed within the auxiliary ball since $K_* \geq 2$, and the smaller spatially varying radius profile is another restriction of the same nonnegative coefficient series. This completes the proof. $\square$

19.6 Construction radii and bounds on the reconstruction majorant

The next lemma collects the conditions on the radii under which the reconstruction maps a product domain strictly inside the analytic domain used at the next step, in

the following sense. For a map with a convergent Taylor series, the absolute coefficient row sum of the Taylor series, evaluated at the radii of the product domain, bounds the image of the domain under the majorant of the map. For the reconstruction we call this bound the image under the reconstruction majorant. For $c \geq 0$ we write S_c for this row sum with the weight $e^{c\ell}$, where ℓ is the length of the connecting network of a monomial including its output; thus S_0 is the unweighted row sum. A bound $S_c < \rho/2$ is stronger than a bound on the values of the map, and it is the hypothesis needed to substitute the reconstruction into a function whose coefficients are controlled at radius ρ . The lemma is applied twice, with the quantum radius ρ and with the classical radius ρ_{cl} .

Lemma 19.7. *Fix the constants of the unperturbed constructions before the later choices of the boundary-layer width, the erosion width, the widths of the protected neighborhoods, and the bound β_{rec} on newly frozen coordinates.*

1. (Construction radius.) Let $v_{\text{free},J} > 0$ be a free construction radius, distinct from the density constant v_J , with

$$Q_J v_{\text{free},J} < \lambda_J/64, \quad I_J Q_J v_{\text{free},J} < r_{z,J}/64.$$

There is a radius $R_{\text{raw}} > 0$, inside both the chart of the block averaging constraint and the chart of Theorem 11.6, which may be chosen below

$$\min \left\{ \frac{\min(r_{\Gamma,J}, r_E)}{64G_J}, \frac{\lambda_J}{64Q_J}, \frac{r_{z,J}}{64I_J Q_J}, \frac{v_{\text{free},J}}{64C_{\text{lift},J}}, 16 \min(\rho_J, \beta_{\text{rec},J}) \right\}.$$

Here ρ_J and $\beta_{\text{rec},J}$ are the coarse and newly frozen parameter radii of Theorem 18.12; thus the lifted arguments of size less than $R_{\text{raw}}/8$ stay inside its parameter product. The quantities G_J and $r_{\Gamma,J}$ are the bound and the radius of the finite constraint words, r_E is the larger radius of the torus action, and $C_{\text{lift},J}$ bounds the block-constant lift and the coarse-field lift, including all embeddings for the initial segments. If

$$C_{\text{lift},J} [K_* \max(\rho, \rho_{\text{cl}}) + r + \beta_{\text{rec}} + q_{\text{raw}}] < R_{\text{raw}}/8,$$

where r is the radius of the small-link domain B_r and q_{raw} is the radius allowed for the fluctuation increments before gauge transformation, then all images of the reconstruction before gauge transformation lie in these fixed larger charts, and their real free Hessians have gap at least $\lambda_J/2$, with the complex inverse margins. No factor M^{-J} is required of the lift.

2. (Zero fluctuations.) For each $\rho_{\bullet} \in \{\rho, \rho_{\text{cl}}\}$, use its own spatially varying radius profile $\phi_{\bullet}$, equal to the boundary radius $\rho_{\partial,\bullet}$ at labels within distance w_{bd} of H and to $\rho_{\bullet}$ elsewhere, the outer profile $K_* \phi_{\bullet}$, and the outer boundary radius $R_{\partial,\bullet} = K_* \rho_{\partial,\bullet}$. For a full renormalization group step, $s = J$ and $B = M^J$, assume the torus contraction inequality (the first inequality below) and impose the four inequalities

$$\begin{aligned} B^{-1} Z_{\bullet}(K_* \rho_{\bullet}) &\leq \rho_{\bullet}/16, & C_{\Delta,\bullet} C_{a,\bullet} R_{\partial,\bullet} &\leq \rho_{\bullet}/16, \\ C_{\Delta,\bullet} C_{a,\bullet} e^{-a_W w_{\text{bd}}/2} K_* \rho_{\bullet} &\leq \rho_{\bullet}/16, & C_{F,\bullet} \beta_{\text{rec}} &\leq \rho_{\bullet}/16, \end{aligned}$$

where $Z_\bullet$ is the classical row majorant of Theorem 13.2, $C_{\Delta,\bullet}$ and $C_{F,\bullet}$ are the constants of Theorem 19.6, and $C_{a,\bullet}$ is a summation constant fixed in the proof. These constants are evaluated on the larger auxiliary construction ball, before the radii and widths are chosen. The sum of the four left sides bounds the row sum S_0 at $q = 0$ by $\rho_\bullet/4$.

3. (Fluctuation increments.) For a radius s_q of the fluctuation increments, the increment of S_0 is bounded by

$$W_\bullet(s_q) = \frac{C_{T,\bullet}s_q}{1 - C_{T,\bullet}(K_*\rho_\bullet + s_q)} + C_{\Delta,\bullet}s_q.$$

Require the denominator to be at least $1/2$ and $W_\bullet(s_q) \leq \rho_\bullet/8$ for both radii. The same rule applies to the product in the jointly normalized coordinates u , with its finite-composition free radius and its own denominator, which includes β_{rec} if it occurs in that majorant. The products for the successive substitution must lie in this joint product.

4. (Spatial weight.) The resulting unweighted absolute row sum satisfies $S_0 \leq 3\rho_\bullet/8$. If $S_c \leq C_J\rho_\bullet$ for a fixed $c > 0$, then there is a fixed $0 < \vartheta < 1$, common to both radii and to the finitely many initial segments, such that

$$S_{\vartheta c} \leq S_0^{1-\vartheta} S_c^\vartheta \leq \rho_\bullet (3/8)^{1-\vartheta} C_J^\vartheta < \rho_\bullet/2.$$

Thus, for both radii, the image under the reconstruction majorant, with connecting-network weight at one positive exponent, is less than half of the radius.

All constants and thresholds have the dependence of Section 19.1 and do not depend on the later widths. For a composition of $s < J$ averaging steps, the conclusion of part (4) requires the torus contraction inequality with $B_s = M^s$ in place of B , which must be checked separately for that composition; the finiteness of the constants at fixed J does not give it.

Proof. We choose the construction radius, bound the unweighted row sum at zero fluctuation and its increment, and then add the spatial weight by interpolation.

Step 1: the construction radius. We choose the finite word and lift constants on the larger products of Theorem 18.12 and Lemma 19.3. We increase $C_{\text{lift},J}$ so that it bounds the block-constant lift of the target, the coarse-field lift, the extensions by zero and by the small newly frozen coordinates, and each of the finitely many embeddings for initial segments. Since a local lift uses only its block stencil of fixed depth, its row and input columns have one finite coefficient bound, independent of the number of components and of the later interface widths. Let $v_{\text{free},J}$ satisfy the two strict inequalities in part (1), and let R_{raw} lie below the listed bounds and below the common radii of the local Haar-density chart and of the compact-overlap chart. The lift inequality then places all images strictly inside the larger inverse and torus action charts, their free arguments inside the radius $v_{\text{free},J}$, and their stationary points inside the solution radius $r_{z,J}$. Indeed, the variation of

the free Hessian is bounded by the sum of its contributions from the free radius and from the lifted arguments, which is at most

$$Q_J v_{\text{free},J} + Q_J R_{\text{raw}} < \lambda_J / 32.$$

Enlarging the fixed lift constant before the radii are chosen accounts for any finite number of such types of arguments. The displacement of the stationary point is bounded by the corresponding inverse times gradient, which is less than $I_J Q_J (v_{\text{free},J} + R_{\text{raw}}) < r_{z,J} / 32$. The real gap λ_J at the identity therefore implies a real gap at least $\lambda_J / 2$ on all these products, with room for the complex margins. The term $E'_{d_0} \Psi_{zz}$, which contains the gradient of the action, is included in the bound on the variation of the Hessian in Theorem 18.12. The conditional stationary points are those of Lemma 19.3.

Step 2: the row sum at zero fluctuation. We carry out the following calculation for each $\rho_\bullet$, with its own profile and constants. The construction radii satisfy $K_* \rho_\bullet \leq U_T$ and $s_q \leq Q_T$, so $K_* \rho_\bullet + s_q < r_s$ and all intermediate margins hold by (19.1); these domain checks precede each rational majorant. At $q = 0$ we use Theorem 19.6. By Theorem 13.2, the torus term has absolute row at most $B^{-1} Z_\bullet(K_* \rho_\bullet)$. We split the regional correction into its value at $\xi = 0$ and its increment in ξ . By Theorem 19.6(2), the latter is bounded by $C_{F,\bullet} \beta_{\text{rec}}$ in the coefficient norm in the target. The former vanishes at zero target, and integrating its differentiated series with a fixed target input bounds it by

$$C_{\Delta,\bullet} \sum_x R_x e^{-a_W [\delta(e) + \text{dist}(e,x)]}, \quad R_x = (K_* \phi_\bullet)(x),$$

where $C_{\Delta,\bullet}$ is increased to include the fixed normalization of the derivative. Terms with $\delta(x) < w_{\text{bd}}$ have $R_x = R_{\partial,\bullet}$, and their sum is at most $C_{a,\bullet} R_{\partial,\bullet}$, where $C_{a,\bullet}$ bounds the sum of the exponentials. For the other terms, by the conventions for endpoints, we have

$$\delta(e) + \text{dist}(e, x) \geq \delta(x) - L'_J \geq w_{\text{bd}} - L'_J.$$

We use half of the exponent for this inequality and sum the other half using the density bound. After $e^{a_W L'_J / 2}$ is included in the fixed constant $C_{a,\bullet}$, these terms are at most $C_{a,\bullet} e^{-a_W w_{\text{bd}} / 2} K_* \rho_\bullet$. This bounds the row at $q = 0$ by the sum of the four terms of part (2). Since each of them is at most $\rho_\bullet / 16$, the row is at most $\rho_\bullet / 4$. All constants in this calculation were evaluated on the larger unperturbed auxiliary ball, before w_{bd} or the smaller boundary radius was chosen.

Step 3: terms of positive degree in the increments. For positive degree in the free coordinates, the torus row is bounded by

$$\frac{C_{T,\bullet} s_q}{1 - C_{T,\bullet} (K_* \rho_\bullet + s_q)},$$

which is the majorant of Theorem 13.2. The row of the correction of positive degree in q is at most $C_{\Delta, \bullet} s_q$, by Theorem 19.6(3). Adding the two, we obtain $W_{\bullet}(s_q)$, and under the two conditions of part (3) the increment is at most $\rho_{\bullet}/8$. Combining this with Step 2, we obtain the absolute coefficient row bound

$$S_0 \leq 3\rho_{\bullet}/8. \quad (19.34)$$

The denominator cannot be discarded: its geometric series contains all higher degrees in the free coordinates and in the target.

Step 4: the jointly normalized coordinates. For the joint coordinate u , we substitute $z = z_0 + K_0^{-1/2}u$ into the maps in the original coordinates before taking the majorant. The successive increments are recovered from z by

$$q_i = K_i(c, z_{>i}; \xi)^{1/2} (z_i - a_i(c, z_{>i}; \xi)),$$

which inverts the unperturbed insertion at scale i . The factor $K_i^{1/2} = K_i K_i^{-1/2}$ has the same larger coefficient bounds. At $u = 0$ all these increments vanish, since the chain at zero fluctuation is the joint stationary point. By finite composition of majorants, there exists a joint radius for which the resulting majorant of the increments is at most Q_T . We take the product of u inside this radius and inside the solution domains of the initial segments. Its target argument is at most U_T , so (19.1) also verifies the torus domain and the denominator for this composition. The finite composition has a rational majorant of the same kind for its increment, with its own constants and argument; in particular, a contribution of the newly frozen coordinates to that argument remains in its denominator. Requiring this majorant to be at most $\rho_{\bullet}/8$ gives (19.34) for the joint product as well. The products for the successive substitution are restricted to the preimage of this product and to the domains of Lemma 19.3. This restricts a coefficient series which is already bounded.

Step 5: the spatial weight. For one output row, we write its terms as $b_v \geq 0$ and let ℓ_v be the length of the attached network, including the output root. For finite partial sums, by Hölder's inequality with exponents $1/(1-\vartheta)$ and $1/\vartheta$, we have

$$\sum_v b_v e^{\vartheta c \ell_v} = \sum_v b_v^{1-\vartheta} (b_v e^{c \ell_v})^{\vartheta} \leq \left(\sum_v b_v \right)^{1-\vartheta} \left(\sum_v b_v e^{c \ell_v} \right)^{\vartheta},$$

and passing to increasing partial sums gives the inequality for the convergent series. Using $S_c \leq C_J \rho_{\bullet}$ and (19.34), we obtain the bound in part (4). The function $\vartheta \mapsto (3/8)^{1-\vartheta} C_J^{\vartheta}$ is continuous at $\vartheta = 0$, with value $3/8 < 1/2$. We fix $\vartheta \in (0, 1)$ at which its value is below $1/2$ for both radii and for the finitely many initial segments. This proves both weighted bounds on the image under the reconstruction majorant at the same positive exponent. Because the weighted row bound is a bound of the unperturbed construction, this choice of exponent uses no later perturbative estimate.

Step 6: shorter compositions. For $s < J$, Step 1 and all estimates for finite compositions still apply, with $B_s = M^s$. The first inequality of part (2), however, becomes $B_s^{-1} Z_\bullet(K_* \rho_\bullet) \leq \rho_\bullet / 16$, which follows only from a separate contraction inequality for that composition. This is the restriction for $s < J$ in the statement. $\square$

19.7 Perturbations of the incoming action

The incoming action of a renormalization group step on a block region is the effective classical action plus a perturbation, defined on the list of variable labels of the previous step. This list and the current block region play different roles; in particular, the protected vertices of the gauge transformation alone do not determine the boundary of the incoming list. We first compare the distance to the boundary of the incoming list with the distance to the current boundary.

Lemma 19.8. *Fix $1 \leq s \leq J$, $B_s = M^s$, a list D_{in} of incoming positions, and a block region Ω . Use the periodic graph metric, rescaled linearly on positions of coordinates introduced at finer lattices, and freeze all components of a label at one position together. Write $\pi_s x = x / B_s$. Suppose that every position $x \notin D_{\text{in}}$ lies in H up to the fixed chart stencil:*

$$\text{dist}(\pi_s x, \Omega^c) \leq L_{\text{buf}, J}.$$

Here $L_{\text{buf}, J}$ and the fixed buffer $c_{\text{end}, J}$ for the convention on link endpoints depend only on the construction geometry, and are fixed before the later widths. Let $d_{D_{\text{in}}}(i)$ be the distance of i to the complement of D_{in} , in input units, and $d_{\text{current}}(e)$ the distance of e to H , in output units. Then for every $i \in D_{\text{in}}$ and every position e of a label in Ω ,

$$d_{D_{\text{in}}}(i) \geq B_s d_{\text{current}}(e) - B_s \text{dist}(e, \pi_s i) - C_{\text{buf}, J}, \quad C_{\text{buf}, J} = \max_{1 \leq s \leq J} B_s (L_{\text{buf}, J} + c_{\text{end}, J}). \quad (19.35)$$

The same conclusion holds with $\text{dist}(e, W)$ in place of $d_{\text{current}}(e)$ after the fixed endpoint buffer is enlarged. It also holds if H is enlarged by additional blocks.

Proof. Suppose that the complement of D_{in} is nonempty, and choose a position $x \notin D_{\text{in}}$ reached by a shortest path from i to this complement; its length is $d_{D_{\text{in}}}(i)$. The scaled path has length $d_{D_{\text{in}}}(i) / B_s$, including the position of x at its own lattice; scale indices and colors give no extra distance. By the triangle inequality and the hypothesis, we have

$$\begin{aligned} d_{\text{current}}(e) &\leq \text{dist}(e, \pi_s x) + L_{\text{buf}, J} + c_{\text{end}, J} \\ &\leq \text{dist}(e, \pi_s i) + d_{D_{\text{in}}}(i) / B_s + L_{\text{buf}, J} + c_{\text{end}, J}. \end{aligned}$$

Multiplying by B_s , we obtain (19.35). The endpoint buffer covers the difference between the distance to the positions in H and the distance to the interface W . Enlarging H can

only decrease its distance from a fixed position in Ω , so additional blocks of H preserve both the hypothesis and the bound. If D_{in} has no complement, the finiteness of the boundary seminorm forces the nonconstant boundary part of the incoming perturbation to vanish, by the convention of Definition 18.1, and there is no incoming boundary dependence to transport. If D_{in} is empty there is no incoming root. $\square$

The constant incoming radius in the next theorem is the radius produced by freezing, in the following form of Lemma 18.9. We start with a classical output profile ϕ of interior radius r_c on a set D of variable labels, keep $I \subset \{i \in D : d_D(i) \geq s_{\text{er}}\}$ with $s_{\text{er}} \geq \max(2, w_{\text{bd}})$, and freeze $F = D \setminus I$, which contains the zone of reduced radius. For an analytic function f and frozen values $c^\sharp$, we write $T_{c^\sharp}f(v) = f(v \oplus c^\sharp) - f(0 \oplus c^\sharp)$. For $|c_j^\sharp| \leq \eta\phi(j)$ with $\eta \leq 1/4$, Lemma 18.9 implies

$$\phi|_I \equiv r_c, \quad \mathcal{X}_I^{r_c}(T_{c^\sharp}f) \leq 2e^{-\alpha_{\text{cl}}(s_{\text{er}}-1)} \mathcal{X}_D^\phi(f). \quad (19.36)$$

The seminorms here have the classical normalization and the stated network and boundary rates. The identity $\phi|_I \equiv r_c$ holds because every remaining position lies outside the zone of reduced radius, and it is needed for the seminorm with constant profile on the left. The frozen values become fixed real parameters, not new variables of radius r_c . Applying (19.36) to the boundary correction produced at the previous step, with $D_{\text{in}} = I$, gives the incoming seminorm at the next step. A new classical boundary-completion difference of the torus action must in addition satisfy its own bound in the seminorm with constant profile, by the separate statement on such differences in Lemma 18.9; erosion alone does not bound such a difference at an additional component of H . The theorem below assumes the bound on the incoming seminorm. That the regions and frozen values of a large-field decomposition satisfy its hypotheses is verified in the construction of that decomposition in Section 21.

Theorem 19.9. *Fix the domains, the protected geometry, the radius R_{raw} , the conditions on the image under the reconstruction majorant for both radii, and the products for the initial segments of Lemmas 19.3, 19.5, and 19.7, with $s = J$; compositions of $s < J$ averaging steps are treated at the end of the statement. Fix K_* , classical input rates $a_{\text{cl}}, \alpha_{\text{cl}} > 0$, and $r_c = \rho_{\text{cl}}$. Let $b = (\xi, b_{\text{rec}})$, where the newly frozen coordinates ξ are small and the previously frozen coordinates b_{rec} are arbitrary in their allowed real set.*

The incoming action. Let $E_{\text{in}}(a; b) = E_{d_0}(\iota_0 a) + P(a; b)$ be holomorphic and real at real arguments in its variable input a , with b_{rec} fixed, where ι_0 extends a by zero. Fix its incoming list D_{in} of variable labels, the positions of these labels at the lattices where they were introduced, and their cell volumes. The incoming seminorm is the seminorm $\mathcal{X}_{D_{\text{in}}, a_{\text{cl}}, \alpha_{\text{cl}}}^{r_c}$ of Definition 18.1, with the constant profile $\phi_{\text{in}} \equiv r_c$ on every label of D_{in} at which Φ_0 below is substituted; the labels of ξ are fixed parameters of this seminorm. The spatially varying radius profiles are used only for the output. Assume that the complement of D_{in} lies in H up to the fixed stencil, as in Lemma 19.8

(H may contain additional blocks), and that, in input units, for every position e of a label in Ω and every incoming position i ,

$$d_{D_{\text{in}}}(i) \geq B_s d_{\text{current}}(e) - B_s \text{dist}(e, i/B_s) - C_{\text{buf}, J}, \quad B_s = M^s,$$

where the positions on the right are in output units and the fixed buffer includes the chart stencils. If D_{in} has no complement, the nonconstant boundary part of the incoming perturbation is zero. All images of Φ_0 and all connecting segments lie strictly inside the ball of the incoming seminorm. A difference of two perturbations uses the same incoming list, weights, and profile.

The exterior extension. If the unperturbed regional objective uses the extension ι_{ξ} , which inserts the newly frozen coordinates into their slots while all previously frozen torus slots remain zero, define the classical boundary-completion difference and the adjusted perturbation

$$S_{\xi}^{\text{cl}}(a) = E_{d_0}(\iota_{\xi}a) - E_{d_0}(\iota_0a), \quad \bar{P} = P - S_{\xi}^{\text{cl}},$$

and assume $\mathcal{X}_{D_{\text{in}}, a_{\text{cl}}, \alpha_{\text{cl}}}^{r_c}(\bar{P}) \leq 2\sigma_*$. If the reference objective already uses the extension by zero, take $S_{\xi}^{\text{cl}} = 0$. All constants of S_{ξ}^{cl} remain in the full classical action. The corresponding difference for the effective interaction, if the reference extension is changed there, is kept with the exact sign of the constant term in Lemma 18.9.

The objective. Let z_0 be the unperturbed joint stationary point, $K_0 = F_{0, \text{zz}}(c, z_0; \xi)$, and $S_0 = K_0^{-1/2}$, and put

$$\begin{aligned} F_0(c, z; \xi) &= E_{d_0}(\iota_{\xi} \Psi_R(c, z; \xi)), \\ A_0(c, u; \xi) &= \Psi_R(c, z_0 + S_0 u; \xi), \\ \Phi_0(c, u; \xi) &= \text{Log}[(\exp A_0(c, u; \xi))^{G^{\Omega}(c)}], \\ \mathcal{F}_{\lambda}(c, u; b) &= F_0(c, z_0 + S_0 u; \xi) + \lambda \bar{P}(\Phi_0(c, u; \xi); b), \quad 0 \leq \lambda \leq 1, \end{aligned}$$

and $\mathcal{Q}_{\lambda}(c, z; b) = \mathcal{F}_{\lambda}(c, S_0^{-1}(z - z_0); b)$. The perturbation is evaluated on the gauge-transformed reconstruction Φ_0 , and the unperturbed Hessian of $\mathcal{F}_{\lambda}$ in u at $u = 0$ is I . Initial segments, the conditional stationary points $a_{\lambda, I}$, and the Hessians $K_{\lambda, i}$ refer to $\mathcal{Q}_{\lambda}$, in the original free coordinates.

Then the following statements hold.

1. (Incoming derivatives.) If the inputs at which $\bar{P}$ is evaluated have absolute coefficient row sums at most $\kappa \rho_{\text{cl}}$, with $0 < \kappa < 1$, then for every $j \geq 1$,

$$\|D^j \bar{P}\|_{\text{root}} \leq \frac{j!}{(1 - \kappa)^{j+1}} (2\sigma_*) \rho_{\text{cl}}^{2-j},$$

with the incoming network and boundary weights, and with any differentiated input as root.

2. (Stationary points.) There exist $\sigma_{0, J}, Q_{0, J} > 0$ and a fixed smaller domain of u such that the following holds for every $0 \leq \sigma_* \leq \sigma_{0, J}$, every $0 \leq \lambda \leq 1$, and every product of increments

with $\max_i \|q_i\|_\infty \leq Q_{0,J}$ whose substitutions stay in that domain. Every initial segment of the original free coordinates has a unique holomorphic stationary point of $\mathcal{Q}_\lambda$ in the domains of Lemma 19.3, and the inequalities of Lemma 19.3(2) on the preconditioned gradient and Hessian hold, not only a spectral bound. The stationary points are real at real data, the real free Hessians satisfy

$$\frac{1}{2}\lambda_J I \leq K_{\lambda,i} \leq \Lambda_J I, \quad \|u_\lambda\|_{\text{coeff}} \leq C_J \sigma_* \rho_{\text{cl}},$$

with the exact factors M^i included in $K_{\lambda,i}$, and for every initial segment I and every $y \in \mathcal{T}_I$,

$$\|a_{\lambda,I}(c, y; b) - a_I(c, y; \xi)\|_{\text{coeff}} \leq C_J \sigma_* \rho_{\text{cl}}.$$

The quantity δ_V of Lemma 19.3 satisfies $\delta_V \leq I_0 C_P \sigma_* \rho_{\text{cl}}$, with constants I_0, C_P depending only on the fixed data, and $\sigma_{0,J}, Q_{0,J}$ are chosen so that the condition of Lemma 19.3(3) holds for all initial segments. The weighted complex resolvents are bounded by $C_J/(\lambda_J + t)$ for $t \geq 0$ at the working rate b_w fixed in part (4).

3. (Differences.) The differences from $\lambda = 0$ of the stationary points, the Hessians, their inverses and inverse square roots, and the successive reconstructions have convergent boundary-localized coefficient bounds in all variables, including every coarse-field input in Ω and every fluctuation increment; their networks carry the incoming boundary dependence. These differences are $O_J(\sigma_*)$, and the displacements of stationary points are $O_J(\sigma_* \rho_{\text{cl}})$ on the fixed product. For example, at fixed coarser coordinates y ,

$$\partial_\lambda z_{I,\lambda} = -\mathcal{Q}_{\lambda,II}^{-1} V_I,$$

where V_I is the I -component of the gradient of $z \mapsto \bar{P}(\Phi_0(c, S_0^{-1}(z - z_0); \xi); b)$. Along the successive chain this derivative contains the additional term $-\mathcal{Q}_{\lambda,II}^{-1} \mathcal{Q}_{\lambda,II^c} \partial_\lambda z_{I^c}$ and the full derivative $(\partial_\lambda K_{\lambda,i}^{-1/2})q_i$. The function $V = \bar{P} \circ \Phi_0$ satisfies $\|V_u\| \leq C_J \sigma_* \rho_{\text{cl}}$ and $\|V_{uu}\| \leq C_J \sigma_*$; the term $\bar{P}' \Phi_{0,uu}$ is kept in the second derivative.

4. (Output rates.) The estimates of part (3) hold at any fixed positive output rates $a_{\text{out}}, \alpha_{\text{out}}$ with

$$a_{\text{out}} + \alpha_{\text{out}} < c_J / C_{\text{geom}}, \quad a_{\text{out}} + \alpha_{\text{out}} < B a_{\text{cl}} / C_{\text{geom}}, \quad \alpha_{\text{out}} < B \alpha_{\text{cl}} / C_{\text{geom}},$$

Before the output rates or the perturbation threshold are chosen, reduce c_J so that $c_J \leq b_{\text{ref}}$ and fix

$$b_w = \frac{\min(b_{\text{ref}}, c_J, B a_{\text{cl}})}{2C_{\text{dim}}}, \quad C_{\text{dim}} \geq 1, \quad C_{\text{geom}} \geq 8C_{\text{dim}}.$$

Here b_{ref} is the unperturbed resolvent exponent fixed above. Then every allowed pair of output rates satisfies $a_{\text{out}} + \alpha_{\text{out}} < b_w / 4$, whereas $C_{\text{dim}} b_w \leq \min(c_J, B a_{\text{cl}}) / 2$. The perturbation

threshold is fixed at this one working rate and need not depend on the subsequently selected output rates. The constant C_{geom} is fixed before J , the rates, and the later widths: it depends only on the dimension, the block geometry, and the allocation of exponents, not on J or on the construction radii. Fixed stencil buffers depending on J affect the amplitudes C_J , not the coefficients of the distances. In particular, the perturbation and every external argument stay in one connected coefficient network.

5. (Two perturbations.) For two perturbations $\bar{P}_0, \bar{P}_1$ as above, the bounds of parts (2) and (3) on the differences hold with σ_* replaced by $\mathcal{X}_{D_{\text{in}}, a_{\text{cl}}, \alpha_{\text{cl}}}^{rc}(\bar{P}_1 - \bar{P}_0)$. The corresponding minimized full actions $V_0^{\text{out}}, V_1^{\text{out}}$ satisfy a Lipschitz bound in the output seminorm with constant K_J^{cl} , namely (19.51).
6. (The full action.) The outgoing full action is $M^s \mathcal{F}_1(c, u_1; b)$, and on the connected chart

$$\mathcal{F}_1(c, u; b) = E_{\text{in}}(\Phi_0(c, u; \xi); b).$$

This identity uses the equality of classical actions on compact gauge overlaps only for the torus term, together with the cancellation of the exact classical boundary-completion difference. It does not assert that $\bar{P}(\Phi_0)$ equals $\bar{P}(A_0)$, and it uses no gauge identity for an incoming branch.

7. (Frozen coordinates.) All perturbed estimates differentiate the coarse-field and free variables at fixed real $b = (\xi, b_{\text{rec}})$; no derivative of a perturbed quantity in ξ or b_{rec} is asserted, and the unperturbed estimates in ξ of Theorem 19.4 are unchanged. If the incoming coefficients and the charts are Borel (respectively continuous) functions of the real parameter b , then the stationary points, the matrix functions, and the minimized values are Borel (respectively continuous) in b .

All constants have the dependence of Section 19.1; they do not depend on the later widths, the number of preceding steps, the shape of Ω , the depth, or the smaller enclosing and fluctuation radii chosen later. For a composition of $s < J$ averaging steps, the same statements hold, with B_s in place of B , whenever the separate conditions of Lemma 19.7 for that composition hold; no repeated reduction of radii is used. The theorem is a local analytic statement and does not compare probability densities.

Proof. We extract the derivatives of the incoming perturbation from the incoming seminorm and transport its boundary dependence to the output. We then construct the stationary points with Lemma 19.3, first jointly and then for the initial segments. The remaining steps treat the differences, two perturbations, the full action, and the dependence on the frozen coordinates.

Step 1: conventions. All functions in this proof are considered at a fixed real $b = (\xi, b_{\text{rec}})$ in the allowed set, including the newly frozen coordinates ξ . Every estimate is uniform over this set, and the supremum over it is taken after the sums over degrees. Only the

coarse field in Ω , the free coordinates, and the interpolation parameter are differentiated, and the labels of ξ are not roots of the incoming seminorm.

Step 2: derivatives of the incoming perturbation. We write $r_c = \rho_{\text{cl}}$ and $\bar{P}(a; b) = \sum_\nu p_\nu(b) a^\nu$, where every multi-index runs over D_{in} , with the positions of its labels at the lattices where they were introduced. Since $\phi_{\text{in}}(i) = r_c$ for every such label, we define, for each degree k ,

$$A_k(b) = r_c^{-2} \sup_{i \in D_{\text{in}}} e^{\alpha_{\text{cl}} d_{D_{\text{in}}}(i)} \sum_{|v|=k} \frac{v_i}{k} |p_\nu(b)| r_c^k e^{a_{\text{cl}} t_{\text{in}}(\nu)}.$$

By hypothesis, $\sup_{b \text{ real}} \sum_{k \geq 1} A_k(b) \leq 2\sigma_*$. This uses the full input radius also at the incoming boundary; a reduced radius near the boundary appears only in the output product. By (19.36), the incoming action produced by freezing has this profile when the hypotheses of Lemma 18.9 on freezing the zone of reduced radius and on the frozen values hold. We consider inputs whose absolute coefficient row sums are at most κr_c , and fix the first differentiated input label i . In degree $k \geq j$, summing over the other $j-1$ ordered differentiated labels gives the factor

$$\frac{v_i}{k} (k)_j, \quad (k)_j = k(k-1) \cdots (k-j+1),$$

and the remaining input powers are bounded by $\kappa^{k-j} r_c^{k-j}$. Keeping the original incoming network keeps all differentiated roots and all contracted input labels in it. Therefore the rooted derivative of order j is at most

$$r_c^{2-j} \sum_{k \geq j} A_k(k)_j \kappa^{k-j} \leq (2\sigma_*) r_c^{2-j} \frac{j!}{(1-\kappa)^{j+1}},$$

where we used $\sum_{k \geq j} (k)_j \kappa^{k-j} = j! / (1-\kappa)^{j+1}$, which follows by differentiating the geometric series j times. Any differentiated root can be chosen first, so this proves part (1), with the incoming network and the boundary weight of its root. Degree one is included. The constant coefficient p_0 does not appear in these derivatives and is not restricted.

Step 3: the normalized objective and the perturbing gradient. Let A_0 , Φ_0 , and $\mathcal{F}_\lambda$ be as in the statement, on the joint product of Lemma 19.7. The image of this product under the reconstruction majorant, with weight at one fixed positive exponent of the connecting network, is less than $r_c/2$, so part (1) applies with a fixed $\kappa < 1$, for example $\kappa = 1/2$. At $u = 0$ we have

$$D_u F_0(c, z_0 + S_0 u; \xi)|_{u=0} = 0, \quad D_u^2 F_0(c, z_0 + S_0 u; \xi)|_{u=0} = S_0^\top K_0 S_0 = I. \quad (19.37)$$

The first identity is the joint stationarity, and the second uses the symmetric inverse square root on the branch connected to the identity. The coefficients of this function, and

those of z_0 , Ψ_R , and G^Ω , are bounded on a larger product by the unperturbed results above.

Let $V = \bar{P} \circ \Phi_0$. By the chain rule, we have

$$V_u = \sum_i \bar{P}_i(\Phi_0)(\Phi_0)_{i,u}, \quad (19.38)$$

$$V_{uv} = \sum_{i,j} \bar{P}_{ij}(\Phi_0)(\Phi_0)_{i,u}(\Phi_0)_{j,v} + \sum_i \bar{P}_i(\Phi_0)(\Phi_0)_{i,uv}. \quad (19.39)$$

By part (1) with $j = 1, 2$, together with the fixed rows and volume-normalized columns of the chart, we obtain

$$\|V_u\| \leq C_P \sigma_* r_c, \quad \|V_{uu}\| \leq C_P \sigma_*. \quad (19.40)$$

The term $\bar{P}'\Phi_{0,uu}$ in (19.39) contributes $C_J \sigma_* r_c$, which is included in $C_P \sigma_*$ since r_c is fixed. We note that the perturbation need not have zero gradient at $u = 0$.

These bounds hold at all orders, as coefficient bounds with spatial weights. Let $\mathfrak{b}_J(t) = C_J t / (1 - C_J t)$ majorize the increment series of the chart in a fixed displacement direction, on a fixed larger domain. After division by the factorials of the derivatives, part (1) and the chain rule over partitions are bounded by the coefficients of

$$2\sigma_*^2 r_c^2 \left\{ \frac{1}{1 - \kappa - \mathfrak{b}_J(t)/r_c} - \frac{1}{1 - \kappa} \right\}. \quad (19.41)$$

Fix $t_* > 0$, a fixed multiple of $\min(r_c, 1)$, with $\mathfrak{b}_J(t_*) \leq (1 - \kappa)r_c/2$. By nonnegativity of its coefficients, the coefficient of degree j of (19.41) is at most $C_J \sigma_*^2 r_c^2 t_*^{-j}$, and restoring $j!$ gives the factorial bounds $j! C_J^{j+1} \sigma_*^2 r_c^{2-j}$, with constants depending on the fixed radius. Fixing a coarse-field or free input uses the column version of the same composition, with the cell volume of that input. Thus all derivatives used here satisfy coefficient bounds on the product.

Step 4: transport of the incoming boundary dependence. A rooted input monomial carries its incoming network and a path from its chosen input root to the complement of D_{in} . The incoming list and its complement are those used in $A_k(b)$. By hypothesis the complement of D_{in} lies in H up to the fixed stencil, so Lemma 19.8 implies (19.35) with its fixed buffer, and hence the comparison assumed in the statement. We scale these paths by B^{-1} to output units and join every network of the chart at its fine input label. The union is connected and contains every external coarse-field and free argument. If the input root lies in Ω , its path to the complement of D_{in} reaches H after at most one additional leg of fixed length. If it lies in a newly removed region, the chart branch joining it to a root in Ω already reaches the boundary. Additional components of H can only shorten the required distance to the boundary.

To make this precise, let ℓ_0 be the length of the incoming network in input units, d_0^{in} the distance of the chosen root to the complement of D_{in} , and ℓ_β the lengths of the branches of the chart and of the free inverse in output units. Let A be the set of external positions and e any root in Ω . There exists a constant C_{dim} , depending only on the dimension and on the comparison of elementary paths, and fixed before J , such that

$$\begin{aligned} a_{\text{out}}t(A) + \alpha_{\text{out}}\delta(e) &\leq C_{\text{dim}}(a_{\text{out}} + \alpha_{\text{out}})\left(\ell_0/B + \sum_{\beta} \ell_{\beta}\right) \\ &\quad + C_{\text{dim}}\alpha_{\text{out}}d_0^{\text{in}}/B + (a_{\text{out}} + \alpha_{\text{out}})L'_J(k+1). \end{aligned} \quad (19.42)$$

Indeed, the scaled incoming network and the chart branches alone join all external positions, and adding the scaled path to the complement of D_{in} also reaches the boundary. The first network bounds $t(A)$, and the second bounds the distance of e to the boundary; this gives the separate coefficients of ℓ_0 and d_0^{in} in (19.42). Here k is the incoming degree, equivalently the number of attached input factors, including factors evaluated at a nonzero frozen value. The last term accounts for their fixed displacements by the stencil. To see why C_{dim} depends only on the dimension, we take shortest paths in the graph metric and their connected union. Using a tree on the external vertices increases the length by at most a factor two, by traversing each edge of a Steiner tree twice. Replacing segments by the fixed paths in coordinate order increases it only by the fixed comparison constant of the three coordinate directions. Scaling is exact, and neither operation depends on the number of scales.

We sum the stencil offsets using the strict image bound, not by shrinking the external Taylor variables. Reduce the intrinsic rate c_J once, before output rates are chosen, so that $e^{c_J L'_J} \leq 4/3$. Put $\beta = (a_{\text{out}} + \alpha_{\text{out}})L'_J$ and $\kappa' = e^{\beta}\kappa$. The reconstruction image has $\kappa \leq 1/2$, so the allowed output rates give $\kappa' \leq 2/3$. Multiplying the degree- k derivative term of Step 2 by $e^{\beta(k+1)}$ gives

$$e^{\beta(k+1)}\kappa^{k-j} = e^{\beta(j+1)}(\kappa')^{k-j}.$$

Thus its degree sum is bounded by

$$(2\sigma_*)r_c^{2-j} \frac{e^{\beta(j+1)}j!}{(1 - \kappa')^{j+1}}.$$

In (19.41) the same calculation replaces κ by κ' and $\mathfrak{b}_J(t)$ by $e^{\beta}\mathfrak{b}_J(t)$, with one additional fixed factor e^{β} . A smaller fixed t_* leaves a positive denominator. These changes alter only the fixed- J amplitudes. They cover degree-zero contributions from frozen coordinates as well, without assuming geometric decay of the incoming sequence A_k . The intrinsic-rate reduction precedes all later profiles and widths and does not change C_{geom} .

We take $C_{\text{geom}} \geq 8C_{\text{dim}}$, increased according to the fixed allocation of exponents among the external network, the route to the boundary, and the internal sums, and we

keep part of the remaining exponent in each chart branch for its density sum $S_J(\varepsilon)$. Under the three strict inequalities of part (4), the incoming factors $e^{a_{\text{cl}}\ell_0}$ and $e^{\alpha_{\text{cl}}d_0^{\text{in}}}$, together with the exponent c_J of the unperturbed branches, dominate the output network and boundary weights in (19.42), and a positive part of the exponent is left for the summations. The exponent in the bound on the image under the reconstruction majorant from Lemma 19.7 is included in the choice of c_J before these rates are specified. The allocation is applied to the whole network, so the exponent is not divided by the Taylor order or by the number of scales. This proves the boundary-localized versions of (19.40) and (19.41) at every allowed output rate.

Step 5: the joint stationary point. We now verify the domain conditions, working first in u near zero. By the unperturbed bound on the third derivative and (19.37), we have

$$\|D_u^2 F_0(c, z_0 + S_0 u; \xi) - I\| \leq C_1 \|u\|_{\text{coeff}}.$$

Fix $u_0 > 0$ inside the joint product with $C_1 u_0 < 1/8$, and fix $\sigma_{0,J} > 0$ with $C_P \sigma_{0,J} < 1/8$ and $C_P \sigma_{0,J} r_c < u_0/8$. Then $u \mapsto u - D_u \mathcal{F}_\lambda$ is a coefficient contraction with factor at most $1/4$ on the domain of radius u_0 , and it maps the ball of radius $u_0/2$ into itself. Picard iteration gives a unique holomorphic joint stationary point, and from the fixed point inequality we obtain

$$\|u_\lambda\|_{\text{coeff}} \leq 2C_P \sigma_* r_c. \quad (19.43)$$

The stationary point is real at real data, since $\mathcal{F}_\lambda$ is real at real arguments and the stationary point is unique.

Step 6: initial segments in the original coordinates. To treat initial segments, we return to the original free coordinates by the invertible affine map $u = S_0^{-1}(z - z_0)$ at fixed c and ξ . The pullback is

$$\mathcal{Q}_\lambda(c, z; b) = F_0(c, z; \xi) + \lambda \bar{P} \left(\text{Log}[(\exp \Psi_R(c, z; \xi))^{G^\Omega(c)}]; b \right). \quad (19.44)$$

All subscripts referring to initial segments refer to this expression, with the original lists of scales. Because the affine map and its inverse have fixed coefficient bounds, (19.40) holds in the original coordinates with larger fixed constants. We take the solution domain of every initial segment from Lemma 19.3 small enough that its image under this affine map stays in the domain of radius u_0 .

At an unperturbed conditional stationary point $a_I(c, y; \xi)$, the preconditioned perturbed gradient is

$$A_I^{-1} \mathcal{Q}_{\lambda, I}(c, a_I, y; b) = \lambda A_I^{-1} V_I(c, a_I, y; b),$$

whose coefficient norm is at most $I_0 C_P \sigma_* r_c$. This holds uniformly for every $y \in \mathcal{T}_I$, not only at the joint stationary point, since all these conditional reference points lie in the

image of the same product under the reconstruction majorant. If δ_V is defined for the pullback of V as in Lemma 19.3, the same calculation yields

$$\delta_V \leq I_0 C_P \sigma_* r_c. \quad (19.45)$$

Across a conditional displacement of size r_I , the derivative bound is at most

$$\|A_I^{-1}(\mathcal{Q}_{\lambda, II} - A_I)\| \leq I_0(C_1 r_I + C_P \sigma_*). \quad (19.46)$$

We first take r_I with $I_0 C_1 r_I < 1/8$, and then reduce $\sigma_{0,J}$ so that, for every initial segment,

$$I_0 C_P \sigma_{0,J} < 1/8, \quad I_0 C_P \sigma_{0,J} r_c < r_I/8.$$

These inequalities are stronger than the two inequalities required in Lemma 19.3(2). We also require that the error of the real Hessian is less than $\lambda_J/8$. The coefficient resolvent condition is imposed at the allocated working rates of Step 4, using the unperturbed resolvents as references, as detailed in Step 8 below. The larger construction products, their radii, and the unperturbed half of the resolvent margin are fixed first; after J and these spatial rates are fixed, all perturbative constants in this condition are fixed before $\sigma_{0,J}$ is decreased and before the later widths and enclosing radii are chosen. Then Lemma 19.3 applies on each domain $\mathcal{T}_I$ of coarser coordinates, and its estimate (19.13) implies that, for every such y ,

$$\|a_{\lambda, I}(c, y; b) - a_I(c, y; \xi)\|_{\text{coeff}} \leq 2\lambda I_0 C_P \sigma_* r_c \leq C_J \sigma_* r_c. \quad (19.47)$$

This is the conditional bound of part (2), with one common $C_J \geq 2I_0 C_P$, and it holds before any moving coarser coordinates are substituted. The mixed coefficient bounds in the original coordinates follow from (19.41) and the affine inverse; together with the free inverse they verify (19.14) on the same products.

Step 7: successive substitution. The recurrence (19.15), with the conditional and matrix-function bounds just obtained, gives a constant $C_{\text{seq}, J}$ for a fixed rough bound on the size of the perturbation. Fix $Q_{0,J} > 0$ and, if necessary, reduce $\sigma_{0,J}$ further, so that

$$C_{\text{seq}, J}(2I_0 C_P \sigma_{0,J} r_c + Q_{0,J}) < \frac{1}{2} \varepsilon_{\text{tube}}, \quad Q_{0,J} \leq r_c,$$

where the conversion between the original and the normalized coordinates is included in the minimal margin. The bound (19.45) verifies the condition of Lemma 19.3(3) for every y . The recurrence bounds the substituted higher arguments and the constructed conditional centers. At each such argument, the Picard iterations stay in the fixed solution balls of Lemma 19.3(2); these balls were chosen inside the larger reconstruction product. Thus the hypothesis on the image under the reconstruction majorant holds throughout the construction. This proves the statements on stationary points in part (2), for all initial segments simultaneously, on the product of increments.

Step 8: the Hessians. The unperturbed real Hessian in the original coordinates is at least $7\lambda_J/8$ on these products, since its variation from the Hessian at the identity is at most $\lambda_J/8$ by Theorem 18.12, and the perturbation changes it by less than $\lambda_J/8$. Its principal blocks and Schur complements have the same lower bound, since for a symmetric matrix $H \geq \lambda I$

$$v^\top (H_{yy} - H_{yI} H_{II}^{-1} H_{Iy}) v = \min_x (x, v)^\top H(x, v) \geq \lambda |v|^2,$$

as in (18.6). After the normalization by $M^i \geq 1$, we obtain $K_{\lambda,i} \geq \lambda_J I/2$. The upper bounds on the tensors and on the inverses give a common upper bound Λ_J on the same product.

The complex resolvent condition is used at the working exponent b_w defined in part (4), with strict reserve for the output network and boundary weights. Its defining inequalities put it below b_{ref} and inside the transported incoming-rate allocation of Step 4, uniformly over every allowed output pair. For each flat reference K_i^b , including the factor M^i when present, the unperturbed bounds give

$$\|(K_i^b + t)^{-1}\|_{b_w} \leq \frac{C_{0,J}}{\lambda_J + t} \quad (t \geq 0).$$

Put $L_i = K_{\lambda,i} - K_i^b$. At each fixed coarser argument on the conditional products, the fixed-reference Picard bounds of Steps 5 and 6 give the coefficient displacement of the conditional center. For this bulk matrix estimate the offset calculation of Step 4 is used with $\beta_w = b_w L'_J$: since $b_w \leq c_J$, one still has $e^{\beta_w} \kappa \leq 2/3$, and $C_{\dim} b_w \leq \min(c_J, Ba_{\text{cl}})/2$ pays the transported network. Together with the unperturbed mixed tensor bounds, this bounds the perturbative matrix difference at b_w by $C_{P,b_w,J} \sigma_*$. For a Schur block, telescope its defining product and use the principal inverses already constructed. This estimate is obtained before successive substitution. Since $b_w < b_{\text{ref}}$, monotonicity preserves the unperturbed half-margin on the fixed products. Fix the perturbative constants at the working rates and then reduce $\sigma_{0,J}$ so that

$$C_{0,J} C_{P,b_w,J} \sigma_{0,J} \leq \lambda_J/8.$$

The two halves give $C_{0,J} \|L_i\|_{b_w} \leq \lambda_J/4$ at every stage. With $R_i^b(t) = (K_i^b + t)^{-1}$, the ordered Neumann expansion

$$(K_{\lambda,i} + t)^{-1} = \sum_{n \geq 0} [-R_i^b(t) L_i]^n R_i^b(t)$$

converges in the coefficient norm at this same working exponent and gives $4C_{0,J}/[3(\lambda_J + t)]$. The bounds with an analytic input fixed follow by inserting the differentiated tensor in this normally convergent series on a fixed smaller product. Thus the inverse uses the intrinsic decay of the unperturbed resolvent and the allocated working decay of the incoming perturbation.

Step 9: the differences carry the perturbation. It remains to prove that the differences carry the incoming boundary dependence, including through the moving coarser arguments. At fixed y , differentiating the gradient equation of an initial segment, we obtain

$$\partial_\lambda z_{I,\lambda} = -\mathcal{Q}_{\lambda,II}^{-1} V_I. \quad (19.48)$$

The free inverse just constructed is a coefficient series at the allocated working rates without boundary weight, and V_I is the boundary-localized series of Steps 3 and 4. These are joint coefficient bounds in the coarse field and the shifted higher parameters: the jet majorant of Step 3, its transported version of Step 4, and the differentiated Picard series give this joint statement on a fixed smaller product. We use these bounds for moving higher arguments; the fixed- y supremum defining δ_V alone is not a substitution estimate. The majorant of the product is (19.4), and it is $O_J(\sigma_* r_c)$; integrating in λ proves the bound on the difference of stationary points. Let Z_λ be the stationary point at which a Hessian is evaluated, and let H denote here this Hessian of $\mathcal{Q}_\lambda$. Its total derivative is

$$\dot{H} = V_{zz} + \mathcal{Q}_\lambda^{(3)}[\dot{Z}_\lambda],$$

which is $O_J(\sigma_*)$ and boundary-localized. For a set L of finer eliminated coordinates, the Schur block of scale i is $S = H_{ii} - H_{iL} R H_{Li}$ with $R = H_{LL}^{-1}$, and its derivative is

$$\dot{S} = \dot{H}_{ii} - \dot{H}_{iL} R H_{Li} - H_{iL} R \dot{H}_{Li} + H_{iL} R \dot{H}_{LL} R H_{Li}. \quad (19.49)$$

Every term has exactly one factor which carries the perturbation, including the term coming from the derivative of the inverse. Multiplication by M^i gives the derivative of $K_{\lambda,i}$. The derivative of the inverse is $-K^{-1} \dot{K} K^{-1}$, and (19.7), or its differentiated integral, gives the derivative of the inverse square root, with integrable majorant $C_J t^{-1/2} (\lambda_J + t)^{-2} \|\dot{K}\|$. These identities prove the stated bounds on all differences of matrix functions.

Along the successive chain, the derivative in (19.48) becomes

$$\dot{z}_I = -\mathcal{Q}_{\lambda,II}^{-1} (V_I + \mathcal{Q}_{\lambda,II^c} \dot{z}_{I^c}).$$

An inserted increment has total derivative

$$\frac{d}{d\lambda} [a_{\lambda,i}(c, z_{>i}) + K_{\lambda,i}(c, z_{>i})^{-1/2} q_i] = \partial_\lambda a_{\lambda,i} + D_y a_{\lambda,i} \dot{z}_{>i} + \frac{d}{d\lambda} K_{\lambda,i}^{-1/2} q_i,$$

where the last derivative includes both the explicit change of the objective and the change of the coarser argument. We start at the coarsest conditional stationary point and descend. Each direct derivative carries the perturbation, and each new contribution from the coarser arguments contains a factor which is already known to carry it. By (19.49) and (19.7) this remains true for every inverse square root used at the next scale. Since $\|q_i\|_\infty \leq Q_{0,J} \leq r_c$, the finite recurrence gives $O_J(\sigma_* r_c)$ for the stationary points and the

reconstructed displacements, and $O_J(\sigma_*)$ for their matrix coefficients. Differentiating in a coarse-field or free input fixes the index of that input in these convergent series, and the allocation of Step 4 keeps the perturbation and every external argument in one connected network. This proves part (3) on the product, and together with Step 4 it proves part (4).

Step 10: two perturbations. For two perturbations we interpolate between them. Since the coefficient bounds are convex and the fixed bound $\sigma_{0,J}$ holds for both, every interpolated objective stays in the same domains. Its derivative in the interpolation parameter is given by (19.48) with $V = (\bar{P}_1 - \bar{P}_0) \circ \Phi_0$. We set

$$\Delta_{\text{in}} = \mathcal{X}_{D_{\text{in}}, a_{\text{cl}}, \alpha_{\text{cl}}}^{r_c}(\bar{P}_1 - \bar{P}_0), \quad \phi_{\text{in}} \equiv r_c. \quad (19.50)$$

Since both perturbations use the same D_{in} , the same positions and cell volumes, and the same incoming boundary, all the preceding difference estimates are linear in Δ_{in} . We write $\mathcal{Q}_\lambda$ also for the interpolated objective. The derivative of the outgoing values in the interpolation parameter is

$$\partial_\lambda [M^s \mathcal{Q}_\lambda(c, z_\lambda; b)] = M^s(\bar{P}_1 - \bar{P}_0)(\Phi_0(c, u_\lambda; \zeta); b).$$

Its nonconstant coefficient networks contain the same incoming difference, also when a derivative hits the stationary point. To display the degree sum, denote the right side by G_λ and use a fixed outer profile $\psi \geq \zeta \phi_{\text{cl}}$, $\zeta > 1$, within the already chosen construction product. The distinguished reconstruction column and the transport estimate of Step 4 give

$$\sup_{i \in \Omega} e^{\alpha_{\text{out}} d_{\Omega}(i)} \sum_{\nu \neq 0} \nu_i |(G_\lambda)_\nu| e^{a_{\text{out}} t(\nu)} \psi^\nu \leq C_J r_c^2 \Delta_{\text{in}}.$$

At degree n , restriction to ϕ_{cl} supplies ζ^{-n} and the root fraction in (19.24) supplies $1/n$. Thus, uniformly in λ and the frozen values,

$$\mathcal{X}_{\Omega, a_{\text{out}}, \alpha_{\text{out}}}^{\phi_{\text{cl}}, \text{cl}}(G_\lambda) \leq C_J \Delta_{\text{in}} \sum_{n \geq 1} \frac{\zeta^{-n}}{n} = -C_J \Delta_{\text{in}} \log(1 - \zeta^{-1}).$$

The fixed profile ratio is used once for this scalar sum. Integrating in λ gives

$$\mathcal{X}_{\Omega, a_{\text{out}}, \alpha_{\text{out}}}^{\phi_{\text{cl}}, \text{cl}}(V_1^{\text{out}} - V_0^{\text{out}}) \leq K_J^{\text{cl}} \Delta_{\text{in}}. \quad (19.51)$$

Here the outgoing classical seminorm uses the output profile ϕ_{cl} with interior radius r_c . The seminorm in (19.50) is the incoming seminorm with constant profile. The output seminorm includes degree one. A difference of incoming constants stays in the full value and has zero seminorm. This proves part (5).

Step 11: the full action. We prove the identity of part (6), with its constants. When Ψ_R is written on its regional list, ι_ξ inserts the newly frozen coordinates once; when it is written as a complete fine field, the same insertion is already included. In either notation, the exact identities of Lemma 19.5 at the endpoints imply the equality of complete group fields

$$\exp(\iota_\xi \Phi_0) = (\exp(\iota_\xi A_0))^{G^{\Omega(c)}},$$

since the basepoint and the fine endpoints of every based transport of a newly frozen coordinate are protected, and the gauge transformation is the identity on the previously frozen extension. For real data both complete fields lie in the common chart of Theorem 11.6: the field before gauge transformation by the choice of R_{raw} , and the gauge-transformed field by the bound on the image under the reconstruction majorant and the bound on the newly frozen coordinates. The gauge transformation is periodic and compact there. Thus Theorem 11.6 applies to the torus classical action, and only to it, and we obtain

$$\begin{aligned} \mathcal{F}_1 &= E_{d_0}(\iota_\xi \Phi_0) + P(\Phi_0; b) - S_\xi^{\text{cl}}(\Phi_0) \\ &= E_{d_0}(\iota_0 \Phi_0) + P(\Phi_0; b) = E_{\text{in}}(\Phi_0; b). \end{aligned}$$

These equalities keep $S_\xi^{\text{cl}}(0)$ as well as all nonconstant coefficients of S_ξ^{cl} ; no constant Taylor coefficient has been subtracted. Both sides are holomorphic on the connected product, and they agree on its real open part. Expanding at a real interior point and differentiating the real identity in each coordinate, we find that all Taylor coefficients of the difference vanish. Continuation through overlapping polydiscs in the connected product gives the equality everywhere on it.

If the reference extension of the effective interaction is changed, the corresponding difference is treated by Lemma 18.9. In particular, replacing a resulting interaction $R(v, c)$ by $R(v, c) - R(0, c)$ changes the normalization constant from Λ to $\Lambda - R(0, c)$ and preserves the density; the constant of the extension is not deleted. The perturbation has been evaluated on Φ_0 throughout.

Step 12: dependence on the frozen coordinates. Fix the geometry, the finite lists of labels, and a strictly smaller analytic product, and suppose that the incoming coefficients and the charts are Borel functions of the real parameter b . A finite Taylor partial sum of any chart or incoming function is then jointly Borel in its analytic arguments and b . Since the common coefficient bounds on the larger radius give a geometric tail on the smaller product, uniformly in b , the limits of the series are jointly Borel. For the scalar incoming function this argument is applied to its part of positive degree, and the unrestricted coefficient $p_0(b)$ is kept as a separate Borel summand.

Every finite-dimensional matrix inverse used in a Picard iterate is a quotient of polynomials in its entries with nonzero determinant on the domain, hence Borel there,

and each Picard iterate is a composition of such operations and of the jointly Borel functions just obtained. The uniform contraction ratio bounds the distance of the iterates to the stationary point by a geometric sequence, uniformly in b , which implies that the stationary point is jointly Borel. This applies first to the joint problem and then to every conditional problem, at arbitrary coarser arguments in the domains, and the finite successive substitution of these maps is Borel.

For matrix functions we use (19.5) and (19.6). Finite Neumann sums are Borel, and their uniform geometric limit gives the resolvent. On a compact interval of integration inside $(0, \infty)$ the Riemann sums of the integral for the inverse square root are Borel, and the uniform bounds on the gap and the resolvent make the tails at zero and at infinity small, uniformly in the parameters; hence the integral is Borel. The same argument applies to the differences and to the logarithmic factors on their fixed nonvanishing branches. Evaluating the full objective at the stationary point gives Borel minimized values, with the incoming constant restored exactly.

If the coefficients and charts are continuous in b , every finite partial sum, inverse, and Picard iterate in this argument is continuous. The common geometric bounds and contraction ratios make all limits uniform on the smaller products, and the tails of the integrals satisfy the same uniform bounds. The limits and the finite successive compositions are therefore continuous in b . This proves part (7) without any derivative in ξ or b_{rec} .

Step 13: shorter compositions. For $s < J$ we replace B by B_s in the scaling and use the separate conditions on the radius and on the image under the reconstruction majorant for that composition; the proof for the initial segments is unchanged, with s in place of J . All constants were computed on fixed larger products, before the eventual free, enclosing, and geometric choices. $\square$

Fluctuation integration and normalization on block regions

In Sections 18 and 19 we set up the averaging steps of one renormalization group step on a block region. There, the coordinates attached to a union Ω of coarse blocks are analytic variables, and the coordinates on the fixed complement of Ω are frozen at real values chosen earlier. Those sections constructed the conditional constrained minimizers of the classical problem on Ω , proved a lower bound on the constrained Hessian independent of the shape of Ω , and controlled the effect of a small perturbation of the classical objective. The goal of this section is to integrate out the normalized fluctuation variables of such a step. We prove that the resulting effective interaction differs from the corresponding torus quantity only by terms that decay exponentially with the distance to the fixed complement, and we keep track of the normalization constant Λ by a separate argument. All estimates are uniform in the shape and topology of Ω , the number of its components, the preceding depth, the number of renormalization group steps already performed, and the values of the frozen coordinates.

Section 20.1 fixes the notation and the coefficient calculus. We then compare the conditional Hessians and Haar densities with their torus counterparts (Section 20.2) and derive the conditional fluctuation potential and its concentration bounds (Section 20.3). We obtain the effective interaction by a cluster expansion (Section 20.4) and control Λ by a separate comparison of real integrals (Section 20.5). Next, we choose the parameters and prove the finite iteration (Theorem 20.14). Finally, we sum the truncation errors for couplings in $(0, \Gamma]$ with any finite Γ (Section 20.7).

We use the norms and the dependence of constants of Section 4, the concentration estimates of Theorem 5.4, and the triangular change of variables and the finite graph inequality of Theorems 12.6 and 12.13. From Section 18 we use the norm with boundary decay, the network estimates, and the substitution identities, and from Section 19 the conditional domains, image bounds, perturbation theorem, and column bounds. The results are used in Section 21, which compares the integrals over the regions of the large-field decomposition with the analytic integrals constructed here, in Section 22, which proves the exponential-moment estimates for blocked actions, and in Section 23, which chooses the parameters.

All statements of this section concern analytic functions on stated domains and real integrals over stated products. We prove in Section 21 that the configurations of the large-field decomposition lie in these domains.

Heuristics. We first explain the orders of magnitude. We write the free logarithmic coordinates of one intermediate scale as $z = z^* + gK^{-1/2}w$, where z^* is the conditional constrained minimizer, K the constrained Hessian, g the coupling, and w the normalized fluctuation variable. Since the classical action enters the density divided by g^2 , and its expansion around the minimizer has no linear term and a quadratic term equal to $|w|^2/2$, the remaining classical terms are

$$g^{-2} O((z - z^*)^3) = O(g|w|^3).$$

The Haar disintegration density, the logarithmic determinants of the higher scales, and the incoming effective interaction are not divided by g^2 , and their dependence on w starts with a term of order $g|w|$. On fluctuation balls of radius $p \geq 1$, the interaction is therefore small when $g[p^3 + (1 + H)p]$ is small, where H bounds the incoming effective interaction; this is the smallness condition of Theorem 20.9. The terms of first order in g are odd under $w \mapsto -w$. Their expectations under a symmetric reference measure vanish, and the fluctuation correction to the effective interaction is of order g^2 . The conditional measure with the frozen coordinates held fixed is interacting and need not be symmetric. For this reason, we bound the means of the fluctuation variables separately in Lemma 20.5.

We control the normalization constant by two identities. First, a missing scalar Gaussian coordinate at an intermediate scale with coupling g_i removes one factor g_i from the density. This contributes $\log(1/g_i)$ to the difference between the regional and the torus normalization. Second, fixing coordinates at values c uses

$$e^{\Lambda - R(v,c)} = e^{\Lambda - R(0,c)} e^{-[R(v,c) - R(0,c)]},$$

which moves the value at $v = 0$ into Λ and leaves a centered function. Since the norms of the effective interaction vanish on constants, they cannot control Λ , and we estimate Λ by comparing real integrals.

Finally, fixing the coordinates of a boundary layer affects the boundary part of the effective interaction in two ways. First, the norm of the old boundary discrepancy is bounded by $2e^{-\alpha_{\text{bd}}(s_{\text{er}} - 1)}$ times its previous norm, where s_{er} is the width of the frozen layer. Second, fixing coordinates at nonzero values creates a new mixed term from the torus interaction, of size at most a constant times $E\beta_{\text{rec}}/\rho$, where E bounds the torus interaction and β_{rec} bounds the frozen values. The first estimate needs a wide layer, the second needs small frozen values, and neither replaces the other.

Attribution. The architecture used here, namely constrained minimization in a gauge-fixed chart, separation of a background field from normalized fluctuations, and integra-

tion of the fluctuations by a cluster expansion, is Balaban's renormalization group for lattice gauge theories [18, 19]. We attribute the individual tools where they are used.

Order of the proofs. The dependencies of the proofs are acyclic. Parts (1)–(2) of Lemma 20.5 use the coefficient, product, and curvature bounds of Lemma 20.3, the density sum (20.2), and Theorem 5.4; they determine the constant C_1 used in a hypothesis of Lemma 20.3. Part (3) refers forward to the independently proved endpoint summation lemma Lemma 20.15. The logarithmic gates are sufficient conditions for the explicit product inclusion used in parts (1)–(2), rather than inputs to that independent summation proof.

20.1 Setting and coefficient calculus

Geometry and labels. Fix an odd integer $M \geq 3$, an integer $J \geq 1$, and $B = M^J$. We consider a composition of s averaging steps, $1 \leq s \leq J$, from a lattice of side $M^s m$ to a lattice of side $m \geq 4$, together with an arbitrary union Ω of blocks of the output lattice. When $s = J$, we speak of a full step; it is one renormalization group step. The intermediate scales are numbered $i = 0, \dots, s - 1$. A label is a scalar coordinate: it specifies a direction, a color, an intermediate scale (or the output lattice), and the lattice on which the coordinate was introduced, and it has an assigned position. A fluctuation block is the three-component vector $w_{i,e} \in \mathbb{R}^3$ of normalized fluctuation variables attached to one free link e at one intermediate scale i ; it is a single variable of the reference measures below, not three independent ones. Scalar labels and fluctuation blocks are counted separately.

The labels attached to Ω are unfixed: their coordinates are analytic variables. These are the coarse background variable $v = \text{Log } c$ on the output lattice and the free logarithmic coordinates z_i at the intermediate scales. We write Ω^c for the fixed complement of the unfixed positions, W for the boundary (the interface between the unfixed positions and Ω^c), and $\Omega_{L_J}^c$ for the fixed enlargement of Ω^c determined by the construction stencils, as in Lemma 18.15. The fixed complement is denoted H in Section 18 and Section 19: in their notation, $\Omega^c = H$ and $\Omega_{L_J}^c = H_{L_J}$. In this section the letter H denotes instead the bound $E + \mathcal{X}^\rho(f)$ on the incoming effective interaction introduced in Lemma 20.7. All distances are graph distances on the refined position graph of Section 18.2, in units of the output lattice, and network lengths use its periodic ℓ^1 metric, with internal vertices allowed; positions of labels introduced on finer lattices are embedded in these units. We write d_g and t_g for this distance and network length, and d_E, t_E for the periodic Euclidean distance and network length of Definition 4.6. Then

$$d_E \leq d_g \leq \sqrt{3} d_E, \quad t_E \leq t_g \leq \sqrt{3} t_E,$$

since $|y|_2 \leq |y|_1 \leq \sqrt{3}|y|_2$ for $y \in \mathbb{R}^3$, applied to each periodic image and to each segment $[p, q]$ of a network, with $y = p - q$. Consequently, a coefficient bound stated with Euclidean distances and network lengths, at spatial exponent c and boundary exponent c' , holds with the graph quantities at exponents $c/\sqrt{3}$ and $c'/\sqrt{3}$. Conversely, a bound stated with the graph quantities implies the Euclidean bound at the same exponents. Every imported Euclidean coefficient bound is converted in this way before the rates of this section are chosen, and the fixed factor $\sqrt{3}$ is included in the geometric constants. Throughout this section, dist and $\mathbf{t}$ denote d_g and $\mathbf{t}_g$, as in Section 18.2. The torus seminorm $\mathcal{B}_{\rho, a_T}$ keeps the Euclidean convention of Definition 4.6. For an unfixed label i we write $d_\Omega(i)$ for its distance to Ω^c . We write Ω' for the set of unfixed scalar output labels. Separately, let $\partial_{\text{blk}}\Omega$ be the set of anchors of retained final blocks that share a face with a final block of the fixed complement. This is a geometric set on the output lattice, counted once per anchor, regardless of directions, colors, or the presence of an output link. In particular it includes an isolated retained block, even though that block has no internal coarse bond and contributes no label to Ω' . It is empty when Ω or its fixed complement is empty. We assume the label-density bound

$$\#\{y : R \leq \text{dist}(x, y) < R + 1\} \leq v_I(R + 2)^3,$$

where v_I accounts for directions, colors, intermediate scales, and the lattices on which labels are introduced.

Frozen coordinates. The frozen coordinates are written $b = (\xi, b_{\text{rec}})$. Here ξ consists of the coordinates frozen at the end of the preceding step, written in the lattice on which they were frozen, and b_{rec} consists of the older frozen coordinates, each on the lattice on which it was frozen. Throughout this section the frozen coordinates are real and fixed, and only the unfixed background and free variables are differentiated or continued to complex values. The supremum over the frozen values is taken after every coefficient sum and every degree sum. We claim no derivative of a perturbed quantity in ξ or b_{rec} .

Norms. For a function $f(v; b_{\text{rec}}) = \sum_\nu f_\nu(b_{\text{rec}})v^\nu$ of the unfixed labels, a radius $\rho > 0$, a positive profile $\phi \leq \rho$, a spatial exponent $a_{\text{sp}} > 0$, and a boundary exponent $\alpha_{\text{bd}} > 0$, we use the coefficient norm with boundary decay of Definition 18.1, which is the graph-metric variant of (4.11):

$$\mathcal{X}_\Omega^\phi(f) = \sup_{b_{\text{rec}} \text{ real}} \rho^{-2} \sum_{k \geq 1} \sup_{i \in \Omega} e^{\alpha_{\text{bd}} d_\Omega(i)} \sum_{|\nu|=k} \frac{v_i}{k} |f_\nu(b_{\text{rec}})| e^{a_{\text{sp}} \mathbf{t}(\text{supp } \nu)} \phi^\nu.$$

Here $\mathbf{t}(S) = \mathbf{t}_g(S)$ is the ℓ^1 length of a shortest connected network containing the positions in S , internal vertices allowed, and d_Ω is the graph distance, as in Definition 18.1. The supremum over the root i precedes the degree sum, and the supremum over the frozen values follows it. Constants and functions on an empty set of unfixed labels have norm

zero, and the degree-one term is included. When Ω^c is empty, a boundary difference has zero nonconstant part and norm zero. We write $\mathcal{X}^\rho$ for the norm with the uniform profile $\phi \equiv \rho$, and $\mathcal{X}^{\text{cl}}$ for the classical version, which has its own radius ρ_{cl} , profiles, and exponents. For a torus function r with $Dr(0) = 0$ we use

$$\mathcal{B}_{\rho,a}(r) = \sum_{k \geq 2} \frac{\rho^{k-2}}{k!} \|D^k r(0)\|_{a,\text{root}}$$

from Definition 4.6. This seminorm does not control the value $r(0)$, which is unrestricted. The three profiles are

$$\phi(i) = \begin{cases} \rho_\partial, & d_\Omega(i) < w_{\text{bd}}, \\ \rho, & d_\Omega(i) \geq w_{\text{bd}}, \end{cases} \quad \psi_{\text{prof}} = \sqrt{K_*} \phi, \quad \hat{\phi} = K_* \phi,$$

where $K_* \geq 2$ is a fixed analytic-radius enlargement factor, $0 < \rho_\partial \leq \rho$, and w_{bd} is the width of the boundary layer with the smaller radius. We call $\hat{\phi}$ the outer profile, ψ_{prof} the intermediate profile, and ϕ the working profile. A bound on a coefficient tensor is rooted at an external index if it holds with that index fixed and all other indices summed, where for an input index the volume-normalized column factor $a_{\text{out}}^3/a_{\text{in}}^3$ of Definition 4.6 is included. A coefficient bound with each external index fixed in turn is a bound rooted at every external index. All coefficient bounds concern absolute Taylor coefficients; we never substitute for them a bound on values of a holomorphic function or on an evaluated Hessian.

Admissible data. We work on the products of Lemmas 19.3 and 19.7: the conditional solution neighborhoods of the finitely many stages of the conditional minimization, the smaller neighborhoods of the higher parameters, and the product of sequential free coordinates contained in all of them. On these products, the images under the absolute reconstruction majorant satisfy strict bounds for the classical and quantum radii. A coarse background, a frozen value, or a product of fluctuation balls is called admissible if it lies in these products and satisfies these strict image bounds and the inequality on the conditional forces in Lemma 19.3. When fixing these analytic products, also fix $h_J > 0$ below the finitely many auxiliary convergence, implicit-function, resolvent, and logarithm radii used below. The outer background products and the sequential free products are restricted inside that reserve; in particular require

$$K_* \max(\rho, \rho_{\text{cl}}) < h_J, \quad Q < h_J \quad \text{for every sequential free radius } Q. \quad (20.1)$$

This restriction is made with the larger construction products, before the preliminary perturbation threshold and the output constants, rather than inferred later from a real Hessian bound.

Dependence of constants. Unless stated otherwise, every constant in this section (C_J , c_J , λ_J , radii, and bounds) has the dependence described in item (3) of Definition 4.14. It depends only on M , J , the fixed larger analytic domains and their radius ratios, the spatial and boundary exponents, the label density v_J , and the fixed torus and perturbation bounds. It does not depend on L , γ , m , N , K , the preceding depth d_0 , the number of renormalization group steps already performed, the shape or topology of Ω , the number of components of Ω or Ω^c , or the values of the frozen coordinates. It is also independent of the widths chosen later (the width of the boundary layer, the width s_{er} of the frozen layer, the widths of protected neighborhoods, and the size of the cells of the large-field decomposition): every constant is evaluated on the fixed larger domains before these widths are chosen. We assert no bound uniform as $J \rightarrow \infty$. Constants in statements about summed errors have the dependence described in item (4) of Definition 4.14.

Density sums and boundary transport. Spatial sums use the label density in units of the output lattice. For $b > 0$ we use

$$S_J(b) = C v_J \sum_{l \geq 0} (l+1)^3 e^{-bl}, \quad \sup_x \sum_y e^{-b \text{dist}(x,y)} \leq S_J(b). \quad (20.2)$$

The constant C accounts for directions, colors, and the finitely many intermediate scales. This is the periodic subset estimate of Lemma 18.15; restricting any of the sets of labels only decreases the sum. Kernels between different meshes keep the volume-normalized column factors of Definition 4.6. For an unfixed root x and a position u in $\Omega_{L_J}^c$ we have

$$d_\Omega(x) \leq d_\Omega(u) + \text{dist}(x, u) + C_J, \quad (20.3)$$

where $d_\Omega(u) = 0$ if $u \in \Omega^c$ and C_J accounts for the fixed enlargement. The weights for network length and for distance to Ω^c are both bounded by one fixed part of the available spatial exponent. The fixed stencil offset in (20.3) contributes a factor e^{bC_J} to the coefficient bound and no factor depending on a later width. We always use the version of Lemma 18.15 with every external index rooted. It performs all contractions with this single allocation of the exponent, independently of the number of vertices and of the Taylor order, instead of reducing an exponent at each contraction.

For a proper nonempty block region, we also use the boundary sum

$$\sum_{u \text{ unfixed}} e^{-c d_\Omega(u)} \leq C_J(c) |\partial_{\text{blk}} \Omega|, \quad c > 0. \quad (20.4)$$

Suppose first that Ω and its fixed complement are nonempty. A shortest route from an unfixed position to the fixed complement crosses an exposed face of a retained final block. The anchor of that block belongs to $\partial_{\text{blk}} \Omega$ and is within a fixed distance C_{face} of the crossing point, in output units. The constant includes the fixed convention for

link midpoints and endpoints. Thus each unfixed label u can be assigned to an anchor $a(u) \in \partial_{\text{blk}}\Omega$ with

$$\text{dist}(u, a(u)) \leq d_\Omega(u) + C_{\text{face}}.$$

The density sum (20.2), applied also at these anchor positions, gives

$$\sum_u e^{-cd_\Omega(u)} \leq e^{cC_{\text{face}}} \sum_{a \in \partial_{\text{blk}}\Omega} \sum_u e^{-c \text{dist}(u, a)} \leq e^{cC_{\text{face}}} S_J(c) |\partial_{\text{blk}}\Omega|.$$

An anchor need not itself carry a variable; the same density bound applies at every position of the refined graph, after a fixed enlargement of v_J if needed. This proves (20.4), without a regularity or component assumption on the boundary. If Ω is empty there is no label to sum. If its fixed complement is empty, boundary differences vanish and the bound is used only for those zero terms.

We also record the comparison with the interface labels used in the normalization count. For a fixed construction width $w_{\text{band}} \geq 1$, let N_{band} count the scalar labels of the regional step whose positions are within that width of W , and set it to zero when W is empty. Then

$$|\partial_{\text{blk}}\Omega| \leq N_{\text{band}} \leq C_J |\partial_{\text{blk}}\Omega|. \quad (20.5)$$

For the upper bound, a position within w_{band} of W is within $w_{\text{band}} + C_{\text{face}}$ of a boundary-block anchor. The label-density bound on this fixed-radius ball proves the claim. For the lower bound, choose one exposed face of each boundary block and a finest-scale M -block touching that face. There is a finest-scale free edge on the exposed face of this smaller block. Indeed, for the coordinate-ordered tree of Definition 18.6, edges in direction 1 are tree edges only on the line where coordinates 2 and 3 equal those of the root, and edges in direction 2 are tree edges only in the plane where coordinate 3 equals that of the root. On a face normal to direction 1, take an edge in direction 2 with coordinate 3 different from that of the root; on a face normal to direction 2 or 3, take an edge in direction 1 off the root line. Such edges exist since $M \geq 3$. Each lies inside a finest-scale block, so it is not an eliminated edge, which joins two such blocks. Its midpoint is within one input mesh unit, hence within $1/B_s \leq 1$, of a link of W . One fixed color component of this edge is therefore counted by N_{band} . The choices for distinct final blocks are distinct scalar labels. This gives the lower bound, including isolated final blocks. All constants depend only on M, J, v_J and the fixed construction width, not on any later width of an excluded or frozen neighborhood.

Two series. For $0 \leq q < 1$ and integers $j \geq 0$ we have

$$\sum_{k \geq 0} (k+1)(k+2)q^k = \frac{2}{(1-q)^3}, \quad \sum_{k \geq j} (k)_j q^{k-j} = \frac{j!}{(1-q)^{j+1}}, \quad (20.6)$$

where $(k)_j = k(k-1) \cdots (k-j+1)$. They follow by differentiating $\sum_{k \geq 0} q^k = (1-q)^{-1}$ twice and j times, respectively, on compact subintervals of $[0, 1)$. In particular a tensor of

order $k + 2$ with factorial bound $C^{k+2}(k + 2)!$ becomes, after division by $k!$, a summable series of degree- k Hessian coefficients. Additional fixed external indices contribute fixed powers of $k + 1$, which are absorbed by passing to one strictly smaller auxiliary product. No smaller radius is chosen at a later renormalization group step.

Coefficient resolution. The following lemma makes precise how the connected networks of coefficients behave under the operations used below. It combines the calculus of coefficientwise majorants of Lemma 5.3 with Definition 18.1. For a tensor V with output index $o(V)$ we write v_i for the cube of the mesh of the lattice of an index i , and $|V|_b$ for the maximum of its output-row norm and its volume-normalized input-column norms at spatial exponent b , as in Definition 4.6.

Lemma 20.1. *Consider a finite family of typed analytic coefficient tensors. Each tensor is supplied with a coefficient resolution whose total variation, weighted by the exponentials of the lengths of its assigned connected networks, satisfies the bounds of Definition 4.6 with each external index fixed in turn on one common larger product. Consider a sum of externally rooted connected contraction diagrams built from these tensors by substitution, contraction at shared indices (represented by equality tensors), finitely many Schur complements of positive constrained free matrices, and positive implicit or Neumann series. Products of factors with disjoint fluctuation supports and no shared index are not connected; they form separate connected components, each with its own network. Assume that every substitution $F \circ G$ is made on a product whose absolute image majorant for G lies strictly inside the coefficient domain of F , that all diagram majorants, including the mesh conversions of part (4), converge absolutely there, and that the inverses and Schur operations satisfy the imported uniform gap and coefficient Neumann conditions on that product. Then the following hold.*

1. *The sum has an exact coefficient resolution indexed by the occurrences of its constituent tensors, its contracted indices, and its derivative partitions. In each connected diagram component, the network of a resolved occurrence is the union of the assigned networks of its constituent occurrences, joined at the shared indices. This union is connected, contains every external position of that component, and*

$$\mathfrak{t}(\text{external positions of the component}) \leq \sum_{\text{constituent occurrences}} \ell(\text{assigned network}).$$

2. *The monomial of a resolved occurrence has an exact multi-index. Its fluctuation support is the set of whole fluctuation blocks on which its multi-index is positive. Background positions need not belong to this support, and no enlargement of the support to a connected geometric set is made.*
3. *If one constituent occurrence of a connected diagram component has a network that visits $\Omega_{L_j}^c$, then the union visits $\Omega_{L_j}^c$, and the visit lies in the same connected network as every external position of that component.*

4. Fix an external root of a connected diagram component. For a vector factor V attached to its parent at an input index j use the counting bound $A_V(j) = (v_j/v_{o(V)})|V|_b$, and if it is attached at its output use $A_V(o(V)) = |V|_b$; for a scalar factor use its counting-root bound. Then the sum over the resolution of the absolute coefficients, weighted by the exponentials of the network lengths, is bounded by the product of these bounds, multiplied at the end by the external normalization v_o/v_{root} when the component has an output o . For a substitution tree the volume ratios cancel exactly and the bound is the product of the norms $|\cdot|_b$, in the row norm and in every volume-normalized column norm. For other diagrams the remaining ratios are kept in the scalar majorant. In particular, a counting scalar outer tensor on an old mesh, rooted at a new input l , carries the factor v_l/v_{old} .
5. If a connected network of length ℓ contains the external positions S and the unfixed index i , and visits $\Omega_{L_1}^c$, then $t(S) \leq \ell$ and $d_\Omega(i) \leq \ell + L_0$, where L_0 is the fixed constant of the enlargement, and consequently

$$e^{at(S)+\alpha d_\Omega(i)} \leq e^{\alpha L_0} e^{(a+\alpha)\ell}.$$

Proof. We first work with finite Taylor polynomials and finite diagrams. For a substitution $U(o; \mathbf{l}_1, \dots, \mathbf{l}_k) = \sum_{j_1, \dots, j_k} T(o; j_1, \dots, j_k) \prod_r S_r(j_r; \mathbf{l}_r)$, the union of the network of T with the networks of the S_r , joined at the indices j_r , connects every external position, so

$$t(o, \mathbf{l}_1, \dots, \mathbf{l}_k) \leq t(o, \mathbf{j}) + \sum_r t(j_r, \mathbf{l}_r).$$

We take absolute values before summing. For an output row we sum the inner factors by their rows and then the outer tensor by its row. If a fixed external input belongs to the factor S_r , we sum the other inner factors by their rows, use the input column of S_r , and use the j_r -column of T . Along this path the volume factors multiply to $(v_o/v_{j_r})(v_{j_r}/v_l) = v_o/v_l$, which is exactly the normalization of the column of the composite at l . This proves the cancellation for substitution trees. Products with a common output are treated in the same way: the networks meet at the output, and an input-rooted estimate uses the column of the factor carrying the input and the rows of the others.

For a general connected diagram, we replace every edge outside a spanning tree by two independently summed copies of its index, which only increases the nonnegative sums. We orient the tree from the factor containing the fixed external root and, starting at the leaves, sum each factor over all its indices except the one attached to its parent. A factor attached at an input index j contributes at most its volume-normalized column bound times $v_j/v_{o(V)}$, since the column norm carries the factor $v_{o(V)}/v_j$; this is the counting bound $A_V(j)$. A factor attached at its output contributes its row bound. Finally summing the root factor over all its indices except the fixed root, we obtain the product of the counting bounds, with no count of the internal indices. The external normalization is applied only at the end. In a substitution tree the ratios on the path from the root

input to the output multiply to v_l/v_o and cancel against this normalization; in a general diagram they are kept. Connectedness is essential: factors with disjoint supports and no shared index have no tree, and their networks remain separate. A cycle without an external index, such as an undifferentiated trace, is not bounded in this way.

For compositions we use the ordered derivative formula

$$D^N(F \circ G)(0)[u_1, \dots, u_N] = \sum_{n=0}^N \frac{1}{n!} \sum_{(B_1, \dots, B_n)} F_n[G_{|B_1|}[u_{B_1}], \dots, G_{|B_n|}[u_{B_n}]],$$

where $G(0) = 0$, the inner sum is over ordered partitions of $\{1, \dots, N\}$ into nonempty blocks, and $F_n = D^n F(0)$, $G_k = D^k G(0)$. If $G(0) \neq 0$, we write $G = G(0) + \tilde{G}$ and apply this formula to $\tilde{G}$, with F_n replaced by $D^n F(G(0))$; the strict image condition justifies this translation and the infinite sums over the constant terms. Dividing by $N!$, we obtain the coefficient of s^N in $\hat{F}(\hat{G}(s))$, where $\hat{F}, \hat{G}$ are the absolute majorants. Fixing an external index chooses its branch and then its position in that branch, as in $(\hat{F} \circ \hat{G})' = (\hat{F}' \circ \hat{G})\hat{G}'$; no second Taylor factorial occurs. We define the resolution in part (1) by retaining a separate signed occurrence for every summand of this formula, every tuple of contracted indices, and every network refinement of the constituent tensors; grouping occurrences can only decrease the total variation. Its monomial is the product of the constituent monomials, which proves part (2). In a connected component the union of networks contains the visit of a constituent occurrence and is connected, which proves part (3). The diagram bound above, applied to every resolved occurrence, proves part (4) for finite truncations; the gap and Neumann conditions cover the Schur complements, Neumann series, and implicit series. Finally, by the image condition the nonnegative majorant sums converge on the common product. Tonelli's theorem then justifies every grouping, contraction, substitution, and derivative, and the resulting analytic functions agree because their coefficients agree.

For part (5), the network contains every external position, so $t(S) \leq \ell$. A route in the network from i to the visited position $h \in \Omega_{L_j}^c$ has length at most ℓ , and at most L_0 more length reaches Ω^c . We conclude that $d_{\Omega}(i) \leq \ell + L_0$, and adding the two inequalities with weights a and α completes the proof. $\square$

When we apply Lemma 20.1, we sum an internal index by keeping a positive part b of the connecting exponent and using (20.2). We convert a visit to $\Omega_{L_j}^c$ by (20.3) into decay in the distance of every unfixed root to Ω^c . In the applications below, the mixed-mesh contractions are substitutions, chains of matrices on a single lattice, and the composition with the incoming interaction, where the remaining ratio is the fixed factor $\Gamma_j \leq M^{3j}$ of Step 4 of the proof of Lemma 20.7; these factors are included in the common majorants.

20.2 Conditional minimizers, constrained Hessians, and Haar densities

We order the intermediate scales from the finest, $i = 0$, to the coarsest, $i = s - 1$. A stage of the conditional minimization is given by an initial segment I of the conditional integrations in this order. At that stage the free coordinates of the scales in I are minimized, with the coordinates $y = z_{I^c}$ of the remaining scales held fixed. The fluctuation increments in the following theorem are the coupling-scaled fluctuation inputs, $\zeta_i = q_i = g_i w_i$ in the notation of Section 19.

Theorem 20.2. *Fix an odd integer $M \geq 3$, an integer $J \geq 1$, $B = M^J$, the larger analytic domains and their strict margins, the label-density bound v_J , and the bounds for the classical and quantum torus norms. Use the conditional solution neighborhoods, the smaller neighborhoods of the higher parameters, and the product of sequential free coordinates of Lemma 19.3, and the perturbed objective of Theorem 19.9. In joint coordinates this objective is*

$$\mathcal{F}(v, u; b_{\text{rec}}) = F_0(v, z_0(v; \xi) + S_0(v; \xi)u; \xi) + \bar{P}(\Phi_0(v, u; \xi); \xi, b_{\text{rec}}), \quad S_0 = K_0^{-1/2}. \quad (20.7)$$

Here $v = \text{Log } c$ is the coarse background variable, z_0 is the unperturbed joint constrained minimizer, K_0 is its constrained Hessian, u is the joint normalized free variable, and Φ_0 is the unperturbed reconstruction followed by the gauge transformation $G^\Omega(c)$ of Lemma 19.5, which depends only on c and is the identity at all protected endpoints. The function $\bar{P} = P - S_\xi^{\text{cl}}$ is the incoming perturbation of the classical action with the classical boundary-completion difference S_ξ^{cl} subtracted, including its classical constant, as in Theorem 19.9. Assume

$$\mathcal{X}^{\text{cl}}(\bar{P}) \leq 2\sigma_*, \quad 0 < \sigma_* \leq \sigma_0,$$

where the preliminary perturbation bound $\sigma_0 > 0$ satisfies the margins of Lemma 19.3 for the preconditioned conditional forces, the Hessians, and the images. Here $\mathcal{X}^{\text{cl}}$ is the incoming classical norm on the incoming set D_{in} of unfixed labels, with the uniform profile ρ_{cl} , as in Theorem 19.9. Assume the inequality on the conditional forces of Lemma 19.3, and assume that the incoming fixed complement lies on the current fixed side, in the sense that for every current unfixed position e and every incoming label i ,

$$d_{D_{\text{in}}}(i) \geq B_s d_{\text{current}}(e) - B_s \text{dist}(e, i/B_s) - C_{\text{buf}, J}. \quad (20.8)$$

Here $B_s = M^s$, $d_{D_{\text{in}}}$ is measured in input units, $d_{\text{current}} = d_\Omega$, i/B_s is the position of i in output units, and $C_{\text{buf}, J}$ is a fixed buffer that includes the chart stencils. Assume the auxiliary convergence-radius restriction in the definition of admissible data, and assume finally that both images of the incoming functions under the absolute reconstruction majorant, for the classical radius and for the quantum radius, are strictly smaller than half of that radius, as in Lemma 19.7. Derivatives are taken at fixed real (ξ, b_{rec}) .

Then the following statements hold for every preceding depth $d_0 \geq 0$, every $1 \leq s \leq J$, every output side $m \geq 4$, every union Ω of output blocks, and every admissible real value of the frozen coordinates.

1. Every stage of the conditional minimization has a holomorphic conditional constrained minimizer, and every scale has a positive constrained Hessian $K_i^{\Omega, P}$, $0 \leq i < s$, which includes the exact factor M^i of its scale. Writing the unnormalized fluctuation increments as $\zeta_i = g_i w_i$, the free logarithmic coordinates at scale i are

$$z_i = z_i^*(c_{i+1}; b_{\text{rec}}) + K_i(c_{i+1}; b_{\text{rec}})^{-1/2} \zeta_i,$$

where c_{i+1} is the reconstruction from the coarser scales, with their fluctuation increments inserted. There are $Q, a_1, \alpha_1, \lambda_J > 0$ and $C_J, \mathcal{L}_{K, J} < \infty$ such that, on the outer profile $\hat{\phi}$ and for $|\zeta_{i,e}| \leq Q$, the real matrices satisfy

$$\frac{1}{2} \lambda_J I \leq K_i^{\Omega, P} \leq \mathcal{L}_{K, J} I, \quad \|(K_i^{\Omega, P} + t)^{-1}\|_{\text{row, col}, a_1} \leq \frac{C_J}{\lambda_J + t} \quad (t \geq 0). \quad (20.9)$$

The complex continuation satisfies the same resolvent bound with a fixed larger C_J .

2. Adjoin to the regional problem, as identical independent auxiliary variables with unit Hessian, the free labels of the matched torus problem that are absent from it, so that both problems have the same sets of free labels. Let $K_i^{\mathbb{T}, \text{cut}}$ be the restriction of the matched torus Hessian to the unfixed free labels, extended by the identity on the auxiliary labels and by zero on the mixed blocks. If

$$\Delta_i(v, \zeta; b_{\text{rec}}) = K_i^{\Omega, P} - K_i^{\mathbb{T}, \text{cut}} = \sum_{v, \kappa} \Delta_{i; v\kappa}(b_{\text{rec}}) v^v \zeta^\kappa,$$

then

$$\sup_{u \text{ unfixed}} e^{\alpha_1 d_\Omega(u)} \sum_{v', v, \kappa} |\Delta_{i; uv', v\kappa}(b_{\text{rec}})| \hat{\phi}^v Q^{|\kappa|} e^{a_1 \mathbf{t}(u, v', \text{supp } v, \text{supp } \kappa)} \leq C_J. \quad (20.10)$$

Each regional or torus quantity is evaluated along its own reconstruction chain, at matched retained data and with missing torus increments set to zero; the two higher arguments need not coincide. The transposed bound holds, and so do the bounds rooted at any unfixed background or free external index, with the volume-normalized column factors of Definition 4.6. The underlying mixed tensors with k additional background or free derivatives have bounds $C_J^{k+2} (k+2)!$ at every external index, on a fixed larger product. The auxiliary blocks have zero defect. If Ω^c is empty, the defect has zero nonconstant part.

3. The same rooted comparisons with boundary decay hold for the conditional minimizers of every stage, the inverse square roots, the logarithms of the Haar disintegration densities, and the diagonal densities $(\log K_i)_{xx}/2$, after composition with the reconstruction from every value of the coarser fluctuation variables. In particular, let L_i be the negative logarithm of

the Haar disintegration density at scale i after its field-independent normalization has been moved into Λ , and put

$$L_{0,i}(c_{i+1}) = L_i(c_{i+1}, z_i^*(c_{i+1})), \quad B_i = L_{0,i} + \frac{1}{2} \log \det K_i.$$

The fluctuation-dependent part of the density contains

$$\sum_{i=0}^{s-1} [L_i(c_{i+1}, z_i) - L_{0,i}(c_{i+1}) + B_i(c_{i+1}) - B_i(\hat{c}_{i+1})], \quad (20.11)$$

where $\hat{c}_{i+1}$ is the reconstruction with every fluctuation increment equal to zero. These differences start at degree one in ζ , whereas the classical remainders start at degree three. The comparison includes the correction $\log(T_{II}) - (\log T)_{II}$ that arises when the logarithm of a torus matrix T is restricted to the list I of unfixed labels. A torus diagonal density that is absent from the regional problem and has an unfixed external index has a network visiting Ω^c and is summable from that index; its constant part remains an explicit sum over omitted labels.

4. All constants have the dependence described in item (3) of Definition 4.14, including independence of the later widths. They are evaluated on the fixed larger products before the final radius for the free coordinates is decreased. In particular they do not change at the renormalization group steps following the creation of a new component of Ω^c .

Proof. Steps 1 and 2 verify that the incoming perturbation satisfies the hypotheses of Theorem 19.9 on all parameter domains. Steps 3 to 5 construct the conditional minimizers by a contraction argument that is uniform in the higher parameters, and check that every sequential substitution stays in the admissible domains. Steps 6 to 9 compare the regional and torus quantities, first without and then with the perturbation. Step 10 treats the matrix functions and the Haar densities, Step 11 the triangular determinant, and Step 12 the dependence of the constants on the frozen values.

We fix the real frozen value $b = (\zeta, b_{\text{rec}})$ and write $\mathcal{Q}_\theta = F_0 + \theta V$, where $V = \bar{P} \circ \Phi_0$ and $0 \leq \theta \leq 1$; here θ is an interpolation parameter. Returning from u to $z = z_0 + S_0 u$ is an invertible affine change of variables at fixed v . In the free logarithmic coordinates, the objective is

$$\mathcal{Q}_\theta(v, z) = F_0(v, z; \zeta) + \theta \bar{P}(\text{Log}[(\exp \Psi_R(v, z; \zeta))^{G^\Omega(v)}]; \zeta, b_{\text{rec}}),$$

where Ψ_R is the reconstruction before the gauge transformation. We use only this change of variables; we do not need the incoming perturbation to take the same value at the reconstruction before and after the gauge transformation.

Step 1: the incoming hypotheses. The incoming analytic labels form the set D_{in} , with their positions on the lattices where they were introduced and their volume factors. On

this set the incoming profile is uniformly ρ_{cl} ; the coordinates ξ are fixed parameters, and the spatially varying profiles are output profiles. The strict bounds on the image under the reconstruction majorant place Φ_0 , on every admissible stage, inside this uniform input ball with a margin of half the radius. The same holds for every segment between two compared images, since coefficientwise a convex combination is bounded by the larger of the two absolute image bounds. In particular, the input jet bounds of Theorem 19.9 apply before any conditional minimization, on all its parameter neighborhoods.

