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The continuum limit of  
three-dimensional  $SU(2)$   
lattice Yang–Mills theory  
on a torus II:  
Fluctuation integration and local  
reconstruction

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### WARNING!

We have extensively audited this manuscript using large language models, including adversarial reviews by multiple models. However, the proof is too complex for Lean formalization at the present time, and it has not yet undergone verification by a human expert.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).

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<https://www.loc.gov/resource/g3200m.gct00126/?sp=106>

#### AUTHORS' NOTE

With the exception of this note, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited.

The manuscript began as an attempt to pursue the program of Magnen, Rivasseau, and Sénéor [132] using large language models. Our hope was that understanding the problem on the torus might offer some insight into the general case. The construction that resulted from this starting point is three-dimensional and lattice-based and draws extensively on Bałaban's renormalization group methods.

As we were verifying these results, we learned that Alpöge and Sussman announced that they appear to have a YM4 existence argument, following Bałaban's program (<https://github.com/ethanwsussman/Yang--Mills-LLM-experiment>). We are thrilled and look forward to learning more about these exciting developments. We hope that this manuscript nonetheless contributes to understanding constructive approaches to three-dimensional Yang–Mills theory. The first version of these results was posted to the first author's GitHub on October 17, 2026 (<https://github.com/pmaronow/3d-ym-torus>). Prior to posting the first version of this work, we did not inspect the ongoing work of Alpöge and Sussman. Further, to the best of our knowledge, the LLMs we used did not access their work during the mathematical development and drafting of this manuscript. This version contains additional clarifications relative to the first version; it has also been partitioned into five volumes.

We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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#### ABSTRACT

We consider the Wilson lattice gauge measure for  $SU(2)$  on a periodic three-dimensional torus of fixed size, with bare coupling  $g^2 = g_3^2 a$  proportional to the lattice spacing  $a$ , as dictated by the superrenormalizability of three-dimensional Yang–Mills theory. Observables are Wilson loops of the lattice configuration evolved by the Wilson flow, the lattice gradient flow of the Wilson action, at positive flow time. We prove that, at every fixed value of the coupling, the joint laws of finitely many flowed Wilson loops converge as the lattice spacing tends to zero along the sequences of lattice sides  $nM^k$ , for fixed odd  $M \geq 3$  and fixed positive  $n$  divisible by four. For each such refinement class, the limits are realized by a unique probability measure on the space of gauge orbits of Yang–Mills heat-flow trajectories at positive times, a version of the state space introduced by Cao and Chatterjee. At each fixed positive time, the energies of the selected trajectories  $\pi_k(U)$ , with  $U$  distributed under the lattice Wilson measures, are tight uniformly in the lattice spacing. The limiting measure is invariant under the lattice symmetries and the center symmetries. We also show that, at short distances, the rescaled deficits  $(1 - W)/(g_3^2 \varepsilon)$  of the normalized fundamental Wilson loops  $W$  of loops  $x + \varepsilon \ell$ , flowed for time  $\sigma \varepsilon^2$  with fixed  $\sigma > 0$ , converge to explicit quadratic functions of the heat-smoothed Gaussian free field. As the coupling tends to zero, the measure concentrates on flat connections, with Gaussian fluctuations of explicit covariance near regular commuting holonomies and fluctuations governed by the constant-mode quartic form near the central holonomies. The proof is a multiscale renormalization group analysis in the tradition of Balaban, carried out on a small-field event whose complement we show to have small probability, uniformly in the lattice spacing, using reflection positivity and chessboard estimates. It is combined with stochastic estimates for the lattice heat flow and a deterministic analysis of the Wilson flow for rough initial data.

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PART IV

Fluctuation integration and conditional  
measures

## Fluctuation integration and cluster expansions

In this section, we integrate out the fluctuations of a bounded number of averaging steps around the classical background field. We consider  $s \leq J$  consecutive averaging steps, from a lattice of side  $M^s m$  to a coarse lattice of side  $m \geq 4$ , starting from the effective classical action  $E_{d_0}$  of some depth  $d_0 \geq 0$ ; for  $s = J$  this is one renormalization group step. Given a coarse field  $c$ , the fine configurations whose average is  $c$  are described by free logarithmic coordinates at each intermediate scale. We center these coordinates at the conditional constrained minimizer, normalize them by the coupling and the constrained Hessian, and integrate the resulting normalized fluctuation variables  $w$  over a fixed product  $\Omega_p$  of three-dimensional balls.

There are three main results. The first is an exact formula for this integral, for one averaging step in Lemma 12.3 and for  $s \leq J$  averaging steps in Theorem 12.13. The second is a convergent polymer expansion of the logarithm of the fluctuation partition function, with volume-independent bounds on all its derivatives of positive order (Theorems 12.6 and 12.7 and Proposition 12.10). The third, Lemma 12.16, concerns an input effective interaction  $R$ , produced by the integrations at finer scales. It shows that integrating out the fluctuations replaces  $R$  by its value on the zero-fluctuation reconstruction plus a correction of second order in the coupling. We do not estimate the probability of the domain of integration under the Wilson measure here; this is done in Section 23.

We use the conventions of Section 4, the absolute coefficient calculus of Lemma 5.3, the constraint coordinates and the exact Haar disintegration of Lemma 6.3, and the conditional constrained minimizers, the nested minimization identity, and the coefficient bounds for  $E_d$  of Theorems 10.10 and 10.11. In Section 13, the results are used to construct the renormalization group recursion on the small-field event and to describe the conditional fluctuation kernel; they are also used in Sections 14, 15, and 23. Sections 18 and 20 reuse the triangular change of variables and the graph combinatorics, and Sections 31, 33, and 35 use the coordinates of Theorem 12.13 and its dimension counts.

**The integral over one averaging step.** We first describe the identity of Lemma 12.3 informally. Let  $b(c, z)$  be the fine configuration with average  $c$  and free logarithmic coordinates  $z$ , and let  $F(c, z) = E_{d_0}(b(c, z))$ . Let  $z^*(c)$  be the minimizer of  $F(c, \cdot)$  and  $K(c) = D_z^2 F(c, z^*(c))$  the constrained Hessian. The substitution

$$z = z^*(c) + gK(c)^{-1/2}w$$

has Jacobian  $g^{3f} \det K(c)^{-1/2}$ , where  $3f$  is the number of real free logarithmic coordinates. Since  $z^*$  is stationary and  $K^{-1/2}KK^{-1/2} = I$ , we have

$$\frac{F(c, z)}{g^2} = \frac{F(c, z^*(c))}{g^2} + \frac{1}{2}|w|^2 + O(g|w|^3),$$

and  $F(c, z^*(c))/g^2 = E_{d_0+1}(c)/(Mg^2)$  by the nested minimization identity. The Haar density of the free logarithmic coordinates is a reference constant times  $e^{-\mathcal{L}_c(z)}$ , and  $\mathcal{L}_c(z) = \mathcal{L}_c(z^*(c)) + O(g|w|)$ . Thus the integral over  $w \in \Omega_p$  is the product of three factors that depend on  $c$  (the exponential of the effective classical action at the coarse scale, the determinant  $\det K(c)^{-1/2}$ , and the Haar density at the minimizer) and of the integral of  $e^{-|w|^2/2 - U(c, w)}$ , where  $U$  is small on  $\Omega_p$ . The determinant and the Haar density at the minimizer become part of the effective interaction as functions of  $c$ , and  $E_{d_0+1}/(Mg^2)$  remains a separate term. We control the integral of  $e^{-|w|^2/2 - U}$  by a polymer expansion about the product of truncated Gaussian measures on the balls. This product is only a reference measure. The conditional law of  $w$ , the conditional fluctuation kernel studied in Section 13, has density proportional to  $e^{-|w|^2/2 - U}$ .

For several averaging steps the change of variables is triangular, from the coarsest intermediate scale to the finest. The minimizers, Hessians, and Haar densities at a finer scale depend on the fluctuations at all coarser scales, and the exact formula of Theorem 12.13 keeps these dependences.

**Organization.** Section 12.1 defines the coefficient norms and proves the exact formula for one averaging step. Section 12.2 proves the cluster expansion in an absolute coefficient algebra, in a form that applies to arbitrary interactions with connected supports. Section 12.3 constructs connected supports for the interaction of one averaging step and estimates the resulting effective interaction. Section 12.4 treats  $s \leq J$  averaging steps, and Section 12.5 treats an input effective interaction. The order in which the constants are chosen is summarized in Remark 12.17.

**Conventions.** All results of this section concern the periodic torus without a fixed complement. Distances are measured in units of the coarse lattice spacing. Unless stated otherwise, constants in the statements about one averaging step have the dependence described in item (2) of Definition 4.14. That is, they depend only on  $M$ , on the fixed larger domains of the classical and Haar disintegration charts, and on fixed radius ratios, spatial decay rates, and norm bounds. They are independent of the depth  $d_0$ , of the side  $m$ , and of the coupling and cutoff parameters satisfying the stated inequalities. For  $s \leq J$  averaging steps, the constants may also depend on  $J$ , on the finitely many changes of scale between the lattices involved, and on the fixed box geometry, and no uniformity as  $J \rightarrow \infty$  is claimed. Every factor that is independent of the field, and every factor that depends only on the coarse field, is kept explicitly in the formulas.

We recall the coefficient norms of Definition 4.6. A tensor is bounded *at every fixed index* if, at the spatial decay rate  $b$ , its output-fixed absolute row sum and each of its input-fixed absolute column sums, the latter with the volume normalization of Definition 4.6, are bounded. Every sum carries the weight  $e^{bt}$ , where  $t$  is the length of a shortest connected network joining the locations of all indices of the coefficient; internal vertices are allowed. For a scalar function, we fix any one input index and sum over the others with the same weight, and we write  $\|\cdot\|_{b,\text{root}}$  for the resulting norm of a derivative tensor. Scalar values are never included in these norms. A *majorant series* of a family of tensors is a power series with nonnegative coefficients that dominate the corresponding sums of absolute coefficients.

**Attribution.** The organization of the integral into an effective classical action and connected local fluctuation terms follows Balaban’s renormalization group analysis of lattice gauge theories [17, 19]; its three-dimensional antecedent is the stability analysis in [15]. We adapt this organization to the block averaging map, the constraint coordinates, and the truncated domains used here, and we prove the expansion below. In particular, we do not use the four-dimensional coupling renormalization of [17]. The cluster expansion combines the Mayer expansion with a convergence criterion of the type of Kotecký and Preiss [120]. The bound on connected graph sums by spanning trees comes from Penrose’s partition [136], in the form described by Fernández and Procacci [87, Section 4.1]; see also Brydges’ exposition [38]. Dimock records the finite graph estimate in the constructive renormalization group setting [68, Appendix B, Theorem 27, p. 39, and (323), p. 42]. His analysis concerns a scalar field, and no other result of [68] is used. The weighted resolvent bounds use the Combes–Thomas method [61], as in Lemma 5.3.

## 12.1 Coefficient norms and normalized fluctuation coordinates

**Definition 12.1.** Let  $\mathcal{L} \geq 1$  be an integer, the box scale; it is unrelated to the side length  $L$  of the torus. Partition each periodic coordinate direction of the coarse lattice into integer intervals of lengths between  $\mathcal{L}$  and  $2\mathcal{L}$ ; a direction that is too short for such a partition forms a single interval. The products of these intervals are the *boxes*. A *support* is a nonempty connected set  $X$  of boxes; sets that wind around the torus are allowed, and  $|X|$  denotes the number of boxes in  $X$ . A function is *supported in  $X$*  if it depends only on coordinates  $\theta$  and  $w$  (or  $\theta$  and  $q$ , when those variables are used) whose locations lie in  $X$ .

For a convergent series  $f(\theta, w) = \sum_\nu f_\nu(w)\theta^\nu$  in the coarse-field coordinates  $\theta$ , whose coefficients are functions of  $w$  on a fixed product  $\Omega_p$ , and for a family  $\{f_X\}$  indexed by supports, set

$$\|f\|_{u,\text{coeff};p} = \sum_\nu u^{|\nu|} \sup_{w \in \Omega_p} |f_\nu(w)|, \quad \|\{f_X\}\|_{u,\kappa;p} = \sup_x \sum_{X \ni x} e^{\kappa|X|} \|f_X\|_{u,\text{coeff};p}, \quad (12.1)$$

where  $x$  ranges over the boxes. The coefficients are taken in the scalar link and color coordinates and include the Taylor factorials, that is,  $f_v$  is the coefficient of the monomial  $\theta^v$ ; replacing the scalar absolute value by a fixed norm on the three color components changes only fixed constants. When the functions do not depend on  $w$  we omit  $p$  from the notation.

For an extensive scalar holomorphic function  $f(\theta, q)$  of the coarse-field coordinates  $\theta$  and of variables  $q$ , with  $q_i = g_i w_i$  in the applications below, set

$$\mathcal{J}_{u,v,b}(f) = \sum_{r+s \geq 1} \frac{u^r v^s}{r! s!} \|D_\theta^r D_q^s f(0,0)\|_{b,\text{root}}, \quad (12.2)$$

where the tensor norm is the scalar counting fixed-input sum: it fixes any one index of  $\theta$  or of  $q$ , sums over all other indices, and uses the weight  $e^{bt}$ , with no mesh-volume factor. This is the scalar convention of Definition 4.6; volume normalization is retained for vector columns. The constant coefficient of  $f$  is not restricted by  $\mathcal{J}_{u,v,b}$ . For a reconstructed field, which is vector valued, we use the analogous majorant series for the output-fixed rows and for all input-fixed volume-normalized columns. In a composition of  $s$  averaging steps the variables  $w_i$  and  $q_i$  of the  $i$ -th step have mesh  $M^{i-s}$ ,  $0 \leq i < s$ , and the number of their indices, for all these steps, located in a unit cube is at most  $C_M s M^{3s}$ .

In (12.1), we take the supremum over  $w \in \Omega_p$  of each coefficient in  $\theta$  and then sum with the weights  $u^{|v|}$ . This norm dominates the supremum of the function. The symbol  $\mathcal{J}_{u,v,b}$  is unrelated to the Haar disintegration densities  $\mathcal{J}_c$  introduced below.

We now fix one averaging step, from side  $Mm$  to side  $m \geq 4$ , with input action  $E_{d_0}$  of depth  $d_0 \geq 0$ . By Lemma 6.3 and Theorem 10.10, the fine configuration with average  $c$  is  $b(c, z) = e^{\Gamma_c(z)} U_0(c)$ , where  $U_0(c)$  is the coarse-field lift,  $\Gamma_c$  solves the block averaging constraint for the eliminated edges  $e_a^*$ ,  $a \in \mathcal{E}_m$ , and  $z$  are the free logarithmic coordinates. We write

$$B(\theta, z) = \text{Log } b(e^\theta, z), \quad F(\theta, z) = E_{d_0}(b(e^\theta, z)), \quad \mathcal{L}(\theta, z) = -\log \frac{\mathcal{J}_{e^\theta}(z)}{\mathcal{J}_I(0)},$$

where  $\mathcal{J}_c(z)$  is the Haar disintegration density of Lemma 12.3 below; when  $c$  rather than  $\theta$  is the variable we also write  $F(c, z)$ . Let  $z^*(c)$  be the conditional constrained minimizer of Theorem 10.10 and  $K(\theta) = D_z^2 F(\theta, z^*(e^\theta))$ .

**Lemma 12.2.** *There exist  $a, A, R > 0$ , with the dependence described in item (2) of Definition 4.14 and independent of  $d_0 \geq 0$  and  $m \geq 4$ , and fixed products of coarse-field coordinates and free logarithmic coordinates, contained with fixed positive margins in the domains of Lemma 6.3 and Theorems 10.10 and 10.11, on which the following bounds hold at the spatial decay rate  $a$ .*

1. *The order- $k$  derivative tensors of  $B(\theta, z)$  have output-fixed absolute row sums and input-fixed volume-normalized column sums bounded by  $A^{k+1} k!$ .*

2. The order- $k$  derivative tensors of the scalar functions  $F(\theta, z)$  and  $\mathcal{L}(\theta, z)$  are bounded by  $A^{k+1}k!$  for every  $k \geq 1$ , with any one input index fixed.
3. The minimizer  $z^*(e^\theta)$  vanishes at  $\theta = 0$  and has the same factorial row and column bounds for its derivative tensors; its normalized Taylor coefficients have convergent majorant series.
4. The coefficients of positive degree in  $\theta$  of  $K(\theta)$  may be bounded with either matrix index fixed, or with one coarse-field index fixed; in the latter case both matrix indices are summed. Their generating series is bounded by  $Ax/(1 - Ax)$  for  $0 \leq x < R$ .

*Proof.* The output side is  $m \geq 4$ , and the input action lives on the side  $Mm \geq 4$  and has depth  $d_0 \geq 0$ , so Theorems 10.10 and 10.11 apply. We fix  $R > 0$  such that the products  $\|\theta\|_\infty \leq 2R$ ,  $\|z\|_\infty \leq 2R$  lie inside the domain of the constraint coordinates and the domain of Theorem 10.10, and such that  $\|B(\theta, z)\|_\infty < r_*/2$  there, where  $r_*$  is the radius of Theorem 10.11. Such a choice exists since  $B(0, 0) = 0$  and by the first-order bounds of Lemma 6.3. The last condition places the argument of  $E_{d_0}$ , and not only the coarse field, in the domain where its coefficient bounds hold. The resulting margins depend only on the data of one averaging step, and all later reductions of  $R$  in this proof are made before  $J$ ,  $g$ , and  $p$  are chosen.

Recall that if  $\sum_\nu a_\nu z^\nu$  converges on a neighborhood of a product of circles of radii  $R_j$ , then by Cauchy's formula  $|a_\mu| \prod_j R_j^{\mu_j}$  is at most the supremum of the function on these circles. The same holds for coefficients with values in a Banach space.

Each component of the solution  $\Gamma$  in Lemma 6.3 depends on a finite stencil of at most  $n_M$  scalar coordinates, of diameter at most  $d_M$  in the output lattice. At fixed  $M$  there are only finitely many local types of such stencils, and the periodic identifications only identify members of this finite list. On a closed complex product strictly inside the domain, the holomorphic solutions are bounded by a fixed constant  $A_0$ . If the circle radius in each scalar coordinate is  $R_0$ , the coefficient of a multiindex  $\nu$  is then at most  $A_0 R_0^{-|\nu|}$ . There are  $\binom{n_M+k-1}{k} \leq (2n_M)^k$  multiindices of degree  $k \geq 1$ . Summing absolute coefficients therefore introduces a factor exponential in  $k$ , and multiplying by  $k!$  gives the bound  $A^{k+1}k!$  for the ordered derivative tensors. A fixed input belongs to at most  $C_M$  stencils, which gives the column bounds. A tree on the  $k+1$  locations of a coefficient has length at most  $d_M k$ , so the exponential weight contributes another factor exponential in  $k$ . This proves the row and column bounds; no bound on directional derivatives is used. Finite products, exponentials, and the logarithm in the identity chart preserve such bounds, since their majorant series are dominated by products, by  $e^x$ , and by  $\sum_{j \geq 1} x^j/j$  on a fixed small disc.

These are volume-normalized column bounds as well as counting bounds. In units of the coarse lattice spacing, the two meshes are 1 and  $M^{-1}$ . The column factor, the output volume divided by the input volume, is therefore 1,  $M^{-3}$ , or  $M^3$ , and we include these fixed factors in  $A$ . In compositions we use these factors as in Lemma 5.3, and the

intermediate volumes cancel at each contraction. Since lengths measured in the units of the input lattice are multiplied by  $M^{-1}$ , the exponential bounds of the input lattice still hold at one fixed positive rate.

The same reasoning applies to each term of  $\mathcal{L}$ . Each nonforest link contributes  $-\log j_H(\Gamma_e)$ , and each coarse link  $a$  contributes  $\log \det(M^2 D_{X_{e_a^*}} C_a)$ . Here  $j_H(0) = 1$ , and the matrix in the second logarithm equals  $I$  at  $z = 0$ . On a smaller product, their differences from 1 and from  $I$  have norm less than  $1/2$ , which ensures that the scalar and matrix logarithm series converge with uniform local coefficient bounds. The constant  $\mathcal{J}_I(0)$  cancels exactly these values at zero. Each index belongs to a bounded number of these local terms, which gives the asserted bounds for their sum with one index fixed. This argument also specifies the real logarithms and their holomorphic continuations.

For  $F$ , the product chosen above places every argument  $B$  inside the radius  $r_*$ , and we use the factorial bounds for  $E_{d_0}$  of Theorem 10.11. In a coefficient of the composition, we join the network of the outer tensor to the networks of the inserted  $B$  tensors. If an input index is fixed, we use the column of the insertion containing it and rows for the other insertions. The contraction rule of Lemma 5.3 then sums every internal index without a volume factor, and the exponential generating series is bounded by the composition of two convergent majorant series. Differentiating twice to form a Hessian introduces a factor  $(k+1)(k+2)$  at coefficient degree  $k$ , which is absorbed on a fixed smaller radius.

To bound all orders of the minimizer, let  $K^0 = D_z^2 F(0,0)$  and write the stationarity equation as the implicit system

$$z = \Phi(\theta, z), \quad \Phi(\theta, z) = -(K^0)^{-1} \{D_z F(\theta, z) - K^0 z\}. \quad (12.3)$$

We have  $\Phi(0,0) = 0$ , and the linear part of  $\Phi$  in  $z$  vanishes at the origin. Its part linear in  $\theta$  is bounded by  $A\|\theta\|$ , and the remaining majorant series is bounded by  $A(\|\theta\| + x)^2 / [1 - A(\|\theta\| + x)]$  when the unknown has majorant  $x$ . The rows and columns of  $(K^0)^{-1}$  are bounded uniformly by Theorem 10.10. We apply the implicit-series clause of Lemma 5.3 with the majorant

$$x = A\|\theta\| + \frac{A(\|\theta\| + x)^2}{1 - A(\|\theta\| + x)},$$

that is, with  $\kappa = 0$  in its notation, and we obtain a convergent majorant at every fixed index. Its unique small solution is the minimizer of Theorem 10.10, by the uniqueness statement there.

Finally we compose the tensor  $D_z^2 F$ , which has two indices of  $z$ , with the minimizer. A coefficient of positive degree in  $\theta$  contains a coarse-field index either in the outer tensor or in an inserted copy of  $z^*$ . Fixing that index and using the appropriate columns along its path, we sum over the two indices of  $z$ . The contractions and the majorant series just described prove the stated bound for  $K - K^0$ . Throughout the contractions we keep the

shortest connected networks, and we replace them by trees on the external indices only once, as permitted by Lemma 5.3. This fixes one positive rate for every order. Finally reduce the evaluation radius  $R$  so that  $AR \leq 1/2$ , keeping the coefficient constant  $A$  fixed from the larger products.  $\square$

In the next lemma,  $U$  denotes a scalar function of the coarse field and of  $w$ , not a configuration.

**Lemma 12.3.** *Fix an odd integer  $M \geq 3$ . There exist  $u_0, v_0, b, m_0, C, \sigma_0 > 0$  and  $p_0 \geq 1$ , with the dependence described in item (2) of Definition 4.14 and independent of  $d_0 \geq 0$  and  $m \geq 4$ , such that the following holds for the reconstruction  $b(c, z)$  of Theorem 10.10. Put*

$$F(c, z) = E_{d_0}(b(c, z)), \quad z^*(c) = z_{d_0}^*(c), \quad K(c) = D_z^2 F(c, z^*(c)), \quad S(c) = K(c)^{-1/2}.$$

1. *There are  $f = 2(M^3 - 1)m^3$  independent  $SU(2)$  variables, hence  $3f$  real free logarithmic coordinates. Their density relative to the product Haar measure on the coarse configuration, including the Haar factors of the eliminated edges, is the Haar disintegration density*

$$\mathcal{J}_c(z) = c_{\text{Haar}}^f \frac{\prod_{e \text{ nonforest}} j_H(\Gamma_{c,e}(z))}{\prod_{a \in \mathcal{E}_m} \det D_{X_{e_a}^*} C_{c,a}(\Gamma_c(z))},$$

where  $j_H(X) = \text{sinc}^2(|X|/2)$ ; its value at  $c = I, z = 0$  is  $\mathcal{J}_I(0) = c_{\text{Haar}}^f M^{6|\mathcal{E}_m|}$ . The integral over the forest variables has mass one. Set  $\mathcal{L}_c(z) = -\log(\mathcal{J}_c(z)/\mathcal{J}_I(0))$  and  $\mathcal{L}_0(c) = \mathcal{L}_c(z^*(c))$ .

2. *For real  $c$  the matrix  $S(c)$  is real, symmetric, and positive definite, and  $S$  extends holomorphically to  $c = e^\theta$ ,  $\|\theta\|_\infty < u_0$ . With  $K^0 = K(I)$ , for every real  $x \geq 0$ ,*

$$\|(K^0 + xI)^{-1}\|_{b, \text{row} + \text{col}} \leq \frac{C}{m_0 + x}, \quad S(c) = \frac{1}{\pi} \int_0^\infty x^{-1/2} (K(c) + xI)^{-1} dx.$$

*The derivative tensors of  $S$  satisfy factorial bounds at every fixed index, with exponentially weighted rows and columns, and each tensor of positive order in  $\theta$  may also be bounded with one coarse-field index fixed and both matrix indices summed.*

3. *There exist  $a_1, A, R > 0$ , chosen before  $J$  and independent of  $d_0, m, g, p$ , and fixed products of radius  $R$  contained with fixed positive margins in the larger classical and Haar disintegration domains, on which  $B(\theta, z)$ ,  $F(e^\theta, z)$ ,  $\mathcal{L}_{e^\theta}(z)$ , and  $z^*(e^\theta)$  satisfy the bounds of Lemma 12.2 at the rate  $a_1$ , with order- $k$  derivative tensors bounded by  $A^{k+1}k!$  for  $k \geq 1$ ; for  $B$  these are output-fixed rows and all input-fixed volume-normalized columns, and for the scalar functions any one input index is fixed. Scalar values are excluded, and no coupling or cutoff enters. The order- $k$  derivative tensors of  $K(e^\theta)$ ,  $K^{1/2} = KS$ , and  $S = K^{-1/2}$  are bounded by  $A^{k+1}k!$  for  $k \geq 1$  with either matrix index fixed, or with a coarse-field index fixed and then both matrix indices and all other inputs summed; their normalized Taylor series of positive*

degree are bounded by  $Au/(1 - Au)$  for  $0 \leq u < R$ , with  $AR \leq 1/2$ . All these bounds also hold after translation to any background point of a fixed smaller product. The volume normalizations are those of Definition 4.6 for this pair of lattices.

4. Let  $g > 0$ ,  $p \geq p_0$ , and  $\|\theta\|_\infty < u_0$ , and assume

$$Cgp < \frac{1}{2}, \quad \sigma = C \left\{ \frac{gp^3}{1 - Cgp} + gp \right\} \leq \sigma_0. \quad (12.4)$$

Then, for every  $w \in \Omega_p = \prod_{e=1}^f \{w_e \in \mathbb{R}^3 : |w_e| \leq p\}$ , the point  $z = z^*(c) + gS(c)w$  satisfies  $\|z\|_\infty < v_0/2$ , and  $b(c, z)$  lies in the domain of the classical coefficient bounds. Let  $z_p = \int_{|w| \leq p} e^{-|w|^2/2} d^3w$ , let  $\pi_p$  be the product reference measure on  $\Omega_p$  with density  $z_p^{-f} \prod_e \mathbf{1}_{|w_e| \leq p} e^{-|w_e|^2/2}$ , and set

$$U(c, w) = \frac{F(c, z^* + gSw) - F(c, z^*)}{g^2} - \frac{1}{2}|w|^2 + \mathcal{L}_c(z^* + gSw) - \mathcal{L}_0(c). \quad (12.5)$$

Then, with  $g_+^2 = Mg^2$ ,

$$\int_{z^* + gS\Omega_p} e^{-F(c, z)/g^2} \mathcal{J}_c(z) dz = \mathcal{J}_I(0) g^{3f} e^{-E_{d_0+1}(c)/g_+^2} \det S(c) e^{-\mathcal{L}_0(c)} z_p^f \Xi(c), \quad (12.6)$$

where  $\Xi(c) = \mathbb{E}_{\pi_p} e^{-U(c, w)}$ . These identities are exact for real  $c$ . For complex  $c$  the left side is defined as the holomorphic continuation obtained by integrating over the same real product of  $w$ .

*Proof.* We take  $p_0 = 1$ ; any fixed larger  $p_0$  also works. We use the products of Lemma 12.2 inside the domain of Theorem 10.10, and we reduce their radii as follows. Every use of the matrix-function coefficient clauses of Lemma 5.3(8)–(9) is on their smaller coefficient ball and retains the positive spatial rate actually supplied by the matrix perturbation. The mesh-volume ratios of one averaging step are fixed powers of  $M$ , so these radius and rate choices are uniform in  $d_0$  and  $m$  and precede  $J$ ,  $g$ , and  $p$ .

**Step 1: the weighted resolvent at the identity.** Let  $\lambda = m_0$  be the uniform lower bound of Theorem 10.10 for the Hessian  $K^0$  on the real space of free logarithmic coordinates; it holds for every  $m \geq 4$ . The indices of  $z$  have bounded density and polynomial growth, with constants depending only on  $M$ . We apply the weighted inverse clause of Lemma 5.3 to  $K^0 + xI$ ,  $x \geq 0$ . Its lower bound is  $\lambda + x$ , and its off-diagonal entries are those of  $K^0$ . We fix  $0 < \beta_0 < a$  by the condition of that clause for the lower bound  $\lambda$ ; this choice works for every  $x \geq 0$ . We obtain

$$|(K^0 + xI)_{ef}^{-1}| \leq \frac{2e^{-\beta_0 \text{dist}(e, f)}}{\lambda + x}.$$

For  $0 < b_0 < \beta_0$ , summing with the weight  $e^{b_0 \text{dist}(e, f)}$  and using the polynomial growth just recalled, we obtain the row and column bound  $C/(\lambda + x)$ . The constant and the rate are independent of  $x$ ,  $d_0$ , and  $m$ .

**Step 2: the inverse square root.** Let  $\Delta K = K(c) - K^0$ . On a smaller coarse-field product, Lemma 12.2(4) implies that  $\|(K^0 + xI)^{-1}\Delta K\|_{b_0} < 1/2$  simultaneously for all  $x \geq 0$ . Hence

$$(K(c) + xI)^{-1} = \sum_{j \geq 0} [-(K^0 + xI)^{-1}\Delta K]^j (K^0 + xI)^{-1} \quad (12.7)$$

converges in the weighted matrix algebra. Let  $C_R \geq 1$  be the resolvent constant of Step 1, and let  $k(u)$  majorize the normalized Taylor coefficients of  $\Delta K$  with either matrix index fixed, with  $k(0) = 0$ . For  $j \geq 0$ ,

$$\frac{1}{\pi} \int_0^\infty x^{-1/2} (\lambda + x)^{-j-1} dx = \lambda^{-j-1/2} 4^{-j} \binom{2j}{j}.$$

Consequently, on a smaller radius with  $z(u) = C_R k(u) / \lambda < 1$ , the integrated row and column majorants are bounded by

$$\frac{C_R}{\sqrt{\lambda}} \sum_{j \geq 0} 4^{-j} \binom{2j}{j} z(u)^j = \frac{C_R}{\sqrt{\lambda}} (1 - z(u))^{-1/2}.$$

This justifies integrating the series coefficient by coefficient. For a real eigenvalue  $k > 0$ , the case  $j = 0$  of the integral identity gives  $\pi^{-1} \int_0^\infty x^{-1/2} (k + x)^{-1} dx = k^{-1/2}$ . Diagonalizing the real symmetric matrix  $K(c)$ , we find that the integral is its positive symmetric inverse square root. The same integral of the normally convergent series defines the holomorphic continuation. By continuation from real data we have  $S^\top = S$  and  $S^\top K S = I$ , with the transpose rather than the Hermitian adjoint.

**Step 3: coefficients of  $S$ .** Each coefficient of positive degree in  $\theta$  of (12.7) differentiates at least one factor  $\Delta K$ . We fix the coarse-field index on that factor, sum both of its matrix indices by Lemma 12.2(4), and contract the remaining resolvents using their rows and columns. If  $k_t(u)$  is the corresponding positive-degree majorant of  $\Delta K$  with this coarse-field index fixed, the  $j$  choices of the marked factor give the convergent bound

$$\frac{C_R^2 k_t(u)}{\lambda^{3/2}} \frac{d}{dz} (1 - z)^{-1/2} \Big|_{z=z(u)} = \frac{C_R^2 k_t(u)}{2\lambda^{3/2}} (1 - z(u))^{-3/2}.$$

This proves the asserted bounds for  $S$ , including the sum over both matrix indices when a coarse-field index is fixed. The undifferentiated term needs only the rows and columns of  $(K^0 + xI)^{-1}$ . All radius reductions and the final rate are chosen here, before  $g$  and  $p$ .

**Step 4: joint bounds and the positive square root.** The tensors of  $B$ ,  $F$ ,  $\mathcal{L}$ , and  $z^*$  in part (3) are those of Lemma 12.2; no fluctuation variable or cutoff enters their construction. For the matrix products, we fix  $a_1 > 0$  such that  $2a_1$  is at most both the rate of Lemma 12.2 and  $b_0$ ; this leaves the factor two needed to replace a shortest connected network by a tree

on the external indices. Since  $S$  is a function of  $K$ , it commutes with  $K$ . The holomorphic matrix

$$P(c) = K(c)S(c) \quad (12.8)$$

equals the positive symmetric square root  $K(c)^{1/2}$  for real data, and  $P^2 = K$  on the complex product since  $S^2K = I$ . Then  $P$  is the continuation of the positive square root, and its row and column bounds follow from submultiplicativity.

We check the coefficient bounds for  $P$ . For a coarse-field multiindex  $\nu$  we have  $P_\nu = \sum_{\mu \leq \nu} K_\mu S_{\nu-\mu}$ , with the Taylor factorials included in the coefficients. If a matrix index is fixed, we use the row or column bound of the factor containing it and the compatible bound of the other factor. If a coarse-field index is fixed, we distinguish its position in the convolution. If it belongs to  $K_\mu$ , we use the bound for  $K$  summing both matrix indices and then a row of  $S$  at the common index. If it belongs to  $S_{\nu-\mu}$ , we use the corresponding bound for  $S$  and the column of  $K$  at the common index. Each contraction joins two networks at the shared index of  $z$ , so the exponential weight and the volume normalization are those of Lemma 5.3.

To treat all orders at once, let  $k(x), s(x)$  be majorant series of the coefficients of  $K$  and  $S$  with a matrix index fixed, including their constant terms, and let  $k_t(x), s_t(x)$  be majorant series of their coefficients of positive degree with a coarse-field index fixed and both matrix indices summed. These four series converge, and the last two have no constant term. The contractions just described bound the coefficients of positive degree of  $P$  by those of

$$k(x)s(x) - k(0)s(0), \quad k_t(x)s(x) + k(x)s_t(x), \quad (12.9)$$

for a fixed matrix index and a fixed coarse-field index, respectively. If such a series is bounded by  $C_0$  on a circle of radius  $r_0$ , its coefficient of degree  $k$  is at most  $C_0 r_0^{-k}$ . Enlarging  $A$  beyond  $\max\{1, (1 + C_0)/r_0\}$  for the finitely many series involved, we bound these coefficients by  $A^k$  for  $k \geq 1$ . This bounds the series of positive degree by  $Au/(1 - Au)$  and the ordered tensors of order  $k$  by  $A^{k+1}k!$ . The same enlargement covers  $B, F, \mathcal{L}, z^*, K$ , and  $S$ .

These bounds persist after translation to any background point of a fixed smaller product. Indeed, suppose that a tensor of order  $n$ , with one index fixed, is bounded by  $C^{n+1}n!$ . Expanding at a background point of radius  $r$ , we obtain for  $k \geq 1$  the order- $k$  bound

$$\sum_{j \geq 0} \frac{r^j}{j!} C^{k+j+1} (k+j)! = \frac{C^{k+1}k!}{(1 - Cr)^{k+1}} \leq (2C)^{k+1}k! \quad (Cr \leq 1/2).$$

The fixed index remains a differentiated input or a matrix index, and the other legs at the background point are summed using the same rows or columns; removing these legs cannot increase the length of the remaining network. Thus one radius reduction and one enlargement of  $A$  cover all orders and all these background points, without changing  $a_1$ .

We take the least common radius  $R$  and the least rate  $a_1$ , then reduce  $R$  so that  $AR \leq 1/2$  without changing the coefficient constant fixed on the larger products. We reduce  $u_0$  to this radius if necessary. This proves part (3), with constants chosen before  $J$ ,  $g$ , and  $p$ .

**Step 5: the Haar disintegration.** By Lemma 6.3 there are  $f = 2((Mm)^3 - m^3) = 2(M^3 - 1)m^3$  independent  $SU(2)$  variables, and the density is exactly  $\mathcal{J}_c$ . At  $c = I$  and  $z = 0$  each of the  $|\mathcal{E}_m|$  determinants  $\det D_{X_{e_a}^*} C_{I,a}(0) = \det(M^{-2}I_3)$  equals  $M^{-6}$ , and each factor  $j_H$  equals one, which gives  $\mathcal{J}_I(0) = c_{\text{Haar}}^f M^{6|\mathcal{E}_m|}$ . The Haar probability measure on the gauge transformations of the block forests has mass one. The forest integral contributes no further factor, since the integral uses the action at the representative in the block axial gauge. This proves part (1).

**Step 6: the domain.** By the bound  $\|z^*(c)\| \leq C\|\theta\|$  and the row bound for  $S$ , we have  $\|z^* + gSw\| \leq C(u_0 + gp)$ . We choose  $u_0$  and the threshold for  $gp$  such that this is less than half of the least chart radius  $v_0$  and the reconstruction stays inside the domain of the classical coefficient bounds. We achieve these requirements by enlarging the constant  $C$  in (12.4) before it is fixed; none of them depends on  $p$ ,  $g$ ,  $d_0$ , or  $m$ .