Step 2: the incoming boundary comparison. For example, the inclusion of fixed complements used in Theorem 19.9 implies (20.8) as follows. Let h be an incoming fixed position nearest to i . A path from e to the current fixed complement can first go to i/B_s , then follow the scaled image of a shortest path from i to h , of length $d_{D_{\text{in}}}(i)/B_s$, and finally cross the fixed buffer between the incoming and the current fixed complements, whose length is included in $C_{\text{buf},J}/B_s$. Multiplying by B_s , we obtain (20.8). This argument uses no regularity or component assumption on the boundary. We keep (20.8) as a hypothesis on the geometry; choosing small radii does not establish it.

In particular, if $\alpha_{\text{out}} \leq B_s \alpha_{\text{cl}}$ and a chart branch has spatial rate $b_{\text{ch}} > \alpha_{\text{out}}$, then

$$\begin{aligned} & e^{\alpha_{\text{out}} d_{\text{current}}(e)} e^{-\alpha_{\text{cl}} d_{D_{\text{in}}}(i)} e^{-b_{\text{ch}} \text{dist}(e, i/B_s)} \\ & \leq e^{\alpha_{\text{out}} C_{\text{buf},J}/B_s} e^{-(\alpha_{\text{cl}} - \alpha_{\text{out}}/B_s) d_{D_{\text{in}}}(i)} e^{-(b_{\text{ch}} - \alpha_{\text{out}}) \text{dist}(e, i/B_s)}. \end{aligned}$$

This is the transport of the input boundary weight used in the perturbation bound. Input network lengths are divided by B_s when expressed in output units. Column normalizations remain the ratios of the cubed mesh sizes of their output and input labels. Local rates may be chosen below the finite minimum of these bounds over $1 \leq s \leq J$; the rates returned after a full step use $s = J$ and $B_s = B$. If there is no incoming boundary, the incoming nonconstant boundary function is zero.

Step 3: conditional minimizers. For each stage I (an initial segment of the conditional integrations, with $y = z_{I^c}$ fixed), let $a_I(v, y)$ be the unperturbed conditional minimizer. By Lemma 19.3, there exist a fixed solution neighborhood, a smaller neighborhood of the higher parameters, and an invertible matrix $A_I = F_{0,II}(v, a_I, y)$. Let I_0 be a common bound for the coefficient norms of the inverses A_I^{-1} on the fixed larger neighborhoods. Using the uniform incoming jet bound with $j = 1$ and the fixed rows of the chart derivatives, we obtain, at every admissible higher parameter y and not only at the joint minimizer,

$$\|V_I(v, a_I, y; b)\|_{\text{coeff}} \leq C_P \sigma_* \rho_{\text{cl}}, \quad \delta_V \leq I_0 C_P \sigma_* \rho_{\text{cl}}, \quad (20.12)$$

where δ_V is the bound on the preconditioned conditional forces of Lemma 19.3. The constant C_P is evaluated on the larger products at the preliminary bound σ_0 and includes the conversion from z to u . The jet bound with $j = 2$ and the chain rule, including the

term $\bar{P}'\Phi_{0,zz'}$, give a uniform bound $C_p\sigma_*$ on the mixed second derivatives. The real gap, the complex coefficient margins, the inequality $\delta_V \leq r_I/4$, where r_I is the radius of the solution neighborhood, and the derivative inequality below are then the hypotheses of Lemma 19.3. On that neighborhood the map

$$h \longmapsto h - A_I^{-1} Q_{\theta,I}(v, a_I + h, y)$$

has derivative norm at most $1/2$ and value at zero of norm at most $r_I/4$. Hence it maps the closed ball of radius $r_I/2$ into itself, and its successive iterates converge: the difference of the iterates $n+1$ and n is at most 2^{-n} times the first difference. The same bounds on absolute coefficients hold for the analytic iterates, which gives normal convergence on every smaller product and a holomorphic limit. Uniqueness follows by subtracting two fixed point equations. Since the objective is holomorphic and real at real arguments, the minimizer is real at real data. The real Hessian and complex resolvent margins are those of Theorem 19.9; its weighted resolvent bounds rest on the exponential conjugation method of Combes and Thomas [61]. Taking the finite minimum over $s \leq J$ and over the stages fixes λ_J and Q , and taking the finite maximum fixes $\mathcal{L}_{K,J}$ and the bound on the inverses. The factors M^i belong to these matrices before the extrema are taken.

Step 4: sequential substitution. For the sequential construction we use the inequality

$$C_{\text{seq},J}(2\delta_V + Q) < \frac{1}{2}\varepsilon_{\text{tube}}, \quad (20.13)$$

where $\varepsilon_{\text{tube}}$ is the minimal margin of the parameter and solution neighborhoods of Lemma 19.3. Let $z_\theta(v)$ and $z_0(v)$ be the perturbed and unperturbed joint stationary points. For the initial segment of all free coordinates there is no higher argument, so the contraction bound gives $\|z_\theta - z_0\|_{\text{coeff}} \leq 2\theta\delta_V \leq 2\delta_V$. The tube gate places every projection of z_θ inside the corresponding parameter and solution domains. Writing $a_{\theta,I}$ for the perturbed conditional minimizer, joint stationarity and conditional uniqueness therefore give $a_{\theta,I}(v, z_{\theta,I^c}) = z_{\theta,I}$. The joint mixed coefficient bounds and the bound on the free inverses give a finite common Lipschitz constant D_J of the conditional minimizers in y , and $\|K_i^{-1/2}\| \leq C_{S,J}$. Let Δ_i^{seq} bound the deviations of the already reconstructed coarser coordinates from z_θ , in the joint coefficient algebra of the background and the remaining increments. Integrating $D_y a_{\theta,I}$ from z_{θ,I^c} to the moving higher argument uses its joint mixed-derivative majorant on the larger product. The coefficient displacement of z_θ permits this recentering on a fixed smaller product; enlarge D_J once to include the composition. The next insertion then satisfies the majorant inequality

$$\Delta_{i-1}^{\text{seq}} \leq (1 + D_J)\Delta_i^{\text{seq}} + C_{S,J}Q, \quad \Delta_s^{\text{seq}} = 0. \quad (20.14)$$

Iterating and adding the joint-center displacement, we find that the total deviation from z_0 is at most $2\delta_V + C_{S,J}Q \sum_{k=0}^{J-1} (1 + D_J)^k$. We choose $C_{\text{seq},J}$ larger than $\max\{1, C_{S,J} \sum_{k=0}^{J-1} (1 +$

$D_I)^k\}$ and include every fixed embedding constant of the stages. Then (20.13) keeps every substituted coarser argument in the admissible domains, even after the deviation is doubled. The mixed bounds and the constant $C_{\text{seq},J}$ are fixed before the later widths. In particular the preliminary bound σ_0 must satisfy the same inequality with $\delta_V \leq I_0 C_P \sigma_0 \rho_{\text{cl}}$.

Step 5: dependence on the coarser coordinates. We explain why this construction controls coefficients when the coarser coordinates move. Differentiating the stationary equation at fixed y , we obtain

$$\partial_\theta z_I = -Q_{\theta,II}^{-1} V_I. \quad (20.15)$$

If y moves, the right side has the additional term $-Q_{\theta,II}^{-1} Q_{\theta,II^c} \partial_\theta y$. A sequential insertion also has the term $(\partial_\theta K_i^{-1/2}) \zeta_i$. All these terms are included in Theorem 19.9; the joint mixed-majorant argument leading to (20.14), together with the joint-center displacement, keeps their evaluations at the moving coarser coordinates inside the same admissible products. At zero free input, conditional uniqueness identifies the sequential chain with z_θ . Decreasing Q therefore does not change the admissible profile of the coarse background.

Step 6: the unperturbed boundary comparison. We use matched reconstruction formulas on the unfixed labels. Away from the fixed construction enlargement of Ω^c , the formulas defining the common unfixed outputs, the block axial gauge equations, and the choices of eliminated edges coincide. Their unique admissible solutions on the unfixed outputs therefore coincide there. This does not assert that the extended fields agree inside a component of Ω^c . The prescribed values in Ω^c are separate bounded sources; a source deep inside a component of Ω^c need not be small at its own position. For the torus action composed with two reconstructions $\Psi_R = \Psi_{\mathbb{T}} + d$, the exact difference is

$$F_R - F_{\mathbb{T}} = \int_0^1 DE_{d_0}(\Psi_{\mathbb{T}} + \tau d)[d] d\tau \quad (20.16)$$

on the common unfixed variables. If auxiliary missing variables are kept, their separate auxiliary quadratic form is kept as a term rooted in Ω^c . Every differentiated coefficient of the integral contains either a derivative of a changed formula or the factor d attached to an action tensor. The action and chart tensors on the larger domain are the tensors of Theorem 19.9, and their contractions with a changed formula are networks covered by Lemma 18.15. Rooting at any unfixed external index connects that index to the change. For example, the second derivative is

$$D_z^2(E_{d_0} \circ \Psi) = \Psi_z^T E_{d_0}''(\Psi) \Psi_z + E_{d_0}'(\Psi) \Psi_{zz}.$$

Both terms are present: stationarity in z does not make $E_{d_0}'(\Psi)$ vanish. Differentiating (20.16) produces derivatives falling on E_{d_0} as well as derivatives falling on d ; the former

remain connected to the inserted change. Summing a bounded source in Ω^c from an unfixed root is justified by (20.2) and (20.3). This proves factorial bounds with a visit to Ω^c for the difference before minimization, rooted at every external index.

Step 7: minimization, Schur complements, and restriction. We pass to the conditional minimizers through their stationary equations. A force carrying a visit to Ω^c is multiplied by the free inverse, as in (20.15). The coefficientwise majorant series of the implicit equation converges on the admissible products: if Y denotes the majorant of the solution without the visit, the variation carrying the visit has the form $H(x, Y)/(1 - A(x, Y))$, where $A \leq 3/4$ after the auxiliary radius has been fixed. This follows by substituting the majorant of the variation into the differentiated fixed point equation and summing $\sum_{n \geq 0} A^n$. The visit and all fixed external indices are kept. For a partial minimum the exact second derivative is the Schur complement

$$\mathcal{Q}_{yy} - \mathcal{Q}_{yI} \mathcal{Q}_{II}^{-1} \mathcal{Q}_{Iy}. \quad (20.17)$$

We subtract two such expressions by replacing the three factors of the product one at a time. An inverse difference is $A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$, so each summand keeps a factor with a visit to Ω^c . Only constrained free blocks are inverted. Positivity passes to the Schur complement by minimizing the positive quadratic form over the coordinates in I .

A separate correction compares restriction with elimination. Let A be a single torus free matrix, and let R and H be the lists of unfixed and of missing labels (in this paragraph R and H denote lists of labels). Block elimination gives

$$(A^{-1})_{RR} - A_{RR}^{-1} = A_{RR}^{-1} A_{RH} (A_{HH} - A_{HR} A_{RR}^{-1} A_{RH})^{-1} A_{HR} A_{RR}^{-1}. \quad (20.18)$$

To check this identity, we solve $Ax = (f_R, 0)$, first for x_R in terms of x_H and then for x_H . Each term on the right of (20.18) passes through a missing index. Its coefficient network from an unfixed root therefore reaches Ω^c , and the part of the exponential decay not used for the network weight sums over that index. Adding this correction to the regional comparison, we obtain the unperturbed defect at each stage. This operation compares restrictions of one torus matrix; it does not identify a regional problem with a torus problem whose complement is set to zero.

Step 8: from factorial bounds to coefficient bounds. The bounds just obtained for the tensors of order $k + 2$, at every external index, are $C_J^{k+2}(k + 2)!$. Dividing by the Taylor factorials and using (20.6), we find that on a fixed smaller auxiliary radius h the summed matrix coefficients are at most

$$C_J^2 \sum_{k \geq 0} (k + 1)(k + 2)(C_J h)^k = 2C_J^2(1 - C_J h)^{-3}.$$

We choose the outer background and free products inside this radius and do not change them later. Since any index may be fixed before the sums are taken, we may fix a background or free input, which proves the column versions.

Step 9: adding the perturbation. We insert the unperturbed regional matrix at its own chain of coarser arguments. The defect in (20.10) is the sum of three terms: the change of the objective and of the minimizer at a common coarser argument, the change of that argument, and the unperturbed regional–torus defect. The first term has the rooted perturbation bound of Theorem 19.9, with its visit to Ω^c . The second term is the integral of an unperturbed matrix derivative against the difference of the coarser reconstructions, which carries a visit to Ω^c ; both are defined on the same products by the statement on the sequential product in Lemma 19.3, and the network joins the new index to the visit. The third term was treated in Steps 6 to 8. The torus quantities follow their own chain throughout. Differences of the intermediate arguments are included by the reconstruction comparison of Theorem 19.6 and by the sequential chain argument of Steps 4–5, with zero missing increments and matched retained data. The sum satisfies (20.10) for fixed smaller $a_1, \alpha_1 > 0$. The real matrices are symmetric, which gives the transposed bound with the same constants. Rooting each contraction at the specified background or free index gives the other bounds, with its volume factor. The auxiliary blocks are identical throughout and have zero difference; no matrix element rooted purely at an auxiliary label is assigned a boundary decay.

Step 10: matrix functions and Haar densities. For positive matrices, integrating the scalar resolvent and applying the resolvent identity, we obtain the ordered identities of holomorphic functional calculus (see [27])

$$A^{-1/2} - B^{-1/2} = -\frac{1}{\pi} \int_0^\infty t^{-1/2} (A+t)^{-1} (A-B) (B+t)^{-1} dt, \quad (20.19)$$

$$\log A - \log B = \int_0^\infty (A+t)^{-1} (A-B) (B+t)^{-1} dt. \quad (20.20)$$

Indeed $A^{-1/2} = \pi^{-1} \int_0^\infty t^{-1/2} (A+t)^{-1} dt$ and $\log A = \int_0^\infty [(1+t)^{-1} I - (A+t)^{-1}] dt$ hold for positive numbers, hence for real symmetric positive matrices by orthogonal diagonalization. The common complex resolvent margins extend these identities holomorphically. The coefficient majorants of the integrands are $C_J t^{-1/2} (\lambda_J + t)^{-2}$ and $C_J (\lambda_J + t)^{-2}$, which are integrable at both endpoints. We may therefore integrate all contractions and derivatives termwise. Each difference keeps its factor $A - B$.

The logarithm of the Haar disintegration density is a sum of logarithms of the non-vanishing Haar factors of finitely many local formulas and of the determinants of the constraints at the eliminated edges, as in (4.4) and Lemma 18.7. On the fixed chart we write each factor as $a(1+h)$, with its nonzero value a at the base point and a fixed strict coefficient margin $|h| < 1$; the series $\log(1+h) = \sum_{k \geq 1} (-1)^{k+1} h^k / k$ gives its absolute

derivatives on every smaller chart. The base factors are moved into Λ . Away from the boundary the local formulas agree. In a difference of compositions, a changed local formula, or a changed minimizer, inverse square root, or coarser argument, is a factor carrying a visit to Ω^c . The preceding calculus therefore yields the comparison of the Haar densities at all orders.

A diagonal matrix-logarithm density keeps its index until an external unfixed index is chosen. Its nonconstant derivatives are chains of resolvents with a differentiated matrix inserted. Rooting at that insertion makes the trace sum summable by (20.2). If a torus diagonal density is missing from the regional problem, its network with any unfixed external index must reach the missing index, and the same argument sums over the missing index from the unfixed index. A pure constant has no such index and remains its exact sum over omitted labels.

To compare with the restriction of the torus logarithm, we set $T = K_i^{\mathbb{T}}$ and $C = T_{II} \oplus T_{HH}$, where $I = I_i$ and $H = H_i$ are the lists of unfixed and missing free labels at scale i . Using the integral representation of the logarithm, we obtain

$$\log T_{II} - (\log T)_{II} = \int_0^\infty [(T+t)^{-1} - (C+t)^{-1}]_{II} dt. \quad (20.21)$$

The integrand equals $-[(T+t)^{-1}(T-C)(C+t)^{-1}]_{II}$. Its middle factor connects unfixed and missing labels, so it has a visit to Ω^c , and it has the same integrable bound as (20.20). Thus this correction, composed with every value of the coarser fluctuation variables, is included in the comparison.

Step 11: the triangular determinant. The determinant of the triangular change of variables of Theorem 12.13, or of the identical finite triangular calculation for the present minimizers, is

$$\prod_i g_i^{3f_i} \det K_i(c_{i+1})^{-1/2},$$

where f_i is the number of free blocks at scale i . Adding and subtracting the logarithm of its value at zero fluctuation, we combine the Haar terms into (20.11). Only one determinant occurs. For an analytic function G on the common product,

$$G(v, \zeta) - G(v, 0) = \int_0^1 D_\zeta G(v, \tau\zeta)[\zeta] d\tau. \quad (20.22)$$

This keeps every positive degree in ζ . Stationarity and the exact normalization by $K_i^{-1/2}$ cancel the terms of degrees one and two of each classical remainder, so its degree is at least three. Composition with the coarser scales does not lower that degree.

Step 12: constants and measurability. The proof uses only margins on fixed domains, finitely many operations on the stages, (20.2), normally convergent coefficientwise majorant series, and the resolvent integrals above. These give the dependence stated in part

(4). A new component of Ω^c changes the set of indices carrying a visit and the frozen values, but no constant. If Ω^c is empty, no formula changes and there is no input carrying a visit. This proves the stated special cases and uniformities.

We also record the regularity in the frozen values needed for finite iterations. If the incoming coefficients and the charts depend on the real value b in a Borel way, then so does every iterate of the contraction in Step 3: finite sums, products, and admissible inverses preserve this property, and a convergent coefficient composition is a limit of its polynomial truncations. The geometric contraction bound is uniform in b , so the coefficients of the limit are Borel. The resolvent integrals for square roots and logarithms are limits of finite Riemann sums with uniformly bounded tails, and preserve this property. With continuous incoming data on a fixed branch, the same uniform convergence gives continuity. None of these arguments differentiates in b . In the finite analytic recursion, the initial torus functions and the local formulas in the identity chart have this regularity, and substitution of a frozen value and the integrals over fixed products below preserve Borel dependence. Thus every coefficient family constructed by the recursion depends on the frozen values in a Borel way. $\square$

20.3 The conditional fluctuation potential

Lemma 20.3. *Use the data and products of Theorem 20.2. Let the incoming effective interaction be*

$$R = r_{d_0}(\cdot \oplus 0) + f, \quad Dr_{d_0}(0) = 0, \quad \mathcal{B}_{\rho, a_{\mathbb{T}}}(r_{d_0}) \leq E, \quad \mathcal{X}^p(f) \leq \sigma_* \leq \sigma_0,$$

where r_{d_0} is the torus effective interaction at depth d_0 , with its torus symmetries, evaluated with the fixed slots set to zero, and f is its boundary part. The value of r_{d_0} at zero is unrestricted. For $0 \leq i < s \leq J$ set

$$t_{\max} = \max_i g_i = \sqrt{\gamma/(Mm)}, \quad g_i = t_{\max} \lambda_i, \quad \lambda_i = M^{(i-s+1)/2} \in [\lambda_*, 1],$$

where $\lambda_* = M^{-(J-1)/2}$. Let F_i be the partially minimized objective at scale i , without the coupling factor and with its exact normalization M^i , and define

$$T_i(c_{i+1}, \zeta_i) = F_i(c_{i+1}, z_i^* + K_i^{-1/2} \zeta_i) - F_i(c_{i+1}, z_i^*) - \frac{1}{2} |\zeta_i|^2.$$

For a fixed admissible real coarse background and fixed real frozen values, the conditional fluctuation potential, with its values at zero fluctuation moved into the prefactor, is exactly

$$\begin{aligned} V(w) = & \frac{1}{2} \sum_i |w_i|^2 + \sum_i g_i^{-2} T_i(c_{i+1}, g_i w_i) \\ & + \sum_i [L_i(c_{i+1}, z_i) - L_{0,i}(c_{i+1}) + B_i(c_{i+1}) - B_i(\widehat{c}_{i+1})] \\ & + R(\text{Log } C(v, (g_i w_i)_i; b_{\text{rec}})) - R(\text{Log } C(v, 0; b_{\text{rec}})), \end{aligned} \quad (20.23)$$

where $C(v, \zeta; b_{\text{rec}})$ denotes the reconstructed configuration on the unfixed labels. Only logarithms of unfixed labels in the identity chart are taken. The Haar disintegration density, the degree-one terms of the incoming interaction, and all determinant terms of the coarser scales occur in (20.23). The separate prefactor contains all constants of the block averaging constraint and of the Haar measure, the Gaussian powers of the couplings, the values at the minimizers, and the external compact factors.

There are $A \geq 1$, $a' > 0$, and $C_{V,J}, C_0 < \infty$ with $C_0 \geq 2A^3$, and a fixed radius $s_W > 0$ for the increments ζ_i , such that on the product $|g_i w_{i,e}| \leq s_W$

$$V(w) = \frac{1}{2}|w|^2 + \sum_{k \geq 3} t_{\max}^{k-2} P_k(w) + \sum_{k \geq 1} t_{\max}^k Q_k(w), \quad \|P_k\|_{a',1} \leq A^k, \quad \|Q_k\|_{a',1} \leq \mathcal{L}_V A^k, \quad (20.24)$$

where P_k, Q_k are homogeneous polynomials of degree k in the scalar coordinates of w and $\mathcal{L}_V = C_{V,J}(1 + E + \sigma_0) \geq 1$. For $T_k(w) = \sum_{|\eta|=k} t_\eta w^\eta$ the norm is

$$\|T_k\|_{a',1} = \sup_j \sum_{\eta_j > 0} \eta_j |t_\eta| e^{a' t(\text{supp } \eta)}.$$

The bounds (20.24) also hold before the background is evaluated, in the coefficient algebra at the outer profile, with the background positions included in the network length.

The following inequalities suffice for a common enclosing product. All constants below are computed at the preliminary bound σ_0 , before σ_* , β_{rec} , r , and the later widths are fixed. Let $r_{q,U}, r_{q,P}, u_0$ be the radii of the unperturbed and perturbed sequential free products and of the joint product of Lemma 19.3, and $c_{\text{tube},J}$ its fixed conversion constant. For $\rho_\bullet \in \{\rho, \rho_{\text{cl}}\}$ let $R_{\text{raw}}, C_{\text{lift},J}, s_{\text{aux},\bullet}, C_{T,\bullet}, C_{\Delta,\bullet}$, and $A_{0,\bullet}$ be the radius of the larger construction domain, the bound on the lifts, the auxiliary radii, the constants of the image increments, and the bound on the image at zero fluctuation in Lemma 19.7. Choose $R_W > 0$ strictly below

$$\min\{s_W/2, r_{q,U}/2, r_{q,P}/2, c_{\text{tube},J}u_0, R_{\text{raw}}/(64C_{\text{lift},J}), 1/(4A), 1/(16C_0), \\ \min_{\rho_\bullet \in \{\rho, \rho_{\text{cl}}\}} \{s_{\text{aux},\bullet}/2, 1/(4C_{T,\bullet}), \rho_\bullet/[8(2C_{T,\bullet} + C_{\Delta,\bullet})]\}\}.$$

Impose the margins of the larger construction domain and of the stages in Lemma 19.7 and, for both radii, $A_{0,\bullet} \leq \rho_\bullet/4$ and $C_{T,\bullet}K_*\rho_\bullet \leq 1/4$. Let $C_{\text{en},J} \geq 1$ be a fixed constant (in the application it is the constant of the enclosure estimate Theorem 21.9; nothing else from that section is used), and require

$$r, \beta_{\text{rec}}, \sigma_*\rho_{\text{cl}} \leq \frac{\lambda_* R_W}{6C_{\text{en},J}}, \quad \delta_{\text{en}} = C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_*\rho_{\text{cl}}), \quad R_{\text{en}} = 2\delta_{\text{en}}/\lambda_*, \quad (20.25)$$

where r is the radius of the small-link domain B_r and β_{rec} is the bound on the frozen values. Then the whole product

$$\mathcal{K}_{\text{en}} = \prod_{i,e} \{|w_{i,e}| \leq R_{\text{en}}/t_{\max}\}$$

is admissible. On it the absolute coefficient increment of either incoming image is bounded by

$$\frac{C_{T,\bullet} R_{\text{en}}}{1 - C_{T,\bullet} (K_* \rho_\bullet + R_{\text{en}})} + C_{\Delta,\bullet} R_{\text{en}} \leq \rho_\bullet / 8,$$

the denominator is at least $1/2$, and the image under the absolute reconstruction majorant at spatial exponent zero is at most $3\rho_\bullet/8$. If these bounds are imposed in a weighted coefficient row, the same bound holds in that row. In the alternative provided by Lemma 19.7, the unperturbed image row $S_c^0 \leq C_J \rho_\bullet$ at an exponent $c > 0$, larger than the working exponent, gives one fixed positive exponent ϑc with

$$S_{\vartheta c}^0 \leq (3\rho_\bullet/8)^{1-\vartheta} (C_J \rho_\bullet)^\vartheta < (\sqrt{3}/4) \rho_\bullet.$$

For the unperturbed construction, this exponent is fixed before the regional rates. For the perturbed reconstruction, impose the output-rate conditions of Theorem 19.9 at the allocated working rates below this exponent and bound its coefficient difference from the unperturbed reconstruction there. Fix the larger-product difference-row constant $C_{P,\bullet}$ before the final perturbation bound and require $C_{P,\bullet} \sigma_* \rho_{\text{cl}} \leq \rho_\bullet/64$ for both radii. The total image row is then less than $(\sqrt{3}/4 + 1/64) \rho_\bullet < \rho_\bullet/2$ at those rates; no bound for the perturbed image at the original higher exponent is required. Thus the margin of half the incoming radius is an absolute weighted coefficient bound, not only a bound on values.

Let C_1 be the constant of Lemma 20.5, which depends only on A , a' , and v_J . For $0 < t_{\text{max}} \leq t_* < e^{-1}$ impose separately

$$C_0 t_*^2 \mathcal{L}_V \leq 1/16, \quad C_1 t_* (\mathcal{L}_V + 2) < \frac{1}{4\sqrt{3}}. \quad (20.26)$$

Then on a neighborhood of the whole real product $\mathcal{K}_{\text{en}}$,

$$\|D_w^2(V - |w|^2/2)\|_{\text{row,col},a'} \leq \frac{C_0(R_{\text{en}} + t_{\text{max}}^2 \mathcal{L}_V)}{(1 - AR_{\text{en}})^3} \leq \frac{8}{27} < \frac{1}{2}. \quad (20.27)$$

If $p(t) = A_w \log(1/t)$, the products of balls with radii $p_i \in [p(t_{\text{max}}), p(t_{\text{max}}) + A_w(J - 1)(\log M)/2]$ lie in $\mathcal{K}_{\text{en}}$ whenever

$$t_{\text{max}} [p(t_{\text{max}}) + A_w(J - 1)(\log M)/2] < R_{\text{en}}.$$

The cutoff of the small-field event (4.6) is the case $p_i = p(t_{\text{max}})$ for all i ; the cutoffs $p_i = p(g_i)$ at the individual scales are a separate case. All constants have the dependence described in item (3) of Definition 4.14, and are also independent of the later widths, of the frozen values, and of the individual radii of the fluctuation balls.

Proof. We keep $b = (\xi, b_{\text{rec}})$ fixed and write $t = t_{\text{max}}$. The data of Theorem 20.2 include the uniform incoming classical profile, (20.8), and (20.12) on all parameter neighborhoods. We choose the radius for the increments ξ below Q ; then (20.13) holds at every stage. These are the hypotheses for the perturbed tensors used here, and no derivative in the frozen values is needed.

Step 1: the exact formula. At scale i , the minimum at fixed coarser arguments is $F_i(c_{i+1}, z_i^*)$. With $S_i = K_i^{-1/2}$, Taylor's formula with integral remainder gives

$$g_i^{-2} T_i(c_{i+1}, g_i w_i) = \frac{g_i}{2} \int_0^1 (1 - \tau)^2 D_z^3 F_i(c_{i+1}, z_i^* + \tau g_i S_i w_i) [S_i w_i, S_i w_i, S_i w_i] d\tau. \quad (20.28)$$

The terms of orders zero, one, and two have been, respectively, subtracted, annihilated by stationarity, and normalized to $|g_i w_i|^2/2$ by $S_i^T K_i S_i = I$. The successive partial minima satisfy $F_{i+1} = M \min F_i$, and the couplings satisfy $g_{i+1}^2 = M g_i^2$, so the values at the minimizers telescope in the exponent. In decreasing order of the scales, the map from w to z is triangular with diagonal blocks $g_i S_i$. This proves the classical part of (20.23) and its Gaussian term, by the same finite algebra as in Theorem 12.13. The Haar disintegration density and the single triangular determinant split as in part (3) of Theorem 20.2. Splitting $R(\text{Log } C)$ at its value at zero fluctuation, we obtain the last two terms. All values removed at zero fluctuation, the Gaussian powers, and the constant Haar densities belong to the prefactor; none of them is set to one.

Step 2: the two series. We fix an auxiliary radius $h_0 > 0$ for the increments ζ , strictly inside every admissible stage product and its joint neighborhood. Theorem 20.2 gives tensors with bounds at every external index on a larger product. Dividing by the Taylor factorials, composing coefficientwise majorants (Lemma 20.1), and using (20.6), we bound the degree- k scalar coefficients, rooted at any free index, by $D_J H_J^k$ for fixed D_J, H_J . An additional multiplicity $\eta_j \leq k$ is bounded by 2^k . For an incoming function, the strict image bound gives the jet bound

$$\|D^j R\|_{\text{root}} \leq (E + \sigma_0) \rho^{2-j} \frac{j!}{(1 - \kappa)^{j+1}}, \quad \kappa \leq \frac{1}{2}, \quad j \geq 1, \quad (20.29)$$

at a point whose image under the absolute reconstruction majorant is at most $\kappa \rho$. To verify it, we fix the first differentiated input index. At original degree l , the fixed-index weight in the input norm and the sum over the other $j - 1$ differentiated positions give the factor $(l)_j$, with the remaining factor $\kappa^{l-j} \rho^{2-j}$, and the second identity in (20.6) sums the result. The torus seminorm starts at degree two and the boundary norm at degree one; the same estimate bounds both. Constants of R disappear only in the fluctuation difference. The input networks are joined to the reconstruction networks, and a prescribed differentiated index stays fixed. This proves (20.29) in the coefficient algebra, before the background is evaluated.

Consider a monomial of T_i , and let η_j be its degree in the free coordinates of scale j . It has at least three indices at scale i , so $\eta_i \geq 3$ and $\sum_{j \geq i} \eta_j = k \geq 3$. After division by g_i^2 its coupling factor is

$$t^{k-2} \lambda_i^{\eta_i-2} \prod_{j>i} \lambda_j^{\eta_j}.$$

The product of ratios is at most one; the weaker bound λ_*^{-2} would also be an admissible fixed constant. By (20.22), every Haar, determinant, and incoming difference has at least one free index. A term of degree k there has the factor $t^k \prod_j \lambda_j^{\eta_j}$, whose product of ratios is again at most one. The values B_i in (20.23) are evaluated at the coarser reconstructions, and their coefficients in the coarser free variables belong to the same series. Summing over at most J scales, increasing $A \geq \max\{1, 2H_J, 2D_J H_J\}$, and decreasing h_0 if necessary, we obtain

$$\|P_k\|_{a',1} \leq A^k, \quad \|Q_k\|_{a',1} \leq C_{V,J}(1 + E + \sigma_0)A^k.$$

The first quantum coefficient Q_1 includes every degree-one term of the incoming interaction. Only powers of the fixed radii were absorbed in these constants. We choose $s_W < h_0$ with a strict margin in the larger domain. The series are then normally convergent on that radius, at the outer profile as well as after evaluation.

Step 3: the enclosing product is admissible. The three inequalities in (20.25) imply that

$$\delta_{\text{en}} \leq \lambda_* R_W / 2, \quad R_{\text{en}} \leq R_W.$$

For $w \in \mathcal{K}_{\text{en}}$ we have $|\zeta_i| = t\lambda_i|w_i| \leq R_{\text{en}}$. The minimum defining R_W places this whole product strictly inside the unperturbed and perturbed sequential products and the joint neighborhood; the constant $c_{\text{tube},J}$ includes every finite sequential embedding. Indeed, the joint-center displacement is still at most $2\delta_V$ at the preliminary bound σ_0 , and the new free radius is at most $R_{\text{en}} \leq Q$. The gate (20.13) therefore gives $C_{\text{seq},J}(2\delta_V + R_{\text{en}}) < \varepsilon_{\text{tube}}/2$. Applying (20.14) with free radius R_{en} and adding the joint-center displacement keeps every moving coarser argument in its admissible domain. The separate margin of the larger construction domain in Lemma 19.7 includes the background, the frozen values, and the free radius; it places all reconstructions before the gauge transformation in the larger charts of the Haar densities and of the action.

Step 4: the image increments. For either radius, we abbreviate $C_T = C_{T,\bullet}$ and $C_\Delta = C_{\Delta,\bullet}$. By hypothesis, we have $C_T(K_*\rho_\bullet + R_{\text{en}}) \leq 1/2$. The rational increment in the statement is therefore at most $(2C_T + C_\Delta)R_{\text{en}} \leq \rho_\bullet/8$. Adding the bound $A_{0,\bullet} \leq \rho_\bullet/4$ at zero fluctuation, we obtain the bound $3\rho_\bullet/8$ at exponent zero. If these inequalities hold in a weighted coefficient row, each calculation holds in that row. Otherwise apply the following interpolation to the unperturbed image row at such an exponent c , with $S_c^0 \leq C_J\rho_\bullet$. For nonnegative coefficient weights b_η and network lengths l_η , Hölder's inequality for finite sums gives

$$\sum_\eta b_\eta e^{\vartheta c l_\eta} \leq \left(\sum_\eta b_\eta \right)^{1-\vartheta} \left(\sum_\eta b_\eta e^{c l_\eta} \right)^\vartheta.$$

Passing to increasing finite sums proves the stated interpolation. Its right side, divided by $\rho_\bullet$, tends to $3/8$ as $\vartheta \downarrow 0$. Hence one positive ϑ works for both radii and the finitely many

stages of the unperturbed construction. Its exponent is fixed before the regional rates are chosen. At allocated working rates below this exponent and satisfying Theorem 19.9, that theorem bounds the coefficient row of the perturbed-minus-unperturbed reconstruction by $C_{P,\bullet}\sigma_*\rho_{\text{cl}}$ on the same enclosing products. Fixing this constant before the final choice of σ_* and making that bound smaller than the remaining half-radius margin gives the asserted perturbed image bound. No interpolation at the unperturbed exponent is applied to inherited perturbative networks.

Step 5: curvature. For the curvature bound, we fix one derivative at the scalar coordinate j and sum over the second differentiated index. Differentiating a homogeneous monomial of degree k twice gives the factor $k - 1$ relative to the norm $\|\cdot\|_{a',1}$. The matrix weight is bounded by the weight of the original network, since the distance between the two differentiated indices is at most the network length. At the radius $\varrho = R_{\text{en}}$ in the variables ζ , the two contributions are bounded by

$$\sum_{k \geq 3} (k-1)A^k \varrho^{k-2} = A^2[(1 - A\varrho)^{-2} - 1] \leq \frac{2A^3\varrho}{(1 - A\varrho)^2},$$

$$t^2\mathcal{L}_V \sum_{k \geq 2} (k-1)A^k \varrho^{k-2} = \frac{A^2 t^2 \mathcal{L}_V}{(1 - A\varrho)^2}.$$

Here we used that on $\mathcal{K}_{\text{en}}$ the coordinates satisfy $|w_j| \leq R_{\text{en}}/t$, so that $t^{k-2}(R_{\text{en}}/t)^{k-2} = R_{\text{en}}^{k-2}$ for the classical terms and $t^k(R_{\text{en}}/t)^{k-2} = t^2 R_{\text{en}}^{k-2}$ for the others. We enlarge $C_0 \geq 2A^3$ to include the fixed conversion between scalar and three-component block norms. The column bound follows from the symmetry of the Hessian. The weaker denominator $(1 - AR_{\text{en}})^{-3}$ in (20.27) applies to both contributions. Since $AR_{\text{en}} \leq 1/4$, $C_0 R_{\text{en}} \leq 1/16$, and $C_0 t^2 \mathcal{L}_V \leq 1/16$, the result is

$$\frac{1/16 + 1/16}{(1 - 1/4)^3} = \frac{8}{27} < \frac{1}{2}.$$

All inequalities have strict margins in the charts, so they hold on a neighborhood of the real product, including its faces and corners. The constant C_1 of Lemma 20.5 is determined from A , a' , the label density, and the curvature bound just proved, so the second inequality in (20.26) is a separate requirement compatible with the present choices.

Step 6: logarithmic cutoffs. A ball of radius p_i lies in $\mathcal{K}_{\text{en}}$ whenever $tp_i < R_{\text{en}}$. The finite logarithmic offset in the statement bounds every such p_i , since $p(g_i) = p(t) + A_w \log(1/\lambda_i)$ and $\lambda_i \geq \lambda_*$. This proves the inclusion both for the common cutoff and for the cutoffs at the individual scales. All constants were computed at σ_0 on fixed larger products, before any final enclosure quantity or width. The proof involves no derivative in the frozen values, no sum over earlier renormalization group steps, and no dependence on the individual radii of the fluctuation balls. $\square$

Remark 20.4. The incoming boundary function f in Lemma 20.3 is measured in the norm $\mathcal{X}^\rho$ with the uniform profile ρ , and (20.29) is stated for this case. For a function measured with a nonuniform profile ϕ , the analogous estimate holds for the rescaled function $\tilde{R}(x) = R((\phi_i x_i)_i)$: if its degree-wise rooted sum at unit profile is at most $H\rho^2$, then at a point whose image under the absolute reconstruction majorant is bounded by $\kappa\phi$, $0 \leq \kappa < 1$,

$$|D^j \tilde{R}|_{\kappa, \text{root}} \leq H\rho^2 \sum_{l \geq j} (l)_j \kappa^{l-j} = H\rho^2 \frac{j!}{(1-\kappa)^{j+1}}, \quad j \geq 1.$$

Thus each differentiated slot carries its own radius ϕ_i . The unscaled form (20.29) does not follow for a spatially varying radius profile. Indeed, fix an unfixed index i at distance d_i from Ω^c . The function $R(v) = H\rho^2 e^{-\alpha_{\text{bd}} d_i} v_i / \phi_i$ has norm H and derivative $H\rho^2 e^{-\alpha_{\text{bd}} d_i} / \phi_i$, which is unbounded as the boundary radius $\phi_i = \rho_\partial$ decreases with d_i fixed. In the coefficient sums of Section 20.4 every differentiated background index is multiplied by its own radius, so no inverse power of the small boundary radius ρ_∂ enters.

Lemma 20.5. *The following statements hold.*

1. Let $\mathcal{K}$ be a finite product of three-dimensional balls, let $S, \mathcal{W} \subset \mathcal{K}$ be Borel sets, let $\chi : \mathcal{K} \rightarrow [0, 1]$ be Borel, let e^{-V} be a positive Borel function with $0 < Z_{\mathcal{K}} = \int_{\mathcal{K}} e^{-V} dw < \infty$, and let $P > 0$ be a constant prefactor (unrelated to the perturbation P of Theorem 20.2). Set

$$G = P \int_S \chi e^{-V} dw, \quad \tilde{F} = P \int_{\mathcal{W}} e^{-V} dw, \quad dv_{\mathcal{K}} = Z_{\mathcal{K}}^{-1} \mathbf{1}_{\mathcal{K}} e^{-V} dw.$$

If $\nu_{\mathcal{K}}(\mathcal{W}^c) \leq \delta < 1$, then $G \leq (1 - \delta)^{-1} \tilde{F}$. Nonnegative functions of the coarse background and external factors may multiply both sides. No inclusion between $\mathcal{W}$ and S is needed. A product with no factors has mass one.

2. Let $\mathcal{K} = \mathcal{K}_{\text{en}}$ be the enclosing product of Lemma 20.3, under its hypotheses with $t_{\text{max}} \leq t_*$, at an admissible real coarse background and fixed real frozen values, and let V be the potential (20.23). There is a constant C_1 , depending only on A, a' , and v_J , such that

$$D_w^2 V \geq I/2, \quad \sup_j |\mathbb{E}_{\nu_{\mathcal{K}}} w_j| \leq C_1 t_{\text{max}} (\mathcal{L}_V + 2),$$

where j runs over the scalar coordinates. For every scalar function F that is 1-Lipschitz for the Euclidean norm and every $v \geq 0$,

$$\nu_{\mathcal{K}}(|F - \mathbb{E}F| \geq v) \leq 2e^{-v^2/4}.$$

Let $p(t) = A_w \log(1/t)$, and choose radii $p_i \geq p(t_{\text{max}})$ with $t_{\text{max}} \max_i p_i < R_{\text{en}}$, so that the product $\mathcal{W} = \prod_{i,e} \{|w_{i,e}| \leq p_i\}$ is contained in $\mathcal{K}$. If $C_1 t_{\text{max}} (\mathcal{L}_V + 2) \leq p(t_{\text{max}}) / (2\sqrt{3})$, which holds for all sufficiently small t_{max} and is one of the requirements on the coupling bound g_* in Theorem 20.14, then

$$\nu_{\mathcal{K}}(\mathcal{W}^c) \leq 6b \exp\{-p(t_{\text{max}})^2/48\}, \quad b \leq C_J m^3,$$

where b counts all fluctuation blocks of the step, jointly over all components of Ω and all scales. These are means under ν_K at fixed frozen values; no symmetry of the frozen values is assumed. The constants have the dependence described in item (3) of Definition 4.14 and are independent of the later widths.

3. Consider the stopping sides $m_t = mB^t$, $t \geq 0$, together with at most one additional side ηmB^j with $j \in \mathbb{N}_0$, where η ranges over a fixed finite set of positive ratios (arising from a final composition of fewer than J averaging steps). Suppose that nonnegative errors at these sides satisfy

$$\delta_t \leq C_J m_t^3 e^{-a_w \log^2 \sqrt{M m_t / \gamma}}, \quad z_t \leq C_J m_t^3 e^{-b m_t / \gamma}, \quad a_w, b > 0,$$

and similarly at the additional side. By part (2), the first bound holds with $a_w = A_w^2/48$ for the comparison errors of this lemma. Then for every fixed $\Gamma \in (0, \infty)$, in particular for every $\Gamma = \gamma_0 \leq 1$, there is a minimum side m_0 , depending only on Γ , the fixed data, and the finite set of ratios, such that, uniformly for $m \geq m_0$ and $0 < \gamma \leq \Gamma$, all errors are at most $1/2$ and

$$\sum_t (\delta_t + z_t) < \infty, \quad \log \prod_t (1 - \delta_t)^{-1} \leq 2 \sum_t \delta_t \leq C_\delta.$$

The sums and the logarithm tend to zero as $m \rightarrow \infty$, uniformly in $0 < \gamma \leq \Gamma$ and in the finite or infinite number of sides. The same conclusion holds for $\epsilon_t = \delta_t / (1 - z_t)$ after increasing m_0 . There is one comparison for the whole product at each stopping side, not a separate factor for each component of Ω^c . The minimum side and C_δ have the dependence described in item (4) of Definition 4.14; none of them depends on the number of steps or on the large-field decomposition.

Proof. Step 1 proves part (1). Steps 2 to 6 prove part (2): we first check that the measure is well defined, then interpolate between the Gaussian product and the potential, bound the force of the interaction using fourth moments, integrate the derivative of the means along the interpolation, and finally bound the probability outside the chosen product. Step 7 reduces part (3) to Lemma 20.15.

Step 1: the abstract comparison. The Borel hypotheses make both integrands measurable, and since $0 \leq \chi \leq 1$ and $0 < Z_K < \infty$, we have $0 \leq G, \tilde{F} \leq PZ_K < \infty$. Since $S \subset K$, we have $G \leq PZ_K$. On the other hand

$$\tilde{F} = PZ_K \nu_K(\mathcal{W}) \geq PZ_K(1 - \delta).$$

Dividing by the positive finite number PZ_K gives part (1); the common prefactor cancels before any estimate. Multiplying by nonnegative functions of the coarse background preserves the inequality pointwise. For a product with no factors we use the unit point mass, and the same proof applies.

Step 2: normalizability on the enclosing product. The potential of Lemma 20.3 is real and smooth on a neighborhood of the fixed compact product $\mathcal{K}_{\text{en}}$. Its ball radii $R_{\text{en}}/t_{\text{max}}$ are positive and finite, so the product has positive finite Lebesgue volume. If V_{min} and V_{max} are the minimum and maximum of V on $\mathcal{K}_{\text{en}}$, then

$$0 < e^{-V_{\text{max}}} \text{vol}(\mathcal{K}_{\text{en}}) \leq Z_{\mathcal{K}_{\text{en}}} \leq e^{-V_{\text{min}}} \text{vol}(\mathcal{K}_{\text{en}}) < \infty.$$

Thus the integrand is positive, Borel, and normalizable. The chosen product $\mathcal{W}$ is closed, hence Borel; the comparison in part (1) keeps its Borel hypotheses on S and χ .

Step 3: convexity and concentration. We write $t = t_{\text{max}}$ and $H_{\text{int}}(w) = V(w) - |w|^2/2$, and interpolate on the same product

$$V_{\theta}(w) = |w|^2/2 + \theta H_{\text{int}}(w), \quad 0 \leq \theta \leq 1.$$

Each V_{θ} is real and smooth on the compact product, so the normalization bounds of Step 2 hold throughout the interpolation. By (20.27) the Hessian perturbation is at most $\kappa = 8/27$, so every V_{θ} has Hessian at least $(1 - \kappa)I \geq I/2$; for $\theta = 1$ this is the first assertion. The absolute coefficient bound in (20.27) bounds the exponentially weighted sums over y of the suprema over the product of $|(V_{\theta})_{xy}|$, in rows and in columns, by the same fixed constant; the three-component block form follows by including the fixed block conversion in C_0 . Fix $0 < b_{\text{cov}} \leq a'$ so small that

$$\max \left\{ \sup_x \sum_{y \neq x} (e^{b_{\text{cov}} \text{dist}(x,y)} - 1) \sup \| (V_{\theta})_{xy} \|, \sup_y \sum_{x \neq y} (e^{b_{\text{cov}} \text{dist}(x,y)} - 1) \sup \| (V_{\theta})_{xy} \| \right\} \leq \frac{1}{4}.$$

Such a choice depends only on the fixed weighted bound, since the convexity of $u \mapsto e^u$ gives $e^{b_{\text{cov}} d} - 1 \leq (b_{\text{cov}}/a')(e^{a' d} - 1)$ for $b_{\text{cov}} \leq a'$. The off-diagonal blocks of the Gaussian part vanish. Thus all hypotheses of Theorem 5.4, which adapts the Bakry–Émery and Helffer–Sjöstrand methods [8, 110] to products of balls with reflecting boundary, hold with convexity constant $1/2$, on a neighborhood of the finite product; its ball faces and product corners are included in that theorem. It yields the scalar tail $2e^{-v^2/4}$ and the covariance bound

$$|\text{Cov}_{\theta}(F, G)| \leq 4 \sum_{x,y} e^{-b_{\text{cov}} \text{dist}(x,y)} \|\nabla_x F\|_2 \|\nabla_y G\|_2, \quad (20.30)$$

where Cov_{θ} and $\|\cdot\|_2$ refer to the normalized measure with density proportional to $e^{-V_{\theta}}$ on $\mathcal{K}_{\text{en}}$. These constants do not depend on the number or the radii of the balls.

Step 4: the force. Let $M(\theta) = \max_j |\mathbb{E}_{\theta} w_j|$, where j is scalar (in this proof $M(\theta)$ is unrelated to the blocking factor M). The tail just obtained implies a universal bound on the fourth moment of $w_j - \mathbb{E}_{\theta} w_j$, by integrating $\int_0^{\infty} 4u^3 \mathbb{P}(|w_j - \mathbb{E}_{\theta} w_j| > u) du$ against the Gaussian majorant. Hence

$$\|w_j\|_4 \leq C_4 + M(\theta).$$

Consider a differentiated classical monomial of degree $k \geq 3$. After fixing its differentiated coordinate, we place two of its remaining $k - 1$ factors in L^4 and the other $k - 3$ factors in the supremum norm R_{en}/t of the product. Hölder's inequality, which applies also when the two factors are the same scalar coordinate, gives the factor

$$t^{k-2}(R_{\text{en}}/t)^{k-3}(C_4 + M(\theta))^2 = tR_{\text{en}}^{k-3}(C_4 + M(\theta))^2.$$

The coefficient sum with the differentiated index fixed is bounded by A^k , and the geometric sum is at most $C(A)t(1 + M(\theta)^2)$ because $AR_{\text{en}} \leq 1/4$. For the quantum terms, including degree one, a direct supremum bound gives

$$\sum_{k \geq 1} \mathcal{L}_V A^k t^k (R_{\text{en}}/t)^{k-1} = \frac{\mathcal{L}_V A t}{1 - AR_{\text{en}}}.$$

Combining the two bounds, we obtain

$$\sup_j \|\partial_j H_{\text{int}}\|_2 \leq C(A)t[\mathcal{L}_V + 1 + M(\theta)^2]. \quad (20.31)$$

Step 5: the means. Differentiating the finite integrals on the fixed compact product, we obtain

$$\frac{d}{d\theta} \mathbb{E}_\theta w_j = -\text{Cov}_\theta(w_j, H_{\text{int}}).$$

Indeed, the derivative of the numerator is $-\int w_j H_{\text{int}} e^{-V_\theta}$ and that of the denominator is $-\int H_{\text{int}} e^{-V_\theta}$, and their quotient is the displayed centered expression. The integrands and their derivatives are bounded on this finite product, so the differentiation is justified directly. This is the calculation of Theorem 13.10 on a fixed product, carried out here for the present measure; we do not use the conclusions of that theorem, whose hypotheses concern the torus products. Combining (20.30), (20.31), and (20.2), we obtain a fixed C_1 with

$$\left| \frac{d}{d\theta} \mathbb{E}_\theta w_j \right| \leq C_1 t[\mathcal{L}_V + 1 + M(\theta)^2]. \quad (20.32)$$

It depends only on A , a' , and the label density; the margin $\kappa < 1/2$ has already been imposed. This is the constant in the hypothesis (20.26) of Lemma 20.3.

At $\theta = 0$ the measure is a product of centered truncated Gaussian measures on balls, so every coordinate has mean zero, for every frozen value. All means are continuous in θ . Suppose that M reached the value one, and let θ_1 be the first time at which it does. Integrating (20.32) for each coordinate on $[0, \theta_1]$, where $M \leq 1$, would give $M(\theta_1) \leq C_1 t(\mathcal{L}_V + 2) < 1/(4\sqrt{3}) < 1$, a contradiction. Inserting $M < 1$ into (20.32) and integrating once more proves $M(1) \leq C_1 t(\mathcal{L}_V + 2)$. This argument integrates the inequalities for the individual coordinates before taking the maximum; it does not use any differentiability of the maximum.

Step 6: the probability outside the chosen product. The explicit hypothesis $t \max_i p_i < R_{\text{en}}$ puts the chosen product strictly inside $\mathcal{K}_{\text{en}}$. The later logarithmic gates (20.48) provide a sufficient choice of this inclusion in Theorem 20.14; the present local argument uses the inclusion itself. If a fluctuation block lies outside its ball of radius $p_i \geq p(t)$, then some scalar coordinate has absolute value larger than $p(t)/\sqrt{3}$. Since the mean of that coordinate is at most $p(t)/(2\sqrt{3})$ in absolute value, the coordinate deviates from its mean by at least $p(t)/(2\sqrt{3})$. By the scalar tail and a union bound over the three coordinates, the probability that a given block lies outside its ball is at most $6e^{-p(t)^2/48}$. The step has at most $b \leq C_J m^3$ blocks: each of the at most J scales has a fixed finite number of free labels per output block, and removing blocks from the region only decreases this number. This proves the bound on $\nu_{\mathcal{K}}(\mathcal{W}^c)$, with no factor depending on the number of components. The enclosure and curvature hypotheses are used on the whole product $\mathcal{K}_{\text{en}}$, which does not depend on the configuration.

Step 7: summation. Part (3) is the case $C = C_J$, $c = b$, $a = a_w$ of Lemma 20.15, with the ratio B and the finite set of additional ratios, since $\log^2 \sqrt{Mm/\gamma} = z^2$ with $z = (1/2) \log(Mm/\gamma)$. With $p(t_{\max}) = A_w \log \sqrt{Mm/\gamma}$, the bound of part (2) is of the stated form with $a_w = A_w^2/48$, since $6b \leq C_J m^3$. $\square$

20.4 Cluster expansion with boundary-localized coefficients

We now expand the logarithm of the fluctuation integral, keeping two structures apart. Probabilistic independence is determined by the set of fluctuation blocks on which a monomial depends: two monomials depending on disjoint sets of blocks are independent under a product reference measure, even if they depend on the same background coordinates. Spatial decay is carried instead by a connected network through all positions of the coefficient, background positions included, which may lie far from the fluctuation blocks. Overlapping monomials share a fluctuation block, so their networks meet there, and the union of their networks is connected without any additional connecting path.

The organization into connected polymer activities and a convergent logarithm follows the constructive cluster expansion; see Balaban [18, 19] for its implementation in lattice gauge theories, Brydges [38] for a general exposition, and Kotecký and Preiss [120] for the abstract polymer criterion. The cancellation of connected graphs against rooted trees used in Theorem 12.6 is Penrose's partition [136]; its construction by generations and parents, and the positive tree recursions used below, are described in [87, Sections 2 and 4.1]. The finite graph inequality is imported in the form of [68, Appendix B, (323), p. 42], through Theorem 12.6; no other result of that paper, which concerns a scalar field theory, is used. We prove directly the features needed here: the two kinds of roots, the exact fluctuation supports, and the visits to the fixed complement.

Complex coupling parameter. In this section we replace the coupling powers in (20.24) by powers of a complex parameter z : the classical powers $t_{\max}^{q-2}$ become z^{q-2} and the other powers $t_{\max}^q$ become z^q . The incoming functions, the couplings of earlier steps, the ratios λ_i , and the normalized symmetric reference measures on balls are kept fixed. The incoming insertion becomes

$$A_R(z) = R(\text{Log } C(v, (z\lambda_i w_i)_i; b_{\text{rec}})) - R(\text{Log } C(v, 0; b_{\text{rec}})).$$

We call the interaction without insertion the sum of all terms of $V - |w|^2/2$ in (20.23) except the incoming insertion, after this substitution. For $p_x > 0$ attached to the fluctuation blocks, with $p_x = p_i$ for a block at scale i , we write $p = \max_i p_i$, and we always assume $p \geq 1$.

Definition 20.6. Let F be an interaction, given with a resolution into scalar monomials

$$U_\lambda = u_\lambda v^{\nu_\lambda} w^{\eta_\lambda}, \quad |\eta_\lambda| \geq 1,$$

as in Lemma 20.1. A fluctuation index is a whole fluctuation block at its scale and position; a background index is an unfixed scalar background coordinate on the lattice where it was introduced. Write S_λ for the set of fluctuation blocks on which η_λ is positive (the fluctuation support of the monomial) and $k_\lambda = |\nu_\lambda|$ for its background degree. Each monomial carries a connected network through all of its fluctuation and background positions; for a monomial whose coefficient has a visit to $\Omega_{L_j}^c$, the network contains this visit. We write ℓ_λ for the total length of the network, including the visit. No enlargement of S_λ is made. Internal indices are summed, using a fixed part of the exponential decay rate, before the resolution into monomials, and repeated occurrences keep their own network refinements. For the exponent $s_0 > 0$ set

$$b_\lambda(F) = |u_\lambda(F)| \psi_{\text{prof}}^{\nu_\lambda} \prod_x p_x^{|\eta_{\lambda,x}|} e^{s_0 \ell_\lambda},$$

and for $h \geq 0$

$$\begin{aligned} W_h(F) &= \sup_x \sum_{\lambda: x \in S_\lambda} e^{h|S_\lambda|} b_\lambda(F), \\ D_h(F) &= \sum_{k \geq 1} \sup_i \sum_{\lambda: k_\lambda = k} v_{\lambda,i} e^{h|S_\lambda|} b_\lambda(F). \end{aligned}$$

In W_h the root x is a fluctuation block, and in D_h the root i is a background index. In D_h the supremum over the root is taken before the sum over the background degree, as in the coefficient norm with boundary decay.

Lemma 20.7. Fix the admissible larger products and the preliminary perturbation ball of Theorem 20.2 and Lemma 20.3, including both strict margins for the images under the reconstruction

majorant and the inequalities on the conditional forces at every stage. Fix the ratio K_* and the profiles $\hat{\phi} = K_*\phi$ and $\psi_{\text{prof}} = \sqrt{K_*}\phi$. For a composition of s averaging steps put $B_s = M^s$, and write $a_{\text{in}}, \alpha_{\text{in}}$ for the incoming rates. Fix an allocation of the intrinsic coefficient exponent and impose the output-rate restrictions of Theorem 19.9(4), with B_s in place of B : in particular, for the fixed geometric allocation constant require $s_0 = a + \alpha < \min(c_J, B_s a_{\text{in}})/C_{\text{geom}}$ and $\alpha < B_s \alpha_{\text{in}}/C_{\text{geom}}$. The output rates may be chosen smaller for a shorter composition; no return of the full-step rates is asserted for it. In the full-step iteration the incoming rates are a, α , and the stronger allocation conditions of Theorem 20.9 apply. Consider the following families of interactions, resolved into monomials as in Definition 20.6: the interaction without insertion, the incoming insertion, the differences between the regional families and the restricted torus families, and the mixed difference of the boundary change and the incoming insertion. Here $\mathcal{X}^p(f)$ is evaluated at the incoming regional rates $a_{\text{in}}, \alpha_{\text{in}}$, whereas E bounds the incoming torus interaction at its fixed torus rate; a_{in} is no larger than that available spatial rate after conversion to the graph metric. The resolved norms W_h, D_h use $s_0 = a + \alpha$ and network lengths in output units. Let $H = E + \mathcal{X}^p(f)$ and set

$$\mathfrak{s}(z) = \frac{C_J |z| (p^3 + p + Hp)}{1 - C_J |z| p},$$

for z in the fixed larger radius with $C_J |z| p < 1$, inside both incoming image margins. Then, uniformly in the fixed real frozen values,

$$W_{16}(F) + D_{16}(F) \leq \epsilon_F$$

for each of these families F . One may take $\epsilon_F \leq C_J \mathfrak{s}(z)$ for the family without insertion and for the boundary differences, and

$$\epsilon_A + \epsilon_{\partial A} \leq \frac{C_J |z| Hp}{1 - C_J |z| p} \quad (20.33)$$

for the incoming insertion A and its boundary difference ∂A . In particular the latter bound holds with a visit to Ω^c and with an arbitrary background root. The constants have the dependence described in item (3) of Definition 4.14 and are independent of the later widths; the profiles and the input radii are not decreased at later steps.

Proof. Step 1 records the coefficient resolution of the terms of the potential and its basic bounds. Steps 2 and 3 write out, with every contraction, the resolution in the two typical cases of a classical monomial and of a monomial of the Haar density. Step 4 treats the incoming insertion and the boundary differences. Step 5 converts the coefficient bounds into the bounds on W_{16} and D_{16} , keeping the order of the supremum and the degree sum.

Step 1: the resolution. The tensors of Theorem 20.2, with bounds at every external index, and the coefficient version of Lemma 20.3 are available on the outer profile, before the background is evaluated. By Lemma 20.1, every term of the potential has an exact

resolution into monomials carrying connected networks. Rooted at an arbitrary free index, this resolution has convergent majorants in the free degree q , namely

$$C_J^q |z|^{q-2} p^q \quad (q \geq 3) \quad \text{and} \quad C_J^q |z|^q p^q \quad (q \geq 1),$$

for the classical terms and for the other terms, respectively. For the incoming insertion the second majorant is multiplied by H . The ratios λ_i have modulus at most one, and J is fixed. For the last assertion the input is (20.29) with $E + \sigma_0$ replaced by H : fixing its first differentiated input and summing over the other inputs gives

$$\|D^j R\|_{\text{root}} \leq H \rho^{2-j} \frac{j!}{(1-\kappa)^{j+1}}, \quad \kappa \leq \frac{1}{2},$$

by applying the factor $(l)_j \kappa^{l-j}$ to the input norm at degree l with that input fixed and summing the geometric series. The constants of R disappear in the fluctuation difference; they are not bounded by H . The composition with every index rooted in Theorem 19.9, with its volume-normalized input columns, keeps a specified external background index as well as a specified free index. At a background root i , the sum of the monomials containing that index is bounded by $\hat{\phi}(i)$ times the absolute coefficient majorant of $\partial_i F$. This produces the factor $\nu_{\lambda,i}$ in D_h . No division by the smaller boundary radius occurs.

Step 2: a classical monomial. Fix a scale i . By the definition of the partial minimization, $F_i(v, c_{i+1}, y)$ is the perturbed objective $\mathcal{Q}_1$ of the proof of Theorem 20.2, in the free logarithmic coordinates and multiplied by the factor M^i , evaluated with the coordinates of the finer scales at their conditional minimizers, at fixed frozen values. We write this as $F_i(v, c_{i+1}, y) = \mathcal{Q}^{(i)}(\Psi_i(v, c_{i+1}, y))$, where $\mathcal{Q}^{(i)} = M^i \mathcal{Q}_1$, y denotes the free logarithmic coordinates of scale i , $c_{i+1} = C_{i+1}(v, \zeta_{>i})$ is the coarser reconstruction, and Ψ_i maps (v, c_{i+1}, y) to the full list of coordinates $(v, z_{<i}^*, y, z_{>i})$, with the finer coordinates $z_{<i}^*$ at their conditional minimizers. The coefficient bounds for $\mathcal{Q}^{(i)}$ on the common product are those of Theorem 19.9, and those for Ψ_i , which contains the conditional minimizers of the finer stages, are those of Theorem 20.2. The factorization only displays the chain rule; the general term can equally be bounded through the tensors $D_y^r F_i$ of that theorem. At fixed (v, c_{i+1}) the minimizer z_i^* satisfies $F_{i,y}(v, c_{i+1}, z_i^*) = 0$, and $K_i = F_{i,yy}(v, c_{i+1}, z_i^*)$, $S_i = K_i^{-1/2}$, $z_i = z_i^* + g_i S_i w_i$. Since $S_i^T K_i S_i = I$, Taylor's formula in y on the common coefficient product gives

$$\frac{F_i(v, c_{i+1}, z_i) - F_i(v, c_{i+1}, z_i^*)}{g_i^2} - \frac{|w_i|^2}{2} = \sum_{r \geq 3} \frac{g_i^{r-2}}{r!} \sum_{a_1, \dots, a_r} \sum_{\beta_1, \dots, \beta_r} F_{i,a_1 \dots a_r}^* \prod_{l=1}^r S_i(a_l, \beta_l) w_{i,\beta_l}.$$

The star denotes evaluation at $y = z_i^*$ (not at zero coarser input), and the β_l are scalar labels at scale i . This expression is analytic at $g_i = 0$: stationarity and the exact subtraction of the quadratic term remove the apparent singularity.

We display the term $r = 3$. For a multi-index $\tau = (\nu, \eta_{>i})$ in the background variables and the coarser free variables, we write $[\cdot]_\tau$ for the coefficient of $v^\nu \zeta_{>i}^{\eta_{>i}}$ after the substitution $c_{i+1} = C_{i+1}(v, \zeta_{>i})$, and set

$$e_{x_1 \dots x_m; \tau}^{(m)} = [(D^m \mathcal{Q}^{(i)})_{x_1 \dots x_m}(\Psi_i^*)]_\tau, \quad P_{x; A; \tau} = [(\partial_{y_A} \Psi_{i,x})^*]_\tau, \quad s_{a\beta; \tau} = [S_i(a, \beta)]_\tau,$$

where $\partial_{y_A} = \partial_{y_a} \partial_{y_b}$ for $A = (a, b)$, and similarly for longer A . The chain rule gives, for $T_{abc; \tau} = [F_{i,abc}^*]_\tau$,

$$\begin{aligned} T_{abc; \tau} &= \sum_{x_1, x_2, x_3} \sum_{\tau_0 + \tau_1 + \tau_2 + \tau_3 = \tau} e_{x_1 x_2 x_3; \tau_0}^{(3)} P_{x_1; a; \tau_1} P_{x_2; b; \tau_2} P_{x_3; c; \tau_3} \\ &+ \sum_{x_1, x_2} \sum_{\tau_0 + \tau_1 + \tau_2 = \tau} e_{x_1 x_2; \tau_0}^{(2)} (P_{x_1; ab; \tau_1} P_{x_2; c; \tau_2} + P_{x_1; ac; \tau_1} P_{x_2; b; \tau_2} + P_{x_1; bc; \tau_1} P_{x_2; a; \tau_2}) \\ &+ \sum_x \sum_{\tau_0 + \tau_1 = \tau} e_{x; \tau_0}^{(1)} P_{x; abc; \tau_1}. \end{aligned}$$

In particular, the Hessian is

$$F_{i,ab}^* = \sum_{x, x'} \mathcal{Q}_{xx'}^{(i)}(\Psi_i^*) \Psi_{i,x,a}^* \Psi_{i,x',b}^* + \sum_x \mathcal{Q}_x^{(i)}(\Psi_i^*) \Psi_{i,x,ab}^*.$$

Stationarity in y does not imply $(\mathcal{Q}^{(i)})'(\Psi_i^*) = 0$, so the term with $(\mathcal{Q}^{(i)})'$ and the last line of the formula for $T_{abc; \tau}$ cannot be deleted. Let η_i be a multi-index of degree three in the fluctuation variables of scale i . After the substitutions $g_l = z\lambda_l$ and $\zeta_{>i} = z\lambda_{>i} w_{>i}$, the coefficient of $v^\nu w_i^{\eta_i} w_{>i}^{\eta_{>i}}$ in this cubic term is

$$\frac{z^{1+|\eta_{>i}|} \lambda_i}{6} \prod_{j=i+1}^{s-1} \lambda_j^{|\eta_j|} \sum_{a,b,c} \sum_{\substack{\beta_1, \beta_2, \beta_3 \\ \mathbf{e}_{\beta_1} + \mathbf{e}_{\beta_2} + \mathbf{e}_{\beta_3} = \eta_i}} \sum_{\tau_0 + \tau_1 + \tau_2 + \tau_3 = (\nu, \eta_{>i})} T_{abc; \tau_0} s_{a\beta_1; \tau_1} s_{b\beta_2; \tau_2} s_{c\beta_3; \tau_3}, \quad (20.34)$$

where η_j is the part of the multi-index $\eta_{>i}$ at scale j . The factor $1/6$ also accounts for repeated β 's; no further factorial occurs. For each summand we take the networks of its coefficients and join them at their common indices x and a , and at the coarser arguments. The network of the resulting monomial is this connected union, and its fluctuation support is

$$S_\lambda = \{\text{blocks of } \beta_1, \beta_2, \beta_3\} \cup \{\text{coarser blocks used by } \eta_{>i}\},$$

with repeated blocks counted once. The background positions in ν belong to the network even if they lie outside S_λ . Let $\mathcal{Q}_m^+$, P_m^+ , and S^+ be the absolute scalar majorants of the jets and maps above on their common product, including the volume ratios of part (4) of Lemma 20.1. The formula for $T_{abc; \tau}$ and Lemma 20.1 give the cubic majorant

$$\frac{1}{6} (\mathcal{Q}_3^+ (P_1^+)^3 + 3 \mathcal{Q}_2^+ P_2^+ P_1^+ + \mathcal{Q}_1^+ P_3^+) (S^+)^3,$$

and the same expression with a derivative falling on one factor bounds the sum rooted at a fluctuation or background index. Each product here corresponds to a connected contraction in the formula for $T_{abc;\tau}$, so no unrooted product over an extensive set appears. For general r the rule is the partition formula

$$\partial_{a_1} \cdots \partial_{a_r} (\mathcal{Q}^{(i)} \circ \Psi_i) = \sum_{\pi} \sum_{(x_B)_{B \in \pi}} (D^{|\pi|} \mathcal{Q}^{(i)})_{(x_B)} (\Psi_i) \prod_{B \in \pi} \partial_{a_B} \Psi_{i, x_B},$$

where π runs over the partitions of $\{1, \dots, r\}$. Together with Lemma 20.1, it gives a convergent coefficientwise majorant on a common smaller free product. Since $r \geq 3$, a term of total free degree $q = r + |\eta_{>i}|$ has the coupling power z^{q-2} , and the ratios of the scales have modulus at most one. This proves the cubic starting order before any expectation over the fluctuation variables is taken.

Step 3: a monomial of the Haar density. Let $\Gamma(v, c_{i+1}, y)$ be the chart of solved links at scale i , let $\mathcal{E}_i^\circ$ be its links outside the block axial gauge trees, and let $\mathfrak{c}$ run over the set $\mathfrak{C}_i$ of constraints solved at scale i , with eliminated edges $e_\mathfrak{c}^*$. We write

$$A_\mathfrak{c} = D_{X_{e_\mathfrak{c}^*}} C_\mathfrak{c}(\Gamma), \quad L_i = - \sum_{e \in \mathcal{E}_i^\circ} \log j_H(\Gamma_e) + \sum_{\mathfrak{c} \in \mathfrak{C}_i} \log \det A_\mathfrak{c},$$

with $j_H(X) = \text{sinc}^2(|X|/2)$ as in (4.2) and Lemma 5.1(2); only field-independent factors have been removed. The fluctuation-dependent part of L_i at scale i is

$$L_i(v, c_{i+1}, z_i) - L_i(v, c_{i+1}, z_i^*) = \sum_{r \geq 1} \frac{g_i^r}{r!} \sum_{a_1, \dots, a_r} \sum_{\beta_1, \dots, \beta_r} L_{i, a_1 \dots a_r}^* \prod_{l=1}^r S_i(a_l, \beta_l) w_{i, \beta_l}.$$

Its linear coefficient is

$$L_{i, a}^* = - \sum_{e \in \mathcal{E}_i^\circ} \sum_{\zeta=1}^3 (\partial_\zeta \log j_H)(\Gamma_e^*) \Gamma_{e\zeta, a}^* + \sum_{\mathfrak{c} \in \mathfrak{C}_i} \sum_{A, B=1}^3 (A_\mathfrak{c}^{-1})_{BA}^* (\partial_{y_a} A_\mathfrak{c})_{AB}^*,$$

where $(\partial_{y_a} A_\mathfrak{c})_{AB} = \sum_{e, \zeta} (\partial_{X_{e\zeta}} A_\mathfrak{c})_{AB} \Gamma_{e\zeta, a}$, and all factors are evaluated at the minimizer and at the coarser reconstruction $C_{i+1}(v, \zeta_{>i})$. Taking the coefficient $[\cdot]_\tau$ of each factor and summing over all allocations of τ gives the coefficients $l_{a;\tau} = [L_{i, a}^*]_\tau$, with every link, constraint, color, integrated coordinate, and multi-index sum displayed. For $|\eta_i| = 1$ the coefficient after the coupling substitution is

$$z^{1+|\eta_{>i}|} \lambda_i \prod_{j=i+1}^{s-1} \lambda_j^{|\eta_j|} \sum_a \sum_{\beta: \mathbf{e}_\beta = \eta_i} \sum_{\tau_0 + \tau_1 = (v, \eta_{>i})} l_{a; \tau_0} s_a \beta; \tau_1.$$

The trace in $L_{i, a}^*$ is rooted at the index a through the derivative $\partial_{y_a} A_\mathfrak{c}$; its coefficient bound sums over $\mathfrak{c}, e, A, B, \zeta$. It is not the unrooted norm of a sum of logarithmic determinants.

The network is the connected union of the networks of the Haar factors, the inverse matrices, and the chart, joined to the network of S_i at a , and the support is the block of β together with the coarser blocks used by $\eta_{>i}$. Lemma 20.1 bounds these coefficients by the products of the majorants of the factors and their derivatives. The series starts at total free degree one, with coupling power equal to the total free degree.

The remaining part of the measure fluctuation in (20.11) is the difference $B_i(c_{i+1}) - B_i(\widehat{c}_{i+1})$, whose fluctuation support may lie entirely in coarser scales; it is not part of the field-independent prefactor. For a coordinate h_γ of the coarser argument, the total derivative is

$$\partial_{h_\gamma} B_i = L_{i,h_\gamma}^* + \sum_a L_{i,a}^* \partial_{h_\gamma} z_{i,a}^* + \frac{1}{2} \sum_{a,b} (K_i^{-1})_{ba} \partial_{h_\gamma} K_{i,ab},$$

where $\partial_{h_\gamma} z_i^* = -K_i^{-1} F_{i,yh_\gamma}^*$ and $\partial_{h_\gamma} K_{i,ab} = F_{i,abh_\gamma}^* + \sum_{a'} F_{i,aba'}^* \partial_{h_\gamma} z_{i,a'}^*$. The difference equals the convergent coefficient expansion of

$$\int_0^1 \sum_{j=i+1}^{s-1} \sum_{\beta \in I_j} \zeta_{j,\beta} \sum_{\gamma} (\partial_{h_\gamma} B_i)(v, C_{i+1}(v, \tau \zeta_{>i})) (\partial_{\zeta_{j,\beta}} C_{i+1,\gamma})(v, \tau \zeta_{>i}) d\tau,$$

since the integrand is the derivative in τ of $B_i(v, C_{i+1}(v, \tau \zeta_{>i}))$; here I_j is the set of unfixed scalar free labels at scale j , and $\partial_{\zeta_{j,\beta}} C_{i+1}$ is evaluated at $\tau \zeta_{>i}$ without an extra factor τ . Each Taylor coefficient of this integral carries the factor $1/(n+1)$ when the remaining integrand has degree n in τ . Likewise, at fixed couplings,

$$dz_i = (\partial_v z_i^*) dv + (\partial_c z_i^*) dc_{i+1} + g_i \{ (\partial_v S_i) dv + (\partial_c S_i) dc_{i+1} \} w_i + g_i S_i dw_i,$$

and none of these terms may be dropped. The inverse square root S_i is treated as a constituent tensor with the bounds of Theorem 20.2; if it is expanded, (20.19) and the Neumann series of the resolvent around $K_i(0,0)$ give a chain of ordinary matrix products whose coefficients are its network refinements. The comparison of determinant matrices with the same indices also includes the correction (20.21), which is a further constituent tensor with a visit to Ω^c .

Step 4: the incoming insertion and the boundary differences. We write $\mathcal{A}(v, \zeta) = \text{Log } C(v, \zeta; b_{\text{rec}})$. Then

$$R(\mathcal{A}(v, \zeta)) - R(\mathcal{A}(v, 0)) = \sum_{n \geq 1} \frac{1}{n!} D^n R(\mathcal{A}(v, 0)) [\mathcal{A}(v, \zeta) - \mathcal{A}(v, 0), \dots, \mathcal{A}(v, \zeta) - \mathcal{A}(v, 0)],$$

and each factor $\mathcal{A}(v, \zeta) - \mathcal{A}(v, 0)$ has positive degree in ζ . The coefficient of $v^\nu \zeta^\eta$ is the sum over n , over the contracted indices $j_1, \dots, j_n$ of the old lattice, and over the allocations of the multi-indices to the n factors with positive free degree in each, of the products of the jet coefficients of R and the coefficients of the factors. No additional

multinomial factor occurs, since each coefficient is already a multi-index coefficient. Each such occurrence keeps the union of the old network of R , scaled by $B_s^{-1} = M^{-s}$ in output units, and the attached reconstruction networks, and its fluctuation support is the set of blocks used by η . By the jet bound of Step 1 with $j = 1$ at an image bounded by $\rho/2$, the derivative series of R at the image is at most $4H\rho$. The column majorant of the reconstruction with a free index fixed vanishes at zero free radius and is linear in the free radius on the fixed product, so it is at most $C_J|z|p$. Fixing an external input in the old counting convention contributes the fixed volume ratio $\Gamma_J = \max_i v_i/v_{\text{old}} \leq M^{3J}$ between the input mesh and the old mesh. We obtain the incoming bound, linear in H and starting at $|z|p$.

For a difference between the regional and the restricted torus constructions we compare factor by factor, before any expansion of logarithms. In each product of coefficients we use the ordered identity

$$\prod_{r=1}^m A_r^1 - \prod_{r=1}^m A_r^0 = \sum_{r=1}^m \left(\prod_{l < r} A_l^1 \right) (A_r^1 - A_r^0) \left(\prod_{l > r} A_l^0 \right),$$

and for the incoming function at two reconstructed arguments C_0, C_1 in the identity chart (here C_0, C_1 denote arguments, not the constants of Lemma 20.3),

$$R(C_1) - R(C_0) = \int_0^1 DR(C_0 + s(C_1 - C_0)) [C_1 - C_0] ds.$$

Both images under the absolute reconstruction majorant lie in the common product with half the incoming radius, so the whole segment does. Terms involving the incoming boundary function use its incoming boundary weight; torus terms acquire the visit carried by the difference of the reconstructions. Fixed input slots are visits to Ω^c . Changed formulas, minimizers, inverse square roots, and Haar densities are treated by the same identity, with $-A^{-1}(A - B)B^{-1}$ for inverses and (20.19) for inverse square roots. Each term contains one changed factor with its visit to Ω^c . The versions with every index rooted in Theorem 19.9 keep every differentiated background or free index, even when the changed factor is not differentiated; the degree-one terms are therefore included.

We join the incoming networks, scaled by B_s^{-1} , to the chart networks at their contracted indices. The transport of the route to the fixed complement is exactly (19.42), and Lemma 18.15 sums the remaining internal indices. We append the resulting route to the network of the monomial. One fixed allocation of the available exponents bounds $e^{s_0 \ell_\lambda}$, including this route and the fixed stencil offsets; the offsets contribute a fixed factor in the degree, absorbed on the margin already provided by the construction. The allocation is the same for every order and every contraction, and no exponent is lost at each vertex. The scaled incoming networks and routes are bounded by the inequalities against $B_s a_{\text{in}}$ and $B_s \alpha_{\text{in}}$ imposed in the statement. Every term still contains an incoming factor, so its

bound is linear in H and starts at $|z|p$. The arguments of Step 5 apply to this majorant with a visit to Ω^c and give (20.33). Finally, a torus monomial deleted because it uses a missing fluctuation block already has a visit at that block; if it has an unfixed root, its network provides the route to Ω^c .

Step 5: conversion to W_{16} and D_{16} . By Steps 1 to 4, there exist a free radius $V_1 > 0$, strictly inside all image and stage margins, and a common bound A_0 for the coefficientwise majorants of the diagrams on the outer profile $\hat{\phi}$ and at free radius V_1 , including each permitted fixed coefficient index and the volume ratios of part (4) of Lemma 20.1. For the incoming family the bound is A_0H , and for a difference it is linear in the bound of the changed factor. We write the resolved coefficients of free degree q and background degree k as $u_\lambda^0 v^{\nu_\lambda} \zeta^{\eta_\lambda}$, after removing the factor z^{q-2} for the classical terms and z^q for the others; the ratios λ_i have modulus at most one. The mixed factorial identity is

$$\frac{1}{k!q!} \sum_{\text{other ordered indices}} |D_v^k D_\zeta^q(u^0 v^\nu \zeta^\eta)| = \begin{cases} (\eta_l/q)|u^0|, & \text{one free input } l \text{ fixed,} \\ (\nu_i/k)|u^0|, & \text{one background input } i \text{ fixed,} \end{cases}$$

for $q \geq 1$ and $k \geq 1$ respectively, and the radii of all inputs multiply these expressions. With $\tilde{C} = \max\{1, V_1^{-1}\}$, Lemma 20.1 and the finite majorant at V_1 give

$$\sup_l \sum_{\lambda: (k_\lambda, q_\lambda) = (k, q)} \frac{\eta_{\lambda, l}}{q} |u_\lambda^0| \hat{\phi}^{\nu_\lambda} e^{s_0 \ell_\lambda} \leq A_0 \tilde{C}^q, \quad \sup_i \sum_{\lambda: (k_\lambda, q_\lambda) = (k, q)} \frac{\nu_{\lambda, i}}{k} |u_\lambda^0| \hat{\phi}^{\nu_\lambda} e^{s_0 \ell_\lambda} \leq A_0 \tilde{C}^q.$$

These bounds concern the total variation of the resolution produced by composition of coefficientwise majorants.

Let $\vartheta = K_*^{-1/2} < 1$. Passing from $\hat{\phi}$ to ψ_{prof} multiplies a term of background degree k by exactly ϑ^k , independently of the positions of its background indices in the spatially varying radius profile. Each term has $|S_\lambda| \leq q$, and selecting a fluctuation block is bounded by the sum of its three scalar multiplicities. At fixed k, q , the preceding bounds therefore contribute at most $3q A_0 \tilde{C}^q \vartheta^k$ for a fluctuation root and $k A_0 \tilde{C}^q \vartheta^k$ for a background root; the latter bound is uniform in the root at each degree. Taking the supremum over the root at each degree and only then summing over k , and using $q \leq 2^q$, we find that the sums in the definitions of W_h and D_h , at fixed q , are at most

$$A_0 \tilde{C}^q e^{hq} \left(\frac{3q}{1-\vartheta} + \frac{\vartheta}{(1-\vartheta)^2} \right) \leq AB^q, \quad A = A_0 \left(\frac{3}{1-\vartheta} + \frac{\vartheta}{(1-\vartheta)^2} \right), \quad B = 2e^h \tilde{C}.$$

Multiplying by $|z|^{q-2} p^q$, $q \geq 3$, or by $|z|^q p^q$, $q \geq 1$, and summing the geometric series, we obtain on $B|z|p < 1$

$$W_h^{\text{cl}} + D_h^{\text{cl}} \leq \frac{AB^3 |z| p^3}{1 - B|z|p}, \quad W_h^{\text{Haar}} + D_h^{\text{Haar}} \leq \frac{AB|z|p}{1 - B|z|p}, \quad W_h^{\text{inc}} + D_h^{\text{inc}} \leq \frac{ABH|z|p}{1 - B|z|p}$$

for the classical, Haar-density, and incoming families. The same bounds hold for the differences with a visit to Ω^c , with their majorants. Note that the classical bound starts with $|z|p^3$: the two coupling powers removed by the division by g_i^2 are not restored. For $h = 16$, enlarging the constant to a common C_J and restricting the disc accordingly gives the assertions of the lemma. Choosing the auxiliary disc smaller by this fixed factor keeps it inside the same larger free radius, and neither an input radius nor the ratio K_* changes. Finitely many additional background marks contribute a polynomial factor in k , which is summable since $\sum_k (k+1)^m \vartheta^k < \infty$. The argument never interchanges $\sup_i \sum_k$ with $\sum_k \sup_i$. The bounds at fixed real frozen values are proved first and are uniform, so taking the supremum over the frozen values afterwards preserves them. All coefficientwise majorant series converge on the same fixed larger products, and finite truncations and the absolute majorants justify the displayed sums. $\square$

Lemma 20.8. *Let $\pi = \otimes_x \pi_x$ be a finite product of probability measures on fluctuation blocks, each π_x supported in the ball $\{|w_x| \leq p_x\}$ with $p_x < \infty$; these are the radii used in b_λ , W_{16} , and D_{16} . Use the same working rates $a, \alpha > 0$ as in $\mathcal{X}^\phi$, and assume $s_0 \geq a + \alpha$. Consider monomial families as in Definition 20.6, with nonempty supports S_λ and networks through all of their positions, and put $\epsilon = W_{16}(U) + D_{16}(U)$. There is an absolute constant $\epsilon_0 > 0$ such that $\epsilon < \epsilon_0$ implies the following.*

1. *The product expansion followed by the hard-core polymer expansion, with incompatibility $S_\lambda \cap S_\mu \neq \emptyset$, gives a normally convergent logarithm of $\int e^{-\sum_\lambda U_\lambda} d\pi$, equal to zero at zero interaction, and this integral does not vanish in the coefficient ball. Neither spatial connectedness of S_λ nor membership of the background positions in S_λ is needed.*
2. *Every resolved term of the logarithm carries the connected union of the networks of its monomials, with no additional length. Its absolute sum rooted at a fluctuation block, with the weight $e^{4|S|}$ where S is the union of the supports of its monomials, is at most $C\epsilon$. After passage from ψ_{prof} to ϕ , the sum over the background degrees of its rooted coefficient suprema is at most $C\epsilon$.*
3. *For one marked family A the same bounds hold with $C\epsilon_A$. Let $U(s, r) = U + sB + rA + srC$, $0 \leq s, r \leq 1$, where B carries the boundary change, A the incoming insertion, and C both (here s, r are interpolation parameters and B, C are families of monomials). Assume that every occurrence in B and C has an assigned network visiting $\Omega_{L_J}^c$, with that visit included in ℓ_λ and in its weighted bound, and that every interpolated family lies in the same fixed small ball. Then the mixed derivative $\partial_s \partial_r$ of the logarithm and its four-corner difference are bounded, in both rooted norms after passage of profile, by*

$$C(\epsilon_{\partial A} + \epsilon_{\partial} \epsilon_A), \quad (20.35)$$

where $\epsilon_{\partial}, \epsilon_A, \epsilon_{\partial A}$ are the values of $W_{16} + D_{16}$ for B, A, C . A term containing the boundary change keeps a visit to $\Omega_{L_J}^c$. Its network can be extended to Ω^c by a segment of length at

most the fixed enlargement constant L_0 of Lemma 20.1(5), whose cost is absorbed in the fixed constants. A term containing both changes keeps an incoming factor. These bounds keep the weight of the network length. In particular, for a difference F of logarithms containing the boundary change, on the working profile,

$$\mathcal{X}^\phi(F) \leq C_J(\epsilon_{\partial A} + \epsilon_{\partial} \epsilon_A)$$

in the case of two changes, and $\mathcal{X}^\phi(F) \leq C_J \epsilon_{\partial}$ in the case of one change.

4. The logarithm and its derivatives in the changes are holomorphic on this ball.

The combinatorial constants are absolute, with the dependence described in item (1) of Definition 4.14; the constants of the passage of profile, of the density, and of the spatial conversion have the dependence described in item (3).

Proof. The proof follows the standard two-stage cluster expansion; we give the estimates so that the root and the support weights stay visible. We first truncate to finitely many monomials.

Step 1: the product expansion. Let $f_\lambda = e^{-U_\lambda} - 1$. Its absolute coefficient majorant is at most $e^{b_\lambda} - 1 \leq 2b_\lambda$ for small ϵ . Indeed $|w_x| \leq p_x$ holds π_x -almost surely, so the polynomial bounds defining b_λ bound the integrands on the support of the product measure. The normalized truncated Gaussian measures on balls used later satisfy this hypothesis. We expand the exact identity

$$e^{-\sum_\lambda U_\lambda} = \prod_\lambda (1 + f_\lambda).$$

We group each selected family of distinct indices by the connected components of the graph in which λ, μ are joined when $S_\lambda \cap S_\mu \neq \emptyset$. Different components depend on disjoint sets of fluctuation blocks, so their expectations factor, even if their background parameters coincide. This is an exact hard-core gas whose polymers are unions of fluctuation supports. Spatially disconnected sets may serve as polymers, because compatibility uses only set intersection and the independence of the reference variables.

Step 2: the tree majorant at the first stage. Let $n_\lambda = |S_\lambda|$ and $a_\lambda = 2b_\lambda e^{10n_\lambda}$. The majorant by rooted trees is the increasing iteration from zero of

$$T_\lambda = a_\lambda \exp\left(\sum_{\mu: S_\mu \cap S_\lambda \neq \emptyset} T_\mu\right). \quad (20.36)$$

Every connected selected family has a spanning tree; expanding the exponential supplies the factorials for unordered children, and dropping the distinctness of the indices only increases this sum of nonnegative terms. If $T_\mu \leq a_\mu e^{n_\mu}$ for all μ , then the exponent in (20.36) is at most

$$n_\lambda \sup_x \sum_{\mu: x \in S_\mu} a_\mu e^{n_\mu} \leq 2\epsilon n_\lambda.$$

For $2\epsilon \leq 1$ the candidate $a_\lambda e^{n_\lambda}$ therefore bounds every iterate. Its sum rooted at a fluctuation block is at most 2ϵ . Thus the polymer activities are absolutely summable with the weight $e^{10|S|}$.

Step 3: the second stage. We apply the finite graph inequality of Theorem 12.6 to this hard-core gas; only its bound on alternating sums over connected graphs is imported. Let z_γ be the absolute polymer activities and $n_\gamma = |S_\gamma|$. Step 2 gives $\sup_x \sum_{\gamma: x \in S_\gamma} z_\gamma e^{10n_\gamma} \leq 2\epsilon$. The second rooted-tree majorant is the increasing iteration from zero of

$$U_\gamma = z_\gamma e^{4n_\gamma} \exp\left(\sum_{\eta: S_\eta \cap S_\gamma \neq \emptyset} U_\eta\right), \quad U_\gamma \leq z_\gamma e^{5n_\gamma}.$$

Indeed, under the displayed candidate bound the child sum is at most

$$n_\gamma \sup_x \sum_{\eta: x \in S_\eta} z_\eta e^{5n_\eta} \leq 2\epsilon n_\gamma \leq n_\gamma \quad (2\epsilon \leq 1),$$

so the candidate is a supersolution and its sum rooted at a fluctuation block is at most 2ϵ . This gives the fluctuation-rooted bound of part (2) with weight $e^{4|S|}$. Repeated polymer occurrences are allowed, as in the finite graph theorem. Thus the input weight is $e^{16|S|}$, the polymer activities have weight $e^{10|S|}$, and the logarithm has weight $e^{4|S|}$.

Step 4: the networks. We keep the resolution into monomials inside each exponential and each polymer. Within a connected family of the first stage, the networks of adjacent factors share a fluctuation position. At the second stage, adjacent polymers share a fluctuation block used by a factor in each of them, so their networks also meet at that position. By induction along a spanning tree at each stage, the union is connected and

$$\ell(\mathcal{N}_{\text{union}}) \leq \sum_{\lambda \text{ occurrences}} \ell_\lambda. \quad (20.37)$$

Coincident positions of different color coordinates have distance zero. Copies of the same monomial in an exponential have the same support and network. A visit to Ω^c remains a visit in the union. Thus the network weights multiply, with no extra connecting length at either stage.

Step 5: the background root. Fix a background coordinate i in a coefficient of a product. The identity $v_i = \sum_\lambda v_{\lambda,i}$ assigns the derivative in i to one monomial occurrence, with its multiplicity. We root the spanning tree of the first stage at this factor and, at the second stage, at the polymer containing it. The estimates for the children just proved bound the remaining part of the expansion by $Ce^{cn_\lambda} b_\lambda$ with $c < 16$, so that the total support weight used is less than e^{16n_λ} . At the distinguished factor we use the degree- k term in D_{16} instead of its norm rooted at a fluctuation block. This works even if the background position is

not a fluctuation position. The other factors are summed through their intersections of fluctuation supports.

Fixing i and summing over all output degrees at the intermediate profile, we obtain the bound $C \sum_k d_k \leq C\epsilon$, where d_k is the degree- k supremum in D_{16} for the distinguished monomial. Indeed, the remaining part of the expansion is summed at its fixed fluctuation root, and its total coefficient majorant there is bounded by the estimates for the children. We thus obtain a bound on $\sup_i \sum_K$ at the intermediate profile, which is not yet the coefficient norm with boundary decay. Let $q_* = K_*^{-1/2} < 1$. At output degree K , passing to the working profile multiplies every coefficient by q_*^K . Thus, taking the supremum separately at each degree,

$$\sum_{K \geq 1} \sup_i \{\text{output coefficients of degree } K \text{ rooted at } i, \text{ at } \phi\} \leq C\epsilon \sum_{K \geq 1} q_*^K.$$

This is the required order of operations; a bound on $\sup_i \sum_K$ at the same radius would not imply a bound on $\sum_K \sup_i$. The second half of the ratio between the profiles is used here, once for each local map on the fixed profiles. The factor v_i/K in the coefficient norm with boundary decay only decreases the multiplicity bound. Repeated copies in an exponential obey the same identity for v_i , and differentiating the exponential accounts for all possible distinguished occurrences.

Step 6: one and two changes. The tree estimates also hold in the rooted norms with changes. Differentiating (20.36) once gives

$$T'_\lambda = e^{\sum_{\mu \sim \lambda} T_\mu} a'_\lambda + T_\lambda \sum_{\mu \sim \lambda} T'_\mu,$$

where $\mu \sim \lambda$ means $S_\mu \cap S_\lambda \neq \emptyset$. In the norm rooted at a fluctuation block, the last operator has norm at most $C\epsilon < 1/2$, since its incidence factor $n_\lambda \leq e^{n_\lambda}$ is bounded by a factor e^{n_λ} of the support weight that is not used in the output bound. Its inverse is a Neumann series with nonnegative terms and norm at most two. Hence a first variation is bounded by the norm of the changed source. Writing $h_\lambda = \sum_{\mu \sim \lambda} T_\mu$, the mixed variation satisfies

$$T_{\lambda, sr} = e^{h_\lambda} a_{\lambda, sr} + e^{h_\lambda} (a_{\lambda, s} h_{\lambda, r} + a_{\lambda, r} h_{\lambda, s}) + T_\lambda h_{\lambda, s} h_{\lambda, r} + T_\lambda \sum_{\mu \sim \lambda} T_{\mu, sr}.$$

Its inhomogeneous terms are a source with both changes, products of the two first variations, and a source with one change times a first variation. Their bounds are $C(\epsilon_{\partial A} + \epsilon_{\partial} \epsilon_A)$, and the same inverse bounds the mixed variation. The polynomial incidence factors have degree at most two, and $\sup_{n \geq 1} n^2 e^{-n} < \infty$. In the bilinear interpolation, terms containing the family C are bounded by a constant times $\epsilon_{\partial A}$, since the other monomial norms are bounded on the fixed small ball. Integrating the mixed

derivative over the unit square, we obtain the four-corner difference, with the same bound. These calculations apply at both stages of the expansion. For a background root, we first sum the resulting rooted expressions at the distinguished factor at the intermediate profile, with the sequence d_k described above, and then apply the factor q_*^K at each output degree. If the background root lies on a factor different from both changes, the remaining expansion has at most one additional distinguished branch. Its cubic incidence factor is bounded by a further factor e^{n_λ} of the support weight, since $\sup_{n \geq 1} n^3 e^{-n} < \infty$; the passages from $e^{16|S|}$ to $e^{10|S|}$ and from $e^{10|S|}$ to $e^{4|S|}$ leave these factors available. The source with both changes keeps its incoming factor, and the product source contains the first variation in the incoming insertion. This proves (20.35) in both rooted norms, including the case of a background root on an unchanged factor.

Step 7: spatial conversion. We keep the joined networks. By the hypothesis on B and C , every term containing the boundary change carries a visit to $\Omega_{L_J}^c$. If a coefficient with unfixed background root i carries such a visit, then by part (5) of Lemma 20.1

$$at(v) + \alpha d_\Omega(i) \leq (a + \alpha) \ell(\mathcal{N}_{\text{union}}) + C_J \alpha.$$

Since $s_0 \geq a + \alpha$, the product of the weights $e^{s_0 \ell_\lambda}$ in b_λ dominates $e^{(a+\alpha) \ell(\mathcal{N}_{\text{union}})}$, by (20.37). The fixed buffer is used only at the final visit to Ω^c . The factor v_i/K in (4.11) is at most the multiplicity of the distinguished index. The bound at the intermediate profile and the degree sum $\sum_{K \geq 1} q_*^K$ just proved, multiplied by the fixed factor ρ^{-2} and the buffer constant, give the stated bounds on $\mathcal{X}^\phi$.

Step 8: the logarithm. The bounds on the coefficientwise majorants prove the normal convergence of every resolved series and of the displayed derivatives. For a small multiple of the interaction, the two series are the Taylor expansion of the logarithm of the finite product integral. Their exponential agrees with the integral throughout the strict ball, by the identity theorem in that multiplier. Hence the integral does not vanish there. The uniform absolute bounds allow us to remove the truncations to finitely many monomials and to differentiate termwise in the parameters, and they prove holomorphy in the absolute coefficient norms. We have not used the hypothesis on background supports in the holomorphy theorem of Section 12. $\square$

Theorem 20.9. Fix the torus rate $a_{\mathbb{T}} > 0$ and the ratio K_* between the profiles $\hat{\phi} = K_* \phi$ and $\psi_{\text{prof}} = \sqrt{K_*} \phi$. There are geometric constants $C_{\text{alloc}} \geq C_{\text{geom}}$ and $C_g \geq 2$, independent of J , with the following property; here C_{geom} includes the transport allocation of Theorem 19.9(4). Choose $C_*^{\text{geom}} \geq C_g$ and choose J once with

$$B = M^J \geq B_{\text{geom}}, \quad \text{where } B_{\text{geom}} = 256 C_{\text{alloc}} (C_*^{\text{geom}} + 1).$$

After J has been fixed, choose positive rates

$$a = a_{\text{cl}} = a_{\text{sp}}, \quad \alpha = \alpha_{\text{cl}} = a/16, \quad m_{\text{rate}} = C_*^{\text{geom}}(a + \alpha), \quad b_1 = 8 C_*^{\text{geom}} \alpha$$

so that

$$m_{\text{rate}} + b_1 < \min(c_{\text{prim},J}, Ba) / C_{\text{alloc}}, \quad b_1 < B\alpha / C_{\text{alloc}}, \quad \sqrt{3}(a + \alpha) < a_{\mathbb{T}},$$

where $c_{\text{prim},J}$ is the fixed positive reserve for the intrinsic tensors and the unperturbed weighted reconstruction images, in the graph metric of Section 20.1, chosen no larger than the intrinsic exponent c_J of Theorem 19.9. The inherited perturbative networks use the separate inequalities against Ba and $B\alpha$ from Theorem 19.9; their images satisfy the half-radius bound at the allocated working rates.

Consider a full renormalization group step ($s = J$) on admissible data. Sum the internal indices first, keep networks through all positions of the monomials, and use the exact nonempty fluctuation supports of Definition 20.6 in both expansions. All factors and polymer occurrences that are joined share a fluctuation position; the union of their networks keeps every background index and every visit to Ω^c , and adds no length. Set $p = \max_i p_i \geq 1$, $x = \mathcal{X}^p(f)$, and $H = E + x$, and substitute the complex parameter z for the coupling powers as described before Definition 20.6, with the insertion $A_R(z)$. Let Ξ_0 and Ξ_R be the integrals of e^{-U} against the product of normalized symmetric reference measures on balls, for the interaction U without insertion and for U plus the insertion, respectively. For the matched torus potential, with the fixed background slots set to zero, define its restriction by deleting the monomials that use a missing fluctuation block, and adjoin the missing blocks with the identical normalized reference measures.

There are $\epsilon_J, c_{z,J} > 0$ and $C_J < \infty$ such that the following holds if

$$\mathfrak{s}(z) = \frac{C_J |z| (p^3 + p + Hp)}{1 - C_J |z| p} \leq \epsilon_J, \quad C_J |z| p < 1.$$

The integrals without and with insertion, the restricted torus integrals, and the linearly interpolated integrals do not vanish, and their logarithms are the normally convergent expansions of Lemma 20.8. The regional and restricted families satisfy $W_{16} + D_{16} \leq C_J \mathfrak{s}(z)$; their boundary-difference families satisfy the same bound with networks visiting Ω^c . The comparisons of the nonconstant parts of the logarithms, including the deleted torus terms with an unfixed background index, satisfy

$$\mathcal{X}^\phi(\log \Xi_\bullet^\Omega - (\log \Xi_\bullet^\mathbb{T})|) \leq C_J \mathfrak{s}(z), \quad \bullet \in \{0, R\},$$

where the bar denotes zero extension of the fixed background slots; constants supported in Ω^c remain separate. The mixed difference of the boundary change and the insertion is bounded by $C_J(\epsilon_{\partial A} + \epsilon_{\partial A})$ in the norms of Definition 20.6, and every term keeps an incoming factor. The family extends holomorphically to $z = 0$, and its compared insertion logarithm F satisfies $F(0) = F'(0) = 0$. In particular, if

$$r_z = \frac{c_{z,J}}{p^3 + p + Hp}, \quad 0 < t_{\max} \leq r_z/2,$$

then

$$\mathcal{X}^\phi(D^\Omega R - (D^\mathbb{T} r_{d_0})|) \leq C_J t_{\max}^2 H(p^4 + Hp^2), \quad DR = -\log \Xi_R + \log \Xi_0.$$

After the whole boundary layer with reduced radius has been frozen as in Lemma 18.9, the classical and quantum comparisons return the same rates a, α and the same input radii; the torus rate remains $a_{\mathbb{T}}$. The constants have the dependence described in item (3) of Definition 4.14 and are independent of the later widths. The return of the same profile is asserted only for full steps with $B \geq B_{\text{geom}}$, not for compositions of fewer than J averaging steps.

Proof. The constants C_{alloc} and $C_g \geq 2$ include the fixed contraction and transport allocation needed to retain a strict positive reserve above $s_0 = a + \alpha$ in the constituent difference rows; in particular $m_{\text{rate}} \geq 2s_0$. They bound the contraction with every index rooted and the transport of the input boundary weight in Theorem 19.9 and Lemma 18.15; they include the factor $\sqrt{3}$ of the conversion between Euclidean and graph exponents in Section 20.1, and hence do not depend on J . The torus coefficients, bounded in $\mathcal{B}_{\rho, a_{\mathbb{T}}}$ with Euclidean network lengths, have graph exponent $a_{\mathbb{T}}/\sqrt{3} > a + \alpha$ by the last inequality in the statement. With $\alpha = a/16$ we have $m_{\text{rate}} + b_1 = (25/16)C_*^{\text{geom}}a$, and both restrictions against Ba and $B\alpha$ follow from the lower bound on B . At this fixed J , we decrease $a > 0$ to satisfy the bounds on the exponents of the coefficients of the potential, of the weighted images, and of the torus, including $\sqrt{3}(a + \alpha) < a_{\mathbb{T}}$. No constant depending on J is used to choose J . The comparison of the classical boundary networks uses the same allocation of the decay rate and no expansion with respect to a reference product. At zero fluctuation it works as follows. Consider a classical comparison, or the comparison of the pullback terms, with the boundary tensors rooted at every index. At background degree $k \geq 1$, its rooted absolute sum on $\hat{\phi}$ is bounded by its tensor majorant for fixed J , and that allocation already accounts for the weights of its output network and of its distance to Ω^c . Passing to ϕ gives the factor K_*^{-k} . Summing over the degree, including any fixed polynomial factor for the choice of a distinguished index, we obtain the bound in the coefficient norm with boundary decay. For a pulled-back incoming boundary term, we use its jet majorant at half the radius and its distinguished input columns before this calculation. Thus the classical coefficients are converted without assigning a fluctuation support to a term at zero fluctuation, and these terms never enter either expansion.

We apply Lemma 20.7 at these rates and then Lemma 20.8. The hypotheses of the latter on the rates and on the boundary-change families hold, since Lemma 20.7 uses $s_0 = a + \alpha$ and gives the boundary differences with their networks visiting Ω^c . Restricting a torus family decreases its norms, which are sums of absolute values. For its difference from the full family, every deleted coefficient with an unfixed background root visits a missing fluctuation position and hence Ω^c , and the same lemma applies. Pure background constants without an unfixed index have zero norm with boundary decay and stay outside this comparison. The estimate with two changes implies the comparison of the logarithms with and without insertion. The logarithms and their derivatives in the parameters are determined by the finite identities and normal convergence proved above, not by a choice of a complex logarithm after integration.

We choose $c_{z,J}$ so that the disc $|z| \leq r_z$ lies strictly inside the margins of Lemma 20.7, of the free product, and of the logarithm. On this disc, (20.33) and the bounds with one and two changes give

$$\|F(z)\|_{\mathcal{X}^\phi} \leq C_J |z| H p.$$

At $z = 0$ every fluctuating monomial vanishes. The coefficient of z is cubic in the fluctuation variables for the classical terms and linear for the quantum, Haar density, determinant, and incoming terms. Each of these is odd under the simultaneous reversal $w \mapsto -w$, so its expectation under the symmetric reference measures on balls vanishes. At first order the logarithm contains a single monomial occurrence, hence $F(0) = F'(0) = 0$ coefficientwise, at every real frozen value.

The normally convergent series $F(z)/z$ is holomorphic, vanishes at zero, and has norm at most $C_J H p$. Cauchy's coefficient bound on $|z| = r < r_z$, followed by $r \uparrow r_z$, gives for $t \leq r_z/2$

$$\|F(t)\|_{\mathcal{X}^\phi} \leq 2C_J t^2 H p / r_z \leq C'_J t^2 H (p^4 + H p^2),$$

where in the last step we used $p \geq 1$. The integral proof of the coefficient bound applies to the supremum over the fixed index at every background degree, and then to their sum, by the triangle inequality.

Finally, Lemma 18.9 restores the incoming radii when the whole zone with reduced radius is frozen with the stated bounds on the frozen values. All conversions of coefficients were made on the fixed outer and intermediate profiles, and none is iterated at later steps. This proves the return of the same profile and the stated independence of the constants. $\square$

20.5 Effective interactions and normalization constants

Theorem 20.10. *Consider a full renormalization group step, $s = J$, on admissible data as in Theorem 20.9. Set*

$$P^\Omega R(v) = R(\text{Log } C^\Omega(v, 0; b_{\text{rec}})), \quad Q^\Omega = B_{\text{ref}}^\Omega - \sum_i f_i \log z_{p_i} - \log \Xi_0^\Omega,$$

$$B_{\text{ref}} = \sum_i B_i(\widehat{c}_{i+1}),$$

where f_i is the number of free blocks at scale i and $z_p = \int_{|w| \leq p} e^{-|w|^2/2} d^3 w$, in the convention of Theorem 12.13. The field-independent factors of the block averaging constraint and of the Haar measure extracted from L_i are kept in Λ . Up to the exact constant kept in Λ , the new effective interaction $M^\Omega R$ produced by the step satisfies

$$M^\Omega R = Q^\Omega + P^\Omega R + D^\Omega R.$$

Here Q^Ω is the newly generated term, $P^\Omega R$ is the pullback of the incoming interaction by the zero-fluctuation reconstruction, and $D^\Omega R$ is the fluctuation correction of Theorem 20.9. For the matched torus reference with zero extension of the fixed slots, there are constants A_J , K_J , and C_J , independent of the later widths, such that

$$\begin{aligned} \mathcal{X}^\phi(Q^\Omega - Q^\mathbb{T}|_{\Omega'} + P^\Omega(r_{d_0}|\Omega) - (P^\mathbb{T}r_{d_0})|_{\Omega'}) &\leq A_J(1 + E), \\ \mathcal{X}^\phi(P^\Omega f) &\leq K_J \mathcal{X}^\rho(f). \end{aligned}$$

Here Ω' is the set of unfixed scalar output labels. Assume that the column bounds of Theorem 19.6 and their perturbed versions hold on the product with doubled output analytic radii, and that the image under the reconstruction majorant on this product is at most half the incoming radius, as in Lemma 19.7. Then one may take $K_J = 4(\log 2)K_{\text{col},J}$. Here $K_{\text{col},J}$ bounds, for one fixed output, the sum over all old input labels, weighted by their boundary decay, of the distinguished derivative series of the perturbed zero-fluctuation reconstruction; it is chosen uniformly on the preliminary perturbation ball by Theorem 19.9. The degree-one terms and every nonconstant dependence of an omitted torus density on the unfixed labels are included.

At every admissible complex value of the coarse background and every fixed real admissible frozen value, let I_i and H_i be the sets of unfixed and of missing scalar free labels at scale i . With matched torus arguments extended by zero,

$$\log \det K_i^\Omega = \sum_{x \in I_i} (\log K_i^\mathbb{T})_{xx} + \mathcal{B}_i, \quad |(\log K_i^\mathbb{T})_{xx}| \leq C_J, \quad |\mathcal{B}_i| \leq C_J |\partial_{\text{blk}} \Omega|.$$

These determinant statements also hold in the coefficient norm before evaluation. Scalar determinant labels and fluctuation blocks are counted separately. All constants have the dependence described in item (3) of Definition 4.14 and are independent of the later widths.

Proof. Every step here is a full step, $s = J$; a final composition of fewer than J averaging steps that is evaluated deterministically adds no fluctuation integral.

Step 1: the exact factorization. The triangular calculation in the proof of Lemma 20.3 gives, at a fixed coarse background and fixed frozen values, the value of the integral

$$\left(\prod_i \mathcal{J}_{i,\text{ref}} g_i^{3f_i} \right) e^{-F_{\text{out}}/g_s^2 - B_{\text{ref}} - R(\text{Log } C(v, 0; b_{\text{rec}}))} \prod_i z_{p_i}^{f_i} \Xi_R(v), \quad (20.38)$$

where F_{out}/g_s^2 is the classical value of the step with the output coupling, and the $\mathcal{J}_{i,\text{ref}}$ are the extracted constant Haar disintegration factors. Inserting the definitions of Q , P , and D into the exponent of (20.38), and writing $\log \Xi_R = \log \Xi_0 - DR$, we obtain $MR = Q + PR + DR$. This proves the recursion, including its signs and its Gaussian convention. Centering moves the value at zero into Λ and changes no factor of this identity.

Step 2: the generated term and the torus pullback. The deterministic part of $Q^\Omega - Q^\mathbb{T}$ is the difference of the Haar densities at the minimizers and of the diagonal matrix-logarithm densities along the two zero-fluctuation chains. Their rooted comparisons with boundary decay are part (3) of Theorem 20.2. A density based at a missing torus label is not discarded: a nonconstant coefficient with an unfixed index has a network to that missing label, and its sum over the missing label is bounded from the unfixed index by (20.2). Its constant is left in Λ . The terms $\log z_{p_i}$ and the constant Haar factors have no nonconstant background coefficient. The difference of the logarithms without insertion has the same fixed rates by Theorem 20.9, with a bounded norm in the cluster expansion. We obtain a finite bound C_J for the comparison of the nonconstant generated terms.

For the torus pullback, denote by a_Ω the zero extension of $\text{Log } C^\Omega(v, 0; b_{\text{rec}})$ and by $a_\mathbb{T}$ the matched torus argument; in this paragraph $a_\mathbb{T}$ is an argument, not the torus rate. We have

$$r_{d_0}(a_\Omega) - r_{d_0}(a_\mathbb{T}) = \int_0^1 Dr_{d_0}(a_\mathbb{T} + \tau(a_\Omega - a_\mathbb{T}))[a_\Omega - a_\mathbb{T}] d\tau. \quad (20.39)$$

Both absolute images are below half the incoming radius, so the whole segment admits the coefficient expansion of the integrand. On an unfixed old input, the difference of the arguments is the unperturbed defect of the reconstruction followed by the gauge transformation (Theorem 19.6) plus the defect of the perturbation (Theorem 19.9). On a fixed old input, the difference is a visit to Ω^c . The action and interaction tensors join every output index to that visit, even if the source does not decay inside its own component of Ω^c . The jet bound (20.29) with $\sigma_0 = 0$ bounds the integrand by $C_J E$, with its visit. The conversion of networks and degrees in the proof of Theorem 20.9 therefore bounds its $\mathcal{X}^\phi$ norm by $C_J E$. Together these bounds prove the first inequality, with A_J fixed on the larger products. No first derivative of the output was canceled.

Step 3: the pullback of the boundary function. For the zero-fluctuation reconstruction $\Phi_e(v) = \text{Log } C_e(v, 0; b_{\text{rec}})$, we define

$$C_{e,x}(2\phi) = \sum_v \nu_x |\Phi_{e,v}| (2\phi)^v e^{a_1 \mathbf{t}(\{e\} \cup \text{supp } v)}.$$

All constants in this step refer to the perturbed map. Let D_J^0 bound the unperturbed distinguished series on the larger product with doubled output analytic radius. The difference of the reconstructions in Theorem 19.9, rooted at every index, with the uniform incoming norm and the inequalities on the conditional forces checked above, bounds the additional distinguished series by $\sigma_* D_J^1$ there. We fix a spatial decay rate $b > \alpha$ and define

$$D_J^{\text{act}} = D_J^0 + \sigma_0 D_J^1, \quad C_{e,x}(2\phi) \leq D_J^{\text{act}}(2\phi_x) e^{-b \text{dist}(e,x)}. \quad (20.40)$$

The factor $2\phi_x$ appears because the input index is distinguished before the sum of absolute coefficients is taken. The rates are below the common exponent available in the coefficient

bounds of the potential, and both D_J^0 and D_J^1 are computed at the preliminary bound on fixed larger products, before the later widths. Here and below, for an old input label e of R (a label of the incoming set D_{in} , on the input lattice of the step), we write $d_{\text{old}}(e) = d_{D_{\text{in}}}(e)$ for its distance to the incoming fixed complement, measured in units of the input lattice as in the incoming norm. For a current unfixed background input x , we write $d_{\text{out}}(x) = d_{\Omega}(x)$ for its distance to Ω^c , in units of the output lattice. Finally, $\text{dist}(e, x)$ is the distance between x and the position e/B_s of e , in units of the output lattice. Applying (20.8) on the old input set of R , which is the same set of unfixed labels, with quantum radius ρ , and dividing by B_s , we obtain

$$d_{\text{out}}(x) \leq d_{\text{old}}(e)/B_s + \text{dist}(e, x) + C_{\text{buf},J}/B_s.$$

In Theorem 19.6 the distance d_{old} is measured in units of the output lattice, and the inequality $d_{\text{out}}(x) \leq d_{\text{old}}(e) + \text{dist}(e, x) + C_J$ is an explicit hypothesis of that theorem. With d_{old} there equal to our $d_{\text{old}}(e)/B_s$, the inequality just displayed verifies this hypothesis, with buffer $C_{\text{buf},J}/B_s$; it holds because the incoming fixed complement lies on the current fixed side. Since $B_s \geq 1$, this also gives the weaker comparison with $d_{\text{old}}(e)$ in place of $d_{\text{old}}(e)/B_s$. The density of the old labels then implies

$$\sum_{e \text{ old unfixed}} e^{-\alpha d_{\text{old}}(e)} C_{e,x}(2\phi) \leq K_{\text{col},J} \rho e^{-\alpha d_{\text{out}}(x)}. \quad (20.41)$$

We define $K_{\text{col},J}$ as a common bound for this inequality, uniformly on the preliminary perturbation ball. One admissible choice is any number at least $2D_J^{\text{act}} e^{\alpha C_{\text{buf},J}/B_s} S_J(b - \alpha)$, with the fixed volume factors included in S_J , and at least the unperturbed column bound; we take the finite maximum over the label types and the stages. The absolute image on the product with doubled output analytic radius is at most $\rho/2$. The construction used a ratio larger than two between the profiles, so no repeated reduction of radii is involved. This is the column calculation of Theorem 19.6, with its constant enlarged by (20.40). The hypotheses on the doubled product and on the incoming image at half the radius are used here.

Let $x_f = \mathcal{X}^\rho(f)$. At an old input root e , the distinguished derivative sum evaluated at $\rho/2$ satisfies

$$\sum_{\eta: \eta_e > 0} \eta_e |f_\eta|(\rho/2)^{|\eta|-1} e^{at(\text{supp } \eta)} \leq 4\rho x_f e^{-\alpha d_{\text{old}}(e)}. \quad (20.42)$$

For degree l this follows by multiplying the normalized coefficient with this input fixed by $l2^{1-l}\rho$, which is at most 4ρ ; the case $l = 1$ is included. The chain rule for coefficientwise majorants, the union of networks, and (20.41) bound the distinguished coefficient sum of the output at 2ϕ by $4K_{\text{col},J}\rho^2 x_f e^{-\alpha d_{\text{out}}(x)}$. At degree k the fixed-index weight of the coefficient norm with boundary decay inserts the factor $1/k$, and evaluation at ϕ instead of 2ϕ inserts 2^{-k} . We take the supremum over the fixed index at each degree and then

sum. Since integrating $\sum_{j \geq 0} u^j = (1 - u)^{-1}$ over $[0, 1/2]$ gives $\sum_{k \geq 1} 2^{-k}/k = \log 2$, we obtain

$$\mathcal{X}^\phi(\mathbf{P}^\Omega f) \leq 4(\log 2)K_{\text{col}, J}x_f.$$

This also shows that a finite K_J exists when no particular column constant is specified.

Step 4: the logarithmic determinants. The diagonal coefficient series of $\log K_i^\mathbb{T}$ is bounded uniformly for each scalar label. To bound the constant as well as the derivatives, we write

$$\log K = \int_0^\infty [(1+t)^{-1}I - (K+t)^{-1}] dt.$$

Near $t = 0$ the gap bounds the integrand. For large t , the resolvent identity with I bounds it by $C_J(1+t)^{-2}$, since $K - I$ has a bounded coefficient row on the larger product. This integrable bound and the derivative bounds with every index rooted allow us to integrate coefficient by coefficient. The exact normalization of the scales adds only the same term $i \log M$ for each scalar label, with $i < J$. Thus $|(\log K_i^\mathbb{T})_{xx}| \leq C_J$ on the stated complex profile, not only at zero background.

The comparison of a regional logarithm with the restriction of the torus logarithm is the sum of (20.20) and (20.21). Its diagonal density at an unfixed root u is bounded in coefficient norm by $C_J e^{-c_J d_\Omega(u)}$. Summing these diagonal defects and using (20.4), we obtain the bound on $\mathcal{B}_i$. If Ω^c is empty, the defect is zero. $\square$

To state the recursion for Λ we fix the counts that enter it.

Definition 20.11. Consider a full step on the block region Ω and the matched torus step. Every scalar label is counted on the lattice on which it is introduced.

1. For each scale i , $|H_i|$ is the number of missing scalar free labels at scale i , that is, free labels of the torus step that are not labels of the regional step.
2. N_{missing} is the number of scalar labels of the torus step (background labels on the output lattice and free labels at the intermediate scales) that are not labels of the regional step. We write $\mathcal{M}_{\text{block}}$ for the set of missing torus fluctuation blocks; since each such block consists of three missing scalar labels, $|\mathcal{M}_{\text{block}}| \leq N_{\text{missing}}$.
3. N_{bd} is the number of scalar labels of the regional step whose positions lie within the fixed distance $w_{\text{band}} \geq 1$ of W , where w_{band} is the width of the interface band determined by the construction stencils; it does not depend on the later widths. This is N_{band} of (20.5), so $|\partial_{\text{blk}} \Omega| \leq N_{\text{bd}} \leq C_J |\partial_{\text{blk}} \Omega|$, including for isolated final blocks. Both counts are zero when W is empty.
4. N_{freeze} is the number of analytic scalar labels that are newly frozen at the end of the step, including, at the first step, the labels frozen at initialization.

When the steps are indexed by t , these counts are written $|H_i|$ for the scales i of step t , and $N_{\text{missing}, t}$, $N_{\text{bd}, t}$, $N_{\text{freeze}, t}$.

Lemma 20.12. *Assume the hypotheses on the products and the small-interaction hypothesis of Theorem 20.9, in a full step, and fix a real admissible coarse background, a real admissible frozen value, and a real coupling $t = t_{\max}$. Use the potential without or with the insertion, from Lemma 20.3, and remove its entire zero-fluctuation part before forming the interaction $U_{\bullet}$, $\bullet \in \{0, R\}$. Let π be the product of the identical normalized three-dimensional Gaussian reference measures on balls, including the reference measures of all missing torus blocks, and write*

$$\Xi_{\bullet}^{\mathbb{T}} = \int e^{-U_{\bullet}^{\mathbb{T}}} d\pi, \quad \Xi_{\bullet}^{\Omega} = \int e^{-U_{\bullet}^{\Omega}} d\pi, \quad \bullet \in \{0, R\}.$$

The regional interaction does not depend on the adjoined variables. With N_{missing} and N_{bd} as in Definition 20.11, there is a restricted interaction $U_{\bullet}^c$ on the same product such that

$$\begin{aligned} \|U_{\bullet}^{\mathbb{T}} - U_{\bullet}^c\|_{\infty} &\leq C_J \mathfrak{s}(t)(N_{\text{missing}} + N_{\text{bd}}), \\ \|U_{\bullet}^{\Omega} - U_{\bullet}^c\|_{\infty} &\leq C_J \mathfrak{s}(t)N_{\text{bd}}. \end{aligned}$$

Consequently, for the positive real logarithms,

$$|\log \Xi_{\bullet}^{\mathbb{T}} - \log \Xi_{\bullet}^{\Omega}| \leq C_J \mathfrak{s}(t)(N_{\text{missing}} + N_{\text{bd}}). \quad (20.43)$$

This includes evaluation at coarse background zero, with the frozen values unchanged. Empty products have integral one.

Proof. The reference measure on a block (i, e) is $z_{p_i}^{-1} \mathbf{1}_{|w| \leq p_i} e^{-|w|^2/2} d^3w$. We take the product of these measures over the union of the two sets of blocks. Adjoining blocks multiplies the regional integral by one, even for blocks at different scales with different radii.

For this comparison of constants only, we resolve the torus monomials as in Definition 20.6, and let $\mathcal{D}_{\text{mon}} = \{\lambda : S_{\lambda} \cap \mathcal{M}_{\text{block}} \neq \emptyset\}$ be the set of deleted monomials. We call the remaining interaction U^c . The bound on W_{16} gives

$$\|U^{\mathbb{T}} - U^c\|_{\infty} \leq \sum_{x \in \mathcal{M}_{\text{block}}} \sum_{\lambda: x \in S_{\lambda}} b_{\lambda} \leq C_J \mathfrak{s} |\mathcal{M}_{\text{block}}|.$$

The set of monomials may be infinite; no bound on its cardinality is used. By Definition 20.11, $|\mathcal{M}_{\text{block}}| \leq N_{\text{missing}} \leq N_{\text{missing}} + N_{\text{bd}}$. This proves the first inequality.

For the regional and the restricted monomials, we compare coefficients on the common set of unfixed fluctuation labels, at the same background argument extended by zero. We use the comparison before the logarithm in Theorem 20.2 and Lemma 20.3: a reconstruction formula, a minimizer, an inverse square root, or a Haar density is replaced one factor at a time. A change of the coarser arguments is the integral of their derivative against the difference of the reconstructions, which carries a visit to Ω^c . For an incoming torus term we integrate Dr_{d_0} on the segment between the two absolute images, which

lie below half the radius; fixed old input slots are visits to Ω^c , and unfixed slots use the defect of the reconstruction in Theorem 19.9. The boundary function f enters with its own incoming boundary weight. By the versions with every index rooted, these comparisons keep a visit to Ω^c and every specified unfixed fluctuation index, including all degree-one terms. They are estimates on monomials before either logarithm is expanded, so the present restriction need not coincide with the one used in the earlier expansions.

Every remaining fluctuation monomial has an unfixed fluctuation index. We assign each compared monomial to its first such index y . The comparison of coefficientwise majorants with every index rooted bounds its absolute sum there by $C_J \mathfrak{s}(t) e^{-\alpha d_\Omega(y)}$. Summing over these indices, we obtain

$$\|U^\Omega - U^c\|_\infty \leq C_J \mathfrak{s}(t) \sum_{y \text{ unfixed}} e^{-\alpha d_\Omega(y)} \leq C'_J \mathfrak{s}(t) |\partial_{\text{blk}} \Omega| \leq C'_J \mathfrak{s}(t) N_{\text{bd}},$$

by (20.4) and Definition 20.11. This proves the second inequality, at every fixed real frozen value.

The zero-fluctuation values of F_i , L_i , B_i , and R in Lemma 20.3 were subtracted before this sum and stay in the prefactor. In particular, a constant depending on the frozen values does not enter the boundary estimate. The convergent coefficientwise majorants justify all assignments and sums, by finite truncation and monotone convergence of absolute sums.

Finally, if real bounded U, V satisfy $|U - V| \leq D$ pointwise, then $e^{-D} e^{-V} \leq e^{-U} \leq e^D e^{-V}$. Integrating against a probability measure and taking logarithms gives

$$\left| \log \int e^{-U} d\pi - \log \int e^{-V} d\pi \right| \leq D.$$

All integrals here are finite and strictly positive because the potentials have bounded majorants. We apply this twice, through U^c , which proves (20.43). For a real interaction, the integral with the interaction multiplied by $0 \leq \theta \leq 1$ is positive and equals one at $\theta = 0$. The logarithm given by the cluster expansion vanishes at $\theta = 0$ and therefore agrees with the positive real logarithm along this path. The comparison therefore also bounds the constant used in the analytic recursion. $\square$

Theorem 20.13. *Use full steps on admissible data with $p_i \geq 1$ at every intermediate scale, the exact extraction of field-independent factors, and the centered torus recursion of Theorem 20.10. At every full step assume the product and rate hypotheses of Theorem 20.9, together with its small-interaction and auxiliary-disc gates for the actual potentials without and, whenever used, with insertion. The bounds in those gates include the incoming torus and boundary interactions; admissibility alone does not impose cluster smallness. Restrict the coarse backgrounds, the couplings, and the frozen values to real data on the chosen products of fluctuation balls. Index the full steps by $t = 0, 1, \dots$, keep the exact incoming value of Λ , and write $\Delta \Lambda_t = \Lambda_t - \Lambda_t^\mathbb{T}$ for the*

difference between the regional and the torus normalization constants before step t . At the end of step t , the newly frozen analytic scalar labels form a set F , frozen simultaneously at the values $c^\sharp$. Assume the following.

- (a0) The incoming centered torus interaction and boundary part have fixed bounds $\mathcal{B}_{\rho,a\mathbb{T}}(r_{\text{in}}) \leq E_{\text{in}}$ and $\mathcal{X}^\rho(f_{\text{in}}) \leq X_{\text{in}}$. In applications of Theorem 20.14, one has $E_{\text{in}} \leq E$ and $X_{\text{in}} \leq \sigma_*$.
- (a) The outgoing centered effective interaction of step t has the form $R^{\text{out}} = r_{\mathbb{T}} + f$ with $R^{\text{out}}(0) = 0$, $Dr_{\mathbb{T}}(0) = 0$, $\mathcal{B}_{\rho,a}(r_{\mathbb{T}}) \leq E$, and $\mathcal{X}_\Omega^\phi(f) \leq Y$, where Y is a fixed bound.
- (b) Every newly frozen value satisfies $|c_j^\sharp| \leq \phi(j)/4$ for $j \in F$.

Then at each step, including the simultaneous restriction to fixed values at its end,

$$\Delta\Lambda_{t+1} \leq \Delta\Lambda_t + \sum_{i \text{ in step } t} |H_i| \log(1/g_i) + C_J(N_{\text{missing},t} + N_{\text{bd},t} + N_{\text{freeze},t}), \quad (20.44)$$

with the counts of Definition 20.11, where C_J may also depend on E , Y , E_{in} , and X_{in} . Each Gaussian dimension occurs once, and the normalized auxiliary variables and the Haar measure of the block axial gauge trees have mass one. The restriction to fixed values is exactly

$$R_{\text{new}}(v) = R^{\text{out}}(v, c^\sharp) - R^{\text{out}}(0, c^\sharp), \quad \Lambda_{\text{new}} = \Lambda^{\text{out}} - R^{\text{out}}(0, c^\sharp),$$

and the frozen constant satisfies

$$|R^{\text{out}}(0, c^\sharp)| \leq \rho^2 |F| (E/16 + Y/4). \quad (20.45)$$

The classical constants remain in the full classical action. A constant of the incoming interaction is first moved into Λ exactly, using $e^{\Lambda-R} = e^{\Lambda-R(0)} e^{-(R-R(0))}$; no nonconstant norm bounds this constant. No additional term proportional to $|\Omega| \log(1/g)$ appears. The restriction at initialization is included, with R^{out} the torus reference and $f = 0$. The constant C_J has the dependence described in item (3) of Definition 4.14 and is independent of the later widths, of the preceding depth, and of the number of steps. No inequality for complex values and no recursion for compositions of fewer than J averaging steps is asserted.

Proof. We compare the logarithms of the regional and the torus factorizations term by term: first the deterministic factors (Step 1), then the fluctuation integrals (Step 2), then the pullback at zero fluctuation (Step 3), and finally the constant produced by fixing coordinates (Step 4).

Step 1: coupling powers, determinants, and Haar factors. At fixed real frozen values, the factorization (20.38) contains exactly one factor g_i for each scalar free coordinate and one triangular determinant. Subtracting the logarithms of the regional and the torus factorizations, we find for the coupling powers

$$\sum_i \left[\sum_{x \in I_i} \log g_i - \sum_{x \in I_i \cup H_i} \log g_i \right] = \sum_i |H_i| \log(1/g_i).$$

The determinant statement of Theorem 20.10 bounds the difference of the logarithmic determinants by C_J for each missing scalar label plus $C_J|\partial_{\text{blk}}\Omega| \leq C_J N_{\text{bd}}$, by (20.5). The same argument applies to the Haar densities at zero fluctuation, by the comparison in part (3) of Theorem 20.2. Their constant normalizations depend only on M and J and contribute C_J for each omitted or interface coordinate. Since $p_i \geq 1$ we have $0 < z_1 \leq z_{p_i} \leq (2\pi)^{3/2}$, so the terms $\log z_{p_i}$ contribute a bounded amount for each missing block. The Haar measure of the block axial gauge trees and the adjoined reference measures contribute no dimension.

Step 2: the fluctuation integrals. Under the expressly assumed product, rate, and small-interaction hypotheses at this step, Lemma 20.12, at coarse background zero, bounds the difference of the fluctuation logarithms without insertion, and of those with insertion whenever the factorization uses them, by $C_J \mathfrak{s}(N_{\text{missing}} + N_{\text{bd}})$. In that proof, each missing fluctuation block is counted among the missing labels and each comparison at unfixed roots in the interface count; no geometric crossing argument is needed. Using $\mathfrak{s} \leq \epsilon_J$, we find that this step contributes $C_J(N_{\text{missing}} + N_{\text{bd}})$.

Step 3: the pullback at zero fluctuation. The zero-fluctuation value of the pullback has not entered Step 2. For its centered torus and boundary parts we assign each monomial of degree $l \geq 1$ to its input roots with the shares η_e/l , whose sum is one. The input norms in hypothesis (a0) and Definition 4.6 give $C_J E_{\text{in}}$ or $C_J X_{\text{in}}$ for each input root, after the common image at half the radius has summed the degree series. Missing roots are counted as missing labels. Differences rooted at unfixed labels carry the visit in (20.39) and are summed by (20.4). The incoming constant was moved into Λ exactly before this calculation.

Step 4: fixing coordinates. Finally we apply Lemma 18.9. To prove (20.45), note that when the unfixed variables are set to zero, only monomials supported in F survive. On such a monomial of degree l we have $\sum_{j \in F} \nu_j/l = 1$, and by hypothesis (b) its frozen value has modulus at most $4^{-l}|f_\nu|\phi^\nu$. Summing over $j \in F$ and using the definition of $\mathcal{X}_\Omega^\phi$ with the weights larger than one dropped, we find that the boundary part contributes at most $|F|\rho^2 Y/4$, since $l \geq 1$. The torus part has $l \geq 2$ and $|c_j^\#| \leq \rho/4$, so the seminorm bounds its contribution by $|F|\rho^2 E/16$. This proves (20.45); the bound contributes C_J for each newly frozen analytic label, degree one included. The value is subtracted from Λ with the sign in the statement, and its absolute value gives the upper bound. The same identity applies at initialization. Data deep inside the fixed complement stay in the external factors, and the classical constants stay in the classical action. Adding the finitely many contributions proves (20.44). Every estimate used a coefficient bound for each label or the boundary sum (20.4), which does not depend on the shape of the region; none of them estimates the incoming constant or introduces a factor depending on the number of

components or of steps. □

20.6 Preservation of the analytic bounds under iteration

The following theorem is the main result of this section.

Theorem 20.14. *Fix the torus analytic radii and rates, the torus support weights and spatial margins, and M , and fix $A_w \geq 1$. Choose J once so that the conditions on J of the torus construction used in Lemma 19.7 hold and $B \geq B_{\text{geom}}$, as in Theorem 20.9. There exist positive fixed base radii ρ , ρ_{cl} , R_{raw} and rates a, α as in Theorem 20.9. At these radii and rates, fix a finite torus ceiling $E \geq 0$ as specified below. For each such fixed E , there exist positive radii R_w , R_{en} , a perturbation bound σ_* , the classical and quantum radius profiles defined in Section 20.1, a width s_{er} of the frozen layer, a bound β_{rec} on the frozen values, a support radius r , and a coupling bound $g_* < e^{-1}$, with the following property. One may take $\rho = \rho_{\text{cl}}$; other fixed ratios are allowed if both image conditions are imposed. The dependent constants and g_* may depend on E , and g_* may also depend on A_w . More generally, after fixing the larger construction data, base radii and rates, and torus ceilings, the conclusions hold for every choice of the remaining parameters made in the order of Table 20.1 and satisfying all requirements in Steps 4–10 of the proof, as well as the explicit hypotheses of the theorem.*

The fixed E is required to be a common ceiling for every incoming and outgoing centered torus reference interaction on the quantum radius and rate used below, including the reference at initialization and the terminal output. It is fixed before all parameter choices whose bounds depend on it, in particular the local output and mean constants and the final coupling ceiling. Consider a full step on an admissible block region satisfying the protected-neighborhood conditions of Lemma 19.5, at any preceding depth and after any number of earlier steps, with a real admissible frozen value and an input satisfying

$$\mathcal{X}^{\text{cl}}(\bar{P}) \leq 2\sigma_*, \quad \mathcal{B}_{\rho, a_{\text{T}}}(r_{d_0}) \leq E, \quad \mathcal{X}^{\rho}(f) \leq \sigma_*, \quad \max_i g_i \leq g_*.$$

Assume the uniform incoming classical norm on D_{in} , the boundary comparison (20.8), and the inequality on the conditional forces of Theorem 19.9 and Lemma 19.3. Set $p(t) = A_w \log(1/t)$ and require at the t -th step, with $t_t = \max_i g_{t,i}$,

$$p(t_t) \leq p_{t,i} \leq p(t_t) + \frac{1}{2}A_w(J-1) \log M.$$

Assume the inclusion of the products, the positivity of the rational image denominators, the small-interaction condition, and the conditions on the auxiliary half-disc of Lemma 20.3 and Theorem 20.9, with the bound chosen for this whole family of radii. The small-field event uses $p_{t,i} = p(t_t)$. All conclusions formulated relative to the fixed torus recursion r_d —the recursion for Λ , the iterated boundary norm bounds, and the resulting oscillation bound—use this common

choice of fluctuation radii. For any other radii in the displayed range, the local single-step comparisons are with the torus update computed on exactly the same fluctuation product; no identification of that update with the fixed r_{d_0+J} is asserted. Under these hypotheses the perturbed objective has all of the following properties: the coefficient bounds of the potential with each index fixed in turn, the gap of the constrained Hessians at every stage, and the strict bounds on the image under the reconstruction majorant, on the outer profile times the whole enclosing product; the comparisons with boundary decay of Theorem 20.2; the potential of Lemma 20.3; the expansions without and with insertion of Theorem 20.9, with input support weight $e^{16|S|}$ and output support weight $e^{4|S|}$ as in Lemma 20.8, and with the same returned rates and radii; the bounds on the generated term, the pullback, and the determinants of Theorem 20.10; and the recursion for Λ of Theorem 20.13. These properties are conclusions of the earlier results.

All output constants are evaluated on the preliminary ball σ_0 and on the fixed boundary-completion ceiling $\mathcal{X}^{\text{cl}}(S_\xi^{\text{cl}}) \leq \sigma_0/4$. Thus one may take $A_J^{\text{cl}} = A_J^{\text{cl},0} + K_J^{\text{cl}}\sigma_0/4$ when passing from the adjusted perturbation to P_t ; this uses no later choice of β_{rec} or σ_* . The final freezing gate implies the stronger ceiling $\sigma_*/4$.

More explicitly, let $x_t = \mathcal{X}(f_t)$ and $x_t^{\text{cl}} = \mathcal{X}^{\text{cl}}(P_t)$ be the incoming boundary norms of the quantum and the classical parts at step t , the boundary-completion adjustment of the classical part being understood. Before coordinates are fixed, the outgoing boundary norms y_t and y_t^{cl} satisfy

$$\begin{aligned} y_t &\leq A_J(1+E) + K_J x_t + C_J t_i^2 (E + x_t) [p_t^4 + (E + x_t)p_t^2], \\ y_t^{\text{cl}} &\leq A_J^{\text{cl}} + K_J^{\text{cl}} x_t^{\text{cl}}, \quad t_t = \max_i g_{t,i}, \quad p_t = \max_i p_{t,i}. \end{aligned}$$

Under the hypotheses on the doubled output analytic radius and on the incoming image at half the radius, $K_J = 4(\log 2)K_{\text{col},J}$, where $K_{\text{col},J}$ is the column bound of Theorem 20.10 for the perturbed zero-fluctuation reconstruction, uniform on the preliminary perturbation ball. The recursion for Λ is Theorem 20.13.

Whenever coordinates are fixed, fix every label with reduced radius and keep the remaining unfixed labels at distance at least s_{er} from the previous boundary. For $\bullet \in \{\text{cl}, \text{q}\}$, with $(\rho_{\text{q}}, E_{\text{q}}) = (\rho, E)$ and classical radius ρ_{cl} , choose

$$E_{\text{cl}} \geq \sup_{d \geq 0} \mathcal{B}_{\rho_{\text{cl}}, a_{\text{T}}}(E_d),$$

where a_{T} is no larger than the classical coefficient rate of Theorem 10.11. The factorial all-root bounds give this finite ceiling after the analytic radius is chosen inside their convergence radius. For both values of $\bullet$, require

$$C_{\text{bc},\bullet} E_{\bullet} \beta_{\text{rec}} / \rho_{\bullet} \leq \sigma_*/4, \quad |c_j^{\#}| \leq \min\{\beta_{\text{rec}}, \phi_{\bullet}(j)/4\}.$$

Here $C_{\text{bc},\bullet} \geq 8$ is fixed before β_{rec} and the later widths, and includes any finite sum over types of boundary-completion differences and over conversions between coordinates; a single simultaneous

restriction of the union of all newly frozen labels to their fixed values uses $C_{bc,\bullet} = 8$ in the coordinates in which the labels were introduced. Every frozen label, whether in the boundary layer, in a protected neighborhood, or among the boundary-crossing coordinates, and including the labels frozen at initialization, is covered by this bound. Here $c^\sharp$ is the value in the analytic coordinates in which the label was introduced; any conversion from group values or transported coordinates is included in the bound by β_{rec} beforehand. Assume $x_0, x_0^{\text{cl}} \leq \sigma_*$. Then, including the new boundary-completion differences of the torus parts,

$$x_{t+1} \leq z_t + 2e^{-\alpha(\text{ser}-1)}y_t, \quad x_{t+1}^{\text{cl}} \leq z_t^{\text{cl}} + 2e^{-\alpha(\text{ser}-1)}y_t^{\text{cl}},$$

where z_t and z_t^{cl} are the norms of the boundary-completion differences of the torus parts created when coordinates are fixed at the end of step t . The choices can be made with

$$x_0, x_0^{\text{cl}} \leq \sigma_*, \quad z_t, z_t^{\text{cl}} \leq \sigma_*/4,$$

and with the last terms of both recursions at most $\sigma_*/2$, uniformly on the input ball. In particular

$$x_t, x_t^{\text{cl}} \leq 3\sigma_*/4 \quad (t \geq 1)$$

after any finite number of steps, with no further loss of radius or rate. For $|v_i| \leq \phi(i)$ the oscillation of the centered effective interaction satisfies

$$|R_t(v; b_{\text{rec}}) - R_t(0; b_{\text{rec}})| \leq \rho^2 \left[E|\Omega_t| + \sigma_* \sum_{j \in \Omega_t} e^{-\alpha d_{\Omega_t}(j)} \right],$$

where, for this count and this sum, Ω_t denotes the set of unfixed scalar labels, directions and colors included. At the output of a step that is not followed by fixing coordinates, σ_* in this formula is replaced by the fixed output bound for y_t .

All constants have the dependence described in item (3) of Definition 4.14 and exclude the later widths. The output bounds are computed first and ser afterwards; J is not enlarged by means of these constants for fixed J . The theorem concerns admissible data; that the configurations of the large-field decomposition are covered by them is proved separately in Section 21. For a composition of $s < J$ averaging steps the estimates at each stage hold on their admissible products, but a conclusion about the two images or about the enclosing product also requires the separate image condition for that shorter composition; the preservation of the profile is asserted only for full steps. An empty set of unfixed labels has nonconstant norm zero, determinant one, logarithmic determinant zero, and fluctuation integral one.

Proof. The proof has three parts. Steps 1 to 3 prove the bounds on y_t and y_t^{cl} before coordinates are fixed. Steps 4 to 10 make the parameter choices in the order summarized in Table 20.1 and check that each choice uses only quantities fixed before it. Steps 11 and 12 prove the iteration and the bound on the oscillation.

Quantity	Chosen after
K_* , torus support weights, and margins	fixed data
J	K_* and the torus data
$\rho, \rho_{\text{cl}}, R_{\text{raw}}$, with $K_* \max(\rho, \rho_{\text{cl}}) < h_J$	J and the fixed auxiliary convergence products
a, α	J and the exponents of the coefficients of the potential
E and the classical ceiling E_{cl}	the base radii and rates
σ_0, r_q	radii, rates, and the fixed ceilings
$A, C_0, C_1, C_{\text{en}, J}, \mathcal{L}_V$, and all output, tensor, and image constants	σ_0
R_W , then σ_*	these constants
$\rho_{\partial, \bullet}, w_{\text{bd}}$	these constants
Y, Y^{cl} , then s_{er}	σ_*
$C_{\text{bc}, \bullet}, C_{\text{word}, J}, C_{\text{conv}, J}$	the larger charts
β_{rec} , then r , then R_{en}	σ_* and the conversion constants
g_*	all of the above, and A_W

Table 20.1. The order of the parameter choices in the proof of Theorem 20.14. The requirements are stated in Steps 4 to 10.

Step 1: the hypotheses at each step. We fix $A_W \geq 1$ before the coupling bound. The coefficient constants of the potential and the torus and geometric conditions used to choose J do not depend on A_W ; only the final coupling bound may depend on it. At each step, the incoming classical norm is the uniform norm on D_{in} , the boundary comparison (20.8) holds by assumption, with all label positions and volume factors on the lattices where the labels were introduced, and the conditional forces satisfy (20.12) and (20.13). The strict bounds on the two images place all segments of evaluation inside the uniform incoming products. Thus, at the fixed frozen value $b = (\xi, b_{\text{rec}})$, both the perturbation and its comparison with zero satisfy the hypotheses of Theorem 19.9.

Step 2: the classical comparison before coordinates are fixed. Let a_R and a_T be the two admissible zero-fluctuation chains before the gauge transformation, the regional one with its missing slots set to zero. The comparison of values, before the common factor M^J , is

$$\begin{aligned}
 F_R(v, a_R) - F_T(v, a_T) &= F_R(v, a_R) - F_T(v, a_R) \\
 &\quad + \int_0^1 D_z F_T(v, a_T + \tau(a_R - a_T)) [a_R - a_T] d\tau.
 \end{aligned}$$

The fixed larger charts contain this segment, since the smaller solution neighborhoods have strict room in them. The first difference has the representation (20.16) through changed formulas. On unfixed free slots, the difference of the chains in the integral is

the comparison of minimizers in Theorem 20.2, here with perturbation zero; on missing free slots it is a bounded source in Ω^c . Every nonconstant coefficient with an unfixed output index therefore has a network to Ω^c , by the action tensors rooted at every index and Lemma 18.15. This includes derivatives falling on the action tensor while the source stays undifferentiated. We sum over the indices in Ω^c from the unfixed index using (20.2). Pure constants supported in Ω^c stay in the classical value. The conversion of coefficients and profiles in the proof of Theorem 20.9, without its expansions, now yields a finite bound A_J^{cl} for the centered comparison. The factor M^J depends only on J and is included in this constant. No interpolation between different fibers is used in this calculation of values.

Step 3: the perturbation and the quantum comparison. Next we interpolate in the adjusted input perturbation. By the envelope formula, the derivative of the minimum value is the evaluation of the perturbation at the minimizer. The strict bound on the image under the reconstruction majorant bounds its input jets, and joining its input boundary network to the reconstruction networks proves the output bound $K_J^{\text{cl}}x^{\text{cl}}$. Equivalently, this is the conclusion of Theorem 19.9 on the minimum values for two perturbations, with its exact boundary-completion adjustment. The fixed small boundary-completion difference is included with the unperturbed term in A_J^{cl} . This proves $y^{\text{cl}} \leq A_J^{\text{cl}} + K_J^{\text{cl}}x^{\text{cl}}$. The coefficient networks in this argument are the classical ones, with the common rates chosen in Theorem 20.9.

For the common cutoff $p_{t,i} = p(t_t)$, we subtract the matched torus recursion, which is the fixed recursion r_d , from the recursion of Theorem 20.10. The difference is the sum of the comparison of the generated terms and torus pullbacks, the term $P^\Omega f$, and the compared fluctuation correction. The first two are bounded in Theorem 20.10 by $A_J(1 + E)$ and $K_J x$, and the third by Theorem 20.9. By the triangle inequality, we obtain the stated bound on y .

Step 4: radii, rates, and the perturbation bound. The fixed products include the auxiliary reserve h_J in the definition of admissibility. Choose the analytic radii within $K_* \max(\rho, \rho_{\text{cl}}) < h_J$, and the free radii below h_J , before fixing the preliminary perturbation or any output coefficient bound. In particular every factorial series and implicit or matrix-function series used for Theorem 20.2 is summed on a strictly smaller product.

We fix K_* before J and choose J for the torus conditions and the geometric condition. We choose the base radii inside the fixed larger construction domains; one may choose $\rho = \rho_{\text{cl}}$, and different fixed ratios simply keep both image conditions. We choose the common rates below the intrinsic coefficient and unperturbed image exponents at fixed J , and impose the separate output-rate conditions against Ba and $B\alpha$ for the inherited perturbation. Its controlled coefficient difference preserves the half-radius image at these working rates after the final perturbation bound is decreased. We decrease R_{raw} and the

base radii if necessary, for the fixed margin of the larger construction domain and for $C_{T,\bullet}K_*\rho_\bullet \leq 1/4$. The torus conditions are used with their stronger torus rate, not through an estimate involving an uncontrolled regional constant.

At the chosen base radii and rates we fix the torus ceiling E in the statement and the classical ceiling E_{cl} furnished by the all-root bounds. Subsequent local constants and the final coupling ceiling may depend on these fixed ceilings. We choose a preliminary bound σ_0 below the admissible inverse, conditional force, and real gap margins, and a free radius r_q with

$$C_{\text{seq},J}(2I_0C_P\sigma_0\rho_{\text{cl}} + r_q) < \varepsilon_{\text{tube}}/2.$$

The constants are those of the larger construction neighborhoods in (20.12), whose uniformity in all higher parameters was checked there. We compute all output, tensor, image, sequential, and enclosure constants uniformly for this bound on the fixed larger products. In particular we fix A , C_0 , C_1 , $C_{\text{en},J}$, and $\mathcal{L}_V$, and the difference-row constants $C_{P,\bullet}$ at the allocated working rates, and choose R_W from the positive minimum in Lemma 20.3. Then we fix $\sigma_* \leq \sigma_0$ so that, for both radii,

$$\sigma_*\rho_{\text{cl}} \leq \lambda_*R_W/(6C_{\text{en},J}), \quad C_{P,\bullet}\sigma_*\rho_{\text{cl}} \leq \rho_\bullet/64.$$

This order is necessary because this summand of the enclosure remains when r and β_{rec} are decreased.

Step 5: the boundary radii. We choose the two reduced boundary radii so that the bounds on the image at zero fluctuation near the boundary in Lemma 19.7 are at most $\rho_\bullet/16$. Then we choose a fixed width w_{bd} of the boundary layer so that the bounds on the tails far from the boundary are at most $\rho_\bullet/16$. The remaining bounds at zero fluctuation are the torus bulk bound and the bound for the frozen values. The first is a torus condition, and the second will be imposed on β_{rec} . All constants in these choices were computed independently of these widths and reduced radii.

Step 6: output bounds and the width of the frozen layer. We set

$$c_J^{\log} = \frac{1}{2}A_w(J-1)\log M, \quad p^+(t) = A_w \log(1/t) + c_J^{\log}, \quad H_* = E + \sigma_*. \quad (20.46)$$

At a step indexed by k , write $t = t_k$. By the hypothesis on the radii, $1 < p(t) \leq p_{k,i} \leq p^+(t)$, since $t \leq g_* < e^{-1}$. On the input ball we impose provisionally the requirement

$$C_J t^2 H_* [(p^+(t))^4 + H_*(p^+(t))^2] \leq 1,$$

which holds for small couplings. We then take the fixed output bounds

$$Y = A_J(1 + E) + K_J\sigma_* + 1, \quad Y^{\text{cl}} = A_J^{\text{cl}} + K_J^{\text{cl}}\sigma_*.$$

We choose the integer width of the frozen layer with

$$s_{\text{er}} \geq \max \left\{ 2, w_{\text{bd}}, 1 + \alpha^{-1} \log(4 \max\{Y, Y^{\text{cl}}, 1\} / \sigma_*) \right\}.$$

The bound may be enlarged to account for the fixed buffer between label positions in Lemma 18.9. Each output restricted to the interior of the frozen layer is then at most $\sigma_*/2$, since $2e^{-\alpha(s_{\text{er}}-1)} \max\{Y, Y^{\text{cl}}\} \leq \sigma_*/2$.

Step 7: the frozen values and the new boundary-completion differences. Whenever coordinates are fixed, let F be the union of all newly frozen analytic labels: the whole zone with reduced radius, the protected neighborhoods, and the boundary-crossing coordinates. The same convention covers the restriction at initialization. In the coordinates in which the labels were introduced, a single application of Lemma 18.9 to this union bounds the torus boundary-completion difference by $8E_{\bullet}\beta_{\text{rec}}/\rho_{\bullet}$. The rate hypothesis of part (2) of that lemma holds with the torus rate $a_{\mathbb{T}}$ and the regional rates a, α , since $\sqrt{3}(a + \alpha) < a_{\mathbb{T}}$ by Theorem 20.9. No separate factor arises for each component of Ω^c . If a fixed finite list of types of boundary-completion differences is kept separately, with conversion factors $L_{k,\bullet}$, the triangle inequality yields instead the bound with $C_{\text{bc},\bullet} = 8 \sum_k L_{k,\bullet}$. We include the classical boundary-completion adjustment among these types whenever it is recorded separately. These constants are fixed on the larger charts, before β_{rec} and all later widths.

The bound concerns the analytic values $c_j^{\sharp}$ in the coordinates in which the labels were introduced. For a frozen group element in the identity chart, $|\text{Log } U| = \rho_G(U)$ in the Euclidean norm of the Lie algebra, so this conversion has constant one. A longer fixed chart formula, in the coordinates before the gauge transformation or in transported coordinates, has a finite bound $C_{\text{word},J}$ on its derivative rows on the larger construction domain. Integrating along the input segment, we bound its logarithm in the identity chart by $C_{\text{word},J}$ times the maximal input norm. These conversion constants are fixed before the bounds on the frozen values are chosen. If a separate estimate for the initial frozen values is provided with a constant $C_{\text{ad},J}$, we choose $C_{\text{conv},J} \geq \max\{1, C_{\text{ad},J}, C_{\text{word},J}\}$ before using a condition relating the support radius to the frozen values. The theorem assumes the bound on every $c_j^{\sharp}$. It does not deduce this bound from a small support radius in an arbitrary gauge representative, and it transports a norm bound between gauge frames for the frozen coordinates only when the change of frame is the norm-preserving adjoint action of a block.

We choose $\beta_{\text{rec}} > 0$ below the bound for the frozen values in the image, below the enclosure bound $\lambda_* R_W / (6C_{\text{en},J})$, and, for both $\bullet \in \{\text{cl}, \text{q}\}$, below

$$\min \left\{ \rho_{\partial,\bullet}/4, \rho_{\bullet}/4, \frac{\sigma_* \rho_{\bullet}}{4C_{\text{bc},\bullet}(E_{\bullet} + 1)} \right\}. \quad (20.47)$$

These are upper bounds on a quantity chosen after their constants, and they remain positive when a bound $E_{\bullet}$ is zero. They imply the two inequalities $C_{\text{bc},\bullet} E_{\bullet} \beta_{\text{rec}} / \rho_{\bullet} \leq \sigma_*/4$

in the statement. The hypothesis $|c_j^\#| \leq \min\{\beta_{\text{rec}}, \phi_\bullet(j)/4\}$ is imposed on all of F , not only on the old boundary zone.

Step 8: initialization. The assumption $x_0, x_0^{\text{cl}} \leq \sigma_*$ of the iteration can be met: starting from the torus reference, both boundary parts are zero before initialization, and the same restriction of the union to fixed values, together with (20.47), gives initial boundary parts of norm at most $\sigma_*/4$. For other prescribed inputs the bounds on the initial norms are kept as hypotheses. The coverage of the configurations and the estimate for the initial frozen values are separate inputs of the later applications.

Step 9: the support radius. We choose $r > 0$ inside all margins of the small-link domain and of the larger construction domain, below $\lambda_* R_W / (6C_{\text{en},J})$, and, when an estimate for the frozen values in terms of the support radius is provided, with $C_{\text{conv},J}r \leq \beta_{\text{rec}}$, imposed after the conversion constant has been fixed. Then $R_{\text{en}} = 2C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_*\rho_{\text{cl}})/\lambda_*$ is positive and at most R_W . The image denominators now have their fixed margins, and Lemma 20.3 supplies the radius-dependent curvature budget on the whole product. The full curvature conclusion also requires the coupling condition $C_0 t^2 \mathcal{L}_V \leq 1/16$, imposed in Step 10.

Step 10: the coupling bound. Fix $C_{\text{gate},J}$ dominating the primitive, small-interaction, and fluctuation-remainder constants of Lemma 20.7 and Theorems 20.9 and 20.10, including their fixed multipliers, while keeping the established $\epsilon_J, c_{z,J}$. Finally, choose $g_* < e^{-1}$ so that the following inequalities hold for every $0 < t \leq g_*$:

$$\begin{aligned} tp^+(t) &< R_{\text{en}}, & C_{\text{gate},J}tp^+(t) &\leq \frac{1}{2}, \\ 2C_{\text{gate},J}tD^+(t) &\leq \epsilon_J, & tD^+(t) &\leq c_{z,J}/2, & D^+(t) &= (p^+(t))^3 + (1 + H_*)p^+(t), \\ C_0 t^2 \mathcal{L}_V &\leq 1/16, & C_1 t(\mathcal{L}_V + 2) &< 1/(4\sqrt{3}), \\ C_{\text{gate},J}t^2 H_* [(p^+(t))^4 + H_*(p^+(t))^2] &\leq 1. \end{aligned} \tag{20.48}$$

The image denominators are at least $1/2$ on the whole enclosing product by the earlier image bounds, which do not involve the coupling. At a step indexed by k , apply these inequalities with $t = t_k \leq g_*$. For any radii in the stated logarithmic range we have $p_k \leq p^+(t)$ and $E + x_k \leq H_*$. The first line implies the inclusion of the products. The next line implies $s(t) \leq \epsilon_J$ and $t \leq c_{z,J}/(2[p_k^3 + p_k + (E + x_k)p_k]) = r_z/2$. The third line implies (20.26); since $p(t) > 1$, it also implies the condition $C_1 t(\mathcal{L}_V + 2) \leq p(t)/(2\sqrt{3})$ on the means in Lemma 20.5. The last line is the provisional requirement of Step 6. The choice is simultaneous, because every left side that has to be small is a fixed combination of $t \log^q(1/t)$ and $t^2 \log^q(1/t)$ for finitely many integers q . Writing $t = e^{-u}$, we have $e^{-u}u^q \rightarrow 0$; for example the power series of $e^{u/2}$ bounds $u^q e^{-u}$ by a constant times $e^{-u/2}$. These upper bounds hold for all sufficiently small positive t , so a single bound works for all scales of every full step in the prescribed family of radii. The common cutoff has

$p_{k,i} = p(t)$, and the cutoffs at the individual scales, $p(g_{k,i}) = p(t) + A_w \log(1/\lambda_i)$, lie in the same range. The expansions are used on these chosen products, not on the enclosing product $\mathcal{K}_{\text{en}}$. This final choice does not change the output bounds or the width s_{er} , which were computed before.

Step 11: the iteration. Here and in the oscillation estimate we use the common fluctuation products, so every torus reference is the fixed recursion r_d . By Lemma 18.9 with $\eta = 1/4$, fixing the coordinates of the union F bounds the old boundary part by $2e^{-\alpha(s_{\text{er}}-1)}y_t$, since $|c_j^\#| \leq \phi_\bullet(j)/4$ and all labels with reduced radius are frozen. Independently, the boundary-completion differences of the torus parts satisfy, by the hypothesis on the frozen values,

$$z_t^\bullet \leq C_{\text{bc},\bullet} E_\bullet \beta_{\text{rec}} / \rho_\bullet \leq \sigma_*/4, \quad (20.49)$$

where $z_t^q = z_t$. Fixing coordinates at an additional interior component of Ω^c satisfies the same estimate, because its values are also bounded by β_{rec} . Combining the two bounds, we obtain

$$x_{t+1} \leq z_t + 2e^{-\alpha(s_{\text{er}}-1)}y_t, \quad x_{t+1}^{\text{cl}} \leq z_t^{\text{cl}} + 2e^{-\alpha(s_{\text{er}}-1)}y_t^{\text{cl}}.$$

These estimates hold for arbitrary additional components of Ω^c , by Lemma 18.9. Its torus term uses the outgoing reference interaction, so the common ceiling E in the hypotheses is needed also at the output of this step, including the terminal step. If both input norms are at most σ_* , then $y_t \leq Y$ and $y_t^{\text{cl}} \leq Y^{\text{cl}}$, and by Step 6 the right sides are at most $3\sigma_*/4$. Starting from the initial bounds, finite induction proves the bounds on x_t, x_t^{cl} after every number of steps. The older frozen values are simply fixed in the next application. The remaining labels keep their input radii because the whole zone with reduced radius was frozen: the conclusion of Lemma 18.9 is the norm with the uniform radius on the remaining labels. This verifies the hypothesis of a uniform incoming profile at the next step. The classical boundary-completion adjustment satisfies

$$\mathcal{X}^{\text{cl}}(\bar{P}) \leq \mathcal{X}^{\text{cl}}(P) + \mathcal{X}^{\text{cl}}(S_\xi^{\text{cl}}) \leq \sigma_* + \sigma_*/4 < 2\sigma_*,$$

by the included bound on the boundary-completion differences. The inclusion of the old fixed complement and the positions of the labels in the next geometry are hypotheses of that step, not conclusions of this norm inequality. The inequality on the conditional forces then holds at every coarser parameter of the next step, by (20.12). Every operation preserves the Borel dependence on the frozen values, as proved in Theorem 20.2. No analytic loss is summed over earlier steps.

Step 12: the oscillation. We write the centered effective interaction as its torus part plus its boundary part. At each degree we assign each monomial to its roots with the shares

$v_i/|v|$, whose sum is one. Evaluating at $|v_i| \leq \phi(i)$ and dropping the network weight, we obtain for the boundary part

$$|f_t(v) - f_t(0)| \leq \rho^2 \mathcal{X}(f_t) \sum_{i \in \Omega_t} e^{-\alpha d_{\Omega_t}(i)}.$$

The degree-one term is included in this assignment. For the torus part the same assignment and its seminorm give $\rho^2 E|\Omega_t|$. These are the stated bounds on the oscillation; before coordinates are fixed we use Y instead of σ_* .

The preceding results, with the choices just made, imply every assertion of the theorem. All local constants were evaluated before the later widths, and none of them was used to choose J . A composition of fewer than J averaging steps keeps its estimates at each stage on its admissible products, but its conclusions about the full radius require the separate image condition. The iteration for Λ and for the norms uses only full steps. A final composition of fewer than J averaging steps may remain as a deterministic function; no recursion for Λ is asserted for it here. The assertions for an empty set of unfixed labels are the usual conventions for empty sums, products, and determinants. This completes the proof. $\square$

20.7 Summation of truncation errors

Lemma 20.15. *Fix $0 < \Gamma < \infty$, $M > 1$, a finite set $\mathcal{R}$ of ratios $R > 1$, a finite set of positive numbers η , and finitely many positive constants a, c, C , and ε .*

1. *For every finite list of nonnegative integers k , there is a minimum side m_0 , independent of $0 < \gamma \leq \Gamma$, such that*

$$C[1 + \log(m/\gamma)]^k \leq \varepsilon m/\gamma \quad (m \geq m_0).$$

2. *With $z = (1/2) \log(Mm/\gamma)$, the functions*

$$X_c(m, \gamma) = m^3 e^{-cm/\gamma}, \quad W_a(m, \gamma) = m^3 e^{-az^2}$$

are decreasing in m and increasing in γ for $m \geq m_0$ and $0 < \gamma \leq \Gamma$, after increasing m_0 , and they satisfy $X_c(Rm, \gamma) \leq X_c(m, \gamma)/2$ and $W_a(Rm, \gamma) \leq W_a(m, \gamma)/2$ for $R \in \mathcal{R}$. Their sums over the sides mR^k , $k \geq 0$, are at most twice their first terms, and tend to zero as $m \rightarrow \infty$, uniformly on $(0, \Gamma]$.

3. *Let $m_k = mR^k$, $k \geq 0$, for some $R \in \mathcal{R}$, together with at most one additional side $\eta m R^j$ with $j \in \mathbb{N}_0$, and let nonnegative errors δ and z_{ext} at these sides satisfy $\delta \leq CW_a$ and $z_{\text{ext}} \leq CX_c$. After increasing m_0 , depending only on $\Gamma, M, \mathcal{R}$, the set of η , and the constants, the following holds uniformly for $m \geq m_0$, $0 < \gamma \leq \Gamma$, and the number of sides, finite or infinite. Every δ*

is at most $1/2$, every z_{ext} is at most $1/4$, and $\epsilon = \delta/(1 - z_{\text{ext}})$ satisfies $\epsilon \leq (4/3)\delta \leq 1/2$. The sums of the δ , of the z_{ext} , and of the ϵ are finite, and

$$\log \prod (1 - \delta)^{-1} \leq 2 \sum \delta, \quad \log \prod (1 - \epsilon)^{-1} \leq 2 \sum \epsilon,$$

also for every finite subproduct. These sums are bounded by any prescribed positive number after increasing m_0 , and they tend to zero as $m \rightarrow \infty$, uniformly on $(0, \Gamma]$.

In particular, part (3) of Lemma 20.5 holds. All constants in this lemma have the dependence described in item (4) of Definition 4.14; the local constants of Lemma 20.5 keep their own dependence.

Proof. Let $x = m/\gamma \geq m_0/\Gamma$. For every fixed k we have $(1 + \log x)^k/x \rightarrow 0$ as $x \rightarrow \infty$: with $y = \log x$, the exponential series bounds $y^k e^{-y/2}$ on a half-line $y \geq y_0$. This proves part (1), for finitely many k simultaneously, and allows us to assume $x \geq e$ throughout.

For part (2), let $\Delta_R = (1/2) \log R > 0$. Since $z(Rm) = z(m) + \Delta_R$, the ratios are

$$\frac{W_a(Rm, \gamma)}{W_a(m, \gamma)} = R^3 e^{-a(2\Delta_R z + \Delta_R^2)}, \quad (20.50)$$

$$\frac{X_c(Rm, \gamma)}{X_c(m, \gamma)} = R^3 e^{-c(R-1)m/\gamma}. \quad (20.51)$$

Their maxima over $m \geq m_0$ and $0 < \gamma \leq \Gamma$ occur at the smallest m and the largest γ , and they tend to zero as m_0 increases, since $z \geq (1/2) \log(Mm_0/\Gamma) \rightarrow \infty$ and $m/\gamma \geq m_0/\Gamma \rightarrow \infty$. We choose m_0 so that both ratios are at most $1/2$ for every $R \in \mathcal{R}$. The logarithmic derivatives in $\log m$ are $3 - cm/\gamma$ for X_c and $3 - az$ for W_a , and those in $\log \gamma$ are cm/γ and az . Hence X_c and W_a are increasing in γ , and they are decreasing in m once $cm_0/\Gamma > 3$ and $(a/2) \log(Mm_0/\Gamma) > 3$. The logarithms of the first terms at $\gamma = \Gamma$ are $3 \log m - cm/\Gamma$ and $3 \log m - a \log^2(Mm/\Gamma)/4$, and both tend to $-\infty$. By monotonicity in γ , the first terms tend to zero uniformly on $(0, \Gamma]$. Summing the series with ratio at most $1/2$ shows that the sums over mR^k are at most twice their first terms, which proves part (2).

For part (3), the additional side $\eta m R^j$ is handled in the same way. Since $j \geq 0$ and $R > 1$, it is at least ηm . Let $m_{\text{low}} = \min\{1, \min \eta\} m$. For sufficiently large m , monotonicity in m bounds the term at the additional side by the corresponding function at m_{low} , whether or not $\eta < 1$. Since m_{low} is a fixed positive multiple of m , the sums over all sides are at most three times the first terms evaluated at m_{low} , and these tend to zero uniformly on $(0, \Gamma]$ as $m \rightarrow \infty$, for finitely or infinitely many sides. We take the maximum of the finitely many minimum sides so obtained. Increasing m_0 once more, we make every δ at most $1/2$, every z_{ext} at most $1/4$, and the sums smaller than any prescribed positive number. Then $\epsilon = \delta/(1 - z_{\text{ext}}) \leq (4/3)\delta$, so the sum and the maximum of

the ϵ can be made as small as desired as well, in particular every ϵ at most $1/2$. For $0 \leq u \leq 1/2$,

$$-\log(1-u) = \int_0^u (1-v)^{-1} dv \leq 2u.$$

Summing this inequality bounds the logarithms of all reciprocal products, including every finite subproduct; the infinite products are defined by the increasing finite sums of logarithms. This proves part (3).

Part (3) of Lemma 20.5 is the case $R = B$, with $C = C_J$, $a = a_w$, $c = b$, and the finite set of additional ratios, since $\log^2 \sqrt{Mm/\gamma} = z^2$; its bound $\epsilon_t \leq 2\delta_t$ follows from $\epsilon_t \leq (4/3)\delta_t$. The local concentration statements of Lemma 20.5 use only its hypotheses on the products and on the coupling at the given step; the restriction to a coupling bound enters only in the summation just proved, so those local statements and their prefactors are unchanged. $\square$

Large-field decompositions and conditional densities

This section connects the renormalization group on unions of blocks, constructed in Sections 18 to 20, with the two measures to which it is applied in Sections 22 and 23: the Wilson measure and the stopped measure. All statements concern a fixed lattice and are finite-dimensional.

The analytic theorems of Section 20 integrate the normalized fluctuation variables of one RG step over a fixed product of balls. They work in local coordinates near the identity, on a block region from which a fixed complement has been removed, with the link variables of that complement held fixed as frozen coordinates. To apply them to the Wilson measure, we must solve three problems. The first is coverage. A fiber of the block averaging map consists of all fine configurations with a given average. It is a Borel subset of the compact configuration space, can contain many gauge-equivalent representatives, and need not lie in a coordinate chart. We show that every point of every fiber carrying mass becomes, after a gauge transformation that fixes all prescribed vertices, a point of the domain of the local inverse coordinates, and that its normalized fluctuation coordinates lie in an enclosing product domain on which the conditional fluctuation potential is uniformly convex. The second problem is the choice of a density. The later estimates evaluate densities at particular gauge representatives. These form sets of Haar measure zero, so a Radon–Nikodym derivative defined only almost everywhere does not suffice. We construct an explicit Borel density from the Haar disintegration formulas and prove that it is invariant at every point under the relevant gauge transformations. The third problem is the preservation of frozen coordinates. When the integration stops near a region of large curvature, the link variables there are kept on the lattice where they were frozen. The gauge transformations used afterward must not change them, and the analytic estimates require their logarithms to be small.

Section 21.1 proves locality of the iterated block averaging map (Lemma 21.1) and a gauge-fixing theorem on arbitrary unions of large closed cells (Theorem 21.3). The last part of that theorem is a rule that protects the endpoints of all frozen coordinates. Section 21.2 defines a deterministic sequence of curvature tests, partitions the configuration space according to their outcomes into large-field decompositions (Theorem 21.5), and verifies the hypotheses of the analytic theorems of Sections 19 and 20 on each decomposition (Lemma 21.7). Section 21.3 proves the coverage of every supported fiber point and constructs, at all stages, the representatives satisfying the reconstruction identities

(Theorem 21.9): they are obtained from the coarsest scale downward by the reconstruction maps, so that each reproduces the output field above it and contains the frozen values. At the first RG step under the Wilson measure, this section uses the lower bound for the constrained Hessian of Lemma 18.14, which is uniform in the region and in the frozen variables. Section 21.4 constructs the invariant densities (Theorem 21.10), and Section 21.5 proves the identity relating the densities before and after freezing (Lemma 21.11). Section 21.6 first compares an integral over a fiber with an integral over the fluctuation cutoff domain (Lemma 21.12), and then combines these results in the induction of Theorem 21.13.

The inputs are the quantitative extension of maps into $SU(2)$ in Lemma 7.5; the stopped measure and its coordinates on the torus (Theorems 14.3 and 14.6); the Haar disintegration on block regions (Lemma 18.7); the gauge transformation fixed at prescribed endpoints and the perturbation theorem of Lemma 19.5 and Theorem 19.9; and the analytic integration, the comparison of normalization constants, and the concentration estimate of Theorems 20.13 and 20.14 and Lemma 20.5. Theorems 21.5, 21.10, and 21.13 are used in Section 22 for the bounds on Wilson densities and the exponential moment bounds, and Theorems 21.3, 21.5, and 21.9 are used in the choice of parameters in Section 23.

The organization by nested regions, fields frozen on their complements, and effective densities on the remaining regions follows Balaban's large-field renormalization [18, 20, 21]. Dimock's account of Balaban's method for the scalar ϕ_3^4 model describes the partition into small-field and large-field regions, the unintegrated complement, and localized block averaging [67, §§1–2]; the summation of the resulting expansion is in [69]. These works concern a different model and different coordinates. The deterministic tests for the block averaging map T_M , the gauge construction on arbitrary unions of cells, the protection of frozen endpoints, the coverage of every fiber point, and the invariant density versions below are proved here.

Conventions. Throughout, $M \geq 3$ is odd, $J \geq 1$, and $B = M^J$. One application of T_M (Definition 4.3), including its Borel extension, is an averaging step, and an RG step consists of J consecutive averaging steps. A group of $1 \leq s \leq J$ consecutive averaging steps with output side m has intermediate scales $0 \leq i < s$ and couplings

$$g_i = t_{\max} M^{(i-s+1)/2}, \quad t_{\max} = \sqrt{\gamma/(Mm)}, \quad \lambda_* = M^{-(J-1)/2} \leq g_i/t_{\max} \leq 1.$$

The angle of $U \in SU(2)$ is $\rho_G(U)$, its distance from I in the bi-invariant metric of Section 4; we have $\rho_G(U_1 \cdots U_l) \leq \sum_j \rho_G(U_j)$ and $|\text{Log } U| = \rho_G(U)$ when $\rho_G(U) < \pi$. We write c for the output field. In analytic functions and in the coordinates of Definition 18.6, the coarse argument is its identity-chart logarithm $\text{Log } c$, which we also denote by c when no confusion can arise; since $|\text{Log } c_e| = \rho_G(c_e)$ for every output link e , bounds on the two agree. A block region Ω is a union of blocks of the output lattice of the current group

of averaging steps; its complement is the fixed complement. The frozen coordinates are written b_{rec} . They consist of frozen links and of based transports of frozen links across the boundary of the block region (defined in Section 21.2), each kept on the lattice on which it was frozen. A link is across the boundary of the block region if exactly one of its endpoints lies in it.

Fix the cutoff amplitude $A_w \geq 1$ at the outset. Every application of Theorem 20.14 below uses the coupling ceiling g_* chosen after this amplitude, and g_* may depend on A_w . The analytic constants and the conditions used to choose J do not depend on A_w .

Unless stated otherwise, constants written $C_J, c_J, u_J, \epsilon_J$, and similar depend only on M, J , the fixed larger construction and analytic domains and their radius ratios, the spatial and boundary decay rates and their margins, the bounded density of labels, and the fixed ceilings on actions and perturbations. They are independent of L, γ , all lattice sides, the total and the preceding depth, the number of RG steps since a coordinate was frozen, the shape, topology, and number of components of the regions, and the values of the frozen coordinates. This is the dependence described in item (3) of Definition 4.14. Where stated, these constants are also independent of the widths of boundary bands, removed bands, protected neighborhoods, and cells chosen later; such widths enter only through explicit geometric counts. No uniformity as $J \rightarrow \infty$ is asserted. The constant C_δ in Theorem 21.13 has instead the dependence described in item (4) of Definition 4.14.

21.1 Local path products and closed cell complexes

The full dependence set of a link of an averaged configuration is the set of links read by the block averaging map, including its Borel extension at exceptional values, by its reference path, and by every path product used to check that the logarithms involved lie in the correct branch. For an iterate we replace every link recursively by its full dependence set, and for a plaquette we take the union of the sets of its four links. These lists are treated as sets when we define excluded label sets; repetitions are kept only to estimate the lengths of path products. Geometric neighborhoods use the periodic maximum coordinate distance. Replacing it by the lattice distance used in the analytic norms changes only dimensional comparison constants.

Lemma 21.1. *Fix M and J . There exist finite constants D_J, L_J, v_J , depending only on M and J , and constants $u_J > 0$ and $C_J < \infty$, with the following properties for every group of $s \leq J$ averaging steps.*

1. *The full dependence set of a link, of a plaquette, or of a prescribed based transport of an averaged configuration is contained in the neighborhood of radius D_J of its output label, in output lattice units, and contains at most v_J input labels per output unit cube. The expanded hierarchical tree paths have length at most L_J .*

2. Suppose that every input link of such a full dependence set has angle at most $u \leq u_J$. Then every link appearing in the recursive computation of the selected output object has angle at most $(10M)^i u$ at level $i \leq s$. The corresponding relative logarithmic coordinates of the block averaging constraint, including the eliminated coordinates, have size at most $C_J u$. After one fixed reduction of u_J , they belong to the domain of the local inverse of the block averaging constraint, with a fixed positive margin.

These conclusions do not depend on the region or on its components.

Proof. An ordered path from the center of an M -block to a vertex of the block has length at most $3(M-1)/2$. The path $F_{c,x}$ of Definition 4.3 has length at most $4M-3$, and $F_{c,x}C_c^{-1}$ has length at most $5M-3$. If $(5M-3)u < 2\pi$, every relative transport $U(F_{c,x})U(C_c)^{-1}$ is in the identity chart, and the exponential of the average of their logarithms has angle at most $(5M-3)u$. Multiplying by $U(C_c)$, we obtain the bound $(6M-3)u \leq 10Mu$. Iteration is permitted when $(10M)^J u$ is smaller than the fixed positive margin of one averaging step. The Borel extension reads the same finite list of links, so the locality of the list does not require this smallness.

The based tree gauge on a block is obtained by multiplying transports along its ordered tree paths. It sets each tree link to I , equals I at the root, and is unique with these properties, since the equation on an oriented tree edge determines the gauge at the child from the gauge at the parent. At the next intermediate scale we make the same construction and lift the new gauge as a constant on each of the previous blocks. Such a lift conjugates each previously fixed tree link and hence preserves the value I . The conditions at coarser scales are fixed before the finer variables are expressed. The resulting hierarchical transports are products of these tree factors at the various scales; no eliminated constraint edge is inserted into a tree. Each factor equals I on the hierarchical tree gauge section. There are at most J scales, each with finitely many paths in a fixed block, and by substitution we obtain a finite maximal path length L_J and a finite maximal radius D_J of the dependence neighborhoods. Equivalently, we may take these constants to be the maxima over the finitely many oriented dependence neighborhoods for $1 \leq s \leq J$ on the infinite cubic lattice: restriction to a region only deletes path products, and passage to a periodic quotient only identifies labels. The maximal number of input labels in the same dependence neighborhoods gives v_J .

After the tree gauge transformation, every link outside the trees is a product of a number, bounded in terms of M and J , of the small links just bounded, and the coarse-field lift of the target is another such product. The relative vector is therefore bounded by $C_J u$. The constraints have the eliminated-edge inverse and the uniqueness neighborhood, with its positive margin, used in Lemma 18.7 and Theorem 20.14. We choose u_J so that the preceding bound is less than half the radius of that neighborhood, taking the minimum over $s \leq J$. Then the relative solution, including its eliminated entries, is the unique solution given by that inverse. This identifies the branch of a given solution; it

does not solve the constraints anew. Every path used above stays in a block of at most J scales, and none follows a tree spanning a whole component. $\square$

The next corollary gives the initial bounds for a stopped measure; it is used in the proofs of Theorems 21.5 and 21.9.

Corollary 21.2. *Before the bound β_{rec} for frozen coordinates and the support radius r are fixed, one can fix $C_{\text{ad},J} < \infty$, depending only on M, J , and the fixed tree conventions, with the following property. Let $U \in D_r$ be an initial configuration of a stopped measure on a lattice of side $M^s m$, with $s \leq J$, $m \geq 4$, and no previously frozen coordinates, and assume that r is below the fixed margin of the finite path-product charts. Then, for every union of blocks, there exists a gauge representative in which the output field and all frozen links and based transports at the input level are bounded by $C_{\text{ad},J} r$. This is a geometric existence statement, uniform in the preceding depth, the side, and the region.*

Proof. A gauge transformation taking U into B_r gives the bound r , not $r/2$, at the input. By Lemma 21.1, all intermediate internal links have angle at most $(10M)^J r$. The expanded hierarchical tree paths have at most L_J factors of these links. A link after the tree gauge transformation is a product of a path, a link, and an inverse path, and has angle at most $(2L_J + 1)(10M)^J r$. A based transport is a path times its link across the boundary of the block region and has the smaller bound $(L_J + 1)(10M)^J r$. Links lying entirely in the fixed complement remain below r , since the regional tree gauge is the identity there. The output roots are fixed by the tree gauge, and the output bound is $(10M)^J r$. We conclude that $C_{\text{ad},J} = 2(2L_J + 3)(10M)^J$ exceeds twice each of these bounds. We fix it now, and only afterward require $C_{\text{ad},J} r$ to be below the fixed identity and inverse chart radii. $\square$

We now turn to gauge fixing on large closed cells. Small curvature does not force small links: a flat configuration with nontrivial holonomy around a noncontractible cycle cannot be gauge transformed to the identity. The following theorem shows that on a union of closed cells of side about R , plaquette angles of order R^{-2} allow a gauge in which all links have angle of order R^{-1} . The term R^{-1} comes from distributing compact gauge values over the edges of the cell grid, which have length at least $R/2$; the curvature contributes $O(\delta R)$ through an axial gauge in each cell. Transitions between the frames of neighboring cells are compared only on common edges and are never multiplied along chains of cells, so the constants do not depend on the topology of the union. We combine standard axial frames and the extension of maps into $\text{SU}(2) \simeq S^3$ with the quantitative patching estimate of Lemma 7.5, preserving all prescribed vertices. The gauge fixing of lattice configurations with small plaquettes goes back to Balaaban [14]; the version below, on arbitrary unions of cells and with prescribed vertex values, is proved here.

Theorem 21.3. *There exist universal constants $C, C', R_0 < \infty$ and $c > 0$ with the following properties. Work in lattice units, let $R \geq R_0$ be an integer and $n \geq 8R$, and partition each*

coordinate circle of the periodic lattice of side n into intervals of lengths in $[R/2, 2R]$. Let P be any union of the resulting closed three-dimensional cells, and let U be a configuration on the links of P such that

$$\rho_G(U_p) \leq \delta \quad \text{for every plaquette of every cell of } P, \quad \delta R^2 \leq 1,$$

including the plaquettes in the boundaries of the cells. We take $\delta \geq 0$ and, in part (2), $\eta \geq 0$. A maximum of link angles over an empty set of links is defined to be zero. These hypotheses are in force in all parts below.

1. There exists a gauge transformation k on the vertices of P with

$$\max_{e \in P} \rho_G(U_e^k) \leq C(\delta R + R^{-1}) \leq C'/R.$$

This holds for disconnected P , for cells meeting only at vertices or edges, and for arbitrary holes, handles, periodic wrapping, and holonomy; the constants depend on none of these features, nor on n , L , the coupling, the depth, or J . If $q(U_p) \leq d_*$ for every such plaquette, one may take $\delta = \pi\sqrt{2d_*}$. For a fixed sufficiently large A_8 , the choice $R = \lceil A_8/r \rceil$ together with $\pi\sqrt{2d_*}R^2 \leq 1$ gives the bound $\rho_G(U_e^k) < r/2$ for every link.

2. Let $\Gamma \subset P$ be a union of closed cells of the grid, and let k_Γ be a gauge transformation on the vertices of Γ with

$$\rho_G(U_e^{k_\Gamma}) \leq \eta \quad (e \subset \Gamma), \quad K_R = 1 + \eta R + \delta R^2, \quad K_R^2/R \leq c.$$

Then there exists an extension k equal to k_Γ at every vertex of Γ with

$$\max_{e \in P} \rho_G(U_e^k) \leq CK_R^{10}/R.$$

Every link contained in a cell with no vertex in Γ satisfies the stronger bound C'/R . In particular, this bound holds after removing two layers of cells around Γ .

3. For fixed M and J , let $C_B \geq (10M)^J$ also bound the lengths of the finite tree path products. Assume $\eta \leq C_B r$, $r \leq 1$, and $R = \lceil A_8/r \rceil$, and set

$$K_B = 2 + C_B(A_8 + 1), \quad C_{\text{col}} = CK_B^{10}/A_8.$$

If $K_B^2 r / A_8 \leq c$ and $C_{\text{col}} r < \beta_{\text{rec}}/2$, then all links of P are smaller than $\beta_{\text{rec}}/2$, and, after one fixed increase of A_8 , the links in cells with no vertex in Γ are smaller than $r/2$. The constants in this part have the dependence described in item (3) of Definition 4.14. These statements assert the existence of gauge transformations; no configuration-dependent gauge transformation is used as a change of variables in an integral.

4. At later scales, use the following endpoint protection rule. On the lattice on which each frozen coordinate was frozen, mark both endpoints of every frozen link, both endpoints of

every based transport, and every prescribed basepoint or root. A block of the current output lattice is excluded whenever its full centered preimage on the lattice of some mark contains that mark. Then every gauge transformation supported on the remaining blocks has a lift, constant on blocks, which equals I at every mark, and this lift preserves every frozen link and every based transport exactly. This holds at every finite depth, for coordinates frozen at any earlier step and at periodic seams, with no sum over the steps at which the coordinates were frozen. A newly chosen root of the Haar disintegration is fixed separately by the current based gauge. For the gauge transformation of Lemma 19.5, the current endpoints of the links across the boundary of the block region are included among the fixed vertices as well.

Proof. We give the absolute and the relative constructions on the same skeleton.

Step 1: axial frames. In each cell Q of P , we fix a corner o_Q , let τ_x be the coordinate-ordered path in Q from o_Q to x , and define $q_Q(x) = U(\tau_x)$. Then $U_{xy}^{q_Q}$ is the transport around the loop $\tau_x(x, y)\tau_y^{-1}$. To bound it, we compare the ordered paths to the adjacent vertices x and y . Moving the last elementary step through the ordered path inserts at most two rectangular strips, each of width one and length at most $2R$. After canceling consecutive inverse edges, the resulting loop is a product of conjugated elementary plaquettes. There are at most CR of them, all in Q , including its boundary. By the product inequality for ρ_G , we have

$$\rho_G(U_{xy}^{q_Q}) \leq C_0 \delta R. \quad (21.1)$$

Here C_0 is absolute, since the aspect ratio of a cell is at most four. This local estimate follows from Lemma 7.1: the cell has an injective periodic lift because $2R \leq n/4$, and the sum of its three side lengths is at most $6R$.

Step 2: transitions. For two cells Q, Q' containing a common lattice edge (x, y) , we define $t_{QQ'}(x) = q_Q(x)q_{Q'}(x)^{-1}$. Direct multiplication gives

$$t_{QQ'}(y) = (U_{xy}^{q_Q})^{-1} t_{QQ'}(x) U_{xy}^{q_{Q'}}, \quad d(t_{QQ'}(x), t_{QQ'}(y)) \leq 2C_0 \delta R, \quad (21.2)$$

where we used (21.1) and bi-invariance. This estimate holds whether the cells share a face or only an edge. When two cells share only a vertex, only the equality of the assigned value at that vertex is required.

Step 3: the edges of the grid. We write the desired gauge in the frame of Q as $s_Q(x) = k(x)q_Q(x)^{-1}$ and set $k = I$ at every vertex of the cell grid. On each edge of the grid, we fix one incident cell frame, interpolate the two endpoint values of s_Q by a shortest geodesic in $SU(2)$, and sample it at the lattice vertices. Since the group has diameter 2π and the edge has length at least $R/2$, the speed is at most $4\pi/R$. Each edge of the grid is assigned once, regardless of the number of incident cells. By (21.2), changing to another incident frame adds at most $C\delta R$ per lattice edge. Since $\delta R^2 \leq 1$, the assigned frame values on every edge of the grid have lattice Lipschitz constant at most C_1/R .

Step 4: the faces of the grid. On a face of the grid, we convert the assigned boundary values into the frame of one incident cell, and we use a single consistent fine triangulation on all faces of the grid. Since the vertices of each fine simplex can be joined by a bounded number of lattice edges in the boundary, their distances are at most C_2/R . We take R_0 so large that the normalized affine interpolation clause of Lemma 7.5 applies, with denominator at least $1/2$. The continuous boundary map then has Lipschitz constant C_3/R . The disk extension clause, rescaled to this rectangle, bounds the Lipschitz constant of its extension by $CR^{-1}(1 + C_3)^2$. Sampling the extension at all lattice vertices of the face, we recover $k = s_Q q_Q$. This is done once for each face, and previously assigned vertices are preserved exactly.

Step 5: the cells. For a cell Q of P , we convert the assigned boundary values into the frame of Q . On every lattice edge in the boundary their difference is at most C_4/R , by the face bound and (21.2). The same consistent triangulation and normalized interpolation give a boundary map from the sphere with Lipschitz constant C_5/R ; we increase R_0 once to satisfy its simplex margin. By the ball extension clause, we obtain a map with Lipschitz constant $CR^{-1}(1 + C_5)^5$, and sampling it gives k at the remaining vertices. For an edge of Q , the final comparison is the sum

$$\begin{aligned} \rho_G(k_x U_{xy} k_y^{-1}) &= \rho_G(s_Q(x) U_{xy}^{q_Q} s_Q(y)^{-1}) \\ &\leq \rho_G(U_{xy}^{q_Q}) + d(s_Q(x), s_Q(y)) \leq C(\delta R + R^{-1}). \end{aligned} \quad (21.3)$$

In the last inequality, we used (21.1) and the Lipschitz bound of the extension. Since $\delta R \leq R^{-1}$, this proves the absolute estimate with $C' = 2C$, after enlarging C to include the finitely many aspect and simplex constants. Since the assignments on the skeleton agree at every common vertex, they can be made directly on the periodic quotient and on any union of cells. Handles, the number of components, and the holonomy have not entered any constant.

Step 6: the relative construction. We keep every lattice vertex of Γ at its prescribed value. In the frame of a cell Q , the prescribed values satisfy, on each prescribed lattice edge,

$$d(k_\Gamma(x) q_Q(x)^{-1}, k_\Gamma(y) q_Q(y)^{-1}) \leq \eta + C_0 \delta R \leq CK_R/R. \quad (21.4)$$

This follows by applying the first inequality in (21.3) in reverse to the two transformed link matrices, whose angles are both known; it imposes no bound on the increments of k_Γ in the original frame. At the vertices of the grid not in Γ , we set $k = I$. An edge of the grid in Γ is kept, and every other edge of the grid is a geodesic in one incident frame between its assigned endpoint values. Since Γ is a union of closed cells of the grid, its intersection with an edge of the grid is either the whole edge or a subset of its endpoints, so this rule respects all prescribed values. Its lattice increments in any cell frame are bounded by CK_R/R .

We keep each face of the grid lying in Γ . On every other face, the boundary interpolation and the disk filling give the bound CK_R^2/R ; the interpolation is legitimate when CK_R/R is below its fixed simplex margin. On the boundary of a cell not in Γ , all assigned frame increments, including those on a face kept from Γ , are now at most CK_R^2/R . The universal c in the hypothesis is taken smaller than both required interpolation margins divided by these fixed constants. Since $K_R \geq 1$, the condition $K_R^2/R \leq c$ implies the first margin as well. The boundary interpolation on the sphere and the ball filling then give

$$\text{Lip}(s_Q) \leq CR^{-1}(1 + CK_R^2)^5 \leq CK_R^{10}/R.$$

All cells of Γ are kept unchanged. Using (21.3) and $\delta R \leq R^{-1} \leq K_R^{10}/R$, we obtain the relative bound.

Step 7: cells away from Γ . If Q has no vertex in Γ , no face or edge of Q is prescribed. Each edge of the grid in the boundary of Q is then a shortest geodesic of speed at most C/R , even if its endpoint values were assigned in the frame of another incident cell, and frame changes add at most C/R . The disk extensions on the faces of Q then have Lipschitz constant C/R , and so does the ball extension in Q . By (21.3), we obtain the stronger conclusion for every edge of Q . Removing two layers of cells around Γ removes every cell with a vertex in Γ .

Step 8: the choice of δ and of A_8 . If $U = e^X$ and $x = |X| \leq 2\pi$, then $q(U) = 2\sin^2(x/4)$ and $\sin(x/4) \geq x/(2\pi)$. Hence $x^2 \leq 2\pi^2 q(U)$, and the stated value of δ is valid. Let A_8 be larger than $4C'$ and R_0 , increased by the fixed geometric factors used later in the excluded sets. The standing hypothesis gives $R \geq R_0$. Also $R \geq A_8/r$, hence $C'/R \leq (C'/A_8)r < r/2$. For part (3) we have $A_8/r \leq R \leq (A_8 + 1)/r$, so

$$K_R \leq 2 + C_B(A_8 + 1) = K_B, \quad CK_R^{10}/R \leq C_{\text{col}}r, \quad K_R^2/R \leq K_B^2r/A_8.$$

The two smallness conditions in part (3) then imply the stated bounds. Since the constant C_B is fixed before r , these conditions are upper bounds on r and a lower bound on the ratio β_{rec}/r , and they are not circular.