**Step 7: the change of variables.** For real  $c$ , the change of variables  $z = z^* + gSw$  has Jacobian  $g^{3f} \det S$ . Stationarity removes the linear term of  $F(c, \cdot)$  at  $z^*$ , and  $S^T K S = I$  turns its quadratic term into  $|w|^2/2$ . By the nested minimization identity  $E_{d_0+1}(c) = MF(c, z^*(c))$  of Theorem 10.10 and  $g_+^2 = Mg^2$ , we have  $F(c, z^*)/g^2 = E_{d_0+1}(c)/g_+^2$ . Substituting  $\mathcal{J}_c(z) = \mathcal{J}_I(0)e^{-\mathcal{L}_c(z)}$  and extracting  $e^{-\mathcal{L}_0(c)}$  leaves the integrand  $e^{-|w|^2/2 - U(c,w)}$ , whose integral over  $\Omega_p$  is  $z_p^f \Xi(c)$  by the definition of  $\pi_p$ . This proves (12.6), with every scalar factor displayed. For complex  $c$ , we integrate over the same real product of  $w$ . Since the uniform chart margins and the convergent series justify integrating coefficient by coefficient, the result is the holomorphic continuation of the real identity.  $\square$

## 12.2 Polymer expansions in the coefficient algebra

In this subsection, we fix a finite set of boxes as in Definition 12.1. A *polymer* is a support, that is, a nonempty connected set of boxes. We prove the cluster expansion of a hard-core polymer gas whose activities take values in the coefficient algebra of (12.1). We then prove a version for arbitrary interactions with connected supports, which includes holomorphic dependence on the interaction and an estimate for the derivative with respect to an inserted interaction. These statements use only the norms and the independence of fluctuation variables in disjoint supports. They are applied to one averaging step in Section 12.3 and to several in Section 12.4. The weighted incidence condition below is a version of the convergence criterion of Kotecký and Preiss [120], and we prove the bound

we need by majorizing with rooted trees. The successive support weights  $e^{8|X|}$ ,  $e^{5|P|}$ , and  $e^{2|Y|}$  below are chosen for convenience; only the gaps between them matter.

**Lemma 12.4.** *Let  $\mathcal{S}$  be a finite family of nonempty sets of boxes, not necessarily connected, with nonnegative weights  $h_X$ ,  $X \in \mathcal{S}$ . Let  $\kappa \geq 0$  and suppose*

$$\eta = \sup_x \sum_{X \ni x} h_X e^{(\kappa+1)|X|} \leq 1.$$

*Put  $a_X = h_X e^{\kappa|X|}$ . For  $X \in \mathcal{S}$  let  $T_X$  be the sum over finite rooted plane trees, with ordered children, whose vertices carry elements of  $\mathcal{S}$ , repetitions allowed, whose root carries  $X$ , and in which the sets carried by adjacent vertices intersect, of the product of the weights  $a_Y$  of all vertices, where a vertex with  $j$  children receives the factor  $1/j!$ . Then  $T_X \leq a_X e^{|X|}$ , and  $\sum_{X \ni x} T_X \leq \eta$  for every box  $x$ . In particular, the sum of  $e^{\kappa|\cup \mathcal{F}|} \prod_{X \in \mathcal{F}} h_X$  over all families  $\mathcal{F}$  of distinct elements of  $\mathcal{S}$  whose union contains  $x$  and whose intersection graph is connected is at most  $\eta$ .*

*Proof.* For trees of height zero, we set  $T_X^{(0)} = a_X$ , and we define iteratively

$$T_X^{(n+1)} = a_X \exp \left( \sum_{Y: Y \cap X \neq \emptyset} T_Y^{(n)} \right). \quad (12.10)$$

Expanding the exponential assigns the factor  $1/j!$  to the  $j$  children of each vertex, and induction on the height shows that  $T_X^{(n)}$  is the sum over the trees of height at most  $n$  described in the statement. Equivalently, for trees with  $k$  vertices and a fixed root, one may label the other vertices  $2, \dots, k$  and divide by  $(k-1)!$ ; grouping the labels among the subtrees of the root gives exactly the same factors  $1/j!$ . This verifies the count also when several vertices carry the same set.

If  $T_Y^{(n)} \leq a_Y e^{|Y|}$  for all  $Y$ , then

$$\sum_{Y: Y \cap X \neq \emptyset} T_Y^{(n)} \leq \sum_{x \in X} \sum_{Y \ni x} a_Y e^{|Y|} \leq \eta |X| \leq |X|,$$

which implies  $T_X^{(n+1)} \leq a_X e^{|X|}$ . Their limit  $T_X$  satisfies the same bound, since the iterates increase. Summing over  $X \ni x$ , we obtain  $\sum_{X \ni x} T_X \leq \sup_x \sum_{X \ni x} h_X e^{(\kappa+1)|X|} = \eta$ . A family  $\mathcal{F}$  of  $k$  distinct sets with connected intersection graph and union containing  $x$  has a member containing  $x$  and an intersection spanning tree rooted at it. Listing its other members in all orders cancels the factor  $1/(k-1)!$ , so the rooted tree sum counts  $\prod_{X \in \mathcal{F}} a_X$  at least once. Finally, the inequality  $|\cup \mathcal{F}| \leq \sum_{X \in \mathcal{F}} |X|$  transfers the weight of the union to the vertex weights.  $\square$

**Lemma 12.5.** *For every finite simple graph  $G$ ,*

$$\left| \sum_H (-1)^{|E(H)|} \right| \leq \#\{\text{spanning trees of } G\},$$

*where the sum is over the connected spanning subgraphs  $H$  of  $G$ .*

This inequality is used by Dimock in [68, Appendix B, Theorem 27, p. 39, and (323), p. 42]. We prove it with the partition of connected subgraphs introduced by Penrose [136]; see also [87, Section 4.1].

*Proof.* For a graph with one vertex both sides equal one, and for a disconnected graph both sides vanish. Otherwise we order the vertices and root the graph at vertex 1. For a connected spanning subgraph  $H$ , let  $d_H(v)$  be the graph distance from  $v$  to 1 in  $H$ , and join each  $v \neq 1$  to its least neighbor in  $H$  in the preceding distance layer. These edges form a spanning tree  $T(H)$  with the same distances to 1. For a spanning tree  $T$ , let  $R(T)$  consist of the edges of  $T$ , of every edge of  $G$  joining two vertices in the same distance layer of  $T$ , and of every edge of  $G$  joining successive layers whose endpoint in the lower layer is not smaller than the parent in  $T$  of its endpoint in the upper layer. No other edge belongs to  $R(T)$ . Then

$$T(H) = T \iff T \subset H \subset R(T).$$

Indeed, an edge of  $H$  cannot skip a distance layer of  $H$ , and it cannot be joined to a smaller vertex of the preceding layer than the chosen parent. Conversely, adding edges of  $R(T) \setminus T$  to  $T$  changes no distance, since each such edge changes the layer by at most one, and it does not change the least parent. Thus the connected spanning subgraphs are partitioned into these intervals, and each interval contributes

$$\sum_{T \subset H \subset R(T)} (-1)^{|E(H)|} = (-1)^{k-1} (1-1)^{|E(R(T)) \setminus E(T)|},$$

where  $k$  is the number of vertices. Its absolute value is at most one, and summing over spanning trees proves the inequality.  $\square$

**Theorem 12.6.** Fix a finite set of boxes and  $u > 0$ . For each polymer  $P$  let  $\mathfrak{z}(P)$  be a convergent power series in the coarse-field coordinates located in  $P$ , with  $\|\mathfrak{z}(P)\|_{u, \text{coeff}} < \infty$ , and assume

$$\sup_x \sum_{P \ni x} e^{5|P|} \|\mathfrak{z}(P)\|_{u, \text{coeff}} \leq 2\sigma, \quad (12.11)$$

where  $\sigma \leq \sigma_0$  for an absolute constant  $\sigma_0 > 0$ . Let

$$\Xi = \sum_{\{P_1, \dots, P_k\} \text{ pairwise disjoint}} \prod_{i=1}^k \mathfrak{z}(P_i)$$

be the partition function of the hard-core polymer gas, in which two polymers are compatible if they are disjoint; the empty family contributes one. Then the following statements hold.

1. We have  $\log \Xi = \sum_Y \Psi_Y$ , where

$$\Psi_Y = \sum_{k \geq 1} \frac{1}{k!} \sum_{P_1 \cup \dots \cup P_k = Y} \varphi^T(P_1, \dots, P_k) \prod_{i=1}^k \mathfrak{z}(P_i).$$

The inner sum is over ordered tuples of polymers, repetitions allowed, and  $\varphi^T(P_1, \dots, P_k) = \sum_H (-1)^{|E(H)|}$ , the sum being over the connected spanning subgraphs  $H$  of the incompatibility graph on  $\{1, \dots, k\}$ , in which  $i \neq j$  are joined if  $P_i \cap P_j \neq \emptyset$ . A single polymer has coefficient one, and  $|\varphi^T|$  is at most the number of spanning trees of the incompatibility graph by Lemma 12.5. All these series converge absolutely in the coefficient algebra.

2. Each  $\Psi_Y$  depends only on the coarse-field coordinates located in  $Y$ , and

$$\sup_x \sum_{Y \ni x} e^{2|Y|} \|\Psi_Y\|_{u, \text{coeff}} \leq C\sigma.$$

3.  $\Xi \neq 0$  on  $\|\theta\|_\infty < u$ , and  $\log \Xi$  is the branch that vanishes at zero activity.  
 4. Suppose that the number of coordinate labels in each box is bounded and that, for every support  $Y$ , all labels located in  $Y$  can be joined by a network of length at most  $C_{\text{geom}}|Y|$ . Then there exists  $b > 0$  such that, on  $\|\theta\|_\infty \leq u/2$ , for every  $s \geq 1$ ,

$$\|D_\theta^s \log \Xi\|_{b, \text{root}} \leq C^s s! \sigma.$$

The constants in (1)–(3) are absolute; in particular they do not depend on the number of polymers. In (4) the constants depend in addition on  $C_{\text{geom}}$ , on the bound on the number of labels per box, and on  $u$ , but not on the number of boxes or on  $s$ .

*Proof.* We take  $\sigma_0 = 1/4$ .

**Step 1: the formal identity.** We give each polymer  $P$  a formal variable  $z_P$ . The partition function is

$$\sum_{k \geq 0} \frac{1}{k!} \sum_{P_1, \dots, P_k} \prod_i z_{P_i} \prod_{i < j} (1 - \mathbf{1}_{P_i \cap P_j \neq \emptyset}),$$

since the last product removes all tuples with an intersecting pair, including repeated polymers. Expanding this product gives a sum over all graphs on  $\{1, \dots, k\}$ , with the weight  $-\mathbf{1}_{P_i \cap P_j \neq \emptyset}$  for each edge. Each graph has a unique partition of its vertices into connected components. For components of sizes  $k_1, \dots, k_l$ , distributing the labels gives the factor  $k! / (l! \prod_j k_j!)$ , where the factor  $1/l!$  accounts for equal components. Hence the sum over all graphs is the exponential of the sum over connected graphs, and comparing coefficients, we obtain the formula for  $\Psi_Y$ .

**Step 2: the bound.** Let  $h_P = \|\mathfrak{z}(P)\|_{u, \text{coeff}}$  and  $a_P = h_P e^{2|P|}$ . For every tuple, the incompatibility graph is a finite simple graph on the distinct indices  $1, \dots, k$ , even if some of the listed polymers coincide, and its edges are the intersecting pairs. Thus Lemma 12.5 applies to every coefficient  $\varphi^T$ . If the union contains  $x$ , we choose a root polymer containing  $x$ , which replaces the factor  $1/k!$  by the larger factor  $1/(k-1)!$ . By Lemma 12.4 with  $\kappa = 2$ , we have

$$\sup_x \sum_{Y \ni x} e^{2|Y|} \|\Psi_Y\|_{u, \text{coeff}} \leq \sup_x \sum_{P \ni x} h_P e^{3|P|} \leq 2\sigma,$$

provided  $2\sigma \leq 1$ . This proves the absolute convergence in the coefficient algebra, the bound in (2), and the fact that  $\Psi_Y$  depends only on coordinates in  $Y$ .

**Step 3: the analytic branch.** We replace every activity by  $\alpha_3(P)$  with  $\alpha \in \mathbb{C}$ . Since  $\sigma \leq 1/4$ , the bound of Step 2 holds for  $|\alpha| \leq 2$ . Since the sum of all these bounds is finite at finite volume, the connected series and its exponential are convergent power series in  $\alpha$  on this disc. By Step 1, their exponential and the polynomial  $\Xi(\alpha)$  have the same Taylor coefficients at  $\alpha = 0$ , which implies that they agree on the whole disc. At  $\alpha = 1$ , this proves  $\Xi = \exp(\sum_Y \Psi_Y)$ , which gives  $\Xi \neq 0$  and identifies the branch. This argument does not use a volume-independent bound on the scalar  $\log \Xi$ .

**Step 4: derivatives.** For a scalar series  $f_Y(\theta) = \sum_\nu f_{Y,\nu} \theta^\nu$  and  $\|\theta\|_\infty \leq u/2$ , the sum over all ordered derivative indices of order  $s$  of the absolute values of the derivatives of  $f_Y$  is bounded by

$$\sum_{n \geq s} \sum_{|v|=n} |f_{Y,\nu}| (n)_s (u/2)^{n-s} \leq 2^{s+1} s! u^{-s} \|f_Y\|_{u,\text{coeff}}, \quad (12.12)$$

where  $(n)_s = n(n-1) \cdots (n-s+1)$ . To check the constant, we write  $n = s + j$  and bound each  $\binom{s+j}{s} 2^{-j}$  by  $\sum_{j \geq 0} \binom{s+j}{s} 2^{-j} = 2^{s+1}$ . Fixing one derivative index only reduces the sum, and makes it zero unless that index is located in  $Y$ . By hypothesis, all distinct labels in  $Y$  can be joined by a network of length  $C_{\text{geom}}|Y|$ , and repeated labels add no length. We choose  $b$  with  $bC_{\text{geom}} \leq 1$ , multiply (12.12) by the weight  $e^{bt} \leq e^{|Y|}$ , and use the bound in (2), in which the fixed index determines a box. This proves (4), with a single factorial. For the boxes of Definition 12.1, whose labels have bounded density, the geometric hypothesis holds. Indeed, we join the centers of adjacent boxes of  $Y$  and, inside each box, join its finitely many labels by a fixed tree; the total length is at most  $C_{M,\mathcal{L}}|Y|$ .  $\square$

In the next theorem, we apply two expansions to an arbitrary interaction with connected supports: first the connected-component expansion of  $\prod_X (1 + (e^{-U_X} - 1))$ , which produces a hard-core polymer gas, and then the cluster expansion of Theorem 12.6.

**Theorem 12.7.** *Fix a finite set of boxes, a fixed product domain  $\Omega$  for the fluctuation variables, each fluctuation variable being located in one box, and a product probability measure  $\pi$  on  $\Omega$ . Fix  $u > 0$ , and let  $\mathcal{A}$  be the algebra of convergent series  $f(\theta, w) = \sum_\nu f_\nu(w) \theta^\nu$  with bounded measurable coefficients and finite norm  $\|f\|_{\mathcal{A}} = \sum_\nu u^{|\nu|} \sup_{w \in \Omega} |f_\nu(w)|$ . For  $h \geq 0$  let  $\mathcal{S}_h$  be the Banach space of families  $\mathbf{U} = (U_X)$ , indexed by supports, such that each  $U_X \in \mathcal{A}$  depends only on the fluctuation and coarse-field coordinates located in  $X$ , with the norm*

$$\|\mathbf{U}\|_{\mathcal{S}_h} = \sup_x \sum_{X \ni x} e^{h|X|} \|U_X\|_{\mathcal{A}}.$$

Constant coefficients are allowed. There exist absolute constants  $\eta_0, C > 0$ , independent of the number of boxes, of the dimension of the fluctuation variables, of  $u$ , and of  $\pi$ , such that the following holds for  $\|\mathbf{U}\|_{\mathcal{S}_8} < \eta_0$ . Define the polymer activities

$$\mathfrak{z}(P) = \sum_{\mathcal{F}} \mathbb{E}_{\pi} \prod_{X \in \mathcal{F}} (e^{-U_X} - 1), \quad (12.13)$$

the sum being over the families  $\mathcal{F}$  of distinct supports whose intersection graph is connected and whose union is  $P$ , and set

$$\mathcal{L}_Y(\mathbf{U}) = - \sum_{k \geq 1} \frac{1}{k!} \sum_{P_1 \cup \dots \cup P_k = Y} \varphi^T(P_1, \dots, P_k) \prod_{i=1}^k \mathfrak{z}(P_i),$$

with  $\varphi^T$  as in Theorem 12.6. Then  $\mathcal{L}_Y(\mathbf{U})$  depends only on the coarse-field coordinates located in  $Y$ , and

$$\sum_Y \mathcal{L}_Y(\mathbf{U}) = - \log \mathbb{E}_{\pi} e^{-\sum_X U_X}, \quad \|\mathcal{L}(\mathbf{U})\|_{\mathcal{S}_2} \leq C \|\mathbf{U}\|_{\mathcal{S}_8}, \quad (12.14)$$

where  $\mathcal{L}(\mathbf{U}) = (\mathcal{L}_Y(\mathbf{U}))_Y$ . Here  $\mathbb{E}_{\pi} e^{-\sum_X U_X} \neq 0$ , and the logarithm is the branch that vanishes at  $\mathbf{U} = 0$ . Thus each connected cluster of interactions is assigned to the union of their supports. The map  $\mathbf{U} \mapsto \mathcal{L}(\mathbf{U})$ , from the open ball of radius  $\eta_0$  in  $\mathcal{S}_8$  to  $\mathcal{S}_2$ , is Fréchet holomorphic. For  $\|\mathbf{U}\|_{\mathcal{S}_8} \leq \eta_0/2$  and every  $\mathbf{A} \in \mathcal{S}_8$ ,

$$\|D\mathcal{L}(\mathbf{U})[\mathbf{A}]\|_{\mathcal{S}_2} \leq C \|\mathbf{A}\|_{\mathcal{S}_8}. \quad (12.15)$$

If the segment  $\mathbf{U} + s\mathbf{A}$ ,  $0 \leq s \leq 1$ , lies in the closed ball of radius  $\eta_0/2$ , then  $\mathcal{L}(\mathbf{U} + \mathbf{A}) - \mathcal{L}(\mathbf{U})$  is the integral of this derivative along the segment, and its  $\mathcal{S}_2$  norm is at most  $C \|\mathbf{A}\|_{\mathcal{S}_8}$ . If  $\mathbf{U}$  depends holomorphically on a parameter in a Banach space, so does  $\mathcal{L}(\mathbf{U})$ . If  $\theta$  is real and all  $U_X$  and  $A_X$  are real valued, then

$$\sum_Y D\mathcal{L}_Y(\mathbf{U})[\mathbf{A}] = \frac{\mathbb{E}_{\pi} [(\sum_X A_X) e^{-\sum_X U_X}]}{\mathbb{E}_{\pi} e^{-\sum_X U_X}}.$$

The supports of the  $A_X$  may lie anywhere relative to the box at which the norm  $\mathcal{S}_2$  is evaluated. The constants are absolute, as in item (1) of Definition 4.14; when these norms are converted to the torus norms of Definition 4.6, the conversion constants have the dependence described in item (2).

*Proof.* By coefficient convolution, we have  $\|fg\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} \|g\|_{\mathcal{A}}$ . The coefficientwise expectation  $\mathbb{E}_{\pi}$  has norm at most one, and functions of the fluctuation variables in disjoint sets of boxes are independent under  $\pi$ , so their product has the product expectation. These are the operations of Lemma 5.3. At a fixed finite number of boxes, the spaces  $\mathcal{S}_h$  are complete, since they are finite products of closed subspaces of weighted  $\ell^1$  spaces of bounded measurable functions, with an equivalent finite maximum norm.

**Step 1: the connected-component expansion.** We write  $F_X = e^{-U_X} - 1$ ,  $h_X = \|F_X\|_{\mathcal{A}}$ , and  $\eta = \|\mathbf{U}\|_{\mathcal{S}_8}$ . For  $\eta_0$  small, the exponential series in  $\mathcal{A}$  implies that  $h_X \leq e^{\|U_X\|_{\mathcal{A}}} - 1 \leq 2\|U_X\|_{\mathcal{A}}$ , and therefore  $\sup_x \sum_{X \ni x} e^{8|X|} h_X \leq 2\eta$ . Since there are finitely many supports, we may expand  $\mathbb{E}_\pi e^{-\sum_X U_X} = \mathbb{E}_\pi \prod_X (1 + F_X)$  as the sum over families of distinct supports of  $\mathbb{E}_\pi \prod_{X \in \mathcal{F}} F_X$ . Each family has a unique decomposition into components whose intersection graphs are connected. Different components have disjoint unions, hence disjoint fluctuation variables, and the expectation factors. Grouping the families according to the unions of their components, we obtain exactly the hard-core gas with the activities (12.13); the empty family contributes one, and the coarse-field variables enter only as parameters. Since the expectation contracts the norm and  $\sup_x \sum_{X \ni x} h_X e^{6|X|} \leq 2\eta \leq 1$ , Lemma 12.4 with  $\kappa = 5$  gives

$$\sup_x \sum_{P \ni x} e^{5|P|} \|\mathfrak{z}(P)\|_{\mathcal{A}} \leq 2\eta.$$

**Step 2: the cluster expansion.** Taking  $\eta_0 \leq \sigma_0$ , we apply Theorem 12.6 to these activities, with  $\sigma = \eta$ . We obtain the absolute convergence of the series defining  $\mathcal{L}_Y$ , the statement about their supports, the nonvanishing of  $\mathbb{E}_\pi e^{-\sum_X U_X}$ , the branch, and (12.14). The repetitions are those of Theorem 12.6. Since each connected union is fixed before the coefficients are summed, these formulas define a family, although its sum could also be distributed over the supports in other ways.

**Step 3: holomorphy.** At a fixed number of boxes the activities are finite sums of products of entire functions of the  $U_X$  in  $\mathcal{A}$ , hence holomorphic. We choose  $\eta_0$  so small that the bounds of Step 2 still hold after every activity is multiplied by 2, uniformly for  $\|\mathbf{U}\|_{\mathcal{S}_8} \leq \eta_0$ . Then the sum of all terms of the logarithm with  $k \geq n$  polymers has  $\mathcal{S}_2$  norm at most  $C\eta_0 2^{-n}$ . Each sum with fixed  $k$  is finite and holomorphic, and by this bound the series converges uniformly on the ball. To see that the limit is holomorphic, we fix around any point of the open ball a closed ball of radius  $r$  inside it. By Cauchy's formula, the homogeneous Taylor polynomial of degree  $j$  of each partial sum is bounded by  $C\eta_0 (\|v\|/r)^j$ , uniformly in the increment  $v$ . The uniform convergence implies that these polynomials converge uniformly on smaller balls, and the limits are continuous homogeneous polynomials, since polarization of the partial sums passes to the limit. For  $\|v\| < r$ , their series converges geometrically, and by the one-variable Cauchy formula on the line through  $v$ , it equals the limit. This proves Fréchet holomorphy, and holomorphy after composition with holomorphic parameter families.

**Step 4: the derivative.** Let  $a = \|\mathbf{A}\|_{\mathcal{S}_8} > 0$  and  $r_v = \eta_0/(4a)$ . For  $|v| \leq r_v$ , the output has norm at most  $C\eta_0$ , since the input  $\mathbf{U} + v\mathbf{A}$  has norm at most  $3\eta_0/4$ . By Cauchy's formula, we have

$$D\mathcal{L}(\mathbf{U})[\mathbf{A}] = \frac{1}{2\pi i} \int_{|v|=r_v} \frac{\mathcal{L}(\mathbf{U} + v\mathbf{A})}{v^2} dv, \quad \|D\mathcal{L}(\mathbf{U})[\mathbf{A}]\|_{\mathcal{S}_2} \leq 4Ca.$$

For  $a = 0$  there is nothing to prove. Renaming  $C$ , we obtain (12.15), and integrating along the segment proves the statement about differences. The norm sums all supports containing the chosen box after the circle integral. An inserted interaction need not contain that box, since the derivative acts at every position of the inserted interaction within a connected cluster.

**Step 5: real data.** At a fixed number of boxes the coefficient sums bound  $\sum_X U_X$  and  $\sum_X A_X$  uniformly on  $\Omega$ . The integrable constant envelope  $|\sum_X A_X|e^{|\sum_X U_X|+v_0|\sum_X A_X|}$  justifies differentiating  $-\log \mathbb{E}_\pi e^{-\sum_X (U_X+vA_X)}$  under the integral at  $v = 0$ , and the sum of the components of  $D\mathcal{L}(\mathbf{U})[A]$  is this derivative, which is the displayed formula. The envelope depends on the volume; we use it only to identify this sum, not in the estimates for the family.  $\square$

The derivative with respect to an inserted interaction gains a second power of a small parameter when the first-order term of the insertion has mean zero under  $\pi$  on every support. This is the mechanism behind Lemma 12.16.

**Corollary 12.8.** *Assume the setting of Theorem 12.7. For  $s \in [0, 1]$  let  $\zeta \mapsto \mathbf{U}_s(\zeta)$  and  $\zeta \mapsto \mathbf{A}(\zeta)$  be holomorphic maps into  $\mathcal{S}_8$  on a neighborhood of the closed disc  $|\zeta| \leq R$  such that, uniformly in  $s$ ,*

$$\|\mathbf{U}_s(\zeta)\|_{\mathcal{S}_8} \leq \eta_0/2, \quad \|\mathbf{A}(\zeta)\|_{\mathcal{S}_8} \leq a|\zeta|, \quad \mathbf{U}_s(0) = \mathbf{A}(0) = 0.$$

*Assume that  $\mathbb{E}_\pi A'_X(0) = 0$  for every support  $X$ , as an identity of coefficients in  $\theta$ , where  $A' = dA/d\zeta$ . Set  $\mathbf{F}_s(\zeta) = D\mathcal{L}(\mathbf{U}_s(\zeta))[\mathbf{A}(\zeta)]$ . Then  $\mathbf{F}_{s,Y}(0) = \mathbf{F}'_{s,Y}(0) = 0$  for every  $s$  and every support  $Y$ , coefficient by coefficient, and*

$$\|\mathbf{F}_s(t)\|_{\mathcal{S}_2} \leq \frac{2Ca t^2}{R}, \quad 0 < t \leq R/2, \quad (12.16)$$

*where  $C$  is the constant of Theorem 12.7. The same conclusion holds if the uniform bounds are assumed only on an open disc.*

*Proof.* By (12.15), we have  $\|\mathbf{F}_s(\zeta)\|_{\mathcal{S}_2} \leq Ca|\zeta|$ . At zero interaction the derivative of the family is  $\mathbb{E}_\pi A_Y$ , support by support: in the first expansion  $e^{-U_X} - 1 = -U_X + O(U_X^2)$ , and in the second only the terms with a single polymer are linear. Terms with two interaction factors are of degree at least two in  $\zeta$ . Since  $\mathbf{U}_s(0) = \mathbf{A}(0) = 0$ , the coefficient of  $\zeta$  in  $\mathbf{F}_s(\zeta)$  is  $(\mathbb{E}_\pi A'_Y(0))_Y = 0$ . Thus  $\mathbf{F}_{s,Y}(0) = \mathbf{F}'_{s,Y}(0) = 0$  for every support and every coefficient in  $\theta$ ; no cancellation between different supports is used.

The family  $\mathbf{B}_s(\zeta) = \mathbf{F}_s(\zeta)/\zeta$  extends holomorphically to  $\zeta = 0$ , vanishes there, and has norm at most  $Ca$ . By Cauchy's formula for Banach-valued functions, we have  $\|[\zeta^k]\mathbf{B}_s\|_{\mathcal{S}_2} \leq CaR^{-k}$ . For  $0 < t \leq R/2$ , summing the series, we find

$$\|\mathbf{F}_s(t)\|_{\mathcal{S}_2} \leq \frac{Ca t^2}{R(1-t/R)} \leq \frac{2Ca t^2}{R}.$$

If the bounds hold only on an open disc, we apply this on smaller circles and pass to the limit. No regularity in  $s$  is used.  $\square$

### 12.3 Connected supports and the effective interaction of one averaging step

We now return to the fluctuation interaction  $U(c, w)$  of one averaging step, defined in (12.5). The coarse-field coordinates and the normalized fluctuation variables have locations on the lattice, and the coefficient bounds of Lemmas 12.2 and 12.3 decay exponentially in the length of a network joining these locations. In the next lemma, we convert this decay into a decomposition of  $U$  into pieces supported on connected sets of boxes, with the summability over supports required by the norm (12.1), as needed to apply Theorems 12.6 and 12.7. A large box scale  $\mathcal{L}$  makes the factor  $e^{8|X|}$ , exponential in the number of boxes met by a network, compatible with the available spatial decay.

**Lemma 12.9.** *Assume the setting of Lemma 12.3. Choose the box scale  $\mathcal{L}$  sufficiently large, using a fixed part of the spatial decay rate of the classical coefficient bounds, and then reduce the coarse-field radius  $u$  and the threshold for  $gp$ , once. Assign every joint Taylor monomial of  $U$ , in  $\theta$  and  $w$ , to the set of boxes met by a rectilinear tree joining the locations of all its coarse-field coordinates and normalized fluctuation variables, chosen by a fixed deterministic rule described in the proof. This yields an exact decomposition  $U = \sum_X U_X$  into functions supported on connected sets of boxes, including sets winding around the torus, with the following properties.*

1. *For degree  $r$  in  $\theta$  and degree  $s$  in  $w$ , the absolute coefficient sums with one index of  $w$  fixed and with the exponential weight of a connecting network at a fixed rate are bounded by*

$$C^{r+s} g^{s-2} \quad (s \geq 3) \text{ for the classical part,} \quad C^{r+s} g^s \quad (s \geq 1) \text{ for the part from } \mathcal{L}.$$

*The classical part of  $U$ , the first two terms of (12.5), has no terms of degree zero, one, or two in  $w$ , and every term of  $U$  has degree at least one in  $w$ .*

2. *The support assigned to a monomial satisfies  $|X| \leq C(r + s + 1 + \mathfrak{t}/\mathcal{L})$ , and*

$$\|\{U_X\}\|_{u,8;p} \leq \sigma = C \left\{ \frac{gp^3}{1 - Cgp} + gp \right\}.$$

3. *Put  $\sigma_{\text{one}} = \min\{\sigma_0, 1/4, \eta_0/2\}$ , where  $\sigma_0$  is the threshold of Lemma 12.3 and  $\eta_0$  is that of Theorem 12.7. For all subsequent uses, reduce the coarea threshold to  $\sigma_{\text{one}}$ . If  $\sigma \leq \sigma_{\text{one}}$ , then, with  $F_X = e^{-U_X} - 1$ , the fluctuation partition function is the hard-core polymer gas*

$$\Xi = \sum_{\{P_1, \dots, P_k\} \text{ disjoint}} \prod_{i=1}^k \mathfrak{z}(P_i), \quad \mathfrak{z}(P) = \sum_{\mathcal{F}} \mathbb{E}_{\pi_p} \prod_{X \in \mathcal{F}} F_X, \quad \|\{\mathfrak{z}(P)\}\|_{u,5} \leq 2\sigma,$$

*where  $\mathcal{F}$  ranges over the families of distinct supports whose intersection graph is connected and whose union is  $P$ . The empty family contributes one, and each activity depends only on coarse-field coordinates located in  $P$ .*

4. Statements (1)–(3) remain valid, with  $g$  replaced by  $|\zeta|$  in all bounds and in (12.4), for the interaction

$$U(c, w; \zeta) = \frac{F(c, z^* + \zeta Sw) - F(c, z^*)}{\zeta^2} - \frac{1}{2}|w|^2 + \mathcal{L}_c(z^* + \zeta Sw) - \mathcal{L}_0(c)$$

with a complex parameter  $\zeta$ , extended holomorphically across  $\zeta = 0$ ; at  $\zeta = g > 0$  this is  $U(c, w)$ .

The constants have the dependence described in item (2) of Definition 4.14; they are independent of  $d_0$ , of  $m$ , of  $g$ ,  $p$  satisfying (12.4), and of the number of supports. The only independence used is that of the fluctuation variables in distinct boxes under the product measure  $\pi_p$ . The exact formula (12.6) concerns real  $g$  only.

*Proof.* We write  $c = e^\theta$  and proceed in four steps.

**Step 1: expansion in the fluctuation variables.** At fixed  $\theta$ , we expand in the increment  $gSw$  and obtain the normally convergent series

$$\frac{F(c, z^* + gSw) - F(c, z^*)}{g^2} - \frac{1}{2}|w|^2 = \sum_{s \geq 3} \frac{g^{s-2}}{s!} D_z^s F(c, z^*)[Sw, \dots, Sw], \quad (12.17)$$

$$\mathcal{L}_c(z^* + gSw) - \mathcal{L}_c(z^*) = \sum_{s \geq 1} \frac{g^s}{s!} D_z^s \mathcal{L}_c(z^*)[Sw, \dots, Sw]. \quad (12.18)$$

The classical terms of degree one and two vanish by stationarity and by  $S^\top KS = I$ . Both series vanish at  $w = 0$  for every coarse field.

**Step 2: coefficient bounds.** We expand the coefficients further in  $\theta$  and apply Lemmas 12.2 and 12.3 and the contraction rule of Lemma 5.3. If an index of  $w$  is fixed, the column of the factor  $S$  containing it carries the fixed index to a tensor of  $F$  or of  $\mathcal{L}$  with one index fixed, and all other insertions use rows. The networks of the factors meet at the contracted indices, so their union joins all external locations. The joint coefficient of degree  $(r, s)$  has the generating factor  $1/(r! s!)$ , and passing from total derivatives gives only the factor  $(r + s)!/(r! s!) \leq 2^{r+s}$ . Summing the composed majorant series proves the two coefficient bounds in (1), at one fixed rate  $a > 0$ . The choice of the fixed occurrence of  $w$ , including multiplicities and colors, contributes at most another factor exponential in  $r + s$ , which is absorbed into  $C$ .

**Step 3: the supports.** Let  $\mathcal{L} \geq 1$  be an integer. If a periodic direction of the coarse lattice has length at least  $2\mathcal{L}$ , we divide it into integer intervals of lengths between  $\mathcal{L}$  and  $2\mathcal{L}$ ; this is possible by distributing the remainder among  $\lfloor m/\mathcal{L} \rfloor$  intervals. A shorter direction forms one interval. The resulting boxes have bounded adjacency degree and contain at most  $D_{M, \mathcal{L}}$  scalar labels.

Consider a monomial with  $k = r + s \geq 1$  coordinate locations. We order all metric spanning trees of these locations and all shortest periodic coordinate paths by one fixed rule, choose a shortest metric spanning tree, and replace each of its edges by the chosen rectilinear path. The resulting network has length at most a fixed multiple of the length  $t$  of a shortest connected network joining the locations. Indeed, traversing every edge of such a network twice gives a closed walk of length  $2t$  through all locations. Shortcutting this walk yields a metric spanning tree of length at most  $2t$ , and replacing an edge by a coordinate path changes its length by at most a factor depending on the dimension. We add the intersection points of the paths as vertices, delete cycles by the fixed rule, and prune branches that contain no location. The result is a rectilinear tree of no greater length. Let  $X$  be the set of boxes met by this tree, including the boxes of the locations. It is connected, since the fixed rule assigns boundary points to adjacent boxes. A coordinate segment of length  $l$  meets at most  $C(1 + l/\mathcal{L})$  boxes. The unpruned paths consist of at most  $3(k - 1)$  segments, and the deletions do not add boxes. Applying the bound to these segments, we obtain

$$|X| \leq C_0(k + 1 + t/\mathcal{L}). \quad (12.19)$$

Supports winding around the torus occur, since the paths are periodic. If  $x \in X$  and  $e$  is the location of the fixed index of  $w$ , then

$$\text{dist}(x, e) \leq C_1\{t + \mathcal{L}(k + 1)\},$$

where the distance to a box is measured to its center; another choice changes the constant by the diameter of a box.

We choose  $\mathcal{L}$  so large that  $8C_0/\mathcal{L} < a/4$ , and then  $b_1 > 0$  so small that  $b_1C_1 < a/4$ . Combining the two geometric bounds, we obtain the support-weight estimate

$$\mathbf{1}_{x \in X} e^{8|X|} \leq CD_{\mathcal{L}}^k e^{at/2} e^{-b_1 \text{dist}(x, e)}, \quad (12.20)$$

with  $D_{\mathcal{L}} = e^{8C_0 + b_1C_1\mathcal{L}}$ . Since the indices of  $w$  have bounded density and polynomial growth, we have  $\sup_x \sum_e e^{-b_1 \text{dist}(x, e)} \leq C_{M, \mathcal{L}, b_1}$ , uniformly in the side of the torus. This includes the case of a short direction, in which the number of labels in that direction is bounded by the box scale.

**Step 4: summation and the polymer gas.** We group the monomials, with their coefficients, according to their supports  $X$ ; the sum  $U_X$  depends only on coordinates located in  $X$ . We apply (12.20) before summing over the location  $e$  of the fixed index of  $w$ , and then the coefficient bounds of Step 2. This bounds the norm with support weight  $e^{8|X|}$  by

$$C \sum_{r \geq 0} (Cu)^r \left\{ \sum_{s \geq 3} C^s g^{s-2} p^s + \sum_{s \geq 1} (Cgp)^s \right\},$$

where the constants include the factors  $D_{\mathcal{L}}^{r+s}$ . We reduce  $u$  until  $Cu \leq 1/2$  and the threshold for  $gp$  until  $Cgp < 1/2$ . Summing the geometric series and enlarging the constant in

(12.4), we obtain the bound  $C\{gp^3/(1 - Cgp) + gp\}$ . The absolute convergence proves both the identity  $U = \sum_X U_X$  and the bound in (2). This estimate controls the sum of the separate coefficient suprema required by (12.1).

The bounds of Steps 1–4 use  $g$  only through the factors  $g^{s-2}$  and  $g^s$  of the series (12.17) and (12.18), and the domain condition  $\|gSw\| \leq C|g|p$ . For a complex parameter  $\zeta$  in place of  $g$ , the same series define  $U(c, w; \zeta)$ , which is holomorphic across  $\zeta = 0$  since the classical series starts at  $\zeta^1$ . The same estimates hold with  $|\zeta|$  in place of  $g$ . This proves (4) for the bounds in (1) and (2), and the argument for (3) below applies without change.