Step 9: endpoint protection. In one coordinate, the balanced-digit identity

$$\left\{ \sum_{j=0}^{s-1} M^j a_j : a_j \in \{-(M-1)/2, \dots, (M-1)/2\} \right\} = \{-(M^s-1)/2, \dots, (M^s-1)/2\}$$

holds: existence follows by successively taking the balanced residue modulo M , and uniqueness by reducing two such expressions modulo M and dividing their difference by M . Taking the product over the three coordinates and then the periodic quotient, we find that each vertex of the lattice of a mark has exactly one ancestor in any current centered partition, also at a seam. A mark in the preimage of a block would exclude that block. A

gauge transformation supported on the remaining blocks is I at every such ancestor, and its lift, constant on blocks, is I at the mark. For any path, the gauge factors at the interior vertices cancel in order, and only the two endpoint factors remain. This proves that every marked frozen link and every based transport is preserved exactly, for all marks at once and with no sum over earlier steps. Finally, the cutoff in Lemma 19.5 vanishes at the prescribed endpoints of the links across the boundary of the block region as well, which proves the last statement. $\square$

21.2 Deterministic large-field decompositions

In this section, we partition the support of the initial measure according to the outcomes of a deterministic sequence of curvature tests, and we show that on each piece every block region admits a representative with small links and small newly frozen coordinates.

Based transports and frozen coordinates. Let $e = (u_e, v_e)$ be a link across the boundary of the block region, oriented from its endpoint u_e in the block region to its endpoint v_e in the fixed complement, and let σ_e be the hierarchical tree path of Lemma 21.1 from the root α_e of the block containing u_e to u_e . The based transport of e is $\widehat{b}_e = U(\sigma_e)U_e$, the transport from α_e to v_e ; its endpoints are α_e and v_e , while the endpoints of the link itself are u_e and v_e . When integration stops at a boundary, such links are frozen through their based transports, and the other links leaving the integration are frozen as links. We write $c^\sharp$ for the analytic coordinates of newly frozen coordinates: in the coordinates we use, the logarithm $\text{Log } b$ of the frozen group value b , or a fixed analytic function of finitely many such logarithms (see (21.5)). Group values of frozen coordinates are never compared with analytic radii directly.

Parameters. We fix M, J , the construction domains of Theorem 20.14, and A_8 sufficiently large as in Theorem 21.3. Then we fix the frozen-coordinate conversion constant $C_{\text{conv}, J}$ of (21.7) below. It is computed from the relative extension of Theorem 21.3, the finite path products of the constraints and trees, the constant $C_{\text{ad}, J}$ of Corollary 21.2, the constant converting the near values at the first step under the Wilson measure, and the constants converting frozen group values into analytic coordinates. It is fixed before the two radii below. Then $r > 0$ and the bound $\beta_{\text{rec}} > C_{\text{conv}, J}r$ for frozen coordinates are taken so small that every newly frozen analytic coordinate satisfies

$$|c_j^\sharp| \leq \min\{\beta_{\text{rec}}, \phi_\bullet(j)/4\}, \quad C_{\text{bc}, \bullet} E_\bullet \beta_{\text{rec}} / \rho_\bullet \leq \sigma_*/4, \quad \bullet \in \{\text{cl}, \text{q}\},$$

for the classical and the quantum spatially varying analytic radii $\phi_{\text{cl}}, \phi_{\text{q}}$, with the radii $\rho_\bullet$, ceilings $E_\bullet$, and boundary-completion constants $C_{\text{bc}, \bullet}$ of Theorem 20.14. These are conditions on deterministic parameters, imposed before the coupling. The deep frozen group variables at the first step under the Wilson measure are arbitrary elements of

SU(2), not small analytic logarithms; they are kept as fixed parameters and are never inserted into this analytic freezing. We set

$$R_{\text{cell}} = \lceil A_8/r \rceil, \quad W = 8BR_{\text{cell}}, \quad \pi\sqrt{2d_*}R_{\text{cell}}^2 \leq 1.$$

The width W is measured in output units of the current RG step; in output units of the next RG step it is $W/B = 8R_{\text{cell}}$. The excluded sets below also contain the band of width s_{er} removed at each boundary (Theorem 20.14) and the protected neighborhoods of the finite path products. We may decrease r initially so that $W \geq s_{\text{er}}$.

At each boundary between RG steps, the previous fixed complement and the earlier frozen basepoints lie outside the next block region, up to the fixed buffer of the dependence neighborhoods of the path products. Equivalently, their incoming and current boundary distances satisfy the comparison required in Theorem 19.9; this is proved as (21.10). The endpoint protection rule of Theorem 21.3 contains these basepoints and the newly frozen labels, and additional excluded regions preserve the inclusion. At the first step, there is no previous fixed complement, and the condition is vacuous.

Cell grids. At each lattice side $n \geq 8R_{\text{cell}}$, we fix the cell grid deterministically. For example, we divide each coordinate circle into $\lceil n/R_{\text{cell}} \rceil$ intervals, distributing the remainder among intervals of the two nearest integer lengths. For $R_{\text{cell}} \geq 2$ these lengths lie in $[R_{\text{cell}}/2, 2R_{\text{cell}}]$. When an input grid must be aligned with the blocks, we perform the same construction in units of the blocking grid and round its scale up to a multiple of the block side. Since B is fixed and R_{cell} may be taken larger than B initially, this changes the scale by at most a fixed factor.

Definition 21.4. Fix a finite range of scales consisting of complete RG steps and an initial configuration U . The ordered tests are the following finite procedure.

1. Before any gauge transformation, compute the iterates $V^\ell = T_M^\ell U$ at all levels of the current RG step, with the Borel extension of T_M . Go through these levels in increasing order and, at each level, through the plaquettes in a fixed order.
2. A plaquette f fails if $q_f = q(V_f^\ell) \geq d_*$ and its full dependence set does not meet the current excluded label set. When f fails, all variables of its full dependence set that are not yet frozen are frozen at the input of its RG step, together with those of a fixed neighborhood of the dependence set. This neighborhood is aligned with the blocks of the output lattice and has output radius $2W$. It includes the fixed margins of the dependence neighborhoods and the protected neighborhood of the endpoint protection rule. Previously frozen variables stay where they are. The neighborhood is added to the excluded label set. A plaquette whose dependence set meets the excluded label set is not tested; it is excluded, not declared good.
3. After the tests of an RG step, retain only the closed cells of the output cell grid whose full dependence sets and protected neighborhoods avoid the excluded label set, and

add every touching excluded cell to the fixed complement. Freeze the full output boundary band: the layer of retained cells next to the excluded set, its incident cells (all retained grid cells meeting that layer, even at a vertex), all labels of reduced analytic radius, the removed band of width s_{er} , and the margins of the dependence neighborhoods. Enlarge the previous fixed complement by an output neighborhood of width W , and then by the protected neighborhood of the endpoint protection rule of Theorem 21.3, using every endpoint and basepoint of every frozen coordinate on the lattice where it was frozen. A cell is retained only if its closure avoids all these exclusions, so a plaquette in the boundary of a cell is tested on the same terms as an interior plaquette.

4. For the next RG step, keep only the blocks that lie, together with their full dependence neighborhoods, in the retained interior. The discarded band is frozen. Excluded regions are combined by taking unions. If the block region becomes empty, its frozen coordinates are kept and the procedure stops.

When the initial measure is the Wilson measure, the first RG step is modified as in Theorem 21.5(4). The outcome of the procedure, consisting of the failing plaquettes, the excluded regions, and the label lists of the coordinates frozen at all steps, is a large-field decomposition, denoted $\mathcal{X}$. The values of these coordinates are variables of the stage measures and are not part of the finite index $\mathcal{X}$. We write $\mathcal{A}_{\mathcal{X}}$ for the set of initial configurations for which the procedure produces $\mathcal{X}$.

Theorem 21.5. *Make the parameter choices above, including the initial smallness reductions and fixed mesh factors specified in the proof below.*

1. *For every finite range of scales, every initial side, every preceding depth $d_0 = aJ$ with $a \geq 0$, and every normalized stopped measure*

$$d\hat{v} = \hat{Z}^{-1} \mathbf{1}_{D_r} e^{-E_{d_0, \text{sat}} / g_{m_{\text{base}}}^2 - [R_{d_0, \text{sat}} - R_{d_0}(I)]} dH_{m_{\text{base}}},$$

the sets $\mathcal{A}_{\mathcal{X}}$ form a finite deterministic Borel partition of the support of $\hat{v}$, provided that the output side of every complete RG step is at least $8R_{\text{cell}}$. The same holds when the initial measure is the Wilson measure, with the first step described in part (4). Every tested plaquette that is not excluded satisfies $q_f < d_*$. The gauge transformations used below to exhibit small representatives do not enter the tests.

2. *At every boundary between RG steps there exists a gauge representative in which the links in the interior of the block region are smaller than $r/2$, and the newly frozen analytic links and based transports at that output boundary are smaller than β_{rec} on the lattice where they are frozen. In the same representative, the ordinary links of the prescribed closed-cell collar Γ used in the relative output extension have the stronger native bound $\rho_G(b_e) \leq C_{\text{Br}}$ on that output lattice. These collar links belong to the full frozen output band, and both of their endpoints are protected at subsequent steps. The first-input deep Wilson group variables of*

part (4) are not among these analytic output coordinates. All previously frozen coordinates are preserved together with their endpoints; no smallness of their later averaged images is asserted. A block region that becomes empty keeps its frozen coordinates, and the procedure stops there. The same representative bounds hold on the closed supports of the stage measures in part (5), by the closed-support argument in Section 21.3.

3. Every added cell and every frozen label can be assigned to a failing plaquette by a local rule. For a failing plaquette f , the radius $R_t(f)$ of the neighborhood assigned to f , measured at the outputs of successive RG steps, satisfies

$$R_{t+1}(f) \leq B^{-1}R_t(f) + C_J(W + R_{\text{cell}} + 1), \quad R_{\text{first}}(f) \leq C_J(W + R_{\text{cell}} + 1),$$

where $R_{\text{first}}(f)$ refers to the RG step in which f fails. The number of scalar labels assigned to f is at most $C_J(BW + R_{\text{cell}} + 1)^3$ in that RG step and at most $C_J(W + R_{\text{cell}} + 1)^3$ in each later RG step. The final plaquettes whose full dependence sets meet a neighborhood assigned to f are at most

$$F_J = C_J(W + R_{\text{cell}} + 1)^3$$

in number, with a fixed enlargement for each of the finitely many remaining numbers of averaging steps after the last RG step (Theorem 21.13). These estimates are subadditive under unions. A set of failing plaquettes that can be produced by the ordered tests determines at most one large-field decomposition. The constants C_J have the dependence stated at the beginning of this section and are independent of the later widths; the displayed polynomials exhibit the dependence of the counts on the widths.

4. For the first RG step under the Wilson measure, the ordered tests also test the plaquettes of the input configuration, before any representative is chosen. On a fixed input cell grid of scale S , aligned with the blocks, with $C_J^{-1}R_{\text{cell}} \leq S \leq C_J R_{\text{cell}}$, retain only the closed cells whose full dependence sets avoid the final excluded set. Move the block region inward by one full layer of cells and by the full dependence neighborhood of the constraint, tree, and basing path products, and require

$$n_{\text{input}} \geq 8S, \quad \pi\sqrt{2d_*}S^2 \leq 1.$$

Then every plaquette of the resulting closed cell complex, including its boundary plaquettes, is a tested plaquette with $q < d_*$. The complex contains every plaquette whose Wilson term depends on an integration variable and every constraint and tree path product of the block region. There are no earlier frozen vertices at this step. The gauge transformation of Theorem 21.3(1) on this complex makes the integration variables and the near values small; it extends to the remaining vertices, whose links are kept as arbitrary elements of $\text{SU}(2)$ (the deep frozen group variables). The relative extension at the output gives the bounds of part (2) for the newly frozen coordinates. All additional excluded sets are assigned to the same failing plaquettes and satisfy the bounds of part (3) after one enlargement of C_J .

5. For either initial measure ν define the stage measures

$$\eta_t^{\mathcal{X}} = (\mathcal{O}_t^{\mathcal{X}})_{\#}(\mathbf{1}_{\mathcal{A}_{\mathcal{X}}} \nu),$$

where $\mathcal{O}_t^{\mathcal{X}}$ maps a configuration to the field on the current block region together with all frozen links and based transports, each on the lattice where it was frozen. All the tests remain in these measures. For every Borel set S (an event, not the grid scale of part (4)), in particular for every small-field event of the form (4.6) on the same finite hierarchy,

$$\mathbf{1}_S = \sum_{\mathcal{X}} \mathbf{1}_S \mathbf{1}_{\mathcal{A}_{\mathcal{X}}} \quad \nu\text{-almost surely.}$$

The partition does not change S and imposes no additional restriction on the Wilson measure. All statements are uniform in the holonomy, the number of RG steps since coordinates were frozen, the topology of the regions, and the deep frozen group variables. A remaining group of fewer than J averaging steps at the end of the range stops before the geometric conditions on the minimal side fail.

The following lemma records which changes of representative preserve the bounds on frozen coordinates; it is used in the proof of Theorem 21.5 and afterward.

Lemma 21.6. *Consider a representative on a block region of a large-field decomposition of Theorem 21.5, with each frozen coordinate kept on the lattice where it was frozen. In part (3), assume that the analytic frozen coordinates satisfy (21.8).*

1. *Every gauge transformation supported on the current block region and lifted under the endpoint protection rule equals I at the endpoints of all earlier frozen links and based transports. It fixes their group values and their logarithms exactly.*
2. *A gauge transformation equal to I at the roots of the block region and at the outer endpoints v_e of the links across the boundary of the block region fixes every based transport $\widehat{b}_e$, although it may move the inner endpoint u_e . The gauge transformation of Lemma 19.5 fixes u_e as well.*
3. *A simultaneous conjugation $b \mapsto k b k^{-1}$ of the frozen coordinates preserves (21.8).*
4. *A change $b_{xy} \mapsto k_x b_{xy} k_y^{-1}$ with independent k_x, k_y need not preserve (21.8).*

In the following, analytic estimates are transported between representatives only by the changes in parts (1)–(3). Changes as in part (4) are used only for the invariance of densities in Theorem 21.10. None of these conventions changes the tests or restricts the initial measure.

Proof. Part (1) is Theorem 21.3(4). For part (2), the gauge factors at the interior vertices of the path σ_e and of the link e cancel, so $\widehat{b}_e$ changes only by the factors at α_e and v_e , which are I ; the factor at u_e cancels. The gauge transformation of Lemma 19.5 is I at the protected endpoints of the links across the boundary of the block region, including u_e . For part (3), we have $\text{Log}(k b k^{-1}) = \text{Ad}_k \text{Log } b$ on the identity chart, and Ad_k is an

isometry of $\mathfrak{su}(2)$. Longer path-product coordinates have the same equivariance because the interior gauge factors cancel. Relative eliminated-coordinate maps are equivariant by covariance of their defining constraint and uniqueness of its local inverse. Thus the fixed analytic coordinates used here transform by the same global adjoint action; mere analyticity of an arbitrary coordinate map would not suffice. For part (4), take $b = I$ and $k_y = I$; then $\rho_G(k_x b k_y^{-1}) = \rho_G(k_x)$ is arbitrary. $\square$

Proof of Theorem 21.5. We first fix all margins and constants that do not depend on a configuration (Steps 1 and 2). We then show that the tested plaquettes that are not excluded are good (Step 3), construct the small representatives by induction over the RG steps (Step 4), and compare boundary distances (Step 5). Steps 6 and 7 prove the counts and the measurability, Step 8 treats the first step under the Wilson measure, and Step 9 identifies the stage measures.

Step 1: fixed margins. All excluded sets in this proof are sets of labels, chosen by their locations and dependence sets; their construction uses no link representative. We fix D_J, L_J, v_J from Lemma 21.1 and $C_{\text{ad},J}$ from Corollary 21.2. The protected neighborhoods include one full copy of each dependence neighborhood, its incident cells, and the removed band of width s_{er} , and A_8 includes the factors required in Theorem 21.3. The support radius will be required to give $R_{\text{cell}} \geq R_0$ and W larger than s_{er} and than the radii of these finitely many dependence neighborhoods. The number of copies and their enlargement factors depend only on M, J , and the fixed construction domains; in particular, they are fixed before W or R_{cell} is evaluated.

Step 2: constants for frozen coordinates. We specify the margins for frozen coordinates before using any representative. On the lattice where it is frozen, the analytic coordinate of a frozen link b_e or based transport $\widehat{b}_e$ is its identity-chart logarithm. If its angle is below π , then

$$|\text{Log } b_e| = \rho_G(b_e), \quad |\text{Log } \widehat{b}_e| = \rho_G(\widehat{b}_e). \quad (21.5)$$

Here $|\cdot|$ is the adjoint-invariant Euclidean norm on $\mathfrak{su}(2)$, and the conversion constant is one. If the fixed coordinate convention also uses a longer path product or a relative eliminated coordinate, let $C_{\text{cr},J} \geq 1$ bound its path length and the row sums of the first derivative of its fixed inverse chart on the larger construction domain. Such maps vanish at the configuration with all links equal to I . By the product inequality for angles, or by integration of the first derivative along the straight segment, their size is at most $C_{\text{cr},J}$ times the bound on their inputs. This constant depends only on the fixed charts and on M, J , not on any width or number of steps.

We fix $C_{\text{conv},J}$ by (21.7) below, after A_8 and before either radius; it includes $C_{\text{ad},J}$ and $C_{\text{cr},J}$. Let $\rho_{\partial,\text{cl}}$ and $\rho_{\partial,\text{q}}$ be the boundary values of the two spatially varying radii, and define

$$b_{\min} = \frac{1}{4} \min\{\rho_{\text{cl}}, \rho, \rho_{\partial,\text{cl}}, \rho_{\partial,\text{q}}\}.$$

We take $\beta_{\text{rec}} < \min\{\pi, \rho/4, \rho_{\text{cl}}/4\}$ satisfying the two conditions $C_{\text{bc},\bullet} E_{\bullet} \beta_{\text{rec}} / \rho_{\bullet} \leq \sigma_*/4$, and then require

$$C_{\text{conv},J} r < \min\{\beta_{\text{rec}}, b_{\min}\}. \quad (21.6)$$

These are the conditions on frozen coordinates in the choice of parameters. All quantities on the right side are fixed positive parameters, so this upper bound on r is compatible with its other upper bounds.

Step 3: the tests and the goodness of retained plaquettes. We first consider a stopped measure. At the input of the first RG step, a gauge transformation into B_r gives all links smaller than r . By Corollary 21.2, which precedes this construction, the initial frozen coordinates at the input are bounded by $C_{\text{ad},J} r$ in a common representative. Since $C_{\text{conv},J} \geq C_{\text{ad},J}$, their analytic coordinates satisfy (21.6) as well. At later inputs, by the induction in Step 4, the links in the interior of the block region are small, with all earlier frozen coordinates fixed. Before any gauge transformation, we compute the iterates at every level of the current RG step and perform the ordered tests of Definition 21.4. The tests always use the values of the iterates themselves, also when a path product that is not used has an exceptional value of the Borel extension.

After the tests, the retained cells are those of Definition 21.4(3). A dependence set avoiding the final excluded set also avoided the smaller excluded set at the time of its test. If that plaquette had value at least d_* , its failure would have put the dependence set into the excluded set. This contradiction shows that every tested plaquette that is not excluded is good, that is, $q_f < d_*$. Adding excluded sets after the tests can only reduce this set of plaquettes.

Step 4: small representatives. We prove part (2) by induction over the boundaries. Let P be the retained closed cell complex. Every link read for an output link of P comes from links of the current interior, before the new freezing at the input. By Lemma 21.1, its angle in the incoming representative is at most $C_B r$, where C_B includes the finite tree path lengths. By Step 3 above and Step 8 of the proof of Theorem 21.3, all plaquettes of P have angle at most $\delta = \pi\sqrt{2d_*}$. Let Γ consist of one layer of retained cells next to the excluded set, enlarged to contain all prescribed boundary vertices. We prescribe the identity gauge on Γ in this representative; its link bound is $\eta \leq C_B r$. After the initial reduction $K_B^2 r / A_8 \leq c$, parts (2) and (3) of Theorem 21.3 apply. The resulting gauge transformation is exactly I at every prescribed lattice vertex, has link bound $C_{\text{col}} r$ on P , and has the bound $< r/2$ in cells without prescribed vertices. It extends by I outside P , because every boundary vertex of P is prescribed. We lift it as a constant through the current RG step. By the endpoint protection rule, the lift is I at all earlier marks and at the variables already frozen at the input in this RG step. Hence every earlier frozen coordinate, its endpoints, and every newly frozen input variable keep their values.

We freeze the full output boundary band, including Γ , its incident cells, all labels of reduced analytic radius, the removed band, and the margins of the dependence neighborhoods. Every ordinary link in Γ is recorded with its pre-extension value, since the extension gauge is I at both endpoints. Its bound $C_B r$ is therefore native to its creation lattice, not an estimate obtained by averaging an earlier frozen value. Every remaining interior link lies in a cell with no vertex in Γ , because every retained cell meeting Γ belongs to the frozen band. Hence its angle is smaller than $r/2$. The newly frozen links have angle at most $\max(C_B, C_{\text{col}})r$. A based transport multiplies such a link by a hierarchical transport with at most L_J factors, and its angle is at most the sum of their angles. As announced before the radii were chosen, we may therefore fix

$$C_{\text{conv},J} > 2C_{\text{cr},J}(1 + L_J) \max\{C_B, C_{\text{col}}, C'/A_8, C_{\text{ad},J}\}, \quad (21.7)$$

with any additional fixed factors of dependence neighborhoods absorbed once. Every analytic coordinate of the union of the newly frozen boundary band, boundary layer, and based transports then has size at most $C_{\text{conv},J}r$. By (21.5) and (21.6), we have simultaneously

$$|c_j^\#| < \beta_{\text{rec}}, \quad |c_j^\#| \leq \phi_\bullet(j)/4 \quad (\bullet \in \{\text{cl}, \text{q}\}). \quad (21.8)$$

This includes every label of reduced radius, not only the boundary links of a layer. The constant is computed from A_8 , C_B , and the universal relative extension before r and β_{rec} are chosen.

For the next RG step, we retain only the blocks that lie, together with their full dependence neighborhoods, in the remaining interior. The discarded band has thickness at most C_J in that input lattice and consists of links that were small before discarding. The fixed factor coming from its dependence neighborhood was included in $C_{\text{conv},J}$ before the radii were fixed, so the band may be frozen with the same bound. The previous excluded region was enlarged by W in the current output lattice. After the next RG step, this is $W/B = 8R_{\text{cell}}$ units, which suffices for the required layer of cells and the fixed incidences; the full dependence neighborhood is included in addition. None of these operations requires a later averaged image of a frozen coordinate to be small, since the coordinate stays on its own lattice. This completes the induction.

Step 5: incoming and current boundary distances. For the next RG step, let D_{in} be its list of remaining analytic input labels after the whole freezing, and let Ω' be the set of its current retained blocks. Each incoming label is kept at its location on the lattice where it was created, with its original volume factor. A current block is retained only when its full input dependence neighborhood is contained in D_{in} . An omitted incoming label lies outside the block region, apart from the fixed displacement between the location of a chart label and its path product. A newly frozen boundary label, or a label of a link across the boundary of the block region, is omitted from D_{in} and satisfies the same

statement. By the endpoint protection rule, an earlier frozen endpoint or basepoint has its current centered ancestor in an excluded block. By the balanced-digit identity, the difference between its rescaled position and that ancestor is less than one current unit, uniformly in the number of steps. Including the finite dependence neighborhoods of the charts, we may take $L_{\text{buf},J} = 3D_J + 6$, so that every omitted input label x satisfies

$$\text{dist}(x/B_s, (\Omega')^c) \leq L_{\text{buf},J}, \quad B_s = M^s. \quad (21.9)$$

In the induction $s = J$. The same inclusion holds for a single group of $s \leq J$ averaging steps when its own domain conditions are imposed. Enlarging the excluded regions can only decrease the distance to $(\Omega')^c$, so it preserves the inclusion.

If the incoming fixed complement is nonempty, let i be an incoming retained label and x a nearest omitted input label. For a current retained label e we use the same distance as in the two boundary-weighted norms; the factor 3 in $L_{\text{buf},J}$ covers the comparison with the maximum-distance convention for dependence neighborhoods. By the triangle inequality and (21.9), we have

$$d_{\text{current}}(e) \leq \text{dist}(e, i/B_s) + B_s^{-1}d_{D_{\text{in}}}(i) + L_{\text{buf},J} + 1,$$

where the last term 1 accounts for the choice of a cell rather than a vertex as location. Rearranging, we obtain

$$d_{D_{\text{in}}}(i) \geq B_s d_{\text{current}}(e) - B_s \text{dist}(e, i/B_s) - C_{\text{buf},J}, \quad C_{\text{buf},J} = B_s(L_{\text{buf},J} + 1). \quad (21.10)$$

Rescaling coordinates in this calculation does not rescale the counting measures of the labels or discard their volume factors in a column norm; these are the same as in the incoming analytic norm. If there is no incoming fixed complement, the boundary-weighted part of the input is zero, and no nearest omitted label is needed. A new excluded region satisfies the inclusion just proved once the initial split has been made. The buffer in (21.10) depends only on M , J , and the fixed charts, not on W , s_{er} , the number of steps, or the number of excluded components.

Step 6: locality and counts. We verify locality and the counts without counting connected components. We label each initially added neighborhood by its failing plaque; when neighborhoods overlap, a label in their intersection is assigned to the first incident failing plaque in the fixed order. The enlargement of a union by a fixed neighborhood is contained in the union of the corresponding enlargements of its members. A cell meeting such an enlargement lies within a further $4R_{\text{cell}}$ of it, and a full dependence neighborhood costs at most $D_J + 1$ more. Projection of a mark to its current ancestor contracts distances by the blocking factor, with a rounding error of at most one current unit for the whole projection, not one for each earlier step. This follows from the balanced-digit identity in the proof of Theorem 21.3, since the whole preimage has radius less than half a block side.

If $R_t(f)$ bounds all current neighborhoods and projected marks assigned to f , the next RG step gives, with a fixed $a_J < \infty$,

$$R_{t+1}(f) \leq B^{-1}R_t(f) + a_J(W + R_{\text{cell}} + 1). \quad (21.11)$$

The neighborhood created when f fails satisfies the same bound in the output units of its RG step. We take $C_J \geq a_J/(1 - B^{-1})$ and at least the corresponding constant for the step in which f fails. Iterating (21.11), we obtain $R_t(f) \leq C_J(W + R_{\text{cell}} + 1)$ at every later boundary between RG steps. A remaining group of fewer than J averaging steps at the end is read as a fixed finite dependence neighborhood at the last boundary; it creates no additional integrated step and no update of the normalization constant. All endpoint marks added when boundary layers are frozen lie in the same neighborhoods, so the protected neighborhoods of the endpoint protection rule obey (21.11) too, even if the frozen coordinates contain marks on many lattices.

A ball of radius a in the maximum distance contains at most $(2a + 3)^3$ vertices. The link direction, the color component, the intermediate scale, and the type of frozen coordinate multiply this by a fixed C_J . In the RG step in which f fails we convert the radius into input units, which costs a factor B ; at later RG steps we use the bound in current units just proved. We obtain the counts

$$C_J(BW + R_{\text{cell}} + 1)^3, \quad C_J(W + R_{\text{cell}} + 1)^3$$

of scalar labels. The lists of missing labels, boundary-band labels, and frozen labels used in Theorem 20.13 are subsets of these lists of bounded density; deterministic images of earlier frozen coordinates are not new independent labels. A final plaquette whose full dependence set meets an excluded region lies in a further fixed dependence neighborhood of one of the assigned sets. Counting the three plaquette directions there, we obtain the bound F_J . The finitely many remaining numbers of averaging steps after the last RG step only enlarge the dependence neighborhood. On a periodic quotient these counts can only decrease. Since we use unions rather than hulls of components, all the bounds are subadditive, including for excluded regions that touch or wrap around the torus.

Step 7: Borel measurability. The outcome of the ordered tests is determined by its set of failing plaquettes: replaying the ordered list and adding the prescribed excluded sets at exactly those failures reproduces it. Hence there is at most one decomposition for each set of failing plaquettes that the procedure can produce. Each event $\mathcal{A}_\chi$ is Borel, since it is a finite Boolean combination of tests of Borel iterates and of membership in deterministic dependence lists. These events are disjoint and exhaust the support. The small representatives were used only after these events had been specified.

Step 8: the first step under the Wilson measure. For the Wilson measure, we perform the same tests, including the plaquettes of the input configuration, without assuming

a small input representative. Let F be the final excluded dependence set. On the fixed input cell grid of scale S , aligned with the blocks, we form the initial good-cell complex P_{in} , consisting of all closed cells whose full dependence sets avoid F . The blocks of the block region are chosen only after moving inward by one full layer of cells and by the full dependence neighborhood of the constraint, tree, basing, and plaquette path products. This choice implies that every input plaquette whose Wilson term depends on an integration variable, and every path product of the block region, is contained in P_{in} . For example, a point of a retained block together with its dependence neighborhood still lies in a retained cell or in its incident layer, and that layer was retained before moving inward. The near links are the links outside the block region read by these path products and plaquettes, as in (18.32), and their values are the near values. All remaining input links are deep links; their values are the deep frozen group variables, that is, the frozen group variables in the fixed complement that are not near values.

An input plaquette of P_{in} , including a boundary plaquette, avoids the final set F ; by the argument of Step 3, it is good. The absolute part of Theorem 21.3 applies with $n_{\text{input}} \geq 8S$ and $\delta S^2 \leq 1$, and we obtain the bound $C(\delta S + S^{-1}) \leq C'/S$. Fixing the mesh factors when A_8 is chosen makes this smaller than the required fixed multiple of r . At this first input there are no earlier fixed vertices, and we extend the resulting gauge transformation arbitrarily to the other vertices. All transformed deep links are kept as arbitrary elements of $\text{SU}(2)$ in the frozen data. The finite path-product estimate and the relative construction at the output of Step 4 give the claimed bounds for the new boundary layers and based transports. The near values are small in this representative, not in the original configuration. Every additional excluded set lies within a neighborhood of radius $C_J S$ of an excluded neighborhood, and we assign it to the same failing plaquette. Since $S \leq C_J R_{\text{cell}}$, the radius and label counts keep their displayed forms after one fixed enlargement. After this first step, the induction of Step 4 applies.

All analytically frozen adjacent variables of this first step lie in the complex P_{in} , and hence obey the same conversion and (21.8). A dependence set frozen because its averaged plaquette fails can contain arbitrary elements of $\text{SU}(2)$. Such deep frozen group variables are fixed parameters, not variables of the analytic freezing, and their plaquettes involving only frozen variables remain in the external Wilson factor. For a stopped measure, the corresponding initial variables lie in its gauge-invariant small-field domain, by Corollary 21.2.

Step 9: stage measures. By Lemma 21.6, every gauge transformation used in this construction fixes the earlier frozen coordinates and their logarithms; this is the form in which the representatives of part (2) are used later. All stage maps are Borel, since they are compositions of restrictions, the block averaging map, and finite transport products; their pushforwards are the measures in part (5). Each frozen coordinate is recorded once, as an independent link or based transport, and carries all its dependence through

the density; no statement about independence under the probability measure is needed. Since the events $\mathcal{A}_\chi$ partition the support of ν , multiplying their pointwise sum by $\mathbf{1}_S$ gives the identity in part (5), and no restriction is added to the Wilson measure. All geometric arguments were finite local constructions; no step involves L , γ , d_0 , the depth, the number of steps since freezing, or the number of components. A final group of fewer than J averaging steps is stopped before its minimal-side condition fails, as stated. $\square$

The next lemma verifies, on every large-field decomposition, the hypotheses of Theorems 19.9 and 20.14 that concern the incoming radius, the boundary distances, the conditional forces, and the cutoff domains. We use the notation of Lemma 19.3. For each initial segment I of the integrated intermediate scales, $y = z_{I^c}$ denotes the remaining free coordinates, $\mathcal{T}_I$ their domain, $a_I(c, y; \xi)$ the unperturbed conditional minimizer, and A_I the unperturbed conditional Hessian at a_I . Further, r_I is the radius of the conditional coordinate neighborhood, r_q the bound on the scaled fluctuations $g_i w_i$, $\varepsilon_{\text{tube}}$ the fixed margin, and $C_{\text{seq}, J}$ the constant of the sequential construction. As in Theorem 19.9, Φ_0 is the reconstruction in the target-dependent gauge and $\bar{P}$ is the classical perturbation adjusted by the boundary-completion difference. Here $\Phi_0(c, u; \xi)$ has the joint normalized argument u used in that theorem: if z_0 is the unperturbed joint stationary point, K_0 the unperturbed Hessian there, and $S_0 = K_0^{-1/2}$, then $z = z_0 + S_0 u$. In the lemma below, the perturbing function in the original free coordinates is

$$V(c, z; b) = \bar{P}(\Phi_0(c, S_0^{-1}(z - z_0); \xi); b). \quad (21.12)$$

Thus V_I and V_{II} below are derivatives in the original free coordinates z , at fixed c and real b , and the constants C_P and C'_P include the fixed coefficient bounds of this affine change of coordinates. For a function f of the labels D and frozen values $c^\sharp$ on $F \subset D$, we write $T_{c^\sharp} f$ for the function of the labels $D \setminus F$ obtained by inserting these values, and $\mathcal{X}_D^\phi$ for the coefficient norm with boundary decay of Definition 18.1 with spatially varying radius ϕ .

Lemma 21.7. *Assume the conditions on frozen coordinates and the construction conditions of Theorem 21.5.*

1. *Freezing at the boundary of a complete RG step removes every label of reduced analytic radius and leaves the incoming perturbation on the remaining labels with a uniform analytic input radius, separately for the classical and the quantum radius. Its incoming and current boundary distances satisfy (21.10).*
2. *Fix real $b = (\xi, b_{\text{rec}})$, and suppose that the adjusted classical perturbation $\bar{P}$ has uniform norm at most $2\sigma_*$. Then the preconditioned conditional force δ_V , taken over every initial segment I and every $y \in \mathcal{T}_I$ and defined by the first expression below with V as in (21.12), satisfies*

$$\delta_V = \max_I \sup_{y \in \mathcal{T}_I} \|A_I^{-1} V_I(c, a_I(c, y; \xi), y; b)\|_{\text{coeff}} \leq I_0 C_P \sigma_* \rho_{\text{cl}}, \quad (21.13)$$

where I_0 and C_P are fixed bounds on the larger domains, independent of the later widths.

3. The enclosing product domain and every conditional coordinate neighborhood lie in the domains of Lemma 19.3 when the construction conditions include

$$\delta_V < \frac{1}{4} \min_I r_I, \quad R_{\text{en}} < r_q, \quad C_{\text{seq},J}(2I_0 C_P \sigma_* \rho_{\text{cl}} + r_q) < \frac{1}{2} \varepsilon_{\text{tube}}, \quad (21.14)$$

together with the fixed Hessian conditions and the two reconstruction-image conditions (21.17).

4. The fluctuation cutoff domain with common radius $p(t) = A_w \log(1/t)$ is one of the cutoff domains allowed in Theorem 20.14. Its containment and expansion conditions hold below a single coupling ceiling, chosen after these fixed positive radii.

None of these statements requires a derivative in ξ or in b_{rec} .

Proof. Steps 1 and 2 prove part (1) and the bound on the new boundary-completion differences, Step 3 proves part (2), Steps 4 and 5 verify the image and convexity conditions behind part (3), and Step 6 proves part (4).

Step 1: uniform radius on the remaining labels. The uniform radii are used in the freezing part of Theorem 20.14. We write D for the old output list, I for the remaining labels, and $F = D \setminus I$. All labels of reduced radius belong to F , and $d_D(i) \geq s_{\text{er}}$ for $i \in I$. If there is no old boundary, the boundary-weighted part of the incoming norm is zero, and the estimate below is immediate. Otherwise, since the full neighborhood of width $s_{\text{er}} - 1$ of the old boundary was removed, a shortest route from i to the old boundary meets F at least $s_{\text{er}} - 1$ steps earlier, and additional excluded regions can only shorten the distance to F . Hence

$$d_D(i) \geq d_I(i) + s_{\text{er}} - 1.$$

For a monomial of degree $k \geq 1$ in the remaining labels and degree $j \geq 0$ in the frozen labels, inserting the frozen values multiplies its absolute coefficient contribution by at most η^j with $\eta \leq 1/4$, by (21.8). The new fraction v_i/k contributes at most a factor $(k+j)/k \leq 1+j$, and the remaining connecting network is no longer than the original one. Since every remaining label had the full radius $\rho_\bullet$, the resulting series is estimated at the uniform radius $\rho_\bullet$ on I , not at a reduced boundary radius. Summing $\sum_{j \geq 0} (1+j)\eta^j = (1-\eta)^{-2} < 2$ and using the distance inequality, we obtain, including degree one,

$$\mathcal{X}_I^{\rho_\bullet}(T_{c^\#} f) \leq 2e^{-\alpha_\bullet(s_{\text{er}}-1)} \mathcal{X}_D^{\phi_\bullet}(f). \quad (21.15)$$

Replacing the radius and rates by the classical radius and rates in this monomial calculation, we obtain the classical statement, with the same factor $(1-\eta)^{-2}$.

Step 2: new boundary-completion differences. The bound $|c_j^\#| \leq \phi_\bullet(j)/4$ controls the inherited boundary term. Freezing also inserts the frozen values into the torus functions, and this produces a new boundary-completion difference. For it we use the bound $|c_j^\#| \leq \beta_{\text{rec}}$ from (21.8). Every contributing monomial has frozen degree $j \geq 1$ and degree $k \geq 1$ in the remaining labels. For a fixed remaining index i , the graph connecting network joins i to a frozen label, so $d_I(i) \leq t(v)$ and $t(v_I) \leq t(v)$. Let t_2 denote the Euclidean network length in the torus coefficient norm, and let $\bar{a}_\bullet$ be its Euclidean rate. The rate choices of Theorem 20.9 give $\bar{a}_\bullet \geq \sqrt{3}(a_\bullet + \alpha_\bullet)$. Therefore

$$a_\bullet t(v_I) + \alpha_\bullet d_I(i) \leq (a_\bullet + \alpha_\bullet) t(v) \leq \sqrt{3}(a_\bullet + \alpha_\bullet) t_2(v) \leq \bar{a}_\bullet t_2(v).$$

With $\eta_\bullet = \beta_{\text{rec}}/\rho_\bullet \leq 1/4$, the ratio of the new and old fractions v_i/k is at most $1 + j$. For each fixed original degree we split its absolute coefficient sum according to the frozen degree j , which is uniquely determined. Taking the supremum over the remaining index before the sum over degrees, we lose at most the factor

$$\sum_{j \geq 1} (1 + j) \eta_\bullet^j = \frac{\eta_\bullet(2 - \eta_\bullet)}{(1 - \eta_\bullet)^2} \leq 4\eta_\bullet.$$

The normalization is $\rho_\bullet^{-2}$ in both norms, and the powers of $\rho_\bullet$ cancel against the factors of the frozen labels. This includes the new degree-one terms, and gives the bound $8E_\bullet \beta_{\text{rec}}/\rho_\bullet$ for one simultaneous freezing of the union of the frozen labels. The fixed constant $C_{\text{bc},\bullet}$ includes every required conversion and any finite sum of such terms. The total contribution therefore satisfies

$$z_t^\bullet \leq C_{\text{bc},\bullet} E_\bullet \beta_{\text{rec}}/\rho_\bullet \leq \sigma_*/4. \quad (21.16)$$

The boundary band, the new boundary layers, the based transports, and the initialization are all included in this single bound. For a new excluded region in the interior, we do not claim that the removed band alone makes this difference small. Equations (21.15) and (21.16) imply the uniform incoming norms in the recursion of Theorem 20.14. The frozen coordinates ξ are parameters of those norms and need not have the full incoming analytic radius.

Step 3: conditional forces. We use the absolute coefficient bounds for the derivatives of $\bar{P}$ in Theorem 19.9, on its uniform input image, which lies strictly inside the input ball. By the fixed construction hypotheses, every conditional coordinate neighborhood has the same image, together with its connecting segments. Composing with the row-sum bounds of the first derivative of Φ_0 and of the affine change of coordinates in (21.12), we find $\|V_I\|_{\text{coeff}} \leq C_P \sigma_* \rho_{\text{cl}}$, uniformly in $y \in \mathcal{T}_I$. Multiplying by the conditional inverse bound I_0 , we obtain (21.13). The second derivative also contains the term $\bar{P}' \Phi_{0,II}$, where $\Phi_{0,II}$ is the second z_I -derivative of $z \mapsto \Phi_0(c, S_0^{-1}(z - z_0); \xi)$; keeping it, we obtain

$\|V_{II}\|_{\text{coeff}} \leq C'_p \sigma_*$. The unperturbed neighborhoods are fixed first, so that their own Hessian variation uses at most half of the preconditioned margin. Reducing the fixed ceiling for σ_* makes $I_0 C'_p \sigma_*$ fit the remaining margin and preserves the real half-gap and the complex inverse margin. At an unperturbed conditional minimizer, we have $F_{0,I} = 0$, so the condition on inverse times force is precisely $\lambda \|A_I^{-1} V_I\| \leq \delta_V < r_I/4$ for $0 \leq \lambda \leq 1$. This verifies the hypothesis of the fixed-parameter contraction at every y , and its solution moves by at most $2\lambda \delta_V$. The uniform mixed coefficient bounds of the same theorem control its dependence on the higher parameters. The sequential conclusion of Lemma 19.3 then applies under (21.14) to all intermediate scales. In particular, a product with $|g_i w_{i,e}| \leq R_{\text{en}} < r_q$ stays inside every parameter and solution neighborhood with margin. This is a check of the conditional force at moving parameters, not only at the joint minimizer.

Step 4: the image conditions. Both inherited image conditions are checked with the bound $s_q = R_{\text{en}}$ for the scaled fluctuations. For each radius, the four zero-fluctuation terms of Lemma 19.7 satisfy

$$\begin{aligned} B^{-1} Z_{\bullet}(K_* \rho_{\bullet}) &\leq \rho_{\bullet}/16, & C_{\Delta, \bullet} C_{a, \bullet} R_{\partial, \bullet} &\leq \rho_{\bullet}/16, \\ C_{\Delta, \bullet} C_{a, \bullet} e^{-a_W w_{\text{bd}}/2} K_* \rho_{\bullet} &\leq \rho_{\bullet}/16, & C_{F, \bullet} \beta_{\text{rec}} &\leq \rho_{\bullet}/16. \end{aligned}$$

Their sum is $A_{0, \bullet}$. The initial good-cell complex at the first step under the Wilson measure and the endpoint protection rule change neither these analytic constants nor the bound on the current data. In the notation of the fixed construction constants of Lemma 19.7, the image conditions are

$$A_{0, \bullet} \leq \rho_{\bullet}/4, \quad 1 - C_{T, \bullet}(K_* \rho_{\bullet} + R_{\text{en}}) \geq \frac{1}{2}, \quad \frac{C_{T, \bullet} R_{\text{en}}}{1 - C_{T, \bullet}(K_* \rho_{\bullet} + R_{\text{en}})} + C_{\Delta, \bullet} R_{\text{en}} \leq \rho_{\bullet}/8. \quad (21.17)$$

Here $R_{\partial, \bullet} = K_* \rho_{\partial, \bullet}$. The margin in the larger construction domain of radius R_{raw} is, explicitly,

$$C_{\text{lift}, J}[K_* \max(\rho, \rho_{\text{cl}}) + r + \beta_{\text{rec}} + q_{\text{raw}}] < R_{\text{raw}}/8,$$

where q_{raw} is the fixed upper bound given by the row sums of the first derivative for the sequential product of scaled fluctuations and for all its initial segments; this bound is fixed before the eventual enclosing radius. Together with this margin, the image conditions give the image $3\rho_{\bullet}/8$ without spatial weights. The fixed interpolation of spatial rates in Theorem 20.14 then gives an absolute weighted image strictly below $\rho_{\bullet}/2$, so all evaluations and segments used for the force estimates above lie strictly in the uniform input ball, for both the classical and the quantum radius. These conditions on the parameters are already required by Theorem 21.5 and the analytic theorem, and they involve no test on the support of the measure.

Step 5: convexity on the enclosing product domain. The radius R_W of the scaled fluctuations is fixed independently of the logarithmic cutoff. We first compute its constants on the fixed larger perturbation domain. Then R_W is taken below the joint and sequential bounds, the bound of the larger construction domain, and the two inherited image bounds of the analytic theorem, as well as below $1/(4A)$ and $1/(16C_0)$, where A and C_0 are the constants of Lemma 20.3. The remaining deterministic choices satisfy

$$r, \beta_{\text{rec}}, \sigma_* \rho_{\text{cl}} \leq \frac{\lambda_* R_W}{6C_{\text{en},J}}, \quad R_{\text{en}} = \frac{2C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_* \rho_{\text{cl}})}{\lambda_*} \leq R_W. \quad (21.18)$$

Here $C_{\text{en},J}$ is evaluated on the fixed larger products, as in the proof of Theorem 21.9, before these radii are reduced. Then $AR_{\text{en}} \leq 1/4$ and $C_0 R_{\text{en}} \leq 1/16$. When $C_0 t^2 \mathcal{L}_V \leq 1/16$, the potential estimate of Theorem 20.14 yields

$$\|D_w^2(V - |w|^2/2)\|_{\text{row,col},a'} \leq \frac{C_0(R_{\text{en}} + t^2 \mathcal{L}_V)}{(1 - AR_{\text{en}})^3} \leq \frac{1/8}{(3/4)^3} = \frac{8}{27} < \frac{1}{2}.$$

This verifies the Hessian margin on the whole enclosing product domain, separately from any expansion on the smaller logarithmic cutoff domain.

Step 6: the logarithmic cutoff domain. Let $p(t) = A_w \log(1/t)$, $c_J^{\log} = A_w(J - 1)(\log M)/2$, and $p^+(t) = p(t) + c_J^{\log}$. The common choice $p_i = p(t)$ lies in $[p(t), p^+(t)]$. Since $g_i/t = M^{(i-s+1)/2}$, the choice $p(g_i)$ at each intermediate scale lies in that interval as well; only the common choice is used here. With the fixed ceiling $H = 1 + E + \sigma_*$ and the construction constants of the analytic theorem, the relevant conditions on the coupling can be checked in the following sufficient form:

$$\begin{aligned} tp^+(t) &< R_{\text{en}}, & C_J tp^+(t) &< \frac{1}{2}, \\ \frac{C_J t[(p^+(t))^3 + p^+(t) + Hp^+(t)]}{1 - C_J tp^+(t)} &\leq \epsilon_J, & 2t[(p^+(t))^3 + p^+(t) + Hp^+(t)] &\leq c_{z,J}, \\ C_0 t^2 \mathcal{L}_V &\leq \frac{1}{16}, & C_1 t(\mathcal{L}_V + 2) &< \min\{1/(4\sqrt{3}), p(t)/(2\sqrt{3})\}. \end{aligned} \quad (21.19)$$

The first line ensures that the chosen radii are enclosed, and the second checks the conditions on the interaction and on the auxiliary half-disc of Theorem 20.14. The last line checks the curvature and mean conditions of Lemma 20.5, in addition to the already fixed small radius R_W . Any needed provisional output remainder has the form $C_J t^2(E + \sigma_*)[(p^+)^4 + (E + \sigma_*)(p^+)^2]$ and fits its fixed bound by the same reduction. Indeed, with $u = \log(1/t)$, each left side required to be small is bounded by a finite sum of terms $e^{-ku}(A_w u + c_J^{\log})^l$ with $k > 0$, or by such a sum divided by a denominator at least $1/2$, and these tend to zero as $u \rightarrow \infty$. With the amplitude A_w fixed, we choose a finite U beyond all these thresholds and $g_* \leq e^{-U} < e^{-1}$. Every condition then holds uniformly

for $0 < t \leq g_*$, and $p(t) \geq 1$. The ceiling may depend on A_w ; it is independent of the side, the preceding depth, the number of steps, and the frozen coordinates. No expansion is applied on the larger radii R_{en}/t : that product is used only for the comparison of Lemma 21.12. $\square$

21.3 Coordinate coverage and representatives satisfying the reconstruction identities

We show that every supported point of every relevant fiber has small relative coordinates in a suitable gauge, and that its normalized fluctuation coordinates lie in the enclosing product domain. We also construct, from the coarsest scale downward, the representatives satisfying the reconstruction identities, at which the densities will be evaluated. The comparison of the corresponding integrals with integrals over the fluctuation cutoff domain is Lemma 21.12 in Section 21.6. That comparison requires an incoming density bound, which is proved there by induction. The coverage statement proved here uses no density bound.

For a group of averaging steps, the conditional minimizers, Hessians K_i , and reconstruction maps C_t are those of Theorems 19.9 and 20.14, with the gauge transformation of Lemma 19.5, which depends only on the target. For a supported fiber point, z^{act} denotes its free logarithmic coordinates and w^{act} its normalized fluctuation coordinates, obtained by inverting the reconstruction. The enclosing product domain is

$$\mathcal{K}_t = \prod_{i,e} \{|w_{i,e}| \leq R_{\text{en}}/t_{\text{max}}\},$$

with R_{en} as in (21.18).

Closed supports and retained transitions. Fix the finite lattice, range of scales, and decomposition $\mathcal{X}$. Before constructing densities, let $\eta_t^{\mathcal{X}}$ be the actual stage measure obtained from the initial measure restricted to $\mathcal{A}_{\mathcal{X}}$ by the integrations and freezings up to stage t . Its variables form the finite compact product X_t of current links and native frozen group coordinates, and put $K_t = \text{supp } \eta_t^{\mathcal{X}}$. If F_t is the retained output map before freezing, a supported fiber over y means exactly $K_t \cap F_t^{-1}(y)$; its free-coordinate set is the inverse image of this intersection under the local reconstruction. At a freezing boundary we use this pre-freeze convention and then the freezing bijection. This definition uses no version of a conditional measure.

The strict passing inequalities of the algorithm become $q(c_p) \leq d_*$ on these closed supports. Here is the local continuity justification. At each fixed finite stage, approach a support point by data actually produced by the restricted measure. The dependence exclusions ensure that every retained transition reads only the good-cell variables, the prescribed near data, and the protected native records; it never inserts exceptional deep

frozen values into a local analytic map. The gauge tuples exhibiting the geometric bounds form a finite product of compact copies of $SU(2)$, with closed endpoint compatibility conditions. A subsequence of such tuples converges. All group path products and native saving maps are continuous, so their bounds pass to the limit, and the plaquette bounds become weak inequalities. The resulting geometric bounds still place every relative logarithm and local constraint inverse strictly inside its fixed larger chart: the final radii were chosen below those margins by fixed factors. In this limiting gauge the retained transition is therefore the ordinary continuous analytic branch, on a neighborhood of the support point. Gauge equivariance identifies it with F_t in the original coordinates. This proves continuity of the retained transitions on the full K_t , by induction over the finitely many stages, starting with the continuous input plaquettes. It asserts no continuity of the globally Borel-extended averaging map on excluded data and chooses no continuous gauge map.

For completeness, this also propagates support through the actual restricted stage measures. If ν_t is the measure at a transition after its retained indicators, then $\text{supp } \nu_t \subset K_t$. Continuity on this compact set gives

$$\text{supp}(F_t)_\# \nu_t = F_t(\text{supp } \nu_t).$$

Indeed, the image is compact and carries the pushforward, while every neighborhood of an image point pulls back to a relative neighborhood of a support point of positive measure. Freezing and its inverse are continuous triangular group path-product maps, so the same assertion applies after freezing. Thus every supported output has supported predecessors, and the geometric and native collar bounds just proved apply to every supported fiber point, including the boundary. The ordered tests themselves retain their strict passing rule.

Lemma 21.8. *There is a fixed $\epsilon_{\text{rel},J} > 0$, depending only on the finite path and local inverse-chart margins fixed before the final radii, with the following property. Consider an inter-step output before its freezing map, with old records $b_{\text{rec}}^{\text{old}}$ and retained closed-cell complex P . Suppose that $q(c_p) \leq d_*$ for every plaquette of P , including its boundary, that the ordinary links of its prescribed collar Γ have their native creation-lattice bound $\rho_G(c_e) \leq C_{Br}$, and that the other analytic output-band and near data have the bounds of Theorem 21.5(2). Assume also $r + \beta_{\text{rec}} < \epsilon_{\text{rel},J}$. Then a gauge transformation K , prescribed to be I on Γ and extended by I outside P , puts the retained interior links below $r/2$ and all links of P below $\beta_{\text{rec}}/2$. It is I at every protected endpoint and basepoint of an old record, including variables already frozen at the input of the current group, preserves $b_{\text{rec}}^{\text{old}}$ and the zero-completed earlier coordinates exactly, and gives a small output datum for the relative-coordinate argument of Theorem 21.9(1). The non-collar links of the current output band may move under K ; at this pre-freeze comparison they belong to the output field, not to $b_{\text{rec}}^{\text{old}}$.*

These hypotheses hold at every datum obtained by inverse freezing from a history-produced representative at the next stage, including every supported fiber point over that datum. Indeed, the ordinary Γ links are recorded literally in the full frozen band, and the inverse freezing map restores them independently of the retained free links. No claim is made that an arbitrary small representative whose saved coordinates are merely below β_{rec} has this native collar property.

Proof. The hypothesis $q(c_p) \leq d_*$ and Step 8 of the proof of Theorem 21.3 give $\rho_G(c_p) \leq \delta = \pi\sqrt{2d_*}$ on every plaquette of P ; on stage supports this weak hypothesis is supplied by the preceding closed-support argument. In Theorem 21.3(2)–(3), take $k_\Gamma = I$, $\eta = C_{Br}$, and $R = \lceil A_8/r \rceil$. The already imposed conditions $\delta R^2 \leq 1$ and $K_B^2 r/A_8 \leq c$ give the relative extension. It is I on Γ and on the boundary vertices of P , so extension by I outside P is consistent. Its link bounds on P and in cells without a vertex of Γ are those stated; every retained cell meeting Γ belongs to the frozen band, so every retained interior link lies in a cell with no vertex in Γ . Analytic near values outside P keep their original bounds, while the finitely many path products crossing the collar have the same fixed L_J conversion as in Step 4 of the proof of Theorem 21.5. The path products and coordinate conversions have fixed length or fixed derivative-row bounds. Their values are at most $C_J(r + \beta_{\text{rec}})$ in the local chart, so choosing $\epsilon_{\text{rel},J}$ below the minimum of the finitely many positive inverse-chart margins divided by this fixed C_J places every coordinate used in the small relative argument inside its chart. This threshold is fixed before the final radii. The endpoint protection rule ensures that the lift of K is I at every mark of $b_{\text{rec}}^{\text{old}}$, including the input variables already frozen in this group. In particular, replacing an earlier frozen value by the identity in the regional completion commutes with K : both endpoints of that completed coordinate are fixed.

For the last assertion, at fixed retained output v the freezing map of Lemma 21.11 records ordinary band links directly. Only a crossing link is replaced by the based expression $U_v(\sigma_e)a_e$. The closed-cell collar lies in the full ordinary frozen band, whose incident retained cells all belong to the frozen band, so no collar link has an endpoint in the active region and its links use the former rule. Its recorded values and both endpoints were protected by Theorem 21.5(2). Consequently every inverse freezing over the same next-stage datum restores the identical collar values, regardless of the supported free coordinates. The comparison gauge K is applied only to this restored pre-freeze output; it does not overwrite the recorded values in the actual downward family. $\square$

Theorem 21.9. *There exist $0 < \epsilon_{\text{en},J} \leq \epsilon_{\text{rel},J}$ and $C_{\text{en},J} < \infty$, with the dependence stated at the beginning of this section and independent of all later widths of boundary bands, removed bands, protected neighborhoods, and cells, such that the following holds for every large-field decomposition of Theorem 21.5. Assume its construction conditions, the two reconstruction-image conditions (21.17), and $r + \beta_{\text{rec}} < \epsilon_{\text{en},J}$.*

1. Let $1 \leq s \leq J$. Let (c, b_{rec}) be the output field and frozen coordinates of a configuration in the support of the stage measure, given either by a small representative of Theorem 21.5(2) (at the first step for a stopped measure, of Corollary 21.2), by the small comparison datum of Lemma 21.8, or by a representative obtained from either such datum by an output gauge change acting on the old frozen coordinates as in Lemma 21.6(1)–(3). Assume that this representative and the associated fields before the target-dependent gauge transformation belong to the common larger construction domain of Theorem 20.14, and that its analytic frozen coordinates satisfy (21.8). Let U be a configuration in the support of the fiber over (c, b_{rec}) . Then there exists a correcting gauge transformation, equal to I at the roots of the block region, at the vertices of the frozen boundary layer on the side of the fixed complement (in particular at the endpoints v_e of the links across the boundary of the block region), and at all protected earlier endpoints and basepoints, with the following property. After the hierarchical tree gauge fixing, every relative constraint vector X_i of U , including its eliminated entries, and its free part z_i satisfy

$$\max_{0 \leq i < s} (\|X_i\|_\infty + \|z_i\|_\infty) \leq C_J(r + \beta_{\text{rec}}).$$

They lie in the common domain of the local inverse of the block averaging constraint, with a fixed positive margin. The correcting gauge transformation is the composition of a smallness gauge and the lift, constant on blocks, of the inverse of its root values; its factors are not separately required to fix the roots. The inner endpoint u_e of a current link across the boundary of the block region may move before the tree gauge fixing, while the based transport is preserved, since its endpoints are fixed.

2. Assume at every integrated stage the adjusted classical perturbation bound $\mathcal{X}^{\text{cl}}(\bar{P}) \leq 2\sigma_*$ and the uniform conditional-force, Hessian, and sequential margins of Theorems 19.9 and 20.14; the initial norm bounds and coupling ceiling of the latter theorem supply these bounds along its analytic recursion. Choose all history-produced small top representatives in the coordinate domains of part (1), retaining the native collar bounds of Theorem 21.5(2). Define the reconstruction maps on their full spatially varying radii first, and descend through them, aligning the target at each step by the accumulated output gauge change. The unperturbed coordinates, conditional minimizers, and inverse square roots of the Hessians transform by blockwise adjoint orthogonal transformations, and the perturbing force is estimated separately. This constructs representatives satisfying the reconstruction identities at all stages, with

$$(T_M^s C_t(c, \zeta; b_{\text{rec}}))|_{\Omega'_t} = c,$$

and for every point of every fiber over every resulting output datum, its normalized fluctuation coordinates satisfy

$$\max_{i,e} g_i |w_{i,e}^{\text{act}}| \leq \delta_{\text{en}} = C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_* \rho_{\text{cl}}).$$

With $R_{\text{en}} = 2\delta_{\text{en}}/\lambda_*$, each such point lies in the interior $|w_{i,e}^{\text{act}}| \leq R_{\text{en}}/(2t_{\text{max}})$ of the enclosing product domain $\mathcal{K}_t$. Each gauge transformation of Lemma 19.5 used here is I at all protected vertices, including the current endpoints of the links across the boundary of the block region. No estimate on the logarithm of the accumulated gauge transformation is used. The statement concerns every supported fiber point, including points near the boundary of the coordinate domains, not one selected point or almost every point under an auxiliary measure.

3. At the first RG step for a stopped measure, every nonempty fiber over an output field and all input frozen coordinates that meets D_r has a representative in which all output links and input frozen coordinates are at most $C_{\text{ad},J}r$. The scaled fluctuations of the unperturbed objective, $K_0^{1/2}(z^{\text{act}} - z_0)$, are bounded by $C_J(r + C_{\text{ad},J}r)$.
4. At the first RG step under the Wilson measure, let b^{near} be the near values, that is, the values on the near links of (18.32), which are the links outside the block region read by a plaquette depending on an integration variable or by a path product of the block region in the initial good-cell complex. Let η_{deep} be the deep frozen group variables, the values on all remaining links. Then we have the exact identity

$$S_W(U) = F_{\text{bare}}(U_{\text{int}}; b^{\text{near}}) + E_{\text{hole}}(b^{\text{near}}, \eta_{\text{deep}}),$$

where U_{int} denotes the integration variables and F_{bare} is the constrained Wilson-action objective of the first block integration, defined in Definition 18.13 and written in these variables; in the coordinates of Definition 18.6 this identity is (18.38). No smallness is imposed on η_{deep} ; its factor $e^{-E_{\text{hole}}/\mathfrak{g}_{\text{base}}^2}$ remains with its original Haar integration and the original partition function. Every conditional minimizer, derivative, square root, and Haar disintegration factor of the integration variables is independent of η_{deep} . A plaquette containing only frozen links, near or deep, belongs to E_{hole} . At this step there is no inherited effective interaction, and the scaled fluctuations are bounded by $C_J(r + \beta_{\text{rec}})$. The deep frozen group variables and exceptional values of the Borel extension in them are never logarithmized or inserted into a torus analytic action.

The constants have the dependence stated at the beginning of this section, and they are uniform in the deep frozen group variables.

Proof. We first work at a small output representative and an arbitrary supported fiber point over it (Steps 1–3). Step 4 transports the estimate to the representatives produced from above, using only the unperturbed objective, and Step 5 estimates the perturbation. Step 6 constructs the representatives satisfying the reconstruction identities, Step 7 converts the bound to the enclosing product domain, and Steps 8 and 9 treat the first step for the two initial measures. For $s < J$, the image hypothesis includes the separately imposed interior image condition of that group of steps. Only its conditional local chart and force estimates are used here; for fewer than J steps there is no restoration of norms and no recursion of normalization constants.

Step 1: a smallness gauge for an arbitrary fiber point. We work at a small output datum (c, b_{rec}) supplied either by Theorem 21.5 or, for a descended inter-step datum, by Lemma 21.8, and keep this notation until Step 4. Its retained interior output links are bounded by $r/2$, while current output-band links and newly used near and based data are bounded by β_{rec} ; its analytic old frozen coordinates obey (21.8). The root-increment argument below uses only $\rho_G(c_{aa'}) \leq r + \beta_{\text{rec}}$ for an internal final link, whether that link is retained or in the current output band. At the relative-admission comparison of Lemma 21.8, the input point is separately put in a smallness gauge as below; the collar estimate alone is not used as an input smallness assumption. An arbitrary analytic background on a larger spatially varying radius need not satisfy the absolute bound on c used in this step. Fix any configuration U in the support of the fiber over this representative. By Theorem 21.5, there exists a smallness gauge S for the input links of the block region. We restrict S to the input vertices of the block region and set it equal to I elsewhere; internal links are unchanged by this restriction. By the endpoint protection rule, its lifts fix every earlier endpoint and basepoint. A current prescribed based transport is not a protected earlier record. It may move under this intermediate operation; the combined correction below fixes its endpoints and restores its prescribed value. We write $V = U^S$ on the block region and let u_{in} be its bound for interior links. At the initial input for a stopped measure we use $u_{\text{in}} = r$; at later inputs $u_{\text{in}} \leq r/2$; at the first input under the Wilson measure $u_{\text{in}} \leq r$, after the fixed choice of A_8 . By Lemma 21.1, for internal dependence neighborhoods reading only interior links, we have

$$\max_{e \text{ internal}} \rho_G((T_M^i V)_e) \leq (10M)^i u_{\text{in}}, \quad 0 \leq i \leq s. \quad (21.20)$$

These dependence sets lie inside full blocks of the region. The display is not used for a path product reading a prescribed near value; such path products are bounded after the alignment below, using their frozen data as well as the interior bound.

Step 2: the root values. For each final root a of the region, let $h_a = S(a)^{-1}$. On an internal final link (a, a') , by the gauge covariance of the block averaging constraint, we have

$$c_{aa'} = h_a (T_M^s V)_{aa'} h_{a'}^{-1}.$$

Multiplying this identity by the appropriate inverse factor and using bi-invariance, we obtain

$$\rho_G(h_a h_{a'}^{-1}) \leq \rho_G(c_{aa'}) + \rho_G((T_M^s V)_{aa'}) \leq r + \beta_{\text{rec}} + (10M)^s u_{\text{in}}. \quad (21.21)$$

We lift h as a constant on each final block of the region, with value I off these blocks, and denote this lift by L (a gauge transformation, not the side of the torus). The total correction is $H = LS$, since $(U^S)^L = U^{LS}$, and it equals I at every final root. At the vertices of the frozen boundary layer on the side of the fixed complement both factors are I , and the endpoint protection rule fixes the earlier frozen endpoints after lifting to

their own lattices. An intermediate internal link either stays in one final block or crosses a single boundary between adjacent final blocks. In the first case L conjugates its value; in the second case its angle increases by at most the root increment in (21.21). Together with (21.20), this bounds all internal iterates by $C_J(r + \beta_{\text{rec}})$. A component consisting of a single block has only the first case, and its root value causes no additional estimate.

The bound $\rho_G(c_{aa'}) \leq r + \beta_{\text{rec}}$ used in (21.21) belongs both to a forward small representative and to the pre-freeze small comparison datum supplied by Lemma 21.8: its retained interior links are below $r/2$, its current analytic output band is below β_{rec} , and any other internal link read here obeys the same sum bound. At the first step for a stopped measure, the representative of Corollary 21.2 has output bound $C_{\text{ad},J}r < \beta_{\text{rec}}$, by (21.6), so the displayed bound holds with $u_{\text{in}} = r$.

Step 3: based transports and the branch. The endpoints of a current based transport are a root of the region and the endpoint of its link in the fixed complement, and the factors of H at both are I . The transport is therefore unchanged, although the endpoint u_e in the region can have moved. We now apply the hierarchical based tree gauge fixing of Lemma 21.1. Since its paths have length at most L_J , all internal links after the tree gauge fixing satisfy the same bound with a larger C_J . A link across the boundary of the block region that is read by a path product of the region is the inverse of such a bounded path times its prescribed based transport. It is therefore bounded by $C_J(r + \beta_{\text{rec}})$ as well. Every path product of the region mixing integration variables and near values has at most L_J factors of these links and of its prescribed frozen data; its bound is $C_J(r + \beta_{\text{rec}})$, including the contribution of the frozen data. We include the initial $C_{\text{ad},J}$ and the constants for these mixed path products in this fixed C_J before choosing $\epsilon_{\text{en},J}$ or the final radii. At the small representative, the relative vector X_i and its free part z_i then satisfy the estimate in part (1). We choose $\epsilon_{\text{en},J}$ so that this bound is less than half the common uniqueness radius and all the fixed construction margins. Since the configuration solves the block averaging constraint, its eliminated entries equal the unique eliminated entries on this branch. This reasoning applies to the arbitrary point U itself, not to another point of its fiber, and neither S nor L has to fix the roots separately.

Step 4: transport to representatives produced from above. We carry the estimate to an output datum produced from above, using only the unperturbed objective with torus input. Let k be the accumulated output gauge change. Its lift, constant on blocks, preserves the tree identities, and the coarse-field lift of the target satisfies $U_0(c^k) = U_0(c)^{\tilde{k}}$. Each relative factor therefore transforms as

$$e^{X_e} \longmapsto e^{\text{Ad}_{k_a} X_e}, \quad z_i \longmapsto A_{k,i} z_i, \quad (21.22)$$

where a is the root of its block and $A_{k,i}$ is block diagonal and orthogonal on the real color vectors. The equation for the eliminated edges transforms in the same way, as we see by

conjugating the constraint and using the uniqueness of its small inverse. We repeat this over the $s \leq J$ intermediate scales. Since the changes at earlier marks are the identity, the prescribed frozen data are preserved. For a comparison of two representatives that are not related by a based change, the same formula uses their correspondingly transformed output data.

For a descended inter-step datum, we first use Lemma 21.8 on the *pre-freeze output* restored from its next-stage records. This produces a small datum by a relative gauge K equal to I at every earlier frozen mark. The inverse change $k = K^{-1}$ carries that small datum back to the given one; its lift is endpoint preserving, so the old records and every coordinate completed by zero agree in both comparisons. The current output band is still part of c at this step. Its non-collar links may change under K , but they are not old records in b_{rec} in the local comparison. Only after the comparison is the actual output frozen by the raw freezing map, which records its actual values. The ordered tests and the initial measure are gauge invariant, so lifting K takes each supported fiber point to a supported point over the comparison datum with the same old records. We apply this change separately to each supported fiber point. Formula (21.22) transports the relative bound and the eliminated-edge inverse back to the given output datum, even if its output links are bounded only by a larger analytic radius; no absolute bound $r/2$ on its iterates is asserted. For part (1), an output datum already related to a small representative by a change of Lemma 21.6(1)–(3) uses that stated change. All old frozen coordinates in the descended comparison satisfy (21.8), and no arbitrary endpoint change of part (4) is used for this analytic estimate.

All fields in this calculation, taken before the target-dependent gauge transformation, lie in the common larger construction domain of Theorem 20.14. In particular, the backgrounds at block boundaries, which need not be of order $M^{-s}\rho_{\text{cl}}$, lie in that domain. The torus classical values used there agree under a compact periodic gauge transformation whenever both representatives lie in the common overlap chart, by Theorem 11.6, as recorded in (14.3). Their membership in that common domain follows from Lemma 19.7(1) and the construction-domain hypothesis of part (1). In a regional completion, the prescribed small frozen data transform at the endpoints of their paths. The earlier frozen coordinates that are completed by zero have both endpoints fixed by the endpoint protection rule, so appending them commutes with this change. For the unperturbed objective F_0 and the product color map A_k ,

$$F_0(c^k, A_k z; b_{\text{rec}}^k) = F_0(c, z; b_{\text{rec}}). \quad (21.23)$$

This identity is compact overlap applied to the torus classical restriction after regional completion. The endpoint-preserving change commutes with zero completion of the old coordinates, and the local constraint inverse transports the remaining coordinates by (21.22). No covariance of the regional perturbation is asserted.

By stationarity in z and the uniqueness of the unperturbed conditional minimizer, we have

$$z_0(c^k; b_{\text{rec}}^k) = A_k z_0(c; b_{\text{rec}}), \quad K_0(c^k; b_{\text{rec}}^k) = A_k K_0(c; b_{\text{rec}}) A_k^{-1}. \quad (21.24)$$

The minimizer stays in the common uniqueness box; if boxes in scalar coordinates are used, the fixed comparison of color norms is included in the margin of the larger construction domain. An orthogonal conjugate of a positive square root is positive and squares to the conjugated matrix, so uniqueness of positive square roots gives the same identity for $K_0^{1/2}$ and $K_0^{-1/2}$. Applying this argument after each conditional minimization, we obtain it for every initial segment. At a small output representative, Theorem 18.12(4) gives

$$\|z_0\|_\infty \leq C_J(\|c\|_\infty + \|\xi\|_\infty) \leq C_J(r + \beta_{\text{rec}}).$$

Equations (21.22) and (21.24) carry the bound

$$\|K_0^{1/2}(z^{\text{act}} - z_0)\|_\infty \leq C_J(r + \beta_{\text{rec}}) \quad (21.25)$$

to every output datum described in part (1). No bound on $\text{Log } k$ is needed, since blockwise adjoint transformations have norm one for every compact value at the roots.

Step 5: the perturbation. We treat the perturbation separately, at fixed real $b = (\xi, b_{\text{rec}})$. The uniform incoming radius, the boundary-distance comparison, and the conditional force estimates are Lemma 21.7; the analytic norm bounds are those of the recursion of Theorem 20.14, constructed without any density inequality. In the notation of Theorem 19.9, the preconditioned force is at most $C_J \sigma_* \rho_{\text{cl}}$, the change of the constrained Hessian is at most $C_J \sigma_*$, and the rows of all square roots for initial segments are bounded by C_J . That theorem estimates the perturbation on the reconstructed field in the target-dependent gauge, including the full boundary-completion difference; it asserts no covariance of the perturbation, and none is needed. We first place the higher coordinates of the arbitrary supported point in the domains where these conditional objects are defined. Put $b_{\text{en}} = r + \beta_{\text{rec}}$. The inverse-square-root row bound of Step 4 and (21.25) give, for the unperturbed *joint* center z_0 ,

$$\|z^{\text{act}} - z_0\|_\infty \leq G_J b_{\text{en}}.$$

Here G_J includes the fixed conversions between color-vector and scalar-coordinate sup norms; it is bounded on the larger construction products before the final radii are chosen. For each initial segment I , put $v_I = y_I^{\text{act}} - z_{0,I^c}$, where $y_I^{\text{act}} = z_{I^c}^{\text{act}}$. At the fixed real coarse field and frozen datum, regard v_I as a constant vector in the shifted parameter chart $y = z_{0,I^c} + v$ of Lemma 19.3(1). Its output-fixed degree-zero coefficient norm is exactly $\|v_I\|_\infty$, with no volume factor, and is at most $G_J b_{\text{en}}$. Let $\eta_{\text{min}} > 0$ be the minimum of the finitely many fixed parameter margins defining $\mathcal{T}_I$, with those same color-norm conversions included. In the already adjustable choice of $\epsilon_{\text{en},I}$ in Step 3, require also

$G_J \epsilon_{\text{en},J} < \eta_{\min}/2$. Since $b_{\text{en}} < \epsilon_{\text{en},J}$, every y_I^{act} lies in $\mathcal{T}_I$ with a strict margin. This membership is established before evaluating any perturbed conditional minimizer or inverse fluctuation coordinate at that argument.

Let $a_i(y_i)$ be the i th component of the unperturbed conditional minimizer a_{I_i} for the prefix $I_i = \{0, \dots, i\}$, with the coarse field and frozen datum suppressed. This is the *conditional* center at scale i , distinct from the joint center $z_{0,i}$. It satisfies $a_i(z_{0,>i}) = z_{0,i}$, and the derivative-row estimate of Lemma 19.3(1) on the fixed larger products gives a constant $D_{a,J}$ such that

$$\|z_i^{\text{act}} - a_i(y_i^{\text{act}})\|_{\infty} \leq (1 + D_{a,J})G_J b_{\text{en}}.$$

We decrease the same $\epsilon_{\text{en},J}$, if necessary, so that this bound is below one quarter of the minimum fixed conditional solution radius. Thus the actual coordinate lies in the common conditional chart. At the already admitted higher argument, (21.13) and (21.14) give

$$\|z_{P,i}^*(y_i^{\text{act}}; b_{\text{rec}}) - a_i(y_i^{\text{act}})\|_{\infty} \leq 2\delta_V \leq C_J \sigma_* \rho_{\text{cl}}.$$

The second point lies in the same conditional chart by the fixed force and solution margins. Only now do we use the exact inverse relation

$$g_i w_i^{\text{act}} = K_{P,i}(y_i^{\text{act}}; b_{\text{rec}})^{1/2} (z_i^{\text{act}} - z_{P,i}^*(y_i^{\text{act}}; b_{\text{rec}})).$$

If S_J bounds the rows of these perturbed positive square roots on the larger products, then, at every scale and for every supported point,

$$g_i \|w_i^{\text{act}}\|_{\infty} \leq S_J [(1 + D_{a,J})G_J b_{\text{en}} + 2\delta_V].$$

The constants $G_J, D_{a,J}, S_J$ and the two fixed positive margins precede the final cutoff and boundary widths. The at-most- J substitutions of higher arguments are already covered by the uniform conditional derivative rows used in $D_{a,J}$; no neighborhood is inferred from a fluctuation bound that has yet to be proved. Absorbing these fixed constants and the bound on δ_V yields

$$\max_{i,e} g_i |w_{i,e}^{\text{act}}| \leq C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_* \rho_{\text{cl}}). \quad (21.26)$$

The construction conditions keep these conditional neighborhoods inside the joint branch; they are not replaced by arbitrary slices of a joint box of free coordinates. No perturbed derivative in a frozen coordinate ξ or in an earlier frozen coordinate is taken; these coordinates enter only as fixed real parameters of the uniform coefficient estimates.

Step 6: representatives satisfying the reconstruction identities. The construction downward is a finite induction. We start with all history-produced small top representatives of Theorem 21.5, including both (21.8) and the native $C_B r$ collar records. At a boundary

where coordinates are frozen, the hierarchical tree transports are I by Lemma 21.1, and the gauge transformation of Lemma 19.5 is I at their endpoints. Undoing the basing therefore inserts the already chosen new frozen values directly. At the next lower group of averaging steps, we lift the accumulated output change, take the free logarithmic coordinates of any configuration in its fiber, and invert the reconstruction defined beforehand. By the branch argument of Steps 1–4 and (21.26), these inverse coordinates lie in its domain. The reconstruction has the same target and the same free coordinates, and uniqueness of the eliminated-edge inverse identifies the reconstructed configuration, before the gauge transformation, with the configuration aligned by the accumulated output gauge change. We then apply the gauge transformation of Lemma 19.5, which depends only on the target; by that lemma, the target is unchanged and every protected vertex is fixed. This produces the present representative and permits the next step of the descent. By Lemma 21.6(1)–(2), the subsequent lifts and the based correcting and tree gauges fix every frozen coordinate chosen at this step, so (21.8) and the native collar records persist throughout the family. At each inverse freezing, the ordinary collar links are restored literally from these records, independently of the newly chosen supported free coordinates, as in Lemma 21.8. Thus the relative-admission argument of Step 4 applies at every lower inter-step output, and all supported points over each fixed output datum are included. An arbitrary small top representative satisfying only (21.8) is not asserted to have this property. Each step uses only the representative above and the analytic maps defined beforehand on their full radii, so the induction assumes no completed family of representatives. Keeping all choices gives all fibers over all resulting output data and the identity for the target in part (2).

Step 7: the enclosing product domain. At output side m , we have $g_i = t_{\max} M^{(i-s+1)/2} \geq \lambda_* t_{\max}$. Dividing (21.26) by g_i , we obtain

$$|w_{i,e}^{\text{act}}| \leq \frac{\delta_{\text{en}}}{\lambda_* t_{\max}} = \frac{R_{\text{en}}}{2t_{\max}}.$$

Every point of $\mathcal{K}_t$ has $g_i |w_{i,e}| \leq R_{\text{en}}$. Both the classical and the quantum image conditions are imposed on the whole product $\mathcal{K}_t$, not only on the supported subset or on the logarithmic cutoff domain.

Step 8: the first step for a stopped measure. We fix one point of a nonempty fiber over an output field and all input frozen coordinates, and apply Corollary 21.2 to it. In the resulting representative, all output links and all based input transports are at most $C_{\text{ad},J}r$, and the remaining input frozen links are smaller than r . For any other point of the same fiber, we take its own gauge transformation into B_r , cancel its root values by the lift of the inverse root values, and cancel its gauge transformation at the vertices of the fixed complement. Equations (21.20) and (21.21) hold with $u_{\text{in}} = r$ and the output increment $C_{\text{ad},J}r + (10M)^s r$. All input variables used analytically, including all input

frozen coordinates, are small after the tree gauge fixing. The unperturbed estimate is $C_J(r + C_{\text{ad},J}r)$. This is the regional form of the coordinates of Theorem 14.3 at the first step; it is uniform in the preceding depth and is not applied to frozen coordinates created at later steps.

Step 9: the first step under the Wilson measure. We use the initial good-cell complex of Theorem 21.5(4), together with the internal coordinates of Definition 18.6 and the completion of Definition 18.13: the links across the boundary of the block region are completed by (18.33), the other links by (18.34), and $\mathcal{P}_{\text{var}}$ is a fixed list as in (18.35)–(18.36), namely the list of the split made in this step. The near links of (18.32) carry the near values b^{near} , and the deep links carry the deep frozen group variables η_{deep} . The hypothesis of Definition 18.13(3) that the near values lie in the identity chart is the conclusion of Theorem 21.5(4), since every near link lies in the initial good-cell complex. Indeed, the links of the path products in $\mathcal{E}_{\Omega}^{\text{dep}}$ lie there because of the inset by the full dependence neighborhood. The plaquettes incident to $\mathcal{L}_0$ lie within one input step of a block of the region, while Theorem 21.5(4) moves inward by a full layer of cells and the full dependence neighborhood. The conversion between the value on a link across the boundary and its based value is included in $C_{\text{conv},J}$.

In the hierarchical tree coordinates, we partition the plaquettes according to whether their Wilson term depends on an integration variable. By the definition of the near links, every term in the first class uses only integration variables and near values, and the sum of these terms is F_{bare} . Every remaining term is a function of frozen variables only and is assigned to E_{hole} , even if its plaquette contains both near and deep links. This partitions the sum $S_W = 4 \sum_p q(U_p)$, and Lemma 18.14(1) implies the exact split (18.38), which is the identity in part (4). In particular, every objective of the integration variables and all its derivatives are independent of η_{deep} , without any estimate of a small remainder. The Haar disintegration path products also avoid the deep links, because of the inset by the full dependence neighborhood. By (18.39), the objective agrees with the objective of Theorem 18.12 at preceding depth zero, up to a function constant in the integration variables. The external factor $e^{-E_{\text{hole}}/g_{\text{base}}^2}$ remains the one in (18.38), with its original Haar variables and partition function.

We apply Lemma 18.14(2)–(3) to the block region of the first step, with the prescribed near completion. Its embedding of the free coordinates is A_{Ω} in (18.15), and its extraction matrix is (18.16). Equations (18.40)–(18.42) apply to the list $\mathcal{P}_{\text{var}}$, including every case near the boundary. By (18.43), the flat joint Hessian in the free coordinates then satisfies

$$K_{\Omega} = \hat{A}_{\Omega}^T d_{\text{var}}^T d_{\text{var}} \hat{A}_{\Omega} \geq \lambda_{M,J} I,$$

It has uniformly bounded absolute rows and columns and the range (18.44), independently of the region and of the deep frozen group variables. By (18.39), this is also the case $d_0 = 0$, $\kappa_* = 1$ of Theorem 18.12. Every principal free matrix and every real Schur

complement of an initial segment keeps a fixed- J gap, by (18.47). No Hessian of a field outside the region is inverted.

All Wilson and constraint derivatives at this step are derivatives of finite local path products. Their absolute row sums are bounded on a fixed smaller identity chart, independently of the deep frozen group variables. We obtain the weighted inverses without a volume estimate. First let K be the flat joint matrix K_Ω , or one of its principal free submatrices, before the minimization over a lower scale. By Lemma 18.14(3), $\lambda_{M,J}I \leq K \leq Q_{M,J}I$, and K has range $D_{M,J}$. We set $a = 2Q_{M,J}$ and $T = I - K/a$. Then T has norm at most $q = 1 - \lambda_{M,J}/a < 1$, and the entries of T^k vanish beyond maximum distance $kD_{M,J}$, hence beyond graph distance $3kD_{M,J}$. By the bound on the density of labels in the proof of Lemma 18.14, each row of T^k has at most $C_J(k+1)^3$ nonzero entries, each of absolute value at most q^k . For $b > 0$ with $qe^{3bD_{M,J}} < 1$, the series (18.48) for K^{-1} and $K^{-1/2}$ have uniformly summable rows and columns, weighted in the graph distance; the second is the scalar binomial series applied to T , whose coefficients are at most one. Small complex perturbations are inverted by the Neumann series about these matrices. For the conditional matrices obtained after minimization over lower scales, we use (18.47) and the bounds for Schur complements in the proof of Lemma 18.14(3), which rest on the argument of Combes and Thomas [61]. These matrices and their inverse square roots have uniform exponential row and column bounds, although the Schur complements need not have finite range. The factors M^i are included in the corresponding matrices.

The equation for the conditional minimizers is a contraction on a fixed smaller box. We first prove this at the small representative used in Steps 1–3. There, its force at zero fluctuation is $C_J(r + \beta_{\text{rec}})$, and by the third-derivative bound in the proof of Lemma 18.14(4) its derivative changes by at most C_J times the radius of the box. At this representative, the coarse argument of (18.45) is $\text{Log } c$, and $\|\text{Log } c\|_\infty$ and $\|\xi\|_\infty$ are at most $C_J(r + \beta_{\text{rec}})$. In the choice of $\epsilon_{\text{en},J}$, we require in addition $C_J(r + \beta_{\text{rec}}) \leq \epsilon_{M,J}$, and we require the radius of the box to be at most $\epsilon_{M,J}$, so that (18.45) applies on the whole box, including the coarse field. We reduce the radius of the box so that the inverse times the change of the derivative is at most $1/2$; the solution then has size $C_J(r + \beta_{\text{rec}})$. The finitely many conditional products and their image margins are those required in Theorems 19.9 and 20.14, and their specialization at this step has no inherited effective interaction. Applying the inverse formula for the normalized fluctuations, we obtain the bound on the scaled fluctuations in part (4) at the small representative, without the perturbative term.

For an output representative transported from above, we use the output gauge change of Step 4, with the frozen values transformed consistently. The Wilson terms are traces of plaquette variables, so the exact split in part (4) gives for F_{bare} the same covariance identity as (21.23). Its conditional minimizers and Hessians therefore transform as in (21.24), and their square roots transform by the same orthogonal conjugations. Hence the minimizers at the transported representative are the images of those at the small

representative, and each normalized fluctuation $w_{i,e}$ is rotated by an orthogonal map, so the bound on the scaled fluctuations carries over. No bound of order $r + \beta_{\text{rec}}$ on the transported coarse links is needed.

All constants arose from local path products, fixed construction products, and their row-sum bounds. $\square$

21.4 Pointwise gauge-invariant densities

The densities of the stage measures will be evaluated at representatives satisfying the reconstruction identities, which form a set of Haar measure zero. We therefore construct an explicit Borel density and prove that it is gauge invariant at every point. The coverage hypothesis of Lemma 18.7 is supplied by Theorem 21.9(1), which uses no density bound.

Theorem 21.10. *Fix a finite lattice, a preceding depth that is a multiple of J , an initial measure (a stopped measure or the Wilson measure), and a large-field decomposition $\mathcal{X}$ of Theorem 21.5 with a range containing at least one integrated full RG step. At stage t , let X_t be the product of the link variables of the current block region and of all independent frozen links and based transports, each on the lattice where it was frozen. Let H_t be the product of the Haar probability measures on these factors, and let $\eta_t^{\mathcal{X}}$ be the stage measure, a subprobability measure. Let Γ_t be the group of consistent gauge assignments on the finite multiscale transport graph formed by these endpoints: repeated physical vertices are identified, and each recorded object transforms at the endpoints of its path. In particular, Γ_t contains arbitrary values at the output roots and their lifts, constant on the blocks of the region, together with the induced changes of earlier frozen coordinates used in Theorem 21.5. For a frozen link on the lattice of level ℓ and for a based transport, the action is*

$$b_{xy} \mapsto k_x^{[\ell]} b_{xy} (k_y^{[\ell]})^{-1}, \quad \widehat{b}_e \mapsto k_{\alpha_e}^{[\ell]} \widehat{b}_e (k_{v_e}^{[\ell]})^{-1}, \quad \widehat{b}_e = U(\sigma_e) b_e.$$

For each fixed assignment, this is a left and right multiplication in the independent group variables, and it preserves H_t .

There exists an explicitly defined nonnegative Borel density p_t such that

$$\eta_t^{\mathcal{X}} = p_t H_t, \quad p_t(x^k) = p_t(x) \quad \text{for every } x \in X_t \text{ and } k \in \Gamma_t.$$

On the gauge-invariant chart domain containing the measure, p_t is given by the parameter integrals of the Haar disintegration formulas, with every indicator of the decomposition and every protected external factor retained, patched by their exact equalities on overlaps of fixed gauge translates. On the invariant complement of this domain, which has zero mass, its value is zero. In particular, the values of p_t at representatives satisfying the reconstruction identities are given by these formulas pointwise.

More explicitly, if $Z_{c,b_{\text{rec}}}$ is the domain of the free coordinates of the fiber over an output datum (c, b_{rec}) satisfying the coordinate conditions of Theorem 21.9, then the outgoing density is

$$p_t^{\text{out}}(c, b_{\text{rec}}) = \int_{Z_{c,b_{\text{rec}}}} \chi_t(c, z, b_{\text{rec}}) p_t(U_{c,b_{\text{rec}}}(z), b_{\text{rec}}) J_{\text{CO}}(c, z; b_{\text{rec}}) dz,$$

where χ_t is the product of the indicators of the decomposition, the incoming support, and the outgoing pre-freeze support, and the normalized integrals over the tree variables are kept until the covariance under based gauges has been applied. The factor at scale i of the Haar disintegration density J_{CO} is exactly

$$c_{\text{Haar}}^{f_i} \frac{\prod_{e \text{ outside the trees}} j_H(\Gamma_{i,e})}{\prod_a \det D_{X_{e_a}^*} C_a(\Gamma_i)}, \quad j_H(X) = \text{sinc}^2(|X|/2),$$

where the product in the denominator runs over the internal constraints, f_i is the number of free group variables at scale i , and the numerator includes the Haar factors of the eliminated edges. The reference measure already contains the Haar measure of the output field; no further coarse Haar density, Gram determinant, or orbit-volume factor occurs. For the stage densities and the tests of the decomposition, the covariance under based tree gauges makes each tree Haar integral equal to one; this elimination is not valid for arbitrary noncovariant Borel densities. Basing at a fixed field on the block region is the triangular Haar translation $b_e \mapsto U(\sigma_e)b_e$, with Jacobian one.

These statements concern the stage measures and the block averaging map with its Borel extension; they assert no independent gauge covariance or overlap identity for the auxiliary regional functions. The torus classical restriction used in the analytic comparisons does use Theorem 11.6 on the common charts, as in (21.23). Deep frozen group variables, including exceptional values of the Borel extension, are allowed. All identities hold in each finite dimension and introduce no factor depending on the size, the number of steps, the shape, or the depth. The analytic chart bounds used in the construction have the dependence stated at the beginning of this section.

For a range with no complete RG step, use the original pointwise invariant initial density, with the retained decomposition and support indicators and zero extension off its support. This gives a Borel density for either initial measure directly; no mass-carrying small-chart domain is asserted. The theorem above includes stage zero of every positive-length decomposition, as well as its empty active regions and empty fluctuation products.

Proof. We construct the density and prove its pointwise invariance together. This avoids evaluating an unspecified Radon–Nikodym derivative on a set of representatives.

Step 1: the endpoint group. We identify repeated physical vertices in the finite list of endpoints. An endpoint assignment is an element of the resulting finite product of copies of $\text{SU}(2)$, and any additional consistency identifications give the corresponding closed

subgroup. Its action on an independent group variable is a left and a right multiplication by fixed matrices. Hence H_t is preserved by each fixed assignment, with no assumption of probabilistic independence of the frozen coordinates under η_t^x .

Step 2: the initial density. At the initial scale we use the pointwise density of the Wilson measure, or the gauge-invariant density of the stopped measure, multiplied by the indicators of the decomposition and of K_0 , and set it to zero off K_0 . At every later stage the support indicator is retained in the defining parameter integral and the version is zero off K_t . The initial density is invariant: the Wilson plaquette variables transform by conjugation, and the density of the stopped measure is the gauge-invariant density of Theorem 14.6. The indicator of the decomposition is invariant as well, since the decomposition is determined by traces of plaquette variables and by deterministic dependence lists. The block averaging map is equivariant also on the exceptional set of its Borel extension: conjugation fixes $-I$, and the replacement reference transport transforms at the same two endpoints. These are pointwise statements. The first freezing is a splitting into independent variables followed by the triangular basing, treated in Step 5.

Step 3: the outgoing density on a chart. Assume that the incoming density has been constructed. At an output datum satisfying the coordinate conditions, we use Lemma 18.7 with all indicators; its coverage hypothesis is part (1) of Theorem 21.9. We keep the normalized tree variables $\mathfrak{f}$ at first. If J_i denotes the factor at scale i , we define the local outgoing value to be

$$\int dH(\mathfrak{f}) \int \mathbf{1}_{Z_{c,b_{\text{rec}}}}(z, \mathfrak{f}) \chi_t(c, z, \mathfrak{f}, b_{\text{rec}}) p_t(U_{c,b_{\text{rec}}}(z, \mathfrak{f}), b_{\text{rec}}) \prod_{i=0}^{s-1} J_i dz. \quad (21.27)$$

An indicator already contained in p_t is not counted again; it may equally be written again, since indicators are idempotent. This convention keeps all tests of the decomposition, including those whose information has already been integrated. By the disintegration identity, testing (21.27) against any nonnegative Borel function of the output gives the pushforward measure on this chart.

Step 4: pointwise covariance. We prove that these integrals are equal under the full group of consistent endpoint assignments. A fixed output assignment has a lift, constant on blocks, through the centered partitions of the block region; at earlier marks we use its prescribed values on the lattices of the marks. There is no conflict, since a mark in the preimage of a block of the region would have excluded that block, and repeated physical values are identified by consistency. Additional changes of the interior frames can be absorbed in the tree variables. The block averaging map and the map recording the frozen coordinates are equivariant, so the lift is a bijection of the fiber over (c, b_{rec})

onto the fiber over (c^k, b_{rec}^k) . It preserves every test, including an equality in a threshold and every condition at earlier scales.

On the hierarchical tree section, the part of the lift that is constant on blocks acts as in (21.22). The adjoint action is orthogonal because it preserves $-2 \operatorname{tr}(XY)$. Its determinant is 1, since it is a continuous function on the connected group $\mathrm{SU}(2) \simeq S^3$ with values in $\{-1, 1\}$ and equals 1 at I . Hence the change of free coordinates preserves dz , and uniqueness of the eliminated-edge inverse transports every eliminated coordinate as well. The factors j_H are radial and unchanged. The derivative matrix of the eliminated edges is multiplied on its row and column frames by these adjoints of determinant one, so its determinant is unchanged, including its positive sign on the real branch. The constants $c_{\text{Haar}}^{f_i}$ are the same. If the lift changes interior tree frames, the tree variables change by left and right Haar translations, before any tree integral is removed. Using the inductive pointwise identity for p_t , we obtain

$$\begin{aligned} p_t^{\text{out}}(c^k, b_{\text{rec}}^k) &= \int_{Z_{c^k, b_{\text{rec}}^k}} \chi_t(c^k, z', b_{\text{rec}}^k) p_t(U_{c^k, b_{\text{rec}}^k}(z'), b_{\text{rec}}^k) J_{c^k, b_{\text{rec}}^k}(z') dz' \\ &= \int_{Z_{c, b_{\text{rec}}}} \chi_t(c, z, b_{\text{rec}}) p_t(U_{c, b_{\text{rec}}}(z), b_{\text{rec}}) J_{c, b_{\text{rec}}}(z) dz = p_t^{\text{out}}(c, b_{\text{rec}}). \end{aligned} \quad (21.28)$$

The normalized tree integrals are understood in both lines when they have not yet been removed. This is an identity of explicit parameter integrals at each output datum; no uniqueness almost everywhere is used.

Step 5: freezing. Freezing has the required covariance too. We orient a link across the boundary of the block region from its endpoint u_e in the region to its frozen endpoint v_e , and let α_e be the root of its hierarchical transport. Cancellation at the interior vertices gives

$$\begin{aligned} U_{v^k}(\sigma_e) a_e^k &= k_{\alpha_e} U_v(\sigma_e) k_{u_e}^{-1} k_{u_e} a_e k_{v_e}^{-1} \\ &= k_{\alpha_e} \widehat{b}_e k_{v_e}^{-1}. \end{aligned} \quad (21.29)$$

At fixed v , the map from the independent variables a_e to their based transports is a product of left Haar translations. Together with the unchanged v , this is a triangular measure-preserving bijection; its inverse multiplies by $U_v(\sigma_e)^{-1}$. Recording any other link as a frozen link is Fubini's splitting of product Haar measure. The endpoint protection rule makes these endpoint changes consistent at every later stage. Equations (21.28) and (21.29) prove the inductive covariance for arbitrary values at the output roots and for all induced changes of earlier frozen coordinates.