For (3), the choice  $\sigma \leq \sigma_{\text{one}}$  places the supported interaction in the ball of radius  $\eta_0/2$  and also meets the hard-core threshold  $1/4$ . We apply Step 1 of the proof of Theorem 12.7 with  $\pi = \pi_p$  and with  $\eta$  replaced by  $\sigma$ . We obtain  $\|F_X\|_{u, \text{coeff}; p} \leq 2\|U_X\|_{u, \text{coeff}; p}$ , the hard-core representation of  $\Xi$  with the displayed activities, and the bound  $\|\{z(P)\}\|_{u, 5} \leq 2\sigma$ . This requires only the independence of the fluctuation variables in disjoint sets of boxes under  $\pi_p$ ; the coarse-field variables are parameters. All constants were chosen using  $M$ , the fixed domains, and the spatial decay rates, so they have the stated dependence.  $\square$

We now combine the exact formula (12.6) with the cluster expansion. We define the effective interaction newly generated by the averaging step as

$$Q_{d_0}^w(c) = \mathcal{L}_0(c) + \frac{1}{2} \log \det K(c) - f \log z_p - \log \Xi(c). \quad (12.21)$$

Since  $\det S(c) = (\det K(c))^{-1/2}$ , Lemma 12.3 can be written as

$$\int_{z^* + gS\Omega_p} e^{-F(c, z)/g^2} \mathcal{J}_c(z) dz = \mathcal{J}_I(0) g^{3f} \exp\left(-\frac{E_{d_0+1}(c)}{g_+^2} - Q_{d_0}^w(c)\right).$$

**Proposition 12.10.** *Assume the setting of Lemma 12.9, with  $\sigma \leq \sigma_{\text{one}}$ . There exist  $u_* \in (0, u_0]$  and  $b > 0$ , with the dependence described in item (2) of Definition 4.14, such that the following holds on  $\|\theta\|_\infty \leq u_*$ , uniformly in  $d_0$ ,  $m$ , and in  $g, p$  satisfying (12.4).*

1. For every  $s \geq 1$ ,

$$\|D_\theta^s Q_{d_0}^w\|_{b, \text{root}} \leq C^s s!.$$

*The value of  $Q_{d_0}^w$  is not bounded uniformly in the volume.*

2. For real  $c$  and  $w \in \Omega_p$ , the Hessian of  $w \mapsto |w|^2/2 + U(c, w)$  is  $I + E$ , where  $E$  is bounded by  $C(gp + g^2)$  in a fixed exponentially weighted row-plus-column norm.

*Proof.* By Lemma 12.9(3), the activities satisfy (12.11), and the labels have bounded density. Thus Theorem 12.6(4) implies that  $\|D_\theta^s \log \Xi\|_{b, \text{root}} \leq C^s s! \sigma$  on  $\|\theta\|_\infty \leq u/2$ .

We next treat the determinant. Let  $A(\theta) = (K^0)^{-1}(K(c) - K^0)$ , and reduce the coefficient radius such that the row and column series of  $A$  have norm less than  $1/2$ . On this radius, we have

$$\frac{1}{2} \log \det K(c) = \frac{1}{2} \log \det K^0 + \frac{1}{2} \sum_{j \geq 1} \frac{(-1)^{j+1}}{j} \text{tr } A(\theta)^j, \quad (12.22)$$

where the logarithm is the branch that is real for real  $c$ . To verify this, we replace  $A$  by  $hA$ . Differentiating the determinant along its columns, we find  $(d/dh) \log \det(I + hA) = \text{tr}[(I + hA)^{-1}A]$ , and integrating the Neumann series from  $h = 0$  to  $h = 1$  gives the formula. A coefficient of positive degree in  $\theta$  differentiates at least one factor  $A$ . We fix the coarse-field index on such a factor and sum over its two matrix indices using Lemma 12.2(4) and the rows and columns of  $(K^0)^{-1}$ ; every other factor is contracted by rows and columns. The choice of the distinguished factor cancels the factor  $1/j$ . If  $a(x) = Cx/(1 - Cx)$  is a majorant series for  $A$ , the trace coefficients with one coarse-field index fixed are bounded by those of  $Ca'(x)/(1 - a(x))$  on a fixed smaller disc, and these coefficients are at most  $C_1^{k+1}$ . Translating to a background point in a fixed smaller product, we obtain factorial bounds for all derivatives of positive order. The network of a trace may contain a cycle. Deleting an edge leaves a network joining the same locations, so the weight of a shortest connected network is still available. The constant  $\log \det K^0$  is extensive and is not estimated. The term  $\mathcal{L}_0$  is the composition of  $\mathcal{L}$  with the minimizer, and by Lemma 12.2 it has the same bounds with one index fixed. The constant  $f \log z_p$  has no derivatives of positive order. Combining these bounds on  $\|\theta\|_\infty \leq u_*$ , where  $u_*$  is the smaller of  $u/2$  and the radius chosen for  $A$ , we obtain (1). Here the determinant term enters (12.21) with the positive sign, and the constant  $-f \log z_p$  is kept.

For (2), we fix a real coarse field. Up to a term independent of  $w$ , the potential is  $g^{-2}F(c, z^* + gSw) + \mathcal{L}_c(z^* + gSw)$ . Since  $S$  depends only on  $c$ , its Hessian minus the identity is

$$S^\top \{D_z^2 F(c, z^* + gSw) - K(c)\} S + g^2 S^\top D_z^2 \mathcal{L}_c(z^* + gSw) S.$$

The third derivative of  $F$  and the second derivative of  $\mathcal{L}_c$  have fixed weighted bounds on the product, and  $\|gSw\| \leq Cgp$  by the row bound for  $S$ . We conclude that the displayed matrix is bounded by  $C(gp + g^2)$  in a fixed weighted row-plus-column norm.  $\square$

The last result for one averaging step concerns symmetries.

**Proposition 12.11.** *In the setting of Lemma 12.3, there exist fixed products of coarse-field logarithms  $\theta$ , free logarithmic coordinates  $z$ , and gauge parameters  $\kappa$ , independent of  $d_0$  and  $m$  and contained in the domains of that lemma, on which both the original and the transformed reconstructed configurations lie in the domains of Lemma 6.3 and Theorems 10.10 and 10.11. We call them the common symmetry domains; they are constructed in the proof. Let  $c = e^\theta$  be real, with  $\theta$  in the common symmetry domain, and let  $\tau$  be one of the following real operations on coarse fields: a translation by a vector of the coarse lattice; the spatial inversion, under which reversed links carry inverted matrices; a constant color conjugation; or a coarse gauge transformation  $c \mapsto c^k$  with  $\text{SU}(2)$ -valued  $k = e^\kappa$  and  $\kappa$  real in the corresponding fixed parameter product. Then there exists an orthogonal map  $O(c)$  of the free logarithmic coordinates, which permutes the*

three-component blocks, acts on each block by an orthogonal matrix, and does not depend on  $z$ , such that

$$z^*(\tau c) = O(c)z^*(c), \quad K(\tau c) = O(c)K(c)O(c)^\top, \quad S(\tau c) = O(c)S(c)O(c)^\top.$$

The induced map  $w \mapsto O(c)w$  of the normalized fluctuation variables preserves  $\Omega_p$ ,  $|w|^2$ , Lebesgue measure, and  $\pi_p$ , and the Haar disintegration density and  $\det K$  are preserved. Consequently the integral (12.6) and  $Q_{d_0}^w$  take the same values at  $c$  and at  $\tau c$ . The resulting identities between analytic functions of the coarse field extend to the corresponding common complex domains.

*Proof.* At depth zero the Wilson action is invariant under periodic gauge transformations, translations, the spatial inversion, and constant color conjugations: the plaquette variables are conjugated, relabeled, or inverted, and  $q(U^{-1}) = q(U)$ . The transformations of the constraint coordinates in Lemma 6.3 intertwine the reconstruction, and on the common symmetry domains they send  $z$  to  $O(c)z$ , where  $O(c)$  is as in the statement. Coarse translations relabel the blocks, roots, trees, and distinguished edges; Proposition 6.2(4) and Lemma 6.3(2)–(3) identify the translated constraints and their coarea factors.

We first prove the invariance of  $E_d$  as an identity of analytic germs at the identity, by induction on the depth  $d$ . If the germ at depth  $d$  is invariant, then the intertwining of the constraint coordinates carries the constrained objective to the transformed objective near  $(c, z) = (I, 0)$ . The transformed stationary point is  $O(c)z^*(c)$ , which lies in the neighborhood where the minimizer of Theorem 10.10 is unique. The two stationary values agree, and multiplying by  $M$  gives the invariance of the germ at depth  $d + 1$ . Only a nonempty neighborhood is used in this induction, and no uniform radius is inferred from it.

To obtain a uniform domain, we use the common holomorphic domains of Lemma 6.3 and Theorems 10.10 and 10.11. We choose fixed smaller products in  $(\theta, z, \kappa)$  on which both the original and the transformed reconstructed fields lie in these domains; these are the common symmetry domains of the statement. The finite-word bounds for the reconstruction and the local gauge word  $e^{\kappa_x} e^{\theta_{xy}} e^{-\kappa_y}$  allow this choice independently of the depth. The differences between the two action values, and between the two constrained objectives, are holomorphic there and have zero germ by the induction. We claim that they vanish on the whole smaller product. Indeed, we restrict such a difference to the complex line  $h \mapsto h(\theta, z, \kappa)$  through a fixed point of the product. The fixed margins allow a disc  $|h| < 1 + \eta$  for some  $\eta > 0$ . Its Taylor series vanishes near  $h = 0$ , and continuation through overlapping discs along  $[0, 1]$  shows that it vanishes at  $h = 1$ . Translations and the spatial inversion are treated in the same way without  $\kappa$ , and a constant color conjugation preserves the radii and is kept fixed in the induction. Thus the identity of the objectives holds on the fixed products, and the uniqueness of the minimizer there implies that  $z^*(\tau c) = O(c)z^*(c)$  on the common domain, without a loss of radius depending on the depth.

Differentiating the objective identity twice, we obtain  $K(\tau c) = OKO^\top$ , and the integral formula for the inverse square root implies that  $S(\tau c) = OSO^\top$ . Then  $w = S^{-1}(z - z^*)/g$  transforms as  $w \mapsto O(c)w$ , which preserves the product of balls,  $|w|^2$ , Lebesgue measure, and the product reference measure  $\pi_p$ . By the exact Haar disintegration of Lemma 6.3 and  $\det(OKO^\top) = \det K$ , every remaining factor of the integral is preserved. This proves the identities for the integral and for  $Q_{d_0}^w$  at real coarse fields, without requiring the individual pieces  $U_X$  to be invariant. The statements about balls, Lebesgue measure, and  $\pi_p$  concern the real change of variables. The identities for  $z^*$ ,  $K$ ,  $S$ , and  $Q_{d_0}^w$  hold between functions that are holomorphic on the common complex symmetry domains, so they extend there by the identity theorem.  $\square$

The gauge parameter  $\kappa$  in Proposition 12.11 ranges over a fixed small product, and Theorem 12.13(6) imposes this restriction at every scale. Transitions between coarse fields in the comparison domains of Section 13 that are related by an arbitrary periodic gauge transformation are given by Theorem 13.4(3).

## 12.4 Several averaging steps

We now compose  $s \leq J$  averaging steps. At each scale, from the coarsest to the finest, we perform the change of variables of Lemma 12.3 with the coarse field of that scale, which depends on the fluctuations already introduced. The following example shows the structure of the resulting change of variables in the simplest case.

**Example 12.12.** Let  $s = 2$ , and let  $c_2 = e^\theta$  be the coarse field. The coordinates of the coarser intermediate scale are  $z_1 = z_1^*(c_2) + g_1 S_1(c_2)w_1$ , and they determine the field  $c_1 = b_1(c_2, z_1)$ , so that  $c_1 = c_1(w_1)$ . The coordinates of the finer scale are

$$z_0 = z_0^*(c_1(w_1)) + g_0 S_0(c_1(w_1))w_0.$$

In the order  $(w_1, w_0)$  the derivative of  $w \mapsto (z_1, z_0)$  is block lower triangular,

$$D_w z = \begin{pmatrix} g_1 S_1(c_2) & 0 \\ D_{w_1} z_0 & g_0 S_0(c_1(w_1)) \end{pmatrix}, \quad D_{w_1} z_0 = D z_0^*(c_1) D_{w_1} c_1 + g_0 D S_0(c_1) [D_{w_1} c_1] w_0.$$

We write  $D_i$  for the number of real coordinates of  $z_i$ ,  $i = 0, 1$ . The off-diagonal block does not change the determinant,  $\det D_w z = g_1^{D_1} \det S_1(c_2) g_0^{D_0} \det S_0(c_1(w_1))$ , but it enters all later derivatives. The determinant of the finer scale depends on  $w_1$ . Let  $\hat{c}_1 = c_1(0) = b_1(c_2, z_1^*(c_2))$  be the zero-fluctuation reconstruction of  $c_1$ . We split

$$\log \det K_0(c_1(w_1)) = \log \det K_0(\hat{c}_1) + [\log \det K_0(c_1(w_1)) - \log \det K_0(\hat{c}_1)].$$

The first term depends only on  $c_2$  and becomes part of the function  $B_{\text{ref}}$  of Theorem 12.13. The second depends on  $w_1$  and is part of the fluctuation interaction  $H_{d_0,2}$ . The negative

logarithmic Haar density at the minimizer of the finer scale is split in the same way. Replacing  $c_1(w_1)$  by  $\hat{c}_1$  in these terms would not give an exact formula. Under the product reference measure,  $w_1$  and  $w_0$  are independent, but the interacting density need not factor. Its classical and Haar-density remainders can involve both variables, while the determinant and minimizer-density corrections displayed here depend only on  $w_1$  and change its marginal weight.

In the theorem below, the field  $c_i$  lives on the lattice of side  $mM^{s-i}$ ,  $0 \leq i \leq s$ , so that  $c_0$  is the finest field and  $c_s = e^\theta$  is the coarse field. The  $i$ -th averaging step, from side  $mM^{s-i}$  to side  $mM^{s-i-1}$ , has input action  $E_{d_0+i}$ ; we write  $b_i(c, z)$  for its reconstruction,  $\mathcal{J}_i(c, z)$  for its Haar disintegration density, and  $z_i^*(c)$  for its conditional constrained minimizer, as in Lemma 12.3. The coupling of the coarsest integrated scale is  $t$ , and the couplings decrease by the factor  $M^{-1/2}$  toward finer scales.

**Theorem 12.13.** *Fix an odd integer  $M \geq 3$  and an integer  $J \geq 1$ , and put  $\sigma_{\text{seq}} = \frac{1}{2} \min\{\sigma_{\text{one}}, \eta_0, 1/4\}$ . There is a box-scale threshold  $\mathcal{L}_J$  such that, for every fixed box scale  $\mathcal{L} \geq \mathcal{L}_J$ , there exist  $u_J, v_J, b_J, C_J > 0$ , uniform in the depth  $d_0 \geq 0$  and in the side  $m \geq 4$ , for which the following holds for every integer  $1 \leq s \leq J$ . For  $0 \leq i < s$  put*

$$f_i = 2(M^3 - 1)(mM^{s-i-1})^3, \quad D_i = 3f_i, \quad g_i = tM^{(i-s+1)/2},$$

and  $g_s^2 = Mg_{s-1}^2$ . Given  $c_s = e^\theta$ , define successively, for decreasing  $i$ ,

$$\begin{aligned} F_i(c, z) &= E_{d_0+i}(b_i(c, z)), & K_i(c) &= D_z^2 F_i(c, z_i^*(c)), & S_i &= K_i^{-1/2}, \\ z_i &= z_i^*(c_{i+1}) + g_i S_i(c_{i+1}) w_i, & c_i &= b_i(c_{i+1}, z_i). \end{aligned}$$

Let  $\Omega_p^{\text{all}} = \prod_{i < s} \prod_e \{w_{i,e} \in \mathbb{R}^3 : |w_{i,e}| \leq p\}$ , and let  $\pi_p^{\text{all}}$  be the product reference measure on it with density proportional to  $e^{-\sum_i |w_i|^2/2}$ . Assume  $p \geq p_0$ ,  $\|\theta\|_\infty < u_J$ , and

$$C_J t p < \frac{1}{2}, \quad \sigma_J = C_J \frac{tp^3 + tp}{1 - C_J t p} \leq \sigma_{\text{seq}}. \quad (12.23)$$

Then the following statements hold.

1. For every  $w \in \Omega_p^{\text{all}}$  the reconstruction lies, with fixed margins, in every classical and Haar disintegration chart, and the recursion is a jointly holomorphic map of  $\theta$  and of the variables  $q_i = g_i w_i$  on a common larger product.
2. In the order from coarser to finer scales,

$$\det D_w z = \prod_{i=0}^{s-1} g_i^{D_i} \det S_i(c_{i+1}).$$

3. Put

$$U_i^{\text{cl}} = \frac{F_i(c_{i+1}, z_i) - F_i(c_{i+1}, z_i^*(c_{i+1}))}{g_i^2} - \frac{1}{2}|w_i|^2.$$

Then, exactly,

$$\frac{E_{d_0}(c_0)}{g_0^2} = \frac{E_{d_0+s}(c_s)}{g_s^2} + \frac{1}{2} \sum_i |w_i|^2 + \sum_i U_i^{\text{cl}}.$$

4. Put  $\mathcal{L}_i(c, z) = -\log(\mathcal{J}_i(c, z) / \mathcal{J}_i(I, 0))$  and

$$B_i(c) = \mathcal{L}_i(c, z_i^*(c)) + \frac{1}{2} \log \det K_i(c).$$

Let  $\widehat{c}_s = c_s$  and  $\widehat{c}_i = b_i(\widehat{c}_{i+1}, z_i^*(\widehat{c}_{i+1}))$  be the zero-fluctuation reconstruction, and set  $B_{\text{ref}}(c_s) = \sum_i B_i(\widehat{c}_{i+1})$ . Define the fluctuation interaction

$$H_{d_0,s}(\theta, w) = \sum_i \left[ U_i^{\text{cl}} + \mathcal{L}_i(c_{i+1}, z_i) - \mathcal{L}_i(c_{i+1}, z_i^*(c_{i+1})) + B_i(c_{i+1}) - B_i(\widehat{c}_{i+1}) \right]. \quad (12.24)$$

If  $\mathcal{D}_s(c_s)$  is the image of  $\Omega_p^{\text{all}}$  under  $w \mapsto (z_i)_{i < s}$ , then

$$\begin{aligned} & \int_{\mathcal{D}_s(c_s)} e^{-E_{d_0}(c_0)/g_0^2} \prod_i \mathcal{J}_i(c_{i+1}, z_i) dz_i \\ &= \left( \prod_i \mathcal{J}_i(I, 0) g_i^{D_i} \right) e^{-E_{d_0+s}(c_s)/g_s^2 - B_{\text{ref}}(c_s)} z_p^{\sum_i f_i} \Xi_{d_0,s}(c_s), \end{aligned} \quad (12.25)$$

where  $\Xi_{d_0,s} = \mathbb{E}_{\pi_p^{\text{all}}} e^{-H_{d_0,s}}$ .

5. For the chosen box scale there is an exact decomposition  $H_{d_0,s} = \sum_X H_X$  into functions supported on connected sets of boxes with  $\|\{H_X\}\|_{u'_J, 8;p} \leq \sigma_J$ , for a fixed coarse-field radius  $u'_J \in (0, u_J]$  that also lies inside the domain of the coefficient series of  $B_{\text{ref}}$  described in the proof. Both expansions of Theorem 12.7 apply;  $\Xi_{d_0,s} \neq 0$  on  $\|\theta\|_\infty < u'_J$ , and the effective interaction

$$Q_{d_0,s}^w = B_{\text{ref}} - \left( \sum_i f_i \right) \log z_p - \log \Xi_{d_0,s}$$

satisfies, on  $\|\theta\|_\infty \leq u'_J/2$ ,  $\|D_\theta^k Q_{d_0,s}^w\|_{b_J, \text{root}} \leq C_J^{k+1} k!$  for every  $k \geq 1$ . The function  $B_{\text{ref}}$  of the coarse field is kept in full.

6. Let  $c_s$  be real, and let  $\tau$  be one of the operations of Proposition 12.11, applied to  $c_s$ . Assume that, for every  $w \in \Omega_p^{\text{all}}$  and every  $i$ , the field  $c_{i+1}$  and the operation induced by  $\tau$  on it satisfy the hypotheses of Proposition 12.11 for the averaging step  $i$ , that is, they lie in the common symmetry domains of that step. Then the induced changes of variables  $w'_i = O_i(c_{i+1})w_i$  are triangular, orthogonal at each scale for fixed coarser variables, preserve every ball, and have absolute joint Jacobian one; their dependence on the coarser fluctuations is retained. They transport every factor of (12.25).

The constants depend only on  $M, J$ , the larger chart margins, the box geometry, and the spatial decay rates; they do not depend on  $d_0, m$ , or on  $t, p$  satisfying (12.23).

In the proof, we first construct one product of coordinates on which all compositions are controlled, before any integration. We then compute the Jacobian and the integral exactly, and finally we estimate the fluctuation interaction and apply the expansions.

*Proof.* For each  $0 \leq i < s$  the averaging step  $i$  has output side  $m_i = mM^{s-i-1} \geq 4$ , input side  $Mm_i$ , and depth  $d_0 + i \geq 0$ . Thus Lemmas 12.2 and 12.3 apply to every step, and the construction below checks their domain conditions at every argument reached by the reconstruction.

**Step 1: a common product.** In logarithmic coordinates, we define the map of one step

$$T_i(x, q) = \text{Log } b_i(e^x, z_i^*(e^x) + S_i(e^x)q).$$

Its value at  $(0, 0)$  is zero. By Lemmas 12.2 and 12.3, its output rows and all its input columns are bounded by one convergent majorant series, which we enlarge to  $P(h) = Ah/(1 - Ah)$ ; these bounds are uniform in  $d_0 + i$  and in  $m_i$ . We choose a smaller domain for each step, with a fixed positive margin  $R_1$  to every classical and Haar disintegration boundary and to every denominator in these series.

For a scalar variable  $h$ , we set  $x_s(h) = h$  and successively  $x_i(h) = P(x_{i+1}(h) + h)$ . There are only  $s \leq J$  compositions, and their denominators equal one at  $h = 0$ . We choose  $r_J > 0$  such that, for every  $s \leq J$  and  $0 \leq i < s$ , all denominators are at least  $1/2$  and  $x_{i+1}(h) + h < R_1/4$  for  $0 \leq h \leq 2r_J$ . These are finitely many continuous inequalities that hold strictly at  $h = 0$ , so such an  $r_J$  exists; it depends only on the data listed in the statement. By coefficientwise composition and the contraction rule of Lemma 5.3, these scalar series majorize the recursion with independent variables  $q_i$ . At a fixed input index, we use the column along its path to the output and the rows on the other branches, and the scalar compositions enumerate the partitions of the other inputs with their Taylor factorials. Thus all joint absolute coefficients are controlled, not only the values of the maps at real arguments.

**Step 2: units and volumes.** The output mesh of step  $i$  is  $M^{i+1-s}$  in units of the coarse lattice spacing. A length  $l$  in the units of step  $i$  therefore becomes  $M^{i+1-s}l$ , and the exponential bound of that step yields a bound at a fixed smaller rate in coarse units. The meshes of all inputs and outputs belong to the finite set  $\{M^{-J}, \dots, 1\}$ . The column factors are the ratios of the corresponding volumes, as in Lemma 5.3. Passing between counting sums and volume-normalized sums introduces at most a factor  $M^{3J}$  for each participating index, hence at most  $M^{3J(k+1)}$  for a coefficient of order  $k$ , which is absorbed by a fixed smaller joint radius. The number of indices of the variables  $q_i$  of the finitely many scales per unit cube is at most  $C_M s M^{3s}$ . In particular, no factor  $M^{d_0}$  and no factor of

the total volume occurs. We keep the composed networks until a single final replacement by trees on the external indices, which gives one fixed rate  $a_J > 0$ .

**Step 3: the domain of the fluctuation variables.** We choose  $u_J$  and  $v_J$  below  $r_J$  and below the last radius reduction. For  $\|\theta\|_\infty < u_J$  and  $\max_i \|q_i\|_\infty \leq tp$ , we enlarge  $C_J$  so that (12.23) implies  $tp < v_J/2$ . By Step 1, all charts then have fixed margins for every  $w \in \Omega_p^{\text{all}}$ , and the recursion is jointly holomorphic in  $(\theta, q)$  on the larger product. Taking maxima over  $1 \leq s \leq J$  fixes all these choices at once. This proves (1).

**Step 4: the Jacobian.** By Lemma 6.3, the scale  $i$  has  $2(M^3 - 1)(mM^{s-i-1})^3 = f_i$  independent  $\text{SU}(2)$  variables, hence  $D_i = 3f_i$  real coordinates. We order the scales as  $s-1, s-2, \dots, 0$ . The field  $c_{i+1}$  depends only on the  $w_l$  with  $l > i$ , so

$$D_{w_i} z_i = g_i S_i(c_{i+1}), \quad D_{w_l} z_i = 0 \quad (l < i).$$

For  $l > i$ , the derivative  $D_{w_l} z_i$  contains both  $Dz_i^*(c_{i+1})D_{w_l} c_{i+1}$  and  $g_i DS_i(c_{i+1})[D_{w_l} c_{i+1}]w_i$ , and we keep these entries. The derivative matrix is block triangular, and its determinant is as stated in (2); for real data, each diagonal determinant is positive. The recursion is one-to-one onto its image, since one recovers  $w_{s-1}$  from  $z_{s-1}$  first and proceeds to finer scales, using the invertibility of each  $S_i$ . Hence it is a change of variables on its image.

**Step 5: the exact integral.** By the nested minimization identity of Theorem 10.10 at scale  $i$ , and since  $g_{i+1}^2 = Mg_i^2$  (including  $g_s = \sqrt{Mt}$ ), we have

$$\frac{E_{d_0+i}(c_i)}{g_i^2} - \frac{E_{d_0+i+1}(c_{i+1})}{g_{i+1}^2} = \frac{F_i(c_{i+1}, z_i) - F_i(c_{i+1}, z_i^*)}{g_i^2} = \frac{1}{2}|w_i|^2 + U_i^{\text{cl}}.$$

Summing over  $i$  proves (3). The product of the Haar disintegration densities and of the Jacobian is

$$\prod_i \mathcal{J}_i(I, 0) g_i^{D_i} \exp \left\{ - \sum_i \left[ \mathcal{L}_i(c_{i+1}, z_i) + \frac{1}{2} \log \det K_i(c_{i+1}) \right] \right\}.$$

For each  $i$  the exponent splits exactly as

$$\begin{aligned} \mathcal{L}_i(c_{i+1}, z_i) + \frac{1}{2} \log \det K_i(c_{i+1}) &= B_i(\widehat{c}_{i+1}) + \mathcal{L}_i(c_{i+1}, z_i) - \mathcal{L}_i(c_{i+1}, z_i^*(c_{i+1})) \\ &\quad + B_i(c_{i+1}) - B_i(\widehat{c}_{i+1}). \end{aligned}$$

We add the terms  $U_i^{\text{cl}}$  from (3). The first terms on the right sum to  $B_{\text{ref}}$ , and all other terms together form  $H_{d_0, s}$ . Integrating the Gaussian factor over each ball contributes  $z_p^{\sum_i f_i}$ , and the remaining expectation is  $\Xi_{d_0, s}$ . This proves (12.25); the off-diagonal derivatives of the triangular change of variables contribute no additional determinant. The fluctuations of the determinants and of the Haar densities at the minimizers of the finer scales are exactly the terms  $B_i(c_{i+1}) - B_i(\widehat{c}_{i+1})$ . For complex coarse fields, the formula again means integration over the same real product of the  $w_i$ .

**Step 6: bounds for the fluctuation interaction.** All coefficients of positive degree of  $B_i$  have factorial bounds with one index fixed; for the determinant we use the series (12.22), and for the negative logarithmic Haar density Lemma 12.2. We compose these scalar series and the other tensors of one step with the maps bounded in Step 1. Fixing a scalar coefficient at any input and propagating the fixed index along columns, we sum all intermediate indices; the bounded density of the indices of the variables  $q_i$  changes only the constants.

By (12.17), a classical term from scale  $i$  has at least three  $w_i$  inputs and the factor  $g_i^{k-2}$  for its degree  $k \geq 3$  in  $w_i$ . Expanding its coarse field  $c_{i+1}$  adds one factor  $g_l w_l$  for each input from a coarser scale  $l > i$ . Since  $0 < g_i \leq t$ , the sum over these degrees and over all coarse-field degrees is bounded, on a smaller fixed coarse-field radius, by  $C_J t p^3 / (1 - C_J t p)$ . By (12.18), a difference of negative logarithmic Haar densities contains at least one factor  $g_i w_i$ . A difference  $B_i(c_{i+1}) - B_i(\hat{c}_{i+1})$  has no term independent of the fluctuations, since at  $q = 0$  both arguments are the zero-fluctuation reconstruction. Each of its terms contains a factor  $g_l w_l$  with  $l > i$ . These two kinds of differences together have the majorant  $C_J t p / (1 - C_J t p)$ . For each of these terms, a fixed index of  $q$  either fixes the outer scalar coefficient or reaches it through the column of an insertion, so these are bounds with one index fixed. Summing over the at most  $J$  scales, we obtain the majorant

$$C_J \frac{t p^3 + t p}{1 - C_J t p}, \quad (12.26)$$

and every monomial of  $H_{d_0,s}$  contains at least one fluctuation variable.

We apply the construction of supports of Lemma 12.9, with (12.19) and (12.20), in units of the coarse lattice spacing, with  $a_J$  in place of  $a$ , and with the density of the indices of  $q$  stated above in the sum over the location of the fixed index. We choose the box scale such that  $8C_0/\mathcal{L} < a_J/4$ , then the rate  $b_1$ , and then we reduce the joint radius to absorb the resulting factor  $D_{\mathcal{L}}^{r+k}$ . The geometric series of Step 4 of that proof now has the majorants (12.26) in place of those of one step. After enlarging  $C_J$ , we obtain an exact decomposition  $H_{d_0,s} = \sum_X H_X$  with norm at most  $\sigma_J$  for the support weight  $e^{8|X|}$ . All the conditions in (12.23) use this final  $C_J$ .

**Step 7: the expansions and the effective interaction.** The connected-component expansion uses only the coefficient bound, the exact supports, and the independence of the fluctuation variables in disjoint supports, which holds for  $\pi_p^{\text{all}}$ . Thus Step 1 of the proof of Theorem 12.7 yields activities with  $\|\{\mathfrak{z}(P)\}\|_{u'_J,5} \leq 2\sigma_J$ , and by Theorem 12.6 we have  $\Xi_{d_0,s} \neq 0$  and the bound with support weight  $e^{2|Y|}$  for the logarithm. The bounded density of the indices also provides a network of length at most  $C_J|Y|$  joining all labels of a support. Then (12.12) implies factorial bounds for the derivatives of  $\log \Xi_{d_0,s}$  on  $\|\theta\|_\infty \leq u'_J/2$ , at one fixed rate  $b_J > 0$ , which we take no larger than  $a_J$ .

The zero-fluctuation reconstruction is the map of Step 1 at  $q = 0$ . We compose its majorant series with the series of the coefficients of positive degree of the  $B_i$ . This bounds the coefficients of positive degree of  $B_{\text{ref}}$ , with one coarse-field index fixed, at the rate  $a_J$  on a fixed product of coarse fields. Translating to background points of a smaller product, we bound every derivative of positive order of  $B_{\text{ref}}$  by  $C_J^{k+1}k!$  there; the value of the extensive trace is not bounded. We choose the radius  $u'_J$  of (5) such that  $\|\theta\|_\infty < u'_J$  lies inside this smaller product, and we restrict the derivative bounds to  $\|\theta\|_\infty \leq u'_J/2$ , where (12.12) applies. Restoring this function and the truncated Gaussian constants in (12.25) proves the formula for  $\mathcal{Q}_{d_0,s}^w$  and its bounds.

**Step 8: symmetries.** For each operation  $\tau$ , the hypothesis of (6) allows us to apply Proposition 12.11 at scale  $i$ , which shows that  $z_i^*(\tau c_{i+1}) = O_i z_i^*(c_{i+1})$ ,  $K_i(\tau c_{i+1}) = O_i K_i O_i^\top$ , and  $S_i(\tau c_{i+1}) = O_i S_i O_i^\top$ , with  $O_i = O_i(c_{i+1})$  independent of  $w_i$ . Proceeding from coarser to finer scales, we obtain  $w'_i = O_i(c_{i+1})w_i$ , which depends on every coarser fluctuation. This map is a bijection of  $\Omega_p^{\text{all}}$ , since one inverts the block of the coarsest scale first and proceeds to finer scales. Its diagonal Jacobian blocks are the  $O_i$ , so its absolute Jacobian is one. Each block preserves its balls and  $\sum_i |w_i|^2$ , and each Haar density and determinant is transported by the identity of one step. Since at zero fluctuations the transformed zero-fluctuation reconstruction is the zero-fluctuation reconstruction of  $\tau c_s$ , the function  $B_{\text{ref}}$  is transported as well. This proves (6). All choices used only the fixed data and finitely many scales, which proves the stated uniformity; nothing is claimed as  $J \rightarrow \infty$ .  $\square$

## 12.5 Perturbation by an input effective interaction

In the recursion of Section 13, the density at the finer scale contains, besides the effective classical action, an input effective interaction  $R$ , produced by the integrations at still finer scales and evaluated at the reconstructed fine field. In this subsection, we fix  $s = J$ , so that the  $J$  averaging steps form one renormalization group step, and we compute the effect of  $R$  on the integral of Theorem 12.13. We set  $\lambda_i = M^{(i-J+1)/2}$ , so that  $g_i = t\lambda_i$ , and let  $\mathcal{C}_{\text{raw}}(\theta, q)$  be the fine field  $c_0$  produced by the recursion of Theorem 12.13 with  $q_i$  in place of  $g_i w_i$ . That is,  $z_i = z_i^*(c_{i+1}) + S_i(c_{i+1})q_i$ , and the matrices  $S_i$  remain in the map. This is the reconstruction before the gauge transformation used in Section 13.

The following heuristic explains why the effect of  $R$ , beyond its value  $R_{\text{ref}}$  on the zero-fluctuation reconstruction, is of second order in the coupling.

*Remark 12.14.* Suppose that  $H = \zeta H_1 + \zeta^2 H_2 + \dots$  and  $A = \zeta A_1 + \zeta^2 A_2 + \dots$  are interactions depending on a small parameter  $\zeta$ , and that  $\mathbb{E}_\pi A_1 = 0$ . Expanding the logarithms to second order gives

$$-\log \mathbb{E}_\pi e^{-H-A} + \log \mathbb{E}_\pi e^{-H} = \zeta^2 \left( \mathbb{E}_\pi A_2 - \text{Cov}_\pi(H_1, A_1) - \frac{1}{2} \text{Var}_\pi(A_1) \right) + O(\zeta^3).$$

For the inserted interaction  $A = R(c_0) - R_{\text{ref}}$ , the first-order term  $A_1$  is linear in the normalized fluctuation variables, and its mean under the symmetric reference measure vanishes. The difference of the logarithms is therefore of order  $\zeta^2$ , and  $\zeta$  will be set equal to  $t$ . Lemma 12.16 proves a uniform version of this statement for the families of Theorem 12.7, using Corollary 12.8 instead of a finite-order expansion.

To apply Corollary 12.8 we vary the size of the fluctuations through a complex parameter  $\zeta$ , which takes the value  $t$  for the integral of Theorem 12.13.

**Definition 12.15.** For a complex parameter  $\zeta$ , substitute  $q_i = \zeta \lambda_i w_i$  in the recursion, and write  $c_i(\zeta)$  and  $z_i(\zeta) = z_i^*(c_{i+1}(\zeta)) + \zeta \lambda_i S_i(c_{i+1}(\zeta)) w_i$  for the resulting fields and coordinates, so that  $c_i(t) = c_i$  and  $c_i(0) = \hat{c}_i$ . The auxiliary fluctuation interaction  $H_{d_0, J}(\theta, w; \zeta)$  is obtained from (12.24) by replacing  $c_{i+1}$  and  $z_i$  by  $c_{i+1}(\zeta)$  and  $z_i(\zeta)$ , and the classical term  $U_i^{\text{cl}}$  by

$$\frac{F_i(c_{i+1}(\zeta), z_i^* + \zeta \lambda_i S_i w_i) - F_i(c_{i+1}(\zeta), z_i^*)}{\zeta^2 \lambda_i^2} - \frac{1}{2} |w_i|^2, \quad (12.27)$$

where  $z_i^*$  and  $S_i$  are evaluated at  $c_{i+1}(\zeta)$ . Thus the explicit inverse square of the coupling varies together with the increments, while the classical functions, the balls, and the cutoff are held fixed. Put

$$\Xi_0(\theta; \zeta) = \mathbb{E}_{\pi_{\text{all}}} e^{-H_{d_0, J}(\theta, w; \zeta)}, \quad (12.28)$$

where  $\log \Xi_0$  is the branch of Theorem 12.7, with  $\log \Xi_0(\theta; 0) = 0$ .

By stationarity and  $S_i^T K_i S_i = I$ , the expression (12.27) equals

$$\sum_{k \geq 3} \frac{\zeta^{k-2} \lambda_i^{k-2}}{k!} D_z^k F_i(c_{i+1}(\zeta), z_i^*) [(S_i w_i)^{\otimes k}],$$

so it extends holomorphically across  $\zeta = 0$ . The differences of negative logarithmic Haar densities and of the functions  $B_i$  in  $H_{d_0, J}(\theta, w; \zeta)$  are evaluated at the same arguments  $c_{i+1}(\zeta)$  and  $z_i(\zeta)$ . At  $\zeta = t$ , the auxiliary interaction is the interaction  $H_{d_0, J}(\theta, w)$  of Theorem 12.13, and  $\Xi_0(\theta; t) = \Xi_{d_0, J}(e^\theta)$ . At  $\zeta = 0$ , every term vanishes, since the classical series starts at  $\zeta^1$ ,  $z_i(0) = z_i^*(\hat{c}_{i+1})$ , and  $c_{i+1}(0) = \hat{c}_{i+1}$ .

**Lemma 12.16.** Fix  $J$  and the larger products and norm conventions of Theorem 12.13 with  $s = J$ , and fix  $u, v, b > 0$ . After fixing these radii and the rate, choose one common box partition using the smaller of the rate available for the baseline and the rate  $b$  of the inserted pullback. Both supported decompositions and all constants below refer to this partition; the baseline bounds are constructed again on it. Let  $R$  be a scalar function of the fine field such that its pullback

$$\hat{R}(\theta, q) = R(\mathcal{C}_{\text{raw}}(\theta, q))$$

is holomorphic on a product  $\{\|\theta\|_\infty < u_2, \max_i \|q_i\|_\infty < v_2\}$  with  $u_2 > u$  and  $v_2 > v$ , contained in the common product of Theorem 12.13(1), and satisfies  $\mathcal{J}_{u,v,b}(\widehat{R}) \leq \epsilon$ . The constant term of  $\widehat{R}$  is unrestricted. Define

$$\begin{aligned} R_{\text{ref}}(\theta) &= \widehat{R}(\theta, 0), & A_R(\theta, w; \zeta) &= \widehat{R}(\theta, (\zeta \lambda_i w_i)_i) - R_{\text{ref}}(\theta), \\ \Xi_{J,R}(\theta; \zeta) &= \mathbb{E}_{\pi_p^{\text{all}}} e^{-H_{d_0,J}(\theta, w; \zeta) - A_R(\theta, w; \zeta)}, & \mathcal{D}_J R(\theta; \zeta) &= -\log \Xi_{J,R}(\theta; \zeta) + \log \Xi_0(\theta; \zeta), \end{aligned}$$

both logarithms being the branches of Theorem 12.7 that vanish at  $\zeta = 0$ . All parameters internal to  $R$ , the balls, and the classical functions are held fixed when  $\zeta$  varies. There exist  $u', b', C_J, c_J > 0$ , depending only on  $M, J, u, v, b$ , the larger margins, and the spatial decay rates, and not on  $d_0, m, t, p$ , or  $\epsilon$ , such that the following holds for every  $p \geq p_0$ , with  $R_\zeta = c_J / (p^3 + \epsilon p)$ .

1. For every complex  $\zeta$  with  $|\zeta| \leq R_\zeta$  and every  $w \in \Omega_p^{\text{all}}$ , the variables  $q_i = \zeta \lambda_i w_i$  satisfy  $\|q_i\|_\infty \leq c_J \leq \min\{v, v_J\}/2$ . Consequently, for  $\|\theta\|_\infty \leq u'$ , the point  $(\theta, q)$  lies in the product on which  $\widehat{R}$  is holomorphic and in the common product of Theorem 12.13(1), and  $H_{d_0,J}(\theta, w; \zeta)$  and  $A_R(\theta, w; \zeta)$  are holomorphic in  $\zeta$  on a neighborhood of the closed disc  $|\zeta| \leq R_\zeta$ .
2. For  $\|\theta\|_\infty \leq u'$ , let  $0 < t \leq c_J / [2(p^3 + \epsilon p)]$ . Then the conditions (12.23) hold, and the integral (12.25) with the additional factor  $e^{-R(c_0)}$  in the integrand equals the integral (12.25) times  $\exp(-R_{\text{ref}}(\theta) - \mathcal{D}_J R(\theta; t))$ .
3. Let  $\mathcal{D}(\zeta) = \mathcal{L}(\mathbf{H}(\zeta) + \mathbf{A}_R(\zeta)) - \mathcal{L}(\mathbf{H}(\zeta))$  be the difference of the families of Theorem 12.7, at the coefficient radius  $u'$ , for the supported decompositions  $\mathbf{H}(\zeta)$  and  $\mathbf{A}_R(\zeta)$  constructed in the proof; its sum over supports is  $\mathcal{D}_J R(\theta; \zeta)$ . Its constant and linear coefficients in  $\zeta$  vanish, support by support. For  $0 < t \leq c_J / [2(p^3 + \epsilon p)]$ ,

$$\|\mathcal{D}(t)\|_{\mathcal{S}_2} \leq C_J t^2 \epsilon (p^4 + \epsilon p^2),$$

and, on  $\|\theta\|_\infty \leq u'$ , for every  $k \geq 1$ ,

$$\|D_\theta^k \mathcal{D}_J R(\cdot; t)\|_{b', \text{root}} \leq C_J^{k+1} k! t^2 \epsilon (p^4 + \epsilon p^2).$$

The hypothesis concerns  $\widehat{R}$  on a product containing all points described in (1). A bound for  $R$  on a smaller set of fine fields, which misses part of the range of the reconstruction over this product, does not suffice. The hypothesis is verified for the effective interactions of the recursion in Theorem 13.5.

*Proof.* We use Definition 12.15 throughout. Since  $H_{d_0,J}(\theta, w; 0) = 0$  and  $A_R(\theta, w; 0) = 0$  for every coarse field, both partition functions equal one at  $\zeta = 0$ .

**Step 1: supported decompositions.** We expand  $\widehat{R}$  in the variables  $\theta$  and  $q$ . Subtracting  $\widehat{R}(\theta, 0)$  removes exactly the terms of degree zero in  $q$ . Every remaining monomial therefore contains a variable  $q_i$ , and its scalar counting coefficient sum with that index

fixed, without a mesh-volume factor, is controlled by  $\mathcal{J}_{u,v,b}(\widehat{R}) \leq \epsilon$ . The support-weight estimate (12.20), in the form used in Step 6 of the proof of Theorem 12.13, yields the bound  $CD^{r+k}e^{bt/2}e^{-b_1 \text{dist}(x,e)}$  for the weight  $e^{8|X|}$  of a monomial of degree  $r$  in  $\theta$ , degree  $k$  in  $q$ , and fixed location  $e$  of an index of  $q$ . The sum over  $e$  contributes only the density of these indices. We choose  $u_1 < u/(2D)$  and a radius  $v_1 < v/(2D)$  for the variables  $q$ , and we reduce them further to absorb the fixed color factors and the choice of the fixed index of  $q$ , which contributes at most a factor  $k \leq 2^k$ . A term of  $\mathcal{J}_{u,v,b}(\widehat{R})$  is then multiplied by at most  $(Du_1/u)^r(C_J|\zeta|p/v)^k$  with  $k \geq 1$ . Summing, we obtain an exact supported decomposition  $A_R(\zeta)$  of  $A_R$  with

$$\|A_R(\zeta)\|_{\mathcal{S}_8} \leq \frac{C_J \epsilon |\zeta| p}{1 - C_J |\zeta| p}, \quad (12.29)$$

an absolute coefficient bound at the coarse-field radius  $u_1$ . We use one box scale, chosen with the smaller of the two available rates, for  $H$  and  $A_R$ . We reduce  $u_1$  also to the baseline and reference coefficient radius for this common partition, so that  $u_1 \leq u'_J \leq u_J$ ; this preserves (12.29). Repeating the calculation of Step 6 in the proof of Theorem 12.13, with  $\zeta$  in place of  $t$ , we obtain

$$\|H(\zeta)\|_{\mathcal{S}_8} \leq \frac{C_J |\zeta| (p^3 + p)}{1 - C_J |\zeta| p}. \quad (12.30)$$

Indeed, the classical terms have degree at least three in the fluctuation variables of their own scale, with the factor  $\zeta^{k-2}$ . The differences of negative logarithmic Haar densities and of the functions  $B_i$  have degree at least one in the fluctuation variables. Joint composition and the support-weight estimate then give exactly these two geometric sums. The assignment of supports does not depend on  $\zeta$  or on the interpolation parameter  $s$  below.

**Step 2: the auxiliary disc.** Let  $R_\zeta = c_J/(p^3 + \epsilon p)$ . Since  $p \geq 1$ , we have  $R_\zeta p \leq c_J$ ,  $R_\zeta(p^3 + p) \leq 2c_J$ , and  $R_\zeta \epsilon p \leq c_J$ . We choose  $c_J$  so small that  $C_J c_J \leq 1/2$ , that  $6C_J c_J \leq \eta_0/2$ , that  $c_J \leq \min\{v_1, v_J\}/2$ , and that  $4C'_J c_J \leq \sigma_{\text{seq}}$ , where  $\eta_0$  is the constant of Theorem 12.7 and  $C'_J$ ,  $\sigma_{\text{seq}}$  are the constants in (12.23). For  $t \leq R_\zeta/2$ , we have  $tp \leq tp^3 \leq c_J/2$ , hence  $C'_J tp < 1/2$  and  $C'_J(tp^3 + tp)/(1 - C'_J tp) \leq 2C'_J c_J \leq \sigma_{\text{seq}}$ , which are the conditions (12.23). Then (1) holds, the denominators in (12.29) and (12.30) are at least  $1/2$  on a neighborhood of the closed disc  $|\zeta| \leq R_\zeta$ , and

$$\|H(\zeta) + sA_R(\zeta)\|_{\mathcal{S}_8} \leq 4C_J c_J + 2C_J c_J \leq \eta_0/2 \quad (0 \leq s \leq 1).$$

**Step 3: cancellation of the first order.** By Theorem 12.7 and (12.29), the derivative family  $D\mathcal{L}(H + sA_R)[A_R]$  has norm at most  $C_J \epsilon p |\zeta|$  on the disc. The coefficient of  $\zeta$  in  $A_R$  is  $\sum_i \lambda_i D_{q_i} \widehat{R}(\theta, 0)[w_i]$ . Each monomial assigned to a support is linear in one fluctuation

block, whose mean under the symmetric reference measure is zero. Thus its expectation vanishes before any sum over supports or coarse-field coefficients, and the hypothesis of Corollary 12.8 holds. Applying that corollary with  $R = R_\zeta$ , and integrating over  $s \in [0, 1]$ , we obtain for the difference  $\mathcal{D}(\zeta)$  the bounds

$$\|\mathcal{D}(\zeta)\|_{\mathcal{S}_2} \leq C_J \epsilon p |\zeta|, \quad \|\mathcal{D}(t)\|_{\mathcal{S}_2} \leq C_J \epsilon p \frac{t^2}{R_\zeta} = C_J t^2 \epsilon (p^4 + \epsilon p^2), \quad (12.31)$$

the second for  $t \leq R_\zeta/2$ . If  $\epsilon = 0$ , then  $\widehat{R}$  is constant,  $A_R = 0$ , and the difference vanishes identically. Both terms  $t^2 \epsilon p^4$  and  $t^2 \epsilon^2 p^2$  in the bound are needed.

**Step 4: derivatives.** With  $u' = u_1/2$ , the bound (12.12) bounds the sum of the derivatives of order  $k$  of each support term by  $2^{k+1} k! u_1^{-k}$  times its coefficient norm. All coarse-field labels in a support  $Y$  can be joined by a network of length at most  $C_{\text{geom}, J} |Y|$ , and we choose  $b'$  with  $b' C_{\text{geom}, J} \leq 1$ . Fixing a derivative index restricts the sum to the supports containing its box, and the factor  $e^{2|Y|}$  dominates the network weight. This proves the bound for the derivatives, with a single factorial. Reducing the coefficient radius from  $u_1$  to  $u'$  preserves the bound for  $\mathcal{D}(t)$ .

**Step 5: proof of (2).** At  $\zeta = t$ , we have  $q_i = g_i w_i$ , so  $R(c_0) = \widehat{R}(\theta, q) = R_{\text{ref}}(\theta) + A_R(\theta, w; t)$ . By Theorem 12.13, the integral with the factor  $e^{-R(c_0)}$ , divided by the integral without it, equals  $e^{-R_{\text{ref}}} \Xi_{J,R}(\theta; t) / \Xi_0(\theta; t)$ , since the two integrals have the same partition function without insertion and the same classical, Haar density, determinant, and Gaussian prefactors. The negative logarithm of this ratio is  $R_{\text{ref}} + \mathcal{D}_J R(\theta; t)$ , which proves (2). The radii and the constants  $C_J, c_J$  were chosen before  $p, \epsilon$ , and  $t$ , which enter only through the conditions  $p \geq p_0$  and  $t \leq c_J / [2(p^3 + \epsilon p)]$ .  $\square$

*Remark 12.17.* The constants of this section are chosen in the following order. We first fix  $M$  and the larger domains of the classical and Haar disintegration charts. Next, we choose the radii and rates of one averaging step, in Lemmas 12.2 and 12.3; these do not depend on  $J$ . Then we fix  $J$ . For this  $J$ , we choose the domain  $r_J$  of the compositions in Theorem 12.13, the box scale  $\mathcal{L}$ , and the rate  $b_1$ , we reduce the coefficient radius, and we fix  $C_J, \sigma_{\text{seq}}$ , and the thresholds in (12.23). Finally,  $d_0, m, t$ , and  $p$  vary subject to the stated inequalities. In Lemma 12.16, the numbers  $u, v, b$  and the larger product of holomorphy are fixed before the common support partition and its constants  $C_J$  and  $c_J$ , and  $\epsilon, p$ , and  $t$  enter only through the final smallness conditions. This section does not close the recursion; Section 13 shows that the hypothesis of Lemma 12.16 is preserved from one renormalization group step to the next, at the same radii.

## Conditional measures and analytic estimates

In this section, we set up the renormalization group recursion for the Wilson measure restricted to the small-field event. We prove that the effective interactions stay in the same ball of an analytic coefficient norm at every scale, and we prove estimates for the conditional law of the normalized fluctuation variables given the coarse field. One renormalization group step consists of  $J$  consecutive applications of the block averaging map  $T_M$ , from a lattice of side  $mM^J$  to a coarse lattice of side  $m \geq 4$ . Its fluctuations are integrated with the results of Section 12. The probability of the restriction event under the Wilson measure and the continuum limit are treated in later sections.

**The exact formula for one step.** The central identity is Theorem 13.4. Let the density at the fine scale be  $\mathbf{1}_{D_r} e^{-E_{d_0, \text{sat}}/g_0^2 - R_{\text{sat}}}$  with respect to the product Haar measure, where  $E_{d_0}$  is the effective classical action of depth  $d_0$  and  $R$  is the input effective interaction, produced by the integrations at finer scales. Restricting to the event on which the step is defined and the fluctuations are cut off, and pushing forward by  $T_M^J$ , we obtain

$$\mathbf{1}_{D_r}(c) A_{\text{coord}} e^{-E_{d_0+J, \text{sat}}(c)/g_J^2 - B_{\text{ref}, \text{sat}}(c) - R_{\text{ref}, \text{sat}}(c)} \left( \int_{\Omega_p} e^{-V_c(w)} dw \right) dH(c). \quad (13.1)$$

Here  $\Omega_p = \prod_{i,e} \{w_{i,e} \in \mathbb{R}^3 : |w_{i,e}| \leq p\}$ , where  $e$  runs over the  $f_i$  normalized fluctuation variables of the intermediate scale  $i$ , each with three components, and  $dH$  is the product Haar measure on the coarse configuration. The factors are defined precisely in Theorem 13.4. Besides the effective classical action  $E_{d_0+J}$  at the coarse scale, the formula involves the reconstructed fine field, written through its logarithm  $\varphi_\theta(w)$  in the gauge chosen in Theorem 13.2. The values at the identity of the Haar disintegration densities and Jacobians are collected in the constant  $A_{\text{coord}}$ . The terms  $B_{\text{ref}}$  and  $R_{\text{ref}}$  are the determinant and Haar density terms and the input effective interaction, respectively, evaluated on the zero-fluctuation reconstruction. The conditional fluctuation potential  $V_c$  contains everything that depends on the fluctuations, and dividing by the integral gives the conditional fluctuation kernel  $k(c, dw)$  of Theorem 13.8. The formula is first proved in local gauge charts and then glued across charts by exact, finite gauge invariance. Iterating it identifies the measures of the recursion with the conditioned Wilson measure, pushed forward to the coarse scales.

**Main results.** In Theorem 13.2, we construct a gauge representative of the reconstruction, depending only on the coarse field, on the whole domain of coarse fields and fluctuations. Theorem 13.4 proves (13.1) and identifies the iterated measures with the conditioned Wilson measure. Theorem 13.5 proves that one step maps the effective interaction into a fixed ball of the norm  $\mathcal{B}_{\rho,a}$ , at the same radius and rate. The new effective interaction is the sum of a newly generated effective interaction, a contracting pullback of the input effective interaction at zero fluctuation, and a fluctuation correction to the pullback, of second order in the coupling. Theorem 13.8 proves convexity and concentration for the conditional fluctuation kernel, and Theorem 13.10 estimates its means, its covariance, and the coefficients of the reconstruction at its mean.

The inputs are the concentration theorem on convex products of balls and the Haar measure identities of Theorem 5.4 and Lemma 5.1, the constrained minimizers and their coefficient bounds of Theorems 10.10 and 10.11, and the gauge invariance of the effective classical action, the contraction estimate, and the background field of Theorems 11.6 and 11.10 and Lemma 11.9. We also use the coordinates, exact integrals, and cluster expansions of Lemmas 12.3 and 12.16 and Theorems 12.6, 12.7, and 12.13. The results are used in Sections 14 and 15 to construct the small-field restriction and the backward reconstruction, and in Sections 16 and 17 for conditional estimates of the reconstruction. They are also used in Sections 19, 20, and 23, and, through the conditional law and its moments, in Sections 26 to 28, 31, and 35.

**Domains.** Several neighborhoods of the identity appear, with different roles.

| Set                    | Role                                                                                                                                                            |
|------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $B_r$                  | coarse fields with link logarithms of norm less than $r$ ; here the integral is identified with the restricted Wilson measure                                   |
| $B_\rho$               | a larger set of coarse fields, $\rho \geq r$ , on which the auxiliary integral and the analytic recursion are defined                                           |
| $B_{\rho/2}$           | contains the gauge-transformed reconstruction for all coarse fields and fluctuations in the enlarged product; the input effective interaction is evaluated here |
| $D_r = \mathcal{G}B_r$ | the gauge-invariant small-field domain; the reconstructed fine field must lie in it                                                                             |

The gauge-transformed reconstruction lies in  $B_{\rho/2}$  by the coefficient bounds of Theorem 13.2. That the reconstructed fine field lies in  $D_r$ , for every coarse field in  $B_r$  and every fluctuation in  $\Omega_p$ , is a separate hypothesis, the support-inclusion hypothesis of Definition 13.3. A sufficient condition for it is  $\mathfrak{M}_J(r, tp) < r/2$ . Since the part of  $\mathfrak{M}_J$  of positive degree in  $q$  does not vanish as  $r \rightarrow 0$ , this condition may require a further reduction of the coupling for each fixed  $r$ .

**Order of choices.** The parameters are chosen in the order

$$M, \text{ initial domains and decay rates } \longrightarrow a, b, \mathcal{L}, K_*, \rho_0 \longrightarrow J \longrightarrow \rho, s_J \longrightarrow E_*, A_w, t_*.$$

The rate  $a$  is taken below  $a_{\text{cen}}/8$ , where  $a_{\text{cen}}$  is the rate of the bounds in Lemma 11.9, and  $b > a$  is taken below the rate  $b_*$  of Theorem 13.2. No constant depending on  $J$  is used to choose  $J$ . The support radius  $r \leq \rho$  and its support-inclusion hypothesis are treated separately.

**Conventions.** All distances within a renormalization group step are measured in units of the coarse lattice spacing. The input mesh is  $\delta = M^{-J}$ , the normalized fluctuation variables  $w_i$  and the variables  $q_i = g_i w_i$  of the intermediate scale  $i \in \{0, \dots, J-1\}$  have mesh  $M^{i-J}$ , and the couplings are  $g_i = t \lambda_i$  with  $\lambda_i = M^{(i-J+1)/2} \in [M^{-(J-1)/2}, 1]$ ; also  $g_J^2 = M g_{J-1}^2$ . The number of scalar indices of  $\theta$  and of  $w$  located in a unit cell is at most  $D_J \leq C_M(1+J)M^{3J}$ ; this is a density bound, not a bound on the number of variables. Unless a bound is stated to hold before  $J$  is chosen, a constant in this section may depend on  $M, J$ , the larger construction domains and their margins, the spatial decay rates,  $\rho, a$ , and the bound  $E_*$  on the effective interaction. Such a constant does not depend on the depth  $d_0$ , the side  $m, L$ , or  $\gamma$ . All results concern the whole periodic torus, not regions with fixed exterior data.

We use ordered scalar tensors with their Taylor factorials divided out; for example, a symmetric scalar tensor  $A$  of degree  $k$  contributes  $A[\theta^{\otimes k}]/k!$ . Fixing one input index and summing over the remaining indices defines the norm of a scalar tensor with one index fixed. For a vector tensor one may instead fix its output, or fix any input and sum over the output with its volume normalization, as in Definition 4.6. Every such sum carries the weight  $e^{bt}$ , where  $t$  is the length of a shortest connected network joining the locations of the indices, with internal vertices allowed. At each degree we take the maximum over the positions of the fixed index. Additive constants are excluded from these norms. As in Section 12, a majorant series is a power series with nonnegative coefficients that dominates the corresponding sums of absolute coefficients.

**Attribution.** The construction follows Balaban's renormalization group method for lattice gauge theories. The logarithmic averaging with reference transports adapts his averaging operations [12]; the use of small gauge representatives belongs to the program of [14]; and the constrained minimizer as a background field is the construction of [16]. The exponential decay of the propagators, used through Sections 10 and 11, adapts [10, 11, 13]. The iteration of effective densities with Gaussian-normalized fluctuations, separated normalization constants, and local effective interactions follows [15, 17, 19]; the four-dimensional coupling renormalization of [17] is not used. None of these works proves the identities for the centered block averaging map and the full periodic reconstruction used here, and we prove them below. The polymer expansions are those of Section 12, which

uses the criterion of Kotecký and Preiss [120] and the finite graph estimate recorded by Dimock [68, Appendix B, Theorem 27, p. 39, and (323), p. 42]. The variance, entropy, and covariance estimates for the conditional law come from Theorem 5.4, which belongs to the framework of Brascamp and Lieb [33], Gross [103], Bakry and Émery [8], and Helffer and Sjöstrand [109, 110]; here we verify its hypotheses for the conditional fluctuation potential.

### 13.1 Weighted coefficient bounds and majorant series

We first describe how the coefficient norms behave under the operations used in this section. Connected networks control the spatial weights; cancellation of mesh-volume factors also requires compatible directed substitutions or common-output products.

**Lemma 13.1.** *Suppose that analytic tensor families have convergent majorant series, at one spatial decay rate  $b$ , for all their output-fixed rows and all their input-fixed volume-normalized columns. Directed vector substitutions and pointwise products at a common output are bounded by the corresponding products and compositions of the majorant series, with their compatible mesh normalizations. For a general connected contraction diagram, use Lemma 5.3(1)–(3): at each attachment through an input port  $j$  of a vector factor with output  $o$ , retain the factor  $v_j/v_o$ , and at an external input  $l$  retain the final normalization  $v_{\text{out}}/v_l$ . These ratios are included in the diagram majorant; connectedness alone does not cancel them. Products of extensive scalar interactions without such a connection are excluded: they require an additional connecting kernel with its own bounds, or separate bounds without a fixed index. To bound a composition with a specified input fixed, we use the derivative of the scalar composition of majorants with respect to the input variable,  $(\widehat{F} \circ \widehat{G})' = (\widehat{F}' \circ \widehat{G}) \widehat{G}'$ , which accounts for the choice of the factor containing that input and of its position there. The volume normalization at the fixed input is that of its own lattice, and no second derivative factorial occurs. On a fixed smaller analytic product, evaluation and any fixed number of derivatives satisfy the same bounds.*

Let  $T(o; i_1, \dots, i_k)$  be a tensor of fixed Taylor degree whose output-fixed rows are bounded by  $A$  at the rate  $2b$ , and let  $v_o$  and  $v_i$  denote the volumes of the meshes of the output and of an input. Then, for every fixed input  $i$ ,

$$\sum_{o, \text{ other inputs}} \frac{v_o}{v_i} e^{bt} |T(o; i, \dots)| \leq \frac{A}{v_i} \sum_o v_o e^{-b \text{dist}(o, i)}. \quad (13.2)$$

If  $i$  is an input of the coarse lattice, then  $v_i = 1$ , and if the outputs have spacing  $\eta \leq 1$  and bounded density, then  $\eta^3 \sum_o e^{-b \text{dist}(o, i)} \leq C_b$ ; thus summing over the output converts the row bound into a column bound at the rate  $b$ , with a constant depending only on  $b$  and the density of the labels. An input of mesh  $\eta_{\text{in}} < 1$  keeps the factor  $\eta_{\text{in}}^{-3}$ , which within one renormalization group step is at most  $M^3$ . Conversions between counting sums at the finitely many meshes of one renormalization group step cost at most a fixed factor depending on  $J$ .

Finally, for an analytic vector map  $F_e(u) = \sum_v f_{e;v} u^v$  and a radius  $r > 0$ , define the entrywise envelope of its derivative

$$\mathcal{C}_{ex}(r) = \sum_{v: v_x > 0} v_x |f_{e;v}| r^{|v|-1}. \quad (13.3)$$

Then  $\sup_{\|u\|_\infty \leq r} |\partial_x F_e(u)| \leq \mathcal{C}_{ex}(r)$ , and the sums of  $\mathcal{C}_{ex}(r)$  over  $x$  at fixed  $e$ , and over  $e$  at fixed  $x$  with the volume factor  $v_e/v_x$ , both with the weight  $e^{bt}$  of the corresponding coefficients, are bounded by the derivatives at  $r$  of the majorant series of the output-fixed rows and of the input-fixed columns at  $x$ , respectively. This is an entrywise bound uniform on the product, not a bound on an evaluated operator norm.

*Proof.* We first consider finite Taylor polynomials. For a pointwise product at a common output, we join the networks of the factors at that output; for a contraction we join them at the contracted index. Repeating these joins along the assumed connected structure, we obtain a connected union joining all uncontracted locations. Its length bounds the external  $t$ , which implies that the external exponential weight is at most the product of the weights of the factors. We fix the external index and choose a spanning tree of the graph of factors, rooted at the factor containing that index. Summation of the other indices is bounded by Lemma 5.3(2), retaining its attachment ratios and final external normalization. For directed substitutions those ratios telescope, and common-output products use compatible ports of the same kind, giving the factor-free product bound. For other connected diagrams the ratios remain in the bound and in its scalar majorant. The argument gives no such bound for two scalar factors with disjoint networks and independent summed indices.

For a composition, we partition the ordered derivative indices among the inner factors. If the parts have sizes  $k_1, \dots, k_l$ , there are  $k!/(k_1! \cdots k_l!)$  assignments, and division by  $k!$  leaves the ordinary coefficients of the composition of exponential generating series. The derivative of the scalar composition bounds every coefficient series with a fixed input, including the columns, since it counts the choice of the inner factor and of the position of the fixed input in it. All bounds concern nonnegative sums, and passing to the limit of finite Taylor truncations gives the statements for the convergent series.

For the radius reduction, if  $A_k \geq 0$  and  $\sum_k A_k R^k < \infty$ , then for  $r < R$  the series obtained by  $l$  derivatives satisfies

$$\sum_{k \geq l} (k)_l A_k r^{k-l} \leq R^{-l} \sup_{k \geq l} (k)_l (r/R)^{k-l} \sum_k A_k R^k < \infty,$$

where  $(k)_l = k(k-1) \cdots (k-l+1)$ . This applies only to series that are already absolute coefficient series; it does not infer absolute coefficients from a bound on directional values.

To prove (13.2), note that  $t \geq \text{dist}(o, i)$  implies  $e^{bt} \leq e^{2bt} e^{-b \text{dist}(o, i)}$ . Applying the row bound separately at each output  $o$ , we obtain the inequality. For an input  $i$  of the coarse lattice, uniformly on periodic tori,

$$\eta^3 \sum_o e^{-b \text{dist}(o, i)} \leq C_b.$$

Indeed, the set of  $o$  with  $k \leq \text{dist}(o, i) < k + 1$  has counting volume, multiplied by  $\eta^3$ , at most  $C(k + 2)^3$ , and  $\sum_{k \geq 0} C(k + 2)^3 e^{-bk}$  converges; a sum over the torus is bounded by the corresponding periodized sum over the infinite lattice. This proves the column bound. An input of mesh  $\eta_{\text{in}} < 1$  keeps its factor  $\eta_{\text{in}}^{-3}$ , and replacing volume-normalized sums by counting sums at any of the fixed meshes changes a measure by at most  $M^{3J}$  for each conversion.

For the envelope,  $|u^{v-e_x}| \leq r^{|v|-1}$  on the product, and the pointwise bound follows. Summing  $\mathcal{C}_{ex}(r)$  over  $x$  at fixed  $e$ , we obtain  $\sum_v |v| |f_{e;v}| r^{|v|-1}$  with the weights, which is the derivative at  $r$  of the majorant for the output-fixed rows. At a fixed input  $x$ , for  $|v| = k$  and  $v_x > 0$ , the remaining  $k - 1$  derivative indices can be ordered in  $(k - 1)! v_x / \prod_j v_j!$  ways, and each ordered derivative of  $f_{e;v} u^v$  at zero has modulus  $|f_{e;v}| \prod_j v_j!$ ; dividing by  $k!$  gives  $(v_x/k) |f_{e;v}|$  (see also (13.11) below). This shows that the column majorant at  $x$  has the coefficients  $c_k = \sum_e \sum_{|v|=k} (v_x/k) |f_{e;v}| (v_e/v_x) e^{bt}$ , and the weighted sum of  $\mathcal{C}_{ex}(r)$  over  $e$  equals  $\sum_k k c_k r^{k-1}$ , the derivative of this majorant at  $r$ .  $\square$

## 13.2 Gauge representatives on the reconstruction domain

The reconstruction  $\mathcal{C}_{\text{raw}d_0,J}(\theta, q)$  of Section 12.5 expresses the fine field in the block axial gauges of the intermediate scales. Even at zero fluctuation it need not lie in a small neighborhood of the identity in which the input effective interaction is defined. We now apply a gauge transformation, depending only on the coarse field, after which the reconstruction of the whole product of coarse fields and fluctuations lies in such a neighborhood. Here  $\Omega(\theta)$  denotes a gauge transformation; it is unrelated to the product of balls  $\Omega_p$ .

**Comparison products.** Fix a comparison radius  $0 < r_{\text{cmp}} < \rho_{\text{ov}}$  inside the common classical and Haar domains, where  $\rho_{\text{ov}}$  is from Theorem 11.6. At each fixed  $J$ , comparison products mean products on which every intermediate coarse target, both before and after the exact transitions between gauge-related charts, lies in  $B_{r_{\text{cmp}}}$  with a common positive margin. We also require common positive margins in each intermediate free-coordinate, constraint, and Haar chart. In a comparison, both coarse representatives are independently required to satisfy  $\|\theta\|_\infty, \|\theta'\|_\infty \leq u$ ; smallness of the transformed representative is not inferred from the first one. Such products exist uniformly in  $d_0$  and  $m$ : the intermediate reconstruction maps, expressed in their own coarse representative and

free inputs, have common absolute coefficient majorants and vanish at zero input. Real transitions are orthogonal block permutations of the free variables, so both reconstructions obey the same majorant when their own coarse inputs are bounded by  $u$ . Shrinking  $u$  and the free-input radius bounds every original and transformed intermediate target strictly below  $r_{\text{cmp}}$ ; the common holomorphic neighborhoods and transition bounds then keep half these margins. Only fixed- $J$  constants enter this construction. These comparison margins are distinct from the final image bound  $B_{\rho/2}$ .

**Theorem 13.2.** *Fix an odd integer  $M \geq 3$ . There exist a spatial decay rate  $b_* > 0$  and a majorant  $Z(u) \leq Cu/(1 - Cu)$ , both chosen before  $J$ , with the following property. Let  $J \geq 1$ ,  $d_0 \geq 0$ , and  $m \geq 4$ , and put  $\delta = M^{-J}$ . There exist  $r_J, C_J > 0$ , independent of  $d_0$ ,  $m$ ,  $L$ , and  $\gamma$ , and a gauge transformation  $\Omega_{d_0,J}(\theta)$  of the fine lattice, holomorphic in  $\theta$ , with values in  $\text{SU}(2)$  for real  $\theta$ , equal to the identity at every vertex of the coarse lattice, and with  $\Omega_{d_0,J}(0) = I$ , such that*

$$\Phi_{d_0,J}(\theta, q) = \text{Log}[\mathcal{C}_{\text{raw}_{d_0,J}}(\theta, q)^{\Omega_{d_0,J}(\theta)}], \quad T_M^J e^{\Phi_{d_0,J}(\theta, q)} = e^\theta, \quad \Phi_{d_0,J}(\theta, 0) = \delta \bar{A}_{d_0,J}(\theta),$$

where  $e^{\delta \bar{A}_{d_0,J}(\theta)}$  is the background field of Lemma 11.9. Here  $q_i$  replaces  $g_i w_i$ , and the matrices  $K_i^{-1/2}$  remain inside the recursion. Moreover the following statements hold.

1. On a fixed larger product with  $u + v < r_J$ , the map  $\Phi_{d_0,J}$  has the majorant series

$$\mathfrak{M}_J(u, v) = \delta Z(u) + \frac{C_J v}{1 - C_J(u + v)}$$

for its output-fixed rows, also when the three output colors at a link are grouped and their coefficient vector is measured in the Euclidean norm. For the input-fixed columns, with the volume normalization of the fixed input, let  $c_\theta(u, v)$  and  $c_q(u, v)$  be the majorant series at a fixed coarse-field input and at a fixed input of  $q$ , in which every input, including the fixed one, is weighted by its radius ( $u$  for coarse-field inputs and  $v$  for inputs of  $q$ ). Then  $c_\theta \leq u \partial_u \widetilde{\mathfrak{M}}_J$  and  $c_q \leq v \partial_v \widetilde{\mathfrak{M}}_J$  for a majorant  $\widetilde{\mathfrak{M}}_J(u, v) = \delta \widetilde{Z}(u) + C_J v / (1 - C_J(u + v))$  of the same form, where  $\widetilde{Z}$  has the properties of  $Z$ ; in particular  $c_q(u, 0) = 0$ . Thus the column bounds are the derivatives of majorants of the same form with respect to the variable of the fixed input, multiplied by its radius. The gauge transformation and its inverse have factorial coefficient bounds at every fixed index on the same larger product of coarse fields. After one fixed choice of radius,  $\Omega$  has the logarithm  $\kappa_\Omega$  in the identity chart, with  $\kappa_\Omega(0) = 0$ ,  $\|\kappa_\Omega(\theta)\|_\infty \leq C_J \|\theta\|_\infty$ , coefficient bounds of the same type, and  $\kappa_\Omega = 0$  at every vertex of the coarse lattice. A positive spatial decay rate and the constants in the bounds for the first three derivatives in  $\theta$  at zero fluctuation can be chosen before  $J$ ; the joint constants and  $r_J$  may depend on  $J$ , the larger construction domains, and the decay rates.

2. For every constant  $u \in \text{SU}(2)$ ,  $\Omega(\text{Ad}_u \theta) = u \Omega(\theta) u^{-1}$  and  $\Phi(\text{Ad}_u \theta, \text{Ad}_u q) = \text{Ad}_u \Phi(\theta, q)$ , on real data and for the holomorphic continuations on the domains above.

3. For every  $K_* \geq 2$ , one can choose  $J$  using only the constants chosen before  $J$ , and then  $\rho, s_J > 0$  such that

$$4K_*\rho < r_J, \quad 4K_*C_J(\rho + s_J) < 1, \quad \mathfrak{M}_J(4K_*\rho, 4K_*s_J) < \rho/2.$$

Then the image of the whole complex product with these radii consists of fields with link logarithms of norm less than  $\rho/2$ , and all intermediate classical and Haar disintegration maps, including the original and transitioned intermediate targets in the comparison products just defined, have common positive margins. For real coarse fields with  $\|\theta\|_\infty \leq r \leq \rho$  and a radius  $Q$  for the variables  $q$  such that  $r + Q < r_J$ ,  $1 - C_J(r + Q) > 0$ , all intermediate margins hold, and  $\mathfrak{M}_J(r, Q) < r/2$ , the image of the entire real product lies in  $B_{r/2}$ . These are statements about the whole product, not only about its center.

4. Suppose that two coarse fields related by a periodic gauge transformation have reconstructions  $C' = \tilde{C}^k$ , related by the exact triangular transition of the fluctuation variables. Then

$$(C')^{\Omega(\theta')} = (C^{\Omega(\theta)})^{\Omega(\theta')\tilde{k}\Omega(\theta)^{-1}}.$$

The gauge transformation changes only the representative at which the input effective interaction is evaluated; it does not change the fluctuation variables, the cutoff domain, the Haar disintegration densities, or the triangular determinant.

*Proof.* We first fix domains on which the maps of each averaging step have coefficient bounds, then construct the gauge transformation from the background field of Lemma 11.9, compose the steps, and finally choose the radii.

**Step 1: the maps of one averaging step.** At the intermediate scale  $i$ ,  $0 \leq i < J$ , the depth is  $d_0 + i$  and the sides are  $mM^{J-i} \rightarrow mM^{J-i-1}$ , with output side at least four. Thus Theorem 10.10 applies at every scale with the same constants  $m_0, C_0, r_c, r_z$ . Statements about order, minima, and positive Hessians concern only real coarse fields and real free logarithmic coordinates. For complex data, we use the holomorphic minimizer and the analytic continuation of the nested minimization identity, without any order statement. The free logarithmic coordinates are the logarithms in the block axial gauge, and

$$m_0I \leq K_i(I) \leq C_0I, \quad \|K_i(I)\|_{\text{row+col}} + \|K_i(I)^{-1}\|_{\text{row+col}} \leq C_0.$$

This concerns the free logarithmic coordinates only; no lower bound in the directions of coarse gauge transformations or of constant harmonic one-forms is used.

We choose smaller products of coarse fields and free logarithmic coordinates, with a common initial radius  $R_{\text{prim}}$  independent of  $d_0$  and  $m$ , strictly inside the domains of Theorem 10.10 and the larger classical and Haar disintegration domains of Lemma 12.3. Part (3) of that lemma applies on these products without a cutoff or a coupling. It provides absolute coefficient bounds at every fixed index for the logarithmic reconstruction  $B_i$ ,

the objective  $F_i$ , the logarithmic Haar density, the minimizers  $z_i^*$ , and  $K_i^{\pm 1/2}$ . For every vector tensor these bounds include all its volume-normalized columns, and a coefficient of positive degree of  $K_i^{\pm 1/2}$  may instead be bounded with a coarse-field index fixed and both matrix indices summed. The scalar tensors have one input fixed and exclude values. The actions in these objectives are the  $E_{d_0+i}$  of Theorem 10.11, which applies at depth  $d_0 + i$  and side  $mM^{J-i} \geq 4$ ; it provides factorial bounds for their derivatives at one rate, and  $DE_{d_0+i}(I) = 0$ . We take a positive rate below the rates of these results. This rate is valid at every scale, since each mesh is at most one in units of the coarse lattice spacing. All later reductions of these products are fixed reductions inside these domains.