Step 6: Borel patching. Before inserting the tests, we take the open chart domains of small block-region variables and small near values, with the positive margins of their inverses; the deep frozen group variables need no logarithm. Their gauge saturation

$\mathcal{O}_t$ is open and invariant, and the proofs of Theorems 21.5 and 21.9 put all the mass in $\mathcal{O}_t$. A finite product of spheres is second countable and has a countable dense subset, for instance obtained by approximating quaternion coordinates by rational ones and normalizing; the finite endpoint subgroup has the same property after the consistency identifications. If a point is moved into an open chart by some fixed gauge assignment, the set of assignments doing so is open and meets this countable dense set. Hence countably many fixed gauge translates of the small charts cover $\mathcal{O}_t$, and their coordinate charts can be chosen countably as well.

On one such chart, the integrand of (21.27), extended by zero outside its domain in a fixed box of free and tree coordinates, is a nonnegative Borel function. Indeed, the Haar disintegration denominators and the reconstruction maps are continuous on their branches, and the tests, including values of the Borel extension at the deep frozen group variables, are Borel. By Tonelli's theorem, its integral with respect to the finite product of Lebesgue and Haar measures on the box is a Borel function of (c, b_{rec}) .

On an overlap, the fixed gauge assignments relating two charts give the change of variables in (21.28). Since both formulas include the same domain, they agree at every point of the overlap. We use the first chart in the countable cover containing a point to define p_t there, retaining the support mask $\mathbf{1}_{K_t}$, and set $p_t = 0$ on $X_t \setminus (\mathcal{O}_t \cap K_t)$. The raw outgoing version likewise retains the indicator of the support before freezing. This function is Borel, it is a density by the disintegration identities, and it does not depend on the chart used. To check invariance under an arbitrary $k \in \Gamma_t$, we take charts containing x and x^k and perform the same change of variables; this proves equality of the two values also when k is not in the countable dense set. Both $\mathcal{O}_t$ and K_t are invariant: invariance of the actual stage measures follows inductively from the equivariant restrictions and pushforwards, so their closed supports are invariant. Their complements have zero mass, so the stated zero extension is invariant there as well. Countability was used only to define a Borel function, not to sum densities or to introduce a multiplicity of charts.

Step 7: removal of the tree integrals. Finally, at fixed target and frozen coordinates, we apply a based tree gauge fixing. The pointwise covariance just proved makes the density and its tests independent of the tree variables in (21.27), so each Haar probability integral over tree variables equals one. Removing them, we obtain the formula in the statement. Its factors at the scales i are exactly those of Lemma 18.7, in particular the numerator with the eliminated edges and the single inverse determinant of the eliminated-edge derivative. The Haar measure of the output is already the reference measure at the next scale, so this change of variables has no room for a further coarse Haar density, a Gram determinant, or an orbit-volume factor. For an arbitrary noncovariant Borel input density, the tree average would remain in (21.27), and its removal uses the covariance of the stage density. The argument uses the Haar disintegration before the change to normalized fluctuation variables. The deep frozen group variables, including exceptional values of

the Borel extension, remain compact Borel data throughout. $\square$

21.5 Freezing and substitution of frozen coordinates

When integration stops on part of the region, the corresponding variables become frozen coordinates. The next lemma shows that, at the representatives of the next stage satisfying the reconstruction identities, the inverse of the freezing map inserts the frozen values directly into the old variables. This identifies the density after freezing with the density before freezing at the same configuration, also on sets of Haar measure zero, and gives the exact substitution rules for the analytic functions.

Lemma 21.11. *At every boundary of a large-field decomposition where coordinates are frozen, use the hierarchical tree transports σ_e of Lemma 21.1, with the tree conditions at coarser scales fixed first. Split the old output into the remaining link variables v , the variables a_e on the links across the boundary of the block region, and the other links to be frozen. The freezing map $\mathcal{A}_t$ replaces each such variable by its based transport*

$$\widehat{b}_e = U_v(\sigma_e)a_e$$

and records the other links as frozen links on that lattice. At fixed v this map preserves the product Haar measure and has Jacobian one.

Let $b^\sharp$ denote the new frozen group values and $\xi^\sharp = \text{Log } b^\sharp$ their logarithms. For every reconstructed configuration of the next stage in the family of representatives satisfying the reconstruction identities,

$$v = C_{t+1}(c', \xi'; b_{\text{rec}}, b^\sharp),$$

the hierarchical tree transport has value I : each of its tree factors is I , and the gauge transformation of Lemma 19.5 is I at its two endpoints. All protected earlier endpoints are fixed as well. Hence

$$a_e = U_v(\sigma_e)^{-1}\widehat{b}_e = \widehat{b}_e, \quad \mathcal{A}_t^{-1}(v, b_{\text{rec}}, b^\sharp) = (v \oplus b^\sharp, b_{\text{rec}}),$$

and the density versions of Theorem 21.10 satisfy the pointwise identity

$$p_{t+1}(v, b_{\text{rec}}, b^\sharp) = p_t^{\text{out}}(v \oplus b^\sharp, b_{\text{rec}}).$$

At this configuration, all omitted old variables are exactly their new frozen values, based transports or links, with the same weights and indicators. In analytic functions, $\xi^\sharp$ is inserted in the corresponding analytic arguments. In particular, the freezing of the effective interaction subtracts only the value $R^{\text{out}}(0 \oplus \xi^\sharp)$ and transfers it with a minus sign to Λ , and the freezing of the classical action is $E_{\text{new}}(v) = E^{\text{out}}(v \oplus \xi^\sharp)$. These are exact finite identities, uniform in the number of steps, the topology, the depth, and the coupling. They compare the same configuration and require no overlap identity between auxiliary functions at different gauges.

Proof. We fix the deterministic geometry of the freezing. The remaining variables and the independent variables on the links across the boundary are different coordinates of the old output product. At fixed v , left multiplication by each fixed matrix $U_v(\sigma_e)$ preserves the Haar probability measure of such a variable, and the inverse multiplication is defined for every value. Integrating first in these variables, we obtain the triangular statement with Jacobian one for every nonnegative Borel integrand; earlier frozen coordinates and the other frozen links do not change.

We verify the value of the transport at a reconstructed configuration. On the based section of one scale, the tree equations set each ordered tree factor to I , and the fixing gauges of the coarser scales, lifted as constants on the finer blocks, conjugate these factors and leave them equal to I . By induction over the scales, with the coarser conditions fixed first, every factor of the prescribed hierarchical transport is I on the hierarchical tree section. This is the finite algorithm of Lemma 21.1, and it uses no path through an eliminated edge. Hence the product has value I for every target and every free coordinate in the domain, not only at zero fluctuation.

The reconstructed configuration is obtained from that section by the gauge transformation of Lemma 19.5. This gauge transformation multiplies a transport from α_e to u_e by its values at exactly these two vertices, and both values are I by that lemma, with the full set of protected vertices supplied by Theorem 21.5. The transport remains I . On the lattices of earlier frozen coordinates, the endpoint protection rule fixes all their endpoints. This proves, at this reconstructed configuration,

$$U_v(\sigma_e)^{-1}\widehat{b}_e = \widehat{b}_e.$$

Inverting the freezing map therefore inserts the new based transports directly into the old variables on the links across the boundary, and every other omitted variable is its new frozen link value. At this configuration, the inverse map is $(v, b_{\text{rec}}, b^\sharp) \mapsto (v \oplus b^\sharp, b_{\text{rec}})$.

The general formula for the density after freezing is the composition of the old outgoing density with the inverse triangular map, since its Haar Jacobian is one. We use the versions of Theorem 21.10 on both sides; that construction uses the same formulas and their pointwise overlap identities. Composing with the inverse, we obtain the pointwise identity, even when the reconstructed configuration lies in a set of representatives of Haar measure zero. All weights and indicators of the decomposition are values of the same functions at the same old configuration.

For a variable that was analytic before this freezing, its new group value lies in the small chart of frozen coordinates, and we insert $\xi^\sharp = \text{Log } b^\sharp$ there. The deep frozen group variables at the first step under the Wilson measure are fixed parameters instead and are not part of this logarithmic substitution. For every old analytic function F we have the exact identity

$$F(v \oplus \xi^\sharp) = [F(v \oplus \xi^\sharp) - F(0 \oplus \xi^\sharp)] + F(0 \oplus \xi^\sharp). \quad (21.30)$$

Applying it to R^{out} , the factor $e^{\Lambda^{\text{out}} - R^{\text{out}}}$ becomes

$$e^{\Lambda^{\text{out}} - R^{\text{out}}(0 \oplus \xi^\sharp)} e^{-[R^{\text{out}}(v \oplus \xi^\sharp) - R^{\text{out}}(0 \oplus \xi^\sharp)]},$$

which proves the minus sign of the transfer to Λ . For the classical function we keep the whole value $E^{\text{out}}(v \oplus \xi^\sharp)$, including its constant. Centering the effective interaction changes no first derivative: the new first derivative in the remaining variables is the old one evaluated at $(0, \xi^\sharp)$, and it is kept. $\square$

21.6 Conditional integration and normalization on large-field decompositions

This section proves the density bounds used in Section 22. The induction has three stages. First, the analytic effective actions, effective interactions, normalization constants, and reconstruction maps are defined from fine to coarse scales, by the recursion of Theorem 20.14, for every value of the frozen coordinates and without reference to any measure. Second, representatives satisfying the reconstruction identities are chosen from coarse to fine scales, using these maps. Third, the stage densities are bounded from fine to coarse scales at exactly these representatives. The construction is not circular, since each stage uses only objects defined in the preceding ones. One step of the third stage is the following comparison, which needs an incoming density bound; the bound is supplied by the induction itself.

Lemma 21.12. *Assume the hypotheses of Theorem 21.9, together with the conditions (21.18) and (21.19), which give the hypotheses of Lemma 20.5 on the whole enclosing product domain. Fix a stage t and an output datum (c, b_{rec}) of the family of representatives satisfying the reconstruction identities in Theorem 21.9(2), with real frozen coordinates, and define the fluctuation cutoff domain*

$$\mathcal{W}_t = \prod_{i,e} \{|w_{i,e}| \leq p(t)\}, \quad t_t = \max_i g_{t,i}, \quad p(t) = A_w \log(1/t). \quad (21.31)$$

Let $V = V_{c,b_{\text{rec}}}$ be the conditional fluctuation potential of Theorem 20.14 at this datum, and let $S_{c,b_{\text{rec}}} \subset \mathcal{K}_t$ be the set of normalized fluctuation coordinates of the supported fiber points over (c, b_{rec}) . For an invariant Borel function $0 \leq \chi \leq 1$, let $G_{t,\chi}(c, b_{\text{rec}})$ be the outgoing density of Theorem 21.10 at (c, b_{rec}) with χ inserted in the integrand. Suppose that there exist a finite prefactor $P_t(c, b_{\text{rec}}) \geq 0$ and a Borel function χ_t on $S_{c,b_{\text{rec}}}$ with values in $[0, 1]$ such that

$$G_{t,\chi}(c, b_{\text{rec}}) \leq P_t(c, b_{\text{rec}}) \int_{S_{c,b_{\text{rec}}}} \chi_t(w) e^{-V(w)} dw. \quad (21.32)$$

Then $S_{c,b_{\text{rec}}}$ is a Borel set, $0 < \int_{\mathcal{K}_t} e^{-V} dw < \infty$, and

$$G_{t,\chi}(c, b_{\text{rec}}) \leq (1 - \delta_t)^{-1} \tilde{F}_t(c, b_{\text{rec}}), \quad \tilde{F}_t = P_t(c, b_{\text{rec}}) \int_{\mathcal{W}_t} e^{-V(w)} dw,$$

with $0 \leq \delta_t < 1$, $\delta_t \leq 6b_t e^{-p(t_t)^2/48}$, and $b_t \leq C_j m^3$, where b_t is the number of three-dimensional free blocks of fluctuation variables of the step. Factors depending only on the output datum multiply both sides. The indicators are removed only in this upper bound. The constants have the dependence stated at the beginning of this section, and they are uniform in the deep frozen group variables.

At the first RG step, for either initial measure, (21.32) holds with equality. In that case P_t consists of the inverse partition function of the initial measure, the external weights, the Haar disintegration constants, the powers of the couplings, the output classical factor, and the zero-fluctuation factors, and χ_t is the product of χ and the indicators composed with the inverse coordinates. At a later stage, (21.32) is obtained by inserting an incoming bound on the density at the representatives satisfying the reconstruction identities, with the incoming multiplicative loss included in the same prefactor P_t ; this incoming bound is proved in Theorem 21.13.

Proof. At an output datum, the set $S_{c,b_{\text{rec}}}$ is defined with all indicators still inserted, and before any upper comparison the Haar disintegration is the exact formula of Lemma 18.7. The radii of $\mathcal{W}_t$ belong to the logarithmic family of Theorem 20.14 and satisfy $t_t p(t_t) < R_{\text{en}}$ by (21.19); for a complete RG step $t_t = t_{\text{max}}$. The product $\mathcal{K}_t$ is used only in this positive comparison, never as the product for the expansion defining the analytic output. At the first step, the Haar disintegration and the change to normalized fluctuation variables give the density integral in the form

$$G_{t,\chi}(c, b_{\text{rec}}) = P_t(c, b_{\text{rec}}) \int_{S_{c,b_{\text{rec}}}} \chi_t(w) e^{-V_{c,b_{\text{rec}}}(w)} dw, \quad \tilde{F}_t = P_t(c, b_{\text{rec}}) \int_{\mathcal{W}_t} e^{-V_{c,b_{\text{rec}}}(w)} dw, \quad (21.33)$$

with the prefactor described in the statement. At a later step the equality in the first formula is replaced by the inequality (21.32).

By Theorem 21.9(2), $S_{c,b_{\text{rec}}} \subset \mathcal{K}_t$ for every such datum. We check the measure-theoretic hypotheses of Lemma 20.5 at this fixed real datum. The finite products $\mathcal{K}_t$ and $\mathcal{W}_t$ are compact Borel sets. The map from free logarithmic coordinates to normalized fluctuation coordinates is a homeomorphism onto its open image, with the continuous inverse already constructed, so the image of the Borel set of free coordinates of supported points is Borel. Composing the Borel tests with that inverse gives a Borel function χ_t , extended by zero outside $S_{c,b_{\text{rec}}}$, with $0 \leq \chi_t \leq 1$. The conditional fluctuation potential is analytic on a neighborhood of $\mathcal{K}_t$ and real at its real points, by Theorem 20.14. Its finite minimum $\underline{V}$ and maximum $\overline{V}$ on this compact product give

$$0 < e^{-\overline{V}} \text{vol}(\mathcal{K}_t) \leq Z_{\mathcal{K}_t} = \int_{\mathcal{K}_t} e^{-V} dw \leq e^{-\underline{V}} \text{vol}(\mathcal{K}_t) < \infty. \quad (21.34)$$

The volume is finite and positive, since every ball has a finite positive radius. For an empty free product its zero-dimensional measure is one and $V = 0$, so $Z_{\mathcal{K}_t} = 1$. No uniform bound on $\underline{V}$ or $\overline{V}$ is needed here. The prefactor is finite at each real datum; if it vanishes, both sides vanish, and otherwise it is positive.

The hypotheses of Lemma 20.5 on the whole product give its potential estimate and, for the normalized enclosing-product comparison measure $\nu_{\mathcal{K}_t}$, the bound

$$\delta_t := \nu_{\mathcal{K}_t}(\mathcal{K}_t \setminus \mathcal{W}_t) \leq 6b_t e^{-p(t_t)^2/48}, \quad b_t \leq C_J m^3.$$

Every ball of $\mathcal{W}_t$ has positive radius, so its volume is positive. Since V is finite on $\mathcal{K}_t$, we have $\int_{\mathcal{W}_t} e^{-V} dw > 0$, and hence $\nu_{\mathcal{K}_t}(\mathcal{W}_t) > 0$ and $0 \leq \delta_t < 1$. This strict inequality concerns the actual missing mass; the displayed union-bound majorant need not be below one under the local hypotheses. For an empty free product, $\delta_t = 0$.

Since $0 \leq \chi_t \leq 1$, the right side of (21.32) is at most $P_t \int_{\mathcal{K}_t} e^{-V}$, and part (1) of Lemma 20.5 bounds the latter by $(1 - \delta_t)^{-1} \tilde{F}_t$. No inclusion of $\mathcal{W}_t$ in $S_{c, b_{\text{rec}}}$ is needed. The same reasoning applies if $P_t = 0$ or if the free list is empty. All constants arose from local path products, the fixed construction products, and their row-sum bounds. $\square$

In the theorem below, E_t , R_t , and Λ_t are the effective classical action, the effective interaction, and the normalization constant at stage t ; the last is the field-independent additive term in the log-density. Further, W_t is the product of the external weights (the factors of frozen variables that are not integrated), C_t is the reconstruction map, and ℓ_t is the level of the output of stage t . For the normalization comparison, $\Lambda_t^{\mathbb{T}}$ denotes the normalization constant of the corresponding recursion on the whole torus. Here H_t denotes the set of scalar Gaussian coordinates of stage t that are integrated in the corresponding renormalization group recursion on the whole torus but absent from the regional free list (it is not the Haar measure of Theorem 21.10), and $d_i(H_t)$ is the number of its elements at scale i . The numbers $N_{\text{missing}, t}$, $N_{\text{bd}, t}$, and $N_{\text{freeze}, t}$ count, on the lattices where they were created, the missing labels of field-independent generated terms, the labels of the fixed interface band, and the newly frozen analytic labels, as in Theorem 20.13.

Theorem 21.13. *Fix a large-field decomposition of Theorem 21.5, an initial measure ν (a stopped measure or the Wilson measure), an initial scale and a stopping scale differing by $T \geq 1$ full RG steps, and the cutoff domains and norm ball of Theorem 20.14. Every integrated group of averaging steps is a complete RG step, $s_t = J$. A remaining number $0 \leq u_{\text{ph}} < J$ of averaging steps within the given hierarchy is kept through the pullback $C \mapsto F(T_M^{u_{\text{ph}}} C)$ of an observable F at the stopping scale; this requires $m_{\text{stop}} = M^{u_{\text{ph}}} m_{\text{rem}}$ with $m_{\text{rem}} \in 4\mathbb{N}$. For a fixed additional number $r_{\text{stop}} \in \mathbb{N}_0$ of RG steps, the same condition is imposed with the total exponent $r_{\text{stop}}J + u_{\text{ph}}$, and all iterates must lie within the finite hierarchy. No recursion for normalization constants is asserted for an integrated group of fewer than J averaging steps. Assume the construction conditions and the two reconstruction-image conditions (21.17) of Theorem 21.9, including $r + \beta_{\text{rec}} < \epsilon_{\text{en}, J}$, the uniform incoming radii obtained by freezing every label of reduced radius, the boundary-distance comparison (21.10), the conditional force conditions (21.13) and (21.14), the cutoff radii of (21.19), the initial norm bounds of Theorem 20.14, and the bounds (21.8) on frozen coordinates. Let $\mathcal{W}_t$*

be the fluctuation cutoff domain (21.31), which is the instance of Theorem 20.14 with common radius $p(t_t)$; the enclosing product domain K_t is used only in Lemma 21.12. Write Z_v for the partition function of the initial measure, without centering: for a stopped measure

$$Z_v = \int_{D_r} e^{-E_{d_0, \text{sat}} / g_{\text{base}}^2 - R_{d_0, \text{sat}}} dH,$$

and $Z_v = Z_{m_{\text{base}}, \gamma}$ for the Wilson measure. Use the stage measures and density versions of Theorem 21.10. Then the following constructions and bounds hold at every finite depth and number of steps.

1. (Analytic functions, from fine to coarse.) For every real value of the frozen coordinates satisfying (21.8), the functions E_t, R_t, Λ_t, W_t , the classical objectives, the reconstruction maps C_t , and the auxiliary fluctuation integrals are defined on the full spatially varying radii by the recursion of Theorem 20.14, with the substitution of frozen coordinates of Lemma 21.11. This definition uses no density inequality and no choice of representatives. The effective interaction is centered only in its constant term, all its first derivatives are kept, and the full classical values are kept. For a stopped measure the normalization constant on the torus is $\Lambda_0^{\mathbb{T}} = -R_{d_0}(I)$, and an initial freezing subtracts in addition the frozen value of the centered effective interaction. For the Wilson measure we use the exact split of Theorem 21.9(4), with its original external factor.
2. (Representatives satisfying the reconstruction identities, from coarse to fine.) Take all history-produced small top representatives in the coordinate domains of Theorem 21.9(1), retaining their native collar records as well as (21.8), and construct such representatives downward with the maps of part (1). At a boundary with freezing, let $\mathcal{D}_{t+1}$ be the set of these representatives at the next stage, and define the set of output data satisfying the freezing and reconstruction identities

$$\mathcal{P}_{t+1} = \mathcal{A}_t^{-1}(\mathcal{D}_{t+1}).$$

For each such datum, include every supported fiber point:

$$\mathcal{D}_t = \{(C_t(c, \zeta; b_{\text{rec}}), b_{\text{rec}}) : (c, b_{\text{rec}}) \in \mathcal{P}_{t+1}, \zeta \in \mathcal{Z}_{c, b_{\text{rec}}}\},$$

where $\mathcal{Z}_{c, b_{\text{rec}}}$ is the set of supported fiber coordinates over (c, b_{rec}) . At the last RG step, define $\mathcal{D}_T$ to be the set of these history-produced small top representatives after terminal freezing, retaining their native collar records; the terminal freezing map is the identity if no coordinates are frozen there. Put $\mathcal{P}_T = \mathcal{A}_{T-1}^{-1}(\mathcal{D}_T)$ when $T > 0$. Every configuration in the support of the terminal stage measure has one of these representatives in its gauge orbit by Theorem 21.5 and the closed-support argument above. At every lower boundary, inverse freezing restores the ordinary collar links literally from the fixed new records, and Lemma 21.8 applies to every supported fiber point. Other representatives are compared only by the changes of Lemma 21.6(1)–(3) when transporting analytic estimates, or by the pointwise invariance of Theorem 21.10 when extending the final density bound. These sets contain all choices allowed

by this history-produced initialization, and no probability law is assigned to a representative map.

3. (Density bounds, from fine to coarse.) For every real value of the frozen coordinates, every $(c, b_{\text{rec}}) \in \mathcal{P}_{t+1}$, and every supported fiber coordinate ζ over it,

$$\begin{aligned} p_t(C_t(c, \zeta; b_{\text{rec}}), b_{\text{rec}}) &\leq A_t Z_v^{-1} e^{\Lambda_t(b_{\text{rec}})} W_t(b_{\text{rec}}) \\ &\quad \times \exp\{-E_t(\text{Log } C_t(c, \zeta; b_{\text{rec}}); b_{\text{rec}}) / g_{\ell_t}^2\} \\ &\quad \times \exp\{-R_t(\text{Log } C_t(c, \zeta; b_{\text{rec}}); b_{\text{rec}})\}. \end{aligned}$$

At the terminal stage the bound is, directly, for every $(v, b_{\text{rec}}) \in \mathcal{D}_T$,

$$p_T(v, b_{\text{rec}}) \leq A_T Z_v^{-1} e^{\Lambda_T(b_{\text{rec}})} W_T(b_{\text{rec}}) \exp\{-E_T(\text{Log } v; b_{\text{rec}}) / g_{\ell_T}^2 - R_T(\text{Log } v; b_{\text{rec}})\}.$$

Here the terminal quantities include the same freezing substitution, if any. Also $A_0 = 1$ and

$$A_{t+1} = A_t(1 - \epsilon_t)^{-1}, \quad \epsilon_t \leq 6b_t e^{-p(t)^2/48}, \quad b_t \leq C_J m_t^3, \quad \log A_T \leq C_\delta,$$

after the uniform choice of the minimal side in Lemma 20.5. Every step uses the exact Haar disintegration and then a single triangular change to normalized fluctuation variables, with determinant

$$\prod_{i=0}^{s_t-1} g_i^{3f_i} \det K_i(c_{i+1}(w_{>i}); b_{\text{rec}})^{-1/2}.$$

The resulting Borel set of fluctuation coordinates of supported points is contained in the enclosing product domain, and Lemma 21.12, with identical prefactors and the smaller cutoff domain, gives the stated loss. The next stage evaluates the same configuration. The indicators are kept until this comparison, and no domination at arbitrary points of the enclosing product domain is required.

4. (Normalization.) With $\Delta\Lambda_t = \Lambda_t - \Lambda_t^{\mathbb{T}}$ in the same centered renormalization group recursion on the whole torus, Theorem 20.13 gives

$$\Delta\Lambda_{t+1} \leq \Delta\Lambda_t + \sum_{i \text{ in } t} d_i(H_t) \log(1/g_i) + C_J(N_{\text{missing},t} + N_{\text{bd},t} + N_{\text{freeze},t}),$$

including the initial freezings. This comparison uses the same centered recursion on the whole torus and the same partition function Z_v throughout.

5. (External classical terms and the measure of frozen coordinates.) For the comparison of the output classical actions, the protected external classical exponents are recombined exactly once. At fixed frozen coordinates, with their original couplings included in $S_{\text{ext}}(b_{\text{rec}})$,

$$e^{-S_{\text{ext}}(b_{\text{rec}})} e^{-E^{\text{int}}/g_t^2} = e^{-E^{\text{full}}/g_t^2}, \quad E^{\text{full}} = E^{\text{int}} + g_t^2 S_{\text{ext}}(b_{\text{rec}}),$$

where E^{int} is the effective classical action of the block region, and this factor is removed from the external weight. Under an RG step both the action and g_t^2 are multiplied by M^{st} , so each such exponent is preserved and counted once. At the top we keep the measure

$$\rho_{\mathcal{X}}(db_{\text{rec}}) = w_{\mathcal{X}}(b_{\text{rec}})H_{\mathcal{X}}(db_{\text{rec}}) \leq H_{\mathcal{X}}(db_{\text{rec}}), \quad w_{\mathcal{X}} = I_{\mathcal{X}}^{\text{act}} I_{\mathcal{X}}^{\text{inv, end}},$$

on the frozen coordinates, where $H_{\mathcal{X}}$ is their product Haar measure. The first indicator keeps every failure $q_f(b_{\text{rec}}) \geq d_*$, which is a function of the frozen coordinates since the whole dependence set of f was frozen. The second keeps any chosen invariant tests depending only on the frozen coordinates at the output. There is no second copy of the frozen-coordinate terms included in the full classical action. Optional noninvariant chart restrictions may be bounded by one.

All local constants have the dependence stated at the beginning of this section, including independence of the later widths, while C_{δ} has the dependence described in item (4) of Definition 4.14. No comparison of regional integrals from below or independent covariance or overlap identity for the auxiliary regional functions is part of this induction. Compact overlap for the torus classical restriction on the common charts is used in (21.23).

For a range containing no complete RG step, this theorem asserts no small terminal representative, for either initial measure. The empty analytic recursion retains its initial values, and estimates for the initial measure are made directly. Empty active regions and empty fluctuation products within a positive-length decomposition retain the conventions of the theorem.

Proof. Every integrated step has $s_t = J$, so $\ell_{t+1} = \ell_t + J$ and $g_{\ell_{t+1}}^2 = Bg_{\ell_t}^2$. We reserve z_i for the free logarithmic coordinates and write $\zeta_i = g_i w_i$ for the scaled fluctuation arguments of C_t . All analytic functions below are functions of the identity-chart variables of the block region at fixed real (ζ, b_{rec}) . We prove the five parts in order, followed by the statement on the remaining averaging steps.

Part (1): initialization. For a stopped measure, we set

$$r_{d_0} = R_{d_0} - R_{d_0}(I), \quad \Lambda_0^{\mathbb{T}} = -R_{d_0}(I).$$

The partition function without centering then gives exactly $Z_v^{-1} e^{\Lambda_0^{\mathbb{T}}} e^{-E_{d_0}/g_{\text{base}}^2 - r_{d_0}}$ at a small representative in D_r . In particular $\widehat{Z} = e^{R_{d_0}(I)} Z_v$ for the centered partition function in Theorem 21.5; we keep Z_v throughout this proof. For an initial freezing at $\tilde{\zeta}^{\sharp}$, we set

$$\begin{aligned} E_0(v; b_{\text{rec}}) &= E_{d_0}(v \oplus \tilde{\zeta}^{\sharp}), \\ R_0(v; b_{\text{rec}}) &= r_{d_0}(v \oplus \tilde{\zeta}^{\sharp}) - r_{d_0}(0 \oplus \tilde{\zeta}^{\sharp}), \\ \Lambda_0(b_{\text{rec}}) &= -R_{d_0}(I) - r_{d_0}(0 \oplus \tilde{\zeta}^{\sharp}). \end{aligned} \tag{21.35}$$

Additional frozen coordinates are held as real parameters. For the Wilson measure, we set $E_0 = F_{\text{bare}}$, $R_0 = 0$, $\Lambda_0 = 0$, and $W_0 = e^{-E_{\text{hole}}/g_{\text{base}}^2}$, together with any kept indicators at the output. The exact split in Theorem 21.9(4) implies this initialization with the partition function of the Wilson measure.

We verify the initial norm bounds. The initial analytic values obey (21.8), because $C_{\text{conv},J} \geq C_{\text{ad},J}$, which was fixed before Theorem 21.5, and the near values at the first step under the Wilson measure satisfy its conversion bound. For a stopped measure, the decomposition with zero completion has classical perturbation $P_0(v) = E_{d_0}(v \oplus \xi^\sharp) - E_{d_0}(v \oplus 0)$, including its constant. With the matched completion $\iota_{\xi^\sharp}$ of Theorem 19.9, this is exactly its boundary-completion difference, and the adjusted perturbation is $\bar{P}_0 = 0$. In the norm with zero completion, its nonconstant part is bounded by $C_{\text{bc,cl}} E_{\text{cl}} \beta_{\text{rec}} / \rho_{\text{cl}} \leq \sigma_*/4$, by (21.16). For the quantum input, splitting the centered function $r_{d_0}(v \oplus \xi^\sharp)$ into its restriction to the torus and its boundary-completion difference gives the bound $C_{\text{bc,q}} E \beta_{\text{rec}} / \rho \leq \sigma_*/4$. The constant in (21.35) is kept exactly and is not bounded by this norm.

At the first step under the Wilson measure, the plaquettes depending on integration variables use only the small near values of the initial good-cell complex. The zero-completed Wilson function evaluated at those data differs from the sum of these plaquette terms by plaquettes involving only frozen variables, which are constant in the integration variables. Hence the adjusted classical perturbation is constant at this step, and its nonconstant boundary-completion difference with zero completion obeys the same classical bound. These classical constants remain in F_{bare} , and the energy of the deep frozen group variables stays in W_0 . The quantum input is zero. This proves $x_0, x_0^{\text{cl}} \leq \sigma_*$ for either initial measure, on uniform radii of the remaining inputs. No deep frozen group variable has entered an analytic logarithm.

Part (1): the analytic recursion. At a later input, let D_{in} be the list of remaining analytic variables of the block region, with their original locations and volume factors; it does not include the frozen parameters. Since the preceding freezing of the union removed every label of reduced radius, the classical and quantum input radii are uniform by (21.15). The previous fixed complement and all earlier frozen basepoints obey (21.10). The initialization satisfies the same statement, with zero boundary-weighted input when there is no incoming fixed complement, and additional excluded regions created by the tests preserve this inclusion. All completion adjustments use only newly frozen small analytic variables; previously nonintegrated torus variables remain zero, and the deep frozen group variables remain fixed external parameters.

We apply Theorems 19.9 and 20.14 inductively, with the fixed amplitude A_w and the coupling ceiling g_* chosen for it, to define the classical objective, the conditional minimizers for initial segments, their positive square roots, and the reconstruction map on the full radii. The hypotheses of each application are checked before any set of representatives

is constructed. The input norm just restored bounds the adjusted perturbation by $2\sigma_*$. The images and every connecting segment lie strictly in the uniform input ball, by (21.17) and the fixed margin of the larger construction domain. The force and Hessian estimates, uniformly for $y \in \mathcal{T}_I$ for each initial segment I , followed by (21.14), give all conditional minimizers and sequential neighborhoods; these checks are Lemma 21.7. All perturbation derivatives are taken in the integration and free variables at fixed real b . The fixed rates and spatial margins are those of the common tuple of Theorem 20.14 and are not changed by freezing. The torus input is the member at accumulated classical depth $d_t = d_0 + tJ$ of its uniform analytic class, with ceiling E ; a new nonconstant boundary term is accounted for by the two input norms rather than added to this ceiling. We include the exact boundary-completion difference when the unperturbed objective uses newly frozen input variables. By Theorem 19.9, this objective agrees on the connected chart with E_t evaluated at its reconstructed field; only the torus term uses compact overlap there, and the perturbation is evaluated at that field from the outset. The full outgoing action is defined to be the value at the conditional minimizer multiplied by M^{s_t} .

We next define the update of the effective interaction by an integral; this also fixes all normalization conventions. Let $J_t^0(c, b_{\text{rec}})$ be the product of the Haar disintegration factors of the intermediate scales evaluated along the zero-fluctuation chain, and define

$$D_t^0(c, b_{\text{rec}}) = J_t^0(c, b_{\text{rec}}) \prod_i g_i^{3f_i} \det K_i(\widehat{c}_{i+1}; b_{\text{rec}})^{-1/2}, \quad (21.36)$$

where $\widehat{c}_{i+1}$ denotes the corresponding higher argument at zero fluctuation. All field-independent constants of the Haar disintegration are included. Sequential minimization and normalization give the exact decomposition of the action: the output value divided by $g_{\ell_{t+1}}^2$, the quadratic term $\sum_i |w_i|^2/2$, and the higher classical remainders add up to the input classical exponent. Indeed, at each intermediate scale we subtract the conditional minimum and its quadratic Taylor term; the latter is $|w_i|^2/2$ by the definition of K_i , and the factor M in the minimum cancels the factor M in the next coupling. Repeating this over the s_t scales, we obtain the decomposition, including its constants. Adding the inherited difference of effective interactions, all logarithmic differences of the Haar disintegration factors, and all logarithmic differences of the determinants that depend on the higher fluctuation variables, we obtain the conditional fluctuation potential $V_t(c, b_{\text{rec}}; w)$, including its quadratic part; it is the potential of Theorem 20.14.

In this integral, we use the fluctuation cutoff domain (21.31): every radius is $p(t_t)$ with $t_t = \max_i g_{t,i}$. This is in the allowed logarithmic family; it is not the radius R_{en}/t_t of the enclosing product. The containment, positive denominator, interaction, and auxiliary half-disc conditions hold by (21.17) and (21.19); they are deterministic conditions on the parameters and on g_* , not restrictions on the support of the measure. On the fixed

product $\mathcal{W}_t$, we define

$$\begin{aligned}\mathcal{I}_t(c; b_{\text{rec}}) &= D_t^0(c, b_{\text{rec}}) e^{-R_t(\text{Log } C_t(c, 0; b_{\text{rec}}); b_{\text{rec}})} \int_{\mathcal{W}_t} e^{-V_t(c, b_{\text{rec}}; w)} dw, \\ Q_t(c; b_{\text{rec}}) &= -\log \mathcal{I}_t(c; b_{\text{rec}}), \\ R_t^{\text{out}}(c; b_{\text{rec}}) &= Q_t(c; b_{\text{rec}}) - Q_t(0; b_{\text{rec}}), \quad \Lambda_t^{\text{out}} = \Lambda_t - Q_t(0; b_{\text{rec}}).\end{aligned}\tag{21.37}$$

The logarithm is real on real data, where it agrees with the real logarithm of the positive integral; its zero-free analytic continuation is supplied by Theorem 20.14, and its sign is unrestricted. The integrand and its domain are defined on the full radii before any support set is considered. We have the exact identity

$$Z_v^{-1} e^{\Lambda_t} W_t e^{-E_t^{\text{out}}(c)/g_{t+1}^2} \mathcal{I}_t(c; b_{\text{rec}}) = Z_v^{-1} e^{\Lambda_t^{\text{out}}} W_t e^{-E_t^{\text{out}}(c)/g_{t+1}^2 - R_t^{\text{out}}(c; b_{\text{rec}})}.\tag{21.38}$$

Changing the distribution of constant factors between D_t^0 and the definition of Q_t leaves this identity unchanged; we use the convention of Theorem 20.13, which assigns each field-independent factor once.

At a boundary we use the substitution of Lemma 21.11. This defines all functions at the next level for every real value of the frozen coordinates, independently of any density bound. The simultaneously frozen list is the union of the full boundary band of reduced radius, the removed band, the new boundary layers, and the based transports; the corresponding initial variables were included at the initial scale. Their analytic values satisfy (21.8) for both radii. At a representative of the next stage satisfying the reconstruction identities, the freezing identity inserts these values, not small values from a different frame, and the total boundary-completion difference obeys (21.16). The resulting norm bounds are exactly those of Theorem 20.14: the output ceilings before freezing are y_t and y_t^{cl} , the boundary-completion differences are at most $\sigma_*/4$, and the removed band makes each remaining term at most $\sigma_*/2$. With $\alpha_q = \alpha$ and the corresponding classical rate, we have for both types

$$x_{t+1}^\bullet \leq C_{\text{bc}, \bullet} E_\bullet \beta_{\text{rec}} / \rho_\bullet + 2e^{-\alpha \cdot (\text{ser}^{-1})} y_t^\bullet \leq \sigma_*/4 + \sigma_*/2,$$

so $x_{t+1}, x_{t+1}^{\text{cl}} \leq 3\sigma_*/4$ whenever the input is in its ball. The output ceilings $y_t^\bullet$ are computed on the input ball before the width s_{er} is chosen. By induction from the initial ball, we obtain the analytic recursion at every finite number of steps. It keeps the terms of degree one; no first derivative has been removed by recentering, and every restoration of the bounds happens at a complete RG step.

It remains to verify the Borel regularity in the real parameters required at each perturbation step. The initial coefficients and chart path products are Borel functions of the protected real frozen coordinates; the deep frozen group variables at the first step under the Wilson measure enter only the Borel external factor. Suppose this holds at

an input. The fixed-point iterations for the conditional minimizers use Borel coefficients and converge uniformly on their fixed neighborhoods, so their limits are Borel in b . The positive matrix square roots and inverses are Borel as uniform limits of the matrix-function constructions. The resulting values, Haar disintegration determinants, and coefficients are Borel. By Tonelli's theorem, integration over the fixed finite product preserves Borel dependence; we apply it to the positive and negative parts and to the real and imaginary parts, which have finite integrals. The logarithm can be chosen jointly Borel without a branch choice depending on the frozen coordinates: on the star-shaped output domain,

$$\log \mathcal{I}_t(c; b) = \log \mathcal{I}_t(0; b) + \int_0^1 \frac{D_c \mathcal{I}_t(sc; b)[c]}{\mathcal{I}_t(sc; b)} ds,$$

where the first term is the real logarithm. Zero-freeness and holomorphy make this the required analytic branch, and the parameter integral is Borel. Finally, the substitution of frozen coordinates evaluates these Borel families at the Borel coordinates of the frozen values. This proves the Borel regularity inductively, without differentiating in a real frozen parameter.

Part (2): representatives satisfying the reconstruction identities. We start from all history-produced small representatives of top configurations, in the coordinate domains of Theorem 21.9(1), retaining both (21.8) and the native C_{Br} collar records; every supported top orbit has such a representative by Theorem 21.5. At a boundary with freezing, we pull the next set of representatives satisfying the reconstruction identities back by the freezing bijection $\mathcal{A}_t$; this defines $\mathcal{P}_{t+1}$, which is not enlarged to an arbitrary comparison set. For each such datum, we include every supported fiber coordinate, as in the definition of $\mathcal{D}_t$. Theorem 21.9, applied at this step of the descent, shows that each of them has its inverse normalized fluctuation coordinate in the interior half of the enclosing product, and that its representative has the correct target and protected frozen coordinates. This defines the whole finite family downward using only the reconstruction maps C_t of part (1). The sets integrated below are the Borel sets of free coordinates of supported fiber points at fixed output data, so no measurable representative map is needed.

The bounds (21.8) persist along this descent, by Lemma 21.6(1)–(3) and, at a new boundary, the direct insertion of Lemma 21.11. Changes as in Lemma 21.6(4) are not used to enlarge these sets. Over each output datum in $\mathcal{P}_{t+1}$, the set still contains every supported fiber point.

These sets meet every supported gauge orbit. Indeed, suppose this holds at stage $t + 1$, and take a supported point at stage t . Map it to its raw output and then through the freezing map. An admissible endpoint gauge sends that next-stage orbit to $\mathcal{D}_{t+1}$. The equivariance of freezing and the fiber bijection used in (21.28) lift this change to the incoming point. Its output then lies in $\mathcal{A}_t^{-1}(\mathcal{D}_{t+1})$, and the resulting supported fiber point is included in $\mathcal{D}_t$ by the all-fiber definition. This proves the assertion by

downward induction from the history-produced top representatives. This orbit selection uses only the exact gauge action on the stage variables and their support; it does not assert covariance of the auxiliary regional perturbations under arbitrary endpoint gauges. Analytic comparisons of the torus classical restriction within the selected family use compact overlap after the endpoint-preserving completions proved above.

Part (3): density bounds. We use the densities of Theorem 21.10. For a stopped measure, the gauge-invariant density agrees with its analytic expression at each representative satisfying the reconstruction identities, and the indicators of the decomposition and of the support are between zero and one. Then (21.35) implies the claimed bound at the initial scale with $A_0 = 1$. For the Wilson measure, the exact split implies the same bound with the kept factor W_0 . At a point outside the support, the density is zero, and no analytic identity is needed there.

Suppose that the inequality holds on $\mathcal{D}_t$, and fix an output datum $(c, b_{\text{rec}}) \in \mathcal{P}_{t+1}$. We apply the exact Haar disintegration to the incoming density, keeping its tree variables at first. By Theorem 21.10, the covariance under based gauges makes their normalized Haar integrals equal to one. For this fixed target, the gauge transformation of Lemma 19.5 is a fixed gauge transformation, independent of the free variables and equal to I at the roots of the target and at every protected vertex. The invariance of the density therefore allows us to evaluate the integrand at $C_t(c, \zeta; b_{\text{rec}})$ instead of at its representative before the gauge transformation, and the tests take the same values there. This changes neither the Haar disintegration nor any determinant, and it is not a covariance statement for the auxiliary functions on the right side of the inequality.

We now perform the single sequential change to normalized fluctuation variables for the classical objective. At scale i its derivative with respect to w_i is $g_i K_i^{-1/2}$, and its other dependence is on higher variables. Ordering the higher variables first makes the derivative block triangular, so its positive real determinant is exactly

$$\prod_{i=0}^{s_t-1} g_i^{3f_i} \det K_i(c_{i+1}(w_{>i}); b_{\text{rec}})^{-1/2}. \quad (21.39)$$

In particular, the determinant at a varying higher argument is not replaced by its value at zero. Equation (21.36) and the corresponding logarithmic differences split this one determinant; they do not add a second copy. Injectivity and the inverse map on the supported domain follow from the branches for initial segments, with their positive margins, established before. As in the proof of Lemma 21.12, the image is a Borel set $S_{c, b_{\text{rec}}}$ in the enclosing product, and the remaining indicators composed with the inverse form a Borel function χ_t on $\mathcal{K}_t$ with values in $[0, 1]$. The potential has a finite nonzero integral over $\mathcal{K}_t$ at each fixed real datum, also for an empty product.

Every supported point of that domain belongs to $\mathcal{D}_t$. We insert the inductive bound there, and let $P_t^0(c, b_{\text{rec}})$ be

$$Z_v^{-1} e^{\Lambda_t(b_{\text{rec}})} W_t(b_{\text{rec}}) D_t^0(c, b_{\text{rec}}) e^{-E_t^{\text{out}}(c; b_{\text{rec}})/g_{t+1}^2 - R_t(\text{Log } C_t(c, 0; b_{\text{rec}}); b_{\text{rec}})}.$$

Tests depending only on the output datum are kept in this common factor. If this factor is zero, the outgoing upper bound is zero; otherwise it is finite and positive. With χ_t denoting the remaining indicators, the exact disintegration followed by the inductive bound gives

$$\begin{aligned} p_t^{\text{out}}(c, b_{\text{rec}}) &\leq A_t P_t^0(c, b_{\text{rec}}) \int_{S_{c, b_{\text{rec}}}} \chi_t(w) e^{-V_t(w)} dw \\ &\leq A_t P_t^0(c, b_{\text{rec}}) \int_{\mathcal{K}_t} e^{-V_t(w)} dw \\ &\leq \frac{A_t}{1 - \epsilon_t} P_t^0(c, b_{\text{rec}}) \int_{\mathcal{W}_t} e^{-V_t(w)} dw. \end{aligned} \tag{21.40}$$

Here ϵ_t is a common upper bound for the datum-dependent δ_t of Lemma 21.12 at stage t , over all admitted output data, after the uniform minimum-side choice in part (3) of Lemma 20.5. One may take $\epsilon_t = 6b_t e^{-p(t_t)^2/48} \leq 1/2$. The first inequality is the hypothesis (21.32) of that lemma with $\chi = 1$ and $P_t = A_t P_t^0$, and the remaining two inequalities are its conclusion. The middle step uses the enclosure of Theorem 21.9, positivity on the whole larger product, and only $0 \leq \chi_t \leq 1$. The last step is Lemma 20.5 with identical prefactors; its radii are the common logarithmic radii, enclosed by $t_t p(t_t) < R_{\text{en}}$, and its Hessian and mean conditions hold by (21.19), in particular with mean at most $p(t_t)/(2\sqrt{3})$. The larger product $\mathcal{K}_t$ is used only in this comparison. This verifies the hypothesis of Lemma 21.12 at every later stage, with the incoming loss A_t in the common prefactor.

By (21.38), the last line of (21.40) is the analytic outgoing bound, defined independently, with the factor $A_{t+1} = A_t/(1 - \epsilon_t)$. By Lemma 21.11, the inverse under freezing of a representative of the next stage is this output configuration. That lemma substitutes the based transports and frozen links directly, transfers $-R_t^{\text{out}}(0 \oplus \xi^\sharp)$ to the normalization constant, and keeps the full classical value. This proves the incoming inequality at stage $t + 1$ on exactly $\mathcal{D}_{t+1}$, including the displayed terminal bound when $t + 1 = T$, and completes the induction.

There is one comparison per output datum, jointly over all components and scales of the step. The probability estimate used here is $\epsilon_t \leq 6b_t e^{-p(t_t)^2/48}$ with $b_t \leq C_J m_t^3$. Ordering the finite list of output data along the sequence of sides and taking the uniform minimal side of Lemma 20.5, we obtain

$$\log A_T = \sum_{t < T} -\log(1 - \epsilon_t) \leq 2 \sum_{t < T} \epsilon_t \leq C_\delta.$$

Empty free products have determinant one, logarithmic determinant zero, and fluctuation integral one, and an empty block region has zero nonconstant norm. Such an empty step contributes no loss.

Part (4): normalization. Since $s_t = J$ and the initial and stopping scales differ by complete RG steps, every application below is within the scope of Theorem 20.13. The recursion just constructed is the same centered recursion, with the same extraction of field-independent factors, as in that theorem. All its norm and domain hypotheses were verified in part (1), including the common logarithmic cutoffs and the freezing of the whole union. The column bound used for the analytic norm propagation in Theorem 20.10 is that of the perturbed zero-fluctuation map. Indeed, the doubled output radius and the inherited half radius are part of the two image conditions, and the differences with each index fixed in Theorem 19.9 give, in the same boundary-weighted sum over distinguished outputs,

$$K_{\text{col},J}^{\text{actual}} \leq K_{\text{col},J}^0 + C_J \sigma_{0,J},$$

where $K_{\text{col},J}^{\text{actual}}$ and $K_{\text{col},J}^0$ are the column ceilings of the perturbed and unperturbed maps. The fixed allocations of dependence neighborhoods and exponents are already included in C_J . More explicitly, the margin in the coefficient exponent, after the transfer of boundary distances in (21.10), leaves a positive exponent c' for summing over the incoming label at a fixed output, and this sum is bounded by $v_J \sum_{q \geq 0} (q+2)^3 e^{-c'q} < \infty$. The volume factors of the lattices where the labels were created are kept in this column sum. Deleting labels or adding excluded regions cannot increase this bound of bounded density. This proves that the enlargement has the dependence stated at the beginning of this section. We fix this enlarged ceiling on the whole larger perturbation domain before choosing s_{er} or later widths. In Theorem 20.10, its coefficient is $K_J = 4(\log 2)K_{\text{col},J}$ for this enlarged ceiling; no column bound for the unperturbed map alone is substituted.

Applying the generated-increment comparison of Theorem 20.13 with the incoming $\Delta\Lambda_t = \Lambda_t - \Lambda_t^{\mathbb{T}}$, we obtain the increment in part (4), including the initial freezings. To check the sign of the dimension term, note that one scalar Gaussian coordinate integrated in the corresponding renormalization group recursion on the whole torus contributes a factor g_i . When that coordinate is absent from the regional free list, the logarithmic difference between the regional and torus terms is $-\log g_i = \log(1/g_i)$. Summing over the missing scalar coordinates, we obtain $d_i(H_t) \log(1/g_i)$, once for each scale. By that theorem, the remaining generated differences of Haar disintegration factors, minimizers, and cluster expansions, and the freezings, are bounded by C_J times the three label counts, taken on the lattices where the labels were created. This comparison is additional analytic information: neither the exact disintegration, nor (21.37), nor normalization would by itself bound these generated differences. The factor A_t remains separate and is not added again to Λ_t .

The remaining averaging steps. Let $a = \ell_T - \ell_0 = JT$ be the number of integrated averaging steps (here a is not a mesh), and let $\ell_{\max}$ be the last level of the given finite hierarchy. A remaining number $0 \leq u_{\text{ph}} < J$ of averaging steps is chosen only when it is available: at the stopping scale we require

$$m_{\text{stop}} = M^{u_{\text{ph}}} m_{\text{rem}}, \quad m_{\text{rem}} \in 4\mathbb{N}, \quad \ell_T + u_{\text{ph}} \leq \ell_{\max}.$$

For a fixed additional number $r_{\text{stop}} \in \mathbb{N}_0$ of RG steps, we set $v = r_{\text{stop}}J + u_{\text{ph}}$ and require

$$m_{\text{stop}} = M^v m_{\text{rem}}, \quad m_{\text{rem}} \in 4\mathbb{N}, \quad \ell_T + v \leq \ell_{\max};$$

the case of u_{ph} alone is $r_{\text{stop}} = 0$. For each $0 \leq k \leq v$, the intermediate side is $M^{v-k} m_{\text{rem}} \in 4\mathbb{N}$, which is at least four. The initial side is $M^a m_{\text{stop}}$, and the bound on the index keeps every intermediate level in the given hierarchy. Hence both T_M^v at the stopping scale and T_M^{a+v} at the initial scale exist; a lower bound on m_{stop} alone would not give these integer conditions. We work on whole configurations, before regional coordinates are introduced. The block averaging map with its Borel extension is Borel, and its iterates are compositions of that map, so for every initial configuration V , we have

$$F(T_M^{a+v} V) = (F \circ T_M^v)(T_M^a V),$$

and the function at the stopping scale is exactly $C \mapsto F(T_M^v C)$, in particular $C \mapsto F(T_M^{u_{\text{ph}}} C)$ for u_{ph} alone. We define the measure $\mu_{a,\mathcal{X}} = (T_M^a)_\#(\mathbf{1}_{\mathcal{A}_\mathcal{X}} \nu)$ on the whole configurations at the stopping scale. For every nonnegative Borel F we have

$$\int \mathbf{1}_{\mathcal{A}_\mathcal{X}}(V) F(T_M^{a+v} V) d\nu(V) = \int (F \circ T_M^v)(C) d\mu_{a,\mathcal{X}}(C), \quad (21.41)$$

by the definition of the pushforward; the identity extends to integrable F through positive and negative parts. Each fixed additional number of steps uses the total exponent v whose existence was just checked, together with its finite enlarged dependence neighborhood. For the local link and plaquette observables of Theorem 21.5, these iterates use its already enlarged final dependence neighborhood. The decomposition keeps this observable on each piece. There is no Haar disintegration, change of variables, restoration of bounds, or update of normalization for v . When the integrated part of the decomposition is empty, the identity is the initial pushforward itself.

Part (5): external classical factors and failure indicators. At fixed frozen coordinates, let $S_{\text{ext}}(b_{\text{rec}})$ be the sum of the protected external classical exponents with their original couplings; for the Wilson measure it includes $E_{\text{hole}}/g_{\text{base}}^2$. The term $g_t^2 S_{\text{ext}}$ is constant in all integration variables, so adding it to the full action of the block region changes no minimizer, Hessian, map, or nonconstant norm. We have the exact identity

$$e^{-S_{\text{ext}}} e^{-E^{\text{int}}/g_t^2} = e^{-(E^{\text{int}} + g_t^2 S_{\text{ext}})/g_t^2}.$$

We delete this external factor from W_t when the full classical action including the frozen-coordinate terms is used. After an RG step, the corresponding constant becomes $M^{s_t} g_t^2 S_{\text{ext}} = g_{t+1}^2 S_{\text{ext}}$, so its exponent is preserved exactly. At each classical freezing, the full value was kept. Induction over the finite decomposition shows that each frozen-coordinate term of the full classical action has one copy in the output action and none in the remaining weight of the frozen coordinates. The constants from the freezing of the effective interaction remain separate exact terms of Λ .

For each failing plaquette f , the whole dependence set was frozen at the input of its RG step. Its value q_f is therefore a Borel function of the frozen coordinates on that lattice, computed with the same path products of the block averaging map, and later operations fix its marked data exactly. Its trace is invariant under the consistent endpoint action. We keep the product of all indicators $\{q_f \geq d_*\}$ as $I_{\mathcal{X}}^{\text{act}}$, and any chosen invariant tests depending only on the frozen coordinates at the output as $I_{\mathcal{X}}^{\text{inv, end}}$. Every other optional coordinate restriction is between zero and one and may be removed in an upper bound. After the classical recombination, this leaves exactly the measure on frozen coordinates in part (5), which is bounded by its product Haar measure. It contains no second copy of an energy that could be used for a lower bound on the output action or for an energy bound on failing plaquettes.

Every local bound has the dependence stated at the beginning of this section on the fixed larger domains, and only C_δ has the dependence of item (4) of Definition 4.14. $\square$

Exponential-moment bounds between renormalization scales

In this section, we prove exponential-moment bounds for sums of plaquette energies of block-averaged configurations. The bounds are uniform in the number of averaging steps between the lattice on which the measure is given and the lattice on which the energies are evaluated. The main result is Theorem 22.1. Under its hypotheses, fix a coupling ceiling $\Gamma > 0$ and take the smallest side $u \geq n_{\min}(\Gamma)$. Let $b \in J\mathbb{N}_0$ with $0 \leq b \leq j-1$, and let ν be either the Wilson measure on the lattice of side n_b or a normalized stopped measure there at a preceding depth in $J\mathbb{N}_0$. The theorem states that the exponential moments under ν of the sum $Q_j^\mathcal{T}$ of the energies of the predecessor plaquettes of a chessboard tiling, (22.2), are bounded by $\exp\{3C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b\}$. The constants are independent of $\gamma \in (0, \Gamma]$, of the lattice sides, and of $j-1-b$, and the expectation contains no small-field or regularity indicator. We also prove a chessboard estimate in which the regularity indicators stay inside the root (Lemma 22.11). Finally, we compare the Wilson measure, restricted to the first a conditions of the small-field event, with the stopped measures; this bounds the probability that the next condition fails, without conditioning on later scales (Lemmas 22.12 and 22.13). The inputs are the large-field decomposition and the density bounds of Section 21 (Theorems 21.5, 21.10, and 21.13); the regional integration estimates of Section 20 (Theorems 20.13 and 20.14 and Lemma 20.5); the classical lower bound of Theorems 11.2 and 11.6; the energy transport and predecessor geometry of Theorem 7.3 and Lemma 7.4; the reference normalization of Theorem 14.8; the chessboard estimate of Theorem 6.6; and, for the comparison of restrictions, Theorems 7.7 and 14.7. The results are used in Section 23 to prove the bound on large fields, Theorem 2.7.

Section 22.1 states the main theorem and outlines its proof, Sections 22.2 to 22.6 prove the estimates for one large-field decomposition, Section 22.7 completes the proof, and Sections 22.8 and 22.9 contain the chessboard estimate and the comparison of restrictions.

22.1 Setting and the main estimate

Scales and couplings. Fix $L > 0$, an odd integer $M \geq 3$, an integer $J \geq 1$, and $B = M^J$. For $u \in 4\mathbb{N}$ and $K \in \mathbb{N}_0$ let $N = uB^K$, $n_i = N/M^i$, and $V^i = T_M^i U$ for $0 \leq i \leq JK$, where T_M is the block averaging map of Definition 4.3, including its Borel extension at

the exceptional values. Increasing i moves toward the infrared, and all the sides n_i are divisible by four. One application of T_M is an averaging step, and J consecutive averaging steps form an RG step. The levels $i \in \mathbb{J}\mathbb{N}_0$ are the boundaries of RG steps. At side n the coupling is $h_n = g_n^2 = \gamma/n$, as in Definition 4.1, so that $h_{n_{i+1}} = Mh_{n_i}$. All Haar measures in this section are probability measures.

Plaquette energies. For $A \in \text{SU}(2)$ let $\rho_G(A) \in [0, 2\pi]$ be the distance from A to I in the bi-invariant metric of Section 4, and recall that $q(A) = 1 - (\text{tr } A)/2$. On the whole group,

$$\frac{\rho_G(A)^2}{2\pi^2} \leq q(A) \leq \frac{\rho_G(A)^2}{8}, \quad 0 \leq q(A) \leq 2. \quad (22.1)$$

Indeed, if $t = \rho_G(A)/4 \in [0, \pi/2]$, then $q(A) = 1 - \cos(2t) = 2\sin^2 t$ by (4.2), and $2t/\pi \leq \sin t \leq t$. For a set $\mathcal{P}$ of positive plaquettes at level i and a configuration V on the lattice of side n_i we write

$$\mathcal{S}_i^{\mathcal{P}}(V) = \frac{4}{h_{n_i}} \sum_{p \in \mathcal{P}} q(V_p).$$

If $\mathcal{P}$ is the set of all positive plaquettes, then $\mathcal{S}_i^{\mathcal{P}}(V) = S_W(V)/h_{n_i}$ is the Wilson exponent.

The plaquette-energy sum. For a set $\mathcal{A}$ of positive plaquettes at level $i+1$, let $\text{Pa}(\mathcal{A})$ be the union, counted once, of the plaquettes at level i in the complete predecessor rectangles of the elements of $\mathcal{A}$ (Theorem 7.3). Let $1 \leq j \leq JK$. For each of the eight tilings $\mathcal{T}$ of the level- j lattice into cells of $2 \times 2 \times 2$ sites in Lemma 7.4, and each cell $C \in \mathcal{T}$, let $\mathcal{P}_{j-1}(C)$ be the union of the complete predecessor rectangles of the six internal level- j plaquettes of C , and let $\mathcal{P}_{j-1}^{\mathcal{T}} = \bigcup_{C \in \mathcal{T}} \mathcal{P}_{j-1}(C)$, counted once. By Lemma 7.4, $|\mathcal{P}_{j-1}(C)| \leq 24M^3$. For a configuration V on the lattice of side n_{j-1} we set

$$Q_j^{\mathcal{T}}(V) = \frac{4}{h_{n_{j-1}}} \sum_{p \in \mathcal{P}_{j-1}^{\mathcal{T}}} q(V_p) = \mathcal{S}_{j-1}^{\mathcal{P}_{j-1}^{\mathcal{T}}}(V). \quad (22.2)$$

This is the function that enters the chessboard estimate of Section 22.8.

Initial measures. Every estimate of this section starts from a probability measure ν on the configurations of some side n_b of the hierarchy, which we call the initial measure. It is one of the following.

1. The Wilson measure $\nu = \mu_{n_b, \gamma}$, with partition function $Z_\nu = Z_{n_b, \gamma}$. It is not restricted to D_r .
2. A normalized stopped measure at a depth $d_0 \in \mathbb{J}\mathbb{N}_0$,

$$d\nu = Z_\nu^{-1} \mathbf{1}_{D_r} e^{-E_{d_0, \text{sat}}/h_{n_b} - R_{d_0, \text{sat}}} dH_{n_b}, \quad Z_\nu = \int_{D_r} e^{-E_{d_0, \text{sat}}/h_{n_b} - R_{d_0, \text{sat}}} dH_{n_b},$$

as in Definition 4.4 and Theorem 14.6. Here $E_{d,\text{sat}}$ is the extension of the effective classical action E_d to D_ρ by gauge invariance (Theorem 11.6), and $R_{d,\text{sat}}$ is the corresponding effective interaction. The partition function Z_ν is not centered: the value $R_{d_0}(I)$ is kept in it.

Large-field decompositions. We use the partition of Theorem 21.5. Given an initial measure ν and a stopping level, the support of ν is partitioned into finitely many Borel sets $\mathcal{A}_\mathcal{X}$, indexed by large-field decompositions $\mathcal{X}$. A decomposition is produced by testing the plaquettes of the iterates $T_M^\ell V$ in increasing order of the level ℓ , and in a fixed order within one level. A plaquette f at level ℓ_f is a recorded bad plaquette of $\mathcal{X}$ if $q_f \geq d_*$ and its full dependence set avoids the set excluded before it. The variables of that dependence set that are not yet frozen are then frozen at the beginning of the RG step containing the level ℓ_f , together with a neighborhood, the excluded neighborhood of f , which is enlarged at each later RG step. We write $n_f = n_{\ell_f}$ for the side on which f is recorded and $x_f = n_f/\gamma$. Away from the excluded regions, the renormalization group is run on block regions, in RG steps of J averaging steps each, from the initial level to the stopping level; both levels are boundaries of RG steps. At the stopping level, of side m , the decomposition leaves a stopping field $c \in \text{SU}(2)^F$ on the set F of links of the block region at the stopping level. We write $H = \mathcal{E}_m \setminus F$ for the other links of the stopping lattice, including the links across the boundary of the block region. For $z \in \text{SU}(2)^H$, we write $c \oplus z$ for the configuration on the stopping lattice that equals c on F and z on H , and we call it a completion of c . The frozen coordinates b_{rec} consist of all frozen group variables, each kept on the lattice of the scale at which it was frozen. The map $V \mapsto \mathcal{O}_T^\mathcal{X}(V) = (c, b_{\text{rec}})$ records the stopping field and the frozen coordinates. For a nonnegative function Q and $s \geq 0$, we write

$$\eta_\mathcal{X} = (\mathcal{O}_T^\mathcal{X})_\#(\mathbf{1}_{\mathcal{A}_\mathcal{X}}\nu), \quad \eta_\mathcal{X}^s = (\mathcal{O}_T^\mathcal{X})_\#(e^{sQ}\mathbf{1}_{\mathcal{A}_\mathcal{X}}\nu).$$

For a Wilson initial measure, the first RG step of Theorem 21.5 separates the frozen group variables adjacent to the block region from those deep inside the excluded regions. The latter are arbitrary elements of $\text{SU}(2)$; they are never inserted into an analytic function and enter only through an exact Wilson factor.

The analytic estimates are applied at gauge representatives of the stopping field and the frozen coordinates that satisfy the domain conditions and the reconstruction identities of Theorem 21.13. They are obtained from the representatives with small frozen coordinates at the stopping level by applying the reconstruction maps of the successive RG steps, and they also satisfy the identities for the insertion of frozen coordinates stated there. Below, we call them representatives satisfying the reconstruction identities.

Table 22.1 lists the measures that occur in the proof of Theorem 22.1, with their total masses and normalizations. The measures $\eta_\mathcal{X}$, $\eta_\mathcal{X}^s$, and ζ_T are not renormalized; they keep the partition function Z_ν of the initial measure.

Table 22.1. Measures used in the proof of Theorem 22.1.

Measure	Space	Total mass	Normalization
Wilson initial measure	$\mathcal{U}_{n_b}$	1	$Z_{n_b, \gamma}$
$\mu_{n_b, \gamma}$			
Stopped initial measure	$D_r \subset \mathcal{U}_{n_b}$	1	Z_ν , not centered
$\eta_{\mathcal{X}}$	stopping fields and frozen coordinates	$\nu(\mathcal{A}_{\mathcal{X}}) \leq 1$	that of ν
$\eta_{\mathcal{X}}^s$	same as $\eta_{\mathcal{X}}$	$\mathbb{E}_\nu[e^{sQ}; \mathcal{A}_{\mathcal{X}}]$	that of ν
Reference measure ζ_T (Theorem 22.4)	$D_r \subset \mathcal{U}_m$	$m_T^{\mathbb{T}} \leq 1$	prefactor $Z_\nu^{-1} e^{\Lambda_T^{\mathbb{T}}}$
Comparison measure $\tilde{\sigma}_{\lambda, d, g}$ (Theorem 22.10)	$\mathcal{U}_m$	finite	$\hat{Z}_d(g)$, an integral over D_r
Frozen-coordinate measure $\rho_{\mathcal{X}} = w_{\mathcal{X}} H_{\mathcal{X}}$	frozen coordinates	≤ 1	Haar

Dependence of constants. The constants of the local analytic estimates have the dependence described in item (3) of Definition 4.14. They depend only on M, J , the larger construction domains and their radius ratios, the spatial and boundary decay rates, the density of variables, and the norm bounds. They are independent of L, γ , the lattice sides, the depth, the number of RG steps since a frozen coordinate was created, the shape and topology of the block regions, and the real values of the frozen coordinates in their domain. Where stated, they are also independent of the widths of the boundary layers and excluded neighborhoods chosen later; these widths enter only through explicit counts. The constants in the probability estimates have the dependence described in item (4) of Definition 4.14. They may in addition depend on L , a fixed coupling ceiling, the fixed parameters, and the finite family of stopping offsets, but not on the coupling below the ceiling, the sides, the depth, or the geometry, age, and frozen values of a decomposition. We claim no useful bound as $J \rightarrow \infty$.

Thresholds. Fix $d_{\text{cut}} > 0$ and set

$$d_* = d_{\text{cut}}/2, \quad c_E = 1 + C_M \pi \sqrt{2d_{\text{cut}}},$$

where δ_* and C_M are the constants of Theorem 22.4(a). For the chessboard estimate of Lemma 22.11 we assume $\pi \sqrt{2d_{\text{cut}}} \leq \delta_*$. In Theorem 22.1, these numbers enter only through the range of s and the definition of the recorded bad plaquettes.

The main estimate. The constants $K_M, K_1, \kappa_*, \kappa_0, c_A^{\text{act}}, \alpha_W$ in the following theorem are defined in Theorem 22.4; they depend only on M and on the fixed classical data.

Theorem 22.1. Fix $L > 0$, an odd integer $M \geq 3$, an integer $J \geq 1$, and $B = M^J$, and fix parameters satisfying the hypotheses of Theorems 20.14, 21.5, and 21.13. These include the coefficient-norm bounds at initialization, both substitution bounds (22.6), the boundary-distance comparison, and the product domains verified in Lemma 22.3. The cutoff amplitude A_W is fixed before the coupling ceiling $g_* < 1$. Fix a finite nonempty set $\mathcal{R} \subset \mathbb{N}_0$ of stopping offsets, all remainders $0 \leq u_{\text{ph}} < J$, the buffers containing their full dependence sets, and the bound F_J of Theorem 21.5 for the number of affected source plaquettes. Fix $\chi_* \geq 1$ such that the ratio of every stopping side of this family to the requested side n_{j-1} is at most χ_* , and such that $n_b/n_{j-1} \leq \chi_*$ whenever the stopping level lies at or below the initial level. Every iterate of T_M that is used must belong to the given finite hierarchy, and only full RG steps are integrated.

Let $K_M, K_1, \kappa_*, \kappa_0, c_A^{\text{act}}, \alpha_W$ be as in Theorem 22.4, and put $K_W = 48K_1/\alpha_W$ and $\theta = 1/2$. Choose $s_0 > 0$, after F_J , such that

$$s_0 \leq \min \left\{ \frac{1}{2}, \frac{\kappa_*}{2} c_E^{-(J-1)}, \frac{1}{4K_W}, \frac{\theta c_A^{\text{act}} d_*}{16F_J}, \frac{\kappa_0}{2K_M} \right\}.$$

Any further fixed decrease of s_0 is allowed. Then for every $0 < \Gamma < \infty$ there exist $n_{\min} \in 4\mathbb{N}$ and finite nonnegative constants C_v, C_s, C_b with $C_s \geq 9\chi_*^2$, chosen before the coupling, the side, and the depth, such that the following holds.

Let $0 < \gamma \leq \Gamma$, $u \in 4\mathbb{N}$ with $u \geq n_{\min}$, $K \in \mathbb{N}_0$, $N = uB^K$, $n_i = N/M^i$, $1 \leq j \leq JK$, and $b \in J\mathbb{N}_0$ with $0 \leq b \leq j-1$. Let ν be either the Wilson measure $\mu_{n_b, \gamma}$ or a normalized stopped measure on $\mathcal{U}_{n_b}$ at a depth aJ , $a \in \mathbb{N}_0$. Then, for each of the eight tilings $\mathcal{T}$ and every $0 \leq s \leq s_0$,

$$\log \mathbb{E}_\nu \exp \{ s Q_j^\mathcal{T} (T_M^{j-1-b} V) \} \leq 3C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b.$$

This covers every remainder u_{ph} , stopping levels at or below the initial level, an empty block region at the stopping level, and decompositions whose block region becomes empty before the stopping level and which are continued by empty RG steps.

More precisely, a recorded bad plaquette f created at side n_f contributes at least $P_0 x_f$ to the classical part of the exponent, where $x_f = n_f/\gamma$ and $P_0 = 4\theta c_A^{\text{act}} d_*$. The plaquette-energy sum costs at most $8sF_J x_f \leq P_0 x_f/4$ per recorded bad plaquette, and all the corrections assigned to f , from the normalization constants, the missing coordinates, the boundary bands, the substitutions, and the integration over the omitted links, are at most

$$C_{\text{ch}, J}^\Gamma (1 + \log x_f)^2 \leq P_0 x_f/4.$$

Here one may take $C_{\text{ch}, J}^\Gamma = (1 + \log_+ \Gamma) C_{\text{ch}, J}^{\text{base}}$, where $C_{\text{ch}, J}^{\text{base}}$ depends only on the fixed parameters and the family of stopping offsets. Each decomposition retains the factor $\prod_f e^{-a_{\text{act}} n_f/\gamma}$, with $a_{\text{act}} = P_0/2$, and the sum of these factors over all decompositions is at most

$$\exp \left\{ \sum_\ell 3n_\ell^3 e^{-a_{\text{act}} n_\ell/\gamma} \right\},$$

which is bounded uniformly in the depth once $u \geq n_{\min}$. At a stopping side $m = \chi n_{j-1}$ one has $g_m = g_{n_{j-1}} / \sqrt{\chi}$, and the only surface term is $9\chi^2 n_{j-1}^2 \log(1/g_{n_{j-1}})$.

The constants $n_{\min}, C_v, C_s, C_b$ have the dependence described in item (4) of Definition 4.14, with the fixed ceiling Γ , the fixed parameters, and the finite family of stopping offsets and remainders. They are independent of the coupling $\gamma \in (0, \Gamma]$, the sides, the depth, the position and the length $j - 1 - b$ of the range of scales, and the age, shape, topology, and frozen values of the decompositions. The constants of the local analytic estimates keep the dependence described in item (3) of Definition 4.14; the widths enter only through the displayed counts.

Remark 22.2. There is no restriction on $j - b$, and the expectation contains no indicator of a cell condition or of a condition of the small-field event. Taking $b = 0$ and $\nu = \mu_{N, \gamma}$ gives the estimate under the Wilson measure with its own partition function. The proof does not use the bounds on the Wilson measure at successive scales of Section 23, where the theorem is used in Theorems 23.2 and 23.6.

Outline of the proof. Fix j, b , and $\mathcal{T}$, and write $n = n_{j-1}$. If the planned stopping level lies at or below the initial level, then only a bounded number of averaging steps separates the initial lattice from level $j - 1$, and the estimate follows from the partition-function bounds on the initial lattice. Otherwise we integrate full RG steps from level b to a stopping level $t_{\text{stop}} \leq j - 1$ and keep the remaining $v = j - 1 - t_{\text{stop}}$ averaging steps inside the function $Q_j^{\mathcal{T}} \circ T_M^v$.

Step 1: classical lower bounds. Along a decomposition $\mathcal{X}$, the density of $\eta_{\mathcal{X}}$ at a representative satisfying the reconstruction identities is bounded by e^{-E_T/g_m^2} times normalization factors, where E_T is the effective classical action of the stopping field with all external classical exponents included (Theorem 21.13). We reconstruct one configuration U_* on the initial lattice with zero fluctuations. Using an energy transport estimate that is uniform in the number of averaging steps, we show that E_T/g_m^2 dominates both $\Psi_F(c)/g_m^2$, where $\Psi_F(c)$ is the minimum of $\alpha_W S_W(c \oplus z)$ over all values z of the links in H , and $A_{\mathcal{X}} = c_A^{\text{act}} \sum_f 4q_f/h_{n_f} \geq 4c_A^{\text{act}} d_* \sum_f x_f$ (Theorem 22.4). Since both lower bounds are gauge invariant, the density bound extends to all points (Theorem 22.10).

Step 2: the plaquette-energy sum. At every point of $\mathcal{A}_{\mathcal{X}}$, the plaquette-energy sum is at most $K_W \Psi_F(c)/g_m^2 + 8F_J \sum_f x_f$ (Lemma 22.8). Multiplication by e^{sQ} therefore leaves the Wilson factor $e^{-\lambda_s \Psi_F(c)/g_m^2}$ with $\lambda_s = 1/2 - sK_W \geq 1/4$, bounded above by $e^{-\Psi_F(c)/(4g_m^2)}$, together with $e^{8sF_J \sum_f x_f}$. Replacing the minimum by a Haar integral over the links in H adds $3|H| \log(1/(c_* g_m)) + 8\lambda_s \alpha_W |H|$ to the exponent (Lemma 22.7).

Step 3: summation over decompositions. Each recorded bad plaquette contributes $-P_0 x_f$ to the exponent. All normalization corrections, boundary terms, substitution terms, and the correction from the integration over the omitted links can be assigned to recorded bad plaquettes, and the corrections assigned to f are at most $C(1 + \log x_f)^2$. For a large minimum side, they are absorbed by $P_0 x_f/4$. This leaves a factor $e^{-P_0 x_f/2}$ per recorded

bad plaquette, and the sum over decompositions converges uniformly in the depth. The remaining ratio of a Wilson integral to the reference partition function on the stopping lattice is at most $e^{Cm^3+9m^2\log(1/g_m)}$ (see Remark 22.5).

The number of ultraviolet scales causes no accumulated loss, for the following reason. A recorded bad plaquette created at side n_f has $q_f \geq d_*$, and therefore contributes at least a fixed multiple of $x_f = n_f/\gamma$ to the classical action. At each later RG step, its excluded neighborhood contains a number of variables bounded independently of the scale, and there are at most $O(1 + \log n_f)$ later RG steps. Each missing Gaussian coordinate changes the normalization by $\log(1/g_i) = \log(n_i/\gamma)/2 \leq (\log x_f)/2 + O(1)$, and each other correction is bounded per variable. The total correction is therefore $O_\Gamma((1 + \log x_f)^2)$, which is small compared to x_f once x_f is large.

We distinguish two sets attached to a recorded bad plaquette f ; see Figure 22.1. The first is its energy-estimate predecessor set $\mathcal{B}_f$ on the initial lattice, which carries the classical lower bound for q_f . These sets are disjoint for recorded bad plaquettes at different levels and overlap at most 48 times at one level. The second is the full excluded neighborhood assigned to f , which is much larger and may overlap with the neighborhoods of other recorded bad plaquettes. Source plaquettes whose full dependence sets meet such a neighborhood are discarded, and their number per recorded bad plaquette is bounded by F_f .

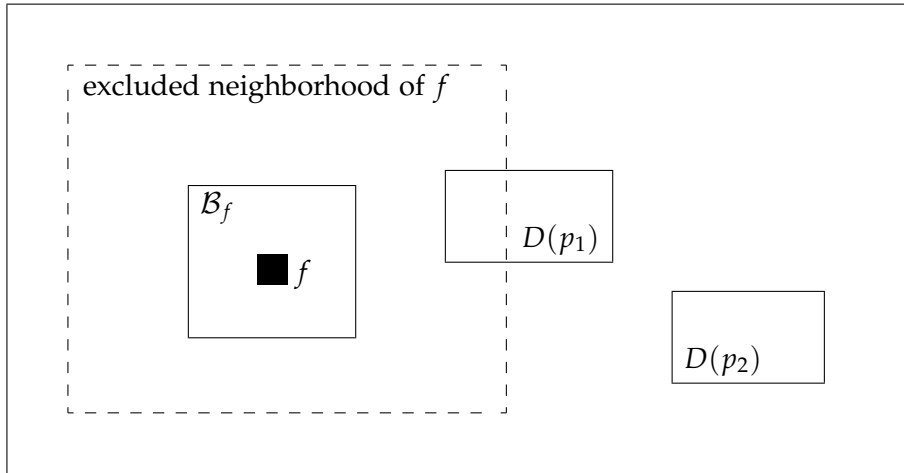


Figure 22.1. Schematic picture of the sets attached to a recorded bad plaquette f , drawn at one scale. The energy-estimate predecessor set $\mathcal{B}_f$ carries the lower bound for the action. The source plaquette p_1 is discarded because its full dependence set $D(p_1)$ meets the excluded neighborhood of f ; the source plaquette p_2 is kept because $D(p_2)$ consists only of links of the block region at the stopping level.

22.2 Product domains along a large-field decomposition

The following lemma checks that the regional integration estimates of Section 20, on which the density bounds of Theorem 21.13 rest, apply along an arbitrary large-field decomposition. Only the small frozen coordinates are substituted into analytic functions (Theorems 21.5 and 21.13).

Lemma 22.3. *Fix $A_w \geq 1$ first, and then fix parameters satisfying the hypotheses of Theorems 20.14 and 21.5, including the product domains and the coefficient-norm bounds at initialization in Theorem 21.13. Assume also the construction and reconstruction-image hypotheses of that theorem, in particular $r + \beta_{\text{rec}} < \epsilon_{\text{en},J}$. Let $\mathcal{X}$ be a large-field decomposition whose initial and stopping levels are boundaries of RG steps, with $T \geq 1$ integrated RG steps. Then every integrated step consists of J averaging steps, the total number of integrated averaging steps is JT , and Theorems 20.13 and 20.14 and Lemma 20.5 apply at the gauge representatives of $\mathcal{X}$ satisfying the domain conditions and reconstruction identities of Theorem 21.13, with the frozen coordinates satisfying the bounds of Theorem 21.5. In each such application the fluctuation cutoff domain is*

$$\mathcal{W}_t = \prod_{i=0}^{J-1} \prod_e \{|w_{i,e}| \leq p(t_i)\}, \quad t_t = \max_i g_{t,i}, \quad p(t) = A_w \log(1/t), \quad (22.3)$$

where $g_{t,i}$ is the coupling of the i -th averaging step of the t -th RG step. The coupling ceiling may depend on the fixed amplitude A_w ; this does not change the constants of the local analytic estimates or the bounds that are uniform before J is chosen. The conclusion is uniform in the number of RG steps since the creation of the frozen coordinates and in the real values of the frozen coordinates in their stated domain, including the first step for a Wilson initial measure. No derivative with respect to a frozen coordinate is involved.

Proof. The cutoff amplitude is fixed before the coupling ceiling of Theorem 20.14 is chosen. By Theorem 21.13, every integrated step has J averaging steps, and (22.3) holds. If m_t is the output side of an RG step, then

$$t_t^2 = \gamma / (M m_t), \quad g_{t,i} = t_t M^{(i-J+1)/2} \quad (0 \leq i < J).$$

In particular, the radii $p_{t,i} = p(t_t)$ used in the small-field event belong to the family of radii allowed in Theorem 20.14,

$$p(t_t) \leq p_{t,i} \leq p(t_t) + c_J^{\log}, \quad c_J^{\log} = \frac{1}{2} A_w (J-1) \log M.$$

The larger enclosing product domain is used only in the comparison of Lemma 20.5; its radii do not replace the radii $p(t_t)$.

We now check the conditions on the coupling for this logarithmic family, which contains the common radius as a special case. Let $p^+(t) = p(t) + c_J^{\log}$. The condition that

the cutoff domain lies in the enclosing product domain is $tp^+(t) < R_{\text{en}}$. With the local constants of Theorem 20.14, and with $E + \sigma_*$ for the norm bound of the input effective interaction, the small-interaction condition and the condition on the auxiliary complex parameter domain required there read

$$C_J tp^+(t) \leq \frac{1}{2}, \quad \frac{C_J t[(p^+(t))^3 + p^+(t) + (E + \sigma_*)p^+(t)]}{1 - C_J tp^+(t)} \leq \epsilon_J, \quad (22.4)$$

$$2t[(p^+(t))^3 + p^+(t) + (E + \sigma_*)p^+(t)] \leq c_{z,J}.$$

These are the expansion conditions of Theorem 20.14, and by the same choice of parameters the denominators of its rational image bounds are positive on the outer product domains. With $v = \log(1/t)$, every term of the form $t^k(p^+(t))^q$ with $k > 0$ equals $e^{-kv}(A_w v + c_J^{\log})^q$, which tends to zero as $v \rightarrow \infty$. Hence one fixed ceiling on t gives the containment, all the inequalities (22.4), and the required smallness of the perturbative term

$$C_J t^2(E + \sigma_*)[(p^+(t))^4 + (E + \sigma_*)(p^+(t))^2].$$

This ceiling is one of the fixed parameters. Since $g_* < e^{-1}$ and $A_w \geq 1$, the radii $p(t_t)$ are at least one. At each fixed value of the frozen coordinates, the bound for the conditional mean in Lemma 20.5 stays uniformly below the cutoff radius: the ratio of the mean bound to $p(t)$ is at most $C_1 t(\mathcal{L}_V + 2)/p(t)$, which tends to zero. Together with the containment, this verifies the hypotheses of the tail estimate of Lemma 20.5 on the chosen cutoff domain.

We also verify the measure-theoretic hypotheses of that comparison, at every fixed real value of the frozen coordinates and every fixed coarse background. The enclosing product domain $\mathcal{K}_t$ is a finite product of closed balls of positive radius in $\mathbb{R}^3$, so it is Borel, compact, and of finite positive Lebesgue volume. The conditional fluctuation potential V given by Theorem 20.14 is real and of class C^2 on a neighborhood of $\mathcal{K}_t$, because the image bounds keep its inverse maps, Jacobians, and positive Hessian determinants within their domains. Thus there exist finite real numbers $v_- \leq v_+$, which may depend on the fixed background and frozen coordinates, such that $v_- \leq V \leq v_+$ on $\mathcal{K}_t$. In particular,

$$0 < e^{-v_+} |\mathcal{K}_t| \leq Z_{\mathcal{K}_t} = \int_{\mathcal{K}_t} e^{-V} dw \leq e^{-v_-} |\mathcal{K}_t| < \infty. \quad (22.5)$$

The set S of supported fluctuation coordinates is the inverse image under a Borel coordinate map of the finitely many tests of the decomposition and of the coordinate domains, and the cutoff domain $\mathcal{W}_t$ is defined by finitely many inequalities; both are Borel. Every indicator or optional test $0 \leq \chi \leq 1$ that is kept is Borel, and e^{-V} is positive and Borel. The Haar disintegration, Gaussian, and constant prefactors are identical on both sides of the comparison. Since these checks are made at each fixed preceding output point and fixed frozen coordinates, no uniform bound on $v_{\pm}$ is needed. If there are no fluctuation

blocks, the product domain is a point of mass one; its partition function is positive and finite, its determinant is one, and its logarithmic determinant is zero.

For the substitution of frozen coordinates into analytic functions, Theorem 21.5 includes the initial bound for finite products of links and all conversion constants in $C_{\text{conv},J} \geq C_{\text{ad},J}$. At the representatives of the lemma, we have, at each scale at which variables are frozen and for both spatially varying radii $\phi_{\text{cl}}, \phi_{\text{q}}$,

$$|c_j^\#| \leq \min\{\beta_{\text{rec}}, \phi_\bullet(j)/4\}, \quad C_{\text{bc},\bullet} E_\bullet \beta_{\text{rec}} / \rho_\bullet \leq \sigma_*/4, \quad \bullet \in \{\text{cl}, \text{q}\}. \quad (22.6)$$

For a frozen group variable b_j kept on the lattice where it was frozen, we have $c_j^\# = \text{Log } b_j$ in the identity chart, and hence $|c_j^\#| = \rho_G(b_j)$. A longer fixed product of links uses the conversion included in $C_{\text{conv},J}$, which is fixed before β_{rec} and r . At initialization, for a stopped initial measure, we use the bound by r coming from D_r , together with $C_{\text{ad},J}r < \beta_{\text{rec}}$; for a Wilson initial measure, the bound for the frozen variables adjacent to the block region is the other initialization clause of Theorem 21.5. The bound (22.6) covers all frozen variables, including the boundary band, the boundary neighborhoods, the links across the boundary of the block region, and the initialization, and its constant $C_{\text{bc},\bullet}$ includes the sum over all these types. The arbitrary frozen group variables deep inside excluded regions are fixed external parameters and are not inserted into analytic functions.

All variables in the band of reduced analytic radius are frozen. Thus every variable that remains in the incoming block region $D_{\text{in},t}$ has the classical radius ρ_{cl} , and the incoming norm of the classical perturbation in the coefficient-norm bounds is evaluated at that radius; the spatially varying radius is used only for the output. By Theorem 21.5, the old fixed complement and the points of creation of frozen variables remain in the fixed complement of the next block region, up to the fixed buffer of the finite link products. In the current units, the incoming and current distances to the fixed complement satisfy

$$d_{D_{\text{in},t}}(i) \geq B d_{\text{current}}(e) - B \text{dist}(e, i/B) - C_{\text{buf},J},$$

where i is the point of creation of an incoming variable, e is a current point, and $C_{\text{buf},J}$ is the fixed buffer constant. This is the boundary-distance comparison required in Theorem 21.13 (compare Theorem 19.9); enlarging the fixed complement preserves the inclusion, and at initialization there is no boundary. The uniform bound on the conditional force kept in that induction is the following condition on the same product domains:

$$\begin{aligned} \delta_{V,t} &= \max_I \sup_{y \in \mathcal{T}_I} \|A_I^{-1} V_I(c, a_I(c, y; \xi), y)\|_{\text{coeff}} \\ C_{\text{seq},J}(2\delta_{V,t} + r_q) &< \frac{1}{2}\varepsilon_{\text{tube}}. \end{aligned} \quad (22.7)$$

Here I runs over the finitely many initial segments of the conditional integrations, A_I is the positive reference block of the segment I , a_I is its conditional constrained minimizer

without perturbation, V_I is the z_I -derivative, in the original free coordinates, of the added force V of (21.12), $\mathcal{T}_I$ is the domain of the remaining parameters, and r_q is the fixed radius of the free coordinates. The coefficient norm includes the common product domain of the coarse fields, uniformly at fixed real frozen coordinates. The condition (22.7) is a hypothesis on the product domains in Theorem 21.13 (see Lemma 19.3), uniform on the prescribed ball of classical perturbations; it is part of the fixed parameters and is verified in Section 23, see (23.29). The supremum is over every remaining parameter used in the sequential construction, not only over the point with zero fluctuations, and all real frozen coordinates are held fixed. Hence the image bounds and the sequential containment used in the induction hold on their stated domains.

Finally, $x_0, x_0^{\text{cl}} \leq \sigma_*$ are the coefficient-norm hypotheses of the induction at initialization, including the substitution at initialization. Its constants are those of the perturbed maps: $K_{\text{col},J}$ in Theorem 20.14 bounds the columns of the perturbed zero-fluctuation map uniformly on the larger perturbation domain, and we use the resulting K_J in the output bound of that theorem and in its choice of the width s_{er} of the boundary layer removed at each step. By the substitution bounds (22.6), we have $z_t, z_t^{\text{cl}} \leq \sigma_*/4$, and the choice of s_{er} gives $2e^{-\alpha(s_{\text{er}}-1)}y_t \leq \sigma_*/2$, with the same inequality for the classical output bound. Thus both incoming norms at the next boundary are at most $3\sigma_*/4 < \sigma_*$, and by induction the coefficient-norm hypothesis holds after any number of RG steps. Since every integrated step is a full RG step, Theorem 20.13 applies at every step, including its incoming value and the initialization.

The sets of such representatives in Theorem 21.13 start from the representatives at which the bounds on the frozen coordinates just checked hold, and they contain every supported fiber point over each preceding output point. A gauge transformation acting independently at the endpoints of the frozen variables need not preserve the bound on their logarithms. We therefore use analytic estimates only on these sets, and we use the pointwise gauge covariance of Theorem 21.10 only after a gauge-invariant upper bound has been obtained. This verifies the hypotheses for both initial measures. $\square$

22.3 Energy transport, feasible configurations, and the reference normalization

The next theorem collects the classical estimates: (a) is used in Sections 22.5 and 22.8, (b) in Section 22.6, and (c) in Section 22.7.

Theorem 22.4. *Fix an odd integer $M \geq 3$.*

- (a) There are $\delta_*(M) > 0$ and $C_M < \infty$, depending only on M , such that for every level i , every set $\mathcal{P}'$ of positive plaquettes at level $i + 1$, every configuration V at level i , and every δ ,

$$\max_{p \in \text{Pa}(\mathcal{P}')} \rho_G(V_p) \leq \delta \leq \delta_* \implies \mathcal{S}_{i+1}^{\mathcal{P}'}(T_M V) \leq (1 + C_M \delta) \mathcal{S}_i^{\text{Pa}(\mathcal{P}')} (V).$$

Under this hypothesis no based loop holonomy in these predecessor rectangles equals $-I$, so all the logarithms in the definition of T_M there are the analytic ones. Define

$$B_* = \min \left\{ \frac{\delta_*^2}{\pi^2}, \left[\frac{(\sqrt{M} - 1) \log 2}{\pi C_M} \right]^2 \right\}, \quad \sigma(b) = \frac{\pi C_M \sqrt{b}}{\sqrt{M} - 1}.$$

Let $\ell \geq 0$ be such that the side at level ℓ is at least four, let $\mathcal{P}_\ell$ be a set of positive plaquettes at level ℓ , and let $\mathcal{P}_i = \text{Pa}(\mathcal{P}_{i+1})$ for $0 \leq i < \ell$; we call the $\mathcal{P}_i$ the energy-estimate predecessor sets of $\mathcal{P}_\ell$. Then for every configuration U at level 0 and every $0 < b \leq B_*$,

$$\mathcal{S}_\ell^{\mathcal{P}_\ell}(T_M^\ell U) \leq \max\{e^{\sigma(b)}, 8|\mathcal{P}_\ell|/b\} \mathcal{S}_0^{\mathcal{P}_0}(U). \quad (22.8)$$

This holds for every holonomy and with the Borel extension of T_M . If $\mathcal{S}_0^{\mathcal{P}_0}(U) \leq b/h_{n_\ell}$, then for $0 \leq i \leq \ell$ every plaquette in $\mathcal{P}_i$ has angle at most $\pi\sqrt{b} M^{-(\ell-i)/2}$, and the factor in (22.8) can be replaced by $e^{\sigma(b)} \leq 2$. If $q < d_{\text{cut}}$ for every plaquette in $\mathcal{P}_\ell$, then $8|\mathcal{P}_\ell|/b$ may be replaced by $4|\mathcal{P}_\ell|d_{\text{cut}}/b$. Let $p_M \leq 24M^3$ be the largest number of plaquettes in a set $\mathcal{P}_{j-1}(C)$, and put

$$K_M = \max(2, 8p_M/B_*), \quad K_1 = \max(2, 8/B_*).$$

Then for $0 \leq k \leq j - 1$ and every configuration V at level k , and for every depth ℓ ,

$$\mathcal{Q}_j^T(T_M^{j-1-k} V) \leq K_M \frac{S_W(V)}{h_{n_k}}, \quad \frac{S_W(T_M^\ell U)}{h_{n_\ell}} \leq 48K_1 \frac{S_W(U)}{h_{n_0}}.$$

These constants are independent of the depth, the side, J , L , γ , and the geometry. The factor K_M may be replaced by 2 if $q < d_{\text{cut}}$ for all the relevant plaquettes at level $j - 1$ and $4p_M d_{\text{cut}} \leq B_*$.

- (b) Let $\mathcal{X}$ be a large-field decomposition as in Theorem 21.13, with ℓ integrated averaging steps and stopping side m , and let $g_{\text{base}}^2 = \gamma/n_b$ be the coupling of the initial lattice. At every representative $(\widehat{c}, \widehat{b_{\text{rec}}})$ satisfying the domain conditions and reconstruction identities of Theorem 21.13, the zero-fluctuation reconstructions of the successive RG steps, with the frozen coordinates inserted at the scales at which they were frozen, give one configuration U_* on the initial lattice such that

$$(T_M^\ell U_*)|_F = \widehat{c}, \quad \frac{E_T(\widehat{c}; \widehat{b_{\text{rec}}})}{g_m^2} = \frac{\mathcal{E}_{\text{base}}(U_*)}{g_{\text{base}}^2}, \quad g_m^2 = M^\ell g_{\text{base}}^2.$$

Here E_T is the effective classical action of the stopping field after all external classical exponents have been incorporated once into the classical exponent. For a stopped initial measure, $U_* \in D_\rho$ and $\mathcal{E}_{\text{base}} = E_{d_0, \text{sat}}$ on this larger gauge-invariant domain; for a Wilson initial measure, U_* is a configuration in $\mathcal{U}_{n_b}$ and $\mathcal{E}_{\text{base}} = S_W$. Consequently

$$E_T(\widehat{c}; \widehat{b_{\text{rec}}}) \geq M^\ell \inf\{\mathcal{E}_{\text{base}}(U) : U \in \mathcal{S}, (T_M^\ell U)|_F = \widehat{c}\},$$

where $\mathcal{S} = D_\rho$ or $\mathcal{S} = \mathcal{U}_{n_b}$, respectively. No minimizer in D_r is asserted.

Let $\kappa_* > 0$ be such that $E_{d, \text{sat}} \geq \kappa_* S_W$ on D_ρ for every depth $d \geq 0$ (Theorems 11.2 and 11.6), and set

$$\kappa_0 = \min(1, \kappa_*), \quad c_A^{\text{act}} = \alpha_W = \frac{\kappa_0}{48K_1}, \quad \mathcal{V}_m = \alpha_W S_W.$$

The two names c_A^{act} and α_W denote the same number, in its two roles below. For $c \in \text{SU}(2)^F$ and for frozen coordinates b_{rec} put

$$\Psi_F(c) = \mathcal{V}_{m;F}(c) = \min_{z \in \text{SU}(2)^H} \mathcal{V}_m(c \oplus z), \quad A_{\mathcal{X}}(b_{\text{rec}}) = c_A^{\text{act}} \sum_{f \in \mathcal{X}} \frac{4q_f(b_{\text{rec}})}{h_{n_f}},$$

where $q_f(b_{\text{rec}})$ is the energy of the recorded bad plaquette f , a function of the frozen coordinates because the full dependence set of f is frozen. Then, at the same representative,

$$\frac{E_T}{g_m^2} \geq \frac{\Psi_F(\widehat{c})}{g_m^2}, \quad \frac{E_T}{g_m^2} \geq A_{\mathcal{X}}(\widehat{b_{\text{rec}}}), \quad (22.9)$$

and

$$A_{\mathcal{X}}(b_{\text{rec}}) \geq 4c_A^{\text{act}} d_* \sum_{f \in \mathcal{X}} \frac{n_f}{\gamma} \quad \text{on } I_{\mathcal{X}}^{\text{act}} = \{q_f(b_{\text{rec}}) \geq d_* \text{ for all } f \in \mathcal{X}\}.$$

- (c) Use the full-torus recursion of the stopped measures (Theorem 14.6) with the same RG steps as the decomposition; we call it the reference recursion. At a depth $d \geq 0$ of the reference recursion on the stopping lattice put

$$\widehat{Z}_d(g) = \int_{D_r} e^{-E_{d, \text{sat}}/g^2 - [R_{d, \text{sat}} - R_d(I)]} dH_m, \quad g^2 = \gamma/m.$$

The restriction of the initial measure to the small-field conditions of the reference recursion and its pushforward to the stopping lattice has the form

$$d\tilde{\zeta}_T = Z_v^{-1} e^{\Lambda_T^{\mathbb{T}}} \mathbf{1}_{D_r} e^{-E_{d, \text{sat}}/g^2 - [R_{d, \text{sat}} - R_d(I)]} dH_m,$$

where $\Lambda_T^{\mathbb{T}}$ is the normalization constant of the reference recursion; for a Wilson initial measure, only this reference measure is restricted to D_r at the start. With $m_T^{\mathbb{T}} = \tilde{\zeta}_T(1) \leq 1$, one has exactly

$$\frac{e^{\Lambda_T^{\mathbb{T}}}}{Z_v} = \frac{m_T^{\mathbb{T}}}{\widehat{Z}_d(g)} \leq \widehat{Z}_d(g)^{-1}. \quad (22.10)$$

All Haar disintegration, Gaussian, cluster expansion, and substitution constants are included in $\Lambda_T^\mathbb{T}$, and for a stopped initial measure $\Lambda_0^\mathbb{T} = -R_{d_0}(I)$.

For every fixed $0 < \lambda_0 \leq \lambda_1 < \infty$ there are $C, C', C'' < \infty$, uniform in the depth d of the reference recursion, in m , and in $0 < g \leq g_* < 1$, such that

$$\widehat{Z}_d(g) \geq e^{-C3m^3} g^{3(2m^3+1)}, \quad \int e^{-\lambda \mathcal{V}_m/g^2} dH_m \leq e^{C'3m^3} g^{3(2m^3-3m^2+1)}$$

for $\lambda \in [\lambda_0, \lambda_1]$ with $4\lambda\alpha_W/g^2 \geq 1$. Consequently the logarithm of the ratio of these two integrals is at most $C''3m^3 + 9m^2 \log(1/g)$. The normalized stopped measure $\widehat{\nu}$ on the stopping lattice, with the same support D_r , satisfies

$$\log \mathbb{E}_{\widehat{\nu}} e^{vS_W/g^2} \leq C''3m^3 + 9m^2 \log(1/g), \quad 0 \leq v \leq \kappa_*/2.$$

For fixed $0 < \lambda_0 \leq 1$, the quantity

$$\widehat{Z}_d(g)^{-1} \int_{D_r} e^{-\lambda(E_{d,\text{sat}}/g^2 + R_{d,\text{sat}} - R_d(I)) + vS_W/g^2} dH_m, \quad \lambda \in [\lambda_0, 1], \quad 0 \leq v \leq \lambda_0 \kappa_*/2,$$

satisfies the same bound with a larger volume constant. In each of these ratios the boundary correction $9m^2 \log(1/g)$ occurs once. The denominator is always the integral over exactly D_r ; the periodic boundary plaquettes are deleted only in the upper bound for the numerator.

The constants of the reconstruction and of the normalization constants have the dependence described in item (3) of Definition 4.14, excluding the widths chosen later. The constants in (c) depend on the fixed domains on the torus, r , the norm bounds, and the interval $[\lambda_0, \lambda_1]$ of action coefficients, not on the depth, the side, or the coupling. Applications to probability estimates have the dependence described in item (4) of Definition 4.14.

Remark 22.5. The factor g^{-9m^2} in (c) comes from a count of independent coordinates. The periodic lattice of side m has $3m^3$ links and m^3 sites, so the complement of a spanning tree has $2m^3 + 1$ links; restricting each of them to a ball of radius proportional to g in $\text{SU}(2)$, a group of dimension three, gives the lower bound $e^{-3Cm^3} g^{3(2m^3+1)}$ for $\widehat{Z}_d(g)$, with the constant of part (c). The open cube with m sites per side, used for the numerator, has $3m^2(m-1)$ links and m^3 sites, hence $2m^3 - 3m^2 + 1$ links outside a spanning tree, which give the upper bound $e^{3C'm^3} g^{3(2m^3-3m^2+1)}$, with the constant of part (c). The difference $3[(2m^3 + 1) - (2m^3 - 3m^2 + 1)] = 9m^2$ is a boundary correction in an inequality; it is not an identification of a physical quantity. At a side $m = \chi n$ it becomes $9\chi^2 n^2 \log(1/g_n) + (9/2)\chi^2 n^2 \log \chi$. Gauge fixing along a tree, deleting nonnegative plaquette terms, and factorizing one-plaquette integrals are elementary stability arguments in the tradition of Faria da Veiga and O'Carroll [64, 65]; the precise periodic and open counts used here are proved in Theorem 14.8, with the open-cube estimate of Lemma 6.10.

Proof of Theorem 22.4(a). We write $h_i = h_{n_i}$ within this proof.

Transport over one averaging step. Multiplying the restricted conclusion of Theorem 7.3 by four and dividing by $h_{i+1} = Mh_i$, we obtain the displayed implication. Its hypotheses are local: the maximum is over the union of the complete predecessor rectangles, and the side at level $i + 1$ is at least four. We may decrease $\delta_*(M)$ so that the exceptional based loop holonomies are excluded as well. To check this, note that a contour $F_{c,x}C_c^{-1}$ in (4.3) is filled by moving its straight segment of length M across the coordinate segments of the path $\sigma_{v,x}$. At most $M \sum_{k=1}^3 |x_k - v_k| \leq 3M(M-1)/2$ elementary moves across one plaquette are needed, and each move inserts a conjugate of a plaquette transport or of its inverse. All these plaquettes belong to the complete predecessor rectangle, so by the bi-invariant triangle inequality the distance of this based loop holonomy from I is at most $3M^2\delta$. If we also require $3M^2\delta_* < 2\pi$, then this holonomy is not $-I$, and its logarithm is the analytic one. This decrease of δ_* changes only a constant depending on M .

Transport over many averaging steps. Let $V^i = T_M^i U$ and write

$$e_i = \mathcal{S}_i^{\mathcal{P}_i}(V^i), \quad \delta_i = \max_{p \in \mathcal{P}_i} \rho_G(V_p^i), \quad \Pi_i = \prod_{k < i} (1 + C_M \delta_k).$$

For an empty set of plaquettes we set the maximum to zero; the corresponding energy is also zero. Suppose first that $e_0 \leq b/h_\ell$. We prove by induction on $i \leq \ell$ that the first i steps satisfy

$$e_i \leq \Pi_i e_0, \quad \Pi_i \leq \exp \left\{ \pi C_M \sqrt{b} \sum_{k < i} M^{-(\ell-k)/2} \right\} \leq 2. \quad (22.11)$$

The case $i = 0$ is immediate. Suppose that (22.11) holds for some i . Since all plaquette energies are nonnegative, (22.1) implies that, for every $p \in \mathcal{P}_i$,

$$\rho_G(V_p^i)^2 \leq 2\pi^2 q(V_p^i) \leq \frac{\pi^2 h_i}{2} e_i \leq \frac{\pi^2}{2} \Pi_i b M^{-(\ell-i)} \leq \pi^2 b M^{-(\ell-i)},$$

where we used $h_i/h_\ell = M^{-(\ell-i)}$ and $\Pi_i \leq 2$. For $i < \ell$ this is at most $\pi^2 b \leq \delta_*^2$. Then the transport over one averaging step applies with $\text{Pa}(\mathcal{P}_{i+1}) = \mathcal{P}_i$; it gives $e_{i+1} \leq (1 + C_M \delta_i) e_i$ and excludes all exceptional values used in that step. Further,

$$\log \Pi_{i+1} \leq C_M \sum_{k \leq i} \delta_k \leq \pi C_M \sqrt{b} \sum_{r \geq 1} M^{-r/2} = \sigma(b) \leq \log 2.$$

This closes the induction (22.11). The bound on individual plaquettes also holds at $i = \ell$, and no further transport step is needed there. This proves all the stated conclusions under the hypothesis $e_0 \leq b/h_\ell$, including the case $\ell = 0$.

If instead $e_0 > b/h_\ell$, then since $q \leq 2$, we have $e_\ell \leq 8|\mathcal{P}_\ell|/h_\ell < (8|\mathcal{P}_\ell|/b)e_0$. If every plaquette of $\mathcal{P}_\ell$ has $q < d_{\text{cut}}$, the upper bound is $4|\mathcal{P}_\ell|d_{\text{cut}}/h_\ell$. These two cases prove (22.8) for arbitrary configurations. Since no logarithm is taken in the second case, the exceptional values of the Borel extension cause no difficulty. If $q < d_{\text{cut}}$ for all the relevant

plaquettes and $4p_M d_{\text{cut}} \leq B_*$, we may choose $b = 4p_M d_{\text{cut}}$; then for every cell the second factor in (22.8) is at most one and the first is at most two.

Cells and all plaquettes. We apply (22.8) with $b = B_*$ separately to the set $\mathcal{P}_{j-1}(C)$ of each cell, with level k in place of level 0. Each such set has at most p_M plaquettes, so the factor is at most K_M . By Lemma 7.4, the energy-estimate predecessor sets at level k of distinct cells of one tiling are disjoint, and the sum of their energies is therefore at most $S_W(V)/h_{n_k}$. This proves the bound with K_M . For an arbitrary set of plaquettes at level ℓ , we apply (22.8) to each plaquette separately, with $|\mathcal{P}_\ell| = 1$ and factor at most K_1 . By the same lemma, a plaquette of the initial lattice lies in at most 48 of these predecessor sets, and summation gives the factor $48K_1$. Both arguments apply with any intermediate level as initial level. $\square$

Proof of Theorem 22.4(b). The effective classical action along the decomposition. By Theorem 21.13, the initial and stopping levels are boundaries of RG steps, and there are T integrated RG steps of J averaging steps each; thus $\ell = JT$. The logarithmic product domains, the bounds on the frozen coordinates that are inserted into analytic functions, the coefficient-norm bounds, and the representatives are those checked in Lemma 22.3. For the t -th RG step, let C_t^0 be its reconstruction map with every normalized fluctuation variable set to zero, that is, its zero-fluctuation reconstruction. By the definition of the classical part of its objective, the outgoing effective classical action is

$$E_{t+1}^{\text{out}}(c; b_{\text{rec}}^{(t)}) = B E_t(C_t^0(c; b_{\text{rec}}^{(t)}); b_{\text{rec}}^{(t)}), \quad B = M^J. \quad (22.12)$$

If new coordinates $c^\sharp$ are frozen at the end of the step, the next incoming action is

$$E_{t+1}(v; b_{\text{rec}}^{(t)}, c^\sharp) = E_{t+1}^{\text{out}}(v \oplus c^\sharp; b_{\text{rec}}^{(t)}).$$

Thus freezing is a direct substitution in this identity; in particular, the value at $v = 0$ is kept together with the nonconstant part. For an external classical exponent S_{ext} , the convention of Theorem 21.13 is the exact identity

$$e^{-S_{\text{ext}}} e^{-E_t^{\text{int}}/g_t^2} = e^{-[E_t^{\text{int}} + g_t^2 S_{\text{ext}}]/g_t^2}.$$

It adds the term once to the effective classical action and removes it from the weight of the frozen coordinates. Under (22.12) the added action and g_t^2 are both multiplied by B , which leaves the exponent unchanged.

One feasible configuration. We start from such a representative at which the bounds on the frozen coordinates hold. We insert the frozen coordinates at each preceding scale at which they were frozen, and apply the maps C_t^0 downward. The reconstruction identities and the protection of all stored endpoints in Theorems 21.5 and 21.13 ensure that each insertion gives back the coordinate of the preceding output point, that each reconstruction

solves its block averaging constraint exactly, and that every older frozen dependence is preserved. The image bounds of Theorem 20.14, applied with the hypotheses verified in Lemma 22.3, keep this downward sequence of reconstructions in its product domains. For a stopped initial measure, the resulting representative on the initial lattice lies in the larger common chart, so its configuration U_* lies in D_ρ ; this uses the larger image domain, not membership in D_r . For a Wilson initial measure, the frozen variables deep inside excluded regions remain arbitrary group elements, and the initial classical exponent is exactly $S_W(U_*)$. Composing the constraint identities, we obtain $(T_M^\ell U_*)|_F = \hat{c}$. Dividing (22.12) by $g_{t+1}^2 = Bg_t^2$ and iterating over the T RG steps gives the identity for E_T/g_m^2 , and $JT = \ell$ implies $g_m^2 = M^\ell g_{\text{base}}^2$. Since U_* belongs to the constrained set, the lower bound by the infimum follows; no interchange of minima is involved.

The first lower bound. By the lower bound of Theorem 11.2 on the common radius ρ , transported by the fixed gauge transformation of a representative on the initial lattice (Theorem 11.6), we have

$$\frac{\mathcal{E}_{\text{base}}(U_*)}{g_{\text{base}}^2} \geq \kappa_0 \frac{S_W(U_*)}{g_{\text{base}}^2}. \quad (22.13)$$

The radius is within the domain of Theorem 11.2, and the constant κ_* there is valid for every integer depth, whether or not an effective interaction is attached to that depth, as also stated in Theorem 14.8; for a Wilson initial measure (22.13) holds with constant one. Combining this with the bound for all plaquettes in (a), we obtain

$$\frac{E_T}{g_m^2} \geq \frac{\kappa_0}{48K_1} \frac{S_W(T_M^\ell U_*)}{g_m^2} \geq \frac{\Psi_F(\hat{c})}{g_m^2}.$$

In the last inequality we used that $T_M^\ell U_* = \hat{c} \oplus z$ for some $z \in \text{SU}(2)^H$, and that Ψ_F minimizes over all such z .

The second lower bound. For a recorded bad plaquette f , let $\mathcal{B}_f$ be the energy-estimate predecessor set of $\{f\}$ on the initial lattice. We first justify disjointness at different levels. On the input lattice of the RG step in which f is recorded, the endpoints of its full dependence list cover every vertex of the four centered blocks obtained by pulling back the corners of f to that input lattice. Indeed, each one-step formula reads the root paths $\sigma_{a,x}$ for every vertex x of each endpoint block; their trees span those blocks, and the same assertion follows recursively from the centered nesting identity. For a plaquette at the input level, its four boundary links already cover its four corner vertices. By Lemma 7.4, every plaquette of $\mathcal{B}_f$ lies in the centered preimages of these blocks.

All these dependence variables are frozen together with their excluded neighborhood by Definition 21.4(2). The neighborhood includes the protected endpoint margins, so it excludes the incident dependence links as well as the marked vertices; the endpoint protection rule of Theorem 21.3(4) preserves this exclusion at later scales. Suppose that f and f' are recorded at levels $\ell_f < \ell_{f'}$ and that their initial energy-estimate predecessor

sets share a plaquette. The centered partitions are nested. A shared vertex therefore places a marked vertex from a corner block of f in a corner block of f' , when both are read on the input lattice where f was recorded. If the two tests belong to the same RG step, the full dependence list of f' meets the excluded neighborhood just added for f . If they belong to different RG steps, the centered ancestor of that mark is excluded by the endpoint protection rule, and again the full dependence list of f' meets the earlier excluded set. Both cases contradict the rule that a recorded test avoid that set. Thus the sets $\mathcal{B}_f$ at different levels are disjoint. This argument uses endpoint coverage and the protected neighborhoods; it does not require the minimal list of read links to contain every link of an energy-estimate patch.

At one level, a plaquette lies in at most 48 of the sets $\mathcal{B}_f$ by Lemma 7.4. Since plaquette energies are nonnegative, we have

$$\frac{S_W(U_*)}{g_{\text{base}}^2} \geq \frac{1}{48} \sum_f S_0^{\mathcal{B}_f}(U_*).$$

The comparison (22.8) for one plaquette and arbitrary configurations, proved in (a), applies at the level ℓ_f of each f with constant K_1 . Since the feasible configuration U_* preserves every frozen dependence, the energy of f in it is $q_f(\widehat{b_{\text{rec}}})$. Thus

$$\frac{S_W(U_*)}{g_{\text{base}}^2} \geq \frac{1}{48K_1} \sum_f \frac{4q_f(\widehat{b_{\text{rec}}})}{h_{n_f}}.$$

Multiplying by κ_0 and using (22.13), we find $E_T/g_m^2 \geq A_{\mathcal{X}}(\widehat{b_{\text{rec}}})$. On $I_{\mathcal{X}}^{\text{act}}$ we have $q_f \geq d_*$ and $h_{n_f} = \gamma/n_f$ for every f , which gives the lower bound for $A_{\mathcal{X}}$. $\square$

Proof of Theorem 22.4(c). The reference identity. We apply the identity at the end of Theorem 14.8 to the same reference operators and the same partition function Z_ν as in Theorem 21.13. Every reference operator is a full RG step on the same levels. The depth d_0 of a stopped initial measure is a multiple of J , and the depth at the stopping level is $d = d_0 + JT$, with $d_0 = 0$ for a Wilson initial measure. Thus the reference recursion lies within the scope of Theorem 14.8. The stopping side is m , the preceding depth is d , and the centered effective interaction is $R_{d,\text{sat}} - R_d(I)$. Restriction and pushforward give a subprobability measure, also for a Wilson initial measure. Its density is the displayed ξ_T , with mass at most one. Integrating it, we obtain (22.10). The initial value $\Lambda_0^{\mathbb{T}} = -R_{d_0}(I)$ for a stopped initial measure holds because Z_ν is not centered. All later constants are those of this one recursion.

The partition-function bounds. These are the bounds of Theorem 14.8 on the lattice of side m , with $D = 3m^3$. The trial configurations of that theorem have $2m^3 + 1$ links outside a spanning tree, each with logarithm of norm at most $a_0g < \min(r/2, 1)$, and the tree

variables have Haar integral one. The bounds on the action and the effective interaction on this set, together with the volume of a ball in $SU(2)$, give

$$\widehat{Z}_d(g) \geq \left(\frac{a_0 g - \sin(a_0 g)}{2\pi} \right)^{2m^3+1} e^{-C_3 m^3} \geq e^{-C_L 3m^3} g^{3(2m^3+1)},$$

where we used $x - \sin x \geq x^3/8$ for $0 \leq x \leq 1$ and included the powers of a_0 in C_L . This lower bound is an integral over exactly D_r . The upper bound for $\int e^{-\lambda \mathcal{V}_m/g^2} dH_m = \int e^{-\lambda \alpha_W S_W/g^2} dH_m$ is the bound for Wilson numerators in Theorem 14.8, with $\alpha_0 = \alpha_W$, $s = 0$, and inverse coupling $4\lambda \alpha_W/g^2 \geq 1$. In its proof, the plaquettes crossing the periodic boundary are deleted from the numerator, the unused links integrate to one, and on the remaining open cube with m sites per side the $2m^3 - 3m^2 + 1$ plaquettes selected along an axial tree have independent Haar-distributed transports; the one-plaquette bound $\int e^{-\beta q} dH \leq \sqrt{2/\pi} \beta^{-3/2}$ of Lemma 6.10 then gives the factor $g^{3(2m^3-3m^2+1)}$. Dividing by the lower bound for $\widehat{Z}_d(g)$ leaves $3[(2m^3+1) - (2m^3-3m^2+1)] = 9m^2$ powers of $1/g$.

The bound for the stopped measure is the exponential-moment bound of Theorem 14.8, with $\lambda_{\text{en}} = \kappa_*/2$. It can also be read off from the same calculation: on D_r its integrand satisfies

$$e^{-E_{d,\text{sat}}/g^2 - [R_{d,\text{sat}} - R_d(I)] + v S_W/g^2} \leq e^{C_3 m^3} e^{-(\kappa_* - v) S_W/g^2}, \quad \kappa_* - v \geq \kappa_*/2,$$

where we used $E_{d,\text{sat}} \geq \kappa_* S_W$ and the bound on the oscillation of the centered effective interaction. For the quantity with the coefficient λ we replace the coefficient by $\lambda \kappa_* - v \geq \lambda_0 \kappa_*/2$, and we bound the centered effective interaction by its uniform oscillation bound from Theorem 14.8. Both estimates then follow from the upper bound for the Wilson numerator and the single lower bound for the denominator. If the ceiling g_* has not yet been reduced so that the inverse coupling of these two numerators is at least one, then on the remaining range of g , below g_* and above that threshold, $\log(1/g)$ is bounded by a fixed constant; enlarging the volume constant gives the same bounds there. Alternatively, one may make the common reduction of the ceiling in Theorem 14.8. $\square$

22.4 The minimum of the Wilson action over omitted links

The density bounds of Section 22.6 contain the factor $e^{-\lambda \Psi_F(c)/g^2}$, a minimum over all values of the links in H , which we compare with a Haar integral over these links. The comparison is a lower bound of Laplace type that uses only stationarity at a minimizer, an upper bound on the second derivative, and the Haar volume of a small ball; no nondegenerate Hessian is needed. We first record the continuity and gauge invariance of the minimum and of the corresponding integral.

Throughout this section $m \geq 4$, $H \subset \mathcal{E}_m$ is a set of positive links, $F = \mathcal{E}_m \setminus H$, and for $c \in SU(2)^F$ and $z \in SU(2)^H$ we write $c \oplus z$ for the configuration on Λ_m that equals c on

F and z on H . We write H_H for the product Haar measure on $\mathrm{SU}(2)^H$, and $\mathcal{V}_m = \alpha_W S_W$ and $\Psi_F(c) = \min_z \mathcal{V}_m(c \oplus z)$ for a fixed $\alpha_W > 0$, as in Theorem 22.4(b).

Lemma 22.6. *For $\lambda > 0$ and $g > 0$ let*

$$I_{\lambda,g}(c) = \int_{\mathrm{SU}(2)^H} e^{-\lambda \mathcal{V}_m(c \oplus z)/g^2} dH_H(z).$$

The functions Ψ_F and $I_{\lambda,g}$ are continuous on $\mathrm{SU}(2)^F$. Let $k \in \mathrm{SU}(2)^{\Lambda_m}$ be a lattice gauge transformation and let c^k be the transformed field on F , which depends only on the values of k at the endpoints of the links in F . Then $\Psi_F(c^k) = \Psi_F(c)$ and $I_{\lambda,g}(c^k) = I_{\lambda,g}(c)$. In particular, Ψ_F and $I_{\lambda,g}$ are invariant under every gauge transformation of the endpoints of the links in F , extended arbitrarily to the other sites, and the result does not depend on the extension.

Proof. For $c, c' \in \mathrm{SU}(2)^F$ we have

$$|\Psi_F(c) - \Psi_F(c')| \leq \alpha_W \sup_{z \in \mathrm{SU}(2)^H} |S_W(c \oplus z) - S_W(c' \oplus z)|.$$

Since S_W is uniformly continuous on the compact group $\mathrm{SU}(2)^{\mathcal{E}_m}$, the right side tends to zero as $c' \rightarrow c$. The same argument, applied to the integrand, shows that $I_{\lambda,g}$ is continuous. Let k be a gauge transformation, and write $z \mapsto z^k$, $(z^k)_{xy} = k_x z_{xy} k_y^{-1}$, for its action on the links in H . This map is a bijection of $\mathrm{SU}(2)^H$ that acts on each factor by a fixed left and a fixed right translation, so it preserves H_H . Since $(c \oplus z)^k = c^k \oplus z^k$ and S_W is gauge invariant, we have

$$\mathcal{V}_m(c^k \oplus z^k) = \mathcal{V}_m(c \oplus z), \quad dH_H(z^k) = dH_H(z).$$

Taking the minimum over z and changing variables in the integral, we conclude that Ψ_F and $I_{\lambda,g}$ are invariant. If a gauge transformation is given only at the endpoints of the links in F , any two extensions to the other sites give bijections of the whole product $\mathrm{SU}(2)^H$ as above, so the result does not depend on the extension. $\square$

Lemma 22.7. *For every $m \geq 4$, every set $H \subset \mathcal{E}_m$ of positive links, $F = \mathcal{E}_m \setminus H$, every $c \in \mathrm{SU}(2)^F$, and every fixed $\alpha_W > 0$, the function $z \mapsto \mathcal{V}_m(c \oplus z) = \alpha_W S_W(c \oplus z)$ has a minimizer z_* on $\mathrm{SU}(2)^H$, and it is stationary at z_* in all directions of the links in H . For $X = (X_e)_{e \in H} \in \mathfrak{su}(2)^H$ and the curve $z_e(t) = z_{*,e} e^{tX_e}$,*

$$\left| \frac{d^2}{dt^2} S_W(c \oplus z(t)) \right| \leq 16 \sum_{e \in H} |X_e|^2.$$

Hence, for every $0 < g \leq 1$, the product neighborhood $\{z_{*,e} e^{X_e} : |X_e| < g \text{ for all } e \in H\}$ has Haar mass at least

$$(c_* g)^{d_H}, \quad d_H = 3|H|, \quad c_* = (16\pi)^{-1/3},$$

and on it $\mathcal{V}_m(c \oplus z) - \Psi_F(c) \leq 8\alpha_W |H| g^2$. For $0 < \lambda_0 \leq \lambda \leq \lambda_1 < \infty$,

$$e^{-\lambda \Psi_F(c)/g^2} \leq (c_* g)^{-d_H} e^{8\lambda \alpha_W |H|} \int_{\mathrm{SU}(2)^H} e^{-\lambda \mathcal{V}_m(c \oplus z)/g^2} dH_H(z),$$

and the exponential factor may be enlarged to $e^{8\lambda_1 \alpha_W |H|}$. These bounds hold for arbitrary links across the boundary of the block region, arbitrary values of c , and arbitrary shape, topology, age, and depth; the displayed constants do not depend on these parameters or on the coupling. If H is empty, equality holds. No regularity, uniqueness, or measurable choice of a minimizer is required.

Proof. The function $z \mapsto \mathcal{V}_m(c \oplus z)$ is continuous on the compact product $\mathrm{SU}(2)^H$, so it attains its minimum. Every curve $t \mapsto z_{*,e} e^{tX_e}$ through the minimizer stays in this product for both signs of t , and differentiating at $t = 0$ proves stationarity in all directions of the links in H .

We extend X by zero on F . Along the curve, the transport around a positive plaquette is a product of four unitary factors. A factor $z_{*,e} e^{tX_e}$ has first and second derivatives of operator norm at most $|X_e|/2$ and $|X_e|^2/4$, since $\|X_e\|_{\mathrm{op}} = |X_e|/2$ with our normalization of the inner product; an inverse factor $e^{-tX_e} z_{*,e}^{-1}$ has the same bounds. By the product rule, $|\mathrm{tr} A|/2 \leq \|A\|_{\mathrm{op}}$, and the Cauchy–Schwarz inequality for four terms,

$$\left| \frac{d^2}{dt^2} q((c \oplus z(t))_p) \right| \leq \frac{1}{4} \left(\sum_{e \in \partial p} |X_e| \right)^2 \leq \sum_{e \in \partial p} |X_e|^2.$$

Here the sum is over the positive links underlying the oriented boundary of p ; for $m \geq 4$ they are distinct. Since each positive link lies in exactly four positive plaquettes and $S_W = 4 \sum_p q(U_p)$, the second derivative of S_W is bounded by $16 \sum_{e \in H} |X_e|^2$ everywhere along the curve. By Taylor’s formula with integral remainder, whose weight $\int_0^1 (1-t) dt$ equals $1/2$, and stationarity at $t = 0$, we have

$$\mathcal{V}_m(c \oplus z_* e^X) - \Psi_F(c) \leq 8\alpha_W \sum_{e \in H} |X_e|^2.$$

Thus on the product of balls of radius g the increase is at most $8\alpha_W |H| g^2$.

The Haar probability measure on $\mathrm{SU}(2)$ has radial density $(2/\pi) \sin^2 \vartheta d\vartheta$ in the angle $\vartheta = |X|/2$. Thus the ball $\{e^X : |X| < g\}$ has mass $(g - \sin g)/(2\pi)$. Taylor’s formula for the sine gives, for $0 \leq g \leq 1$,

$$g - \sin g \geq g^3/6 - g^5/120 \geq g^3/8.$$

The exponential map is injective on this ball since $g \leq 1 < 2\pi$. By translation invariance of Haar measure and the product structure, the product neighborhood has mass at least $(g^3/(16\pi))^{d_H} = (c_* g)^{d_H}$. On this neighborhood the integrand $e^{-\lambda \mathcal{V}_m(c \oplus z)/g^2}$ is at least $e^{-\lambda \Psi_F(c)/g^2 - 8\lambda \alpha_W |H|}$. Integrating over the neighborhood, enlarging the domain of

integration to the whole product, and rearranging, we obtain the displayed inequality. Replacing λ by λ_1 in the positive exponential only enlarges the right side. If H is empty, all products have mass one and the additional exponents vanish. Each bound is uniform on the whole compact group, and a minimizer was used separately for each c , not as a measurable function of c . $\square$

22.5 Domination of the plaquette-energy sum by the Wilson minimum

Along a large-field decomposition, the function $Q_j^T \circ T_M^{j-1-b}$ need not be a function of the stopping field and the frozen coordinates alone: some requested plaquettes may have dependence sets that meet excluded neighborhoods. The next lemma shows that this does not matter for an upper bound: the bound holds for every completion $c \oplus z$, and we may then take the infimum over completions.

Lemma 22.8. *Fix an odd integer $M \geq 3$, an integer $J \geq 1$, $B = M^J$, and the fixed analytic, geometric, and buffer data of Theorems 21.5 and 21.13. Fix a finite family of stopping offsets and remainders, with total exponents $v = r_{\text{stop}}J + u_{\text{ph}}$ and $0 \leq u_{\text{ph}} < J$, and fix the corresponding enlargement F_J of the bound for the number of affected source plaquettes in Theorem 21.5, before the interval of s or any minimum side is chosen. Let $K_1, \alpha_W > 0$ be as in Theorem 22.4 and put $K_W = 48K_1/\alpha_W$.*

Let $L > 0$ and $\gamma > 0$, and let a finite hierarchy of sides in $4\mathbb{N}$, an initial measure (stopped, at a depth $d_0 \in J\mathbb{N}_0$, or Wilson), a large-field decomposition $\mathcal{X}$ from its initial level to its stopping level with $\ell \in J\mathbb{N}_0$ averaging steps, and an exponent v of the family be given. Write $m = M^v n_{\text{req}}$ for the stopping side, where $n_{\text{req}} \in 4\mathbb{N}$ is the requested side; every iterate that is used must belong to the finite hierarchy. Let F be the set of links of the block region at the stopping level, $H = \mathcal{E}_m \setminus F$, and let $\mathcal{P}_{\text{orig}}$ be the requested list of plaquettes at side n_{req} , a union of the sets $\mathcal{P}_{j-1}(C)$ of cells, counted once. Set

$$Q_{\text{orig}}(V) = \frac{4}{h_{n_{\text{req}}}} \sum_{p \in \mathcal{P}_{\text{orig}}} q((T_M^{\ell+v} V)_p), \quad g_m^2 = \gamma/m, \quad D_{\mathcal{X}} = 8F_J \sum_{f \in \mathcal{X}} \frac{n_f}{\gamma}.$$

Then every point $V \in \mathcal{A}_{\mathcal{X}}$, with $(c, b_{\text{rec}}) = \mathcal{O}_T^{\mathcal{X}}(V)$, and every $z \in \text{SU}(2)^H$ satisfy

$$Q_{\text{orig}}(V) \leq 48K_1 \frac{S_W(c \oplus z)}{g_m^2} + D_{\mathcal{X}}. \quad (22.14)$$

Consequently, at every point of $\mathcal{A}_{\mathcal{X}}$,

$$Q_{\text{orig}}(V) \leq \frac{K_W}{g_m^2} \min_{z \in \text{SU}(2)^H} \{\alpha_W S_W(c \oplus z)\} + 8F_J \sum_{f \in \mathcal{X}} \frac{n_f}{\gamma} = \frac{K_W \Psi_F(c)}{g_m^2} + D_{\mathcal{X}}. \quad (22.15)$$

The minimum Ψ_F is continuous and invariant under gauge transformations of the endpoints of the links in F (Lemma 22.6). For either initial measure ν , with its partition function Z_ν , which is not centered, let η_χ and η_χ^s be as in Section 22.1, with $Q = Q_{\text{orig}}$. For every $s \geq 0$,

$$\eta_\chi^s(dc db_{\text{rec}}) \leq e^{sK_W \Psi_F(c)/g_m^2 + sD_\chi} \eta_\chi(dc db_{\text{rec}}). \quad (22.16)$$

This includes decompositions without recorded bad plaquettes, an empty set F , and decompositions whose block region becomes empty before the planned stopping level, continued by empty RG steps. No small-field event, support, or partition function is changed, and no smallness of the coupling is needed beyond the existence of the decomposition. The constant K_W depends only on M and on the fixed classical lower bound and its domain; it does not depend on the side, L , the coupling, the depth, the geometry, the age, or the frozen values. The constant F_f keeps the explicit polynomial dependence on the widths in Theorem 21.5, and its coefficient has the dependence described in item (3) of Definition 4.14, independently of the widths chosen later. The constants of later probability estimates have the dependence described in item (4) of Definition 4.14.

Proof. We fix the decomposition and the requested list first.

Step 1: dependence sets. For each requested plaquette p , let $D_m(p)$ be the full set of links of the stopping lattice on which $C \mapsto q((T_M^v C)_p)$ depends. It contains the links of the four boundary transports of p and, recursively, all reference paths, relative contours, filling plaquettes, and branch tests of every entry read in the ν averaging steps; the clause on complete predecessor sets in Lemma 7.4 supplies these finite dependence sets. We also trace them to every scale of the integrated decomposition at which variables were frozen. We keep p only if $D_m(p) \subset F$ and the traced dependence sets avoid all the full excluded neighborhoods assigned in Theorem 21.5: the neighborhoods of the recorded bad plaquettes with their buffers, the adjacent cells, the frozen boundary bands, the links across the boundary of the block region, the enlarged earlier excluded regions, the buffers protecting earlier stored endpoints, and the protective sets of the first step for a Wilson initial measure. Their levels of creation and their assignment to recorded bad plaquettes are kept. The smaller energy-estimate predecessor sets $\mathcal{B}_f$ would not suffice for this definition.

Step 2: kept plaquettes are determined by the stopping field. If two configurations on the stopping lattice agree on $D_m(p)$, then for every entry needed in the first averaging step they read identical path products and identical branch tests. A regular entry has the same analytic logarithm in both configurations, and an exceptional entry has the same reference-path value. By induction over the ν averaging steps, the final plaquette transports agree, and hence so do their traces. This proves locality for all group values, including the exceptional branches; for $\nu = 0$ it is the equality of the four boundary links.

By the identity for the observation map in Theorems 21.5 and 21.10, we have $(T_M^\ell V)|_F = c$, and therefore, for every completion $C = c \oplus z$,

$$\frac{4}{h_{n_{\text{req}}}} \sum_{p \text{ kept}} q((T_M^{\ell+v} V)_p) = \frac{4}{h_{n_{\text{req}}}} \sum_{p \text{ kept}} q((T_M^v C)_p).$$

This equality concerns the point V itself; no auxiliary feasible configuration or representative has been substituted.

Step 3: discarded plaquettes. We assign each discarded requested plaquette to the first full excluded neighborhood that its dependence set meets, ordering first by the level of creation and then by the fixed order of the tests. If $D_m(p) \not\subset F$, an omitted link lies in an excluded neighborhood at the stopping level; otherwise the exclusion comes from an earlier traced intersection. The counting clause of Theorem 21.5, enlarged for the finite family of stopping offsets and remainders, bounds the number of plaquettes assigned to each recorded bad plaquette f by F_f . The deterministic assignment to a single f is needed only because the neighborhoods may overlap; we never use disjointness of the full neighborhoods. Since $n_{\text{req}} \leq n_f$ and $0 \leq q \leq 2$, each discarded plaquette contributes at most $8n_{\text{req}}/\gamma \leq 8n_f/\gamma$ to Q_{orig} . Summing over the discarded plaquettes, we find, for every $C = c \oplus z$,

$$Q_{\text{orig}}(V) \leq \frac{4}{h_{n_{\text{req}}}} \sum_{p \text{ kept}} q((T_M^v C)_p) + D_{\mathcal{X}} \leq \frac{4}{h_{n_{\text{req}}}} \sum_p q((T_M^v C)_p) + D_{\mathcal{X}},$$

where the last sum is over all positive plaquettes at side n_{req} .

Step 4: the Wilson action of a completion. We apply the bound for all plaquettes and arbitrary configurations in Theorem 22.4(a) with initial configuration C at side m and depth v . Its initial exponent is $S_W(C)/g_m^2$, and $h_{n_{\text{req}}} = M^v g_m^2$. Together with Step 3, this gives (22.14), with no additional factor m/n_{req} and no regularity test at the requested level. The bound applies to every completion, whatever its holonomies or exceptional values. Taking the infimum over the whole compact product $\text{SU}(2)^H$, we obtain (22.15). By continuity of S_W and compactness, the infimum is attained, but no measurable choice of a minimizer is needed. Continuity and invariance of Ψ_F are proved in Lemma 22.6.

Step 5: empty cases. If $\mathcal{X}$ has no recorded bad plaquettes, there are no excluded neighborhoods and $D_{\mathcal{X}} = 0$. If F is empty, then no plaquette is kept, since the dependence set of every plaquette is nonempty on sides at least four, and the assignment of Step 3 gives $Q_{\text{orig}} \leq D_{\mathcal{X}}$ directly; in this case $\Psi_{\emptyset} = 0$, attained by the identity configuration. If the block region becomes empty before the planned stopping level, we carry the frozen coordinates unchanged through the remaining RG steps. For these steps the fluctuation integral and the determinant equal one, the nonconstant norm is zero, and the simultaneous replacement $E \mapsto BE$, $g^2 \mapsto Bg^2$ leaves the exponent unchanged. The recurrence

$R_{t+1}(f) \leq B^{-1}R_t(f) + C_J(W + R_{\text{cell}} + 1)$ for the radii of the excluded neighborhoods in Theorem 21.5 still assigns every later omitted link to the same neighborhoods, even when their union is the whole torus. The finite dependence range of the requested plaquettes at the stopping level is included in F_J . Thus the same assignment and the same bound hold at the planned stopping level. These empty steps use only the clauses on empty products and on normalization constants in Theorems 20.13 and 20.14 and the partition of Theorem 21.5.

Step 6: integration. For a nonnegative Borel function $\varphi(c, b_{\text{rec}})$, we integrate $\varphi(\mathcal{O}_T^{\mathcal{X}}(V))e^{sQ_{\text{orig}}(V)}$ against $\mathbf{1}_{\mathcal{A}_{\mathcal{X}}} d\nu(V)$ and insert (22.15). The right side becomes the integral of $\varphi e^{sK_W \Psi_F / g_m^2 + sD_{\mathcal{X}}}$ against $\eta_{\mathcal{X}}$, which proves (22.16). Both sides carry the same factor Z_V^{-1} . All integrands are nonnegative, the spaces are finite-dimensional, and Q_{orig} is bounded on them. The dependence of the constants follows from the bound for all plaquettes and from the explicit count F_J . $\square$

The same argument applies to the Wilson exponent of all plaquettes at the requested side, with a larger bound for the number of affected plaquettes. This version is used in Section 23 for exponential moments of the full blocked action.

Corollary 22.9. *In the setting of Lemma 22.8, let the requested function at side n instead be $Q_n(V) = S_W(T_M^{\ell+v}V) / g_n^2$, where $m = M^v n$, v belongs to a fixed finite family of exponents, and all iterates are available. Enlarge the bound for the number of affected plaquettes to a bound F_J^A for the list of all positive plaquettes, before choosing any interval of s . Then at every point of $\mathcal{A}_{\mathcal{X}}$ and for every completion z ,*

$$Q_n(V) \leq 48K_1 \frac{S_W(c \oplus z)}{g_m^2} + 8F_J^A \sum_f \frac{n_f}{\gamma}, \quad \text{hence} \quad Q_n(V) \leq \frac{K_W \Psi_F(c)}{g_m^2} + 8F_J^A \sum_f \frac{n_f}{\gamma}.$$

The corresponding inequality for the pushforward measures, analogous to (22.16), holds with the same partition function. This includes Wilson initial measures, empty sets F , and decompositions continued by empty RG steps. One may take $F_J^A = C_J^A(W + R_{\text{cell}} + 1)^3$; this bound has the same polynomial dependence on the widths, and its coefficient the same dependence, as F_J , and K_W is unchanged.

Proof. Steps 1, 2, and 4–6 of the proof of Lemma 22.8 apply verbatim with the list of all positive plaquettes at side n in place of $\mathcal{P}_{\text{orig}}$, with n in place of n_{req} , and with F_J^A in place of F_J . In Step 3, each discarded plaquette still contributes at most $8n/\gamma \leq 8n_f/\gamma$, since the recorded bad plaquette to which it is assigned lies at a level finer than the requested side. In Step 4, the Wilson exponent of $T_M^v(c \oplus z)$ at side n is bounded by Theorem 22.4(a) with no factor depending on the ratio of the sides.

It remains to justify the count in Step 3. We check explicitly that the counting clause of Theorem 21.5 extends to the list of all plaquettes. The family of exponents v

is finite, and each exponent has a finite fixed dependence range. A neighborhood of radius R is met by the dependence sets of at most a fixed multiple of $(R + 1)^3$ positive plaquettes. The recurrence for the radii in Theorem 21.5 bounds R by a fixed multiple of $W + R_{\text{cell}} + 1$ at later outputs, and the enlargement at the input of the RG step of creation is already included in the count at creation, which carries the factor B . Thus passing to all plaquettes changes only this incidence constant, and unions and overlaps are handled by subadditivity and by the assignment to a single recorded bad plaquette. This proves the bound $F_J^A = C_J^A (W + R_{\text{cell}} + 1)^3$, with the enlargement for the finite family of exponents; the number of RG steps of the decomposition does not occur. $\square$

22.6 Gauge-invariant domination of the densities

We now combine the density bound of Theorem 21.13, which holds at representatives satisfying the reconstruction identities, with the classical lower bounds of Theorem 22.4(b). Since the resulting bound has only gauge-invariant factors, the pointwise gauge covariance of the density in Theorem 21.10 extends it to every point.

Theorem 22.10. *Let $\mathcal{X}$ be a large-field decomposition for a stopped or a Wilson initial measure with at least one integrated full RG step, as in Theorem 21.13, with stopping side m , $g^2 = \gamma/m$, and $0 < g \leq 1$. Whenever the uniform bounds of Theorem 22.4(c) on the stopping lattice are used, impose also their fixed ceiling on g . Let F be the set of links of the block region at the stopping level, H the set of all other links of the stopping lattice, including the links across the boundary of the block region, and $d_H = 3|H|$. Let $p_{\mathcal{X}}(c, b_{\text{rec}})$ be the density of $\eta_{\mathcal{X}}$ with respect to $H_F \otimes H_{\mathcal{X}}$ given by Theorem 21.10, where H_F and $H_{\mathcal{X}}$ are the product Haar measures on the stopping field and on the frozen coordinates. Let $w_{\mathcal{X}}$ be the gauge-invariant weight of the frozen coordinates and $\rho_{\mathcal{X}} = w_{\mathcal{X}} H_{\mathcal{X}}$, as in Theorem 21.13; they keep the indicator $I_{\mathcal{X}}^{\text{act}}$ of the tests of all recorded bad plaquettes. Fix $0 < \theta < 1$, put $\lambda = 1 - \theta$, and use the constants of Theorem 22.4.*

At every representative $(\widehat{c}, \widehat{b}_{\text{rec}})$ in the support satisfying the domain conditions and reconstruction identities of Theorem 21.13, the density bound of Theorem 21.13 and its estimate for the normalization constant give

$$p_{\mathcal{X}}(\widehat{c}, \widehat{b}_{\text{rec}}) \leq \widehat{Z}_d(g)^{-1} e^{L_{\mathcal{X}}(\widehat{b}_{\text{rec}})} w_{\mathcal{X}}(\widehat{b}_{\text{rec}}) e^{-E_T(\widehat{c}; \widehat{b}_{\text{rec}})/g^2}, \quad L_{\mathcal{X}} \leq \overline{L}_{\mathcal{X}}.$$

One may take

$$L_{\mathcal{X}} = \log A_T + \Lambda_T - \Lambda_T^{\mathbb{T}} + C_{\text{osc}, J} m^3, \quad \overline{L}_{\mathcal{X}} = C_{\delta} + 3C_J m^3 + C_{\partial} \text{Surf}_{\mathcal{X}} + \sum_{f \in \mathcal{X}} \overline{\text{ch}}_f.$$

Here Λ_T is the normalization constant of the regional recursion along $\mathcal{X}$, $\Lambda_T^{\mathbb{T}}$ is that of the reference recursion, and A_T is the accumulated comparison constant of Theorem 21.13. Further, $\text{Surf}_{\mathcal{X}}$ is the number of variables in the boundary bands that enter the normalization constant, and

$\overline{\text{ch}}_f$ is the sum of the nonnegative upper bounds, assigned to f by Theorem 21.5, for the corrections from missing Gaussian coordinates, from missing constant terms of fluctuation integrals, and from substitutions of frozen coordinates. Each missing Gaussian coordinate contributes its own $\log(1/g_i)$, and each other such variable contributes a fixed bound C_J , including the variables at initialization. These are deterministic bounds in terms of the geometry and the couplings, independent of the chosen representative. The identity (22.10) supplies only the prefactor $\widehat{Z}_d(g)^{-1}$; the bound for $\Lambda_T - \Lambda_T^{\mathbb{T}}$ comes from the increment estimate of Theorem 20.13.

Applying both lower bounds (22.9) at the same representative gives a gauge-invariant upper bound which is defined everywhere. At every point (c, b_{rec}) , with the density of Theorem 21.10, which vanishes off the gauge saturation of these representatives,

$$p_{\mathcal{X}}(c, b_{\text{rec}}) \leq \widehat{Z}_d(g)^{-1} e^{\overline{L}_{\mathcal{X}}} w_{\mathcal{X}}(b_{\text{rec}}) \exp\{-\lambda \Psi_F(c)/g^2 - \theta A_{\mathcal{X}}(b_{\text{rec}})\}.$$

The minimum Ψ_F and the corresponding integral over H are invariant under every gauge transformation of the endpoints of the links in F , extended to the other sites. The tests of the recorded bad plaquettes and the other tests in $w_{\mathcal{X}}$ are invariant under the action of consistent endpoint gauge transformations at all scales. Hence the pointwise covariance of Theorem 21.10 applies to this bound. Define the finite measure, which is in general not normalized,

$$d\tilde{\sigma}_{\lambda, d, g}(C) = \widehat{Z}_d(g)^{-1} e^{-\lambda \mathcal{V}_m(C)/g^2} dH_m(C),$$

and let $\tilde{\sigma}_{\lambda, d, g; F}$ be its marginal on the links of F . Integrating the Wilson factor over the links in H gives

$$\begin{aligned} \eta_{\mathcal{X}}(dc db_{\text{rec}}) &\leq \exp\{\overline{L}_{\mathcal{X}} + d_H \log(1/(c_*g)) + 8\lambda\alpha_W |H| - \theta A_{\mathcal{X}}(b_{\text{rec}})\} \\ &\quad \times [\tilde{\sigma}_{\lambda, d, g; F} \otimes \rho_{\mathcal{X}}](dc db_{\text{rec}}). \end{aligned}$$

For $\theta = \lambda = 1/2$, let Q_{orig} be a requested plaquette-energy sum as in Lemma 22.8, and let $\eta_{\mathcal{X}}^s = (\mathcal{O}_T^{\mathcal{X}})_{\#}(e^{sQ_{\text{orig}}} \mathbf{1}_{A_{\mathcal{X}}} \nu)$. For $0 \leq s \leq (4K_W)^{-1}$ and $\lambda_s = 1/2 - sK_W \in [1/4, 1/2]$,

$$\eta_{\mathcal{X}}^s \leq \exp\{\overline{L}_{\mathcal{X}} + sD_{\mathcal{X}} + d_H \log(1/(c_*g)) + 8\lambda_s\alpha_W |H| - A_{\mathcal{X}}(b_{\text{rec}})/2\} [\tilde{\sigma}_{\lambda_s, d, g; F} \otimes \rho_{\mathcal{X}}]. \quad (22.17)$$

The function Q_{orig} need not be a function of the observation (c, b_{rec}) , and the same partition function Z_{ν} and the same centered denominator $\widehat{Z}_d(g)$ are used for all s .

All external classical exponents occur once in E_T and are absent from $\rho_{\mathcal{X}}$. The intermediate auxiliary actions and effective interactions need only be defined at the analytic arguments given by these representatives; no global auxiliary density is asserted or integrated. The local constants in the bounds for the normalization constant and for the oscillation have the dependence described in item (3) of Definition 4.14, independently of the widths chosen later. The explicit counts contain the dependence on the geometry, and uniform probability constants have the dependence described in item (4) of Definition 4.14.

Proof. We first check the hypotheses of the imported estimates, then bound the density at such a representative, extend the bound to every point by gauge invariance, and finally integrate over the omitted links.

Step 1: hypotheses. The decomposition in Theorem 21.13 has its initial and stopping levels at boundaries of RG steps and integrates at least one full RG step. The zero-step case for either initial measure is treated directly in Step 7 of the proof of Theorem 22.1; it requires no small terminal representative. By Lemma 22.3, its logarithmic product domains satisfy the containment and the expansion conditions, its frozen coordinates satisfy the one-quarter radius bounds and the substitution bounds, and its coefficient norms at initialization remain within the same bounds. Here A_w is fixed before the ceiling, and (22.5) gives measurability and a positive finite partition function for every comparison with the enclosing product domain. This verifies the hypotheses of the integration estimate and of the comparison at every step, with all frozen coordinates real.

Step 2: the bound at a representative. We choose such a representative at which the bounds on the frozen coordinates hold, and let Z_v be the partition function of Theorem 21.13. At this representative, after the external classical exponents have been incorporated once, the induction of that theorem gives

$$p_{\mathcal{X}}(\widehat{c}, \widehat{b_{\text{rec}}}) \leq A_T Z_v^{-1} e^{\Lambda_T(\widehat{b_{\text{rec}}})} w_{\mathcal{X}}(\widehat{b_{\text{rec}}}) e^{-E_T/g^2 - R_T(\widehat{c}; \widehat{b_{\text{rec}}})}.$$

The regional effective interaction R_T is centered at the origin of the block region at the stopping level. By Theorem 20.14, its part coming from the full torus and its boundary correction satisfy the oscillation bound

$$|R_T(v; b_{\text{rec}})| \leq \rho^2 \left[E|\Omega_T| + y_* \sum_{i \in \Omega_T} e^{-\alpha d_{\Omega_T}(i)} \right] \leq C_{\text{osc},J} m^3,$$

where Ω_T is the list of scalar variables of the block region at the stopping level, $d_{\Omega_T}(i)$ is the distance of i to its complement, and y_* is the fixed output bound of Theorem 20.14, which is used because no boundary layer is removed after the last RG step. There are at most $9m^3$ scalar variables. In the last inequality, we used $e^{-\alpha d} \leq 1$, which avoids any assumption on the shape of the boundary. The linear terms are included in this oscillation bound. The constants are evaluated on the fixed analytic domains, before the widths are chosen.

Step 3: the normalization constants. Inserting (22.10), that is $e^{\Lambda_T^{\mathbb{T}}}/Z_v \leq \widehat{Z}_d(g)^{-1}$, into this density bound, we obtain the stated expression for $L_{\mathcal{X}}$. To bound it, we sum over the RG steps the increment inequality of Theorem 20.13, as recorded in Theorem 21.13. Every summand comes from a full RG step, and the column bounds of the perturbed maps and all the product conditions have been checked in Lemma 22.3. At initialization the normalization constant equals the constant of the reference recursion plus the

increment from the substitution at initialization. Each missing scalar Gaussian coordinate contributes $\log(1/g_i)$, and each missing constant term of a fluctuation integral and each substituted variable contributes its fixed bound C_J . Using Theorem 21.5, we assign these nonnegative bounds to the recorded bad plaquette whose assigned neighborhood contains the variable. Variables in the boundary bands contribute $C_{\partial} \text{Surf}_{\mathcal{X}}$. A variable that lies in several neighborhoods is assigned to the first recorded bad plaquette in the fixed order of the tests, and is therefore counted once, or overcounted by a bounded positive amount. Variables at initialization are assigned in the same way. Thus

$$\Lambda_T - \Lambda_T^{\mathbb{T}} \leq C_{\partial} \text{Surf}_{\mathcal{X}} + \sum_f \overline{\text{ch}}_f,$$

after a fixed bound for the bulk at the stopping level has been absorbed into $3C_J m^3$; the same C_J may include $C_{\text{osc},J}/3$. Finally, $\log A_T \leq C_{\delta}$ by Theorem 21.13 and Lemma 20.5. These estimates prove the bound by $\bar{L}_{\mathcal{X}}$, in which only counts of geometric objects and the given couplings g_i occur, and no value at the representative.

Step 4: the classical lower bounds. At the same representative, by Theorem 22.4(b), we have $E_T/g^2 \geq \Psi_F(\hat{c})/g^2$ and $E_T/g^2 \geq A_{\mathcal{X}}(\widehat{b_{\text{rec}}})$. Multiplying these inequalities by λ and θ , respectively, adding them, and using $\lambda + \theta = 1$, we obtain

$$\frac{E_T}{g^2} \geq \frac{\lambda \Psi_F(\hat{c})}{g^2} + \theta A_{\mathcal{X}}(\widehat{b_{\text{rec}}}).$$

Inserting this into the bound of Step 2, we find the pointwise upper bound at every such representative.

Step 5: extension to every point. By Lemma 22.6, Ψ_F and the integral $I_{\lambda,g}$ are continuous and invariant under gauge transformations of the endpoints of the links in F , extended to the other sites. Each value $q_f(b_{\text{rec}}) = 1 - \text{tr}(U_f(b_{\text{rec}}))/2$ of a recorded bad plaquette is computed from the transport $U_f(b_{\text{rec}})$ around f , using its frozen full dependence set and the Borel extension of T_M ; under an endpoint gauge transformation the intermediate endpoint factors cancel and the final transport is conjugated. Thus $A_{\mathcal{X}}$ and all tests in $w_{\mathcal{X}}$ are invariant under the action of consistent endpoint gauge transformations in Theorem 21.10. The bound $\bar{L}_{\mathcal{X}}$ is gauge invariant, since it depends only on the geometry and the couplings. It follows that the right side of the bound is a gauge-invariant Borel function G defined everywhere. For a point $x = (c, b_{\text{rec}})$ in the gauge saturation of the history-produced reconstruction family of Theorem 21.13(2), we choose a representative $\hat{x} = x^k$ in that family; it exists for every supported orbit by Theorem 21.5 and Lemma 21.8 and the all-fiber descent of Theorem 21.9(2). Its native collar records and analytic frozen bounds justify Steps 2–4. One such representative per supported orbit suffices here, because only the final compact density and the right side G are extended by pointwise

gauge invariance. The endpoint action on arbitrary frozen variables need not preserve the bound on their logarithms, but G is defined by invariant factors only. By the pointwise covariance of the density in Theorem 21.10, we have

$$p_{\mathcal{X}}(x) = p_{\mathcal{X}}(\hat{x}) \leq G(\hat{x}) = G(x).$$

This is a pointwise statement for each x , and the gauge transformation k may depend on x ; neither a random choice according to Haar measure nor an integration over the set of representatives is needed. Off the gauge saturation the density vanishes, and the inequality holds there as well.

Step 6: integration over the omitted links. It remains to integrate the Wilson factor, which is now defined globally. The hypothesis of Lemma 22.7 is $0 < g \leq 1$, which we have assumed; hence that lemma applies with the fixed λ . This condition at the stopping side is separate from the ceiling on the largest coupling of an integrated RG step: at the output of a full RG step, $g = \sqrt{M} t_{\max}$. The reference identity used in Step 3 needs no uniform bound on the ratio of partition functions; whenever that bound is used, its fixed ceiling is imposed as well, as required in the statement. Multiplying by $\hat{Z}_d(g)^{-1} dH_F(c)$ gives exactly

$$\hat{Z}_d(g)^{-1} \left[\int_{\mathrm{SU}(2)^H} e^{-\lambda \mathcal{V}_m(c \oplus z)/g^2} dH_H(z) \right] dH_F(c) = d\tilde{\sigma}_{\lambda,d,g;F}(c).$$

All these are finite nonnegative integrals over compact products. Multiplying the pointwise inequality by $dH_F(c) dH_{\mathcal{X}}(b_{\mathrm{rec}})$ proves the inequality for $\eta_{\mathcal{X}}$, with $w_{\mathcal{X}} dH_{\mathcal{X}} = d\rho_{\mathcal{X}}$. Every external classical exponent has entered E_T exactly once. The invariant indicators of the recorded bad plaquettes remain in $\rho_{\mathcal{X}}$; hence $q_f \geq d_*$ holds at every point of integration used later. No global value or covariance property of E_T or R_T was used. The dependence of the constants follows from the local bounds for the oscillation and the normalization constant and from the displayed counts.

Step 7: the weighted measures. For the last assertion, we take $\theta = \lambda = 1/2$ and apply (22.16) to the bound for $\eta_{\mathcal{X}}$ obtained in Step 5, before the integration over H . This changes $-\Psi_F/(2g^2)$ into $-\lambda_s \Psi_F/g^2$ and adds $sD_{\mathcal{X}}$ to the exponent. We then apply Lemma 22.7 to this Wilson factor with $\lambda = \lambda_s$. Its constants are uniform for $\lambda_s \in [1/4, 1/2]$, and the marginal is exactly $\tilde{\sigma}_{\lambda_s,d,g;F}$. This proves (22.17). Since the multiplication by $e^{sQ_{\mathrm{orig}}}$ was made before the integration over H , no two partition functions are compared. $\square$

22.7 Summation over large-field decompositions

We now prove Theorem 22.1, following the outline in Section 22.1. The method of absorbing the normalization losses of excluded regions into the action of the large fields

goes back to Bałaban's treatment of large fields [20, 21]; the decomposition, the frozen coordinates, and the constants used here are those of Section 21.

Proof of Theorem 22.1. Steps 1 and 2 fix the ceilings and the stopping level, Steps 3–6 treat a stopping level above the initial level, and Step 7 treats the remaining case.

Step 1: ceilings and the minimum side. We write $n = n_{j-1}$. The parameters, the buffers, and the count F_J are fixed first. We choose the coupling bound $g_{\text{anc}} > 0$ for the bounds in Theorem 22.4(c), uniformly for $\lambda \in [1/4, 1/2]$, and impose

$$g_{\text{end}} \leq \min\{1, g_*, g_{\text{anc}}, \sqrt{\alpha_W}, \sqrt{2\kappa_*}\}, \quad n_{\min} \geq \max\{e\Gamma, \Gamma/g_{\text{end}}^2, \Gamma/(Mg_*^2), m_0(\Gamma)\}. \quad (22.18)$$

Here $m_0(\Gamma)$ is the uniform threshold of Lemma 20.5(3), which supplies $\log A_T \leq C_\delta(\Gamma)$; this dependence is included in C_b . We also include every fixed minimum side required by the geometry and by the first step for a Wilson initial measure in Theorem 21.5. Then, for every stopping side $m \geq n_{\min}$, we have $g_m \leq 1$, as required in Lemma 22.7. Each ratio of partition functions satisfies its own coupling bound, the inverse-coupling conditions $4\lambda\alpha_W/g_m^2 \geq 1$ for $\lambda \geq 1/4$ and $2\kappa_*/g_m^2 \geq 1$ hold, and a full RG step with output side m has $t_{\max}^2 = \gamma/(Mm) \leq g_*^2$. The identity $g_m = \sqrt{M}t_{\max}$ shows why the condition at the stopping side is checked separately. No restriction on γ enters a local estimate.

Step 2: the stopping level. Let $t_0 = J \lfloor (j-1)/J \rfloor$ and $u_{\text{ph}} = j-1-t_0$. For a given $r_{\text{stop}} \in \mathcal{R}$, we set $t_{\text{stop}} = t_0 - r_{\text{stop}}J$. If $t_{\text{stop}} > b$, we integrate the $(t_{\text{stop}} - b)/J$ full RG steps of the decomposition. The function at their endpoint is exactly

$$C \mapsto Q_j^T(T_M^v C), \quad v = r_{\text{stop}}J + u_{\text{ph}} = j-1-t_{\text{stop}}, \quad m = n_{t_{\text{stop}}} = M^v n. \quad (22.19)$$

Here $0 \leq b < t_{\text{stop}} \leq j-1 < JK$, and therefore all v iterates are available and all their sides are multiples of four. The depth $aJ + t_{\text{stop}} - b$ of the reference recursion is a multiple of J . If $t_{\text{stop}} \leq b$, we use the initial measure directly: $j-1-b$ is an available nonnegative exponent, bounded by $J \max \mathcal{R} + J-1$, and $n_b/n \leq \chi_*$. This treats all remainders without integrating a partial RG step. Steps 3–6 concern the case $t_{\text{stop}} > b$.

Step 3: one decomposition. We use the Borel partition of Theorem 21.5 and the density induction of Theorem 21.13. For a Wilson initial measure, the first step covers all of $\mathcal{U}_{n_b}$ and does not first restrict the Wilson measure to D_r . By Lemma 22.3, the product domains, the coefficient norms, the bounds on the frozen coordinates, and the boundary comparisons hold for every finite decomposition, without a bound on its number of RG steps.

Fix one decomposition $\mathcal{X}$. Let $\Psi_F = \min_z \alpha_W S_W(c \oplus z)$, and let $H_{\mathcal{X}} = \sum_f x_f$. In Steps 3–6, $H_{\mathcal{X}}$ denotes this scalar sum, not the product Haar measure on the frozen coordinates. At the observation, by Lemma 22.8, we have $Q_{\text{orig}} \leq K_W \Psi_F / g_m^2 + 8F_J H_{\mathcal{X}}$. We apply

the bound of Theorem 22.10 with $\theta = 1/2$, before the integration over H . Multiplication by $e^{sQ_{\text{orig}}}$ is justified by the inequality (22.16) between pushforward measures, without assuming that Q_{orig} is a function of the observation. With $\lambda_s = 1/2 - sK_W \in [1/4, 1/2]$, the exponent before the integration over H satisfies

$$sQ_{\text{orig}} - \frac{\Psi_F}{2g_m^2} - \frac{A_{\mathcal{X}}}{2} \leq -\frac{\lambda_s \Psi_F}{g_m^2} - (P_0 - 8sF_J)H_{\mathcal{X}}.$$

Indeed, by Theorem 22.4(b) and the invariant indicators of the recorded bad plaquettes, we have $A_{\mathcal{X}}/2 \geq 2c_A^{\text{act}}d_*H_{\mathcal{X}} = P_0H_{\mathcal{X}}$, and the condition $s \leq \theta c_A^{\text{act}}d_*/(16F_J)$ gives $8sF_J \leq P_0/8 \leq P_0/4$. We integrate the Wilson minimum once over H by Lemma 22.7, which adds $3|H| \log(1/(c_*g_m)) + 8\lambda_s\alpha_W|H|$ to the exponent. The centered reference denominator and the partition function Z_ν of the initial measure are the same for every s .

Step 4: assignment of the corrections. We enlarge the local counting constant to account for the directions, the colors, the links across the boundary of the block region, and the finite family of remainders, and set

$$D_0 = C_J(BW + R_{\text{cell}} + 1)^3, \quad D_1 = C_J(W + R_{\text{cell}} + 1)^3.$$

By Theorem 21.5, every variable outside the bulk that enters the normalization constant is assigned to a recorded bad plaquette at whose creation it was produced, and the assignment is subadditive when neighborhoods merge. At most D_0 variables are assigned to f at its creation, and at most D_1 at each later RG step. At creation their integration side is at most Bn_f , and the sides at later steps decrease by the factor B . We absorb the finite factors coming from the stopping offsets and the finite link products into a constant $\chi_{\text{lay}} \geq 1$. For every missing Gaussian coordinate assigned to f ,

$$\log(1/g_i) = \frac{1}{2} \log(n_i/\gamma) \leq \frac{1}{2} \log x_f + \frac{1}{2} \log \chi_{\text{lay}}. \quad (22.20)$$

By Theorems 20.13 and 21.13, all other corrections per variable have fixed bounds. Thus a constant C_{lab} , with the dependence described in item (3) of Definition 4.14, bounds the correction of each variable by $C_{\text{lab}}(1 + \log x_f)$. The dependence on the widths is entirely in D_0 , D_1 , and the fixed counts of affected plaquettes.

Let H_f be the number of RG steps after the creation of f , including empty steps. Counting the decreasing sides down to four, we find

$$H_f + 1 \leq C_{\text{lay},J} + \#\{q \in \mathbb{N}_0 : 4B^q \leq \chi_{\text{lay}}n_f\} \leq C'_{\text{lay},J}(1 + \log n_f).$$

The finite additive term includes the conventions at creation and at the stopping level; the number of RG steps of the decomposition does not occur. Since $x_f \geq e$ and $\gamma \leq \Gamma$, we have $1 + \log n_f \leq (1 + \log_+ \Gamma)(1 + \log x_f)$. We write H_f^{term} for the set of omitted links at the stopping level assigned to f . The deterministic assignment of Theorem 21.5, to the

first recorded bad plaquette in the fixed order of the tests, partitions H into these sets, and $|H_f^{\text{term}}| \leq D_1$. The corrections assigned to f are listed in Table 22.2. For the integration over H we also used that $m \leq n_f$, so that $\log(1/(c_*g_m)) \leq \log(1/c_*) + (\log x_f)/2$.

Table 22.2. All corrections assigned to one recorded bad plaquette f created at side n_f . Here $L_f = 1 + \log x_f$, and the later RG steps include empty steps.

Stage	Corrections included, and where they are bounded	Bound
Creation	Missing Gaussian coordinates, missing constant terms of the Haar disintegration and fluctuation integrals, substitutions at initialization and at creation; Theorems 20.13 and 21.13	$D_0 C_{\text{lab}} L_f$
Later nonempty RG steps	Missing coordinates and constant terms, boundary bands, and every later substitution, at the couplings of their integration; the same increment estimate	$D_1 C_{\text{lab}} L_f$ per step
Integration over H	Omitted links, including links across the boundary of the block region: $3 H_f^{\text{term}} \log(1/(c_*g_m)) + 8\lambda_s \alpha_W H_f^{\text{term}} $; Lemma 22.7	$D_1 C_{\text{com}} L_f$, once
Empty RG steps	Determinant and fluctuation integral equal one; logarithmic determinant and centered effective interaction vanish. Missing reference variables use the same increment estimate	at most $D_1 C_{\text{lab}} L_f$ per empty step
External exponents	Incorporated once into the classical exponent; the weight of the frozen coordinates is a subprobability; Theorem 21.13	no further positive correction
Plaquette-energy sum	Discarded plaquettes keep the coefficient $4/g_n^2$; Lemma 22.8	$8sF_j x_f \leq P_0 x_f/4$

The numbers of nonempty and of empty later RG steps in the table add up to at most H_f , so no step contributes twice. At the first RG step at which the block region is empty, all omitted variables are covered by assigned neighborhoods, and the recurrence $R_{t+1}(f) \leq B^{-1}R_t(f) + C_J(W + R_{\text{cell}} + 1)$ of Theorem 21.5 continues to cover their coarse images, including the full finite dependence range, even when the union of the neighborhoods is the whole torus. We carry the frozen coordinates by the identity through the remaining full RG steps; both the effective classical action and g^2 acquire the

factor B , which leaves each external exponent unchanged. The clauses on empty products in Theorems 20.13 and 20.14 therefore justify the row for empty steps.

Combining the first four rows of the table, we find

$$C_\partial \text{Surf}_\mathcal{X} + \sum_f \overline{\text{ch}}_f + 3|H| \log(1/(c_* g_m)) + 8\lambda_s \alpha_W |H| \leq \sum_f C_{\text{ch},J}^\Gamma (1 + \log x_f)^2. \quad (22.21)$$

For example, one may take for $C_{\text{ch},J}^{\text{base}}$ a fixed multiple of $(D_0 + D_1)(1 + C'_{\text{lay},J})(C_{\text{lab}} + C_{\text{com}})$. The bulk term $3C_J m^3$ and the single comparison constant C_δ of Theorem 22.10 remain as explicit prefactors at the stopping level; they are not generated again at each step.

Step 5: absorption and the bound for one decomposition. For $x \geq e$, the function $(1 + \log x)^2/x$ decreases to zero. We increase the minimum side so that $C_{\text{ch},J}^\Gamma (1 + \log x)^2 \leq P_0 x/4$ for every $x \geq n_{\min}/\Gamma$. Since $n_f \geq m \geq u \geq n_{\min}$ and $\gamma \leq \Gamma$, every x_f is in this range, and by Steps 3 and 4 at least $P_0 x_f/2$ of the classical lower bound remains for each recorded bad plaquette. By Theorem 21.13, the weight of the frozen coordinates, after the removal of the incorporated external exponents, has mass at most one. The marginal $\tilde{\sigma}_{\lambda_s, d, g_m; F}$ has the mass of the whole Wilson numerator, $\hat{Z}_d(g_m)^{-1} \int e^{-\lambda_s \mathcal{V}_m / g_m^2} dH_m$. Using the bound for the single ratio in Theorem 22.4(c), uniform for $\lambda_s \in [1/4, 1/2]$, we conclude that

$$\mathbb{E}_\nu[e^{S_{\text{Orig}}}; \mathcal{A}_\mathcal{X}] \leq e^{3C_T m^3 + 9m^2 \log(1/g_m) + C_\delta} \prod_f e^{-a_{\text{act}} n_f / \gamma}, \quad (22.22)$$

where C_T collects C'' of Theorem 22.4(c) and C_J . Only the reference integral in the denominator is restricted to D_r .

Step 6: summation over decompositions. Since the sets $\mathcal{A}_\mathcal{X}$ partition the support of ν , $\mathbb{E}_\nu e^{S_{\text{Orig}}}$ is the sum of the left sides of (22.22) over all decompositions. By Theorem 21.5, there is at most one decomposition for each attainable set of recorded bad plaquettes, and there are at most $3n_\ell^3$ positive plaquettes at level ℓ . Hence

$$\sum_{\mathcal{X}} \prod_f e^{-a_{\text{act}} n_f / \gamma} \leq \prod_\ell (1 + e^{-a_{\text{act}} n_\ell / \gamma})^{3n_\ell^3} \leq \exp \left\{ \sum_\ell 3n_\ell^3 e^{-a_{\text{act}} n_\ell / \gamma} \right\},$$

which is the bound in the theorem. To bound its logarithm, we include all levels, also those inside RG steps, so that consecutive sides differ by the factor M , and note that

$$\frac{(Mm')^3 e^{-a_{\text{act}} M m' / \gamma}}{m'^3 e^{-a_{\text{act}} m' / \gamma}} = M^3 e^{-a_{\text{act}} (M-1) m' / \gamma} \leq \frac{1}{2}$$

for all sides m' above a fixed further minimum side, uniformly for $\gamma \leq \Gamma$. We also require $n_{\min} \geq 3\Gamma/a_{\text{act}}$, so that $x \mapsto x^3 e^{-a_{\text{act}} x / \gamma}$ decreases for $x \geq n_{\min}$, and $6n_{\min}^3 e^{-a_{\text{act}} n_{\min} / \Gamma} \leq 1$. Since all sides n_ℓ that occur are at least $u \geq n_{\min}$, the sum in the exponent is at most twice

its first term, hence at most one, for every finite depth. Since $m = \chi n$, the remaining exponent is

$$3C_T\chi^3n^3 + 9\chi^2n^2\log(1/g_n) + \frac{9}{2}\chi^2n^2\log\chi + C_\delta + 1.$$

The fixed term containing $\log\chi$ is absorbed into the volume constant, and the condition $C_s \geq 9\chi_*^2$ accommodates the single surface term.

Step 7: stopping level at or below the initial level. In this case $Q_{\text{orig}} = Q_j^{\mathcal{T}} \circ T_M^{j-1-b}$, and by the bound for cells in Theorem 22.4(a), we have $Q_{\text{orig}} \leq K_M S_W(V)/h_{n_b}$. For a stopped initial measure, the bound for stopped measures in Theorem 22.4(c) applies, because $sK_M \leq \kappa_0/2 \leq \kappa_*/2$. For a Wilson initial measure, we have $sK_M \leq 1/2$, and the moment is at most

$$\frac{\int e^{-(1-sK_M)S_W/h_{n_b}} dH_{n_b}}{\int e^{-S_W/h_{n_b}} dH_{n_b}}.$$

We use the integral over D_r of the reference recursion at depth zero only to bound the denominator from below, and the Wilson integral over the whole configuration space, with coefficient $1 - sK_M \in [1/2, 1]$, in the numerator. The same calculation of Theorem 22.4(c), that is, of Theorem 14.8, bounds this ratio by $e^{3C_T n_b^3 + 9n_b^2 \log(1/g_{n_b})}$. We convert units using $n_b/n \leq \chi_*$, as in Step 6. Finally, we enlarge the constants C_v, C_s, C_b to cover the finite family of stopping offsets and remainders, which proves the theorem. All increases of the minimum side are made after Γ is fixed and before the coupling is chosen. No step used a bound on $j - b$ or a bound on the probability, under the Wilson measure, of the complement of a small-field event. $\square$

22.8 Chessboard estimates with regularity indicators

The chessboard estimate converts the exponential-moment bound of Theorem 22.1 into a bound for the probability that a given cell contains a bad plaquette at level j , given regularity at finer levels inside the cells. Reflection positivity of the Wilson measure with respect to reflections in lattice planes is due to Osterwalder and Seiler [134], and systematic treatments of reflection positivity and chessboard estimates were given by Fröhlich, Israel, Lieb, and Simon [92, 93]. We use the functional form of the chessboard estimate in [29, Theorem 5.8, (5.34)–(5.35)]. It applies to the algebra of bounded gauge-invariant functions of the links on one side of a lattice plane through links, and to the pullbacks of such functions by the block averaging maps with their Borel extension, as verified in Theorem 6.6. In the form proved there, it states the following. Let $\mu^{(\ell)}$ be the pushforward of the Wilson measure to level ℓ , tile the lattice at level ℓ into cells Q_t of $2 \times 2 \times 2$ sites, m in number, and let ϑ_t denote the reflected placements in the cells. Then

for nonnegative bounded gauge-invariant functions $f_1, \dots, f_k$ of the links of one cell and distinct cells $t_1, \dots, t_k$,

$$\mathbb{E}_{\mu^{(\ell)}} \prod_{i=1}^k \vartheta_{t_i} f_i \leq \prod_{i=1}^k \left[\mathbb{E}_{\mu^{(\ell)}} \prod_t \vartheta_t f_i \right]^{1/m},$$

and the same holds for each of the eight shifted tilings. Reflection positivity is used only for the Wilson measure and its pushforwards under the block averaging maps; a measure restricted to a fiber of the block averaging map or to a small-field event need not be reflection positive.

Lemma 22.11. *Let $N = uB^K$ with $u \in 4\mathbb{N}$, and let $1 \leq j \leq JK$. For each of the eight tilings at level j , let $\mathfrak{m}_j = n_j^3/8$ be the number of its disjoint cells of $2 \times 2 \times 2$ sites. All events below are gauge-invariant functions of the links of one cell, at the respective levels, pulled back to the fine lattice by the block averaging maps with their Borel extension, and they are considered under the Wilson measure $\mu = \mu_{N,\gamma}$. For two such events R_0 and I_0 attached to one cell, let $\vartheta_t R_0$ and $\vartheta_t I_0$ be their reflected placements in the cells t of the tiling, and set*

$$R = \bigcap_t \vartheta_t R_0, \quad I_{\text{all}} = \bigcap_t \vartheta_t I_0.$$

Then

$$\mu(R \cap I_0) \leq \mu(R \cap I_{\text{all}})^{1/\mathfrak{m}_j} \mu(R)^{1-1/\mathfrak{m}_j}, \quad \mu(R_0 \cap I_0) \leq \mu(R \cap I_{\text{all}})^{1/\mathfrak{m}_j}.$$

If $\mu(R) > 0$, this gives $\mu(I_0 \mid R) \leq \mu(I_{\text{all}} \mid R)^{1/\mathfrak{m}_j}$.

Now let $R_0 = R_{<j}(C; d_{\text{cut}})$ be the event of Lemma 7.4 that $q < d_{\text{cut}}$ for exactly those plaquettes at finer levels whose full dependence sets lie in the cell C , and let $I_0 = I_j(C; d_*)$ be the event that at least one internal level- j plaquette of C has $q \geq d_*$, where $d_* = d_{\text{cut}}/2$. The event R imposes no condition on plaquettes whose dependence sets cross the boundaries of the cells. Assume $\pi\sqrt{2d_{\text{cut}}} \leq \delta_*$, and let $c_E = 1 + C_M \pi\sqrt{2d_{\text{cut}}}$. On $R \cap I_{\text{all}}$, the function (22.2) of the configuration at level $j-1$ satisfies

$$Q_j^{\mathcal{T}} \geq \frac{4d_* \mathfrak{m}_j}{c_E h_{n_j}}.$$

Consequently, for every $s > 0$,

$$\log \mu_{N,\gamma}(R_0 \cap I_0) \leq -\frac{4sd_* n_j}{c_E \gamma} + \mathfrak{m}_j^{-1} \log \mathbb{E}_{\mu_{N,\gamma}}[e^{sQ_j^{\mathcal{T}}}; R].$$

The expectation on the right is restricted to the event R and taken with the partition function of the Wilson measure. If its logarithm is at most $3C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b^\circ$, then dividing this bound by $\mathfrak{m}_j$ gives, at $m = n_j$, exactly

$$\mathcal{A}_\gamma(m) = 24C_v M^3 + \frac{4C_s M^2}{m} \log(Mm/\gamma) + \frac{8C_b^\circ}{m^3}.$$

No division by the probability of a regularity event occurs. Regularity of all plaquettes at finer levels implies the event R , but R implies neither regularity of the plaquettes whose dependence sets cross cell boundaries nor any condition of the small-field event. The inequalities contain no constant depending on the depth, the geometry, or the coupling, apart from c_E .

Proof. Each indicator of an event in a cell used below is bounded by one and Borel: its list of plaquettes is finite, q is continuous, and every iterate of the block averaging map with its Borel extension is Borel. Since it is gauge invariant and depends only on links internal to its cell, it belongs to the algebra of Theorem 6.6; the support condition, the reflection geometry for odd M , and the exact number of cells needed there are given by Lemma 7.4. In the functional inequality we place $\mathbf{1}_{R_0 \cap I_0}$ in the distinguished cell and $\mathbf{1}_{R_0}$ in every other cell. Placing the first function at all reflected translates gives $\mathbf{1}_{R \cap I_{\text{all}}}$, and placing each of the other functions at all reflected translates gives $\mathbf{1}_R$. The product on the left is $\mathbf{1}_{R \cap I_0}$. This proves the first inequality, including the case of zero probabilities. Applying the same inequality to the single function $\mathbf{1}_{R_0 \cap I_0}$, with the constant function one in the other cells, proves the second inequality. If $\mu(R) > 0$, dividing the first inequality by $\mu(R)$ gives the conditional form, with the normalization still inside the root.

For the event R , every plaquette in a complete predecessor rectangle of an internal level- j plaquette is tested, since these lists are included in the tests of Lemma 7.4. By (22.1), the angles of these plaquettes at level $j - 1$ are at most $\pi\sqrt{2d_{\text{cut}}} \leq \delta_*$. Applying the transport estimate over one averaging step in Theorem 22.4(a) to the union of the internal level- j plaquettes, we obtain

$$\frac{4}{h_{n_j}} \sum_{\text{internal } P} q(V_P^j) \leq c_E Q_j^T.$$

On I_{all} every cell has at least one internal plaquette with $q \geq d_*$. The internal plaquettes of distinct cells are distinct, so the sum on the left is at least $4d_* m_j / h_{n_j}$. This proves the lower bound for Q_j^T . For $s > 0$, the exponential Markov inequality under the Wilson measure implies that

$$\mu(R \cap I_{\text{all}}) \leq \exp\{-4sd_* m_j / (c_E h_{n_j})\} \mathbb{E}_{\mu_{N,\gamma}}[e^{sQ_j^T}; R].$$

Taking the m_j -th root, using the second chessboard inequality, and taking logarithms, with $\log 0 = -\infty$, we obtain the stated bound, since $h_{n_j} = \gamma / n_j$. Only one factor $d_* = d_{\text{cut}}/2$ occurs.

Finally, we have $n_{j-1} = Mm$, $m_j = m^3/8$, and $\log(1/g_{n_{j-1}}) = \log(Mm/\gamma)/2$. Dividing the three terms of the assumed bound by $m^3/8$ gives $24C_v M^3$, $4C_s M^2 \log(Mm/\gamma)/m$, and $8C_b^\circ/m^3$, respectively, which is $\mathcal{A}_\gamma(m)$. Regularity of all plaquettes at finer levels implies the event R by inclusion of the lists of plaquettes, while R omits every finer plaquette whose full dependence set crosses the boundary of its cell. $\square$

22.9 Domination of successive small-field restrictions

The next lemma is the input for the second term in the bound on large fields, Theorem 2.7, which concerns configurations whose curvature is small at every scale but for which some fluctuation cutoff or coordinate-domain condition fails. The probability of the preceding conditions is multiplied, never divided, and no condition at later scales enters. The restricted measures and their dominations are indexed by $0 \leq a \leq K$, and quantities involving the next RG step by $0 \leq a < K$.

Lemma 22.12. *Fix the small-field event of Definitions 4.4 and 14.4, with its prefactors and its fixed local parameters, together with $L > 0$, M , J , and $0 < \gamma_0 \leq 1$. For $N = uB^K$ and $0 \leq a \leq K$, let S^a be the event consisting of the initial condition $U \in D_r$ and precisely the conditions of the first a RG steps in (4.6), and put*

$$\eta_a = (T_M^a)_\#(\mathbf{1}_{S^a} \mu_{N,\gamma}), \quad \zeta_a = (T_M^a)_\#(\mathbf{1}\{V^{bJ} \in D_r \text{ for } 0 \leq b \leq a\} \mu_{N,\gamma}).$$

For $0 \leq a < K$, at the output side $m = n_{(a+1)J}$ of the next RG step, set

$$\begin{aligned} t_a^2 &= \gamma / (Mm), & \delta_a &= 6b_a e^{-m_w p(t_a)^2/24}, & z_a &= 6b_a e^{-m_w Q_{\text{DB}}^2 / (24t_a^2)}, \\ \epsilon_a &= \delta_a / (1 - z_a), & b_a &\leq C_J m^3, \end{aligned}$$

where $p(t) = A_w \log(1/t)$ and $a_w = m_w A_w^2 / 24$. Here b_a is the number of three-dimensional fluctuation blocks of the RG step, $Q_{\text{DB}} > 0$ is the radius denoted Q_{phys} in Definition 14.4, whose product of balls of radius Q_{DB}/t_a lies inside the supported fiber coordinates, and $m_w > 0$ is the lower bound for the Hessian of the conditional fluctuation potential; neither Q_{DB} nor m_w is related to the plaquette-energy sum or to the logarithmic cutoff. Above the common minimum side described below, the comparison of Theorem 14.7, with identical unnormalized prefactors, gives

$$\mathbf{1}_{D_r}(T_M^J)_\# \eta_a \leq (1 - \epsilon_a)^{-1} \eta_{a+1}, \quad \zeta_a \leq \prod_{b < a} (1 - \epsilon_b)^{-1} \eta_a.$$

It also gives the following bound for the failure of the next condition:

$$\mu_{N,\gamma}(S^a \cap A_m \cap W_m^c) \leq \epsilon_a \mu_{N,\gamma}(S^a) \leq C_J m^3 e^{-a_w \log^2 \sqrt{Mm/\gamma}},$$

where $A_m = \{V^{(a+1)J} \in D_r\}$ and $W_m = \mathcal{A}_{a,J} \cap \{\max_{i,e} |w_{a,i,e}| \leq p(t_a)\}$ is the condition of the next RG step on the coordinate domains and the fluctuation cutoff, so that $S^{a+1} = S^a \cap A_m \cap W_m$. This bound keeps only the preceding conditions and the condition on the target; there is no conditioning on regularity at later scales.

There is one minimum side, valid simultaneously for all $0 < \gamma \leq \gamma_0$ and every finite depth, denoted $n_{\min}$, above which $z_a, \epsilon_a \leq 1/2$ and the product is at most e^{C_δ} ; the logarithm of the product can be made at most one. Every event S^a has positive probability. Let $\hat{v}_a = \eta_a / \mu_{N,\gamma}(S^a)$ be the normalized stopped measure. If $G_{\leq aJ}$ is an event of regularity of the curvature which, by

Theorem 7.7, implies the conditions $V^{bJ} \in D_r$ for $0 \leq b \leq a$, then for every nonnegative Borel function F of the configuration at level aJ ,

$$\mathbb{E}_{\mu_{N,\gamma}}[F(V^{aJ}); G_{\leq aJ}] \leq e^{C_\delta} \mu_{N,\gamma}(S^a) \mathbb{E}_{\hat{\nu}_a} F \leq e^{C_\delta} \mathbb{E}_{\hat{\nu}_a} F.$$

This is a multiplicative domination, valid also for test functions with exponentially large values on the finite lattice; the probability $\mu_{N,\gamma}(S^a)$ is not inverted. It does not identify a stopped measure with a marginal of the conditioned Wilson measure.

The same minimum side can be chosen so that, in addition, any fixed finite collection of the following inequalities holds: for $k \in \{1, 2\}$, fixed $C < \infty$, and fixed $\varepsilon > 0$,

$$C[1 + \log(m/\gamma)]^k \leq \varepsilon m/\gamma \quad (m \geq n_{\min}, 0 < \gamma \leq \gamma_0).$$

For the event $\text{FF}_0 = \{\max_p q(U_p) \geq d_*\}$ that some plaquette of the fine lattice has $q(U_p) \geq d_*$, the following bound holds at every $N \in 4\mathbb{N}$ with $\beta_N = 4N/\gamma \geq 2$:

$$\mu_{N,\gamma}(\text{FF}_0) \leq 3N^3 \exp\left\{\frac{C_{\text{LT}} + 36 \log \beta_N/N}{8} - \frac{\beta_N d_*}{16}\right\},$$

where $C_{\text{LT}} = \max\{C_0 + 24 \log 2, 0\}$, with C_0 the constant of Theorem 6.6, is independent of N , β_N , and γ . Since

$$\sup_{\beta \geq 4N/\gamma_0} \frac{C_{\text{LT}} + 36 \log \beta/N}{\beta} \leq \frac{C_{\text{LT}}\gamma_0}{4N} + \frac{9\gamma_0 \log(4N/\gamma_0)}{N^2} \rightarrow 0,$$

one obtains, above the common minimum side,

$$\mu_{N,\gamma}(\text{FF}_0) \leq 3N^3 e^{-d_* N/(8\gamma)}.$$

The bound for the supremum is used for $4N/\gamma_0 \geq e$. The numbers of free blocks can always be kept in the form $\mathbf{b} \leq C_J m^3$; in terms of $t^2 = \gamma/(Mm)$ this also gives $\mathbf{b} \leq C_J \gamma_0^3 t^{-6}/M^3$. All minimum sides and all constants of absorption and of the products have the dependence described in item (4) of Definition 4.14; they are independent of the coupling, the side, the depth, and the geometry and age of decompositions. The correction $36 \log \beta_N/N$ is kept exactly until the minimum side, uniform in the coupling, has been chosen.

Proof. Restricted measures are indexed by $0 \leq a \leq K$, and quantities of a transition by $0 \leq a < K$. We first choose the common minimum side, so that all positivity conditions on denominators hold before a quantitative comparison is applied.

Step 1: measure-theoretic hypotheses. The amplitude $A_w \geq 1$ is fixed before its coupling ceiling and before $n_{\min}$. The product domains on the torus here are the instances of Lemma 22.3 without boundary. In particular, their sets of supported fluctuation coordinates, their cutoff domains, and their tests are Borel, the conditional fluctuation potential

is smooth on a neighborhood of the compact enclosing product domain, and (22.5) gives a positive finite partition function at each fixed target satisfying the coordinate-domain conditions. The initial Wilson density is positive, and every identical prefactor carries the normalization of the Wilson measure. These facts verify the measure-theoretic hypotheses of the comparison in Lemma 20.5 whenever its conclusion for products over endpoints is used below.

Step 2: the common minimum side. We choose all minimum sides uniformly in the coupling. In particular, take $n_{\min} \geq \gamma_0/(Mt_*^2)$, so $t_a^2 = \gamma/(Mm) \leq t_*^2$ at every transition $m \geq n_{\min}$ and $\gamma \leq \gamma_0$, as required by Theorem 14.7. Set

$$z(m, \gamma) = \frac{1}{2} \log(Mm/\gamma), \quad a_w = m_w A_w^2/24 > 0.$$

Since $p(t) = A_w \log(1/t)$ and $\log(1/t_a) = z(m, \gamma)$, the errors satisfy

$$\delta_a \leq C_J m^3 e^{-a_w z(m, \gamma)^2}, \quad z_a \leq C_J m^3 e^{-b_{DB} m/\gamma}, \quad b_{DB} = m_w M Q_{DB}^2/24 > 0.$$

For any fixed $R > 1$, the exponential envelope has ratio

$$R^3 e^{-b_{DB}(R-1)m/\gamma} \leq R^3 e^{-b_{DB}(R-1)n_{\min}/\gamma_0}$$

between the sides Rm and m , for $m \geq n_{\min}$ and $\gamma \leq \gamma_0$. For the envelope $W(m, \gamma) = C_J m^3 e^{-a_w z(m, \gamma)^2}$, let $\Delta_R = (\log R)/2$. Since $z(Rm, \gamma) = z(m, \gamma) + \Delta_R$, its ratio is exactly

$$\frac{W(Rm, \gamma)}{W(m, \gamma)} = R^3 e^{-a_w (2\Delta_R z + \Delta_R^2)},$$

and the right side decreases in z . Also

$$\frac{\partial \log W}{\partial \log m} = 3 - a_w z, \quad \frac{\partial \log W}{\partial \log \gamma} = a_w z > 0 \quad \text{when } z > 0.$$

Thus, for $m \geq n_{\min}$ and $0 < \gamma \leq \gamma_0$, one sufficiently large $n_{\min}$ makes both ratios at $R = B$ at most $1/2$. Both envelopes then decrease in m , they are maximized in the coupling at γ_0 , and their first terms tend to zero as $n_{\min} \rightarrow \infty$. The same holds for each ratio in a fixed finite family of remainders, after taking the maximum of the required minimum sides.

We increase this minimum side until $z_a \leq 1/2$ for every a and $W(n_{\min}, \gamma_0) \leq 1/4$. Then

$$z_a \leq \frac{1}{2} < 1, \quad \delta_a \leq \frac{1}{4} < 1, \quad \epsilon_a = \frac{\delta_a}{1 - z_a} \leq \frac{1}{2} < 1 \quad (0 \leq a < K). \quad (22.23)$$

Thus the hypotheses $d_t < 1$ and $\epsilon_t < 1$ of Theorem 14.7 both hold before any error bound or reciprocal bound is used. We set $D_a = \prod_{0 \leq b < a} (1 - \epsilon_b)^{-1}$ for $0 \leq a \leq K$, with

$D_0 = 1$. The output sides of the RG steps of a finite sequence, ordered increasingly, form a segment of the sequence $m, mB, mB^2, \dots$, where $m \geq n_{\min}$. Using $\epsilon_a \leq 2\delta_a$ and $-\log(1-x) \leq 2x$ for $0 \leq x \leq 1/2$, we obtain

$$\sum_a \epsilon_a \leq 4W(n_{\min}, \gamma_0), \quad \log D_a \leq 2 \sum_b \epsilon_b \leq 8W(n_{\min}, \gamma_0).$$

Increasing $n_{\min}$ makes the last quantity at most one, or arbitrarily small. This proves the assertion on C_δ and the bound by the exponential envelope, with an enlarged fixed C_J . The argument is an explicit instance of the conclusion for products over endpoints in Lemma 20.5.

Step 3: positivity. The events S^a have positive probability. Initially, D_r contains a product neighborhood of the identity of positive Haar measure, on which the Wilson density is positive. At each further RG step on the torus, the cutoff domain has positive volume and the integrand is real and positive. The recursion for the stopped measures used in Theorems 14.8 and 21.13 therefore multiplies a positive constant by a positive integral at the stopping level, which is bounded below by its strictly positive trial integral over D_r . By induction, $\mu_{N,\gamma}(S^a) > 0$ for every $0 \leq a \leq K$. The normalized pushforward is exactly the stopped measure $\hat{v}_a$, at the preceding depth aJ .

Step 4: one RG step. At the initial side, $\eta_0 = \zeta_0 = \mathbf{1}_{D_r} \mu_{N,\gamma}$. The density of η_a is the unnormalized density of the stopped measure, including the probability of the preceding conditions and all its constant factors. For $0 \leq a < K$, at a target of side $m = n_{(a+1)J}$ satisfying the coordinate-domain conditions, we apply Theorem 14.7 to this density. Its functions G and $\tilde{F}$ have identical Haar disintegration, Gaussian, and constant prefactors, and multiplying both by the probability of the preceding conditions preserves their ratio. The quantities Q_{phys} and m_{cv} of that theorem are the present Q_{DB} and m_{w} . Its number of blocks is the number $b_a \leq C_J m^3$ of blocks of the RG step, its largest integrated coupling is $t_a^2 = \gamma/(Mm)$, and its logarithmic cutoff is the common $p(t_a)$ at every averaging step of this RG step. The condition $t_a p(t_a) < R_{\text{en}}$ follows from the fixed ceiling for the logarithmic family, and the enclosing product domain is that of the fixed parameters of the stopped measures. The exact identity $\tilde{F}_a/G_a = \kappa_c^{\text{old}}(\Omega_p^{\text{w}})$ needs no smallness of the errors. Its consequence for the tail uses $d_{t_a} = z_a < 1$ from (22.23) and gives, on D_r ,

$$\tilde{F}_a/G_a \geq 1 - \epsilon_a, \quad \epsilon_a = \delta_a/(1 - z_a), \quad (22.24)$$

with the displayed δ_a and z_a . The reciprocal bound also uses $\epsilon_a < 1$ from (22.23). The ratio bound holds at every target satisfying the coordinate-domain conditions, and integrating it, we obtain $\mathbf{1}_{D_r}(T_M^J)_{\#} \eta_a \leq (1 - \epsilon_a)^{-1} \eta_{a+1}$. This comparison includes the coordinate-domain conditions of the next RG step, as required in $\tilde{F}_a$.