**Step 2: the zero-fluctuation reconstruction and the gauge.** By Lemma 11.9, the background field  $C(\theta) = C_{d_0,J}(\theta) = e^{\delta \bar{A}_{d_0,J}(\theta)}$  satisfies  $T_M^J C(\theta) = e^\theta$ , and the output rows of  $(\bar{A}_{d_0,J}, \nabla_\delta \bar{A}_{d_0,J})$  have the majorant  $Z(u) = Cu/(1 - Cu)$ , uniformly before  $J$ . Each column at a coarse input, with output volume  $\delta^3$ , is bounded by the corresponding derivative of  $Z$ . By the same lemma, for fixed  $J$ , there exists a block axial gauge  $F(\theta) = F_J(\theta)$ , fixed successively on the intermediate scales, such that

$$C(\theta)^{F(\theta)} = \mathcal{C}_{\text{raw } d_0, J}(\theta, 0) \quad (13.4)$$

on a complex ball of fixed radius. The gauge transformation  $F$  takes values in  $SU(2)$  for real data, equals the identity at the vertices of the coarse lattice and at  $\theta = 0$ , and  $F$  and  $F^{-1}$  have factorial coefficient bounds at every fixed index. These bounds are uniform in  $d_0, m, L$ , and  $\gamma$  for fixed  $M$  and  $J$ , and they are not used to choose  $J$ .

We set  $\Omega = F^{-1}$ . Since  $\Omega$  equals the identity at the vertices of the coarse lattice, it preserves the coarse field. By (13.4) and the rule  $(U^k)^l = U^{lk}$ , we have  $\mathcal{C}_{\text{raw}}(\theta, 0)^\Omega = C(\theta)$  and  $\text{Log}[\mathcal{C}_{\text{raw}}(\theta, 0)^\Omega] = \delta \bar{A}_{d_0, J}(\theta)$ . We reduce the radius so that the values of  $\Omega$  lie in one identity chart. The absolutely convergent series  $\text{Log}(I + X) = \sum_{l \geq 1} (-1)^{l+1} X^l / l$  and Lemma 13.1 give the bounds for  $\kappa_\Omega$ . Its constant coefficient vanishes, and its majorant is  $C_J u / (1 - C_J u)$  after increasing  $C_J \geq 1$ . At the vertices of the coarse lattice the matrix is  $I$ , and the logarithm vanishes there.

**Step 3: composition of the steps.** The recursion uses the independent variables  $q_i = g_i w_i$  in

$$z_i = z_i^*(c_{i+1}) + K_i(c_{i+1})^{-1/2} q_i,$$

that is, the square root is not absorbed into  $q_i$ . We apply the bounds of Step 1 at every scale. If  $X_{i+1}$  is a majorant for the rows of  $\text{Log } c_{i+1}$ , the majorants of the minimizer, of  $S_i$ , and of  $B_i$  give

$$X_i \preceq G(X_{i+1}, v), \quad G(x, y) = \frac{A(x + y)}{1 - A(x + y)}, \quad X_J = u,$$

after one enlargement of the constant  $A$ . All maps vanish at  $(\theta, q) = (0, 0)$ , except for the bounded constant matrix  $S_i(I)$  multiplying  $q_i$ . There are only  $J$  compositions, and

we choose  $r_J > 0$  so that these scalar compositions converge and all intermediate values stay below the radii of Step 1. The networks are connected, since each composition joins the outputs of one step to the inputs of the next and the factor  $S_i q_i$  contracts its index of  $z$ . The derivative of the same composition with respect to a fixed input controls each column at the mesh of that input. We obtain the joint absolute coefficient series of the reconstruction, independently of  $d_0$  and  $m$ ; the constants and the radius may depend on  $J$ .

To include the factors of the gauge transformation at the endpoints of each link, we transfer arrays indexed by vertices to the start or to the end of each link by the local incidence maps. These maps have one entry in each output row and a bounded number of entries in each input column. The corresponding segments have length at most  $\delta$ , and these maps therefore have bounded series at the fixed rate. The three resulting factors indexed by links have a common output, and their connected pointwise product is the matrix

$$\Omega_x(\theta) \mathcal{C}_{\text{raw}_{xy}}(\theta, q) \Omega_y(\theta)^{-1}$$

on the link  $(x, y)$ . Reducing  $r_J$  so that the logarithm of the identity chart applies, we obtain a joint absolute series for its logarithm at every fixed index. At  $q = 0$  this series is exactly  $\delta \bar{A}$ . Subtracting this series coefficient by coefficient leaves only terms of positive degree in  $q$ . Any convergent nonnegative series in two variables with this property is bounded, on a smaller fixed product, by

$$\frac{C_J v}{1 - C_J(u + v)}.$$

Indeed, its coefficient of degree  $(r, s)$ ,  $s \geq 1$ , is at most  $AR^{-r-s}$  on a smaller disc of absolute convergence, and if  $C_J R \geq \max(1, A)$ , then

$$[u^r v^s] \frac{C_J v}{1 - C_J(u + v)} = C_J^{r+s} \binom{r+s-1}{r} \geq AR^{-r-s} \quad (r \geq 0, s \geq 1),$$

including the coefficient of  $v$  itself. Grouping the three output colors costs at most a factor three in the absolute coefficient rows. We absorb this fixed factor into the pre- $J$  majorant  $Z$  and the post- $J$  constant  $C_J$  before fixing their final values. Thus evaluation of  $\mathfrak{M}_J$  controls the Euclidean norm of a complex link logarithm, as well as each scalar component. We reduce the final radius so that  $C_J r_J < 1/2$ . The part of degree zero in  $q$  has the separate majorant  $\delta Z(u)$ . This proves the majorant  $\mathfrak{M}_J$  as a statement about coefficients on the larger product.

**Step 4: columns and rates.** Every input is kept as a fixed input in this composition, and its column has the volume normalization of its own lattice. Weighting each input, including the fixed one, by its radius turns the derivative of the composed majorant

with respect to the variable of the fixed input into  $u \partial_u$  or  $v \partial_v$  of that majorant. This gives the column majorants  $c_\theta$  and  $c_q$  of (1). Since every term of the part of positive degree in  $q$  contains a variable  $q_i$  and the part of degree zero does not depend on  $q$ , we have  $c_q(u, 0) = 0$ . The sum over the output carries the factor  $\delta^3$ ; converting it to a sum with a coarse-field index fixed therefore introduces no factor of the total volume. Summing over the kinds of indices of  $q$  costs only  $D_J$ . The spatial decay rate in coarse units can be fixed before  $J$ . Indeed, a step of mesh  $\eta \leq 1$  turns a rate  $b_0$  in its own units into the rate  $b_0/\eta \geq b_0$  in coarse units, and contractions use the union of networks at this one rate. The block forests of all scales together extend over a distance at most  $C_M \sum_{l=1}^J M^{-l} < C_M/(M-1)$ . A fixed part of the decay rate is used once for the conversion from rows to columns in Lemma 13.1, not at every scale. The constants for the first three derivatives at zero fluctuation, which are uniform before  $J$ , are those of Lemma 11.9.

**Step 5: color equivariance.** All operations in the construction of  $F$  commute with constant color conjugation: ordered transports, logarithms in the identity chart, the minimizers, and the positive square root. For the square root, conjugation carries the positive square root to the positive square root of the conjugated matrix, by uniqueness. Equivariance for real data extends to the holomorphic continuations by comparing Taylor coefficients. This proves (2).

**Step 6: the choice of radii.** We first take  $\rho_0$  with  $4K_*C\rho_0 < 1/2$  for the uniform majorant  $Z$ , so that  $Z(4K_*\rho) \leq 8K_*C\rho$  for  $\rho \leq \rho_0$ . We choose  $J$  with  $8K_*C\delta < 1/4$ . At this fixed  $J$ , let  $G_{\text{cmp},J}(u, v)$  be one convergent nonnegative row majorant for the finitely many intermediate reconstruction maps in their own coarse and free inputs, with  $G_{\text{cmp},J}(0, 0) = 0$ . For a real comparison set  $u = \max(\|\theta\|_\infty, \|\theta'\|_\infty)$ ; the orthogonal free-coordinate transition preserves the bound  $v$ , so the same majorant bounds both sets of intermediate targets. This does not assert that a fixed nonconstant gauge sends the zero coarse input to zero. We first reduce the admissible coarse-field radius and then the free-input radius so that  $G_{\text{cmp},J}(4K_*\rho, 4K_*s_J) < r_{\text{cmp}}/2$ , retaining strict free-coordinate, constraint, and Haar margins. These reductions are possible by continuity of this common scalar majorant at zero. After this choice we take  $\rho < \min(\rho_0, r_J/(8K_*), (32K_*C_J)^{-1})$ , and finally  $0 < s_J < r_J/(8K_*)$  with  $4K_*C_Js_J < \rho/16$ , reduced further if needed for the fixed construction margins. Then the denominator in  $\mathfrak{M}_J(4K_*\rho, 4K_*s_J)$  is larger than  $1/2$ , the first term is less than  $\rho/4$ , and the second term is less than  $\rho/8$ . Together, these bounds show that the majorant is less than  $\rho/2$ . The coefficient bound holds on the entire complex product, since also  $4K_*(\rho + s_J) < r_J$ . For real radii  $(r, Q)$  we require  $r + Q < r_J$ ,  $1 - C_J(r + Q) > 0$ , and the intermediate classical and Haar disintegration margins on this product; only then is  $\mathfrak{M}_J(r, Q)$  a convergent majorant. If its value is less than  $r/2$ , domination of the coefficients places the image of the entire real product in  $B_{r/2}$ . This

proves (3).

**Step 7: overlaps.** If  $C' = C^{\tilde{k}}$ , then by the rule  $(U^k)^l = U^{lk}$  we have

$$(C')^{\Omega(\theta')} = C^{\Omega(\theta')\tilde{k}} = (C^{\Omega(\theta)})^{\Omega(\theta')\tilde{k}\Omega(\theta)^{-1}}.$$

No fluctuation variable changes in this calculation. The coordinate map, the cutoff domain, and the triangular determinant are therefore those of Theorem 12.13. This proves (4) and completes the proof.  $\square$

### 13.3 Cutoffs and gauge invariance on overlapping charts

We compute the integral over one renormalization group step in a chart, that is, for a chosen small representative of the coarse field. Different representatives are related by gauge transformations whose logarithms need not be small. The following definition states the invariance property used to glue the local formulas and the two hypotheses on the reconstruction.

**Definition 13.3.** Fix the larger construction domains and the coefficient bounds of Theorem 13.2.

1. A scalar function  $R$  on configurations of a periodic lattice is *gauge invariant on  $B_\rho$*  if it is real analytic on  $B_\rho$  and

$$R(U^k) = R(U)$$

for every periodic  $SU(2)$ -valued gauge transformation  $k$  of that lattice with  $U, U^k \in B_\rho$ . Its gauge-invariant extension is  $R_{\text{sat}}(U) = R(U^k)$  whenever  $U^k \in B_\rho$ ; it is defined on  $D_\rho = \mathcal{G}B_\rho$ .

2. The *enlarged-domain reconstruction hypothesis* consists of the coefficient inequality  $\mathfrak{M}_J(4K_*\rho, 4K_*s_J) < \rho/2$ , including the column bounds at every fixed input, together with fixed margins for every intermediate classical, constraint, and Haar disintegration map, and the comparison-product conditions above, including the common margin inside  $B_{r_{\text{cmp}}}$  for every original and transitioned intermediate target. It does not depend on the support radius.
3. Let  $0 < r \leq \rho$ . A fixed real product  $\Omega_p$  satisfies the *support-inclusion hypothesis* if, for every  $c = e^\theta \in B_r$  and every  $w \in \Omega_p$ , the variables  $q_i = g_i w_i$  lie in the enlarged product of Theorem 13.2, all intermediate domain conditions hold, and  $\mathcal{C}_{\text{rawd}_0, J}(\theta, q) \in D_r$ . The condition  $\mathfrak{M}_J(r, tp) < r/2$ , together with the margins of the enlarged product, is sufficient.

Both hypotheses are required at every depth that is used. Neither imposes a condition  $D_r$  at an intermediate scale of the renormalization group step.

The *restriction event* of a renormalization group step, at a given small representative of the coarse field, is the image of  $\Omega_\rho$  under the sequential reconstruction of Theorem 12.13, together with the domain conditions of all intermediate classical, constraint, and Haar disintegration maps, the condition that the fine field lie in  $D_r$ , and the condition that the coarse field lie in  $D_r$ . The global event is its full periodic fine-gauge saturation; saturation by gauges fixing coarse vertices describes only the fiber over one coarse representative. We write  $\mathbf{1}_{\text{cut}}$  for the global indicator.

**Theorem 13.4.** *Let  $0 < r \leq \rho$ , and suppose that both hypotheses of Definition 13.3 hold for the renormalization group step under consideration. Let  $R$  be gauge invariant on  $B_\rho$ , with  $DR(I) = 0$  and  $B_{\rho,a}(R) \leq E_*$ , and invariant under translations, the spatial inversion, and constant color conjugations. Then the following statements hold for every depth  $d_0$ .*

1. *The extension  $R_{\text{sat}}$  is well defined and real analytic on  $D_\rho$ , and*

$$|R_{\text{sat}}(U) - R(I)| \leq C|\mathcal{E}|E_*\rho^2,$$

*where  $|\mathcal{E}|$  is the number of links. The input density*

$$P_{\text{in}}(U) = \mathbf{1}_{D_r}(U) e^{-E_{d_0,\text{sat}}(U)/g_0^2 - R_{\text{sat}}(U)}$$

*has a finite and strictly positive integral on every finite torus.*

2. *At a chosen representative of the coarse field, the local restriction set is exactly the image of the fluctuation product described above. Saturation by fine gauge transformations that equal the identity at coarse vertices describes its fiber over that representative. The global restriction event is the saturation of these sets by the full periodic fine gauge group; it is Borel and independent of the chosen representative.*
3. *Whenever two coarse fields related by a periodic gauge transformation lie, with margins, in the comparison domains, the exact transitions  $w'_i = O_i(c_{i+1})w_i$  are triangular from coarser to finer scales, with orthogonal block permutations on the diagonal; they preserve every ball and the Gaussian term and have absolute Jacobian one. They identify the full integrals, including all Jacobians, Haar disintegration densities, and values of  $R$ . In particular the output density defined by this integral is a pointwise defined function, consistent on overlapping charts, and positive at every  $c \in B_r$ . Its effective interaction, obtained by subtracting  $E_{d_0+J}/g_J^2$  from the negative logarithm of the density, is gauge invariant in the same sense, has the same symmetries, and has vanishing derivative at the identity. The auxiliary integral on  $B_\rho$  satisfies the same identities, but it is identified with the integral of the restricted input measure only on  $B_r$ .*
4. *Let  $B_{\text{ref}}$  and  $H_{d_0,J}$  be as in Theorem 12.13 with  $s = J$ , let  $\varphi_\theta(w) = \Phi_{d_0,J}(\theta, (g_i w_i)_i)$  and  $R_{\text{ref}}(c) = R(e^{\varphi_\theta(0)})$ , and set*

$$V_c(w) = \frac{1}{2}|w|^2 + H_{d_0,J}(\theta, w) + R(e^{\varphi_\theta(w)}) - R_{\text{ref}}(c), \quad A_{\text{coord}} = \prod_i \mathcal{J}_i(I, 0) g_i^{3f_i}.$$

With respect to the product Haar measure  $dH$  on the coarse configuration,

$$\begin{aligned} & (T_M^J)_*(\mathbf{1}_{\text{cut}} P_{\text{in}} dH) \\ &= \mathbf{1}_{D_r}(c) A_{\text{coord}} e^{-E_{d_0+J,\text{sat}}(c)/g_f^2 - B_{\text{ref,sat}}(c) - R_{\text{ref,sat}}(c)} \left( \int_{\Omega_p} e^{-V_c(w)} dw \right) dH(c). \end{aligned} \quad (13.5)$$

The right side is defined by gluing the whole expression across charts. The terms  $B_{\text{ref}}$  and  $R_{\text{ref}}$  need not be separately gauge invariant, and the subscript  $\text{sat}$  on them is notation for the glued expression. Every indicator in  $\mathbf{1}_{\text{cut}}$  is kept in its definition; the support-inclusion hypothesis shows that the indicator of  $D_r$  for the fine field equals one on the domain of integration.

5. Start with  $d_0 = 0$ ,  $R_0 = 0$ , and the measure  $\mathbf{1}_{D_r} e^{-S_W/g_N^2} dH_N$ . Then any finite iteration of these positive operators gives the pushforward of the Wilson measure restricted to the initial condition  $D_r$  and to the intersection of the pulled-back restriction events of the steps. After normalization this is the Wilson measure conditioned on this event, pushed forward to the current scale.

The constants depend only on  $M, J, \rho, a, E_*$ , and the fixed construction margins; they do not depend on the side, the depth,  $L, \gamma$ , or the number of steps.

*Proof.* We glue local formulas, using the gauge transformation and the coefficient bounds of Theorem 13.2 and the two hypotheses of Definition 13.3. The main point is that the transition between two charts is an exact, orthogonal, and triangular change of the fluctuation variables, so that the full integrals agree pointwise and not only almost everywhere.

**Step 1: gluing and normalization.** If  $U^k, U^l \in B_\rho$ , then  $U^l = (U^k)^{lk^{-1}}$ , and gauge invariance on  $B_\rho$  implies that  $R(U^l) = R(U^k)$ . Thus  $R_{\text{sat}}$  is well defined. The set  $D_\rho$  is open, as the union of the images of  $B_\rho$  under fixed gauge transformations. On the open set where  $U^k \in B_\rho$  the function  $U \mapsto R(U^k)$  is real analytic. These functions agree on intersections, and together they define a real analytic function on  $D_\rho$ . Translations and the spatial inversion preserve  $B_\rho$  and intertwine the gauge action, and the corresponding identities glue as well. For the effective classical action, the gluing is Theorem 11.6 on the classical domain.

We write  $R(e^x) = R(I) + \sum_{k \geq 2} \sum_{|\alpha|=k} r_\alpha x^\alpha$  near zero, in scalar coordinates. The degree- $k$  sum of  $|r_\alpha|$  is at most  $3|\mathcal{E}|$  times the corresponding sum with one index fixed, and the norm  $\mathcal{B}_{\rho,a}(R)$  sums these at radius  $\rho$ . This implies that the Taylor series converges absolutely on the complex product  $\|x\|_\infty < \rho$ . It agrees with  $R$  on the real ball, since both are real analytic on this connected ball and agree near zero. This also specifies the complex continuation used below. By Euler's identity,

$$\sum_{|\alpha|=k} |r_\alpha| = \sum_i \sum_{|\alpha|=k} \frac{\alpha_i}{k} |r_\alpha|.$$

There are  $3|\mathcal{E}|$  scalar indices  $i$ , and the inner sum is the sum of the ordered derivatives with  $i$  fixed, divided by  $k!$ , without its spatial weight, which is at least one. Summing over the degrees, we obtain

$$\sup_{B_\rho} |R - R(I)| \leq 3|\mathcal{E}|\rho^2 \mathcal{B}_{\rho,a}(R). \quad (13.6)$$

The same bound holds for  $R_{\text{sat}}$ , which takes the same values. The effective classical action is nonnegative, since it is a minimum of the Wilson action multiplied by a positive normalization. Hence  $P_{\text{in}} \leq \exp[-R(I) + 3|\mathcal{E}|E_*\rho^2]$ , and since the Haar measure has mass one, the integral of  $P_{\text{in}}$  is finite. The classical action and the effective interaction are finite and continuous near the identity, and  $D_r$  contains an open neighborhood of the identity, of positive Haar measure. Integrating over a smaller closed neighborhood shows that the integral is strictly positive. This argument keeps the constant  $R(I)$ . This proves (1).

**Step 2: the exact transition between charts.** Fix real coarse fields  $c$  and  $c' = c^k$  in the comparison domains, with margins. The overlap statement of Theorem 11.6 applies on every comparison domain required by the enlarged-domain reconstruction hypothesis. In a block forest, we transport each link variable to the root before taking its logarithm. Changing the frame at the root by  $k$  conjugates the transported variables by the value of  $k$  at the root, and the factors of the paths at intermediate vertices cancel in pairs. Thus a free logarithmic coordinate changes by the orthogonal adjoint action on its block. A translation or the spatial inversion in addition relabels the oriented blocks, and the reversal of an orientation acts by an orthogonal map on the color components. We denote by  $O_i(c_{i+1})$  the resulting orthogonal map of the free logarithmic coordinates of scale  $i$ , which permutes the blocks. It depends on the coarser field and not on the free logarithmic coordinates of scale  $i$ . These are the transformations of Proposition 12.11 and Theorem 12.13.

By Theorem 11.6 the two classical objectives agree on these domains. Their minimizers are unique, and the transformation of the coordinates carries one to the other. Differentiating the objective in the orthogonal coordinates, we obtain

$$z_i^{*'} = O_i z_i^*, \quad K_i' = O_i K_i O_i^T, \quad S_i' = O_i S_i O_i^T;$$

the last identity follows by taking the positive square root of real matrices, and it is continued analytically only on the fixed neighborhoods. Substituting in  $z_i = z_i^* + g_i S_i w_i$  gives  $w_i' = O_i(c_{i+1}) w_i$ . We proceed from scale  $J - 1$  to finer scales; when  $O_i$  is evaluated, every coarser field has already been transformed. This proves  $C' = C^{\tilde{k}}$  for the reconstructions, with an  $\text{SU}(2)$ -valued gauge transformation  $\tilde{k}$  of the fine lattice, as well as the triangular transition of the fluctuation variables.

In this order the derivative matrix of the transition is triangular. Its diagonal blocks are the  $O_i$ , and its off-diagonal blocks contain all derivatives of the  $O_i$  with respect to the coarser fluctuations. Therefore

$$|\det D_w w'| = \prod_i |\det O_i| = 1, \quad \sum_i |w'_i|^2 = \sum_i |w_i|^2,$$

and each ball is carried onto the corresponding ball. The identity for block forests in Lemma 5.1 preserves the product Haar measure, gives mass one to the forest variables, and keeps every vertex of the coarse lattice fixed. The changes of the free logarithmic coordinates and of the determinants of the constraints are exactly the changes in the Haar disintegration formula of Lemma 6.3; since that formula is the disintegration of the Haar measure, the product of these factors is preserved. Equivalently, we may substitute  $z'_i = O_i z_i$  in each integral over the free logarithmic coordinates and use the conjugation identity for  $S_i$ ; the diagonal determinants and all Haar disintegration factors agree. No off-diagonal derivative is dropped in either computation.

**Step 3: values of the input effective interaction.** The two gauge-transformed reconstructions lie in  $B_{\rho/2}$ , and they are related by the gauge transformation  $\Omega(\theta') \tilde{k} \Omega(\theta)^{-1}$  of Theorem 13.2(4). The values of  $R$  agree by gauge invariance on  $B_\rho$ , not by smallness of the logarithm of this gauge transformation. In particular,

$$R_{\text{sat}}(\mathcal{C}_{\text{raw}}(\theta, (g_i w_i)_i)) = R(e^{\varphi_\theta(w)}). \quad (13.7)$$

For the auxiliary coarse fields in  $B_\rho$  this identity uses  $D_\rho$  only. For coarse fields in  $B_r$ , the support-inclusion hypothesis also places the reconstructed fine field in  $D_r$ . This removes the indicator of  $D_r$  for the fine field from the integral over the fluctuation product.

**Step 4: the restriction event.** On a chart of the coarse field, we take the image of the fluctuation product under the triangular reconstruction, with its classical, constraint, and Haar disintegration domain conditions. We keep the conditions  $D_r$  for the fine field and for the coarse field. We take its orbit under the gauge transformations of the fine lattice that equal the identity at the vertices of the coarse lattice. Changing the small representative of the coarse field changes this local set by the bijective triangular transition of Step 2. Extending also by the gauge transformations of the coarse lattice defines one event, independent of the chart; representatives are not counted.

This event is Borel. To see this, we exhaust the open ball of logarithms of the coarse field by smaller closed balls, and each construction margin by closed positive margins. On each such compact set the finite reconstruction and its domain data are continuous. The product of such a set with the closed balls of fluctuations and the compact group of gauge transformations therefore has a compact image in the finite-dimensional configuration manifold. By Step 2, the union of the local events over all charts coincides with the

local event on every chart; as the countable union of these compact images, it is Borel. Intersecting with the Borel conditions on the fine and the coarse field yields a Borel event. No measurable global choice of gauge is used. This proves (2).

**Step 5: the density.** We use the algebraic action identity and triangular change-of-variables identity proved in Theorem 12.13(2)–(4), which require only the domain margins assumed here, and insert (13.7). The cluster-expansion smallness assumptions of that theorem are not needed for this identity. The normalization of the Haar disintegration is  $A_{\text{coord}} = \prod_i \mathcal{J}_i(I, 0) g_i^{3f_i}$ . The exponent depending on  $w$  is

$$\frac{1}{2}|w|^2 + H_{d_0, J}(\theta, w) + R(e^{\varphi_\theta(w)}) - R(e^{\varphi_\theta(0)}),$$

and the terms removed from it are exactly  $E_{d_0+J}/g_J^2 + B_{\text{ref}} + R_{\text{ref}}$ . Including them, we obtain (13.5) with respect to the product Haar measure on the coarse configuration. There is no additional Haar density of the coarse field and no additional factor for the volume of gauge orbits.

The integral is finite and strictly positive at every coarse field satisfying the hypotheses, since the integrand is positive and continuous on the compact product of balls, which has positive Lebesgue measure. By Step 2, the entire integral, including the terms evaluated at the zero-fluctuation reconstruction and the constant  $A_{\text{coord}}$ , takes the same value in overlapping charts. It is therefore a pointwise defined version of the density, which is stronger than an identity almost everywhere. Taking the real logarithm and subtracting the gauge-invariant classical term proves the gauge invariance of the effective interaction on overlapping charts. The same argument for translations, the spatial inversion, and constant color conjugations proves its other symmetries. If  $\ell$  is its derivative at the identity, color symmetry shows that  $\ell$  is a linear functional on each link variable that is invariant under all rotations of the color components. Rotations by  $\pi$  about two coordinate axes force all three components to vanish. Hence  $DR_+(I) = 0$ , where  $R_+$  denotes the output effective interaction.

On  $B_\rho$  the formula over the fixed product is the auxiliary integral. All comparisons of charts used above stay inside the enlarged construction domains, so the same identities hold for it. The condition  $D_r$  for the fine field was used only on  $B_r$ ; outside  $B_r$  the auxiliary integral is not identified with the integral of the restricted input measure. The subscript sat on the terms evaluated at the zero-fluctuation reconstruction is only notation for the glued expression, and the argument does not require their separate invariance. This proves (3) and (4).

**Step 6: iteration.** Let  $A_j$  be the restriction event of the step applied to the configuration obtained after  $j$  completed steps, and let  $\eta_j$  be the unnormalized measure after  $j$  steps. For every nonnegative Borel function  $F$ , by the definition of the pushforward, we have

$$\int F d\eta_{j+1} = \int \mathbf{1}_{A_j}(U) F(T_M^J U) d\eta_j(U).$$

Starting from the restricted Wilson density and substituting this identity repeatedly, we obtain, since all integrands are nonnegative,

$$\int F d\eta_k = \int \mathbf{1}_{D_r}(U) \prod_{j=0}^{k-1} \mathbf{1}_{A_j}(T_M^{jJ} U) F(T_M^{kJ} U) e^{-S_W(U)/g_N^2} dH_N(U).$$

Every normalization is positive by Step 5. Dividing this identity by its value at  $F = 1$ , we obtain exactly the Wilson measure conditioned on the stated intersection of events. The constants used above depend only on the data stated in the theorem, and the extensive factor in (13.6) is explicit. This proves (5).  $\square$

### 13.4 Preservation of analytic bounds under iteration

We now show that one renormalization group step maps an effective interaction with  $\mathcal{B}_{\rho,a}(R) \leq E_*$  to one with the same bound, at the same radius  $\rho$  and rate  $a$ . The new effective interaction has three parts: the newly generated effective interaction  $Q_{d_0,J'}^W$ , which is bounded by Theorem 12.13; the pullback  $\mathcal{P}_\delta R$  of  $R$  at zero fluctuation, which contracts by Theorem 11.10; and the fluctuation correction  $\mathcal{D}_J R$  to the pullback, which is of second order in  $t$  by the argument of Lemma 12.16. Two points require care. First, the supported estimates must be converted back to the norm  $\mathcal{B}_{\rho,a}$  without reducing  $\rho$ , since otherwise the radius would shrink at every step; this uses the translation invariance of the total effective interaction. Second, the composition of  $R$  with the reconstruction must be estimated at the level of coefficients, on the whole range of the reconstruction.

**Theorem 13.5.** *Fix an odd integer  $M \geq 3$  and the bounds for the classical coefficients and for the reconstruction. Then the following statements hold.*

1. *There exist a spatial decay rate  $a > 0$ , a larger rate  $b > a$ , a box scale  $\mathcal{L}$ , a factor  $K_* \geq 2$ , and  $\rho_0 > 0$ , all chosen before  $J$ ; the density of the coefficient indices affects only constants that depend on  $J$ . Let  $C_{\text{geom}} \geq 1$  be a constant, depending only on the metric conventions, that bounds the lengths of the rectilinear networks used to construct supports by the lengths of shortest connecting networks. The rate  $a$  is chosen below  $a_{\text{cen}}/8$ , where  $a_{\text{cen}}$  is the rate of the bounds of Lemma 11.9. Choose  $J$  such that*

$$q_P = C_P(a)M^{-J} < \frac{1}{4}, \quad M^{-J} \leq \delta_0(a), \\ M^{-J}Z(4K_*\rho) \leq \rho/4 \quad (0 < \rho \leq \rho_0), \quad C_{\text{geom}}bM^{-J} \leq a.$$

where  $C_P(a)$  and  $\delta_0(a)$  are the constant and mesh threshold of Theorem 11.10, and  $Z$  is the majorant of Theorem 13.2. The pre- $J$  radius  $\rho_0$  is also chosen within the radius allowed by that contraction theorem. Then choose  $0 < \rho \leq \min(\rho_0, r_J/(8K_*))$ , with  $\rho C_B \leq 1/2$  for a constant  $C_B$  depending only on  $M, J$ , and the larger domains (defined in the proof), and

$s_J > 0$  such that the enlarged-domain reconstruction hypothesis of Definition 13.3 holds strictly. These choices use only the fixed larger domains, not the support radius  $r$ .

2. Let  $\{\Psi_Y\}$  be a family of functions of the coarse field indexed by supports, with coefficient norms at radius  $\rho$ , whose sum  $f = \sum_Y \Psi_Y$  is invariant under the translations of the coarse lattice. Assume also that  $Df(0) = 0$ . Suppose that at  $\leq 2|Y|$  for the coarse-field labels of each coefficient of  $\Psi_Y$ . Then

$$\mathcal{B}_{\rho,a}(f) \leq 9\rho^{-2} \sup_x \sum_{Y \ni x} e^{2|Y|} \|\Psi_Y\|_{\rho, \text{coeff}}$$

in scalar orientation and color coordinates. The individual  $\Psi_Y$  need not be translation invariant. Without translation invariance of  $f$ , a bound of this kind requires a strictly larger coefficient radius.

3. There exist  $C_G, C_T, c_* > 0$ , depending on  $M, J, \rho, a, b, K_*, s_J$ , and the larger domains, and independent of the side and of  $d_0$ , such that the following holds. Let  $R$  be gauge invariant on  $B_\rho$ , invariant under translations, the spatial inversion, and constant color conjugations, with  $DR(I) = 0$  and  $\mathcal{B}_{\rho,a}(R) \leq \epsilon$ . Then the pullback  $R(e^{\Phi_{d_0,J}(\theta,q)})$  has the joint coefficient bound  $C_{J,\rho,a}\epsilon$  at every fixed index on the enlarged product of coarse fields and variables  $q$ . Define the auxiliary recursion by the integral over the fixed product of fluctuation variables and the formula (13.5), using this pullback on the gauge-transformed image in  $B_{\rho/2}$ , and without any indicator of  $D_r$  at coarse fields outside  $B_r$ . Let  $p \geq p_0$ , and assume

$$0 < t \leq \frac{c_*}{2(p^3 + \epsilon p)}. \quad (13.8)$$

Then, with  $R_{\text{aux}} = c_*/(p^3 + \epsilon p)$ , for every complex  $\zeta$  with  $|\zeta| \leq R_{\text{aux}}$  and every  $w \in \Omega_p$  the variables  $\zeta \lambda_i w_i$  lie in the enlarged product of Theorem 13.2, the conditions (12.23) hold at  $(t, p)$ , and the supported families of the auxiliary interaction and of the inserted interaction constructed in the proof lie in the ball of radius  $\eta_0/2$  of Theorem 12.7. Under these conditions, apart from the field-independent factor  $A_{\text{coord}}$ , the output effective interaction is

$$R_+ = Q_{d_0,J}^w + \mathcal{P}_\delta R + \mathcal{D}_J R, \quad \mathcal{B}_{\rho,a}(R_+) \leq C_G + q_P \epsilon + C_T t^2 \epsilon (p^4 + \epsilon p^2). \quad (13.9)$$

Here  $\mathcal{P}_\delta R(\theta) = R(e^{\Phi_\theta(0)}) = R(e^{\delta \bar{A}_{d_0,J}(\theta)})$  is the pullback at the zero-fluctuation reconstruction,  $\mathcal{D}_J R$  is the difference of the logarithms of the fluctuation partition functions without and with the inserted interaction, as in Lemma 12.16 at the parameter  $t$ , and  $Q_{d_0,J}^w$  is the effective interaction of Theorem 12.13. The output  $R_+$  is gauge invariant on  $B_\rho$ , invariant under translations, the spatial inversion, and constant color conjugations, and  $DR_+(I) = 0$ . Coupling parameters internal to  $R$  are held fixed when the auxiliary parameter varies.

4. Fix  $E_* \geq 2C_G/(1 - q_P)$  and  $p(t) = A_w \log(1/t)$  with  $A_w \geq 1$ . There exists  $t_* > 0$ , depending only on the fixed data, such that for every  $0 < t \leq t_*$  we have  $p(t) \geq p_0$ ,  $tp(t) < s_J$ , the enlarged-domain hypotheses, and

$$t \leq \frac{c_*}{2(p(t)^3 + E_* p(t))}, \quad C_T t^2 (p(t)^4 + E_* p(t)^2) \leq \frac{1 - q_P}{2}.$$

Consequently  $\mathcal{B}_{\rho,a}(R) \leq E_*$  implies  $\mathcal{B}_{\rho,a}(R_+) \leq E_*$ .

5. The auxiliary recursion starting at  $R_0 = 0$  exists, with the same  $\rho$ ,  $a$ , and  $E_*$ , through every finite chain in which each renormalization group step has largest integrated coupling at most  $t_*$  and output side at least four. For any  $r \leq \rho$  for which the support-inclusion hypothesis holds at every step, the restricted densities after  $k$  steps are exactly

$$Z_k^{-1} \mathbf{1}_{D_r} e^{-E_{kJ,\text{sat}}/g_{kJ}^2 - R_{k,\text{sat}}}, \quad \mathcal{B}_{\rho,a}(R_k) \leq E_*,$$

for every  $k$  for which the output sides are at least four and the couplings of the integrated scales are at most  $t_*$ . Nothing is asserted about the integral of the restricted input measure at coarse fields outside  $B_r$ .

*Proof.* We use the coefficient bounds of Theorem 13.2. We first fix the rates and the supports, then prove (2), then estimate the three parts of  $R_+$ , and finally choose the coupling threshold.

**Step 1: rates, supports, and the order of choices.** At an intermediate scale of mesh  $\eta_i \leq 1$ , a weight  $e^{A\mathbf{t}}$  with  $\mathbf{t}$  measured in the units of that scale equals  $e^{(A/\eta_i)\mathbf{t}}$  with  $\mathbf{t}$  measured in coarse units. In particular, it gives any fixed rate below  $A$  in coarse units. The union of networks in Lemma 13.1 keeps one rate under finite composition and does not divide it by the number of scales. Sums over the kinds of coordinates change constants by  $D_J$ .

We separate these two effects with the following geometric bounds. For a monomial with  $n$  coordinate locations, we choose a rectilinear network of length at most  $C_{\text{geom}}\mathbf{t}$ . To that end, we first replace a shortest connected network by a tree on the locations, of at most twice its length, by traversing its edges and shortcutting the walk. Then we replace each edge of the tree by its three coordinate segments. This also shows that the number of segments is at most a fixed multiple of  $n + 1$ . We partition the torus into boxes with integer side lengths between  $\mathcal{L}$  and  $2\mathcal{L}$ , a shorter direction forming one interval. Since a coordinate segment of length  $l$  meets at most  $2 + l/\mathcal{L}$  intervals in its direction, the connected set  $X$  of boxes met by the network satisfies

$$|X| \leq c_0(n + 1 + C_{\text{geom}}\mathbf{t}/\mathcal{L}). \quad (13.10)$$

All constructions are periodic and include sets winding around the torus, and  $c_0$  depends only on the dimension, not on the density of labels.

We choose  $\mathcal{L}$  so large that  $8c_0C_{\text{geom}}/\mathcal{L}$  is at most one quarter of the available rate. The factor  $e^{8c_0(n+1)}$  in  $e^{8|X|}$  is absorbed by enlarging  $K_* \geq 2$  once, since the radii carry the factor  $K_*^n$ . A second quarter of the available rate is used for the sums over the location of the fixed index. If  $x \in X$  and  $u$  is any location of the monomial, the network from  $u$  to  $x$  has length at most  $C_{\text{geom}}t + C_{\mathcal{L}}$ , which leaves a factor that decays exponentially in  $\text{dist}(x, u)$ . Summing over  $u$  then costs only  $D_J$ . These choices give the support weights used in Theorem 12.6 without a factor depending on  $J$  in the degree.

Each box contains at most  $C\mathcal{L}^3$  coarse-field labels. Joining them inside each box and then joining adjacent boxes of a connected set  $Y$ , we obtain

$$t(\text{coarse-field labels in } Y) \leq C_{\mathcal{L}}|Y|,$$

and repeated labels do not increase this length. The bounds of Lemma 11.9 for the background field  $e^{\delta\bar{A}_{d_0,J}}$  hold at a rate  $a_{\text{cen}}$ , and for this representative Theorem 11.10 provides the contraction at every output rate  $0 < a < a_{\text{cen}}/8$ , and at no larger rate. We therefore choose  $a$  with  $0 < a < a_{\text{cen}}/8$  and  $aC_{\mathcal{L}} \leq 2$ ; such an  $a$  also satisfies  $a < a_*/8$ , where  $a_*$  is the rate of Theorem 10.11. We choose  $b > a$  below the rate  $b_*$  of Theorem 13.2, with enough room for the support and location estimates above. This is possible since  $b_*$  is a fixed positive rate chosen before  $J$ , and  $a$  may be taken as small as needed. The condition  $C_{\text{geom}}bM^{-J} \leq a$  is then imposed when  $J$  is chosen, after  $a$  and  $b$  are fixed. All these choices precede  $J$ , and the metric conventions are included once in  $C_{\text{geom}}$ .