Step 5: induction and the failure bound. For $0 \leq a < K$, if $\zeta_a \leq D_a \eta_a$, then, since pushforward and restriction preserve inequalities between measures,

$$\zeta_{a+1} = \mathbf{1}_{D_r}(T_M^J)_\# \zeta_a \leq D_a \mathbf{1}_{D_r}(T_M^J)_\# \eta_a \leq D_a (1 - \epsilon_a)^{-1} \eta_{a+1}.$$

By induction, the stated domination holds for every $0 \leq a \leq K$. For $0 \leq a < K$, the failure bound follows from the other consequence of (22.24), namely $G_a - \tilde{F}_a \leq \epsilon_a G_a$ on D_r . Integrating over the targets in D_r , we obtain

$$\mu_{N,\gamma}(S^a \cap A_m \cap W_m^c) \leq \epsilon_a \mu_{N,\gamma}(S^a \cap A_m) \leq \epsilon_a \mu_{N,\gamma}(S^a).$$

Step 6: regular events and test functions. For $0 \leq a \leq K$, by the hypothesis on $G_{\leq aJ}$, the deterministic inclusion of Theorem 7.7 places every configuration V^{bJ} , $0 \leq b \leq a$, on this event in D_r . Its hypotheses include the fixed threshold for the plaquette energies and the minimum side of that theorem, and arbitrary holonomies of flat connections are allowed. Thus the pushforward of the restriction of $\mu_{N,\gamma}$ to $G_{\leq aJ}$ is bounded by ζ_a . For every nonnegative bounded Borel function F , the domination implies that

$$\mathbb{E}_{\mu_{N,\gamma}}[F(V^{aJ}); G_{\leq aJ}] \leq D_a \int F d\eta_a = D_a \mu_{N,\gamma}(S^a) \mathbb{E}_{\hat{\nu}_a} F.$$

For an arbitrary nonnegative Borel function F , we apply this to $\min(F, k)$ and let $k \rightarrow \infty$. This proves the stated inequality also when the test function has exponentially large values on the finite lattice. The probability $\mu_{N,\gamma}(S^a)$ is at most one and may be dropped, without being inverted. When $K = 0$ there is only $\eta_0 = \zeta_0 = \mathbf{1}_{D_r} \mu_{N,\gamma}$ and $D_0 = 1$, and no transition error, next target, or failure event occurs.

Step 7: absorption of logarithms. For the remaining minimum sides, we set $x = m/\gamma \geq n_{\min}/\gamma_0$. For $k = 1, 2$, we have

$$\frac{d}{dx} \frac{(1 + \log x)^k}{x} = \frac{(1 + \log x)^{k-1} (k - 1 - \log x)}{x^2}.$$

Thus this ratio decreases to zero once $x \geq e^{k-1}$. For a given finite collection of C , k , and ϵ , we choose $n_{\min}$ so that the ratio at $n_{\min}/\gamma_0$ is at most ϵ/C for each entry. This proves all the displayed absorptions simultaneously.

Step 8: the fine lattice. Finally, we use the bound of Theorem 6.6 for exponential moments of arbitrary sets of plaquettes at the fine scale, with a single plaquette and its parameter $s = 1/2$. At $\beta_N = 4N/\gamma \geq 2$ it reads

$$\mathbb{E}_{\mu_{N,\gamma}} e^{\beta_N q(U_p)/16} \leq \exp\{[C_0 + 24 \log 2 + 36 \log \beta_N / N] / 8\}.$$

By the exponential Markov inequality and a union bound over the $3N^3$ positive plaquettes, we obtain the displayed bound with $C_{\text{LT}} = C_0 + 24 \log 2$, enlarged to be nonnegative

if necessary. For $4N/\gamma_0 \geq e$, both $1/\beta$ and $(\log \beta)/\beta$ decrease on the ray $\beta \geq 4N/\gamma_0$; evaluating each at the left endpoint gives the displayed bound for the supremum. Its right side tends to zero as $N \rightarrow \infty$, independently of the coupling. We choose the common minimum side so that it is at most $d_*/4$ for every larger N . Then the positive part of the exponent is at most $\beta_N d_*/32$, which leaves

$$-\frac{\beta_N d_*}{16} + \frac{\beta_N d_*}{32} = -\frac{\beta_N d_*}{32} = -\frac{d_* N}{8\gamma}.$$

This is the asserted bound, in which the exact correction $36 \log \beta_N / N$ was kept until this absorption, which is uniform in the coupling.

Solving $t^2 = \gamma/(Mm)$ gives $m^3 = \gamma^3 t^{-6}/M^3 \leq \gamma_0^3 t^{-6}/M^3$, which proves the last counting formula. Taking the maximum of the finitely many minimum sides constructed above, and rounding up to an element of $4\mathbb{N}$, proves the claim on the common minimum side, with the dependence stated in the lemma. $\square$

The next lemma extends these conclusions to an arbitrary finite coupling ceiling Γ , which is the form used in Theorem 23.5.

Lemma 22.13. *Fix $0 < \Gamma < \infty$ and the parameters, the buffers, the count of affected plaquettes, the finite family of stopping offsets and remainders, and the interval of s of Theorem 22.1. The exponential-moment bounds and the conclusions on decompositions of that theorem hold for $0 < \gamma \leq \Gamma$. Above one further minimum side, all conclusions of Lemma 22.12 on successive restrictions, on nonnegative test functions, and on the fine lattice also hold for $0 < \gamma \leq \Gamma$, with the same small-field event and identical unnormalized prefactors. One may require*

$$m \geq \max\{e\Gamma, \Gamma/g_{\text{end}}^2, \Gamma/(Mg_*^2)\},$$

with g_{end} as in (22.18), and $C_{\text{ch},J}^\Gamma = (1 + \log_+ \Gamma) C_{\text{ch},J}^{\text{base}}$. In particular, for $0 \leq a < K$, with $m = n_{(a+1)J}$ and with A_m and W_m as in Lemma 22.12,

$$\mu_{N,\gamma}(S^a \cap A_m \cap W_m^c) \leq \epsilon_a \mu_{N,\gamma}(S^a) \leq C_J m^3 e^{-a_w \log^2 \sqrt{Mm/\gamma}}.$$

There is no conditioning on the curvature at later scales. All constants have the dependence described in item (4) of Definition 4.14, with the fixed ceiling Γ ; the constants of the local estimates and the intervals of s and of the action coefficients do not change when the ceiling is increased.

Proof. The assertion on exponential moments was proved in Theorem 22.1 for the whole interval $0 < \gamma \leq \Gamma$. For the assertions on successive restrictions, we use Theorem 14.7 with the same unnormalized prefactor and the count $b_a \leq C_J m^3$. With $t_a^2 = \gamma/(Mm)$, its cutoff and exterior errors are exactly $\delta_a = 6b_a e^{-m_w p(t_a)^2/24}$ and $z_a = 6b_a e^{-m_w Q_{\text{DB}}^2/(24t_a^2)}$. The bounds for a finite coupling ceiling in Lemma 20.15 make $z_a \leq 1/4$, $\epsilon_a = \delta_a/(1 - z_a) \leq 1/2$, and the logarithm of every reciprocal product at most one, above a minimum side

independent of $\gamma \in (0, \Gamma]$. All transitions have $a < K$. Every finite sequence of satisfied conditions has positive probability. Indeed, at the identity configuration its coordinate-domain conditions hold strictly and its normalized fluctuation variables vanish; thus a finite intersection of neighborhoods of the identity on which the conditions hold has positive Haar measure, and the Wilson density is positive. Iterating the comparison of one RG step as in Steps 4–6 of the proof of Lemma 22.12, we obtain the domination and every consequence for nonnegative test functions. Multiplying the failure bound for one RG step by the probability of the preceding conditions proves the displayed bound; the event keeps only the preceding conditions and the target.

The estimate on the fine lattice from Theorem 6.6, in the form used in Step 8 of the proof of Lemma 22.12, is

$$\mu(\text{FF}_0) \leq 3N^3 \exp\{(C_{\text{LT}} + 36 \log \beta/N)/8 - \beta d_*/16\}, \quad \beta = 4N/\gamma.$$

For $4N/\Gamma \geq e$, the monotonicity of $1/\beta$ and $\log \beta/\beta$ implies that

$$\frac{C_{\text{LT}} + 36 \log \beta/N}{\beta} \leq \frac{C_{\text{LT}}\Gamma}{4N} + \frac{9\Gamma \log(4N/\Gamma)}{N^2} \longrightarrow 0.$$

We choose the minimum side so that this is at most $d_*/4$. The exponent is then at most $-\beta d_*/32 = -d_*N/(8\gamma)$, which proves the same bound on the fine lattice uniformly for $0 < \gamma \leq \Gamma$. $\square$

Uniform bounds for the small-field event

This section proves that the small-field event $S_{N,u,\gamma}$ of Definition 4.4 has probability close to one under the Wilson measure, uniformly in the number of renormalization group (RG) steps and in the coupling below a fixed ceiling. The main result is Theorem 23.10. For every $0 < \Gamma < \infty$ it provides an integer J , with $B = M^J$, and all remaining parameters of the small-field event, chosen before the coupling, such that

$$\mu_{uB^K,\gamma}(S_{uB^K,\gamma}^c) \leq C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}} \quad (23.1)$$

for all $0 < \gamma \leq \Gamma$, all coarse sides $u \in 4\mathbb{N}$ above a threshold n_Γ , and all $K \geq 0$. The first term bounds the probability that some averaged configuration $V^j = T_M^j U$ has a plaquette whose energy exceeds a fixed threshold d_* . The second term bounds the probability that the curvature is small at every scale, but the coordinate-domain conditions or the cutoffs $|w_{k,i,e}| \leq A_w \log(1/t_k)$ on the normalized fluctuation variables fail in some RG step. This is the bound on large fields, Theorem 2.7, which lets us pass from the conditioned Wilson measure $\nu_{N,u,\gamma}$ to the Wilson measure by (3.1).

Two regimes of (23.1) are used later. For a fixed coarse side u , the probability of S^c is smaller than every power of γ as $\gamma \rightarrow 0$, uniformly in K (Corollary 23.11); this is used in the weak-coupling analysis. For a fixed ceiling Γ , it tends to zero as $u \rightarrow \infty$ through the sides nB^a , uniformly in $\gamma \in (0, \Gamma]$ and K ; this is used to remove the conditioning in the construction of the continuum measure. The logarithmic cutoff limits the rate: by Theorem 23.12, at a fixed coarse side, $\sup_K \mu(S^c)$ is of order $\exp(-c \log^2(1/\gamma))$, and a matching lower bound holds already for one RG step.

The principal antecedent of the present construction is Bałaban's proof of ultraviolet stability for three-dimensional lattice gauge theories [15]. The estimate proved here concerns the probability of the specific event $S_{N,u,\gamma}$. Its proof has two independent parts: a bound on large curvature under the Wilson measure itself, and a bound on the first failure of a coordinate-domain condition or of a cutoff. Both are described at the beginning of Section 23.4.

The other task of this section is the choice of parameters. The integer J , the analytic radii, the support radius r of B_r and D_r , the widths of the boundary layers of the large-field decompositions, the cutoff amplitude A_w , the coupling bound g_* , and the minimum coarse side must satisfy all hypotheses of the results of Sections 13 to 15 and 18 to 22, and these hypotheses refer to one another. Theorem 23.2 lists the choices in an order in

which each choice uses only quantities fixed before it, and Table 23.1 summarizes this order. The main point is that J is chosen using only constants that are uniform in J , and is never enlarged afterward.

Sections 23.1 and 23.2 contain the elementary inequalities and the statements on the choice of parameters, Sections 23.3 to 23.5 the probability estimates, Sections 23.6 and 23.7 the proofs of the parameter theorems, and Section 23.8 the extension to every sufficiently large J , which is needed in Section 29 to pass from the subsequences $n_* B^K$ to the full sequence nM^k .

The main inputs are the moment bounds for coarse actions and the comparison for initial segments of Section 22; the large-field decompositions and their pointwise support bounds for the normalized fluctuation variables in Section 21; the regional estimates of Sections 18 to 20; the recursion for the effective interaction on the torus and the reconstruction maps of Sections 13 to 15; and the chessboard and curvature estimates of Sections 6 and 7. The results of this section are used in Sections 28 and 29 to remove the conditioning and to construct the continuum measure, in Sections 31 and 32 at fixed coupling, and in Sections 34 to 36 at weak coupling.

Throughout, $L > 0$ and an odd integer $M \geq 3$ are fixed. We write $h_n = g_n^2 = \gamma/n$ for the lattice coupling at side n . The letter p with an argument denotes the cutoff radius $p(t) = A_w \log(1/t)$ (written $p_w(t)$ in Definition 4.4); a subscript p on a configuration denotes a plaquette.

23.1 Elementary inequalities

We use the following lemma repeatedly to choose a coupling bound and a minimum side. It states that powers of logarithms are dominated by positive powers, and that sums over geometrically growing sides are dominated by their first term.

Lemma 23.1. *Fix $0 < \gamma_0 < \infty$, a finite set of ratios $R > 1$, finitely many positive constants a, c, C, ε , and finitely many nonnegative integers k . Then there exists $0 < g_* < e^{-1}$ such that each of the finitely many inequalities $Ct^\varepsilon[1 + \log(1/t)]^k \leq c$ holds for all $0 < t \leq g_*$. There exists m_0 , independent of $\gamma \in (0, \gamma_0]$, such that each of the finitely many inequalities*

$$C[1 + \log(m/\gamma)]^k \leq \varepsilon m/\gamma$$

holds for all $m \geq m_0$ and $0 < \gamma \leq \gamma_0$. For a fixed $M > 1$ define

$$X_c(m, \gamma) = m^3 e^{-cm/\gamma}, \quad W_a(m, \gamma) = m^3 e^{-az(m, \gamma)^2}, \quad z(m, \gamma) = \frac{1}{2} \log(Mm/\gamma).$$

After increasing m_0 , each of these functions is decreasing in $m \geq m_0$ and increasing in $\gamma \in (0, \gamma_0]$, and its value at Rm is at most half of its value at m . Consequently

$$\sum_{h \geq 0} X_c(mR^h, \gamma) \leq 2X_c(m, \gamma), \quad \sum_{h \geq 0} W_a(mR^h, \gamma) \leq 2W_a(m, \gamma),$$

and the right sides tend to zero as $m \rightarrow \infty$, uniformly in $\gamma \in (0, \gamma_0]$. Finally, let $\delta > 0$, and let numbers $\epsilon_h \geq 0$, attached to the sides mR^h , be bounded by fixed finite sums of these functions. Then, after increasing m_0 , we have $\log \prod_h (1 - \epsilon_h)^{-1} \leq \delta$ for all $m \geq m_0$. The thresholds g_* and m_0 depend only on the data fixed in the statement.

Proof. Set $y = \log(1/t)$. For $\varepsilon > 0$ and an integer $l > k$ we have $e^{\varepsilon y/2} \geq (\varepsilon y/2)^l/l!$ for $y > 0$, which implies $e^{-\varepsilon y}(1+y)^k \rightarrow 0$ as $y \rightarrow \infty$. Then each inequality holds on a half-line $y \geq y_*$, that is, for $0 < t \leq g_*$, and we take the smallest of the finitely many thresholds. The same calculation with $y = \log x$ shows that $(1 + \log x)^k/x \rightarrow 0$ as $x \rightarrow \infty$. Since $x = m/\gamma \geq m_0/\gamma_0$, the supremum of this ratio over the relevant half-line tends to zero as $m_0 \rightarrow \infty$, which gives the second assertion.

With $\Delta_R = (1/2) \log R > 0$, the two ratios are

$$\frac{X_c(Rm, \gamma)}{X_c(m, \gamma)} = R^3 e^{-c(R-1)m/\gamma}, \quad \frac{W_a(Rm, \gamma)}{W_a(m, \gamma)} = R^3 e^{-a(2\Delta_R z + \Delta_R^2)}. \quad (23.2)$$

Their suprema over $m \geq m_0$ and $\gamma \leq \gamma_0$ are attained at $m = m_0$ and $\gamma = \gamma_0$, and tend to zero as $m_0 \rightarrow \infty$. The derivatives of $\log X_c$ and $\log W_a$ with respect to $\log m$ are $3 - cm/\gamma$ and $3 - az$, which are negative once m_0 is large; their derivatives with respect to $\log \gamma$ are cm/γ and az , which are positive in the same range. Taking logarithms shows that $m^3 e^{-cm/\gamma_0} \rightarrow 0$ and $m^3 e^{-a \log^2(Mm/\gamma_0)/4} \rightarrow 0$. The sums are bounded by geometric series with ratio $1/2$. Finally, for $0 \leq \epsilon \leq 1/2$ we have

$$-\log(1 - \epsilon) = \int_0^\epsilon \frac{dv}{1-v} \leq 2\epsilon.$$

We increase m_0 so that every ϵ_h is at most $1/2$ and $2\sum_h \epsilon_h \leq \delta$. This proves the last assertion, also for finite sublists of the ϵ_h taken in either order. All thresholds depend only on the fixed data of the lemma. $\square$

23.2 Choice of the parameters

The constants in the next theorem are those of the cited earlier results, evaluated on their larger construction domains before any of the later restrictions. In particular, q_{CA} is the contraction factor for the zero-fluctuation pullback of the effective interaction on the torus in Theorem 13.5, C_G is the constant bounding the part of the effective interaction generated by one RG step, and $Z_\bullet$ are the majorants for the absolute coefficient sums of the background-field map at zero fluctuation from Theorem 13.2. In Step 8 of Section 23.6, we use Theorem 14.6 to identify the recursion controlled by these constants with the recursion defining the small-field event. The number $q_{\text{raw}} > 0$ is a radius reserved for the free logarithmic coordinates. The symbols A_j, R_j denote the coefficient bounds for the reconstruction maps $\hat{\mathcal{R}}$ in Lemma 15.16; output constants denoted by the same letters

elsewhere can be enlarged to a common bound at fixed J . The normalization constant Λ is kept separate from the bound $\mathcal{L}_V$ for the conditional fluctuation potential. A subscript $\bullet$ stands for either of the two analytic radii, ρ for the effective interaction and ρ_{cl} for the effective classical action, and every condition carrying $\bullet$ is imposed for both.

The radii play different roles. The support radius r defines B_r and D_r ; $R_{\text{buf}} = C_{\text{map},J}r$ is the larger radius of coarse fields on which the analytic recursion is defined; ρ and ρ_{cl} are analytic radii of coefficient norms; R_{raw} is the radius of the larger domain of free logarithmic coordinates; r_{eff} is the working radius for the zero-fluctuation reconstruction; $0 < P < Q_{\text{DB}}$ are radii, proportional to r , for the unscaled fluctuation coordinates $\zeta_i = g_i w_i$ in the small-field construction; R_W bounds the unscaled fluctuation coordinates allowed in the regional estimates, and $R_{\text{en}} \leq R_W$ is the radius of the enclosing product domain; β_{rec} bounds the frozen coordinates of the large-field decompositions.

In applying Section 15.3, its generic torus interaction exponent a_{sp} is instantiated by $a_{\mathbb{T}}$. Its contraction and the construction and convergence of the torus effective interactions are used at that fixed exponent. The smaller regional exponent $a = a_{\text{sp}}$ chosen below is used for boundary estimates; bounds and convergence of the already constructed torus interactions pass to it by monotonicity of the coefficient norms. No contraction on an entire ball at the weaker exponent is inferred from this monotonicity.

Theorem 23.2. *Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \gamma_0 \leq 1$. There exist an integer $J \geq 1$, with $B = M^J$, a tuple Θ of the parameters listed in items (1)–(10) below, a number $0 < g_* < e^{-1}$, and an integer $n_{\text{min}} \in 4\mathbb{N}$, all chosen before the coupling, with the following property. For every*

$$0 < \gamma \leq \gamma_0, \quad u \in 4\mathbb{N}, \quad u \geq n_{\text{min}}, \quad K \in \mathbb{N}_0, \quad N = uB^K,$$

the same J , Θ , and reconstruction maps satisfy the local, moment, and scalar conditions of the results cited below; only the running couplings and the cutoff radii $p(t)$ are evaluated at γ . In particular, the parameters satisfy the hypotheses of Theorem 19.9 on uniform incoming radii, on boundary compatibility, and on conditional displacements, and the bounds of Theorem 20.14 on logarithmic fluctuation radii and on frozen coordinates. In the large-field decompositions only complete renormalization group steps are integrated; remainders modulo J and short scale intervals are treated by the composed insertions and the bounded side-ratio clauses of Theorem 22.1. The coefficient majorants on the torus and the recursion for the effective interaction are those of Theorems 13.2 and 13.5, and Theorem 14.6 identifies this recursion with the effective interactions of the stopped measures used in Theorem 22.1. The geometric hypotheses of Theorem 21.5 hold, including those of Theorem 21.3 for the neighborhoods in which gauge transformations are extended, the closed complex of tested cells at the initial step, and the protection of all older endpoints and tree roots attached to frozen records.

The parameters can be chosen in the order of items (1)–(10). Each item uses only constants provided by the cited results and quantities fixed in earlier items.

- (1) The number J . Fix the spatial decay rate $a_{\mathbb{T}}$ of the torus estimates with $0 < a_{\mathbb{T}} < a_{\text{cen}}/8$ and no larger than the classical coefficient rate of Theorem 10.11, where a_{cen} is the J -uniform rate of the bounds of Lemma 11.9 (the contraction of Theorem 11.10 for the aligned representative holds only at such output rates). Fix also all larger spatial, analytic, and differentiation domains and the analytic-radius enlargement factor K_* needed for the connected-network estimates of Theorem 20.9. Enlarge the J -uniform constant C_{CP} of part (3) of Theorem 15.13 to include the reconstruction tolerances. In the notation of that theorem, it bounds the value and the first three coarse-field derivatives of the pair formed by $B\hat{\Phi}(\theta, 0)$ and its first spatial differences $\nabla_{1/B}B\hat{\Phi}(\theta, 0)$, with the row sums, volume-normalized column sums, and polydisc of coarse fields specified there. Bounds for derivatives with respect to the fluctuation inputs are chosen after J is fixed. Let $Z_{\bullet}(t) \leq C_{\bullet}t/(1 - C_{\bullet}t)$ be the J -uniform majorants for the absolute coefficient sums of the background-field map at zero fluctuation on the torus. Choose J so that

$$B \geq B_{\text{geom}}, \quad q_{\text{CA}} < 1/4, \quad q_{\text{CP}} \leq 1/4, \quad M^{-J} \leq \delta_0(a_{\mathbb{T}}),$$

where $q_{\text{CA}} = C_P(a_{\mathbb{T}})/B$ (denoted q_P in Theorem 13.5), $q_{\text{CP}} = C_{\text{CP}}^2/B$, and $C_P(a_{\mathbb{T}})$ and $\delta_0(a_{\mathbb{T}})$ are the constants of Theorem 11.10, and so that the finitely many linear image conditions on the torus hold with strict inequality. These include $C_{\bullet}K_*/B < 1/32$, which permits the later bound $B^{-1}Z_{\bullet}(K_*\rho_{\bullet}) \leq \rho_{\bullet}/16$, and the two conditions (23.25) on spatial rates. Here B_{geom} is the constant of Theorem 20.9. None of these lower bounds on B uses a constant of the regional estimates or of the effective-interaction estimates computed at fixed J .

- (2) Coordinate domains at fixed J . Choose the product domains of Lemmas 19.3 and 19.7. In their notation choose $v_{\text{free},J} > 0$ with

$$Q_J v_{\text{free},J} < \lambda_J/64, \quad I_J Q_J v_{\text{free},J} < r_{z,J}/64,$$

and choose R_{raw} strictly below the radii in those results for the identities on overlapping gauge charts and for the local constraint coordinates, and strictly below

$$\min\{\min(r_{\Gamma,J}, r_E)/(64G_J), \lambda_J/(64Q_J), r_{z,J}/(64I_J Q_J), v_{\text{free},J}/(64C_{\text{lift},J})\}.$$

Fix the auxiliary reserve $h_J > 0$ of (20.1) below its finite convergence, implicit-function, resolvent, and logarithm radii. Restrict the outer background and sequential free products inside this reserve, with every free radius $Q < h_J$. For each initial segment of conditional integrations, fix first a neighborhood of its conditional constrained minimizer, then a smaller domain for the parameters of the later integrations, and then a common smaller product domain for the free logarithmic coordinates of the successive substitutions. These domains preserve real values on real arguments, the conditional displacement bound $r_I/4$, and the bound $1/2$ for the error of the preconditioned Hessian. All inverses are inverses of constrained Hessians in the free coordinates.

- (3) *Analytic radii and coefficient bounds.* Fix on the larger torus construction product a common image-majorant constant $C_{\text{tor},J} \geq \max_{\bullet} C_{T,\bullet}$, before choosing $\rho, \rho_{\text{cl}} > 0$. Choose both radii within the fixed inner, middle, and outer domains of the reconstruction maps and require $K_* \max(\rho, \rho_{\text{cl}}) < h_J$, $C_{\bullet} K_* \rho_{\bullet} \leq 1/2$, and $4C_{\text{tor},J} K_* \rho_{\bullet} < 1/4$ for both $\bullet$. These are radius restrictions at fixed J . Fix $E^{\text{cl}} \geq \sup_{d \geq 0} \mathcal{B}_{\rho_{\text{cl}}, a_{\mathbb{T}}}(E_d)$ for the effective classical action on the torus; the all-root factorial bounds make this finite at the chosen radius. Fix also a bound

$$E \geq 2C_G / (1 - q_{\text{CA}})$$

for the effective interaction. Set aside the inequality

$$C_{\text{lift},J} [K_* \max(\rho, \rho_{\text{cl}}) + r + \beta_{\text{rec}} + q_{\text{raw}}] < R_{\text{raw}} / 8,$$

in which the first summand, fixed now, is required to be below $R_{\text{raw}} / 64$. Choose now

$$0 < q_{\text{raw}} \leq R_{\text{raw}} / (64C_{\text{lift},J})$$

inside the common free-coordinate reserve. It measures displacement from the unperturbed joint center in the raw free logarithmic coordinates. The remaining summands are made small by β_{rec} in item (5) and r in item (6). Compute the coefficient constants A_J, R_J of the reconstruction maps on these larger domains, after one fixed enlargement, before the working and support radii are chosen; these constants and the bounds for the limiting maps do not depend on those radii. Then choose the regional rates as in Theorem 20.9: $a = a_{\text{sp}} > 0$, $\alpha = \alpha_{\text{bd}} = a/16$, equal classical rates, $m_{\text{rate}} = C_*^{\text{geom}}(a + \alpha)$, and $b_1 = 8C_*^{\text{geom}}\alpha$, with

$$m_{\text{rate}} + b_1 < \min(c_{\text{prim},J}, Ba) / C_{\text{alloc}}, \quad b_1 < B\alpha / C_{\text{alloc}}, \quad \sqrt{3}(a + \alpha) < a_{\mathbb{T}}.$$

Here $c_{\text{prim},J}$ is the reserve for the intrinsic tensors and the unperturbed weighted reconstruction images in the graph metric of Section 20.1, obtained from the Euclidean exponents by dividing by $\sqrt{3}$. It includes the positive exponent obtained once from interpolation of the unperturbed coefficient bounds and is chosen no larger than the intrinsic exponent c_J of Theorem 19.9. Inherited perturbative networks use the separate inequalities against Ba and $B\alpha$. Decreasing a changes neither $a_{\mathbb{T}}$ nor J .

- (4) *Perturbation bounds and the fluctuation radius.* Choose a preliminary bound $\sigma_0 > 0$ for perturbations that are real on real arguments, on the larger product domains. Compute all local constants of the output, sequential, image, curvature, and mean estimates, and of the estimates of the support of the normalized fluctuation variables, uniformly for $\mathcal{X}^{\text{cl}}(\bar{P}) \leq 2\sigma_0$ and $\mathcal{X}(f) \leq \sigma_0$. In particular fix

$$A \geq 1, \quad C_0 \geq 2A^3, \quad C_1, \quad C_{\text{en},J}, \quad \mathcal{L}_V = C_{V,J}(1 + E + \sigma_0) \geq 1,$$

the already bounded torus image constants $C_{T,\bullet} \leq C_{\text{tor},J}$, the remaining image constants $C_{\Delta,\bullet}, C_{F,\bullet}, C_{P,\bullet}$, and all radii for free logarithmic coordinates that occur in Lemma 20.3. Fix

also the geometric constants $A_8, C_B, K_B, C_{\text{col}}, C_{\text{conv},J}$, and the mesh and interior-restriction factors of the initial step. The fixed path constant L_J includes the three-block cutoff band and the fine-link endpoint buffer of Theorem 19.6, before any later width is selected. None of these constants depends on the widths chosen later (of boundary layers, removed layers, protected neighborhoods, and large cells). On the preliminary larger products fix $D_{\text{raw},J} \geq 1$ bounding, in absolute coefficient rows, the derivative of the raw sequential reconstruction with respect to all its scaled fluctuation inputs, uniformly for the preliminary bound σ_0 . Choose $R_W < q_{\text{raw}}/(2D_{\text{raw},J})$ and below the full minimum in Lemma 20.3, including $1/(16C_0)$, both image bounds, both product domains, and the conversion to the joint solution neighborhood. Then fix

$$0 < \sigma_* \leq \sigma_0, \quad \sigma_* \rho_{\text{cl}} < \lambda_* R_W / (6C_{\text{en},J}), \quad \lambda_* = M^{-(J-1)/2},$$

all margins for the preconditioned forces and displacements, $2\delta_V < q_{\text{raw}}/2$ for the preconditioned force bound at σ_* , and $C_{P,\bullet} \sigma_* \rho_{\text{cl}} < \rho_\bullet / 64$. Thus the shift of the center plus the fluctuation displacement is less than q_{raw} . This choice precedes item (5), and it is not replaced by taking r and β_{rec} small.

- (5) Boundary radii, removed layers, and frozen coordinates. Fix the outer boundary radius $R_{\partial,\bullet}$, the inner radius $\rho_{\partial,\bullet} = R_{\partial,\bullet} / K_*$, and the width w_{bd} of the boundary layer so that the zero-fluctuation image bound satisfies $A_{0,\bullet} \leq \rho_\bullet / 4$. Specifically, impose

$$\begin{aligned} B^{-1}Z_\bullet(K_*\rho_\bullet) &\leq \rho_\bullet / 16, & C_{\Delta,\bullet}C_{\rho,\bullet}R_{\partial,\bullet} &\leq \rho_\bullet / 16, \\ C_{\Delta,\bullet}C_{\rho,\bullet}e^{-a_W,\bullet}w_{\text{bd}}/2K_*\rho_\bullet &\leq \rho_\bullet / 16, & C_{T,\bullet}K_*\rho_\bullet &\leq 1/4, \end{aligned}$$

where the first and fourth inequalities were already imposed in item (3), together with the full condition

$$\begin{aligned} A_{0,\bullet} &\leq B^{-1}Z_\bullet(K_*\rho_\bullet) + C_{\Delta,\bullet}C_{\rho,\bullet}[R_{\partial,\bullet} + e^{-a_W,\bullet}w_{\text{bd}}/2K_*\rho_\bullet] \\ &\quad + C_{F,\bullet}\beta_{\text{rec}} + C_{P,\bullet}\sigma_*\rho_{\text{cl}} \leq \rho_\bullet / 4. \end{aligned}$$

Here $C_{F,\bullet}$ and $C_{P,\bullet}$ are the image constants for frozen coordinates and for perturbed conditional constrained minimizers on the larger domains. Choose the width $s_{\text{er}} \geq w_{\text{bd}}$ of the layer removed after each renormalization group step so that

$$2e^{-\alpha(s_{\text{er}}-1)}[A_J(1+E) + K_J\sigma_* + 1] \leq \sigma_*/2, \quad 2e^{-\alpha(s_{\text{er}}-1)}[A_J^{\text{cl}} + K_J^{\text{cl}}\sigma_*] \leq \sigma_*/2.$$

Choose the bound $\beta_{\text{rec}} > 0$ for frozen coordinates below all coordinate-domain, product, and boundary radii, below one quarter of $\min\{\rho_\bullet, \rho_{\partial,\bullet}\}$, and below $\lambda_* R_W / (6C_{\text{en},J})$. Impose the bounds of Lemma 18.9 for the substitution of frozen coordinates and for the boundary-completion differences, in particular

$$C\beta_{\text{rec}}E_\bullet/\rho_\bullet \leq \sigma_*/8, \quad |\xi_j^\sharp| \leq \phi_\bullet(j)/4, \quad z_t, z_t^{\text{cl}} \leq \sigma_*/4,$$

including the substitution at the initial step, where C denotes the larger-domain constants of Lemma 18.9. Require also $C_{F,\bullet}\beta_{\text{rec}} < \rho_\bullet / 64$.

(6) Working radius and support radius. Set $\alpha_* = C_{CP}/B$ and choose

$$\beta_J \geq \max\{1, R_J^{-1}, A_J/(\alpha_* R_J^2)\}.$$

Choose a working radius $r_{\text{eff}} > 0$ inside the domains of item (3) with $\beta_J r_{\text{eff}}/(1 - \alpha_*) < 1/8$. Then choose the support radius $r > 0$ so that

$$r < r_{\text{eff}}/4, \quad r < \beta_{\text{rec}}/C_{\text{conv},J}, \quad r < \lambda_* R_W/(6C_{\text{en},J}), \quad (5M - 3)(10M)^{J-1}r < \pi,$$

together with the fixed coordinate-domain and chart margins of the earlier constructions, in particular those of Definition 14.4. Set

$$R_{\text{buf}} = C_{\text{map},J}r, \quad C_{\text{map},J} \geq (10M)^J, \quad r < R_{\text{buf}} < \rho.$$

There exist radii $0 < P < Q_{\text{DB}}$, proportional to r , such that every reconstruction from a coarse field e^v with $|v| \leq r$ and unscaled fluctuation coordinates with $\max_i |\zeta_i| \leq Q_{\text{DB}}$ has all link logarithms smaller than $r/2$. Import the full gates of Definition 14.4: its torus enclosing radii are $\mathcal{L}_{\text{en}}r$ and the larger comparison radius $\mathcal{L}'_{\text{en}}r$, distinct from the regional R_{en} below, and require

$$r < \min\{\rho/8, r_{\text{geom}}\}, \quad R_{\text{buf}} < \min\{r_{\text{geom}}, \rho/2, 4K_*\rho\}, \quad \mathcal{L}'_{\text{en}}r < 4K_*s_J, \\ R_{\text{buf}} + \mathcal{L}'_{\text{en}}r < r_J, \quad C_{\text{tor},J}(R_{\text{buf}} + \mathcal{L}'_{\text{en}}r) < 1/2, \quad \frac{C\mathcal{L}'_{\text{en}}r}{(1 - C\mathcal{L}'_{\text{en}}r)^3} < 1/4.$$

Both torus physical products lie strictly inside the earlier free-coordinate products, and the zero-fluctuation majorant obeys $B^{-1}Z(r) < r/4$. The condition including the coupling is completed in item (8). Define

$$\delta_{\text{en}} = C_{\text{en},J}(r + \beta_{\text{rec}} + \sigma_*\rho_{\text{cl}}), \quad R_{\text{en}} = 2\delta_{\text{en}}/\lambda_* \leq R_W.$$

On the entire enclosing product domain,

$$\frac{C_{T,\bullet}R_{\text{en}}}{1 - C_{T,\bullet}(K_*\rho_{\bullet} + R_{\text{en}})} + C_{\Delta,\bullet}R_{\text{en}} \leq \rho_{\bullet}/8, \quad 1 - C_{T,\bullet}(K_*\rho_{\bullet} + R_{\text{en}}) \geq 1/2.$$

Together with item (5), the image under the reconstruction majorant at spatial exponent zero is at most $3\rho_{\bullet}/8$. The margin of half the radius in the weighted norm is kept, either by the same bounds for weighted sums or by the interpolation of coefficient bounds in Lemma 20.3, whose positive exponent was included before a and α were chosen.

(7) Geometric widths and the curvature threshold. Set

$$R_{\text{cell}} = \lceil A_8/r \rceil, \quad W = 8BR_{\text{cell}}, \quad q = \max(s_{\text{er}}, W), \quad r_F = \lceil q/B \rceil.$$

The choice of r in item (6) requires $W > \max\{s_{\text{er}}, D_{\text{dep},J}\}$, where $D_{\text{dep},J}$ bounds the finitely many full dependence radii fixed before the widths, as in Step 1 of the proof of Theorem 21.5.

The protected sets contain all lifted endpoints of older frozen links and transports and all tree roots, in the lattices where they were introduced. At the initial renormalization group step, which starts from the Wilson measure, use the block-aligned mesh $S \asymp_J R_{\text{cell}}$, one full layer of large cells (the closed cells of side R_{cell} of Theorem 21.3), and the interior restriction by the full dependence radius of the constraint, tree, and transport paths. These choices fix a preliminary affected-plaquette count F_J^{pre} ; its enlargement for the stopping family and the charge constant $C_{\text{ch},J}$ are fixed in item (9). Here a threshold failure is a bad plaquette, that is, a plaquette selected by the tests of a large-field decomposition because its energy is at least d_* , together with the neighborhood excluded because of it. Fix $0 < \delta_* \leq 1$ for the curvature threshold of Theorem 7.3, as in its proof, and then choose

$$0 < d_{\text{cut}} \leq \min\{d_{\text{SG}}, B_*/(96M^3), \delta_*^2/(2\pi^2)\}, \quad d_* = d_{\text{cut}}/2, \quad \pi\sqrt{2d_*}S^2 \leq 1.$$

The threshold d_{SG} , with its side condition, gives membership in D_r with link bound $r/2$ by Theorem 7.7. Use the constants $K_M, K_1, \kappa_0, c_A^{\text{act}}, \alpha_W$ of Theorem 22.4 for arbitrary fields, $K_W = 48K_1/\alpha_W$, and $\theta = \lambda = 1/2$.

- (8) Cutoff amplitude and coupling bound. Fix $A_W \geq 1$ in the range allowed by Theorem 20.14 (no earlier choice depends on A_W , so it may be fixed at any earlier stage), and set $p(t) = A_W \log(1/t)$ and $p^+ = p(t) + A_W(J-1) \log M/2$. Choose $g_* \leq t_* < e^{-1}$ such that, for every $0 < t \leq g_*$,

$$C_0 t_*^2 \mathcal{L}_V \leq 1/16, \quad C_1 t_*(\mathcal{L}_V + 2) < 1/(4\sqrt{3}), \quad tp(t) < P, \quad tp^+ < R_{\text{en}},$$

$$C_T t^2 [p(t)^4 + Ep(t)^2] \leq (1 - q_{\text{CA}})/2, \quad C_J t^2 (E + \sigma_*) [p(t)^4 + (E + \sigma_*)p(t)^2] \leq 1,$$

the same two inequalities with p^+ in place of $p(t)$,

$$t \leq \frac{c_{z,J}}{2[(p^+)^3 + p^+ + (E + \sigma_*)p^+]}, \quad \mathfrak{s}(t) \leq \epsilon_J, \quad \frac{C[\mathcal{L}'_{\text{en}} r + t^2(1 + E)]}{(1 - C\mathcal{L}'_{\text{en}} r)^3} \leq \frac{1}{2},$$

and the strict interaction and Jacobian margins of the earlier product theorems with their finitely many bounds for free coordinates and for parameter derivatives of the normalized log density. Here the parameter derivatives are coarse-field derivatives on a fixed fluctuation product: the running couplings, the cutoff radii, and the real frozen data are held fixed. For such a derivative ∂ and $k = Z^{-1}e^{-V}$ on that product, $\partial \log k = -\partial V + \mathbb{E}_k[\partial V]$; the applicable bounds are those of Theorem 13.10 and their regional versions. Each remaining bound is a fixed estimate of the form $Ct^\epsilon \log^k(1/t) \leq c$ with $\epsilon, c > 0$. Require in addition

$$g_*^2 \leq \alpha_W, \quad 2\kappa_*/g_*^2 \geq 1, \quad a_0 g_* < \min(r/2, 1)$$

for a fixed positive constant a_0 of the tree trial configuration. The total error in the Hessian of the conditional fluctuation potential on the enclosing product domain is then at most $8/27$.

- (9) Stopping levels, the insertion interval, and the moment constants. Fix the finite family of stopping levels, the treatment of each of the J possible remainders modulo J , their side-ratio bound χ_* , and the conversion constant K_{src} . Include both functions whose exponential moments are bounded in Theorem 22.1 and Corollary 22.9, namely the predecessor action Q_j^T of Section 23.3 and the total Wilson action S_W/g^2 ; we call them the predecessor insertion and the total-action insertion. Enlarge the geometric counting constant and F_j^{pre} for this fixed family and both insertions, call the resulting count F_J , and then fix $C_{\text{ch},J}$ from these enlarged counts. Only then choose $s_0 > 0$ in the interval of Theorem 22.1, imposing also

$$s_0 \leq \frac{\kappa_0}{2 \max\{K_M, 48K_1\}}.$$

The coefficient of the Wilson action ranges over $[1/4, 1/2]$, and the bounds for the stopping partition-function ratio and its numerator are fixed for that interval. Set

$$s_b = s_0/2, \quad a_b = 4s_b d_*/c_E, \quad 0 < \bar{c} \leq \min\{d_*/8, a_b/2\}.$$

Compute C_v, C_s, C_b from Theorem 22.1, which places no restriction on the number of averaging steps, keeping $C_s \geq 9\chi_*^2$; these constants may be enlarged for the total-action insertion by Corollary 22.9 and the same proof. Fix the value $C_\delta = 1$ for the bound on the logarithm of the reciprocal products in Lemma 22.12; it is achieved by the minimum side in item (10). Set $C'_b = C_b + 1 + \log 2$; a larger fixed choice is allowed.

- (10) The minimum side. Finally choose $n_{\min} \in 4\mathbb{N}$ above all fixed geometric, curvature-threshold, mesh, flat, and scalar thresholds, with

$$n_{\min} \geq \max\{4, e\gamma_0, \gamma_0/g_{\text{end}}^2, 8R_{\text{cell}}, 8S\}, \quad 0 < g_{\text{end}} \leq \min\{1, g_*, g_{\text{anc}}, \sqrt{\alpha_W}, \sqrt{2\kappa_*}\},$$

where g_{anc} is the coupling bound for the stopping partition-function ratio of Theorem 22.4, uniform for coefficients in $[1/4, 1/2]$. Keep every fixed margin near flat configurations, in particular $2\pi/n_{\min} < \min\{r/2, r_{\text{flat}}\}$ with strict inequality, where r_{flat} is fixed by Lemma 14.9. Choose $n_{\min}$ so that the ratios of successive terms in the error sums at the endpoints and in the sums over large-field decompositions are at most $1/2$, the reciprocal products have logarithm at most $C_\delta = 1$, and

$$\mathcal{A}_\gamma(m) = 24C_v M^3 + \frac{4C_s M^2}{m} \log(Mm/\gamma) + \frac{8C'_b}{m^3} \leq \frac{a_b m}{2\gamma} \quad (m \geq n_{\min}, 0 < \gamma \leq \gamma_0). \quad (23.3)$$

Impose also $M^3 e^{-\bar{c}(M-1)n_{\min}/\gamma_0} \leq 1/2$ and the threshold at the initial scale in Lemma 22.12. All these are lower bounds on $n_{\min}$ in terms of previously fixed constants. No bound on the number of averaging steps is imposed.

The reconstruction maps of Theorem 15.13 are fixed once on the countable family of pairs of finite lattices. Their joint coefficient bound and β_J are obtained before r_{eff} and r ; a later

refinement of thresholds only increases the finite deterministic thresholds of these maps and does not enlarge that bound. The prescription for the entire indexed family is fixed once and does not depend on γ , on the field, on observations, or on a later comparison index. Its members and their deterministic thresholds are indexed by the pair of finite lattices and the preceding classical depth; no choice is repeated when the coarse side changes. Write $\hat{C}$ for the group-valued reconstruction and $\hat{P}(v) = \text{Log } \hat{C}(e^v, 0)$ for the logarithm of its value at zero fluctuation on the active variables. At every preceding classical depth these maps are majorized by

$$\varphi(z) = \frac{\alpha_* z}{1 - \beta_J z}, \quad \sup_{|v| \leq r_{\text{eff}}} \|D\hat{P}(v)\|_{\text{root}} \leq a_{\text{map}} = \frac{\alpha_*}{(1 - \beta_J r_{\text{eff}})^2}, \quad a_{\text{map}} \sqrt{B} \leq \frac{32}{49} < 1. \quad (23.4)$$

Here the row sums and volume-normalized column sums follow the conventions of Lemma 15.16. We use its common majorant as a_{map} , which gives $q_{\text{anc}} = B\alpha_*^2 \leq Ba_{\text{map}}^2 < 1$ as required in the later geometric summations. All constants of the full joint estimates and of the estimates for the gauge transformations of the reconstruction maps are constants at fixed J . The choice of J uses only the J -uniform geometric constants and torus rates, the J -uniform series Z of Lemma 11.9, and the finitely many coarse-field derivative bounds of the reconstruction maps at zero fluctuation.

For every $0 \leq a < K$, on the event that the initial condition and the first a renormalization group steps pass the tests of the small-field event, the next J averaging steps, from side uB^{K-a} to side uB^{K-a-1} , avoid the exceptional set of T_M : no based loop holonomy in (4.3) equals $-I$, even if a later test fails. No assertion is made about a further averaging step at $a = K$. Indeed, every closed path of such a holonomy has length at most $5M - 3$, and one averaging step increases a bound on link distances by at most the factor $10M$. Consequently every coarse field produced by the coordinate construction of the small-field event equals the iterate T_M^J of the finer field. The gauge transformations of the reconstruction maps depend only on the coarse field and are the identity at the tree roots; they preserve this identity and all conditional kernels, cutoffs, and Jacobians.

The local constants have the dependence described in item (3) of Definition 4.14, and do not depend on the widths chosen later. The probability constants and $n_{\min}$ may depend on L , γ_0 , M , J , Θ , and the fixed family of stopping levels, as in item (4) of Definition 4.14; they do not depend on γ , u , N , K , the preceding classical depth, or the regions, frozen coordinates, and numbers of elapsed renormalization group steps of a large-field decomposition. Neither J nor r is changed at a different coarse side. The analytic radii are restored to their full values, after removal of a boundary layer, only at the end of complete renormalization group steps; remainders modulo J are treated by the fixed insertions and side-ratio bounds.

We prove Theorem 23.2 in Section 23.6. The parts of the inequalities in items (3) and (6) reserved for r , β_{rec} , and the coupling are filled by later upper bounds; q_{raw} is chosen in item (3), so no cycle arises.

Item	Quantities chosen	Determined by
(1)	$J, B = M^J$	J -uniform constants $a_{\mathbb{T}} < a_{\text{cen}}/8$, $\delta_0(a_{\mathbb{T}}), K_*, C_{\text{CP}}, C_{\bullet}, C_P(a_{\mathbb{T}}), B_{\text{geom}},$ $C_{\text{geom}}^{\mathbb{T}}, b_{\mathbb{T}}, C_{\text{cen}}$
(2)	$v_{\text{free},J}, R_{\text{raw}}, \text{solution neighborhoods}$	J ; constants of Section 19 at fixed J
(3)	$\rho, \rho_{\text{cl}}, E^{\text{cl}}, E, q_{\text{raw}}; A_J, R_J; a, \alpha, m_{\text{rate}}, b_1$	items (1)–(2); $C_G, q_{\text{CA}}, c_{\text{prim},J}, C_{\text{alloc}},$ C_*^{geom}
(4)	σ_0 ; local and geometric constants; $D_{\text{raw},J};$ $R_W; \sigma_*$	items (1)–(3); Sections 20 and 21
(5)	$R_{\partial,\bullet}, \rho_{\partial,\bullet}, w_{\text{bd}}; s_{\text{er}}; \beta_{\text{rec}}$	items (1)–(4); Lemma 18.9
(6)	$\beta_J, r_{\text{eff}}; r, R_{\text{buf}}; P, Q_{\text{DB}}; R_{\text{en}}$	items (1)–(5)
(7)	$R_{\text{cell}}, W, \varrho, r_F, S; F_J^{\text{pre}}; \delta_*, d_{\text{cut}}, d_*$	items (1)–(6); Sections 7, 21, and 22
(8)	A_w, t_*, g_*	items (1)–(7); Lemma 23.1
(9)	stopping levels, χ_* ; final $F_J, C_{\text{ch},J}; s_0; s_b,$ $a_b, \bar{c}; C_v, C_s, C_b, C_b'$	items (1)–(8); Section 22
(10)	$g_{\text{anc}}, g_{\text{end}}, n_{\text{min}}$	items (1)–(9); Lemmas 23.1 and 23.4

Table 23.1. The order of the choices in Theorem 23.2. Each row uses only the quantities of the earlier rows and the constants of the cited results, evaluated on their larger construction domains. The amplitude A_w may be fixed at any stage before item (8).

Remark 23.3. The parameters of Theorem 23.2 also allow one to fix, before the coupling, the exponents $\alpha_{\text{fl}} = 7/12$ and $\sigma_{\text{sm}} = 5/24$ of Definition 4.8 and

$$\tau_{\text{tail}} = \frac{1}{2} \min \left\{ \frac{5}{24}, \frac{1}{2}, -\frac{\log q_{\text{CP}}}{\log B} \right\} > 0,$$

together with smaller positive losses in the heat-kernel estimates. These numbers are positive since $q_{\text{CP}} \leq 1/4$ and $B > 1$. They satisfy the conditions of Definition 4.8 and the tail condition $0 < \tau_{\text{tail}} < \min\{\sigma_{\text{sm}}, 1/2, -\log q_{\text{anc}}/\log B\}$ of Section 4: indeed $1/2 < 7/12 < 2/3$ and $5/24 < 1 - 7/12$, and the contraction factor of the reconstruction amplitude is $a_{\text{cap}} = C_{\text{CP}}/B$ in the amplitude bound of Lemma 15.10, so that $q_{\text{anc}} = Ba_{\text{cap}}^2 = q_{\text{CP}}$ and the factor $1/2$ makes the inequality strict. This is a numerical choice only; it asserts no stochastic estimate for the flow. The deterministic sequence of matching levels in Theorem 29.19 is chosen only after the convergence at fixed coupling.

Lemma 23.4. *Fix the parameters of items (1)–(8) of Theorem 23.2. As in item (9), fix the family of stopping levels with its remainders, the full sets of plaquettes affected by both insertions with the enlarged count F_J , and s_0 in the interval of Theorem 22.1. Fix $0 < \Gamma < \infty$, and set*

$$s_b = s_0/2, \quad a_b = 4s_b d_*/c_E, \quad 0 < \bar{c} \leq \min\{d_*/8, a_b/2\}.$$

Let C_v, C_s, C_b be the constants of Theorem 22.1 for the ceiling Γ , and let $C'_b \geq C_b$ (for example $C'_b = C_b + 1 + \log 2$). Then there exists $n_\Gamma \in 4\mathbb{N}$, chosen before γ , the coarse side, and the number of renormalization group steps, and larger than every previous geometric, flat, and local threshold, with the following properties. The conclusions of Theorem 22.1 and Lemma 22.13 hold for all coarse sides $u \geq n_\Gamma$ and $0 < \gamma \leq \Gamma$, and

$$\mathcal{A}_\gamma(m) \leq a_b m / (2\gamma) \quad (m \geq n_\Gamma, 0 < \gamma \leq \Gamma),$$

with $\mathcal{A}_\gamma(m)$ as in (23.3). For $m \geq n_\Gamma$, the ratios of successive terms in the exponential sums with exponents $a_{\text{act}}, \bar{c}$, and $c_* = \min\{a_b, d_*/8\}$ over sides with ratio M or B , and in the sums for the fluctuation cutoffs over sides with ratio B , are at most $1/2$; and every product $\prod(1 - \epsilon)^{-1}$ required in Lemma 22.12 has logarithm at most one. No restriction on the number of averaging steps is imposed. The new constants depend only on L, Γ, M, J, Θ , and the family of stopping levels, as described in item (4) of Definition 4.14.

Proof. Theorem 22.1 and Lemma 22.13 provide the thresholds for the endpoints, for the factors attached to threshold failures, and for the initial segments at the fixed parameters. None of their hypotheses is a bound on the probability of a failure at some scale. For $x = m/\gamma \geq e$ and $m \geq 4$ we have

$$\mathcal{A}_\gamma(m) \leq D(1 + \log x), \quad D = 1 + 24C_v M^3 + C_s M^2(1 + \log M) + C'_b/8.$$

Since $(1 + \log x)/x \rightarrow 0$, we may choose a half-line $x \geq n_\Gamma/\Gamma$ on which $D(1 + \log x) \leq a_b x/2$. The ratio of successive terms with exponent c is $R^3 e^{-c(R-1)m/\gamma}$, with $R = M$ or $R = B$ as appropriate. For the sums of the fluctuation cutoffs we write $z = (1/2) \log(Mm/\gamma)$ and $\Delta = (1/2) \log R$; the ratio is $R^3 e^{-a(2\Delta z + \Delta^2)}$. Both ratios are at most $1/2$ after a further increase of the minimum side, uniformly for $\gamma \leq \Gamma$, and the corresponding sums tend to zero. If $\epsilon \leq 1/2$, then $-\log(1 - \epsilon) \leq 2\epsilon$, and the products of reciprocal errors have logarithm as small as prescribed; this is also the content of Lemma 20.15. We take the maximum of these thresholds, of the threshold at the initial scale, and of all earlier geometric and flat thresholds, rounded up to an element of $4\mathbb{N}$; all of them are monotone conditions on the minimum side.

The structural parameters fixed in the hypotheses involve only the local running couplings and fixed larger products, and none of them is recomputed from C_v, C_s, C_b or from Γ . The same argument works for any prescribed J for which the parameters of items (1)–(8) have been chosen, and it keeps that J . This proves the lemma, with the stated quantifier order and with the margin $a_{\text{map}} \sqrt{B} \leq 32/49$ of the reconstruction maps unchanged. $\square$

Theorem 23.5. Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$. There exist an integer $J \geq 1$, with $B = M^J$, a tuple Θ , reconstruction maps, a number $0 < g_* < e^{-1}$, and an integer $n_\Gamma \in 4\mathbb{N}$,

all chosen before γ , the coarse side, and the number of renormalization group steps, such that every structural, local, moment, scalar, flat, and reconstruction conclusion of Theorem 23.2 holds for all

$$0 < \gamma \leq \Gamma, \quad u \in 4\mathbb{N}, \quad u \geq n_\Gamma, \quad K \in \mathbb{N}_0, \quad N = uB^K,$$

with γ_0 replaced by Γ in its conclusions and side conditions, and with the moment constants of Theorem 22.1 and Lemma 22.13 for the ceiling Γ . In particular $a_{\text{map}}\sqrt{B} \leq 32/49$, every coarse field produced by the coordinate construction is the corresponding iterate of T_M , and the reconstruction maps, the support radius, and the local constants are fixed before the coupling. The minimum side satisfies

$$n_\Gamma \geq \max\{4, e\Gamma, \Gamma/g_{\text{end}}^2, 8R_{\text{cell}}, 8S\}$$

and the coupling condition of Lemma 22.13; no restriction on the number of averaging steps is imposed. The parameters satisfy $0 < \delta_* \leq 1$, fixed before $d_{\text{cut}}, g_*^2 \leq \alpha_W$, and

$$0 < s_0 \leq \min\left\{\frac{1}{2}, \frac{\kappa_*}{2}c_E^{-(J-1)}, \frac{1}{4K_W}, \frac{c_A^{\text{act}}d_*}{32F_J}, \frac{\kappa_0}{2\max\{K_M, 48K_1\}}\right\}, \quad K_W = 48K_1/\alpha_W,$$

and the stopping partition-function ratio is controlled uniformly for coefficients of the Wilson action in $[1/4, 1/2]$. The construction can be carried out with any prescribed J for which parameters as in Theorem 23.2 with $\gamma_0 = 1$ have been chosen, without changing J . The local constants have the dependence described in item (3) of Definition 4.14, and the probability constants the dependence described in item (4), with the fixed ceiling Γ . No uniformity as $\Gamma \rightarrow \infty$ is asserted.

We give the proof in Section 23.7. When $\Gamma = \gamma_0 \leq 1$, the parameters of Theorem 23.2 satisfy the conclusions of Theorem 23.5, and in that case we write $n_{\min}$ for n_Γ .

23.3 Bounds under the Wilson measure

In this subsection, we bound the probability that some averaged configuration has a plaquette of large energy, and we bound exponential moments of coarse actions under the Wilson measure. We recall the notation of Theorem 22.1 and Lemma 22.11. For $N = uB^K$ and $n_j = N/M^j$, fix one of the eight tilings $\mathcal{T}$ of the lattice of side n_j into $r_j = n_j^3/8$ disjoint cells with two sites in each direction (this number is denoted m_j in Lemma 22.11). Let $\mathcal{P}_{j-1}^\mathcal{T}$ be the union, counted once, of the complete predecessor patches at side n_{j-1} of the internal plaquettes of the cells, and for a configuration V at side n_{j-1} set

$$Q_j^\mathcal{T}(V) = \frac{4}{h_{n_{j-1}}} \sum_{p \in \mathcal{P}_{j-1}^\mathcal{T}} q(V_p).$$

For a cell C , let $R_{<j}(C; d_{\text{cut}})$ be the local regularity event of Lemma 22.11, which tests at the threshold d_{cut} exactly the plaquettes of the finer scales whose full dependence sets

lie in C ; let $I_j(C; d_*)$ be the event that some internal plaquette of C at scale j has energy at least d_* ; and let $R_{<j}^{\mathcal{T}}(d_{\text{cut}})$ be the intersection of the events $R_{<j}(C; d_{\text{cut}})$ over the cells of $\mathcal{T}$. These local events do not test the plaquettes on the boundaries of the cells. By Lemma 7.4, every plaquette at scale j is internal to a cell of one of the eight tilings.

The probability bounds below use the disjoint link-reflection chessboard estimate proved in Lemma 6.9, following the reflection-positive method of Fröhlich, Israel, Lieb, and Simon [92, 93] and Biskup [29, Theorem 5.8]. The Wilson measure is reflection positive [134]. Part (2) of Theorem 6.6 gives the chessboard inequality, for each of the eight shifted tilings, under the image measure $(T_M^j)_* \mu_{N,\gamma}$ for nonnegative observables supported in the coarse cells, and more generally under $\mu_{N,\gamma}$ for nonnegative observables supported in the fine cells formed by the blocks B_{Mj} of the vertices of a coarse cell. The side n_j is divisible by four, as that theorem requires. The event $I_j(C; d_*)$ is a function of $T_M^j U$ on the coarse cell C , and the event $R_{<j}(C; d_{\text{cut}}) \cap I_j(C; d_*)$ is a gauge-invariant function of the fine links in the fine cell of C , since it tests only plaquettes whose full dependence sets lie in C ; this is the form in which Lemma 22.11 applies the theorem. All estimates in this subsection are under the Wilson measure $\mu_{N,\gamma}$, normalized by its own partition function; no conditioned measure occurs.

Theorem 23.6. *Use the parameters and the minimum side n_Γ of Lemma 23.4; for $\Gamma \leq 1$ these are the parameters of Theorem 23.2 with $\gamma_0 = \Gamma$, and for general finite Γ those of Theorem 23.5. Let $0 < \gamma \leq \Gamma$, $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$, and $K \in \mathbb{N}_0$, and let $N = uB^K$, $D = JK$, $n_j = N/M^j$, and $V^j = T_M^j U$. Define*

$$\text{FF}_0 = \{\max_p q(U_p) \geq d_*\}, \quad \text{FF}_j = \{q(V_p^\ell) < d_* \text{ for all } \ell < j \text{ and all } p, \max_p q(V_p^j) \geq d_*\}$$

for $1 \leq j \leq D$. Then, simultaneously for all $0 \leq j \leq D$,

$$\mu_{N,\gamma}(\text{FF}_j) \leq 3n_j^3 e^{-\bar{c}n_j/\gamma}. \quad (23.5)$$

For $1 \leq j \leq D$, each of the eight tilings $\mathcal{T}$, and $0 \leq s \leq s_0$,

$$\log \mathbb{E}_{\mu_{N,\gamma}} [e^{sQ_j^{\mathcal{T}}(V^{j-1})}; R_{<j}^{\mathcal{T}}(d_{\text{cut}})] \leq 3C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C'_b.$$

The same bound holds without the indicator of $R_{<j}^{\mathcal{T}}(d_{\text{cut}})$, with C_b in place of C'_b . The constants depend only on L, Γ, M, J, Θ , and the family of stopping levels, as described in item (4) of Definition 4.14, and are uniform in γ , the side, the number of renormalization group steps, the scale j , and the tiling.

Proof. At $j = 0$, by the threshold at the initial scale in item (10) of Theorem 23.2 (for general Γ , in Lemma 23.4 and Lemma 22.13), we have

$$\mu_{N,\gamma}(\text{FF}_0) \leq 3N^3 e^{-d_* N/(8\gamma)} \leq 3N^3 e^{-\bar{c}N/\gamma}, \quad (23.6)$$

where we used $\bar{c} \leq d_*/8$. For a scale $j \geq 1$, we apply Theorem 22.1 with the Wilson measure at side N as the starting measure (that is, with $b = 0$) and the same j . This gives the bound without the indicator directly at that scale, whatever its distance from the finest lattice. Since the integrand is nonnegative and $C_b \leq C'_b$, the bound with the indicator follows. This proves both moment bounds, without any assumption on earlier failures.

Let $m = n_j$. We divide the logarithm of the moment bound with the indicator by the number $r_j = m^3/8$ of cells. Since $n_{j-1} = Mm$ and $\log(1/g_{Mm}) = (1/2) \log(Mm/\gamma)$, we have

$$\begin{aligned} \frac{3C_v(Mm)^3 + C_s(Mm)^2 \frac{1}{2} \log(Mm/\gamma) + C'_b}{m^3/8} &= 24C_v M^3 + \frac{4C_s M^2}{m} \log(Mm/\gamma) + \frac{8C'_b}{m^3} \\ &= \mathcal{A}_\gamma(m). \end{aligned}$$

This identity explains why a moment bound with a volume term and a surface term carrying a logarithm suffices. After the reciprocal-cell-count exponent in the chessboard estimate is applied, the entropy per cell is of order $1 + \log(m/\gamma)/m$, which is dominated by the action $a_b m/\gamma$ of a cell containing a bad plaquette. Then, by the estimate of Lemma 22.11, which combines Markov's inequality at $s = s_b$ with the chessboard estimate, we have

$$\log \mu_{N,\gamma}(R_{<j}(C; d_{\text{cut}}) \cap I_j(C; d_*)) \leq -a_b m/\gamma + \mathcal{A}_\gamma(m) \leq -\bar{c} m/\gamma, \quad (23.7)$$

where in the last inequality we used $\mathcal{A}_\gamma(m) \leq a_b m/(2\gamma)$ and $\bar{c} \leq a_b/2$.

Since $d_* < d_{\text{cut}}$, regularity at the threshold d_* at all earlier scales implies every local event $R_{<j}(C; d_{\text{cut}})$. Every plaquette of scale j with energy at least d_* is internal to a cell of one of the eight tilings, by Lemma 7.4. Then FF_j is contained in the union of the $8r_j = m^3$ events in (23.7), and by a union bound we have $\mu_{N,\gamma}(\text{FF}_j) \leq m^3 e^{-\bar{c} m/\gamma}$. This implies (23.5), where the constant three is kept only for a common formula with (23.6). The argument applies to each j separately, without induction over scales; if $K = 0$, only the assertion at $j = 0$ remains. $\square$

The next theorem gives exponential moments of coarse actions without any indicator. It is not needed for the bound on S^c , but it is a quantitative consequence of the same estimates.

Theorem 23.7. *Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$. Use parameters as in Theorem 23.5, in which, as in item (9) of Theorem 23.2, the count F_j includes the plaquettes affected by both the predecessor insertion and the total-action insertion before s_0 is chosen. There exist $s_* = s_0 > 0$, $n_\Gamma \in 4\mathbb{N}$, and finite nonnegative constants C_v, C_s, C_b , chosen before γ , the*

side, and the number of renormalization group steps, such that for every $0 < \gamma \leq \Gamma$, $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$, $K \in \mathbb{N}_0$, $N = uB^K$, $n_j = N/M^j$, and $0 \leq s \leq s_*$,

$$\log \mathbb{E}_{\mu_{N,\gamma}} e^{sQ_j^\mathcal{T}(T_M^{j-1}U)} \leq 3C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b, \quad 1 \leq j \leq JK, \quad (23.8)$$

$$\log \mathbb{E}_{\mu_{N,\gamma}} e^{sS_W(T_M^j U)/g_{n_j}^2} \leq 3C_v n_j^3 + C_s n_j^2 \log(1/g_{n_j}) + C_b, \quad 0 \leq j \leq JK. \quad (23.9)$$

The first inequality holds for each of the eight tilings $\mathcal{T}$. All expectations are under the Wilson measure, normalized by its partition function, and all iterates of T_M that occur are defined on the given finite sequence of lattices. The constants depend only on L, Γ, M, J, Θ , and the family of stopping levels, as described in item (4) of Definition 4.14; they are uniform in γ , the side, the scale, and the number of renormalization group steps, and they do not depend on the large-field decompositions used in the proof. Moreover, with $\mathcal{B}_\gamma(n) = 3C_v n^3 + C_s n^2 \log(1/g_n) + C_b$, for every integer $p \geq 1$,

$$\mathbb{E}_{\mu_{N,\gamma}} \left[(S_W(T_M^j U)/g_{n_j}^2)^p \right] \leq p! s_*^{-p} e^{\mathcal{B}_\gamma(n_j)},$$

and the analogous polynomial moment bound holds for $Q_j^\mathcal{T}(T_M^{j-1}U)$, with $\mathcal{B}_\gamma(n_{j-1})$.

Proof. By the premise, the final count F_j already covers both insertions and both finite families of remainders; put $F'_j = F_j$. The fixed interval with $s_* = s_0$ satisfies the conditions of item (9) of Theorem 23.2, including

$$0 < s_* = s_0 \leq \frac{\kappa_0}{2 \max\{K_M, 48K_1\}}.$$

All entries are positive constants at this stage. The bound (23.8) is Theorem 22.1 with the Wilson measure as the starting measure ($b = 0$). For (23.9), we choose the stopping level $t = J \lfloor j/J \rfloor$, with $v = j - t \in \{0, \dots, J-1\}$, so that $m = n_t = M^v n_j$. All iterates from level t to level j are defined, including for $j = JK$, where $v = 0$ and $t = JK$.

Suppose first that $t > 0$. We integrate exactly t/J complete RG steps, starting from the Wilson measure, and use the bound of Corollary 22.9 for the total-action insertion inside the invariant density bound without weight of Theorem 22.10. The coefficient of the Wilson action remains $\lambda_s = 1/2 - sK_W \in [1/4, 1/2]$, and the plaquettes affected by the insertion contribute at most $8sF'_j \sum_f n_f/\gamma \leq P_0 \sum_f n_f/(4\gamma)$. The factors listed in Table 22.2 are unchanged except for their fixed incidence constants: the normalization terms generated by the integration and the completion terms concern the same large-field decompositions, and the finite dependence set of the insertion was enlarged before the interval was chosen. The quantitative conclusion of Theorem 22.1 on these factors applies, with the scale n_j and enlarged fixed constants. We complete the density bound by Lemma 22.7 and use the single stopping partition-function ratio of Theorem 22.4. Then each large-field decomposition contributes at most

$$e^{3C_T m^3 + 9m^2 \log(1/g_m) + C_\delta} \prod_f e^{-(P_0/2)n_f/\gamma}.$$

The bound of Theorem 22.1 for the sum over decompositions controls the sum of these contributions uniformly at the common minimum side. Converting from $m = \chi n_j$, with $\chi \leq M^{J-1}$, leaves exactly a volume term, the surface term $9\chi^2 n_j^2 \log(1/g_{n_j})$, and a fixed volume contribution from $\log \chi$. We take the maximum of the constants for the two insertions.

Suppose now that $t = 0$. Then $0 \leq j < J$ and $N/n_j = M^j$ is bounded. By the comparison of Theorem 22.4 for arbitrary fields, valid at all depths, we have $S_W(T_M^j U)/g_{n_j}^2 \leq 48K_1 S_W(U)/g_N^2$. Since $48sK_1 \leq \kappa_0/2 \leq 1/2$, the ratio of the full compact numerator to the periodic trial denominator at depth zero, used for short scale intervals in Theorem 22.1, applies. This ratio has the required surface-order correction in the partition-function comparison at side N , and the bounded ratio $N/n_j \leq M^{J-1}$ implies the stated bound at n_j . At $j = 0$ the deterministic inequality holds with the smaller constant one; this case includes $K = 0$. In both cases we bounded the integrals without indicators directly; no indicator of a regularity event was dropped from an upper bound.

Finally, for $x \geq 0$ we have $x^p \leq p! s_*^{-p} e^{s_* x}$, by the exponential series. Inserting the exponential moment bounds, we obtain the last assertion. $\square$

Corollary 23.8. *Use the parameters and constants of Theorem 23.7, increasing only the minimum side if necessary. There exists $c_{\text{bad}} > 0$ with the following property. Let $0 < \gamma \leq \Gamma$, $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$, $K \in \mathbb{N}_0$, $N = uB^K$, and $0 \leq j \leq JK$, and let $\mathcal{T}$ be one of the eight tilings at scale j . We set*

$$r_j = n_j^3/8, \quad q_j = e^{-c_{\text{bad}} n_j/\gamma},$$

and for a cell C of $\mathcal{T}$ let I_C be the event that some internal plaquette P of C satisfies $q((T_M^j U)_P) \geq d_*$. Then for every deterministic set A of distinct cells of $\mathcal{T}$,

$$\mu_{N,\gamma} \left(\bigcap_{C \in A} I_C \right) \leq q_j^{|A|}. \quad (23.10)$$

For $Y_j = \sum_{C \in \mathcal{T}} \mathbf{1}_{I_C}$, every integer $0 \leq k \leq r_j$, and every $\vartheta \geq 0$,

$$\mathbb{E}_{\mu_{N,\gamma}} \binom{Y_j}{k} \leq \binom{r_j}{k} q_j^k, \quad (23.11)$$

$$\mathbb{E}_{\mu_{N,\gamma}} e^{\vartheta Y_j} \leq [1 + (e^\vartheta - 1)q_j]^{r_j} \leq \exp\{r_j q_j (e^\vartheta - 1)\}. \quad (23.12)$$

Equivalently, $\mathbb{E}(Y_j)_k \leq (r_j)_k q_j^k$ for the falling factorials. The constants depend only on L, Γ, M, J, Θ , and the family of stopping levels (item (4) of Definition 4.14), and are uniform in γ , the side, the number of renormalization group steps, the scale, the tiling, and A .

The bounds (23.11) and (23.12) are those of a sum of independent Bernoulli variables with parameter q_j , but they are proved from the joint bounds (23.10) alone; no independence or stochastic domination is asserted.

Proof. Fix j and $\mathcal{T}$, and let $m = n_j$. The indicator of I_C is a nonnegative, bounded, gauge-invariant function of the configuration $T_M^j U$ on the internal plaquettes of the cell C , that is, an observable supported in the coarse cell C . Part (2) of Theorem 6.6, applied to the image measure $(T_M^j)_* \mu_{N,\gamma}$ and the tiling $\mathcal{T}$, applies to it; equivalently, its pullback is supported in the fine cell formed by the centered blocks of the vertices of C . The reflection positivity in part (1) of that theorem includes the configurations at which the Borel extension of T_M is used. Each I_C is the reflected translate of the corresponding event for the base cell of $\mathcal{T}$, since the energy of a plaquette is invariant under reversal of orientation. For nonempty A , the chessboard inequality with all observables equal to the indicator of this event gives

$$\mu_{N,\gamma} \left(\bigcap_{C \in A} I_C \right) \leq \mu_{N,\gamma}(I_{\text{all}})^{|A|/r_j}, \quad I_{\text{all}} = \bigcap_{C \in \mathcal{T}} I_C.$$

On I_{all} , we choose one internal plaquette with energy at least d_* in each cell, in a fixed deterministic order. The chosen plaquettes are distinct, since the cells have disjoint vertex sets and only internal plaquettes are used. This gives, pointwise on I_{all} ,

$$S_W(T_M^j U) / g_m^2 \geq 4d_* r_j m / \gamma.$$

By Markov's inequality and (23.9) at $s = s_*$, we have

$$\frac{1}{r_j} \log \mu_{N,\gamma}(I_{\text{all}}) \leq -4s_* d_* m / \gamma + 24C_v + \frac{4C_s}{m} \log(m/\gamma) + \frac{8C_b}{m^3}.$$

Here the entropy terms are in the units of scale j itself, so no factor M^3 appears. With $x = m/\gamma \geq e$ and $m \geq 4$, their sum is at most $D_A(1 + \log x)$ for a fixed D_A . We increase the minimum side so that $D_A(1 + \log x) \leq 2s_* d_* x$ on the half-line $x \geq n_\Gamma/\Gamma$. Then $c_{\text{bad}} = 2s_* d_* > 0$ works. If $\mu_{N,\gamma}(I_{\text{all}}) = 0$, the chessboard estimate already implies the result; otherwise we exponentiate the bound on its logarithm. For $A = \emptyset$ both sides of (23.10) equal one.

Pointwise we have $\binom{Y_j}{k} = \sum_{|A|=k} \prod_{C \in A} \mathbf{1}_{I_C}$, including for $k = 0$, and taking expectations proves (23.11). For $\vartheta \geq 0$ we expand the finite product

$$e^{\vartheta Y_j} = \prod_{C \in \mathcal{T}} [1 + (e^\vartheta - 1) \mathbf{1}_{I_C}].$$

Since every coefficient in the expansion is nonnegative, the joint bounds give (23.12); its last inequality uses $1 + y \leq e^y$ for $y \geq 0$. The case $j = 0$, including $K = 0$, is covered by the instance $j = 0$ of (23.9). $\square$

23.4 Probability of the small-field event

We now bound the probability of the complement of the small-field event. Recall from Definitions 4.4 and 14.4 that, for $N = uB^K$ and $V^\ell = T_M^\ell U$,

$$S_{N,u,\gamma} = \{U \in D_r\} \cap \bigcap_{k=0}^{K-1} \left[\mathcal{A}_{k,J}(U) \cap \{\max_{i,e} |w_{k,i,e}(U)| \leq p(t_k)\} \cap \{V^{(k+1)J} \in D_r\} \right],$$

where $t_k^2 = \gamma / (Mn_{(k+1)J})$ and $p(t) = A_w \log(1/t)$. Here $\mathcal{A}_{k,J}(U)$ is the event that, in the inverse coordinates of the k -th RG step at preceding classical depth kJ , each pair (c_{i+1}, z_i) lies in the coordinate domain of Definition 14.4, with its conditional constrained minimizer, its positive constrained Hessian, and a positive constraint Jacobian; and $w_{k,i,e}(U)$ are the normalized fluctuation variables of that step, defined by these inverse coordinates. There is no condition of the form $V^\ell \in D_r$ at levels ℓ that are not multiples of J . For $K = 0$ the event is $\{U \in D_r\}$.

The proof splits S^c according to the event G that every plaquette at every scale has energy below d_* . The probability of G^c is the sum of the probabilities of the first failures FF_j , which Theorem 23.6 bounds under the Wilson measure. On G , Theorem 7.7 implies all conditions $V^{kJ} \in D_r$, and S can fail only through the coordinate-domain conditions or the cutoffs of some RG step. We bound the first such failure using the comparison of Lemma 22.12, which keeps only the passing initial segment and the condition on the next coarse field. In this comparison, the probability of the initial segment is multiplied, never divided, and we never condition on regularity at later scales, which would change the conditional law.

Theorem 23.9. *Use the parameters, the renormalization group steps, and the reconstruction maps of Theorem 23.2. There exist constants $C', C, C_w, a_w > 0$ such that for every $u \in 4\mathbb{N}$ with $u \geq n_{\min}$, every $K \in \mathbb{N}_0$, and every $0 < \gamma \leq \gamma_0$,*

$$\mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}$$

and

$$\mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C u^3 e^{H_0 + H_2 \log(1/\gamma) - c_* u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}},$$

where

$$H_0 = 24C_v M^3 + \frac{4C_s M^2}{n_{\min}} \log(Mn_{\min}) + \frac{8C'_b}{n_{\min}^3},$$

$$H_2 = \frac{4C_s M^2}{n_{\min}}, \quad c_* = \min\{a_b, d_*/8\} > 0.$$

One may take a_w to be the smallest of the finitely many fixed positive exponents in the fluctuation bounds for the auxiliary measures and the stopped measures. The constants may depend on

$L, \gamma_0, M, J, \Theta$, and the family of stopping levels, but not on γ, u, N, K , the preceding classical depth, or the large-field decompositions. The event, and hence these bounds, are the same when the inverse coordinates are computed with the reconstruction maps of Theorem 15.13.

The two theorems have a common proof.

Theorem 23.10. *Use the parameters of Theorem 23.5 for a fixed $0 < \Gamma < \infty$, and the small-field event $S_{N,u,\gamma}$ defined with them, for $0 < \gamma \leq \Gamma$, $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$, $K \in \mathbb{N}_0$, and $N = uB^K$. There exist constants $C', C, C_w, a_w > 0$ such that*

$$\mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}, \quad (23.13)$$

$$\mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C u^3 e^{H_0^\Gamma + H_2 \log(\Gamma/\gamma) - c_* u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}, \quad (23.14)$$

where

$$H_0^\Gamma = 24C_v M^3 + \frac{4C_s M^2}{n_\Gamma} \log(Mn_\Gamma/\Gamma) + \frac{8C'_b}{n_\Gamma^3} \geq 0,$$

$$H_2 = \frac{4C_s M^2}{n_\Gamma} \geq 0, \quad c_* = \min\{a_b, d_*/8\} \geq \bar{c} > 0.$$

The constants may depend on the fixed Γ (item (4) of Definition 4.14), and do not depend on γ , the side, the number of renormalization group steps, the preceding classical depth, or the large-field decompositions. Moreover, for every fixed $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$ and every $q > 0$,

$$\sup_{K \geq 0} \mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C_{u,q,\Gamma} \gamma^q \quad (0 < \gamma \leq \Gamma),$$

with the admissible constant

$$C_{u,q,\Gamma} = C u^3 e^{H_0^\Gamma} \Gamma^{H_2} \left(\frac{H_2 + q}{c_* u e} \right)^{H_2 + q} + C_w u^3 (Mu)^{-q} e^{q^2/a_w}.$$

Finally, for every fixed $n \in 4\mathbb{N}$ with $n \geq n_\Gamma$ and $u_a = nB^a$,

$$\lim_{a \rightarrow \infty} \sup_{0 < \gamma \leq \Gamma} \sup_{K \geq a} \mu_{nB^K, \gamma}(S_{nB^K, u_a, \gamma}^c) = 0.$$

Proof of Theorems 23.9 and 23.10. For $0 \leq a \leq K$, let S^a be the event formed by the initial condition $U \in D_r$ and the conditions of the first a RG steps in the definition of $S_{N,u,\gamma}$, and let

$$G = \{q(V_p^j) < d_* \text{ for every } 0 \leq j \leq JK \text{ and every plaquette } p\}.$$

The disjoint events FF_j , $0 \leq j \leq JK$, exhaust G^c . By Theorem 23.6 and the condition $M^3 e^{-\bar{c}(M-1)n_\Gamma/\Gamma} \leq 1/2$, summing over the sides uM^h in increasing order, we have

$$\mu(G^c) \leq 3 \sum_{h \geq 0} (uM^h)^3 e^{-\bar{c}uM^h/\gamma} \leq 6u^3 e^{-\bar{c}u/\gamma}. \quad (23.15)$$

The sum over sides starts at the coarse side u , so no factor depending on K appears.

To keep the dependence on γ before absorption, we use (23.7) in the form $\log \mu(R_{<j}(C; d_{\text{cut}}) \cap I_j(C; d_*)) \leq -a_b m/\gamma + \mathcal{A}_\gamma(m)$. Since $\log(Mm/\gamma) = \log(Mm/\Gamma) + \log(\Gamma/\gamma)$ and $\log(Mm/\Gamma)/m$ is decreasing for $m \geq n_\Gamma \geq e\Gamma$, we have

$$\begin{aligned} \mathcal{A}_\gamma(m) &\leq H_0^\Gamma + H_2 \log(\Gamma/\gamma), \\ H_0^\Gamma &= 24C_v M^3 + \frac{4C_s M^2}{n_\Gamma} \log(Mn_\Gamma/\Gamma) + \frac{8C'_b}{n_\Gamma^3}, \quad H_2 = \frac{4C_s M^2}{n_\Gamma}, \end{aligned} \quad (23.16)$$

and this bound is nonnegative. Combining the bound at the initial scale with the bounds at positive scales, and summing the same geometric list of sides, we obtain

$$\mu(G^c) \leq 6u^3 e^{H_0^\Gamma + H_2 \log(\Gamma/\gamma) - c_* u/\gamma}, \quad c_* = \min\{a_b, d_*/8\} \geq \bar{c}. \quad (23.17)$$

For Theorem 23.9 we use instead $H_0 + H_2 \log(1/\gamma)$, which follows from the same calculation with Γ replaced by one in the logarithm; the function $\log(Mm)/m$ is decreasing for $m \geq 4$.

On G , the condition $U \in D_r$ and every condition $V^{(k+1)J} \in D_r$ hold by Theorem 7.7. By the identity of coarse fields in Theorems 23.2 and 23.5, on a passing initial segment each coarse field produced by the coordinate construction is the corresponding iterate of T_M . We conclude that

$$G \cap S^c \subset \bigcup_{0 \leq a < K} (S^a \cap A_{n_{(a+1)J}} \cap W_{n_{(a+1)J}}^c), \quad (23.18)$$

where, for $m = n_{(a+1)J}$, A_m is the event that the coarse field $V^{(a+1)J}$ lies in D_r , and W_m is the event that all coordinate-domain conditions and fluctuation cutoffs of the $(a+1)$ -th RG step hold. After this inclusion we discard G before estimating each term. By the comparison with identical unnormalized prefactors in Lemma 22.12, in its form for a fixed upper coupling bound Γ in Lemma 22.13, we have

$$\mu(S^a \cap A_m \cap W_m^c) \leq \epsilon_a \mu(S^a) \leq C_{\text{rej},J} m^3 e^{-a_w \log^2 \sqrt{Mm/\gamma}}. \quad (23.19)$$

This is the comparison for coarse fields in D_r at preceding depth aJ ; only $a < K$ is used, and the probability $\mu(S^a) \leq 1$ of the initial segment is multiplied, never inverted. Summing the bound over the sides $u, uB, uB^2, \dots$ and using the ratio condition for the fluctuation cutoffs, we obtain

$$\mu(G \cap S^c) \leq 2C_{\text{rej},J} u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}}. \quad (23.20)$$

Adding (23.15) or (23.17) proves both bounds of each theorem, for example with $C = C' = 6$ and $C_w = 2C_{\text{rej},J}$. For $K = 0$ the sum (23.20) is empty, and the initial condition

holds on the event that there is no failure at the initial scale. By Theorem 15.13, the reconstruction maps preserve the event, the transitions, the cutoffs, the Jacobians, and the kernels, so the same bounds hold in the coordinates of the reconstruction maps.

For the bound by powers of γ , we divide the curvature term of (23.14) by γ^q . For $\kappa = H_2 + q > 0$ and $A = c_* u > 0$ we have $\sup_{\gamma > 0} \gamma^{-\kappa} e^{-A/\gamma} = (\kappa/(Ae))^\kappa$. For the fluctuation term we write $z = (1/2) \log(Mu/\gamma)$, and then

$$\gamma^{-q} e^{-a_w z^2} = (Mu)^{-q} e^{2qz - a_w z^2} \leq (Mu)^{-q} e^{q^2/a_w}.$$

These give the constant $C_{u,q,\Gamma}$. For $u_a = nB^a$, both terms of (23.13) are bounded by their values at $\gamma = \Gamma$ for sides above the minimum side; this uses monotonicity in γ with $Mu/\gamma > 1$, as in Lemma 23.1. Those values tend to zero as $a \rightarrow \infty$, uniformly in $K \geq a$, since $nB^K = u_a B^{K-a}$. This proves the last assertion of Theorem 23.10. $\square$

Corollary 23.11. *Use the parameters of Theorem 23.2, and fix $u \in 4\mathbb{N}$ with $u \geq n_{\min}$ before varying the coupling. For every real $q > 0$ there exists a finite $C_{u,q}$ such that*

$$\sup_{K \geq 0} \mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c) \leq C_{u,q} \gamma^q \quad (0 < \gamma \leq \gamma_0).$$

With the constants of the second bound of Theorem 23.9, an admissible constant is

$$C_{u,q} = Cu^3 e^{H_0} \left(\frac{H_2 + q}{c_* u e} \right)^{H_2 + q} + C_w u^3 (Mu)^{-q} e^{q^2/a_w},$$

or any larger finite constant. In particular, if

$$F_u(\gamma) = \limsup_{K \rightarrow \infty} \mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c),$$

then $F_u(\gamma) = o(\gamma^\kappa)$ as $\gamma \rightarrow 0$ for every fixed $\kappa > 0$. Besides the probability constants, $C_{u,q}$ depends only on u and q ; it does not depend on K .

Now fix $n \in 4\mathbb{N}$ with $n \geq n_{\min}$, and set $u_a = nB^a$ and $N_K = nB^K$. For all integers $a \geq 0$ and $K \geq a$,

$$e_{K,a}(\gamma) = \mu_{N_K, \gamma}(S_{N_K, u_a, \gamma}^c) \leq \mathcal{F}(u_a, \gamma), \quad \mathcal{F}(u, \gamma) = C' u^3 e^{-\bar{c}u/\gamma} + C_w u^3 e^{-a_w \log^2 \sqrt{Mu/\gamma}},$$

and

$$\lim_{a \rightarrow \infty} \sup_{0 < \gamma \leq \gamma_0} \mathcal{F}(u_a, \gamma) = 0.$$

These bounds use the same small-field events and reconstruction maps for all coarse sides; no radius, cutoff, map, or threshold is chosen again at a larger coarse side.

At each fixed positive coupling, the probability of the complement tends to zero along the sides u_a , while at each fixed side it is smaller than every power of the coupling. The corollary gives no rate, uniform in the coupling, for the convergence of flowed observables, and it does not interchange a limit in the cutoff or in an observable with the limit $a \rightarrow \infty$.

Proof. We keep u fixed and let $\tau = \log(1/\gamma)$ and $d_u = \log(Mu)$. Dividing the second bound of Theorem 23.9 by γ^q , its curvature term becomes

$$Cu^3 e^{H_0} \exp\{(H_2 + q)\tau - c_* u e^\tau\}.$$

For $\kappa = H_2 + q > 0$, the derivative of $v^\kappa e^{-c_* uv}$ on $v > 0$ has the sign of $\kappa - c_* uv$. This function has the global maximum $[\kappa/(c_* u e)]^\kappa$, attained at $v = \kappa/(c_* u)$. Bounding the value at $v = e^\tau = 1/\gamma$ by this maximum, we obtain the first summand of $C_{u,q}$. For the fluctuation term, completing the square gives the identity

$$q\tau - \frac{a_w}{4}(\tau + d_u)^2 = -\frac{a_w}{4}\left(\tau + d_u - \frac{2q}{a_w}\right)^2 + \frac{q^2}{a_w} - qd_u,$$

and dropping the negative square gives the second summand. Both constants are finite for fixed u and q , and the bound of Theorem 23.9 is uniform in K . This proves the bound on the supremum. Given $\kappa > 0$, we choose $q > \kappa$; then

$$0 \leq \gamma^{-\kappa} F_u(\gamma) \leq C_{u,q} \gamma^{q-\kappa} \longrightarrow 0,$$

and no limit in the cutoff has been interchanged with the limit in the coupling.

Now fix n and set $u_a = nB^a$. For $K \geq a$ we have $N_K = u_a B^{K-a}$, and Theorem 23.9 applies directly with coarse side u_a and $K - a$ RG steps. No parameter of the event, of a local reconstruction, or of its finite thresholds is chosen again. We obtain $e_{K,a}(\gamma) \leq \mathcal{F}(u_a, \gamma)$.

Since both terms of $\mathcal{F}(u, \gamma)$ increase with γ for $Mu/\gamma \geq Mu/\gamma_0 > 1$, we have

$$\sup_{0 < \gamma \leq \gamma_0} \mathcal{F}(u_a, \gamma) \leq C' u_a^3 e^{-\bar{c} u_a / \gamma_0} + C_w u_a^3 e^{-a_w \log^2(Mu_a / \gamma_0) / 4}.$$

Up to fixed constants, the logarithms of these two terms are $3 \log u_a - \bar{c} u_a / \gamma_0$ and $3 \log u_a - a_w \log^2(Mu_a / \gamma_0) / 4$, and both tend to $-\infty$ as $a \rightarrow \infty$, since $u_a \rightarrow \infty$. $\square$

23.5 Two-sided bounds of logarithm-squared order

The fluctuation term in Theorem 23.9 has a simple heuristic origin. The cutoff radius is $p(t) \asymp \log(1/t)$, and a fluctuation variable with a Gaussian-type tail exceeds it with probability of order $\exp(-c \log^2(1/t))$. At a fixed coarse side u this gives an upper bound of order $\exp(-c \log^2(1/\gamma))$. The next theorem shows that this order is correct for the supremum over the number of RG steps. For the lower bound we fix one RG step, so that the problem is finite dimensional, and we exhibit a set of configurations of controlled Haar measure and controlled action on which one normalized fluctuation variable lies just beyond its cutoff while all other conditions hold.

Theorem 23.12. *Use the small-field event and the parameters of Theorems 23.2 and 23.9. For each fixed $u \in 4\mathbb{N}$ with $u \geq n_{\min}$ set*

$$E_u(\gamma) = \sup_{K \in \mathbb{N}_0} \mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c).$$

There exist $c_u, C_u > 0$ and $0 < \gamma_u \leq \min\{\gamma_0, e^{-1}\}$, depending on u and the probability constants but not on γ or on the number of renormalization group steps, such that

$$e^{-C_u[1+\log(1/\gamma)]^2} \leq \mu_{uB, \gamma}(S_{uB, u, \gamma}^c) \leq E_u(\gamma) \leq e^{-c_u \log^2(1/\gamma)} \quad (0 < \gamma \leq \gamma_u). \quad (23.21)$$

In particular, for fixed positive c, ε , and D , the bound $E_u(\gamma) \leq D \exp(-c\gamma^{-\varepsilon})$ does not hold for all sufficiently small $\gamma > 0$. No lower bound on $\limsup_{K \rightarrow \infty} \mu_{uB^K, \gamma}(S_{uB^K, u, \gamma}^c)$ is asserted.

Proof. Upper bound. Let $x = \log(1/\gamma)$ and $\ell_u = (1/2) \log(Mu) > 0$. The first bound of Theorem 23.9, which is uniform in K , reads

$$E_u(\gamma) \leq C' u^3 e^{-\tilde{c}_u e^x} + C_w u^3 e^{-a_w(x/2 + \ell_u)^2}.$$

For all sufficiently large x , the first term is at most $e^{-(a_w/8)x^2}/2$, since $e^x/x^2 \rightarrow \infty$; the second term is also at most this quantity, since its exponent is a negative quadratic in x with leading coefficient $a_w/4$. This proves the upper bound with, for example, $c_u = a_w/8$, after decreasing γ_u .

A box in the coordinates of one RG step. Fix $K = 1$, $N = uB$, and preceding classical depth zero. For the rest of the proof this is a fixed finite-dimensional problem. We set

$$t = \sqrt{\gamma/(Mu)}, \quad p = p(t) = A_w \log(1/t), \quad g_i = t\lambda_i, \quad \lambda_i = M^{(i-J+1)/2} \quad (0 \leq i < J).$$

The recursion of Theorem 12.13 is

$$z_i = z_i^*(c_{i+1}) + g_i S_i(c_{i+1}) w_i, \quad c_i = b_i(c_{i+1}, z_i), \quad c_J = c = e^\theta, \quad S_i = K_i^{-1/2}.$$

We use it as a classical change of coordinates, before any fluctuation cutoff is imposed. This is valid beyond the region $\max|w| \leq p$: Lemma 12.3 gives product domains of fixed positive radius for the coarse fields and the free coordinates, independent of g and p , and Theorems 12.13 and 13.2 give the composition of the J averaging steps on a common such domain, with the unscaled inputs $q_i = g_i w_i$. The conditional constrained minimizer at the identity is zero by Theorem 10.10, and the gauge transformation of Theorem 13.2 is the identity there. All intermediate Jacobian determinants are positive on a sufficiently small closed product domain.

Fix $0 < \varepsilon < 1/4$ and one free link (i_*, e_*) of the step, with its three-dimensional fluctuation variable; such a link exists since $M \geq 3$ and $u \geq 4$. We identify the corresponding

copy of $\mathfrak{su}(2)$ with $\mathbb{R}^3$ through the orthonormal basis of Section 4, and let e_1 be its first unit vector. We define the box $\mathcal{B}_\gamma$ of parameters by

$$\begin{aligned} |\theta_e| &< \epsilon\sqrt{\gamma} \quad (e \in \mathcal{E}_u), \\ |w_{i,e}| &< \epsilon \quad ((i,e) \neq (i_*,e_*)), \quad |w_{i_*,e_*} - (p+1)e_1| < \epsilon. \end{aligned} \quad (23.22)$$

Each ball for a fluctuation variable has the same positive Lebesgue volume, independent of its center and of γ . The largest unscaled fluctuation input is at most $t(p+1+\epsilon) \rightarrow 0$, and $\|\theta\|_\infty \rightarrow 0$. Then the closure of the box lies inside the fixed larger coordinate domain, where no cutoff is imposed, for all sufficiently small γ . We do not assign a Gaussian law to fluctuation variables beyond the cutoff. Explicitly, one may take $\tilde{p} = p+2$ in Theorem 12.13: the box is contained in $\Omega_{\tilde{p}}^{\text{all}}$, while $C_J t \tilde{p} < 1/2$ and $C_J(t\tilde{p}^3 + t\tilde{p})/(1 - C_J t \tilde{p}) \rightarrow 0$. All hypotheses of that theorem on the coordinate domains and on the triangular Jacobian hold for this larger domain at small coupling. The radius $\tilde{p}$ is used only to check the domain; the small-field event still uses exactly $p(t)$.

For the coordinate-domain conditions, we note that at small coupling the unscaled fluctuation inputs lie in the product domain of radius Q_{phys} of Definition 14.4 and Theorem 13.2, and the coarse field is closer to the identity than r . The reconstruction then has all input logarithms smaller than $r/2$, with the intermediate margins for the classical action, the constraint coordinates, and the Jacobians. Then $c_0 \in D_r$, $c \in D_r$, and $\mathcal{A}_{0,J}$ holds, and the averaged field is $T_M^J c_0 = c$. By the uniqueness and the identities on overlapping charts in Theorem 14.3 and Definition 14.4, the constructed w_i are the normalized fluctuation variables defined by the inverse coordinates, up to their norm-preserving triangular orthogonal changes. Let $\mathcal{D}_\gamma \subset \mathcal{U}_N$ be the set of configurations obtained from the configurations with coordinates in $\mathcal{B}_\gamma$ by all gauge transformations of the block axial gauge that equal the identity at the prescribed tree roots, that is, the union over the tree gauge variables of the images of $\mathcal{B}_\gamma$. Every configuration in $\mathcal{D}_\gamma$ fails the small-field event, since

$$|w_{i_*,e_*}| > p+1-\epsilon > p,$$

while it satisfies all other conditions of the step. In particular, $\mathcal{D}_\gamma$ is a subset of the complement of the small-field event itself.

Haar measure of the box. By repeated use of Lemma 6.3, followed by the triangular substitution from free coordinates to normalized fluctuation variables, we obtain the exact formula

$$\begin{aligned} H_N(\mathcal{D}_\gamma) &= \int_{\mathcal{B}_\gamma} dH_u(c) dw \prod_{i=0}^{J-1} [\mathcal{J}_i(c_{i+1}, z_i) g_i^{D_i} \det S_i(c_{i+1})], \\ D_i &= 6(M^3 - 1)(uM^{J-i-1})^3. \end{aligned} \quad (23.23)$$

The integral is over the Haar box of coarse fields and the balls of fluctuation variables; the tree gauge variables of $\mathcal{D}_\gamma$ are integrated out with Haar probability mass one. The

determinant identity is that of Theorem 12.13, and its off-diagonal dependence on the coarser fluctuation variables is kept. The formula is a change of variables on the open coordinate domain, and its restriction to the box needs no indicator of the cutoff product; equivalently, it follows from successive Haar disintegration formulas for single averaging steps and Fubini's theorem for the nonnegative integrand. There is no additional density for the logarithm of the coarse field, no Gram determinant, and no gauge multiplicity. Here D_i is the number of real free coordinates of the i -th averaging step: besides the eliminated edges there are $3(M^3 - 1)(uM^{J-i-1})^3$ fine links, with three real coordinates each, and the gauge transformations equal to the identity at the tree roots remove $3(M^3 - 1)(uM^{J-i-1})^3$ of these coordinates.

On the fixed small closed product domain, all $\mathcal{J}_i$ and $\det S_i$ are continuous and strictly positive, by Lemmas 6.3 and 12.3. Their finite product has a positive lower bound depending on u and the fixed chart, but not on γ . Since $g_i = \sqrt{\gamma} (Mu)^{-1/2} \lambda_i$, and the box of coarse fields is a product over the $|\mathcal{E}_u| = 3u^3$ links, Lemma 5.1 and (23.23) imply

$$H_N(\mathcal{D}_\gamma) \geq c(u) \gamma^{\kappa_u}, \quad \kappa_u = \frac{9u^3 + \sum_{i=0}^{J-1} D_i}{2} > 0, \quad (23.24)$$

where $\sum_i D_i = 6[(uB)^3 - u^3]$. This lower bound keeps the Haar densities and all powers of the couplings from the Jacobians.

The Wilson action on the box. By the majorant of Theorem 13.2, or its finite-dimensional derivative bound on the same smaller product, we have, in the gauge representative of that theorem,

$$\max_e |\text{Log } c_{0,e}| \leq C_u \{ \epsilon \sqrt{\gamma} + t(p + 1 + \epsilon) \} \leq C'_u \sqrt{\gamma} (1 + p).$$

For each plaquette, the triangle inequality for the group distance in Lemma 5.1 bounds its distance from the identity by four times this quantity, and by the same lemma we have $q \leq \rho_G^2/8$. There are $3N^3$ plaquettes, so for a fixed finite A_u ,

$$\frac{4N}{\gamma} \sum_p q((c_0)_p) \leq A_u (1 + p)^2 \quad (c_0 \in \mathcal{D}_\gamma).$$

By gauge invariance, the same bound holds on all of $\mathcal{D}_\gamma$. The partition function of the Wilson measure satisfies $Z_{N,\gamma} \leq 1$, by Lemma 5.1. Combining these bounds, we obtain

$$\mu_{uB,\gamma}(S_{uB,u,\gamma}^c) \geq \mu_{uB,\gamma}(\mathcal{D}_\gamma) \geq c(u) \gamma^{\kappa_u} \exp\{-A_u(1 + p)^2\}.$$

Finally, $p = (A_w/2)[x + \log(Mu)]$ for $x = \log(1/\gamma) \geq 1$. Then $1 + p \leq C''_u(1 + x)$ and $\kappa_u x + |\log c(u)| \leq C'''_u(1 + x)^2$. After increasing C_u and decreasing γ_u to include every threshold of the finite chart and of the upper bound, this is the lower bound in (23.21). Since $e^{\epsilon x}/(1 + x)^2 \rightarrow \infty$, comparing logarithms excludes the bound $D \exp(-c\gamma^{-\epsilon})$. $\square$

Corollary 23.13. *Use the parameters of Theorem 23.5. For each fixed $u \in 4\mathbb{N}$ with $u \geq n_\Gamma$ there exist $c_{u,\Gamma}, C_{u,\Gamma} > 0$ and $0 < \gamma_{u,\Gamma} \leq \min\{\Gamma, e^{-1}\}$ such that*

$$\begin{aligned} e^{-C_{u,\Gamma}[1+\log(1/\gamma)]^2} &\leq \mu_{uB,\gamma}(S_{uB,u,\gamma}^c) \leq \sup_{K \geq 0} \mu_{uB^K,\gamma}(S_{uB^K,u,\gamma}^c) \\ &\leq e^{-c_{u,\Gamma} \log^2(1/\gamma)} \quad (0 < \gamma \leq \gamma_{u,\Gamma}). \end{aligned}$$

No lower bound is asserted for $\limsup_{K \rightarrow \infty} \mu_{uB^K,\gamma}(S_{uB^K,u,\gamma}^c)$.

Proof. The upper bound follows from (23.13) exactly as in the proof of Theorem 23.12: with $x = \log(1/\gamma)$, the curvature term decays like e^{-ce^x} , while the fluctuation term has an exponent that is a negative quadratic in x with leading coefficient $a_w/4$. After decreasing $\gamma_{u,\Gamma}$, the sum is at most $e^{-(a_w/8)x^2}$.

For the lower bound we keep u and J fixed and use the coordinates of one RG step as in the proof of Theorem 23.12. Their coordinate domains, the classical change of coordinates, the Jacobians, and the cutoff of the small-field event are the same for the parameters of Theorem 23.5, since that theorem enlarges only the probability constants and the minimum side. On the box $|\theta_e| < \epsilon\sqrt{\gamma}$ of coarse fields, we place all fluctuation variables in balls of radius $\epsilon < 1/4$, except one, whose ball is centered at $(p(t) + 1)e_1$, where $t^2 = \gamma/(Mu)$. Since $t(p(t) + 2) \rightarrow 0$, all unscaled inputs and all Jacobian determinants lie in the same fixed domain without cutoff for sufficiently small γ . Every such point satisfies the other conditions and fails the cutoff. The exact formula (23.23) bounds the Haar measure of the corresponding set of configurations below by $c(u, \Gamma)\gamma^{\kappa_u}$, with $\kappa_u = (9u^3 + \sum_i D_i)/2$ and $D_i = 6(M^3 - 1)(uM^{J-i-1})^3$. The Wilson exponent on this set is at most $A(u, \Gamma)(1 + p(t))^2$, and the partition function is at most one. The probability of the complement is then at least $c\gamma^{\kappa_u}e^{-A(1+p(t))^2}$. Since $p(t) = A_w[x + \log(Mu)]/2$, we obtain the claimed lower bound. All domains and constants are fixed before γ ; the restriction $\gamma \leq \Gamma$ is obtained by decreasing $\gamma_{u,\Gamma}$ only. $\square$

23.6 Proof of the choice of parameters

In this subsection we prove Theorem 23.2, verifying the choices in the order of the items and using each earlier estimate on its domains and with the quantifiers of its statement. No probability estimate is used. Steps 1–7 make the choices of items (1)–(8), Steps 8 and 9 verify the hypotheses of the cited results without new choices, Steps 10 and 11 make the choices of items (9) and (10), and Step 12 fixes the reconstruction maps and proves the identity of the coarse fields with the iterates of T_M .

We fix $A_w \geq 1$ in the range allowed by Theorem 20.14 at the start. This precedes every use of the existence of a coupling bound in that theorem. Its constants and the J -uniform conditions do not depend on A_w ; the coupling bound g_* may depend on it.

Step 1: the choice of J . We apply Theorem 13.2 at the fixed odd M , and denote its J -uniform spatial exponent by b_* and its majorant for the conditional constrained minimizers at zero fluctuation by $Z(v) \leq Cv/(1 - Cv)$. First we fix a rate $0 < b_{\mathbb{T}} < b_*$ below the remaining margin of the classical and image estimates. Next we fix a finite scale $\mathcal{L}_{\mathbb{T}}$ for boxes covering supports, and its support-counting factor, as in Theorem 13.5, so that the entropy of supports is compensated by the rate $b_{\mathbb{T}}$. At this stage we enlarge the entropy and support constants to cover the margin for the free degrees of freedom in Theorem 20.9. Only after fixing these boxes do we choose the rate $a_{\mathbb{T}} \in (0, b_{\mathbb{T}})$, with $a_{\mathbb{T}} < a_{\text{cen}}/8$ and small enough for the network bound below. Here a_{cen} is the J -uniform rate of the bounds of Lemma 11.9; for the aligned representative, Theorem 11.10 gives the contraction at every output rate below $a_{\text{cen}}/8$ and at no larger rate, so this restriction is necessary. It also gives $a_{\mathbb{T}} < a_*/8$ for the rate a_* of Theorem 10.11, as observed in Theorem 13.5. These choices precede J , and decreasing $a_{\mathbb{T}}$ does not change the boxes.

We check the support condition of the conversion from supported coefficient bounds to coefficient norms at the same analytic radius in Theorem 13.5. A connected set Y of boxes joins the coefficient locations of the background in its boxes by a network of length at most $C_{\mathcal{L}_{\mathbb{T}}}|Y|$, with $C_{\mathcal{L}_{\mathbb{T}}}$ independent of J . Indeed, we connect the centers of adjacent boxes along a spanning tree of Y , and connect the finitely many locations in each box by the coordinate grid inside that box. The same construction covers a connected union of intersecting boxes, and shortest periodic connections give the same bound in directions that wrap around the torus. The bound also covers the supports of the logarithms in the cluster expansion and the larger output radius. Repeated locations add no length. There are only nine types of orientation and color at a site, and the density of free variables does not change this count. With this $C_{\mathcal{L}_{\mathbb{T}}}$ we take $a_{\mathbb{T}}C_{\mathcal{L}_{\mathbb{T}}} \leq 2$, so that $a_{\mathbb{T}}\mathbb{T} \leq 2|Y|$ for every support used in the conversion. The effective interaction and its output as a whole, rather than the individual polymer activities, have the translation symmetry required by the conversion, and their color and inversion identities are those kept by Theorem 13.5.

Let $C_{\text{geom}}^{\mathbb{T}}$ be the network comparison constant of that theorem. We include the conditions

$$C_{\text{geom}}^{\mathbb{T}}b_{\mathbb{T}}/B \leq a_{\mathbb{T}}, \quad C_{\text{cen}}/B < 1/8 \quad (23.25)$$

in the choice of J . The first compensates the conversion from the trees on the finer lattice to the trees on the coarse lattice, while keeping the rate $a_{\mathbb{T}}$ on the torus; it is separate from the smaller regional rates of item (3). By part (3) of Theorem 15.13, one enlarged constant $C_{\text{CP}} \geq 1$ bounds the value, the first spatial differences, and the first three coarse-field derivatives of the zero-fluctuation reconstruction, with all row sums and all volume-normalized column sums, uniformly before J is chosen; the enlargement includes the prescribed reconstruction tolerances. Derivatives with respect to the fluctuation inputs use the fixed- J joint bounds of Lemma 15.16, not this uniformity. The same results provide the signed conservation identities for the volume-normalized derivative kernels and the

full bound at fixed J . The series Z of Lemma 11.9 and its spatial rate are J -uniform, as are these finite derivative bounds and the geometric constants. Neither the full joint norm nor the full norm of the gauge transformations of the reconstruction maps is J -uniform. We enlarge the J -uniform constant C_{alloc} of Theorem 20.9, which controls the distribution of the spatial decay rates, so that it also covers the corresponding geometric constant in Theorem 19.9, choose $C_*^{\text{geom}} \geq \max(1, C_g)$, and keep $B_{\text{geom}} = 256C_{\text{alloc}}(C_*^{\text{geom}} + 1)$ from Theorem 20.9. All lower bounds on M^J in item (1) now have fixed finite right sides, and they hold simultaneously for large J , since $M^J \rightarrow \infty$.

It remains to separate the linear conditions that choose J from the later choice of radii. For existence we may take $\rho_{\text{cl}} = \rho$, but we keep the subscript $\bullet$ to check both functions. The same classical map evaluates both the effective interaction and the effective classical action, and the series Z provides both Z_q and Z_{cl} ; fixed different ratios of radii would only require the two restrictions of the same series. We choose a J -uniform bound ρ_0 with $4K_*C\rho_0 \leq 1/2$. With $Z_\bullet(v) \leq C_\bullet v / (1 - C_\bullet v)$, the conditions

$$C_\bullet K_*/B < 1/32, \quad C_\bullet K_*\rho_\bullet \leq 1/2$$

give $B^{-1}Z_\bullet(K_*\rho_\bullet) < \rho_\bullet/16$, and at the enlarged radii, uniformly for $0 < \rho_\bullet \leq \rho_0$,

$$B^{-1}Z_\bullet(4K_*\rho_\bullet) \leq (8C_\bullet K_*/B)\rho_\bullet < \rho_\bullet/4. \quad (23.26)$$

Only the first condition chooses J ; the second is an upper bound on a later radius. Together with $q_{\text{CA}} < 1/4$, the threshold $M^{-J} \leq \delta_0(a_{\mathbb{T}})$ of Theorem 11.10, and (23.25), these are the J -uniform conditions of Theorem 13.5, whose contraction constant $C_P(a_{\mathbb{T}})$ is uniform over the later radii $0 < \rho \leq \rho_0$. Since $a_{\mathbb{T}}$ is fixed before J and depends only on J -uniform data, these conditions are lower bounds on B with fixed right sides. The finitely many additional conditions on classical first derivatives use the same J -uniform series and C_{CP} , and we impose them with strict inequality in the same choice. No constant at fixed J , no radius for free coordinates, and no constant of the effective-interaction estimates has entered; no later step changes J .

Step 2: larger domains, solution neighborhoods, and rates. At this J , all entries in the minima defining $v_{\text{free},J}$ and R_{raw} in Lemma 19.7 are positive and depend only on the larger construction data, and we choose each radius strictly below its minimum. For each of the finitely many initial segments of conditional integrations, we choose by Lemma 19.3 a neighborhood of the conditional constrained minimizer and then a smaller domain for the later parameters. At zero perturbation we choose, with a positive margin, a smaller product domain for the free coordinates whose successive substitutions stay inside all the previously chosen domains; the condition with a perturbation is checked at every value of the later parameters in Step 3. Taking the minimum of the finitely many margins at zero perturbation preserves them. These substitution products are chosen with fixed

absolute-coefficient image margins inside the next factor's coefficient radius, using the common convergent composed majorants of Lemmas 18.3 and 18.15; their distinguished-input bounds are fixed on these larger products as well. This requires coefficient-image control, not merely pointwise image inclusion. In particular, the conditional displacement bound and the bound on the preconditioned Hessian hold on these products, and a slice of a larger joint solution domain is never used in their place. The real lower bound for the constrained Hessian and the complex domain of its inverse in the two cited lemmas apply on the resulting product domains.

We choose the inner, middle, and outer domains of the reconstruction maps strictly inside these construction domains; Theorem 15.13 provides its bounds on these domains independently of the later support and working radii. We choose $\rho, \rho_{\text{cl}} > 0$ inside them, with (20.1), $4C_{\text{tor},J}K_*\rho_{\bullet} < 1/4$, and the fixed outer and sequential free products inside h_J , small enough that the first summand in the inequality of item (3) is below $R_{\text{raw}}/64$ and that both denominators of the torus images are positive. All coefficient sums used here can be bounded on the larger domains before the radii are restricted, since a restriction can only decrease a sum of absolute coefficients; this is the convention of Lemma 19.7. We also impose the hypotheses of Theorems 13.2 and 13.5 on the larger product. With r_J and the torus majorant $C_{\text{tor},J}$ fixed on that product, we choose

$$\begin{aligned} 0 < \rho &\leq \min\{\rho_0, r_J/(16K_*)\}, & 4K_*C_{\text{tor},J}\rho &< 1/4, \\ 0 < s_J &< \min\{r_J/(16K_*), 1/(16K_*C_{\text{tor},J}), \rho/(64K_*C_{\text{tor},J})\}. \end{aligned} \quad (23.27)$$

These restrictions are compatible with the bounds on the coordinate and reconstruction domains just chosen. We have $4K_*(\rho + s_J) < r_J$ and $1 - 4K_*C_{\text{tor},J}(\rho + s_J) > 1/2$. Then the majorant of the free part is at most $8K_*C_{\text{tor},J}s_J < \rho/8$. Together with (23.26), the enlarged image is strictly below $3\rho/8 < \rho/2$. All intermediate margins for the classical action and for the constraint Jacobians are the strict margins of Theorem 13.2, since we work on a restriction of its product. We keep this larger product after r is fixed.

The two majorants for the input functions used here concern the torus map of Theorem 13.2. We do not assert a J -uniform bound of all orders for the gauge transformations of the reconstruction maps. The full bound at fixed J and the strict image margins of Theorem 15.13 apply on nested subproducts of the same domains, and their limiting maps and later refinements of thresholds preserve these margins. The only constants of the reconstruction maps used before J were the finitely many coarse-field derivative bounds at zero fluctuation.

Theorem 13.5 now provides finite constants $C_G, C_T > 0$, and the constant $c_* > 0$ for its auxiliary discs (not to be confused with the exponent $c_* = \min\{a_b, d_*/8\}$ of the probability bounds), on the fixed data $(M, J, \rho, a_{\mathbb{T}}, b_{\mathbb{T}}, K_*, s_J)$. They do not depend on the side, the preceding depth, or the later support radius. We fix $E^{\text{cl}} \geq \sup_{d \geq 0} \mathcal{B}_{\rho_{\text{cl}}, a_{\mathbb{T}}}(E_d)$, finite by Theorem 10.11, and take a finite $E \geq 2C_G/(1 - q_{\text{CA}})$; the denominator is positive

by item (1). We verify the invariance of this bound under the recursion in Step 8, after the coupling bound has been chosen; it is not assumed as a hypothesis of a regional estimate. We now choose $q_{\text{raw}} > 0$ at most $R_{\text{raw}}/(64C_{\text{lift},J})$ and inside the common free-coordinate reserve. The later r and β_{rec} are bounded by the same number; then the sum in item (3) is strictly below $R_{\text{raw}}/8$.

The interpolation of Lemma 19.7 provides a positive bound on the spatial rate of the unperturbed coefficients on these larger products, and we include it in $c_{\text{prim},J}$ after the conversion of Section 20.1 to the graph metric. Its finite bound for sums with the remaining exponent can be written as $C_{\text{int},J}\rho_\bullet$: we divide a common bound on the larger product by the smaller of the two positive radii and take $C_{\text{int},J} \geq 1$. This constant is chosen after the radii and is distinct from the earlier torus majorant. The bound is uniform over the later restrictions, which stay inside the larger product. For its remaining exponent $c > 0$ we choose $0 < \vartheta < 1$ with

$$\vartheta \max\{1, \log(8C_{\text{int},J}/3)\} < \frac{1}{2} \log(4/3), \quad (23.28)$$

and take the minimum of the resulting positive exponents over the two radii and the finitely many initial segments. By Lemma 19.7, the bounds on the larger product are uniform in the later smaller radii and widths, so this choice precedes the choice of r .

We now set $\alpha = a/16$, $m_{\text{rate}} = C_*^{\text{geom}}(a + \alpha)$, and $b_1 = 8C_*^{\text{geom}}\alpha$. Then $m_{\text{rate}} + b_1 = (25/16)C_*^{\text{geom}}a$, and the inequalities of item (3) containing Ba and $B\alpha$ reduce to

$$\frac{25}{16}C_*^{\text{geom}} < B/C_{\text{alloc}}, \quad 8C_*^{\text{geom}} < B/C_{\text{alloc}},$$

which follow from $B \geq B_{\text{geom}}$. We decrease $a > 0$ further so that $\sqrt{3}(a + \alpha) < a_{\mathbb{T}}$ and the conditions involving $c_{\text{prim},J}$ and the unperturbed image exponent hold, including $m_{\text{rate}} < \vartheta c$ for every radius and initial segment. The cluster expansion with exact supports imposes no further restriction on lengths. The rate $a_{\mathbb{T}}$ and J are unchanged.

Step 3: perturbation bounds, the fluctuation radius, and the final perturbation bound.

We choose a positive preliminary bound σ_0 for perturbations that are real on real arguments, inside the margins for the force, for the lower bound of the constrained Hessian, and for the domain of its complex inverse in Lemma 19.3 and Theorem 19.9. Here and in every later application, the incoming classical norm is taken on the incoming list D_{in} of active labels, with the lattices where its coefficient locations were introduced and the volumes of its variables, at the uniform radius ρ_{cl} . The spatially varying radii are output radii. In Step 9 we verify, using the coverage by large-field decompositions and the substitution of frozen coordinates, that these are the incoming radii and compatible complements after any number of RG steps.

The perturbation must be controlled at every value of the later parameters, not only at the joint constrained minimizer. Let $\varepsilon_{\text{tube}} > 0$ be the minimum of the margins of the

unperturbed solution and parameter domains, and let I_0 and $C_{P,J}^{\text{force}}$ be the constants on the larger product for the inverse and for the conditional force in Theorem 19.9; here V_I is the z_I -derivative, in the original free coordinates, of the function V in (21.12). The estimate of that theorem, uniform over all $y \in \mathcal{T}_I$, gives

$$\delta_V = \max_I \sup_{y \in \mathcal{T}_I} \|A_I^{-1} V_I(c, a_I(c, y; \xi), y)\|_{\text{coeff}} \leq I_0 C_{P,J}^{\text{force}} \sigma_0 \rho_{\text{cl}}, \quad (23.29)$$

$$C_{\text{seq},J}(2\delta_V + r_{q,P}) < \varepsilon_{\text{tube}}/2.$$

To obtain the last strict inequality, we first decrease σ_0 so that $2C_{\text{seq},J}I_0C_{P,J}^{\text{force}}\sigma_0\rho_{\text{cl}} < \varepsilon_{\text{tube}}/8$ and $I_0C_{P,J}^{\text{force}}\sigma_0\rho_{\text{cl}} < \min_I r_I/4$. Then we choose $0 < r_{q,P} < \varepsilon_{\text{tube}}/(8C_{\text{seq},J})$ inside the fixed margin of the joint solution domain; the sum is then even less than $\varepsilon_{\text{tube}}/4$. We decrease σ_0 also for the bound $1/2$ on the error of the preconditioned Hessian, for half of the real lower bound of the constrained Hessian, and for the domain of its complex inverse; these are the coefficient bounds of Theorem 19.9 on the same larger product. Its contraction at fixed parameters bounds every conditional displacement by $2\delta_V$, and the sequential conclusion of Lemma 19.3 applies at all such y with a margin of a factor two. In particular, the chain of conditional constrained minimizers at zero fluctuation satisfies the coordinate-domain conditions before the fluctuation radius is chosen. A smaller final σ_* preserves all these conditions.

All derivatives here are taken in the active background variables and the free coordinates, at fixed real frozen data $b = (\xi, b_{\text{rec}})$. The incoming coefficients and charts are Borel functions of b . At the initial step, the input consists of finite products of link variables and the analytic charts of the classical action; at a step starting from a stopped measure, it is the effective interaction, which is invariant on overlapping charts; and each later input is obtained by substitution, finite products, and convergent operations on coefficients. Integration over a fixed compact product of fluctuation balls preserves Borel dependence on b . Indeed, after rescaling the balls to unit balls the measure is a fixed finite measure, and the integral of a nonnegative jointly Borel function against a fixed finite measure is a Borel function of the parameters; this is the monotone class argument given in the proof of Theorem 21.10. Then the uniformly convergent Picard iterations in Theorem 19.9 preserve Borel dependence on b . No derivative in a frozen real parameter is needed.

On these larger fixed products, Lemma 20.3 and Theorems 20.14 and 21.9 provide finite constants for the output, sequential, image, potential, and mean estimates, and for the estimates of the support of the normalized fluctuation variables, uniformly for the doubled classical bound $2\sigma_0$. Their dependence, described in item (3) of Definition 4.14, excludes all later widths and all values of frozen coordinates. We use these bounds, rather than bounds computed on the eventual small cutoff domain. The threshold $\epsilon_{\text{en},J}$ and the constant $C_{\text{en},J}$ of Theorem 21.9 are fixed at this stage as well; the threshold already contains the smallness $C_J(r + \beta_{\text{rec}}) \leq \epsilon_{M,J}$ that its proof needs for the Hessian bound

(18.45) of Lemma 18.14, at the small representative, before the orthogonal transport to the representatives used in the descent. The radius of the auxiliary box in that proof is chosen there. The constant $C_{\text{en},J}$ is a bound for every covered large-field decomposition; its conclusion for every point is applied only after the coverage is established in Step 9.

We fix all geometric constants at this stage. Let C, C', R_0, c be the constants of Theorem 21.3. We take $A_8 \geq R_0 + 1$ so large that $C'/A_8 < 1/4$, and enlarge it for the fixed mesh at the initial step and the fixed margins for path lengths. This is a lower bound independent of r and β_{rec} . Let $C_B \geq (10M)^J$ cover the lengths of all tree and transport paths within one RG step, and set

$$K_B = 2 + C_B(A_8 + 1), \quad C_{\text{col}} = CK_B^{10}/A_8. \quad (23.30)$$

Every small frozen coordinate created at the first step after a stopped measure is a product of a fixed number of current small links and of their at most J iterates under T_M . By repeated use of the triangle inequality for the group distance and the factor $10M$ per averaging step, it is bounded by $C_{\text{ad},J}r$, where $C_{\text{ad},J}$ depends only on these fixed products. The frozen coordinates at the protected initial step satisfy the same bound. Older long transports are frozen group variables, not new small analytic coordinates. Let $C_{\text{an},J} \geq 1$ bound all fixed conversions from group or transported coordinates to the analytic coordinates at the lattice where a frozen coordinate was introduced. For a link variable in the identity chart this constant is one, since $|\text{Log } U| = \rho_G(U)$; a fixed longer product is bounded by its fixed length and its analytic derivative bound. We fix $C_{\text{conv},J}$ large enough for these conversions, including

$$C_{\text{conv},J} \geq C_{\text{ad},J}, \quad C_{\text{conv},J} > 2C_{\text{an},J}C_{\text{col}}, \quad C_{\text{conv},J} > C_{\text{an},J}C_{\text{ad},J},$$

and enlarge it for the fixed transport paths and the protected frozen coordinates of Theorem 21.5. We fix the mesh multiplier $C_{\text{mesh},J}$ of the initial step and the interior restriction by the full dependence radius at the same stage. None of these constants depends on a later width.

Every entry in the minimum defining R_W in Lemma 20.3 is positive at this J . On these preliminary products and uniformly for σ_0 , the common absolute coefficient majorant gives a finite $D_{\text{raw},J} \geq 1$ for the derivative of the raw sequential reconstruction in its scaled fluctuation inputs. Choose also $R_W < q_{\text{raw}}/(2D_{\text{raw},J})$. Radial integration of this derivative bounds displacement from the perturbed zero-input joint center by $D_{\text{raw},J}R_W$. We choose R_W strictly below the preceding minimum as well, including the conversion to the joint solution neighborhood and both product domains. Include in $C_{P,\bullet}$ the larger-product absolute coefficient row constant for the difference of the perturbed and unperturbed reconstructions at output rates (m_{rate}, b_1) in Theorem 19.9. These rates

satisfy its conditions by item (3), since C_{alloc} includes its geometric constant. Then we fix $\sigma_* > 0$ below σ_0 , below all preconditioned displacement margins, and below

$$\frac{\lambda_* R_W}{6C_{\text{en},J}\rho_{\text{cl}}}, \quad \min_{\bullet} \frac{\rho_{\bullet}}{64C_{P,\bullet}\rho_{\text{cl}}}.$$

Require in addition $2I_0C_{P,J}^{\text{force}}\sigma_*\rho_{\text{cl}} < q_{\text{raw}}/2$. For the final inputs, $\delta_V \leq I_0C_{P,J}^{\text{force}}\sigma_*\rho_{\text{cl}}$. The perturbed joint center is within $2\delta_V$ of its unperturbed counterpart by the joint case of Theorem 19.9. Hence

$$2\delta_V + D_{\text{raw},J}R_W < q_{\text{raw}},$$

which is the raw-coordinate reserve used in item (3), including displacement of the center. If a constant vanishes, the corresponding upper bound is not needed. Only fixed positive upper bounds occur. In particular, the perturbation term in the radius R_{en} of the enclosing product domain is accounted for before the removed layer is chosen.

Step 4: boundary radii, removed layers, and frozen coordinates. We choose each outer boundary radius so that its contribution is at most $\rho_{\bullet}/16$, and set $\rho_{\partial,\bullet} = R_{\partial,\bullet}/K_*$. Since $a_{W,\bullet} > 0$, a finite common width w_{bd} makes each contribution from distant parts of the boundary at most $\rho_{\bullet}/16$. The torus contribution was made smaller than this in Step 1. Once β_{rec} is chosen below with $C_{F,\bullet}\beta_{\text{rec}} < \rho_{\bullet}/64$, and since $C_{P,\bullet}\sigma_*\rho_{\text{cl}} < \rho_{\bullet}/64$ by Step 3, the sum at zero fluctuation satisfies

$$A_{0,\bullet} < \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{64} + \frac{1}{64} \right) \rho_{\bullet} = \frac{7}{32} \rho_{\bullet} < \rho_{\bullet}/4. \quad (23.31)$$

The individual bounds give the full sum condition of item (5), with every summand counted.

The brackets in the two inequalities for s_{er} are fixed finite numbers. For example, if $Y = A_J(1 + E) + K_J\sigma_* + 1$, then $s_{\text{er}} \geq 1 + \alpha^{-1} \log(4Y/\sigma_*)$ suffices for the first inequality, and the same calculation with $Y^{\text{cl}} = A_J^{\text{cl}} + K_J^{\text{cl}}\sigma_*$ yields the second. We take an integer above these bounds and above $\max(2, w_{\text{bd}})$. If a bracket vanishes, its inequality holds already.

We now choose $\beta_{\text{rec}} > 0$ below the minimum of the coordinate-domain, product, and boundary radii and of the bound $\lambda_* R_W / (6C_{\text{en},J})$ used for the radius of the enclosing product domain. On every label with reduced radius, β_{rec} below one quarter of the corresponding inner boundary radius makes the frozen fraction at most $1/4$. The bound for the boundary-completion difference in Lemma 18.9 is $8\beta_{\text{rec}}E_{\bullet}/\rho_{\bullet}$ at uniform radius; the fixed conversion to the spatially varying radius changes only its constant on the larger domain. This is part (2) of that lemma. Its hypothesis on the torus rate holds for both functions, since their torus norms have the rate $a_{\mathbb{T}}$ and $\sqrt{3}(a + \alpha) < a_{\mathbb{T}}$ by item (3). We fix a constant $C_{\text{bc},\bullet} \geq 8$ for the simultaneous substitution of all frozen coordinates and every required conversion, including the boundary layer, the neighborhoods where

gauge transformations are extended at the lattices of introduction, the based transports of frozen links across the boundary, and the initial step. If these substitutions are made in several fixed stages, we take the sum of their converted bounds in this constant. The number of frozen labels, of components, and of earlier RG steps does not enter, since the estimate of Lemma 18.9 already bounds the union. We require

$$C_{bc,\bullet} E_{\bullet} \beta_{\text{rec}} / \rho_{\bullet} \leq \sigma_*/8, \quad \beta_{\text{rec}} < \frac{1}{4} \min\{\rho_{\bullet}, \rho_{\partial,\bullet}\} \quad (\bullet \in \{q, \text{cl}\}). \quad (23.32)$$

Then the sum of these differences is below the bound $\sigma_*/4$ required in item (5). We also require $\beta_{\text{rec}} < \min(\pi/4, \epsilon_{\text{en},J}/4)$, where $\epsilon_{\text{en},J}$ is the fixed margin of Theorem 21.9. The geometric argument of Step 6 provides these bounds in the analytic coordinates at the lattice of introduction. We include the bound for frozen coordinates used in (23.31). All these choices are compatible, since each is an upper bound with a fixed positive right side.

For a large-field decomposition with the hypotheses on incoming radii and boundaries verified in Step 9, these conditions preserve the initial norm bound. If the input norm is at most σ_* , the last nonlinear term in Theorem 20.14 is at most one after the coupling bound is chosen in Step 7. Then its output is bounded by Y , and its output after removal of the layer of width s_{er} is bounded by $\sigma_*/2$. Adding the boundary-completion difference, we obtain at most $3\sigma_*/4$. The classical calculation uses Y^{cl} and gives the same bound $3\sigma_*/4$. The initial norm is at most σ_* once the initial substitution has been checked. Step 9 provides this initialization and the input hypotheses at every step, so no errors are summed over earlier RG steps.

Step 5: the majorant of the reconstruction maps, the working radius, and the support radius. The coefficient constants of Lemma 15.16 were computed on the fixed larger domains. Recall that $\alpha_* = C_{\text{CP}}/B$. Since $B \geq 3$ and $C_{\text{CP}}^2/B \leq 1/4$, we have $0 < \alpha_* \leq 1/(2\sqrt{B}) < 1$. For the coefficients in that lemma, the choice of β_J gives, for every $m \geq 2$,

$$\alpha_* \beta_J^{m-1} \geq \alpha_* \frac{A_J}{\alpha_* R_J^2} R_J^{-(m-2)} = A_J R_J^{-m},$$

where we used $\beta_J \geq A_J/(\alpha_* R_J^2)$ and $\beta_J \geq R_J^{-1}$. The linear term is also bounded by α_* . This proves the majorant φ in (23.4) and its derivative bound at every fixed input, in the volume normalization of that lemma. We choose $r_{\text{eff}} > 0$ inside the domains of item (3) and below $(1 - \alpha_*)/(8\beta_J)$. Then $\beta_J r_{\text{eff}} < 1/8$, and we have

$$\sup_{|v| \leq r_{\text{eff}}} \|D\hat{P}(v)\|_{\text{root}} \leq \varphi'(r_{\text{eff}}) = \frac{\alpha_*}{(1 - \beta_J r_{\text{eff}})^2} \leq \frac{64}{49} \alpha_*, \quad a_{\text{map}} \sqrt{B} \leq \frac{32}{49}. \quad (23.33)$$

We choose r below all positive bounds displayed in item (6), including the bound from path lengths, the bound from frozen coordinates, and $r_{\text{eff}}/4$. Further upper bounds used

for the protected neighborhoods below are imposed in the same choice. In particular we require $8B \lceil A_8/r \rceil > \max\{s_{\text{er}}, D_{\text{dep},J}\}$, so the width W in item (7) covers the removed band and all fixed dependence radii in the stated W -based counts. We include $r < \min(1, \epsilon_{\text{en},J}/4)$ and

$$K_B^2 r / A_8 < c, \quad C_{\text{an},J} C_{\text{col}} r < \beta_{\text{rec}} / 2, \quad C_{\text{ad},J} r < \beta_{\text{rec}}, \quad (23.34)$$

which hold after decreasing r , since all constants in (23.30) are fixed. In particular, $\beta_{\text{rec}} > C_{\text{conv},J} r$. With $R_{\text{buf}} = C_{\text{map},J} r$ and a fixed $C_{\text{map},J} \geq (10M)^J$, the requirement $R_{\text{buf}} < \rho$ and the margins of the larger charts are again upper bounds on r . No lower bound on r occurs.

We also choose the radii P and Q_{DB} explicitly. On a fixed smaller joint product, the bound of Lemma 15.16 provides a finite $C_{\text{free},J} \geq 1$ such that the free increment of the reconstruction is at most $C_{\text{free},J} \max_i |\zeta_i|$. For $|v| \leq r$ we have

$$|\hat{P}(v)| \leq \frac{\alpha_* r}{1 - \beta_J r} \leq \frac{32}{31} \cdot \frac{r}{2\sqrt{B}} < \frac{r}{3},$$

where we used $\beta_J r < 1/32$ and, in the last inequality, $B \geq 3$. For the torus map of Theorem 13.2 we also take $Cr < 1/2$ and $r < \min(r_J/4, 1/(4C_J))$; by the J -uniform condition we have $B^{-1}Z(r) \leq 2Cr/B < r/3$. We choose

$$0 < Q_{\text{DB}} = q_{\text{DB},J} r, \quad q_{\text{DB},J} \leq \min\{1/2, 1/(12C_{\text{free},J}), 1/(24C_J)\}. \quad (23.35)$$

Then $r + Q_{\text{DB}} < r_J$ and $1 - C_J(r + Q_{\text{DB}}) > 1/2$. The majorant of the free part of the torus map is then below $2C_J Q_{\text{DB}} \leq r/12$, and the hypotheses of Theorem 13.2 for coarse fields in B_r hold, including the domain before the pole of its majorant and $\mathfrak{M}_J(r, Q_{\text{DB}}) < r/2$. The same Q_{DB} works for the free part of the reconstruction maps, since $C_{\text{free},J} Q_{\text{DB}} \leq r/12$. We take $P = Q_{\text{DB}}/2$, decreasing the fixed proportionality factor if necessary so that every intermediate coordinate-domain condition holds. Then every input logarithm is smaller than $r/3 + r/12 < r/2$. At zero coarse field and zero fluctuation all these intermediate maps vanish, and their finite coefficient bounds at fixed J and the positive margins of the coordinate and product domains justify this simultaneous decrease. It is made when r is chosen, before the coupling. Both torus enclosing constants $\mathcal{L}_{\text{en}}$ and $\mathcal{L}'_{\text{en}}$ of Definition 14.4 are fixed on the larger domains. Decreasing r enforces every imported gate displayed in item (6), including the quarter-Hessian margin at the larger torus comparison radius. These choices are separate from the regional enclosing radius R_{en} . The remaining term $t^2(1 + E)$ is controlled in Step 7.

The three contributions to the radius of the enclosing product domain give

$$r + \beta_{\text{rec}} + \sigma_* \rho_{\text{cl}} \leq \frac{\lambda_* R_W}{2C_{\text{en},J}}, \quad R_{\text{en}} = \frac{2C_{\text{en},J}}{\lambda_*} (r + \beta_{\text{rec}} + \sigma_* \rho_{\text{cl}}) \leq R_W. \quad (23.36)$$

For either radius, each of the two terms subtracted in the denominator of item (6) is at most $1/4$, since $C_{T,\bullet}K_*\rho_\bullet \leq 1/4$ and, by the choice of R_W , $C_{T,\bullet}R_W \leq 1/4$. The denominator is then at least $1/2$, and the increment from the fluctuations is at most

$$(2C_{T,\bullet} + C_{\Delta,\bullet})R_{\text{en}} \leq (2C_{T,\bullet} + C_{\Delta,\bullet})R_W \leq \rho_\bullet/8.$$

Together with (23.31), this gives the bound $3\rho_\bullet/8$ at spatial exponent zero. Write S_b^0 for the absolute image row of the unperturbed reconstruction at exponent b , and ΔS_b for the absolute coefficient row of the difference of the perturbed and unperturbed reconstructions on the same products. The interpolation of Lemma 19.7 and (23.28), applied to the unperturbed coefficients, gives

$$S_{m_{\text{rate}}}^0 \leq S_{\vartheta c}^0 < (\sqrt{3}/4)\rho_\bullet,$$

where the choice in Step 2 includes $m_{\text{rate}} < \vartheta c$. Applying Theorem 19.9 at output rates (m_{rate}, b_1) and the bound on σ_* in Step 3 now gives

$$\Delta S_{m_{\text{rate}}} \leq C_{P,\bullet}\sigma_*\rho_{\text{cl}} \leq \rho_\bullet/64, \quad S_{m_{\text{rate}}} \leq S_{m_{\text{rate}}}^0 + \Delta S_{m_{\text{rate}}} < \left(\frac{\sqrt{3}}{4} + \frac{1}{64}\right)\rho_\bullet < \rho_\bullet/2.$$

This is the required absolute weighted image bound for both input functions on the entire enclosing product. The higher exponent included in $c_{\text{prim},J}$ belongs to the unperturbed construction; inherited perturbative networks use the separate inequalities against Ba and $B\alpha$.

Step 6: widths, the curvature threshold, and the large-field decompositions. Let R_{cell} , W , ϱ , and r_F be as in item (7). The initial choice of r already requires $W > \max\{s_{\text{er}}, D_{\text{dep},J}\}$, so the removed layer and the fixed dependence neighborhoods fit the W -based coverage counts. Enlarging the removed width to ϱ also decreases the left sides of its exponential inequalities. We now verify the geometric hypotheses of Theorem 21.5, using Theorem 21.3 for the neighborhoods in which gauge transformations are extended. Here $R_{\text{cell}} \geq R_0$, and the output side after each RG step must be at least $8R_{\text{cell}}$; both follow from the choice of A_8 and from the minimum side in Step 11. Whenever $\delta R_{\text{cell}}^2 \leq 1$ and $\eta \leq C_B r$, the constants of the estimate on large cells satisfy

$$\begin{aligned} K_R &= 1 + \eta R_{\text{cell}} + \delta R_{\text{cell}}^2 \leq K_B, \\ K_R^2/R_{\text{cell}} &\leq K_B^2 r/A_8 < c, \quad CK_R^{10}/R_{\text{cell}} \leq C_{\text{col}} r < \beta_{\text{rec}}/(2C_{\text{an},J}), \end{aligned} \tag{23.37}$$

where we used $rR_{\text{cell}} \leq A_8 + r \leq A_8 + 1$. Away from the two prescribed boundary layers of cells, the bound is $C'/R_{\text{cell}} < r/4$. Then the fixed conversion for products of links and (23.34) give the margin $r/2$ in the bulk and the bound for frozen analytic coordinates required by Theorem 21.5. These small representatives are used only for their existence, not for a change of variables in a probability.

To protect older data, we mark, in each lattice where data were introduced, both endpoints of every older frozen link, both endpoints of every transported frozen coordinate, and all prescribed tree roots and coefficient locations attached to frozen records. We exclude every current block of the coarse lattice whose centered lift to any of these lattices meets a mark. A gauge transformation supported in the remaining active blocks has a lift that is constant on blocks and equals the identity at every mark, by the corresponding clause of Theorem 21.3; hence every older frozen link and transport is preserved exactly. The newly chosen roots of the current constraint coordinates are fixed separately by the current based gauge, as in that theorem; they are not added to the persistent exclusion marks unless attached to a frozen record. The endpoints of the current links across the boundary are fixed by the target-dependent gauge of Lemma 19.5, and their records are protected at subsequent steps if those links are frozen. The union of the persistent exclusions protects the data of all earlier RG steps at once, without a product of losses over these steps. Every protected frozen-record vertex on the current output lattice is either in the fixed complement H or in its two-block neighborhood: blocks whose lifts meet an older mark were excluded, and the current boundary endpoints lie in the adjacent protected band. Hence the cutoff of Lemma 19.5 vanishes at all such vertices outside H ; its three-block transition band and the fine-link endpoint buffer were included in L_J before the widths were chosen.

We fix the block-aligned input mesh $S = B\lceil C_{\text{mesh},J} R_{\text{cell}}/B \rceil$, where $C_{\text{mesh},J} \geq 1$ was fixed in Step 3, large enough for the layer of large cells around the active blocks and for the full dependence radius of the constraint, tree, and transport paths. Then $R_{\text{cell}} \leq S \leq (C_{\text{mesh},J} + B)R_{\text{cell}}$, so $S \asymp_J R_{\text{cell}}$. At an initial RG step from the Wilson measure, the input side n_{input} is divisible by B . When $n_{\text{input}} \geq 8S$, this mesh partitions each coordinate circle into block-aligned intervals with lengths in $[S/2, 2S]$: we distribute the n_{input}/B blocks among $\lfloor n_{\text{input}}/S \rfloor$ intervals as evenly as possible.

At the initial step, the procedure of Theorem 21.5 tests the input plaquettes before a representative is chosen, and keeps only those closed large cells whose full dependence sets avoid the excluded dependencies. Every plaquette of the resulting complex, including each boundary plaquette, is tested good. We restrict the active blocks to the interior by one full layer of large cells and by the fixed dependence radius. This places every plaquette that depends on an active variable, and every relevant constraint, tree, and transport path, inside this closed complex. There are no older protected data at the initial step. The curvature condition below implies the estimate on large cells there, and the remaining complement consists of arbitrary compact data. This is the initial covering construction of Theorem 21.5; it is not a restriction of the Wilson measure to D_r .

The three upper bounds d_{SG} , $B_*/(96M^3)$, and $\delta_*^2/(2\pi^2)$ for d_{cut} are fixed and positive; the first is given explicitly below. Since $d_* = d_{\text{cut}}/2$, the mesh condition $\pi\sqrt{2d_*}S^2 \leq 1$ is the additional bound $d_{\text{cut}} \leq 1/(\pi^2 S^4)$. Since $S \geq R_{\text{cell}}$, it also implies $\pi\sqrt{2d_*}R_{\text{cell}}^2 \leq 1$

for the estimate on large cells. For the link bound $r/2$ we use Theorem 7.7 with support radius $r/2$; one may take

$$d_{SG} = \min\{\delta_{SG}^2/(2\pi^2), r^4/(8192\pi^2 C_{SG}^4)\}, \quad m \geq 8C_{SG}/r,$$

where the second condition is a condition on the side m . Strict inequalities in the threshold and in the side give the link bound $r/2$. The constants $K_M, K_1, \kappa_0, c_A^{\text{act}}, \alpha_W, K_W$ are those of Theorem 22.4 for arbitrary fields. Its hypotheses require the Borel extension of T_M and complete predecessor patches, both of which we use. We use no improvement of K_M based on tests at an untested stopping level; such an improvement would require an energy hypothesis at the base and tests at the top plaquettes.

With these conditions, Theorem 21.5 gives, for each finite sequence of RG steps starting from a stopped measure or from the protected initial step, a finite Borel partition of the support of the starting measure into large-field decompositions. Its tests use the iterates of T_M , including its Borel extension, and every test is kept. Its geometric constants $C_{\text{cov},J}^{\text{pre}}$ do not depend on the widths, and the counts keep the widths explicitly:

$$\begin{aligned} R_{t+1}(f) &\leq B^{-1}R_t(f) + C_{\text{cov},J}^{\text{pre}}(W + R_{\text{cell}} + 1), & R_{\text{first}}(f) &\leq C_{\text{cov},J}^{\text{pre}}(W + R_{\text{cell}} + 1), \\ N_{\text{first}}(f) &\leq C_{\text{cov},J}^{\text{pre}}(BW + R_{\text{cell}} + 1)^3, & N_{\text{later}}(f) &\leq C_{\text{cov},J}^{\text{pre}}(W + R_{\text{cell}} + 1)^3, \\ F_J^{\text{pre}} &= C_{\text{cov},J}^{\text{pre}}(W + R_{\text{cell}} + 1)^3. \end{aligned} \tag{23.38}$$

Here, for a threshold failure f , $R_t(f)$ is the radius of the neighborhood attributed to f at the output of the t -th following RG step, and $R_{\text{first}}(f)$ is its radius at creation. The counts $N_{\text{first}}(f)$ and $N_{\text{later}}(f)$ bound the numbers of labels of normalization factors attributed to f at its creation and at each later step, and F_J^{pre} is the preliminary affected-plaquette count before the stopping family is chosen. The affected sets include the full excluded neighborhoods: the protected neighborhoods, the touching cells, the boundary layers of frozen coordinates, the links across the boundary, the cores enlarged by the fixed layer, and the attachments at the lattices of introduction, together with the closed excluded sets at the initial step. They are not just the smaller predecessor sets used in the energy estimate. These are preliminary geometric bounds. Step 10 will enlarge them after fixing the finite stopping family, to count discarded insertion plaquettes whose full dependence sets meet these excluded sets. The family of remainders is bounded independently of the number of averaging steps. The enlargement preserves the displayed polynomial dependence on widths and subadditivity under unions; those final counts determine F_J and $C_{\text{ch},J}$ without changing the earlier geometry or local analytic constants. The only side conditions used for this cover are $8R_{\text{cell}}$ for outputs and $8S$ for the input at the initial step; both are imposed in Step 11. If the active region is empty, the construction stops, and all its frozen data are kept.

Step 7: the coupling bound. With A_w fixed, we set $c_J^{\log} = A_w(J-1) \log M/2$, so that $p^+ = A_w \log(1/t) + c_J^{\log}$. All fluctuation products used with Theorem 20.14 have radii $p(t) \leq p_i \leq p^+$. Let $C_{\text{rem},J}$ be the fixed quantum remainder constant in the outgoing bound of Theorem 20.14, distinguished from $C_{\text{tor},J}$ and $C_{\text{int},J}$. In the choice of the coupling bound we impose the two remainder conditions also with p^+ :

$$\begin{aligned} C_T t^2 [(p^+)^4 + E(p^+)^2] &\leq (1 - q_{\text{CA}})/2, \\ C_{\text{rem},J} t^2 (E + \sigma_*) [(p^+)^4 + (E + \sigma_*)(p^+)^2] &\leq 1, \quad tp^+ < R_{\text{en}}. \end{aligned} \quad (23.39)$$

These imply the conditions of item (8) with the common radius $p(t)$ and the conditions for every product in the family of logarithmic radii. No radius of size R_{en}/t enters a cluster expansion: the enclosing product domain is used only for the Hessian, for the mean, and for the comparison with the auxiliary measure. Every inequality of item (8) involving t , after multiplication by a fixed positive denominator if necessary, is bounded by finitely many terms $Ct^\varepsilon[1 + \log(1/t)]^k$. For the interaction radius, for example, we first impose $C_J tp^+ \leq 1/2$; then its denominator is at least $1/2$, and

$$s(t) \leq 2C_J t [(p^+)^3 + p^+ + (E + \sigma_*)p^+] \rightarrow 0.$$

The other bound for discs has the same limit. We apply Lemma 23.1 to all these inequalities, to $tp(t) < P$ and $tp^+ < R_{\text{en}}$, and to every fixed bound for free coordinates, for these coarse-field derivatives of the normalized log density on the fixed fluctuation product, and for Jacobians. We choose t_* for the two conditions on the potential and the mean, and then $g_* \leq t_*$ for the remaining inequalities. These include the curvature condition of item (8) involving $t^2(1 + E)$, the conditions $g_*^2 \leq \alpha_W$, $g_*^2 \leq 2\kappa_*$, and $a_0 g_* < \min(r/2, 1)$, and the bound $t[p(t)^3 + Ep(t)] \leq c_*/2$ for the auxiliary discs, with the constant c_* of Theorem 13.5. For the bound on the reconstruction amplitude in Lemma 15.16, let τ be the fixed ceiling of (15.36) and require $g_* \leq \tau$. In particular $tp(t) \leq v_0$ and the forcing bound below hold on the same joint-product domain. We require in addition

$$C_{\text{free},J} tp(t) < (1 - a_{\text{map}})r_{\text{eff}}/4 \quad (0 < t \leq g_*), \quad (23.40)$$

which is possible by the same argument. Together with $b_0 \leq r < r_{\text{eff}}/4$ and $a_{\text{map}} < 1$, this is the literal forcing and input-domain hypothesis of Lemma 15.16; its induction keeps the amplitudes below $r_{\text{eff}}/2$. All these conditions hold on the interval $0 < t \leq g_*$.

Step 8: the effective interaction on the torus. We now attach the constants to the recursion that defines the small-field event. We start with $R_0 = 0$ and use the common fluctuation product $p_i = p(t)$ and the full formula for the auxiliary integral in Theorem 13.5. At every step the input effective interaction is invariant on overlapping charts, has vanishing first derivatives at the identity, and has the total color, translation, and

inversion symmetries; these hold at $R_0 = 0$ and are preserved by that theorem. Its hypothesis on the enlarged image is (23.27), and the unscaled fluctuation inputs lie in the product domain by the coupling conditions of Step 7. The conversion of coefficients at the same radius is valid by the checks of supports and symmetries in Step 1. If $\mathcal{B}_{\rho, a_T}(R_k) \leq E$, then we have

$$\begin{aligned} \mathcal{B}_{\rho, a_T}(R_{k+1}) &\leq C_G + q_{CA}E + C_T t^2 E[p(t)^4 + E p(t)^2] \\ &\leq \frac{1}{2}(1 - q_{CA})E + q_{CA}E + \frac{1}{2}(1 - q_{CA})E = E, \end{aligned} \quad (23.41)$$

where we used $C_G \leq (1 - q_{CA})E/2$ and item (8). The norm does not change when a constant is subtracted. This proves the bound, at the fixed radius and with the torus rate, for every incoming and outgoing centered torus reference interaction, including initialization and the terminal output of every finite chain. By monotonicity it also holds at the smaller regional rate; thus the ceiling required by Theorems 20.13 and 20.14 covers both ends of every step. The constants C_G and C_T were chosen after J and the outer radii, and neither was used to choose J .

We apply Theorem 14.6 to the same parameters. The condition on the product for coarse fields in B_r was checked in (23.35), and $tp(t) < P$ and the intermediate margins for coordinates, the classical action, and the Jacobians hold. The support of the starting measure is D_r , the fluctuation cutoff is the same common radius, the constrained Hessians are the same positive Hessians of the classical action, and the triangular determinant and the constant A_{coord} are unchanged. At every coarse field in D_r , the integral of the Wilson density over the fiber of the block averaging map, restricted by the conditions of the small-field event, equals the auxiliary integral just used. At preceding classical depth aJ we write $r_{aJ} = R_a - R_a(I)$. The stopped measure used as a starting measure in Theorem 22.1 is then

$$\widehat{Z}^{-1} \mathbf{1}_{D_r} e^{-E_{aJ, \text{sat}}/g^2 - r_{aJ, \text{sat}}} dH, \quad \mathcal{B}_{\rho, a_T}(r_{aJ}) \leq E, \quad (23.42)$$

where the subscript sat denotes the gauge-invariant extension to D_r . All removed values $R_a(I)$ and all constants from coordinates and Gaussian integrals stay in the normalization constant. With the convention that unnormalized densities are written $\exp(\Lambda - E/g^2 - R)$, the update is $\Lambda_{a+1} = \Lambda_a + \log A_{\text{coord}} - R_+(I)$. The same holds at every allowed starting side and preceding depth, since the two cited theorems are uniform in these parameters. In particular, this is the recursion on the torus used in Theorem 22.1; no new analytic class is assumed.

The reconstruction maps give the same effective interaction. At coarse fields in B_r , invariance on overlapping charts and the gauge transformations depending only on the coarse field preserve the full fiber integral, including its constant, by Theorem 15.13. On the common outer domain both auxiliary formulas are holomorphic. Equality on a real

open ball of coarse fields implies equality of all real Taylor derivatives there, and hence of the holomorphic series. Their nonconstant parts, and the bound (23.41), coincide; no J -uniform bound of all orders for the reconstruction maps is used in this identification.

Step 9: the hypotheses of the regional estimates. We take the large-field decompositions of Theorem 21.5, with the geometric conditions verified above. The controlled analytic functions are evaluated in the simultaneous gauge representatives, provided by the cover, in which all frozen coordinates are small. The protection of Step 6 preserves all older frozen data and their attachments exactly. The new analytic coordinates in the neighborhoods where gauge transformations are extended have the bound checked in Step 6, and $C_{\text{conv},J} \geq C_{\text{ad},J}$ covers the initial representative of size r (not $r/2$) after a stopped measure. Frozen logarithms of single links have Euclidean norm equal to their group distance, and the larger constant $C_{\text{an},J}$ covers the other fixed products. We conclude that every substitution of analytic frozen coordinates, including the initial one, satisfies

$$|c_j^\sharp| \leq \min\{\beta_{\text{rec}}, \phi_\bullet(j)/4\} \quad \text{for both spatially varying radii.} \quad (23.43)$$

Plaquettes with energy of order one and arbitrary distant data are frozen group variables, and they are never substituted into an analytic coordinate. We do not claim that an arbitrary independent gauge transformation at the endpoints preserves the norm of a frozen logarithm. The bounds for small frozen coordinates are used in the simultaneous representatives provided by the cover, with older marks fixed; for a positive-length decomposition, the later comparison of measures follows from Theorem 22.10, after the density bound at representatives satisfying the reconstruction identities has been replaced by its gauge-invariant upper bound. Color rotations at the roots, without perturbation, preserve the relevant free norms, since they act by isometries of the adjoint representation.

At a step starting from a stopped measure, the incoming action on the torus and (23.42) have no boundary perturbation before the substitution. At a step starting from the Wilson measure, $R_0 = 0$ and the active Wilson action is the exact restriction; the action of the fixed complement stays with its compact variables and its normalizing constant. The first substitution of analytic coordinates creates only the stated boundary-completion differences. The initial boundary norms are at most σ_* , by (23.32), with their constants kept in the full action and in the normalization constant. This proves the initialization $x_0, x_0^{\text{cl}} \leq \sigma_*$ for both kinds of starting measure.

After each substitution at the output, the zone of reduced radius is removed, and all remaining labels have distance at least $s_{\text{er}} \geq w_{\text{bd}}$ from the fixed complement. Lemma 18.9 then gives the uniform incoming radii ρ and ρ_{cl} on the next list D_{in} of active labels, with the lattices of introduction of its coefficient locations and the volumes of its variables. This is the uniform input norm required in Theorem 19.9; no derivative is estimated with a reduced radius at the boundary of the output. Its adjustment for the completion adds only the boundary-completion difference already accounted for, and we have $\mathcal{X}^{\text{cl}}(\bar{P}) \leq 2\sigma_*$.

The input for the effective interaction has $\mathcal{X}(f) \leq \sigma_*$, and its part on the torus is the effective interaction in (23.41).

Theorem 21.5 also places every older incoming complement and its older coefficient locations in the next current fixed complement, up to its fixed buffer for the dependence of paths. In input units this gives, with $B_s = B$ for each integrated step,

$$d_{D_{\text{in}}}(i) \geq B d_{\text{current}}(e) - B \text{dist}(e, i/B) - C_{\text{buf},J}. \quad (23.44)$$

Indeed, a point of the older complement nearest to i , divided by B , lies in the current fixed complement up to $C_{\text{buf},J}/B$, so $d_{\text{current}}(i/B) \leq (d_{D_{\text{in}}}(i) + C_{\text{buf},J})/B$; the triangle inequality from e to i/B and a rearrangement yield (23.44). Additional fixed complements only enlarge the current complement and preserve this implication. At an initialization without boundary, the input marked by the boundary vanishes. The input norm and the norm of differences use this same list and uniform radius, and every input image and connecting segment lies strictly inside it, by the weighted image bound of half the radius and the product domains of Step 2.

We conclude that the hypotheses of Theorem 19.9 on conditional displacements, frozen real data, uniform radii, and boundary compatibility hold after any number of RG steps. The inequalities for the rates required there are included in the choice of rates in Step 2 and in $m_{\text{rate}} + b_1 < c_{\text{prim},J}/C_{\text{alloc}}$. For each integrated complete RG step and its fixed torus reference we use the common radii $p_i = p(t)$. Products with other logarithmic radii are used only for local auxiliary comparisons with the same product in the regional and torus integrals; they are not iterated as the fixed torus reference recursion. By the bounds (23.29), (23.32), (23.39), and (23.43), all hypotheses of Theorem 20.14 hold. Combining its output estimate, the removal of the boundary layer, and the new boundary-completion difference, we obtain the recursion with bound $3\sigma_*/4$ of Step 4. By induction, these hypotheses hold again at the next complete RG step with the common cutoff product. A local auxiliary matched-product comparison supplies no additional fixed-reference equality or iterated ceiling.

We now apply Theorem 21.9 to each covered large-field decomposition. Its construction hypotheses and its hypotheses on the two images have been checked, and $r + \beta_{\text{rec}} < \epsilon_{\text{en},J}$ by our choices. For every point of every fiber in the support, over compatible fixed data, we have $g_i |w_i| \leq \delta_{\text{en}}$ by that theorem. Since $g_i \geq \lambda_* t$, this implies

$$|w_i| \leq \delta_{\text{en}}/(\lambda_* t) = R_{\text{en}}/(2t). \quad (23.45)$$

Conversely, every point of the enclosing product domain satisfies $|g_i w_i| \leq R_{\text{en}}$. The full coordinate, sequential, and image margins apply on the enclosing product domain, while the points of the support keep a margin of half the radius.

By Lemma 20.3, on the enclosing product domain we have

$$\|D_w^2(V - |w|^2/2)\|_{\text{row,col}} \leq \frac{C_0(R_{\text{en}} + t^2 \mathcal{L}_V)}{(1 - AR_{\text{en}})^3} \leq \frac{1/16 + 1/16}{(1 - 1/4)^3} = \frac{8}{27} < \frac{1}{2}, \quad (23.46)$$

where we used $R_{\text{en}} \leq R_W \leq 1/(16C_0)$, $C_0 t_*^2 \mathcal{L}_V \leq 1/16$, and $AR_{\text{en}} \leq A/(16C_0) \leq 1/4$, the last since $C_0 \geq 2A^3$ and $A \geq 1$. The separate condition for the mean also holds. Convexity is asserted only in the fluctuation variables; all inverses above are inverses of constrained Hessians in the free coordinates.

We also check the hypotheses of the comparison on the enclosing product domain in Lemma 21.12, which compares the outgoing density with the integral over the fluctuation cutoff domain. Its hypotheses are those of Theorem 21.9, verified above, the conditions on the enclosing product domain verified in Steps 5 and 7, and an incoming density bound (21.32). At the first RG step after either kind of starting measure, this bound holds with equality. At a later step, it is the incoming bound at the representatives satisfying the reconstruction identities provided by the induction from fine to coarse scales in Theorem 21.13, whose hypotheses are the conditions verified in this step. We next check the measurability required by the comparison. For fixed finite geometry and fixed real frozen data, the enclosing product domain $\mathcal{K} = \prod_i \bar{B}_{\mathbb{R}^3}(0, R_{\text{en}}/t)$ is compact and has finite positive Lebesgue measure when an active fluctuation coordinate is present. The support and the fluctuation product are Borel sets: the tests of the decomposition use finitely many Borel iterates of T_M and finitely many inequalities, and the inverse coordinates and their dependence on frozen data are Borel by Step 3. Every indicator of a support or of an event of the decomposition is a Borel function χ with values in $[0, 1]$. The conditional fluctuation potential V is real and smooth on a neighborhood of $\mathcal{K}$, by the margins on the full product and the positive real constrained Hessians. In particular, for fixed data there exist finite numbers $V_{\min}$ and $V_{\max}$ bounding V on $\mathcal{K}$, and

$$0 < e^{-V_{\max}} |\mathcal{K}| \leq Z_{\mathcal{K}} = \int_{\mathcal{K}} e^{-V(w)} dw \leq e^{-V_{\min}} |\mathcal{K}| < \infty. \quad (23.47)$$

No uniform bound on these extrema is needed for the normalization. The integrand is jointly Borel, and integration over each fixed compact product, after rescaling its balls to unit balls, preserves Borel dependence on the parameters, by the fact on parameter integrals recalled in Step 3. The comparison then applies to a measurable finite normalizing constant. For an empty active region the product has mass one, the nonconstant norm vanishes, the determinant is one, and the fluctuation integral is one. All factors independent of the field keep their places in the normalization constant.

Step 10: stopping levels and the insertion interval. We fix the finite family of stopping levels and remainders of Theorem 22.1, with finite constants for side ratios and conversions. Explicitly, for a scale j and a starting level b divisible by J , let $t_{\text{align}} = J \lfloor (j-1)/J \rfloor$ and $u_{\text{ph}} = j-1-t_{\text{align}} \in \{0, \dots, J-1\}$. For each fixed offset $r_{\text{stop}} \in \mathbb{N}_0$ of the family, put $t_{\text{stop}} = t_{\text{align}} - r_{\text{stop}}J$. When $t_{\text{stop}} > b$, integrate the full RG steps from b to this stopping level. Every integrated step consists of J averaging steps, and in this case the remaining

insertion at the stopping level is

$$C \mapsto Q_j^{\mathcal{T}}(T_M^{r_{\text{stop}}J+u_{\text{ph}}}C), \quad n_{t_{\text{stop}}} = M^{r_{\text{stop}}J+u_{\text{ph}}}n_{j-1}, \quad n_{j-1} \in 4\mathbb{N}. \quad (23.48)$$

Indeed $j-1-t_{\text{stop}} = r_{\text{stop}}J+u_{\text{ph}}$, and every level between t_{stop} and $j-1$ belongs to the given finite sequence of lattices, whose last side is $n_{j-1} \geq u \geq 4$. These iterates of T_M are defined, with the required pairs of integer lattices; a large side alone would not justify this. We use the conversion of predecessor sets for arbitrary fields, with its fixed constant K_{src} , without adding a regularity test at the stopping level. The side ratio to n_{j-1} is $M^{r_{\text{stop}}J+u_{\text{ph}}} \leq \chi_*$, after taking the maximum over the fixed finite family. If $t_{\text{stop}} \leq b$, we apply the clause of the same theorem for a bounded side ratio at the starting measure, with the exponent $j-1-b$ and the side identity $n_b = M^{j-1-b}n_{j-1}$; in this case the exponent is bounded by the same fixed bound. Every remainder is kept as part of the insertion, and none requires an integrated step of fewer than J averaging steps, a short recursion for the normalization constant, or a new output radius.

We include both the predecessor insertion and the total-action insertion of Theorem 22.1 and Corollary 22.9. For this now fixed finite stopping family, enlarge $C_{\text{cov},J}^{\text{pre}}$ and F_J^{pre} to constants $C_{\text{cov},J}$ and F_J covering every discarded insertion plaquette and its full dependence set. The enlarged constants still satisfy the bounds of (23.38) with the preliminary constants replaced by the final ones. They also fix the label counts D_0, D_1 and the layer factors $\chi_{\text{lay}}, C'_{\text{lay},J}$ in the proof of Theorem 22.1. Choose $C_{\text{ch},J}$ to dominate the fixed multiple of $(D_0 + D_1)(1 + C'_{\text{lay},J})(C_{\text{lab}} + C_{\text{com}})$ in (22.21). These are enlargements of bounds on the existing sets, and change no earlier analytic constant. With the final count F_J and charge constant already fixed, choose $s_0 > 0$ in the full interval of Theorem 22.1, in particular with $s_0 K_W \leq 1/4$ and $s_0 \leq \theta c_A^{\text{act}} d_*/(16F_J)$, where $\theta = 1/2$, and we impose $s_0 \leq \kappa_0/(2 \max\{K_M, 48K_1\})$ for the total-action insertion. Each bound is fixed and positive before the coupling.

If $t_{\text{stop}} > b$, fix a covered large-field decomposition of length $\ell = t_{\text{stop}} - b$, with the set F of coarse links kept at the stopping level. We use the list $\mathcal{P}_{\text{orig}} = \mathcal{P}_{j-1}^{\mathcal{T}}$ at scale $j-1$ and $Q_{\text{orig}}(V) = Q_j^{\mathcal{T}}(T_M^{j-1-b}V)$, with the coefficient $4/h_{n_{j-1}}$. The exponent and the side identity in (23.48) are the hypotheses of Lemma 22.8. Using the full excluded sets and the enlarged count in (23.38), we obtain, at every value (c, b_{rec}) of the coarse field and the frozen coordinates,

$$Q_{\text{orig}}(V) \leq K_W \mathcal{V}_{m;F}(c)/g_m^2 + 8F_J \sum_{f \in \mathcal{X}} n_{\ell_f}/\gamma, \quad m = n_{t_{\text{stop}}}, \quad K_W = 48K_1/\alpha_W, \quad (23.49)$$

where $\mathcal{X}$ is the set of threshold failures of the decomposition, n_{ℓ_f} is the side at which f was created, and $\mathcal{V}_{m;F}$ is the compact minimal action of Lemma 22.8. For $0 \leq s \leq s_0$, write $\Psi_F = \mathcal{V}_{m;F}$, $\lambda_s = 1/2 - sK_W \in [1/4, 1/2]$, $P_0 = 4\theta c_A^{\text{act}} d_*$, and $H_{\mathcal{X}} = \sum_f n_{\ell_f}/\gamma$. At

every value of the coarse field and the frozen coordinates, the action bound and the gauge-invariant threshold tests imply

$$sQ_{\text{orig}}(V) - \frac{\Psi_F(c)}{2g_m^2} - \theta A_{\mathcal{X}}(b_{\text{rec}}) \leq -\lambda_s \frac{\Psi_F(c)}{g_m^2} - \frac{3P_0}{4} H_{\mathcal{X}}. \quad (23.50)$$

Indeed $\theta A_{\mathcal{X}} \geq P_0 H_{\mathcal{X}}$ by the threshold tests, and $8sF_J \leq P_0/8 \leq P_0/4$. We use this inequality in the sense of image measures, as in (22.16); no factorization of the full insertion through the coarse variables is asserted. The invariant density bound without weight has been extended to the whole configuration space before this multiplication. We complete its Wilson factor once, at the coefficient λ_s , using (22.17). The scalar and completion factors attributable to the threshold failures are at most $P_0 H_{\mathcal{X}}/4$ after the same choice of minimum side, which leaves $a_{\text{act}} H_{\mathcal{X}}$ with $a_{\text{act}} = P_0/2$. The constants for the stopping partition-function ratio are chosen uniformly for $\lambda_s \in [1/4, 1/2]$, with $g_m^2 \leq \alpha_W$ and the fixed coupling bound g_{anc} . The denominator is the same centered trial integral on the torus, and the partition function Z_ν of the starting measure is unchanged. This positive-length argument uses only the positive action bound, and it adds no constant and no further surface logarithm. It includes decompositions that acquire an empty active region after initialization. When $t_{\text{stop}} \leq b$, use instead the direct initial-measure bound of Step 7 in the proof of Theorem 22.1, for both stopped and Wilson initial measures; no small terminal representative or reconstruction-family envelope is invoked.

We define s_b , a_b , and $\bar{c}$ as in item (9). The full affected sets were fixed before this sequence of choices: an enlargement for the finite family affects only these later choices and $C_v, C_s, C_b, n_{\min}$, never J or an earlier analytic constant. We apply Lemma 23.4 with $\Gamma = \gamma_0$ to obtain the moment constants and their common minimum side; its hypotheses on the local product domains, the protection of older data, the norm bounds, and the insertions for the remainders have been checked above. Its surface coefficient is kept with $C_s \geq 9\chi_*^2$.

Step 11: the minimum side. All geometric thresholds, including $8R_{\text{cell}}$, $8S$, $8C_{\text{SG}}/r$, and every fixed margin near flat configurations, are finite lower bounds on the side. We also fix a coupling bound $0 < g_{\text{anc}} < 1$ for the stopping partition-function ratios of Theorem 22.4, on the fixed intervals of coefficients of the Wilson action used in Theorem 22.1, whose lower endpoint is $1/4$. It depends on the fixed torus domains, r , the norm bounds, and that interval, and it is uniform in the depth, the side, and γ . We take $g_{\text{end}} = \min\{1, g_*, g_{\text{anc}}, \sqrt{\alpha_W}\} > 0$ and impose

$$n_{\min} \geq \gamma_0/g_{\text{end}}^2, \quad g_m = \sqrt{\gamma/m} \leq g_{\text{end}} \quad (m \geq n_{\min}, 0 < \gamma \leq \gamma_0). \quad (23.51)$$

This verifies the completion at the stopping level, the uniform bound on the stopping partition-function ratio, and every coupling bound of an integrated averaging step. At the output of a complete RG step we have $g_m = \sqrt{M} t_{\text{max}}$, and the coupling restriction

at the stopping level is checked directly from the bound on the side; it is not inferred from $t_{\max} \leq g_*$ alone. The two side conditions of the cover hold separately: an output after an RG step has side at least $u \geq n_{\min} \geq 8R_{\text{cell}}$, and an input at the initial step has side at least $u \geq n_{\min} \geq 8S$. They need no congruence condition on u beyond $u \in 4\mathbb{N}$; an integrated step has input side Bm with $m \in 4\mathbb{N}$, and every intermediate side is at least four.

We fix $r_{\text{flat}} > 0$ from Lemma 14.9, below r and below all fixed margins of one RG step for the real classical action, the constraint, and the gauge transformations; these are uniform in the preceding depth and are fixed before γ . We impose, with strict inequality,

$$2\pi/n_{\min} < \min\{r/2, r_{\text{flat}}\}. \quad (23.52)$$

Lemma 14.9 then applies at every $u \geq n_{\min}$ and every finite K for the same parameters: constant commuting links have distance at most $2\pi/u$ from the identity, the averaging steps preserve their holonomies, all free normalized fluctuation variables vanish, and all coordinate-domain conditions hold. This uses the uniform margin of that lemma near all flat configurations, not an open neighborhood of the flat set that depends on the cutoff.

The factors attached to threshold failures and the ratios of the sums over decompositions are controlled by Lemma 23.1, with $x = m/\gamma$ and $k \leq 2$. The errors at the endpoints in Lemma 22.12 have the forms $C_J X_c$ and $C_J W_a$ with fixed positive exponents; the finite family of remainders adds only finitely many constants and ratios of scales. Explicitly, for every transition $0 \leq a < K$, we take the minimum side so large that the exterior error z_a and the cutoff error δ_a of that lemma are at most $1/4$; the exterior error is the positivity condition on d_t in the comparison of Theorem 14.7. Then $z_a < 1$ and $\epsilon_a = \delta_a/(1 - z_a) \leq 1/3 < 1$, and every rejection estimate and every domination by reciprocals applies. The error ϵ_t of the regional comparison on the enclosing product domain in Lemma 21.12 is likewise made at most $1/4$; in particular, every reduction through the auxiliary measure and its annular region has error strictly below one. We make the ratios of successive envelopes at most $1/2$, and the sum of the ϵ_a so small that every reciprocal product has logarithm at most one; this achieves $C_\delta = 1$ before the final enlargement of $n_{\min}$. The measures for the initial segments are defined up to $a = K$, but no quantity of a next transition is used at $a = K$; when $K = 0$ these products and lists are empty.

For the entropy, at $m \geq 4$ and $x = m/\gamma \geq e$, the quantity $\mathcal{A}_\gamma(m)$ of (23.3) satisfies

$$\mathcal{A}_\gamma(m) \leq D_0(1 + \log x), \quad D_0 = 1 + 24C_v M^3 + C_s M^2(1 + \log M) + C'_b/8,$$

since $4/m \leq 1$, $8/m^3 \leq 1/8$, and $\log(Mm/\gamma) = \log M + \log x$. Increasing $n_{\min}$ makes this at most $a_b x/2$, uniformly in γ . The ratio with exponent $\bar{c}$ is made at most $1/2$ by (23.2). These conditions become stronger as the minimum side increases.

Finally we keep the exact correction at the initial scale from Lemma 22.12. With $\beta = 4N/\gamma \geq 4N/\gamma_0 \geq e$, eight times its positive term divided by β is at most

$$\frac{C_{\text{LT}}\gamma_0}{4N} + \frac{9\gamma_0 \log(4N/\gamma_0)}{N^2} \longrightarrow 0,$$

where we used that $1/\beta$ and $\log \beta/\beta$ are decreasing for $\beta \geq e$. We require this to be at most $d_*/4$. The positive term divided by β is therefore at most $d_*/32$, and the exponent at the initial scale is at most $-\beta d_*/32 = -d_*N/(8\gamma)$, at all larger sides and all couplings in the interval. We choose $n_{\min} \in 4\mathbb{N}$ above this threshold and all preceding ones. The logarithmic correction at the initial scale is not replaced by a bound along a fixed sequence of sides.

Step 12: the reconstruction maps and the identity of coarse fields. The nested domains and the tolerances for coefficients and derivatives used in Theorem 15.13 are fixed. Its thresholds at fixed lattices can now be increased for the smaller inner image margins, by Lemma 15.16: below the threshold the earlier bound holds, and above it the bound for the limit plus the same fixed tolerance holds. This increase therefore cannot enlarge C_{CP} , A_J , R_J , or β_J . There are countably many pairs of finite lattices and preceding depths, and we fix the maps on all of them at once, rather than choosing a new limit or threshold at some coarse side. Each choice concerns classical maps on domains of free coordinates and has no argument depending on the coupling, the field, or an observation. The coarse fields, kernels, cutoffs, and Jacobians are preserved, by Theorem 15.13.

The bound on the reconstruction amplitude in Lemma 15.16 applies to these maps. To check its parameters, we fix $u \geq n_{\min}$ and let $H = L/u$, $a_j = HB^{-j}$, and $g_3^2 = \gamma/L$. Then

$$t_j = M^{(J-1)/2} g_3 \sqrt{a_j}, \quad t_1^2 = \gamma/(Mu) \leq g_*^2, \quad t_{j+1} = t_j/\sqrt{B},$$

where $t_1^2 \leq \gamma_0/n_{\min} \leq g_{\text{end}}^2 \leq g_*^2 \leq \tau^2$ by (23.51) and Step 7. Here $p(t_j) \geq 1$ and $t_j p(t_j) \leq v_0$. The free input is at most $t_j p(t_j)$, all products remain in the common working ball by (23.40), and $b_0 \leq r$. Then the amplitude bound and its sum give

$$\begin{aligned} b_j &\leq C_{\text{cap},J} [r\alpha_*^j + g_3 \sqrt{a_j} p(t_j)], \\ \sum_{j=1}^k b_{j-1} &\leq C_{\text{cap},J} [r/(1-\alpha_*) + g_3 \sqrt{H} (p(t_1) + 1)]. \end{aligned} \tag{23.53}$$

The constant does not depend on H , u , γ , or the length of the finite chain. We used the deterministic bound of Lemma 15.16, not a stochastic bound at fixed H , and made no new choice at a coarse side.

Finally, the path-length condition gives the identity of coarse fields, even before a test that may fail. In a representative whose link distances are at most s , a product of l links

has distance at most ls , by repeated use of the triangle inequality for the bi-invariant distance. The two tree paths have total length at most $3(M-1)$; adjoining the straight segment of M links and the inverse of the reference path of M links gives the closed paths of the based loop holonomies, of length at most $5M-3$. If $(5M-3)s < \pi$, all these based loop holonomies have principal logarithms. In (4.3), the exponential of the average logarithm then has distance at most $(5M-3)s$, and multiplication by the reference transport gives the bound $(6M-3)s \leq 10Ms$. Starting from $s = r$, induction gives the link bound $(10M)^i r$ before averaging step $i < J$, so the condition $(5M-3)(10M)^{J-1}r < \pi$ covers each of the next J averaging steps. For a next RG step with $0 \leq a < K$, the input side is $uB^{K-a} = M^J m$ with $m = uB^{K-a-1} \geq n_{\min} \geq 4$, so each of these J averaging steps has a valid pair of input and output lattices. There is no next step at $a = K$.

For an arbitrary input in D_r , we apply this argument to a small periodic representative. Under a gauge transformation, the based loop holonomies are conjugated at their basepoint, and their distances and their membership in the exceptional set are unchanged. Conjugation commutes with the principal logarithm and with the exponential, and the reference transport transforms in the usual way at its endpoints. The same covariance holds for the Borel extension, since $-I$ is central. Then T_M and all its iterates are gauge covariant, and the conclusion transfers back without taking a logarithm of the gauge transformation. On an initial segment that passes the tests, the current coarse field lies in D_r , and the same argument applies to the next coarse field. The inverse coordinates in Definition 14.4 satisfy the identity for one averaging step on this coordinate domain; composing the J identities shows that the coarse field produced by the construction equals T_M^J of the finer field, even when a later cutoff fails. A gauge transformation depending only on the coarse field and equal to the identity at the tree roots preserves this identity, as well as the kernels and Jacobians, by Theorem 15.13.

All local estimates were evaluated before the later widths, and all side conditions used fixed constants and the bound γ_0 . This proves the stated dependences, and the absence of any dependence on the number of elapsed RG steps, on shapes, or on depths. The single choice of J used only the J -uniform geometric constants and torus rates, the J -uniform series Z , and the finitely many coarse-field derivative bounds of the reconstruction maps at zero fluctuation; the full joint norms and the norms of the gauge transformations of the reconstruction maps were fixed after J . The full radii were restored only after complete RG steps, and the remainders were included in the fixed family of insertions. This completes the proof of Theorem 23.2.

23.7 Parameters below an arbitrary coupling ceiling

In this subsection we prove Theorem 23.5.

Proof of Theorem 23.5. We start with parameters given by Theorem 23.2 with $\gamma_0 = 1$. The argument works for every J obtained from that construction; it does not select a new J . We keep its J , the analytic domains and rates on the torus and on block regions, the margins of the product domains for free coordinates, the cutoff amplitude, the widths of boundary layers, removed layers, and protected neighborhoods, the margins near flat configurations, g_* , the interval for s , the family of remainders, and the numbers s_0 , d_* , a_b , and $\bar{c}$. Before the curvature threshold is chosen, δ_* is normalized to $(0, 1]$, which keeps the range of Theorem 7.3. The parameters include K_W and $g_*^2 \leq \alpha_W$, and the stopping partition-function ratio is controlled for coefficients in $[1/4, 1/2]$ before the side is enlarged. The deterministic column bounds come from Lemma 15.16; no stochastic estimate is used in this choice. All these quantities are constructed from the local results cited in Theorem 23.2, whose hypotheses concern g_i , t , and fixed products and contain no upper bound on γ . In particular, the classical constants, the coefficient constants of the reconstruction maps, and the J -uniform constants are not chosen again using a later probability constant. The reconstruction maps of Lemma 15.16 and Theorem 15.13 are fixed on the same countable family of pairs of finite lattices; they depend on deterministic domains, not on the coupling. Their exact identities and the margin $32/49$ are unchanged.

At these parameters we apply Theorem 22.1 and Lemma 22.13 for the ceiling Γ , without a restriction on the number of averaging steps. We obtain C_v, C_s, C_b , and we set $C'_b = C_b + 1 + \log 2$. Only the later probability constants and thresholds use these constants, so their possible dependence on Γ creates no cycle with an earlier choice. We choose the final side at least Γ/g_{end}^2 and above every earlier geometric and flat threshold. For every side $m \geq n_\Gamma$ of a stopping lattice we have $g_m^2 = \gamma/m \leq \Gamma/n_\Gamma \leq g_{\text{end}}^2$, and also $t_k^2 = \gamma/(Mn_{(k+1)J}) \leq \Gamma/(Mn_\Gamma) \leq g_{\text{end}}^2 \leq g_*^2$. These inequalities verify the coupling bounds at the stopping level and for the integrated steps, respectively. The separately imposed flat thresholds give $2\pi/n_\Gamma < \min(r/2, r_{\text{flat}})$. All inequalities involving local small parameters hold on $0 < t \leq g_*$, including every fixed power of t multiplied by a logarithmic cutoff, and they hold at these sides on the new coupling interval.

We apply Lemma 23.4. At $x = m/\gamma \geq e$ we have

$$\mathcal{A}_\gamma(m) = 24C_v M^3 + \frac{4C_s M^2}{m} \log(Mm/\gamma) + \frac{8C'_b}{m^3} \leq D_0(1 + \log x),$$

with the same $D_0 = 1 + 24C_v M^3 + C_s M^2(1 + \log M) + C'_b/8$ as in Step 11 of Section 23.6, since $m \geq 4$. Lemma 20.15 makes this at most $a_b x/2$ uniformly on $x \geq n_\Gamma/\Gamma$. The same lemma makes all ratios at the endpoints at most $1/2$ and the exterior errors at most $1/4$, and it makes the cutoff errors and the logarithms of the reciprocal products as small as prescribed; in particular $C_\delta \leq 1$. The ratio over scales satisfies $M^3 e^{-\bar{c}(M-1)n_\Gamma/\Gamma} \leq 1/2$. The exact positive term at the initial scale, divided by β , is bounded by the vanishing expression in Lemma 22.13. Finally, the constant $C_{\text{ch},J}^\Gamma$ for the factors attached to threshold failures at the ceiling Γ is absorbed there, before the moment constants and the final side

are used. All these requirements become stronger as the minimum side increases, and we take their maximum, rounded up to an element of $4\mathbb{N}$.

The amplitude bounds of Lemma 15.16 require only $t_1^2 = \gamma/(Mu) \leq g_*^2$ and the common working ball. The coupling condition just verified provides this, and under this hypothesis the bound is uniform in $H = L/u$ and g_3 . The conditions from path lengths and the coordinate-domain conditions are deterministic in r , M , and J , and they are unchanged. The conditions listed in this proof are the only places where the proof of Theorem 23.2 uses the bound $\gamma \leq \gamma_0$. Hence every conclusion of that theorem holds, with the same local dependences and with the fixed Γ entering only the probability constants. No support radius or reconstruction map changes at a larger coarse side. $\square$

23.8 Every sufficiently large number of averaging steps

Section 29 compares the limits obtained along the sequences $n_* M^{JK}$ for two consecutive values of J , so it needs the construction of Theorem 23.2 for every sufficiently large J . No uniformity of the constants in J is asserted.

Lemma 23.14. *Fix $L > 0$, an odd integer $M \geq 3$, a ceiling $0 < \Gamma \leq 1$, and the J -uniform constants in item (1) of Theorem 23.2. There exists a finite J_0 such that, for every integer $J \geq J_0$, there exist a tuple Θ_J , a positive local coupling bound $g_{*,J}$, and $n_{\text{sf},J} \in 4\mathbb{N}$, chosen before the coupling and the side, for which all conclusions of Theorems 23.2 and 23.9 hold with $B = M^J$, simultaneously for*

$$0 < \gamma \leq \Gamma, \quad u \in 4\mathbb{N}, \quad u \geq n_{\text{sf},J}, \quad N = uB^K, \quad K \geq 0.$$

In particular, two consecutive values of J can be fixed before γ is varied. Neither the tuples nor any later constants or thresholds are asserted to be uniform in J .

Proof. We verify the additional quantifier “for all $J \geq J_0$ ” in the proof of Theorem 23.2; it is not inferred from the existence statement. We fix the outer rate on the torus, the boxes, and the analytic-radius enlargement factor K_* . The series for the conditional constrained minimizers at zero fluctuation of Lemma 11.9 and Theorem 13.2 has a J -uniform majorant $Z(v) \leq Cv/(1 - Cv)$ and a J -uniform spatial rate. The finitely many coarse-field derivative bounds of the reconstruction maps at zero fluctuation, as opposed to the full analytic norm of their gauge transformations, are also J -uniform, by Theorem 15.13 and Lemma 15.16. The additional identity at fixed J for hierarchical trees in Lemma 11.9 does not change its J -uniform series, and only that series is used here. Thus all constants in the following list are fixed before J :

$$\begin{aligned} B \geq B_{\text{geom}}, \quad C_P(a_{\mathbb{T}})/B < 1/4, \quad C_{\text{CP}}^2/B \leq 1/4, \\ C_{\bullet}K_*/B < 1/32, \quad C_{\text{geom}}^{\mathbb{T}}b_{\mathbb{T}}/B \leq a_{\mathbb{T}}, \quad C_{\text{cen}}/B < 1/8, \quad B^{-1} \leq \delta_0(a_{\mathbb{T}}). \end{aligned} \tag{23.54}$$

These are the conditions of item (1) of Theorem 23.2 and (23.25); the rate $a_{\mathbb{T}} < a_{\text{cen}}/8$ and the threshold $\delta_0(a_{\mathbb{T}})$ are fixed before J . The finitely many additional image conditions on first derivatives use the same series and derivative bounds; they are also strict lower bounds on B with fixed finite right sides. The maximum of their thresholds is exceeded by M^J for every $J \geq J_0$. This uses the exception for J -uniform constants in item (2) of Definition 4.14, not a bound on a joint free map.

The constants C_{alloc} and C_g in Theorem 20.9 do not depend on J . We choose $C_*^{\text{geom}} \geq \max(1, C_g)$ and keep $B_{\text{geom}} = 256C_{\text{alloc}}(C_*^{\text{geom}} + 1)$. For each fixed J above the threshold we set $\alpha = a/16$, $m_{\text{rate}} = C_*^{\text{geom}}(a + \alpha)$, and $b_1 = 8C_*^{\text{geom}}\alpha$. The inequalities against Ba and $B\alpha$ reduce to

$$\frac{25}{16}C_*^{\text{geom}} < B/C_{\text{alloc}}, \quad 8C_*^{\text{geom}} < B/C_{\text{alloc}}.$$

We decrease $a > 0$ below the positive bounds at fixed J from the initial spatial decay rate and the image exponent, and so that $\sqrt{3}(a + \alpha) < a_{\mathbb{T}}$. No further inequality on supports or connecting lengths remains.

We also check the later choices. At each fixed J there are finitely many product domains for initial segments. The positive lower bounds for the constrained Hessians and the radii of the inverses in Lemmas 19.3 and 19.7 have a positive finite minimum, and the coordinate, analytic, and reconstruction domains are chosen within them. The contraction on the torus satisfies $q_{\text{CA}} < 1/4$, and the bound $E \geq 2C_G/(1 - q_{\text{CA}})$ is finite by Theorem 13.5. The preliminary perturbation bound, the radius of the enclosing product domain, and the image bounds of Lemma 20.3 and Theorem 20.14 can be chosen positive on the fixed domains at this J . The widths of boundary layers and removed layers can be increased after their output constants are computed, and the bound for frozen coordinates, the working radius, and the support radius can then be decreased. In particular, $(5M - 3)(10M)^{J-1}r < \pi$ is an upper bound on r , not on J .

The bound on derivatives follows the same order of choices. Its linear coefficient is $\alpha_* = C_{\text{CP}}/B$, and by (23.54) we have $\alpha_*\sqrt{B} \leq 1/2$. At this fixed J , β_J is finite by Lemma 15.16. The restriction $\beta_J r_{\text{eff}}/(1 - \alpha_*) < 1/8$ implies $\beta_J r_{\text{eff}} < 1/8$, and hence

$$a_{\text{map}}\sqrt{B} \leq \frac{1/2}{(1 - \beta_J r_{\text{eff}})^2} \leq \frac{32}{49}.$$

This requires no new choice of J . The remaining conditions for the fluctuations and the interaction are finitely many inequalities $C_J t^\varepsilon \log^b(1/t) \leq c_J$ with $\varepsilon, c_J > 0$, including the fixed additive term $A_w(J - 1) \log M/2$ in a logarithmic radius. They hold after decreasing $g_{*,J} > 0$, by Lemma 23.1.

We next check items (7)–(10). We normalize $0 < \delta_* \leq 1$ before fixing d_{cut} . The product domains, the block-aligned initial mesh S , the classical lower bound $\alpha_w > 0$, the count

F_J , and the constant for the normalization factors are all fixed at this J . By Lemma 22.8, $K_W = 48K_1/\alpha_W < \infty$ and the insertion satisfies

$$Q_{\text{orig}} \leq K_W \mathcal{V}_{m;F} / g_m^2 + 8F_J \sum_f n_{\ell_f} / \gamma.$$

We enlarge F_J for the finitely many remainders. The interval of Theorem 22.1 allows

$$0 < s_0 \leq \min \left\{ \frac{1}{2}, \frac{\kappa_*}{2} c_E^{-(J-1)}, \frac{1}{4K_W}, \frac{\theta c_A^{\text{act}} d_*}{16F_J}, \frac{\kappa_0}{2 \max\{K_M, 48K_1\}}, s_{\text{short},J} \right\},$$

where $s_{\text{short},J} > 0$ is the minimum of its finitely many conditions for short scale intervals and $\theta = 1/2$. Every entry is positive once J is fixed, and $\lambda_s = 1/2 - sK_W \in [1/4, 1/2]$. The action bound keeps at least $3P_0/4$ of the coefficient of the threshold failures, with $P_0 = 4\theta c_A^{\text{act}} d_*$, and the later scalar threshold uses at most $P_0/4$, so that $P_0/2$ remains. The compact minimum in the action bound uses the partition function of the starting measure once; no factorization of the insertion through the coarse variables is needed. We decrease $g_{*,J} > 0$ to impose $g_{*,J}^2 \leq \alpha_W$ and every local condition already listed. We choose the bounds for the stopping partition-function ratio and its numerator on the fixed interval $[1/4, 1/2]$, and let $g_{\text{end},J} > 0$ be their minimum with 1 and $g_{*,J}$. Finally, we increase the minimum side to include $n_{\text{sf},J} \geq \Gamma / g_{\text{end},J}^2$ and all fixed geometric and scalar thresholds and the thresholds for initial segments. The latter are conditions on the minimum side, by Lemma 22.12; for $\Gamma \leq 1$ their uniform scalar thresholds also follow from Lemma 23.1.

These are exactly the later choices in Theorem 23.2, and none of them requires enlarging J using a constant already computed at that J . By Theorems 23.6 and 23.9, we then obtain the bound on the small-field failure probability on $0 < \gamma \leq \Gamma$. The positive exponent $\min(5/24, 1/2, -\log q_{\text{CP}} / \log B) / 2$ of Remark 23.3 is also available at each fixed J . $\square$

Corollary 23.15. *Fix $L > 0$, an odd integer $M \geq 3$, and $0 < \Gamma < \infty$. There exists a finite J_0 such that, for every $J \geq J_0$, there are a tuple Θ_J , $g_{*,J} > 0$, and $n_{\Gamma,J} \in 4\mathbb{N}$, fixed before γ , the side, and the number of renormalization group steps, for which every conclusion of Theorems 23.5 and 23.10 holds on $0 < \gamma \leq \Gamma$ with this prescribed J . In particular, consecutive values of J can be fixed before the coupling. No constant or threshold is asserted to be uniform in J .*

Proof. We apply Lemma 23.14 with the ceiling one. Its check of the J -uniform conditions provides J_0 and parameters for $\gamma \leq 1$ for every $J \geq J_0$. The constant $K_W = 48K_1/\alpha_W$ is fixed after the classical lower bound α_W and before s_0 , and the normalization $\delta_* \leq 1$ is fixed before the curvature threshold; neither changes a J -uniform lower bound on $B = M^J$ or imposes an upper bound on J . The interval for s is a finite minimum of positive numbers at fixed J , and the coupling bound at the endpoints is positive. For

each prescribed $J \geq J_0$, we apply the construction in the proof of Theorem 23.5 to these parameters. That construction changes only the later probability constants and the final minimum side, and it keeps this J and the deterministic bounds of Lemma 15.16. By Theorem 23.10, we then obtain the bounds for all finite numbers of RG steps on the interval $0 < \gamma \leq \Gamma$. For two values J and $J + 1$ one may take the larger of the two minimum sides, with no uniformity as J varies. $\square$

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