The coefficient bounds for the reconstruction  $\Phi_{d_0,J}$ , for the auxiliary interaction  $H$ , and for  $B_{\text{ref}}$  hold at a common rate  $b_* > 0$  in coarse units, chosen before  $J$ . Indeed, the bounds of one averaging step in Lemma 12.3 hold at a rate chosen before  $J$ . This rate does not decrease when lengths are measured in coarse units, since every mesh is at most one, and composition through connected networks keeps one rate, by Lemma 13.1. The rate  $b_*$  of Theorem 13.2 is chosen below these rates. The inserted interaction is treated separately in Step 3: the network of  $R$  has the rate  $a/\delta$  in coarse units, the condition  $C_{\text{geom}}b\delta \leq a$  supplies the rate  $b$  needed for the support estimates, and the attached networks of the reconstruction have rate at least  $b_* > b$ . Since

$$0 < a < b < b_*,$$

all four families of bounds hold at the rate  $a$  of the norm  $\mathcal{B}_{\rho,a}$  and at the rate  $b$  used for the supports. The rate  $b_J$  of Theorem 12.13(5), which may depend on  $J$ , is not used; this is why the bound on  $Q_{d_0,J}^w$  in Step 5 is proved directly.

The constant  $C_P(a)$  of Theorem 11.10 does not change under later reductions of the radius. We choose  $\rho_0$  in the range allowed by that theorem and in the range of convergence of  $Z$ , so that  $Z(4K_*\rho) \leq C_1\rho$  for  $\rho \leq \rho_0$ . We then take  $J$  so large that  $q_P < 1/4$ , that  $M^{-J}Z(4K_*\rho) \leq \rho/4$ , that  $C_{\text{geom}}bM^{-J} \leq a$ , and that  $M^{-J}$  is below the threshold  $\delta_0(a)$  of Theorem 11.10. This is the only choice of  $J$ . Next we choose  $\rho$  and  $s_J$  by

the fixed- $J$  construction in Theorem 13.2(3), including  $G_{\text{cmp},J}(4K_*\rho, 4K_*s_J) < r_{\text{cmp}}/2$  and every comparison-product margin, with strict image inequalities and with  $\rho C_B \leq 1/2$ . Here  $C_B$  is the constant in the coefficient bound for  $B_{\text{ref}}$  of Step 5, which depends only on  $M, J$ , and the larger domains. Reducing these radii does not change  $q_P$ , and none of these choices uses  $r$ . This proves (1).

The image-majorant constants  $C_J, C_{\text{col},J}$ , and the radii just fixed are now frozen. Later supported estimates use the separately named constant  $C_{\text{sup},J}$ ; enlarging it changes coupling gates only and does not change an established image inequality.

**Step 2: conversion at the same radius.** For a monomial  $c_\alpha \theta^\alpha$  of degree  $k$ , the ordered derivatives with a scalar input  $i$  fixed satisfy

$$\frac{1}{k!} \sum_{i_2, \dots, i_k} |D_i D_{i_2} \cdots D_{i_k} (c_\alpha \theta^\alpha)(0)| = \frac{\alpha_i}{k} |c_\alpha|. \quad (13.11)$$

Indeed, the remaining indices can be ordered in  $(k-1)!/((\alpha_i-1)! \prod_{j \neq i} \alpha_j!)$  ways, and each derivative equals  $c_\alpha \prod_j \alpha_j!$ . For sums of supported terms the triangle inequality gives the corresponding upper bound. A term differentiated at  $i$  has  $i \in Y$ , and  $e^{at} \leq e^{2|Y|}$  by hypothesis.

At each degree the translation-invariant function  $f$  has the same norm with a fixed index at all translates of a given scalar orientation and color type. We choose one reference index for each of the  $3 \cdot 3 = 9$  types, replace the maximum of the nine nonnegative degree sums by their sum, and only then sum over the degrees. Applying (13.11) at these fixed indices, we obtain

$$\mathcal{B}_{\rho,a}(f) \leq 9\rho^{-2} \sup_x \sum_{Y \ni x} e^{2|Y|} \|\Psi_Y\|_{\rho, \text{coeff}}.$$

The individual  $\Psi_Y$  need not be translation invariant. Without translation invariance of  $f$ , a coefficient norm at a radius  $u > \rho$  bounds the degree- $k$  norm with a fixed index by  $u^{-k}$  times the norm at  $u$ , and summing  $\rho^{k-2}u^{-k}$  over  $k \geq 2$  gives  $u^{-2}(1 - \rho/u)^{-1}$ . This explains the need for a strictly larger radius in that case. A supremum over the fixed index cannot be interchanged with the infinite sum over degrees at the same radius. This proves (2).

**Step 3: the coefficients of the composed input interaction.** Let  $\epsilon = \mathcal{B}_{\rho,a}(R)$ . We write the logarithms of the fine field as  $A = (A_j)$ , where the labels  $j$  of the fine lattice have volume  $v_{\text{old}} = \delta^3$ , and we expand

$$\begin{aligned} R(e^A) &= R(I) + \sum_{n \geq 2} \frac{1}{n!} \sum_{j_1, \dots, j_n} R_n(j_1, \dots, j_n) \prod_{r=1}^n A_{j_r}, \\ P_j(\theta, q) &= \Phi_{d_0, J}(\theta, q)_j = \sum_{|\xi| + |\eta| \geq 1} p_{j, \xi \eta} \theta^\xi q^\eta; \end{aligned} \quad (13.12)$$

there is no term with  $n = 1$  since  $DR(I) = 0$ . For a multiindex  $(\nu, \eta)$  of positive total degree  $N = |\nu| + |\eta| \geq 1$ , the coefficient of  $\theta^\nu q^\eta$  in  $R(e^P)$  is

$$[\theta^\nu q^\eta]R(e^P) = \sum_{n=2}^N \frac{1}{n!} \sum_{j_1, \dots, j_n} \sum_{\substack{\sum_r \xi_r = \nu, \sum_r \eta_r = \eta \\ |\xi_r| + |\eta_r| \geq 1}} R_n(j_1, \dots, j_n) \prod_{r=1}^n p_{j_r, \xi_r \eta_r}, \quad (13.13)$$

where the inner sum is over ordered allocations. No further multinomial factor occurs, since each  $p_{j, \xi \eta}$  is already the coefficient of a monomial. For example, the allocation  $(\xi_1, \eta_1) = (\mathbf{e}_i, 0)$ ,  $(\xi_2, \eta_2) = (0, \mathbf{e}_l)$  with  $n = 2$  contributes  $(1/2) \sum_{j_1, j_2} R_2(j_1, j_2) p_{j_1, \mathbf{e}_i, 0} p_{j_2, 0, \mathbf{e}_l}$ , and the exchanged allocation contributes the other half. We keep each summand of (13.13) with the union of the network of  $R_n$  and the networks of the attached coefficients of the reconstruction. These are connected through the indices  $j_r$ , and Lemma 13.1 applies; no product of independent extensive scalars occurs.

Let  $r_n = |R_n|_a / n!$ , where  $|R_n|_a$  is the norm with one index fixed, and  $F(s) = \sum_{n \geq 2} r_n s^n$ . Then  $\sum_n r_n \rho^{n-2} = \epsilon$ , so each  $r_n \rho^{n-2} \leq \epsilon$ , and for  $0 \leq m \leq \rho/2$ ,

$$F'(m) \leq \epsilon \rho \sum_{n \geq 2} n 2^{-(n-1)} = 3\epsilon \rho. \quad (13.14)$$

At the radii  $U_0 = 4K_*\rho$  and  $V_0 = 4K_*s_J$  the majorant for the output-fixed rows satisfies  $\mathfrak{M}_J(U_0, V_0) < \rho/2$ ; that is, every attached factor has majorant less than  $\rho/2$ . Fix an external input  $i$ . We choose the factor containing it and its position there, and we use the column of that factor and rows for all other factors. The column carries the factor  $v_{\text{old}}/v_i$ , while the norm of  $R_n$  with a fixed index has no volume factor, and the bound for the composed scalar with  $i$  fixed therefore contains  $v_i/v_{\text{old}} \leq \Gamma_J$ , where

$$\Gamma_J = \max_i \frac{v_i}{v_{\text{old}}} \leq M^{3J},$$

the maximum being over indices of  $\theta$  and of  $q$ . Let  $c_\theta(U, V)$  and  $c_q(U, V)$  be the majorant series of Theorem 13.2(1) for the columns at a fixed coarse-field input and at a fixed input of  $q$ , in which the fixed input carries its radius  $U$  or  $V$ . In particular,  $c_q(U, 0) = 0$ . Then the series with the input  $i$  fixed is bounded by

$$\Gamma_J F'(\mathfrak{M}_J(U, V)) (c_\theta(U, V) + c_q(U, V)), \quad (13.15)$$

and by (13.11) the series without a distinguished input is no larger. The column majorants are finite at  $(U_0, V_0)$ . Combining (13.14) and (13.15), we obtain the joint coefficient bound  $C_{J, \rho, a} \epsilon$  on the enlarged product. It controls coefficients with a coarse-field index fixed, with an index of  $q$  fixed, and mixed ones, before any evaluation, and the constant  $R(I)$  never enters.

A network of locations of the fine lattice has length  $\delta^{-1}$  times its length in coarse units. Its weight at the rate  $a$  therefore gives the rate  $a/\delta$  in coarse units, and the condition

$C_{\text{geom}} b \delta \leq a$  provides the rate  $b$  on this part of the composed network. Each attached network of the reconstruction has its own rate, which is at least  $b$ . Their union provides the rates used for the supports and for the location sums in Step 1, while the output norm still has the rate  $a$ . Conversions between the fixed meshes cost factors depending only on  $J$ , not volume factors.

For real coarse fields and fluctuations, the value of this pullback is  $R_{\text{sat}}(\mathcal{C}_{\text{raw}})$ , by (13.7). Thus  $R(e^{\Phi_{d_0, J}(\theta, q)})$  is the holomorphic pullback required in Lemma 12.16, and its hypothesis is verified on the whole range of the reconstruction, not only at its center. This proves the first assertion of (3).

**Step 4: the inserted interaction at the radius  $\rho$ .** We now show that the estimate of Lemma 12.16 holds at the radius  $\rho$  itself, rather than at a smaller radius. We fix  $p$ , every parameter internal to  $R$ , the balls, and the classical functions, and we substitute  $q_i = \zeta \lambda_i w_i$ . For each summand  $o$  of (13.13) with  $|\eta| \geq 1$ , where  $o$  specifies  $n$ , the indices  $j_r$ , the allocation, and the refinements of the networks, we set

$$a_o(\zeta) = \frac{\zeta^{|\eta|}}{n!} R_n(j_1, \dots, j_n) \prod_{r=1}^n p_{j_r; \zeta_r \eta_r} \prod_l \lambda_{s(l)}^{\eta_l},$$

where  $s(l)$  is the scale of the index  $l$  of  $q$ . Let  $Y_o$  be the set of boxes assigned to the external locations of  $o$  by the construction of Lemma 12.9, and set  $(A_R)_Y(\theta, w; \zeta) = \sum_{o: Y_o=Y} a_o(\zeta) \theta^v w^\eta$ . Then, exactly,

$$\sum_Y (A_R)_Y = R(e^{\Phi(\theta, (\zeta \lambda_i w_i)_i)}) - R(e^{\Phi(\theta, 0)}),$$

supports winding around the torus being included; the construction is linear in  $R$ . We replace the supremum over  $w$  in the norm  $\mathcal{S}_8$  of Theorem 12.7, at the coefficient radius  $\rho$ , by the sum of absolute coefficients at  $|w_l| \leq p$ . We then bound each monomial by the sum over its variables  $w_l$  of the corresponding terms with that variable fixed. The support-weight estimate (12.20) costs a fixed constant and a factor  $K^N$  for a monomial with  $N$  locations, which we absorb by multiplying all radii by  $K$ . This enlargement is included in  $K_*$ , so that  $K\rho \leq U_0$  and  $K|\zeta|p \leq V_0$  in the range used below. The sum over the location of the fixed index of  $w$  costs a constant depending on the density. In this supported norm we count the chosen variable with its multiplicity  $\eta_l$ . A branch monomial of total degree  $k$  contributes only  $\eta_l/k$  times its absolute coefficient to the normalized input column of Step 3, so we must use the degree-marked column

$$c_q^{\text{mark}}(U, V) = (U\partial_U + V\partial_V)c_q(U, V).$$

The derivative  $F'$  in (13.15) accounts separately for the choice of the outer branch. To bound the marked column, let  $C_{\text{col}, J}$  be the constant of the rational part of  $\widetilde{\mathfrak{M}}_J$  in The-

orem 13.2(1), and put  $\eta_{\text{col}} = 1 - C_{\text{col},J}(U_0 + V_0) > 0$ . Applying degree marking to the coefficientwise bound  $c_q \preceq V \partial_V \widetilde{\mathfrak{M}}_J$ , and writing  $C = C_{\text{col},J}$ , gives

$$c_q^{\text{mark}}(U, V) \leq \frac{CV(1 - CU + CV)}{[1 - C(U + V)]^3} \leq 2CV\eta_{\text{col}}^{-3} \quad (U \leq U_0, V \leq V_0).$$

Thus  $L_q^{\text{mark}} = \sup_{0 < V \leq V_0} c_q^{\text{mark}}(U_0, V)/V$  is finite. Using the branch sum of (13.15) with this marked column and (13.14), we obtain, with a separately chosen supported-estimate constant  $C_{\text{sup},J}$ ,

$$\|A_R(\zeta)\|_{\mathcal{S}_8} \leq \frac{C_{\text{sup},J}|\zeta|\epsilon p}{1 - C_{\text{sup},J}|\zeta|p} \quad (C_{\text{sup},J}|\zeta|p < 1). \quad (13.16)$$

The bound is linear in  $\epsilon$ , since it comes from the single tensor family  $(R_n)$ . In the same way, repeating Step 6 of the proof of Theorem 12.13 at the coefficient radius  $\rho$  and with the boxes of Step 1, we obtain for the auxiliary interaction of Definition 12.15

$$\|H(\zeta)\|_{\mathcal{S}_8} \leq \frac{C_{\text{sup},J}|\zeta|(p^3 + p)}{1 - C_{\text{sup},J}|\zeta|p},$$

since its classical terms contain at least three fluctuation variables of their own scale and its Haar density and determinant terms at least one.

Let  $s_{\text{aux}} > 0$  be such that  $Ks_{\text{aux}} \leq V_0$ , and choose

$$0 < c_* < \min\left\{s_{\text{aux}}, \frac{1}{2C_{\text{sup},J}}, \frac{\eta_0}{12C_{\text{sup},J}}, \frac{\sigma_{\text{seq}}}{2C_{\text{sup},J}}\right\}, \quad R_{\text{aux}} = \frac{c_*}{p^3 + \epsilon p}, \quad (13.17)$$

where  $\eta_0$  is the constant of Theorem 12.7,  $\sigma_{\text{seq}}$  is the constant of Theorem 12.13, and  $C_{\text{sup},J}$  is chosen to dominate the constant of that theorem. Since  $p \geq 1$ , we have  $R_{\text{aux}}p \leq c_*$ , so for  $|\zeta| \leq R_{\text{aux}}$  the substituted variables satisfy  $|\zeta\lambda_i w_i| \leq c_* \leq s_{\text{aux}}$ , the denominators above are at least  $1/2$ , and, since  $p^3 + p + \epsilon p \leq 2(p^3 + \epsilon p)$ ,

$$\|H(\zeta) + sA_R(\zeta)\|_{\mathcal{S}_8} \leq 2C_{\text{sup},J}R_{\text{aux}}(p^3 + p + \epsilon p) \leq 4C_{\text{sup},J}c_* < \eta_0/2 \quad (0 \leq s \leq 1).$$

Under (13.8) we have  $tp \leq tp^3 \leq c_*/2$ , which implies  $C_{\text{sup},J}tp < 1/2$  and  $C_{\text{sup},J}(tp^3 + tp)/(1 - C_{\text{sup},J}tp) \leq 2C_{\text{sup},J}c_* \leq \sigma_{\text{seq}}$ ; these are the conditions (12.23). This proves the statements about the domain in (3).

By Theorem 12.7 and (13.16), the family  $F_s(\zeta) = D\mathcal{L}(H(\zeta) + sA_R(\zeta))[A_R(\zeta)]$  satisfies  $\|F_s(\zeta)\|_{\mathcal{S}_2} \leq C|\zeta|\epsilon p$  on the disc. Every term of  $H$  and of  $A_R$  vanishes at  $\zeta = 0$ . Their coefficients of order one in  $\zeta$  are cubic polynomials in the fluctuation variables for the classical terms, and linear polynomials for the Haar density and determinant terms and for the terms from  $R$ . Since each of them changes sign under  $w \mapsto -w$ , while the reference product  $\pi_p^{\text{all}}$  is invariant under this reflection, each supported integral of

first order vanishes. Terms with two interaction factors are of order at least two. Thus  $F_s(0) = F'_s(0) = 0$  on every support. The family  $F_s(\zeta)/\zeta$  is holomorphic, vanishes at zero, and is bounded by  $C\epsilon p$ . Cauchy's formula then gives, for  $0 < t \leq R_{\text{aux}}/2$ ,

$$\|F_s(t)\|_{\mathcal{S}_2} \leq C\epsilon p \frac{t^2}{R_{\text{aux}}(1 - t/R_{\text{aux}})} \leq \frac{2C}{c_*} t^2 \epsilon (p^4 + \epsilon p^2).$$

Integrating over  $s \in [0, 1]$ , we find that the difference  $\mathcal{D}(t) = \mathcal{L}(H(t) + A_R(t)) - \mathcal{L}(H(t))$  satisfies

$$\|\mathcal{D}(t)\|_{\mathcal{S}_2} \leq C\epsilon p \frac{t^2}{R_{\text{aux}}} \leq Ct^2 \epsilon (p^4 + \epsilon p^2), \quad (13.18)$$

at the coefficient radius  $\rho$ , without differentiation in the coarse field and without reduction of the radius. The dependence on  $\epsilon^2$  comes from the radius of the disc.

The auxiliary integral with the inserted interaction, the one without it, and the pullback at the zero-fluctuation reconstruction have the identities of Theorem 13.4 under gauge transformations, translations, the spatial inversion, and color conjugations. Subtracting, we find that the total difference of logarithms  $\mathcal{D}_J R$  is translation invariant. Applying part (2) to (13.18), we obtain

$$\mathcal{B}_{\rho,a}(\mathcal{D}_J R) \leq C_T t^2 \epsilon (p^4 + \epsilon p^2).$$

The constant term of  $\mathcal{D}_J R$  is not controlled by this norm; it is transferred to the normalization constant  $\Lambda$ , with densities written as  $\exp(\Lambda - E/g^2 - R)$ .

**Step 5: the pullback at zero fluctuation and the newly generated effective interaction.** Gauge invariance on  $B_\rho$  implies the local gauge-invariance identities required in Theorem 11.10. Indeed, for a fixed real small field, we take a small gauge variation at one vertex whose endpoints remain in the chart and differentiate the exact invariance identity. The other symmetries and  $DR(I) = 0$  are hypotheses. The background field  $e^{\delta \bar{A}_{d_0,J}(\theta)}$  has the first-derivative bounds of Lemma 11.9, and by the choice of  $J$  its image under the reconstruction majorant on  $\|\theta\|_\infty \leq 2\rho$  lies in the ball of radius  $\rho/2$ , since  $\delta Z(2\rho) \leq \delta Z(4K_*\rho) \leq \rho/4$ . Theorem 11.10, applied to this representative at the rate  $a < a_{\text{cen}}/8$  chosen in Step 1, with  $2\rho_0$  below the radius of Lemma 11.9, implies that

$$\mathcal{B}_{\rho,a}(\mathcal{P}_\delta R) \leq q_p \epsilon.$$

Its value is the value of  $R$  at the zero-fluctuation reconstruction that appears in the integral. By gauge invariance on  $B_\rho$ , this identification holds even if the representative in the block axial gauge is not in  $B_\rho$ .

The newly generated effective interaction, from Theorem 12.13, is

$$Q_{d_0,J}^w = B_{\text{ref}} - \left( \sum_i f_i \right) \log z_p - \log \Xi_{d_0,J}.$$

We bound its three terms in the norm  $\mathcal{B}_{\rho,a}$  directly, using only rates at least  $b_*$ ; the rate  $b_J$  of Theorem 12.13(5), which may depend on  $J$ , is not used.

The function  $B_{\text{ref}}(c) = \sum_i B_i(\widehat{c}_{i+1})$  depends on the coarse field; it is not a constant. Its coefficients of positive degree are obtained by composing the coefficient series of the terms  $B_i$  with the zero-fluctuation reconstruction  $\theta \mapsto \widehat{c}_{i+1}$ , that is, with the reconstruction of Step 3 of the proof of Theorem 13.2 at  $q = 0$ . The coefficients of positive degree of  $B_i$ , with one index fixed, are bounded by the logarithmic determinant series of the proof of Proposition 12.10 and by the bounds for the logarithmic Haar density and the minimizer of Lemma 12.3(3), at the rate of one averaging step. The reconstruction has coefficient bounds at the rate  $b_*$ , with its columns. The networks of the composed coefficients are connected through the indices of the fields  $\widehat{c}_{i+1}$ , and Lemma 13.1 applies. The conversion from the column normalization of these indices to the counting normalization of the scalar norm costs at most the factor  $\Gamma_J \leq M^{3J}$ , as in Step 3. We conclude that there exists  $C_B$ , depending only on  $M$ ,  $J$ , and the larger domains, such that

$$\frac{1}{k!} \|D_\theta^k B_{\text{ref}}(0)\|_{b_*, \text{root}} \leq C_B^{k+1} \quad (k \geq 1),$$

uniformly in  $d_0$  and  $m$ . Since  $a < b_*$  and  $\rho C_B \leq 1/2$  by the choice of  $\rho$  in Step 1,

$$\mathcal{B}_{\rho,a}(B_{\text{ref}}) \leq \sum_{k \geq 2} \rho^{k-2} C_B^{k+1} \leq 2C_B^3.$$

For the logarithm of the fluctuation partition function, Step 4 gives an exact supported decomposition  $\mathbf{H}(t)$  of  $H_{d_0,J}$ , at the coefficient radius  $\rho$  and with the boxes of Step 1, with  $\|\mathbf{H}(t)\|_{\mathcal{S}_8} \leq \eta_0/2$  under (13.8). By Theorem 12.7,  $-\log \Xi_{d_0,J} = \sum_Y \mathcal{L}_Y(\mathbf{H}(t))$  with  $\|\mathcal{L}(\mathbf{H}(t))\|_{\mathcal{S}_2} \leq C\eta_0$ , and each  $\mathcal{L}_Y$  depends only on the coarse-field coordinates in  $Y$ . The integral (12.25), the action  $E_{d_0+J}$ , and the function  $B_{\text{ref}}$  are invariant under the translations of the coarse lattice, by Theorem 12.13(6) and its proof, so  $\log \Xi_{d_0,J}$  is translation invariant. Since  $aC_{\mathcal{L}} \leq 2$ , the support condition of part (2) holds, and by part (2) we have

$$\mathcal{B}_{\rho,a}(\log \Xi_{d_0,J}) \leq 9C\eta_0\rho^{-2}.$$

The Gaussian normalization  $(\sum_i f_i) \log z_p$  is a constant; it has zero norm but remains in the density. Altogether  $\mathcal{B}_{\rho,a}(Q_{d_0,J}^w) \leq C_G$  with  $C_G = 2C_B^3 + 9C\eta_0\rho^{-2}$ , uniformly in the side, in  $d_0$ , and in  $t, p$  satisfying (13.8). Taking the negative logarithm of the integral formula (13.5) at the parameter  $t$ , and separating  $-\log A_{\text{coord}}$ , we obtain

$$R_+ = Q_{d_0,J}^w + \mathcal{P}_\delta R + \mathcal{D}_J R.$$

Adding the three bounds proves (13.9). The symmetries of the integral give the stated symmetries of  $R_+$ , and the color argument of Step 5 of the proof of Theorem 13.4 shows that  $DR_+(I) = 0$ . This proves (3).

**Step 6: one coupling threshold and the induction.** Let  $E_*$  and  $p(t) = A_w \log(1/t)$  be as in (4). For  $k > 0$  and  $l \geq 0$  we have  $t^k \log(1/t)^l \rightarrow 0$  as  $t \downarrow 0$ . There are finitely many strict requirements: the image conditions, the cluster conditions, the condition  $t \leq c_*/(2(p(t)^3 + E_* p(t)))$ , the condition on  $C_T t^2(p(t)^4 + E_* p(t)^2)$ , the bound  $tp(t) < s_J$ , and  $p(t) \geq p_0$ . For  $r \leq \rho$ , the bound  $tp(t) < s_J$  implies  $r + tp(t) < r_J$  and a positive denominator in  $\mathfrak{M}_J(r, tp(t))$ , by the inequalities for the enlarged product, and the intermediate maps have the margins already fixed. The additional inequality  $\mathfrak{M}_J(r, tp(t)) < r/2$  is still required whenever the support-inclusion hypothesis is invoked; it is not implied by a coupling threshold independent of  $r$ . We choose  $t_* > 0$  below the thresholds of all these requirements. For  $\epsilon \leq E_*$ , condition (13.8) holds, and by the recursion we have

$$C_G + q_P \epsilon + C_T t^2 \epsilon (p^4 + \epsilon p^2) \leq \frac{1 - q_P}{2} E_* + q_P E_* + \frac{1 - q_P}{2} E_* = E_*.$$

This proves (4). Starting with  $R_0 = 0$ , we construct the next effective interaction at the same  $\rho$ ,  $a$ , and  $E_*$  from the auxiliary integral, the nonvanishing of its continuation, and this bound, and we repeat this through every finite chain satisfying the coupling ceiling and output-side restriction at each step. Where the separate support-inclusion hypothesis holds at every step, Theorem 13.4 identifies each restricted density with the displayed expression, and the induction keeps every initial and pulled-back indicator. This proves (5).  $\square$

### 13.5 The conditional fluctuation potential

Dividing (13.5) by the integral over the fluctuations gives the conditional law of the normalized fluctuation variables given the coarse field. We show that its potential is uniformly convex, which implies dimension-free concentration and covariance estimates. The heuristic reason is the following count of powers of the coupling. By stationarity and the normalization by the constrained Hessian,

$$\frac{F(z^* + gK^{-1/2}w) - F(z^*)}{g^2} = \frac{1}{2}|w|^2 + O(g|w|^3),$$

while the Haar density terms, the determinant terms, and the input effective interaction contribute  $O(g|w|) + O(g^2|w|^2) + \dots$ . Substituting the fluctuations of coarser scales into the backgrounds of finer scales adds one factor  $g_l w_l$  for each such variable and preserves these powers. Thus the Hessian of the interaction is small when  $tp$  is small, while the complex polymer expansion of Section 13.4 needs  $tp^3$  small. On a larger product with radius  $P = q_*/t$ , in which only  $tP = q_*$  is small, real convexity still holds although the polymer expansion is not available.

We first record some facts about probability measures on a fixed product of balls. Here the product does not move when parameters vary. Differentiation of a family of

observables means differentiation of continuous functions with continuous derivatives on a neighborhood of the product, and the same formulas hold for families with dominated difference quotients and integrable derivatives.

**Lemma 13.6.** *Let  $\Omega$  be a fixed product of closed balls, let  $I \subset \mathbb{R}$  be an interval, and let  $(V_u)_{u \in I}$  be real functions such that  $V_u(w)$  and  $\partial_u V_u(w)$  are continuous on  $I \times N$  for a neighborhood  $N$  of  $\Omega$ . Let  $\mathbb{E}_u, \text{Cov}_u$  refer to the probability measure on  $\Omega$  with density proportional to  $e^{-V_u}$ , and let  $F_u$  be a family of observables as above.*

1. We have

$$\partial_u \mathbb{E}_u F_u = \mathbb{E}_u \partial_u F_u - \mathbb{E}_u (F_u \partial_u V_u) + \mathbb{E}_u F_u \mathbb{E}_u \partial_u V_u = \mathbb{E}_u \partial_u F_u - \text{Cov}_u(F_u, \partial_u V_u), \quad (13.19)$$

$$\text{and } \partial_u \log \int_{\Omega} e^{-V_u} dw = -\mathbb{E}_u \partial_u V_u.$$

2. Let  $m_u = \mathbb{E}_u w$ ,  $\xi = w - m_u$ ,  $\Sigma_u = \mathbb{E}_u \xi \xi^T$ , and  $Q_u = \partial_u V_u - \mathbb{E}_u \partial_u V_u$ . Then

$$\partial_u \Sigma_u(i, j) = -\mathbb{E}_u \xi_i \xi_j Q_u, \quad (13.20)$$

and

$$\mathbb{E} \xi_i \xi_j Q_u = \text{Cov}(\xi_i \xi_j, Q_u) = \text{Cov}(\xi_i, \xi_j Q_u) = \text{Cov}(\xi_j, \xi_i Q_u), \quad (13.21)$$

including the case  $i = j$ .

3. For a matrix  $A$ ,  $\|A\|_{\text{op}} \leq (\|A\|_{\text{row}} \|A\|_{\text{col}})^{1/2}$ , where  $\|A\|_{\text{row}} = \sup_i \sum_j |A_{ij}|$  and  $\|A\|_{\text{col}} = \sup_j \sum_i |A_{ij}|$ . If an array  $(b_{eij})$  satisfies  $\sup_e \sum_{i,j} |b_{eij}| \leq C$  and  $\sup_{i,j} \sum_e |b_{eij}| \leq C$ , then  $\|\sum_e a_e b_{e..}\|_{\text{HS}} \leq C \|a\|_2$  for every array  $(a_e)$ .

*Proof.* On a compact interval of  $u$ , the functions  $e^{-V_u}$ ,  $F_u$ , and their  $u$ -derivatives are bounded on  $\Omega$ . The fundamental theorem of calculus applied to difference quotients, followed by integration over the fixed set  $\Omega$ , differentiates the integrals without a boundary term, and the quotient rule gives (13.19). Taking  $F = 1$  before normalizing gives the formula for the logarithm. No integration by parts in  $w$  is used. For families with dominated difference quotients, the same argument applies with the integrable dominating function.

For (2), the explicit  $u$ -derivative of the observable  $\xi_i \xi_j$  has expectation  $-(\partial_u m_i) \mathbb{E} \xi_j - (\partial_u m_j) \mathbb{E} \xi_i = 0$ . Then (13.19) implies (13.20). Since  $\xi_i$ ,  $\xi_j$ , and  $Q_u$  are centered, expanding the definition of the covariance gives (13.21).

For (3), by the Cauchy–Schwarz inequality, we have

$$\sum_i \left| \sum_j A_{ij} v_j \right|^2 \leq \left( \sup_i \sum_j |A_{ij}| \right) \sum_{i,j} |A_{ij}| |v_j|^2 \leq \|A\|_{\text{row}} \|A\|_{\text{col}} \sum_j |v_j|^2.$$

Regard  $(b_{eij})$  as a matrix  $T$  with rows indexed by  $e$  and columns by pairs  $(i, j)$ . Since its row sums and column sums are at most  $C$ , we have  $\|T\|_{\text{op}} \leq C$ , and the same holds for

its transpose. The array  $\sum_e a_e b_{e..}$  is  $T^\top a$ , and its Hilbert–Schmidt norm is  $\|T^\top a\|_2 \leq C\|a\|_2$ . This uses the column sums; a bound for each output  $e$  separately would not suffice.  $\square$

The next lemma bounds conditional means along an interpolation that starts from a measure with zero means. We use it twice below.

**Lemma 13.7.** *Let  $\Omega$  be a fixed product of balls in the scalar coordinates  $w_i$ , and for  $s \in [0, 1]$  let  $\pi_s$  be the probability measure on  $\Omega$  with density proportional to  $e^{-V_s}$ , where  $V_s$  satisfies the hypotheses of Lemma 13.6 in  $s$ , with continuous derivatives  $\partial_{w_j} \partial_s V_s$ . Let  $m(s) = \max_i |\mathbb{E}_{\pi_s} w_i|$ . Assume that*

1.  $m(0) = 0$ ;
2. *there is a matrix  $L = (L_{ij})$  with nonnegative entries such that, for every  $s$  and  $i$ ,  $|\text{Cov}_{\pi_s}(w_i, \partial_s V_s)| \leq \sum_j L_{ij} \|\partial_{w_j} \partial_s V_s\|_{L^2(\pi_s)}$ , with  $\sup_i \sum_j L_{ij} \leq \Lambda_0$ ;*
3. *for every  $s$ ,  $\sup_j \|\partial_{w_j} \partial_s V_s\|_{L^2(\pi_s)} \leq \beta(1 + m(s)^2)$ .*

*If  $2\Lambda_0\beta < 1$ , then  $m(s) < 1$  and  $m(s) \leq 2\Lambda_0\beta s$  for every  $s \in [0, 1]$ .*

*Proof.* By (13.19) with  $F = w_i$ , we have  $d\mathbb{E}_{\pi_s} w_i / ds = -\text{Cov}_{\pi_s}(w_i, \partial_s V_s)$ . Since the entries of  $L$  are nonnegative, (2) and (3) give  $|d\mathbb{E}_{\pi_s} w_i / ds| \leq \Lambda_0\beta(1 + m(s)^2)$ . Integrating from 0 and using (1), we obtain  $m(s) \leq \Lambda_0\beta \int_0^s (1 + m(v)^2) dv$ . The function  $m$  is continuous. If  $m$  reached the value one, then at the first such time  $s_1 \leq 1$  we would have  $1 = m(s_1) \leq 2\Lambda_0\beta s_1 < 1$ , which is impossible. Thus  $m < 1$  on  $[0, 1]$ , and inserting this in the integral, we obtain  $m(s) \leq 2\Lambda_0\beta s$ .  $\square$

**Theorem 13.8.** *Fix the parameters of Theorem 13.5, with  $p = p(t) = A_w \log(1/t)$ , and reduce its coupling threshold  $t_*$  once to enforce the Hessian and mean-absorption bounds in the proof below. This reduction depends only on the fixed data, including  $A_w$ , and on the fixed  $q_*$  when part (3) is used. Let  $0 < t \leq t_*$  and let  $R$  be as in part (4) of that theorem, with  $\mathcal{B}_{p,a}(R) \leq E_*$ . Consider a renormalization group step whose restriction event satisfies both hypotheses of Definition 13.3. Then the following statements hold for every  $c = e^\theta \in B_r$ , every depth, and every side  $m \geq 4$ .*

1. *The conditional law of the normalized fluctuation variables given the coarse field is the conditional fluctuation kernel*

$$k(c, dw) = \frac{\mathbf{1}_{\Omega_p}(w) e^{-V_c(w)} dw}{\int_{\Omega_p} e^{-V_c(w)} dw}, \quad V_c = \frac{1}{2}|w|^2 + H_{d_0, J} + R(e^{\varphi_\theta(w)}) - R(e^{\varphi_\theta(0)}).$$

*The domain is the fixed product of three-dimensional balls, independent of  $c$ . The conditional fluctuation potential  $V_c$  contains all classical terms, the differences of Haar densities, the dependence of the determinants of finer scales on coarser fluctuations, and the full chain rule for the input effective interaction. If the whole hierarchy of restrictions is conditioned on this coarse field, the conditional law of the fluctuations of this step is the same kernel: the restrictions at coarser scales depend only on  $c$  and cancel in the normalization.*

2. There exist  $b, C_H, C > 0$ , depending only on  $M, J, \rho, a, E_*$ , and the larger domains, such that the entrywise envelope  $\mathcal{H}_{ij} = \sup_{w \in \Omega_p} |\partial_{w_i} \partial_{w_j} (V_c - |w|^2/2)|$  satisfies

$$\|\mathcal{H}\|_{b, \text{row} + \text{col}} \leq \frac{C_H [tp + t^2(1 + E_*)]}{(1 - C_H tp)^3} \leq \frac{1}{2}.$$

In particular  $D_w^2 V_c \geq I/2$  on the closed product, and  $V_c$  is smooth on a neighborhood of it. For the Neumann realization on the convex product  $\Omega_p$ , every real  $F \in H^1(k)$  satisfies

$$\text{Var}_k F \leq 2\mathbb{E}_k |\nabla F|^2, \quad \text{Ent}_k(F^2) \leq 4\mathbb{E}_k |\nabla F|^2,$$

and for real or complex  $F, G$ , with the covariance taken bilinear,

$$|\text{Cov}_k(F, G)| \leq C \sum_{i,j} e^{-b' \text{dist}(i,j)} \|\partial_i F\|_2 \|\partial_j G\|_2$$

for a fixed  $b' > 0$ . The constants do not depend on the dimension, on the radii of the balls, on the side, on the depth, or on the coupling below the reduced threshold  $t_*$ .

3. Choose  $0 < q_* < 4K_* s_J$  strictly inside the common enlarged product, with its image and denominator conditions and every intermediate comparison margin, and put  $P(t) = q_*/t$ . On  $\Omega_{P(t)}$  define the same smooth potential, whether or not this larger product satisfies the support-inclusion hypothesis. Let  $C_H$  be the Hessian coefficient constant in part (2). If  $C_H q_* < 1$  and

$$\frac{C_H [q_* + t^2(1 + E_*)]}{(1 - C_H q_*)^3} \leq \frac{1}{2},$$

then for each  $s \in [0, 1]$  the potential  $V_{s,c}^P = |w|^2/2 + s(V_c - |w|^2/2)$  has Hessian at least  $I/2$  on  $\Omega_{P(t)}$  and satisfies the inequalities of (2). If  $\pi_{s,c}^P$  denotes the normalized measure, then, for a sufficiently small fixed  $t_*$ ,

$$\sup_{s,c,i} |\mathbb{E}_{\pi_{s,c}^P} w_i| \leq Ct(E_* + 2), \quad \|w_i\|_{L^4(\pi_{s,c}^P)} \leq C(1 + \sup_j |\mathbb{E}_{\pi_{s,c}^P} w_j|).$$

*Proof.* We use the coefficient bounds of Theorem 13.2 and estimate the potential directly, before taking any logarithm of a partition function.

**Step 1: the conditional kernel.** In the density of Theorem 13.4, the factors  $A_{\text{coord}}$  and  $\exp[-E_{d_0+J}/g_J^2 - B_{\text{ref}} - R_{\text{ref}}]$  do not depend on  $w$ . Dividing the joint density by its positive integral over  $w$  cancels exactly these factors, and we obtain  $k(c, dw)$ . The domain is the product of balls of the normalized fluctuation variables, which does not depend on  $c$ . The terms  $B_i(c_{i+1}) - B_i(\hat{c}_{i+1})$  in  $H_{d_0,J}$  keep the dependence of the determinants and of the Haar densities at the minimizers on the coarser fluctuations, and its other terms are the differences of the Haar densities. Thus  $k(c, dw)$  is the full conditional law, not a perturbation of a Gaussian reference measure.

The same calculation can be expressed without choosing a version of the conditional law. To see this, we multiply the joint density by a bounded Borel function of  $c$  and a bounded Borel function of  $w$  and integrate first in  $w$  and then in  $c$ . The result is the joint integral of Theorem 13.4. This verifies the identity for the conditional kernel, and its pointwise version is the positive continuous integral. In a complete hierarchy, every restriction coarser than this step is a function of  $c$  and of its further averages. Such factors multiply both the numerator and the normalization of the conditional law by the same function of  $c$  and cancel. They do not cancel from the unnormalized law of the coarse field. This proves (1).

**Step 2: powers of the coupling.** Let  $U = V_c - |w|^2/2$ . We fix the coarse field in its larger ball but do not freeze the coarser fluctuations. At scale  $i$  the classical Taylor remainder has degree  $s \geq 3$  in the variables of its own scale, and the normalization of the action gives the factor  $g_i^{s-2}$ . Substituting the coarser fluctuations into the background adds  $r_l$  fluctuation variables from each coarser scale  $l > i$ . With total degree  $n = s + \sum_{l>i} r_l$  in  $w$ , the coupling factor is exactly

$$g_i^{s-2} \prod_{l>i} g_l^{r_l} = t^{n-2} \lambda_i^{s-2} \prod_{l>i} \lambda_l^{r_l}, \quad n \geq 3. \quad (13.22)$$

No negative power of a ratio  $\lambda_i \leq 1$  occurs, since  $s \geq 3$ . Centering at the minimizer removes the term of degree one in the variables of its own scale. The normalization by the positive square root of the Hessian removes the quadratic term, at every value of the coarser fluctuations. Thus no classical term of total degree one or two in  $w$  survives the substitution.

Each difference of Haar densities and each difference of determinant terms has positive total degree  $n$  in  $w$  and the coupling factor  $t^n$ . The composed difference of the input effective interaction obeys the same rule, with amplitude bounded by  $C\epsilon \leq CE_*$ . The joint coefficient bounds, evaluated on a slightly larger coarse-field radius and a slightly larger product of the variables  $q$ , therefore give the following bounds for the absolute coefficients with one index fixed:

| contribution                             | degree in $w$ | coefficient bound |
|------------------------------------------|---------------|-------------------|
| classical remainder                      | $n \geq 3$    | $C^n t^{n-2}$     |
| Haar density and determinant differences | $n \geq 1$    | $C^n t^n$         |
| difference of $R \circ e^\varphi$        | $n \geq 1$    | $C^n E_* t^n$ .   |

(13.23)

These bounds include every substitution of a background and every derivative of the matrices  $S_i$ . The fixed ratios  $\lambda_i$  and the fixed radii are included in  $C$ .

**Step 3: the Hessian.** We take two derivatives in  $w$  and sum their weighted rows. The multiplicities of the two derivatives are at most  $n(n-1)$ , and the other coordinates have

absolute value at most  $p$ . With  $x = Ctp < 1$ , the classical sum is

$$\sum_{n \geq 3} n(n-1)C^n t^{n-2} p^{n-2} = 2C^2 \{(1-x)^{-3} - 1\} \leq \frac{6C^3 tp}{(1-Ctp)^3},$$

where we used  $1 - (1-x)^3 \leq 3x$ . The sums for the Haar density and determinant terms and for the terms from  $R$  are bounded by

$$(1 + E_*) \sum_{n \geq 2} n(n-1)C^n t^n p^{n-2} = \frac{2C^2 t^2 (1 + E_*)}{(1 - Ctp)^3},$$

and the terms of degree one contribute nothing. The network weights give the decay in the distance between the two remaining indices, and by symmetry the column sums have the same bound. These sums of absolute coefficients, evaluated at the radius  $p$ , are uniform in  $w \in \Omega_p$ , so they bound the entrywise envelope  $\mathcal{H}_{ij}$ ; this is the bound (13.3) of Lemma 13.1 applied to the first derivatives of  $U$ . Choose  $C_H$  once to dominate the coefficient constants and fixed multipliers in these two sums. Then, on the branch  $C_H tp < 1$ , they give the stated bound for  $\|\mathcal{H}\|_{b, \text{row} + \text{col}}$ . For the part coming from  $R$ , both terms of

$$D^2(R \circ e^\varphi) = D^2(R \circ \exp)(\varphi)[D\varphi, D\varphi] + D(R \circ \exp)(\varphi)[D^2\varphi]$$

are included in its series of degree  $n$ . In particular, the first derivative of  $R$  multiplying the second derivative of the reconstruction is kept.

We reduce the coupling threshold of Theorem 13.5 so that  $C_H tp < 1/2$  and this bound is at most  $1/2$ . By Lemma 13.6(3), the operator norm is bounded by the row and column sums, and we obtain  $D_w^2 V_c \geq I/2$ . The margin in the coefficient bounds provides a smooth extension of the potential to an open neighborhood of the closed product.

**Step 4: the inequalities of the Neumann realization.** The balls are convex, have dimension three, and have radii independent of  $c$ , and the potential has the smoothness and the lower bound on the Hessian just proved. By Theorem 5.4 with  $m = 1/2$ , we have the variance constant 2 and the entropy constant 4, including the corners of the product. To check its spatial hypothesis, recall that the weighted row-plus-column norm of  $\mathcal{H}$  at the rate  $b$  is at most  $1/2$ . The hypothesis concerns the operator norms of the mixed blocks of the three-dimensional balls, so we group the scalar indices into their three-component blocks. The operator norm of a block is at most the sum of the absolute values of its scalar entries. Hence the weighted row and column sums of the operator norms of the mixed blocks are at most three times the scalar sums, that is, at most  $3/2$ . We set  $b' = b/8$ . For  $z \geq 0$ , convexity gives

$$e^{z/8} - 1 \leq \frac{1}{8}(e^z - 1) \leq \frac{1}{8}e^z.$$

Both block sums with the weight  $e^{b' \text{dist}(x,y)} - 1$  are therefore at most  $3/16$ , which is below  $m/2 = 1/4$ ; the diagonal Gaussian blocks have no mixed contribution. Thus the spatial

hypothesis of Theorem 5.4, which involves the suprema over  $\Omega_p$  of the mixed blocks, holds at the rate  $b'$ . Expanding the block gradient norms into their scalar components, we obtain the stated scalar covariance inequality, with a fixed change of its constant. Distinct blocks with the same location can be treated by separating them by an arbitrarily small distance, applying the inequality at a slightly smaller rate, and passing to the limit; the strict inequality  $3/16 < 1/4$  leaves room for this perturbation. All sums are finite at this stage, and the weighted bounds remain uniform in the limit. This proves (2).

**Step 5: the larger product.** The proof of the Hessian bound evaluated the majorant series at the radius  $tp$  of the variables  $q$ ; it did not use smallness of  $tp^3$ . Let  $0 < q_* < 4K_*s_J$  lie strictly inside the common enlarged product, retain its image and denominator conditions, every intermediate comparison margin, and the displayed Hessian smallness condition, and set  $P = q_*/t$ . Then  $\rho + q_* < r_J$ , the denominator of the reconstruction majorant is positive, and all intermediate margins follow from the enlarged product. The gauge-transformed reconstruction lies in  $B_{\rho/2}$ , where the input effective interaction is defined, and no inclusion in  $D_r$  is needed. If desired, we reduce the coupling threshold so that  $tp(t) < q_*$ ; then this product contains  $\Omega_p$ . We also impose  $C_H q_* < 1$ . Thus the two geometric series remain convergent after replacing  $p$  by  $P = q_*/t$ , and with the same  $C_H$  we obtain

$$\|\mathcal{H}^P\|_{b,\text{row}+\text{col}} \leq \frac{C_H[q_* + t^2(1 + E_*)]}{(1 - C_H q_*)^3},$$

for the entrywise envelope  $\mathcal{H}^P$  of  $D_w^2 U$  on  $\Omega_P$ . The potential on this product is the smooth potential of the auxiliary integral, and no indicator of  $D_r$  is inserted. Under the displayed condition,  $|w|^2/2 + sU$  has Hessian at least  $I/2$  for every  $s \in [0, 1]$ , its mixed blocks satisfy the same bound, and Step 4 applies on this product. No complex expansion at the radius  $P$  is used.

**Step 6: the means on the larger product.** It remains to bound the means without a loss of a factor  $P^2$ . By the concentration statement of Theorem 5.4, we have for each scalar coordinate  $\mathbb{P}(|w_i - \mathbb{E}w_i| > u) \leq 2e^{-u^2/4}$ , hence  $\mathbb{E}|w_i - \mathbb{E}w_i|^4 \leq \int_0^\infty 8u^3 e^{-u^2/4} du \leq C$ . Thus, with  $m(s) = \max_i |\mathbb{E}_{\pi_{s,c}^P} w_i|$ ,

$$\|w_i\|_{L^4(\pi_{s,c}^P)} \leq C(1 + m(s)), \quad (13.24)$$

and the bound for three-component blocks changes only  $C$ .

If we differentiate a classical monomial of degree  $n$  in  $U$  with respect to one fluctuation variable, there remain  $n - 1 \geq 2$  fluctuation factors. We place two of them in  $L^4$  and the others in  $L^\infty$ , and Hölder's inequality bounds their product in  $L^2$  by  $C(1 + m(s)^2)^{P^{n-3}}$ . The coupling factor becomes

$$t^{n-2} P^{n-3} = tq_*^{n-3}. \quad (13.25)$$

The absolute coefficient series converges at  $q_*$ , and the margin in the radius absorbs the multiplicities of the derivative and the choice of the two factors placed in  $L^4$ . The derivatives of the Haar density and determinant terms and of the terms from  $R$  of degree  $n$  give  $t^n P^{n-1} = tq_*^{n-1}$ , hence the bound  $Ct(1 + E_*)$ . It follows that

$$\sup_j \|\partial_{w_j} U\|_{L^2(\pi_{s,c}^P)} \leq Ct(1 + E_* + m(s)^2) \leq Ct(1 + E_*)(1 + m(s)^2). \quad (13.26)$$

These bounds hold with one index of  $w$  fixed, with a profile that decays exponentially in the distance to that index. Summing the covariance kernel of Step 4 over its second index costs only a constant depending on the density. We apply Lemma 13.7 to  $V_s = |w|^2/2 + sU$ , for which  $\partial_s V_s = U$ . At  $s = 0$  the measure is the even product of truncated Gaussian measures, and  $m(0) = 0$ . Hypothesis (2) of the lemma holds with the nonnegative kernel  $L_{ij} = Ce^{-b' \text{dist}(i,j)}$  of the covariance bound of Step 4, whose row sums are at most a constant  $\Lambda_0$  depending only on the fixed data, and hypothesis (3) holds with  $\beta = Ct(1 + E_*)$  by (13.26). After one fixed reduction of the coupling threshold,  $2\Lambda_0\beta < 1$ , and the lemma implies that  $m(s) \leq Ct(1 + E_*) \leq Ct(E_* + 2)$ , uniformly in  $s$ ,  $c$ , and the volume. Together with (13.24) this proves (3). All bounds are bounds at a fixed coordinate, and the total dimension does not appear.  $\square$

### 13.6 Conditional means and coarse-field derivatives

We now study how the conditional fluctuation kernel depends on the coarse field. The main point is that its means are small, of order  $t\|\theta\|_\infty$ , even though the kernel is interacting. At the identity coarse field they vanish by the global color symmetry. We also control the derivative of the conditional fluctuation covariance, by a three-point decay estimate, and the first two Taylor coefficients of the reconstruction at the conditional mean, which are used in the later reconstruction estimates. We first prove the three-point estimate under abstract hypotheses.

**Lemma 13.9.** *Let  $\Omega$  be a fixed product of balls of radii at most  $p$ ,  $p \geq 1$ , in scalar coordinates  $w_i$ , and let  $V_\theta$  be potentials depending on parameters  $\theta_u$ , with the probability measures of density proportional to  $e^{-V_\theta}$  on  $\Omega$ , along the segment  $\{s\theta : 0 \leq s \leq 1\}$ . Assume that  $V_\theta(w)$  and  $\partial_{\theta_u} V_\theta(w)$ , for every parameter coordinate  $u$ , are jointly continuous on  $\mathcal{U} \times N$ , where  $\mathcal{U}$  is an open neighborhood of the segment and  $N$  is a neighborhood of the fixed compact product  $\Omega$ . Let  $\Sigma_\theta$  be the covariance matrix of  $w$ . Suppose that  $D_0 \geq 2$ ,  $A_{sc} \geq 0$ , and that along the segment, for*

every location  $x$  and every  $R \geq 0$ ,

$$\begin{aligned} \#\{i : \text{dist}(i, x) \leq R\} + \#\{u : \text{dist}(u, x) \leq R\} &\leq D_0(1 + R)^3, \\ |\text{Cov}(F, G)| &\leq \sum_{i,j} L_{ij} \|\partial_i F\|_2 \|\partial_j G\|_2, \quad L_{ij} \leq D_0 e^{-\text{dist}(i,j)/D_0}, \\ \sup_{\Omega} |\partial_{w_i} \partial_{\theta_u} V| &\leq A_{\text{sc}} D_0 e^{-\text{dist}(i,u)/D_0}. \end{aligned}$$

Let  $\tau_3(i, j, u)$  be the sum of the two smallest of the distances  $\text{dist}(i, j)$ ,  $\text{dist}(i, u)$ , and  $\text{dist}(j, u)$ . Then, for an absolute constant  $C$ ,

$$|\partial_{\theta_u} \Sigma_{\theta}(i, j)| \leq C p A_{\text{sc}} D_0^6 e^{-\tau_3(i, j, u)/(4D_0)}, \quad \|\Sigma_{\theta} - \Sigma_0\|_{\text{op}} \leq C p A_{\text{sc}} D_0^{14} \|\theta\|_{\infty}.$$

*Proof.* The joint continuity assumptions bound the potential and each parameter derivative on compact sub-neighborhoods of the fixed product, so differentiation under the integral is justified and the covariance has continuous parameter derivatives. We write  $D = D_0$  and  $A = A_{\text{sc}}$ . The counting hypothesis implies, for each fixed  $c > 0$ ,

$$\sup_x \sum_i e^{-\text{dist}(i,x)/(cD)} + \sup_x \sum_u e^{-\text{dist}(u,x)/(cD)} \leq C_c D^4. \quad (13.27)$$

To prove it, let  $N_x(R) = \#\{i : \text{dist}(i, x) \leq R\}$  and use  $e^{-r/(cD)} = (cD)^{-1} \int_r^{\infty} e^{-R/(cD)} dR$ ; summing this identity over  $i$  bounds the first sum by

$$\frac{1}{cD} \int_0^{\infty} e^{-R/(cD)} D(1 + R)^3 dR \leq C_c D^4,$$

and the sum over  $u$  is bounded in the same way. The argument also works on small periodic tori. By the triangle inequality,

$$\sum_l e^{-\text{dist}(i,l)/D - \text{dist}(l,u)/D} \leq C D^4 e^{-\text{dist}(i,u)/(2D)}, \quad (13.28)$$

indeed, we extract the exponential on the right using half of the two distances and sum the remaining factor  $e^{-\text{dist}(i,l)/(2D)}$  by (13.27).

Since each coordinate and its mean have absolute value at most  $p$ , we have  $|\xi_i| \leq 2p$ , where  $\xi = w - \mathbb{E}w$ . Moving between two points of the convex product along a segment and using the bound on the mixed derivatives, we find that the oscillation of  $\partial_{\theta_u} V$  is at most  $2pAD \sum_l e^{-\text{dist}(l,u)/D}$ . Subtracting the expectation does not increase this bound, and the centered potential derivative  $Q_u = \partial_{\theta_u} V - \mathbb{E} \partial_{\theta_u} V$  satisfies

$$\|Q_u\|_{\infty} \leq C p A D^5, \quad |\partial_l Q_u| \leq A D e^{-\text{dist}(l,u)/D}. \quad (13.29)$$

By (13.20),  $\partial_{\theta_u} \Sigma(i, j) = -\mathbb{E} \xi_i \xi_j Q_u$ , and we bound this by each of the three covariances in (13.21). For the first, the gradient of  $\xi_i \xi_j$  is  $\partial_l(\xi_i \xi_j) = \delta_{li} \xi_j + \delta_{lj} \xi_i$ . Using  $\|\xi_i\|_2 \leq 2p$ , the amplitudes  $D$  and  $AD$  of the two kernels, and (13.28), we obtain

$$|\mathbb{E} \xi_i \xi_j Q_u| \leq C p A D^6 \{e^{-\text{dist}(i,u)/(2D)} + e^{-\text{dist}(j,u)/(2D)}\}.$$

For the second, the gradient of  $\xi_i$  is the coordinate vector at  $i$ , and  $\partial_l(\xi_j Q_u) = \delta_{lj} Q_u + \xi_j \partial_l Q_u$ . The first term costs  $L_{ij} \|Q_u\|_2 \leq CpAD^6 e^{-\text{dist}(i,j)/D}$ , and the second costs the convolution (13.28) times  $2pAD^2$ . We obtain

$$|\mathbb{E} \xi_i \xi_j Q_u| \leq CpAD^6 \{e^{-\text{dist}(i,j)/(2D)} + e^{-\text{dist}(i,u)/(2D)}\}.$$

For the third, the coordinate vector is at  $j$ , the gradient of the product is  $\delta_{lj} Q_u + \xi_j \partial_l Q_u$ , and the two resulting terms are bounded by

$$CpAD^6 \{e^{-\text{dist}(i,j)/(2D)} + e^{-\text{dist}(j,u)/(2D)}\}.$$

Using only the first of these three bounds would not control the separation of  $i$  from  $j$ .

We order the three distances as  $x \leq y \leq z$ . The smallest of the three braces is at most  $e^{-y/(2D)} + e^{-z/(2D)} \leq 2e^{-y/(2D)}$ , and since  $\tau_3 = x + y \leq 2y$ , this is at most  $2e^{-\tau_3/(4D)}$ . This proves the bound on  $\partial_{\theta_u} \Sigma(i, j)$ .

Since  $\tau_3$  is the smallest length of the three trees with two edges on the points  $i, j, u$ , the factor  $e^{-\tau_3/(4D)}$  is at most the sum of the three corresponding products of exponentials. For fixed  $i$ , summing one such tree over  $(j, u)$  removes its two leaves successively, and (13.27) gives a factor at most  $CD^8$ . Hence

$$\sup_i \sum_{j,u} |\partial_{\theta_u} \Sigma(i, j)| \leq CpAD^{14}.$$

Integrating along the segment  $s\theta$ ,  $0 \leq s \leq 1$ , and using  $|\theta_u| \leq \|\theta\|_\infty$ , we bound the row sums of  $\Sigma_\theta - \Sigma_0$ . By symmetry in  $i, j$ , the column sums have the same bound, and Lemma 13.6(3) implies the operator bound.  $\square$

In the following theorem the conditional law is the kernel  $k(c, dw)$  of Theorem 13.8 on the product  $\Omega_p$ , and the derivatives are taken with the couplings, and hence the radii of the balls, held fixed. We write  $\mathbb{E}_\theta, \text{Cov}_\theta$  for this kernel at  $c = e^\theta$ .

**Theorem 13.10.** *Use the conditional fluctuation kernel of Theorem 13.8 with  $p = A_w \log(1/t)$ , and reduce the coupling threshold once so that  $tp^3$  is small. Let  $m_\theta = \mathbb{E}_\theta w$ ,  $\xi = w - m_\theta$ ,  $\Sigma_\theta = \mathbb{E}_\theta \xi \xi^\top$ , and  $\varphi_\theta(w) = \Phi_{d_0, J}(\theta, (g_i w_i)_i)$ . There exist  $C, b > 0$  such that the following statements hold uniformly for all  $\|\theta\|_\infty < r \leq \rho$  for which the support-inclusion hypothesis holds, all depths, and all sides.*

1. *The conditional means satisfy*

$$\|m_\theta\|_\infty \leq Ct \|\theta\|_\infty, \quad \sup_e |\mathbb{E}_\theta \varphi_{\theta, e}(w) - \varphi_{\theta, e}(0)| \leq Ct^2 \|\theta\|_\infty,$$

*and both vanish at  $\theta = 0$  by the global color symmetry.*

2. For  $0 \leq s \leq 1$  let  $\dot{V}_s = dV_{s\theta}/ds$  on the same product. Then, uniformly in  $s$ ,

$$\sup_l \|\partial_{w_l} \dot{V}_s\|_{L^2(k(e^{s\theta}, \cdot))} \leq Ct \|\theta\|_\infty.$$

This bound holds at each fixed index of  $w$ , with the exponentially summable profile of the joint coefficient bounds; it is not a bound on a norm of the extensive function  $\dot{V}_s$ . It includes the classical, Haar density, and determinant terms of the potential, and the terms from  $R$ .

3. For every family  $F_\theta$  of observables on this fixed product satisfying the differentiability conditions stated before Lemma 13.6 (continuity with continuous derivatives on a neighborhood of the product, or dominated difference quotients with integrable derivatives), and every coordinate  $\theta_u$ , the identities (13.19), (13.20), and (13.21) hold; in particular

$$\partial_u \mathbb{E}_\theta F_\theta = \mathbb{E}_\theta \partial_u F_\theta - \text{Cov}_\theta(F_\theta, \partial_u V_\theta), \quad \partial_u \log \int_{\Omega_p} e^{-V_\theta} dw = -\mathbb{E}_\theta \partial_u V_\theta,$$

and  $\partial_u \Sigma_\theta(i, j) = -\mathbb{E}_\theta \xi_i \xi_j Q_u$  with  $Q_u = \partial_u V_\theta - \mathbb{E}_\theta \partial_u V_\theta$ . The domain does not move, so there is no boundary term.

4. The hypotheses of Lemma 13.9 hold with a fixed  $D_0$  and with  $A_{sc} = Ctp^2$ ; more precisely,

$$\sup_{\Omega_p} |\partial_{w_i} \partial_{\theta_u} V_\theta| \leq Ctp^2 e^{-b \text{dist}(i, u)},$$

$$\|\Sigma_\theta - \Sigma_0\|_{b', \text{row} + \text{col}} \leq Ctp^3 \|\theta\|_\infty, \quad \|\Sigma_\theta\|_{\text{op}} \leq 2,$$

for a fixed  $b' > 0$ . Here  $\Sigma_0 = \Sigma_{\text{sp}} \otimes I_3$  is the covariance of the conditional fluctuation kernel at the identity coarse field; this kernel is interacting and truncated, and  $\Sigma_{\text{sp}}$  includes all its spatial covariances.

5. Put  $g = g_0$ ,  $E_\theta = g^{-1} D_w \varphi_\theta(m_\theta)$ , and  $B_\theta = (2g^2)^{-1} D_w^2 \varphi_\theta(m_\theta)$ . For every  $w = m_\theta + \xi \in \Omega_p$ ,

$$\varphi_\theta(m_\theta + \xi) = \varphi_\theta(m_\theta) + g E_\theta \xi + g^2 B_\theta(\xi, \xi) + \mathcal{R}_\theta(\xi), \quad \|\mathcal{R}_\theta\|_\infty \leq Cg^3 p^3.$$

Moreover  $\|E_\theta\|_{\text{op}} \leq C$ , the scalar coefficients  $b_{eij}$  of  $B_\theta$  satisfy

$$\sup_e \sum_{i,j} |b_{eij}| + \sup_i \sum_{e,j} |b_{eij}| \leq C, \quad \left\| \sum_e a_e b_{e..} \right\|_{\text{HS}} \leq C \|a\|_2$$

for every array  $(a_e)$ , and

$$\|E_\theta - E_0\|_{\text{op}} + \|B_\theta - B_0\|_{\text{abs,root}} \leq C \|\theta\|_\infty,$$

$$\|E_\theta \Sigma_\theta E_\theta^\top - E_0 \Sigma_0 E_0^\top\|_{\text{op}} \leq C \|\theta\|_\infty (1 + tp^3), \quad \sup_e |\mathbb{E}_\theta B_{\theta,e}(\xi, \xi)| \leq C \|\theta\|_\infty,$$

where  $\|\cdot\|_{\text{abs,root}}$  is the sum of absolute coefficients with one index fixed. At  $\theta = 0$  the matrix  $E_0$  is scalar in the color components, and the contraction of the color-equivariant quadratic tensor  $B_0$  with  $\Sigma_0$  vanishes.

6. For the unnormalized integral of one step,

$$\tilde{F}(\theta) = A_{\text{coord}} e^{-E_{d_0+J}(e^\theta)/g_J^2 - B_{\text{ref}}(\theta) - R_{\text{ref}}(\theta)} \int_{\Omega_p} e^{-V_\theta(w)} dw,$$

we have

$$D \log \tilde{F} = -D(E_{d_0+J}/g_J^2 + B_{\text{ref}} + R_{\text{ref}}) - \mathbb{E}_\theta DV_\theta.$$

The derivatives of the terms evaluated at the zero-fluctuation reconstruction disappear only from normalized conditional expectations.

The constants depend only on  $M, J, \rho, a, E_*, A_w$ , the ratios of the couplings, and the fixed domains; they are independent of the side,  $L, \gamma$ , the depth, and the coarse field satisfying the hypotheses. Converting volume-normalized column bounds into bounds for operators in counting coordinates costs a fixed factor depending on  $J$ .

*Proof.* We use the coefficient bounds of Theorem 13.2 throughout; the conditional law is always the kernel on  $\Omega_p$  of Theorem 13.8. We first prove (3) and (4), then the bounds on the means and on the derivative of the potential by Lemma 13.7, and finally the statements about the reconstruction.

**Step 1: differentiation and the mixed derivatives.** With the couplings, and hence the radii of the balls, fixed, the potential  $V_\theta$  and its derivatives in  $\theta$  are continuous on a neighborhood of the closed product, and Lemma 13.6 applies and proves (3).

We differentiate the series of (13.23) once in a coordinate  $\theta_u$  of the coarse field and once in a fluctuation variable  $w_i$ . A derivative in  $\theta$  on the larger product of coarse fields preserves the powers of the coupling and the coefficient bounds. A classical term of degree  $n$  leaves  $n - 1$  fluctuation factors after the derivative in  $w_i$ , so its sum is bounded by

$$Ctp^2 \sum_{n \geq 3} n(Ctp)^{n-3} \leq \frac{Ctp^2}{(1 - Ctp)^2}.$$

The Haar density and determinant terms and the terms from  $R$  give at most  $C(1 + E_*)t/(1 - Ctp)^2$ , which is absorbed in the same bound since  $E_*$  is fixed and  $p \geq 1$ . Keeping both indices  $i$  and  $u$  in the network of each coefficient gives the factor  $e^{-b \text{dist}(i,u)}$ . Reducing the coupling threshold so that  $Ctp$  is below a fixed margin, we obtain

$$\sup_{\Omega_p} |\partial_{w_i} \partial_{\theta_u} V_\theta| \leq Ctp^2 e^{-b \text{dist}(i,u)}. \quad (13.30)$$

The Gaussian term contributes nothing to this derivative in  $\theta$ , taken with all couplings fixed, since the normalization by the constrained Hessian made it equal to  $|w|^2/2$  for every coarse field.

The density of the indices and the covariance bound of Theorem 13.8 satisfy the hypotheses of Lemma 13.9 with a fixed  $D_0$  depending only on the fixed data, and by

(13.30) we may take  $A_{\text{sc}} = Ctp^2$ . The proof of that lemma is linear in  $A_{\text{sc}}$ , and it yields the bound  $Ctp^3e^{-b_1\tau_3}$  for the derivative of  $\Sigma$ . We keep an exponential weight in the final sums. Since  $\text{dist}(i, j) \leq \tau_3$  by the triangle inequality, we may choose  $0 < b' < b_1/2$  and absorb  $e^{b'\text{dist}(i, j)}$  into half of the exponential of the tree. The remaining sums over two-edge trees converge by the fixed density. Integrating along the segment of coarse fields, we obtain

$$\|\Sigma_\theta - \Sigma_0\|_{b', \text{row} + \text{col}} \leq Ctp^3\|\theta\|_\infty. \quad (13.31)$$

Applying the variance bound of Theorem 13.8 to  $\sum_i v_i w_i$ , we find  $v^\top \Sigma_\theta v \leq 2|v|^2$ , hence  $\|\Sigma_\theta\|_{\text{op}} \leq 2$ . This proves (4), except for the structure of  $\Sigma_0$ , which is proved in Step 5.

**Step 2: the conditional means.** The concentration bound used in (13.24) implies, on  $\Omega_p$ , that  $\|w_i\|_{L^4} \leq C(1 + m(s))$ , where now  $m(s) = \max_i |\mathbb{E}_{s\theta} w_i|$ . We set  $\dot{V}_s = \sum_u \theta_u \partial_{\theta_u} V_{s\theta}$  and take one derivative in  $w$  in its joint series. Every coarse field  $s\theta$  lies in  $B_r$ , so the support-inclusion hypothesis holds along the whole segment. The larger product of coarse fields of Theorem 13.5 controls the derivative in  $\theta$ , also when  $\|\theta\|_\infty$  is close to  $r$ . The couplings, the radii of the balls, and the parameters internal to  $R$  are fixed while  $s$  varies. From a classical term of degree  $n \geq 3$  we keep two of the remaining factors in  $L^4$  and bound the other  $n - 3$  by  $p$ , which gives the coupling factor  $t^{n-2}p^{n-3} = t(tp)^{n-3}$ . The derivative in  $\theta$  and its column sum give the factor  $\|\theta\|_\infty$ . Thus

$$\sup_j \|\partial_{w_j} \dot{V}_s\|_2 \leq Ct\|\theta\|_\infty(1 + E_* + m(s)^2), \quad (13.32)$$

with all norms in  $L^2(k(e^{s\theta}, \cdot))$ . More precisely, we keep the index  $j$  of  $w$  and a scalar coarse-field index  $v$  in the network of each coefficient before contracting  $v$  with  $\theta_v$ . The exponential weight then gives the factor  $e^{b\text{dist}(j, v)}$ , while the two remaining fluctuation factors are estimated in  $L^4$ . The same summation shows that

$$\|\partial_{w_j} \dot{V}_s\|_{L^2(k(e^{s\theta}, \cdot))} \leq Ct(1 + E_* + m(s)^2) \sum_v e^{-b\text{dist}(j, v)} |\theta_v|, \quad \sup_j \sum_v e^{-b\text{dist}(j, v)} \leq C, \quad (13.33)$$

the last sum being over coarse-field indices of fixed density, bounded uniformly in the side by the count used in (13.27). The Haar density and determinant terms and the terms from  $R$  contribute  $Ct(1 + E_*)\|\theta\|_\infty$ , since one derivative in  $w$  of a term of degree  $n \geq 1$  leaves  $t^n p^{n-1} = t(tp)^{n-1}$ . The margins in the coefficient bounds absorb all multiplicities of the derivatives. Summing the exponential covariance kernel against this profile costs only a constant depending on the density.

At the identity coarse field, a constant color conjugation rotates every three-component block of  $w$  and preserves the balls and the potential. A vector fixed by all such rotations vanishes, so  $m(0) = 0$ . We apply Lemma 13.7 to  $V_s = V_{s\theta}$ , with the nonnegative exponential kernel and the constant  $\Lambda_0$  of the covariance bound of Theorem 13.8 and

with  $\beta = Ct\|\theta\|_\infty(1 + E_*)$ , by (13.32) and  $1 + E_* + m^2 \leq (1 + E_*)(1 + m^2)$ . Since  $\|\theta\|_\infty < \rho$ , after one fixed reduction of the coupling threshold we have  $2\Lambda_0\beta \leq Ct\rho(E_* + 2) < 1$ , and the lemma implies that  $m(s) < 1$  and  $m(1) \leq Ct\|\theta\|_\infty$ . This proves the first bound in (1), for the interacting conditional means. Inserting  $m(s) < 1$  in (13.32) and (13.33), and absorbing the factor  $E_* + 2$  into  $C$ , we obtain

$$\sup_j \|\partial_{w_j} \dot{V}_s\|_{L^2(k(e^{s\theta}, \cdot))} \leq Ct\|\theta\|_\infty \quad (0 \leq s \leq 1),$$

which is (2). It uses the same coupling threshold as the bound on the means, and no norm of the extensive function  $\dot{V}_s$ .

**Step 3: the mean of the reconstruction.** Let  $G_e(\theta, w) = \varphi_{\theta,e}(w) - \varphi_{\theta,e}(0)$ . Its coefficients of degree  $n$  in  $w$  carry the factor  $t^n$ , and every term has positive degree in  $w$ . The same holds after a derivative in  $\theta$  on the larger product. Its rows in the fluctuation variables are bounded by  $Ct$  on  $\Omega_p$ . For  $\dot{G}_e = dG_e(s\theta, w)/ds$ , the expectation of the part of degree one is at most  $Ct\|\theta\|_\infty m(s)$ . For degrees at least two we keep two fluctuation factors in  $L^4$  and bound the others by  $p$ . Summing the series with one index fixed, we obtain

$$|\mathbb{E}_{s\theta} \dot{G}_e| \leq Ct\|\theta\|_\infty \{tm(s) + t^2(1 + m(s)^2)\}. \quad (13.34)$$

The covariance of  $G_e$  with  $\dot{V}_s$  is at most  $Ct^2\|\theta\|_\infty(1 + E_* + m(s)^2)$ , by the bound  $Ct$  on the rows of  $G_e$ , the profile (13.32), and the summable covariance kernel. By Step 2,  $tm(s) \leq Ct^2\|\theta\|_\infty$ . Applying (13.19) to  $G_e$ , we find

$$\left| \frac{d}{ds} \mathbb{E}_{s\theta} G_e(s\theta, w) \right| \leq Ct^2\|\theta\|_\infty.$$

At the identity coarse field  $\Omega(0) = I$ , and the reconstructed logarithm is color equivariant by Theorem 13.2(2). Its expectation is then a vector invariant under all rotations of the color components, which must vanish. Integrating from 0 to 1 proves the second bound in (1). The reference value  $\varphi_\theta(0) = \delta \bar{A}_{d_0, J}(\theta)$  is the zero-fluctuation reconstruction.

**Step 4: the expansion at the conditional mean.** Since each block of the mean lies in its closed convex ball, the segment  $m_\theta + v\tilde{\zeta}$ ,  $0 \leq v \leq 1$ , lies in  $\Omega_p$  whenever  $m_\theta + \tilde{\zeta} = w \in \Omega_p$ . Three derivatives in  $w$  of the reconstruction have row sums at most  $Cg^3$ , since the ratios  $g_i/g$  are fixed. By Taylor's formula, we have

$$\mathcal{R}_\theta(\tilde{\zeta}) = \frac{1}{2} \int_0^1 (1-v)^2 D_w^3 \varphi_\theta(m_\theta + v\tilde{\zeta})[\tilde{\zeta}, \tilde{\zeta}, \tilde{\zeta}] dv,$$

and since  $\|\tilde{\zeta}\|_\infty \leq 2p$ , we obtain  $\|\mathcal{R}_\theta\|_\infty \leq Cg^3p^3$ . The first two Taylor coefficients are exactly  $E_\theta$  and  $B_\theta$ , including the factor  $1/2$  in the latter.

The row bounds and the volume-normalized column bounds of  $E_\theta$  and  $B_\theta$  follow from the absolute coefficient series of the reconstruction. Replacing the volume factors by

counting measures costs only a fixed factor depending on  $J$ . By Lemma 13.6(3), we have  $\|E_\theta\|_{\text{op}} \leq C$ . The quadratic tensor satisfies  $\sup_e \sum_{i,j} |b_{eij}| \leq C$  and  $\sup_i \sum_{e,j} |b_{eij}| \leq C$ ; the second bound implies  $\sup_{i,j} \sum_e |b_{eij}| \leq C$ , and Lemma 13.6(3) implies the Hilbert–Schmidt bound.

The normalized first and second derivatives of the reconstruction have bounded first derivatives, in their coefficient norms with one index fixed, on a still larger product. Comparing them at  $(\theta, (g_i m_{\theta,i})_i)$  and at  $(0,0)$  along the straight segment and integrating their derivatives, we obtain the bound

$$C\{\|\theta\|_\infty + g\|m_\theta\|_\infty\} \leq C\|\theta\|_\infty.$$

The same bound holds for their rows and their columns, so the first derivative also satisfies the operator bound in (5). No derivative of the mean as a function of the coarse field is needed for this comparison.

**Step 5: color structure and contractions.** At the identity coarse field, the potential and the gauge-transformed reconstruction commute with every constant rotation of the color components, and  $m_0 = 0$  by Step 2. Therefore each  $3 \times 3$  block of  $\Sigma_0$  and of  $E_0$  commutes with every rotation. Such a matrix is a multiple of the identity: rotations by  $\pi$  about the coordinate axes force its off-diagonal entries to vanish, and rotations exchanging two axes equate its diagonal entries. This proves  $\Sigma_0 = \Sigma_{\text{sp}} \otimes I_3$  and that  $E_0$  is scalar in the color components.

We expand

$$E_\theta \Sigma_\theta E_\theta^T - E_0 \Sigma_0 E_0^T = (E_\theta - E_0) \Sigma_\theta E_\theta^T + E_0 (\Sigma_\theta - \Sigma_0) E_\theta^T + E_0 \Sigma_0 (E_\theta - E_0)^T.$$

The bounds of Step 4,  $\|\Sigma_\theta\|_{\text{op}} \leq 2$ , and (13.31) give the bound  $C\|\theta\|_\infty(1 + tp^3)$ .

For one scalar output component  $e$ , let  $B_{\theta,e}$  be the symmetric matrix of the quadratic form  $B_{\theta,e}$ . Each scalar matrix unit has nuclear norm one, so the sum of the absolute coefficients bounds the nuclear norm, and

$$\|B_{\theta,e}\|_* \leq C, \quad \|B_{\theta,e} - B_{0,e}\|_* \leq C\|\theta\|_\infty.$$

The contraction of the color-equivariant quadratic tensor at the identity coarse field with the color-scalar matrix  $\Sigma_0$  is a vector invariant under all rotations of the color components, hence zero. This includes all spatial covariances of the kernel at the identity. Subtracting this zero contraction, we obtain

$$|\mathbb{E}_\theta B_{\theta,e}(\xi, \xi)| \leq \|B_{\theta,e} - B_{0,e}\|_* \|\Sigma_\theta\|_{\text{op}} + \|B_{0,e}\|_* \|\Sigma_\theta - \Sigma_0\|_{\text{op}} \leq C\|\theta\|_\infty(1 + tp^3) \leq C\|\theta\|_\infty,$$

where in the last inequality we used the reduced coupling threshold. This proves (5).

**Step 6: the unnormalized integral.** The constant  $A_{\text{coord}}$  does not depend on the coarse field. Differentiating  $\log \tilde{F}$  and using the formula for the derivative of the logarithm of the normalization in (3), we obtain (6). It keeps the derivatives of the classical, determinant, and Haar density terms and of  $R_{\text{ref}}$ , evaluated at the zero-fluctuation reconstruction, which cancel only in normalized conditional expectations. All estimates used sums with one index fixed, at a fixed density and fixed  $J$ . The constants therefore have the stated dependence and do not depend on the depth or the volume.  $\square$

## Small-field conditioning and partition-function bounds

This section connects the analytic recursion of Section 13 with the Wilson measure conditioned on the small-field event. We prove four results. First, consider one RG step and a fine configuration such that it and its coarse field both lie in the gauge-invariant small-field domain  $D_r$ . After a hierarchical tree gauge fixing, every such configuration has unique normalized fluctuation coordinates, whether or not it satisfies the fluctuation cutoff (Theorem 14.3). Second, with the parameters chosen in Definition 14.4, the recursion of Theorem 13.5, started at the Wilson action, produces effective interactions  $R_k$  whose densities are exactly those of the pushforwards of the Wilson measure restricted by the successive tests of the small-field event  $S_{N,u,\gamma}$ . Every normalization constant is accounted for (Theorem 14.6). Third, for coarse fields in  $D_r$ , if  $d_t < 1$  and  $\epsilon_t < 1$ , then imposing the fluctuation cutoff changes the unnormalized density by a factor whose reciprocal lies between 1 and  $(1 - \epsilon_t)^{-1}$ , with explicit errors  $d_t$  and  $\epsilon_t$  (Theorem 14.7). Fourth, on the stopping lattice of side  $n$ , we bound the partition function of the stopped measure from below by a tree-gauge trial integral over the same domain  $D_r$ . We also bound exponential moments of the Wilson action under the stopped measure with a single boundary term  $9n^2 \log(1/g)$  (Theorem 14.8). Finally, every flat configuration, including those whose holonomies form one of the eight central triples, belongs to the small-field event once the coarse side is large enough (Lemma 14.9).

Two points need care. First, the effective interactions are holomorphic on the ball  $B_\rho$  of configurations with link logarithms of norm less than  $\rho$ , while the small-field event uses a smaller radius  $r < \rho/8$ . On all of  $B_\rho$ , the recursion is defined by an auxiliary integral over the fluctuation cutoff domain. This integral is defined because the reconstruction maps the whole cutoff domain into  $B_{\rho/2}$ , where the incoming effective interaction is analytic. For coarse fields in  $B_r$ , the auxiliary integral coincides with the fiber integral of the restricted Wilson density. For coarse fields in  $B_\rho \setminus B_r$ , it is only an analytic continuation, with no probabilistic meaning. Second, the fiber of the incoming restricted measure over a coarse field need not be convex. However, it is contained in a product domain  $\mathcal{K}_t$ , independent of the coarse field, on which the conditional fluctuation potential is uniformly convex. Tail bounds for the normalized measure on  $\mathcal{K}_t$  pass to the fiber by dividing by its conditioning probability; this does not assert a logarithmic Sobolev or

covariance-decay inequality for the nonconvex fiber.

In the partition-function bounds, the denominator is bounded below on the same domain  $D_r$  as the stopped measure. Only the numerator is enlarged, to an integral over an open cube. After tree gauge fixing, the periodic lattice has  $2n^3 + 1$  independent link variables and the open cube has  $2n^3 - 3n^2 + 1$ . Since each variable contributes three powers of  $g$ , the comparison leaves exactly the factor  $g^{-9n^2}$ .

We use the following inputs. Sections 5 to 7 supply the coefficient calculus, tree gauges, the Haar disintegration over the block averaging constraint, the partition-function bound on an open box, and the local inverse branch of the block averaging map. Sections 10 and 11 supply the classical background fields, the nested minimization identity for the effective classical action, its comparison with the Wilson action, and its gauge invariance on chart overlaps. Section 12 supplies the joint coordinate calculus and the sequential density formula. Section 13 supplies the reconstruction gauge and its coefficient majorant, the invariant analytic ball of the recursion, the cutoff identity for the restricted density, and the convexity of the conditional fluctuation potential. In turn, Section 15 uses the exact recursion and the conditional fluctuation kernels; Sections 18 and 21 to 23 use the fiber coordinates, the classical lower bound, the normalization, and the cutoff comparison; and Sections 33 to 36 use the estimates on the stopping lattice and the flat configurations. This section does not show that the small-field event has probability close to one; that is done in Section 23. In particular, we do not estimate here the ratio of auxiliary integrals that bounds the exit probability of the coarse field in Theorem 14.7.

Separating the field into regular gauge-field coordinates and then performing successive convergent integrations is the constructive lattice gauge program of Bałaban [14, 18]. Within this framework, we prove the coordinates for every point of the small-field fiber and the normalization identities below for the specific recursion and event used in this paper.

Section 14.1 fixes notation. Section 14.2 collects the classical estimates on a common domain and derives the exact density of one RG step in fluctuation coordinates. Section 14.3 proves the coordinate theorem and states the choice of parameters and the small-field event. Section 14.4 identifies the recursion with the restricted Wilson measure, and Section 14.5 proves the cutoff comparisons. Section 14.6 contains the partition-function bounds, and Section 14.7 treats flat configurations.

## 14.1 Conventions

**One RG step.** Fix an odd integer  $M \geq 3$  and an integer  $J \geq 1$ , and set  $B = M^J$ . One RG step maps a configuration on the lattice of side  $Bm = M^J m$  to a configuration on the lattice of side  $m \geq 4$  by  $T_M^J$ . Its levels are indexed by  $i = 0, \dots, J$ ; the integrated intermediate scales have  $i < J$ . The lattice at level  $i$ , which we also call a scale, has side

$M^{J-i}m$ . We write  $c_i$  for the configuration at level  $i$ , so that  $c_0$  is the fine configuration and  $c_J = c$  is the coarse one, and we write  $c = e^\theta$  when  $c$  lies in a chart of the identity. If the fine lattice is at classical depth  $d_0$ , the averaging step from scale  $i$  to scale  $i+1$  is performed at depth  $d_0 + i$ . The couplings satisfy  $g_{i+1}^2 = Mg_i^2$ . The integrated couplings are  $g_0, \dots, g_{J-1}$ , and we write

$$t = \max_{0 \leq i < J} g_i = g_{J-1}, \quad t^2 = \frac{\gamma}{Mm}, \quad \lambda_* = M^{-(J-1)/2} \leq \frac{g_i}{t} \leq 1, \quad g_J^2 = Mt^2 = \frac{\gamma}{m}.$$

The normalized fluctuation variables  $w_i$  of the step from scale  $i$  to scale  $i+1$  are defined by

$$z_i = z_i^*(c_{i+1}) + g_i K_i(c_{i+1})^{-1/2} w_i,$$

where  $z_i$  are the free logarithmic coordinates,  $z_i^*$  is the conditional constrained minimizer, and  $K_i$  is the constrained Hessian (Section 14.2). A block index  $(i, e)$  labels one vector  $w_{i,e} \in \mathbb{R}^3$ ; a scalar index also specifies its color component. Distances are measured in units of the coarse lattice of side  $m$ . For fixed  $M$  and  $J$ , the number of coordinate labels in a ball of radius  $R$  is at most  $C_{M,J}(1+R)^3$ , uniformly on every periodic lattice, since the smallest label spacing is  $M^{-J}$  and there are finitely many scale and orientation types. This bound controls all sums of exponential kernels below.

**Norms.** For a matrix  $A$  indexed by labels and a rate  $b > 0$ , the weighted row-and-column norm is

$$\|A\|_{b, \text{row}+\text{col}} = \max \left\{ \sup_x \sum_y e^{b \text{dist}(x,y)} |A_{xy}|, \sup_y \sum_x e^{b \text{dist}(x,y)} |A_{xy}| \right\}.$$

By Schur's test, it bounds the operator norm of  $A$  on  $\ell^2$ . This is the maximum of the row bound and the counting column bound. In Section 10, the same two bounds are added instead. That sum lies between this norm and twice it, so the bounds imported from there hold for this norm with the same constants. No column-volume factor enters this comparison, since both conventions use counting columns. For a tensor  $T$ ,  $\|T\|_{b, \text{root}}$  denotes the weighted absolute coefficient sum with one index fixed, maximized over the choice and position of the fixed index, with spatial weight  $e^{bt}$ , where  $t$  is the length of a shortest connecting network of the coefficient locations (Definition 4.6). Input-fixed sums are volume-normalized unless we say that they are counting sums. For a function  $R$  of the link logarithms with  $DR(I) = 0$ , the weighted analytic coefficient norm is

$$\mathcal{B}_{\rho, a_{\text{sp}}}(R) = \sum_{k \geq 2} \frac{\rho^{k-2}}{k!} \|D^k R(I)\|_{a_{\text{sp}}, \text{root}},$$

as in Definition 4.6; it does not control the constant  $R(I)$ .

**Gauge-invariant extensions.** Let  $B_\rho = \{\exp A : \max_e |A_e| < \rho\}$  and  $D_\rho = \mathcal{G}B_\rho$ , the set of configurations gauge equivalent to a configuration in  $B_\rho$ . A real analytic function  $R$  on  $B_\rho$  is *gauge invariant on overlaps* if  $R(U^k) = R(U)$  whenever  $U$  and  $U^k$  lie in  $B_\rho$  for a periodic lattice gauge transformation  $k$  (Definition 13.3). Then  $R_{\text{sat}}(U) = R(U^k)$ , for any  $k$  with  $U^k \in B_\rho$ , is a well-defined function on  $D_\rho$ , which is gauge invariant and real analytic. The same subscript denotes the gauge-invariant extension  $E_{d,\text{sat}}$  of the effective classical action from its chart (Theorem 11.6). Global color rotations act by  $U_e \mapsto uU_eu^{-1}$  with constant  $u \in \text{SU}(2)$ , and spatial inversion is  $x \mapsto -x$  with the induced action on oriented links.

**Constants.** Unless a parameter is displayed separately, the constants  $C, c, b, \dots$  of this section depend only on  $M$ , the fixed  $J$ , the larger classical and analytic domains fixed below and the positive distances between them, the positive spatial rates, and the fixed bounds on coefficients and effective interactions. They do not depend on the lattice side, on  $L$  or  $\gamma$ , on the depth, or on the number of RG steps. All estimates concern the whole torus; we make no statement about block regions. The value  $R(I)$  of an effective interaction at the identity is always kept separately. Only constants explicitly identified as uniform before the choice of  $J$  are used to choose  $J$ .

**Radii.** Several radii occur. The following table lists them, and Definition 14.4 fixes the inequalities between them.

| Radius                                                                  | Role                                                                |
|-------------------------------------------------------------------------|---------------------------------------------------------------------|
| $r_C$                                                                   | larger domain of the classical estimates (Section 14.2)             |
| $\rho$                                                                  | effective interactions are holomorphic on $B_\rho$                  |
| $r < \rho/8$                                                            | radius of the small-field domain $D_r$                              |
| $R_{\text{buf}} = (10M)^J r$                                            | $T_M^J(D_r) \subset D_{R_{\text{buf}}}$                             |
| $P_{\text{phys}} < Q_{\text{phys}}$                                     | physical radii of products of fluctuations contained in every fiber |
| $R_{\text{en}} = \mathcal{L}_{\text{en}} r, \mathcal{L}'_{\text{en}} r$ | physical radii of the enclosing products                            |
| $p_w(t) = A_w \log(1/t)$                                                | radius of the fluctuation cutoff in the variables $w$               |

A product of fluctuation balls of radius  $a/t$  in the variables  $w$  is said to have physical radius  $a$ ; on it  $|g_i w_{i,e}| \leq a$  for all  $i, e$ , since  $g_i \leq t$ .

## 14.2 Classical coordinates and the conditional density

This subsection puts the classical estimates of Sections 10 to 12 on one common domain, proves the coordinate calculus needed below, and derives the density of one RG step in

the normalized fluctuation variables. We prove the classical and coarea identities on the whole branch of the local constraint coordinates, before imposing any fluctuation cutoff. The density formula involving an incoming interaction is restricted to reconstructed fine fields in  $D_\rho$ , with its reference value also defined there.

### Classical estimates on a larger domain

For an averaging step at depth  $d$ , with coarse field  $c$  and free logarithmic coordinates  $z$ , we write

$$b(c, z) = e^{\Gamma_c(z)} U_0(c), \quad B(c, z) = \text{Log } b(c, z), \quad F_d(c, z) = E_d(b(c, z)).$$

Here  $U_0(c)$  is the coarse-field lift, which is the identity on block interiors and has the prescribed coarse links on the interface links, and  $\Gamma_c$  are the local constraint coordinates solving the block averaging constraint for the eliminated edges (Lemma 6.3). We always write the lift with its argument, to distinguish it from a fine configuration  $U_0$ .

Fix a larger classical ball of radius  $r_C$  strictly inside the radii of Theorems 10.10, 10.11, 11.2, and 11.6; in particular  $r_C < \min(r_{CK}, \rho_{ov}, r_*)$ , where  $r_{CK}$ ,  $\rho_{ov}$ , and  $r_*$  are the radii of Theorem 11.2, Theorem 11.6, and Theorem 10.11. We take the products of coarse fields and free coordinates of one averaging step strictly inside the joint products of Lemma 12.3, and small enough that  $B(c, z)$  and every representative related to it by a gauge transformation used below stay in  $B_{r_C}$ . These reductions depend only on the fixed construction domains, not on the side or the depth. On these products, we use the following data:

$$m_0 I \leq K_d^0 \leq C_0 I, \quad \|K_d^0\|_{a_0, \text{row}+\text{col}} + \|(K_d^0)^{-1}\|_{a_0, \text{row}+\text{col}} \leq C_0, \quad (14.1)$$

$$\|D^s E_d\|_{a_0, \text{root}} \leq C_0^{s+1} s! \quad (s \geq 2), \quad E_d(I) = D E_d(I) = 0, \quad (14.2)$$

$$E_{d+1}(c) = M F_d(c, z_d^*(c)), \quad E_d(b^k) = E_d(b), \quad (14.3)$$

the last identity on the larger overlaps, for periodic gauge transformations  $k$  with  $b, b^k \in B_{r_C}$ . Here

$$K_d^0 = (D_z B(I, 0))^T D^2 E_d(I) D_z B(I, 0),$$

and  $z_d^*(c)$  is the conditional constrained minimizer, an interior stationary point. The constants  $m_0, C_0$  and the rate  $a_0 > 0$  are independent of  $d$  and the side. This spatial rate  $a_0$  is local to the present subsection and unrelated to the trial amplitude  $a_0$  of Theorem 14.8. The lower bound in (14.1) bounds the Hessian in the free coordinates  $z$  of the constraint; we assert no lower bound for the quadratic form in the coarse variables.

These data come from the cited results as follows. Theorem 10.10 gives (14.1), the conditional constrained minimizer, and the first identity in (14.3), including the factor  $M$ , on the whole real product of free coordinates of that theorem. The minimizing property and the positivity of Hessians are used only for real coarse fields and free

coordinates. The stationary equation and the identity for the stationary value then extend holomorphically to smaller complex products, where no minimum is asserted. The sides in that theorem are  $Mm$  and  $m$  with  $m \geq 4$ , and we infer nothing from it about iterated steps. Theorem 10.11 gives the tensors in (14.2) at one rate, and its bound on the first derivative gives  $DE_d(I) = 0$ . We take only the derivatives of the scalar  $E_d$  from that theorem, not its counting input columns for the coordinate maps. The upper bound of Theorem 11.2 at the identity gives  $E_d(I) = 0$ . We take one common smaller rate  $a_0$  and enlarge  $C_0$  to cover both results. Since the two representatives of each classical overlap lie in  $B_{r_C} \subset B_{\rho_{\text{ov}}}$ , Theorem 11.6 implies the second identity in (14.3) for every periodic gauge transformation, without a global logarithm of the gauge transformation.

Finally, Lemma 12.3 supplies joint tensor bounds for  $B$ ,  $F_d$ , the negative logarithmic Haar density  $\mathcal{L}$  defined below,  $z_d^*$ ,  $K_d$ , and  $K_d^{\pm 1/2}$  on the chosen products. For a matrix-valued derivative with a fixed coarse-field index, it sums both matrix indices, and its input-fixed column sums are volume-normalized. Since the volume ratios for one averaging step, or for the  $J$  steps of one RG step, are fixed powers of  $M$ , passing to counting column sums costs only a constant depending on  $M$  and  $J$ . These bounds control no field-independent value. The exact determinant and the fluctuation interaction  $H_{d_0,J}$  on cutoff products are given by Theorem 12.13; below we also derive the underlying pointwise identities on the larger coordinate domain just fixed.

**Lemma 14.1.** *On the larger domains just fixed, there are smaller radii  $r_c, r_z > 0$  with the following properties, for every depth  $d \geq 0$  and every averaging step from side  $Mm$  to side  $m \geq 4$ . The conditional constrained minimizer  $z_d^*(c)$  is unique,  $\|z_d^*(e^\theta)\|_\infty \leq C\|\theta\|_\infty$ , and throughout the real product  $\|\theta\|_\infty < r_c$ ,  $\|z\|_\infty \leq r_z$  we have*

$$\begin{aligned} D_z^2 F_d &\geq m_0 I / 2, & \|D_z^2 F_d\|_{b, \text{row} + \text{col}} &\leq C, \\ \|K_d^{1/2}\|_{b, \text{row} + \text{col}} + \|K_d^{-1/2}\|_{b, \text{row} + \text{col}} &\leq C, & K_d &= D_z^2 F_d(c, z_d^*(c)). \end{aligned}$$

*The minimizer  $z_d^*$  and the two square roots extend holomorphically to a common smaller complex product, and their weighted absolute Taylor coefficients of degree  $s$ , for output rows and for volume-normalized input columns, are bounded by  $C^{s+1}s!$ . Compositions of these maps over the  $J$  averaging steps of one RG step have the same property, with constants depending on  $J$ .*

*Proof.* For every  $d \geq 0$  and every averaging step  $Mm \rightarrow m$  with  $m \geq 4$ , we apply Theorem 10.10 on the intersection of its product with the larger products fixed above. This gives the unique real interior minimizer, its linear bound in the coarse field, and the Hessian estimate on the whole real product. Keeping the local labels, we have

$$D_z^2 F_d = (D_z B)^\top D^2 E_d(B) D_z B + DE_d(B) D_z^2 B, \quad (14.4)$$

and

$$\|D_z^2 F_d - K_d^0\|_{b, \text{row} + \text{col}} \leq C(\|\theta\|_\infty + \|z\|_\infty), \quad (14.5)$$

with  $D_z^2 F_d \geq m_0 I/2$  for real arguments. The second term in (14.4), which contains the first derivative of  $E_d$ , is kept. Here  $b$  is a positive rate smaller than the rates of Theorem 10.10 and of the joint calculus. Combining the upper bound in (14.1) with (14.5), we obtain the upper bound for the Hessian.

Next, we use the clause of Lemma 12.3 for coordinates of fixed physical radius, which does not involve the coupling. It supplies the holomorphic minimizer and the matrices  $K_d^{1/2}$  and  $K_d^{-1/2}$ , with factorial bounds on their weighted absolute coefficients for rows, columns, and coarse-field indices, summing both matrix indices. Its radii and constants are chosen before the number  $J$  of averaging steps in an RG step, and they are uniform in  $d$  and  $m$ . We choose  $r_z$  inside those products and then reduce  $r_c$  so that  $Cr_c < r_z/8$ , keeping all earlier strict inclusions. For one averaging step, passing from volume-normalized to counting columns costs the factor  $M^3$ , which is included in  $C$ . The two square roots are the positive square roots for real coarse fields and their holomorphic continuations; no positivity is asserted for complex coarse fields.

For fixed  $J$ , we compose these tensors using Lemma 5.3, with a volume-normalized input column along the branch that contains the fixed input index and output rows elsewhere. Since each contraction attaches an output to an input, the coefficient networks remain connected. We pass to trees on the external indices only at the end. The finite list of meshes in one RG step has label density bounded in terms of  $M$  and  $J$ , and the volume ratios cancel at contracted indices. A fixed loss in the spatial rate and one fixed smaller radius, both depending on  $J$ , then give the factorial bounds for all the compositions. The resulting constant  $C_J$  is independent of the side, the depth,  $L$ , and  $\gamma$ .  $\square$

### Haar disintegration and the fluctuation potential

Fix a coarse field  $c$  and the block forests rooted at its coarse vertices. By the block version of Lemma 5.1, the passage to forest-fixed coordinates is a unique based change of Haar variables; the tree variables have total mass one, and there is no variable at a coarse vertex. On the branch of the local constraint coordinates, Lemma 6.3, or equivalently the coarea clause of Lemma 12.3, gives the Haar disintegration density

$$\mathcal{J}_c(z) = c_{\text{Haar}}^f \frac{\prod_{e \text{ not in the forest}} j_H(\Gamma_{c,e}(z))}{\prod_a \det D_{X_{e_a^*}} C_{c,a}(\Gamma_c(z))}, \quad j_H(X) = \text{sinc}^2(|X|/2). \quad (14.6)$$

Here  $f = 2(M^3 - 1)m^3$  is the number of free group variables,  $c_{\text{Haar}} = 1/(16\pi^2)$ ,  $a$  runs over the coarse links,  $e_a^*$  is the eliminated edge of the constraint  $C_{c,a} = 0$  at  $a$ , and  $X_{e_a^*}$  is its logarithmic coordinate. The density is taken relative to the product Haar measure on coarse configurations and includes the Haar factor of every eliminated edge. It holds for nonnegative Borel integrands on every measurable subset of this branch, with all indicators of the branch conditions kept. It requires neither smallness of the interaction nor nonvanishing of any partition function.

For the sequential change of variables, we set  $S_i = K_i^{-1/2}$  and order the variables from scale  $J - 1$  down to scale 0. Since  $c_{i+1}$  depends only on the fluctuation variables of the coarser scales  $i + 1, \dots, J - 1$ , the Jacobian matrix is block triangular and

$$\det D_w z = \prod_{i=0}^{J-1} g_i^{3f_i} \det S_i(c_{i+1}). \quad (14.7)$$

Here  $f_i$  is the number of free group variables at scale  $i$ ; the off-diagonal derivatives of  $S_i$  and  $z_i^*$  do not change the triangular determinant. We define

$$\begin{aligned} \mathcal{L}_i(c, z) &= -\log[\mathcal{J}_i(c, z) / \mathcal{J}_i(I, 0)], \\ B_i(c) &= \mathcal{L}_i(c, z_i^*(c)) + \frac{1}{2} \log \det K_i(c), \\ \widehat{c}_J &= c, \quad \widehat{c}_i = b_i(\widehat{c}_{i+1}, z_i^*(\widehat{c}_{i+1})), \quad B_{\text{ref}}(c) = \sum_i B_i(\widehat{c}_{i+1}), \\ U_i^{\text{cl}} &= g_i^{-2} [F_i(c_{i+1}, z_i) - F_i(c_{i+1}, z_i^*(c_{i+1}))] - \frac{1}{2} |w_i|^2, \\ H_{d_0, J} &= \sum_i [U_i^{\text{cl}} + \mathcal{L}_i(c_{i+1}, z_i) - \mathcal{L}_i(c_{i+1}, z_i^*(c_{i+1})) + B_i(c_{i+1}) - B_i(\widehat{c}_{i+1})]. \end{aligned}$$

The fields  $\widehat{c}_i$  form the zero-fluctuation reconstruction of  $c$ . The determinant term  $B_i$  of a finer scale is evaluated at the coarse field  $c_{i+1}$  obtained from the given fluctuation variables of the coarser scales. Only its value along the zero-fluctuation reconstruction is subtracted. The nested minimization identity (14.3) and  $g_{i+1}^2 = M g_i^2$  give the telescoping identity

$$\frac{E_{d_0}(c_0)}{g_0^2} = \frac{E_{d_0+J}(c_J)}{g_J^2} + \frac{1}{2} |w|^2 + \sum_i U_i^{\text{cl}}. \quad (14.8)$$

Indeed  $F_i(c_{i+1}, z_i) = E_{d_0+i}(c_i)$  and  $F_i(c_{i+1}, z_i^*(c_{i+1})) = E_{d_0+i+1}(c_{i+1})/M$ , so the  $i$ th term of the sum equals  $E_{d_0+i}(c_i)/g_i^2 - E_{d_0+i+1}(c_{i+1})/g_{i+1}^2 - |w_i|^2/2$ .

Let  $\Phi(\theta, q) = \Phi_{d_0, J}(\theta, q)$  be the logarithm of the fine configuration reconstructed from  $c = e^\theta$  and the variables  $q_i = g_i w_i$ , in the gauge given by the gauge transformation of Theorem 13.2. This gauge transformation depends only on  $c$  and is the identity at the coarse vertices, so  $T_M^J e^{\Phi(\theta, q)} = e^\theta$ . Write  $C_c(w) = \mathcal{C}_{\text{rawd}_0, J}(\theta, (g_i w_i)_i)$  and  $g w = (g_i w_i)_i$ . For an incoming effective interaction  $R$ , assume  $C_c(0) \in D_\rho$  and set  $R_{\text{ref}}(c) = R_{\text{sat}}(C_c(0))$ . Where the gauge-transformed reference lies in  $B_\rho$ , this equals  $R(e^{\Phi(\theta, 0)})$ . Multiplying (14.6) over the scales and by (14.7), and then inserting (14.8), we find the following. On every measurable set of fluctuation variables whose reconstructed configurations lie in  $D_\rho$ , the density of the measure  $e^{-E_{d_0}/g_0^2 - R_{\text{sat}}} dH$  in the coordinates  $(c, w)$ , relative to  $dH(c) dw$ , has a common prefactor, independent of the fluctuation variables and of the integration subset, namely

$$\prod_i \mathcal{J}_i(I, 0) g_i^{3f_i} \exp[-E_{d_0+J}(c)/g_J^2 - B_{\text{ref}}(c) - R_{\text{ref}}(c)], \quad (14.9)$$

multiplied by  $e^{-V_c(w)}$ , where  $V_c(w) = |w|^2/2 + H_{d_0,J}(\theta, w) + R_{\text{sat}}(C_c(w)) - R_{\text{ref}}(c)$ . When both gauge-transformed images lie in  $B_\rho$ , its interaction difference is also  $R(e^{\Phi(\theta, gw)}) - R(e^{\Phi(\theta, 0)})$ . To check this, we write  $\mathcal{J}_i(c_{i+1}, z_i) = \mathcal{J}_i(I, 0)e^{-\mathcal{L}_i(c_{i+1}, z_i)}$  and  $\det S_i = e^{-(\log \det K_i)/2}$ , and regroup  $\sum_i [\mathcal{L}_i(c_{i+1}, z_i) + (\log \det K_i(c_{i+1}))/2]$  as  $H_{d_0,J} - \sum_i U_i^{\text{cl}} + B_{\text{ref}}$ . The remaining interaction term is  $R_{\text{sat}}(C_c(w)) - R_{\text{ref}}(c)$ ; its expression through  $\Phi$  uses the just stated image condition. A geometric branch alone does not extend  $R$  beyond  $D_\rho$ . Theorem 13.4 states exactly this formula on the cutoff product whenever its hypothesis on the image of the coarse fields in  $B_r$  holds, and there the identities agree term by term with Theorem 12.13. The classical and coarea derivation uses only the branch of the local constraint coordinates; the interaction-dependent formula additionally uses membership in  $D_\rho$  and the defined reference value. It does not extend the small-interaction or nonvanishing conclusions of Theorem 12.13 to a product of fixed physical radius.

The next lemma counts powers of the coupling in the Taylor coefficients of the potential. We use it for the convexity of the potential on the enclosing product and for the mean bound in Theorem 14.7, where one derivative in the coarse field is needed.

**Lemma 14.2.** *Fix the ball  $B_\rho$  and a function  $R$  holomorphic there, with  $DR(I) = 0$  and  $\mathcal{B}_{\rho, \text{asp}}(R) \leq E_*$ , gauge invariant on overlaps, and invariant under global color rotations, lattice translations, and spatial inversion. Use fixed smaller products of coarse fields and of free coordinates of fixed physical radius on which the coefficient majorant of Theorem 13.2 holds with image in  $B_{\rho/2}$ , and on which the classical and coarea conditions hold with strict inequality. Define*

$$V_\theta(w) = \frac{1}{2}|w|^2 + H_{d_0,J}(\theta, w) + R(e^{\Phi(\theta, (g_i w)_i)}) - R(e^{\Phi(\theta, 0)}), \quad W_\theta = V_\theta - \frac{1}{2}|w|^2.$$

*After the Taylor series in the coarse field has been summed on a strictly smaller radius, the following holds on these domains. The classical terms of  $W_\theta$  of total degree  $k$  in  $w$  vanish for  $k \leq 2$ , and for  $k \geq 3$  they have the form  $t^{k-2}$  times coefficients whose weighted absolute sums, with any free or coarse-field index fixed and with a fixed positive rate, are at most  $C_J^{k+1}$ . The coarea, determinant, and effective-interaction terms of total degree  $k \geq 1$  have the form  $t^k$  times coefficients with the corresponding bound  $C_J^{k+1}(1 + E_*)$ . The same statements hold for one derivative in the coarse field, with a further fixed constant. Consequently, on the product  $\{|w_{i,e}| \leq p\}$  with  $tp \leq q$ , where  $q$  is at most the fixed physical radius of these domains and  $Cq < 1$ ,*

$$\|D_w^2 W_\theta\|_{b, \text{row} + \text{col}} \leq \frac{C[q + t^2(1 + E_*)]}{(1 - Cq)^3}. \quad (14.10)$$

*Proof.* For real coarse fields, the Taylor expansion of  $F_i - F_i(z_i^*)$  at the conditional constrained minimizer has no zeroth or first order term, and normalization by the positive square root of  $K_i$  makes its second order term exactly  $g_i^2 |w_i|^2/2$ . By the stationary equation and the square-root identity of Lemma 14.1, these coefficient identities extend holomorphically to the smaller complex products. After division by  $g_i^2$ , a term of order

$k$  in  $w_i$  therefore has  $k \geq 3$  and the factor  $g_i^{k-2}$ . Each dependence on the fluctuation variables of a coarser scale  $j$  through the field  $c_{i+1}$  carries an additional factor  $g_j w_j$ . Since  $\lambda_* \leq g_i/t \leq 1$ , dividing by two such couplings changes constants by at most the fixed factor  $\lambda_*^{-2}$ . Every classical monomial of total degree  $k$  thus carries  $t^{k-2}$ . Each difference of  $\mathcal{L}_i$  or of  $B_i$  has positive degree in  $w$  and no inverse power of the coupling, so it carries  $t^k$ .

We control these compositions by the weighted coefficient bounds of Lemma 14.1: we fix the external index, sum the indices of the innermost differentiated map, and proceed outward, using the volume-normalized column at an input index. By the triangle inequality, the union of the intermediate connecting networks bounds the external network length. A fixed loss in the spatial rate absorbs the bounded label density of each of the  $J$  kinds of labels. For  $\log \det K$ , we write, on a smaller domain,

$$\log \det K = \log \det K^0 + \sum_{h \geq 1} \frac{(-1)^{h+1}}{h} \operatorname{tr} [((K^0)^{-1}(K - K^0))^h].$$

Here the perturbation  $(K^0)^{-1}(K - K^0)$  has weighted norm below  $1/2$ . A derivative of positive order fixes one factor of this cycle, and its two matrix indices can then be summed by the weighted bounds. The constant trace belongs to  $B_{\text{ref}}$  and is not bounded by this calculation. For the effective interaction, since the image of the coefficient majorant lies in  $B_{\rho/2}$ , the joint pullback bound of Theorem 13.5 applies. Its constant term is subtracted as  $R_{\text{ref}}$ , and each free input supplies its own factor  $g_i$ . The same weighted contractions give the factor  $t^k$ .

We divide the factorial Taylor bounds by the Taylor factorial and sum the series in the coarse field on a fixed, strictly smaller radius. This leaves geometric majorants  $C_J^{k+1}$ , and one derivative in the coarse field is controlled on the same smaller radius. Two derivatives in  $w$  cost at most  $k(k-1)$ . For the classical part we sum  $C^k t^{k-2} p^{k-2}$  over  $k \geq 3$ , and for the other part we sum  $C^k t^k p^{k-2}$  over  $k \geq 2$ . The identity  $\sum_{h \geq 0} (h+1)(h+2)x^h = 2/(1-x)^3$ , obtained by differentiating the geometric series twice, then gives (14.10). This is a convergent Taylor calculation at the small physical radius  $q$ ; it does not assert a cluster expansion on a fluctuation product of radius  $q/t$ .  $\square$

### 14.3 Coordinates for the small-field fiber and the choice of parameters

We now show that the normalized fluctuation variables describe every point of the fiber of one RG step whose fine and coarse fields lie in  $D_u$ , and not only the points near the conditional constrained minimizers or inside the fluctuation cutoff domain. We then fix the parameters of the construction in a definite order and define the small-field event.

**Theorem 14.3.** *Fix an odd integer  $M \geq 3$ , an integer  $J \geq 1$ , and the larger classical domains and charts of local constraint coordinates of Section 14.2. By the common branch we mean*

the intersection of the branch of the local constraint coordinates (Lemma 7.2) with these classical domains. There exist  $r_{\text{geom}}, C_{\text{TG}} > 0$ , and a positive distance of all coordinates below from the boundary of the common branch, independent of the depth  $d_0 \geq 0$ , the coarse side  $m \geq 4$ ,  $L$ ,  $\gamma$ , and the couplings, such that the following holds for every  $0 < u < r_{\text{geom}}$ . Let  $U_0$  be a configuration on the lattice of side  $M^J m$ , let  $U_i = T_M^i U_0$ , a configuration on the lattice of side  $M^{J-i} m$ , and suppose that  $U_0 \in D_u$  and  $U_J \in D_u$ . Let  $c \in B_u$  be any configuration gauge equivalent to  $U_J$ .

1. There is a periodic gauge transformation of the fine lattice, equal to the identity at every coarse vertex when  $U_J = c$ , after which the intermediate configurations have link logarithms of norm at most  $C_{M,J}u$  and every averaging step lies in the common branch. One may take  $10M$  as the growth factor of link logarithms in one averaging step, and  $C_{M,J} = 2(10M)^J + 1$  before the constants of the forest coordinates are included.
2. After the unique hierarchical fixing of the block forests rooted at the coarse vertices, the free logarithmic coordinates and the normalized fluctuation variables

$$z_i = z_i(U_0), \quad w_i = g_i^{-1} K_i(c_{i+1})^{1/2} (z_i - z_i^*(c_{i+1})), \quad g_i \|w_i\|_\infty \leq C_{\text{TG}} u,$$

reconstruct every such fiber point, including points outside the fluctuation cutoff domain. They use one based Haar disintegration and the triangular determinant of Theorem 12.13; no sum over choices of the auxiliary gauge transformation and no multiplicity factor occurs.

3. For any two configurations  $c, c' \in B_u$  gauge equivalent to  $U_J$ , the transition between the two coordinate systems is triangular, its diagonal blocks are orthogonal on each three-dimensional color block, and it preserves every  $|w_{i,e}|$ .
4. If  $K_i \leq C_K I$  and  $D_z^2 F_i \geq m_0 I/2$  are the fixed classical bounds on the chart, then

$$\frac{E_{d_0, \text{sat}}(U_0)}{g_0^2} - \frac{E_{d_0+J, \text{sat}}(c)}{g_J^2} \geq \frac{m_0}{4C_K} \sum_i \|w_i\|_2^2.$$

These classical values are evaluated on the larger classical chart, irrespective of the radius  $\rho$  of the effective interactions.

5. Let  $R$  be gauge invariant on overlaps, with  $DR(I) = 0$  and  $\mathcal{B}_{\rho, a_{\text{sp}}}(R) \leq E_*$ . For  $0 < \eta < 1$  and  $\|\tilde{\xi}\|_\infty \leq (1 - \eta)\rho$ ,

$$\|DR(e^{\tilde{\xi}})\|_{a_{\text{sp}}, \text{root}} \leq CE_* \rho \eta^{-1}, \quad \|D^2 R(e^{\tilde{\xi}})\|_{a_{\text{sp}}, \text{root}} \leq CE_* \eta^{-2}.$$

Suppose that  $U_0$  is as above and also lies in  $D_{(1-\eta)\rho}$ , fix one gauge transformation taking it into  $B_{(1-\eta)\rho}$ , and hold this gauge transformation fixed while differentiating. Let  $U_0(w)$  be the fine configuration reconstructed from  $c$  and  $w$ . Then, for one fixed smaller rate  $b > 0$ ,

$$\|D_w^2 [R_{\text{sat}}(U_0(w))]\|_{b, \text{row} + \text{col}} \leq C_J E_* t^2 (\eta^{-2} + \rho \eta^{-1}), \quad t = \max_i g_i.$$

For the domain  $D_r$  one may take  $\eta = 1 - r/\rho$ .

The bounds in (5) hold at each point. They do not assert that the fiber is convex, and they do not place the whole fiber in one chart of the effective interaction in the gauge of Theorem 13.2.

The point of (2) is that the coordinates cover the whole fiber over  $c$  of the measure restricted to  $D_u$ , so that its classical and coarea formulas apply at every point of this fiber. Its interaction-dependent density formula applies where the reconstructed field and reference value lie in  $D_\rho$ , in particular in the later application  $u = r < \rho$ . Part (4) is a quadratic lower bound for the excess classical action in terms of the normalized fluctuation variables.

*Proof.* Steps 1 and 2 construct the gauge transformation in (1), using only the local inverse-branch estimate of Lemma 7.2 and the nesting of centered blocks; no classical estimate is used there. Steps 3 to 6 fix the forests, bound the fluctuation variables, and prove (2) and (3). Steps 7 and 8 prove (4) and (5).

**Step 1: a small representative and its averages.** Let  $L_M = 10M$ , and fix a periodic gauge transformation  $k_0$  such that  $V_0 = U_0^{k_0} \in B_u$ . The fine side is  $M^J m$  with  $m \geq 4$ , and scale  $s$  has side  $M^{J-s} m$ . Each of the  $J$  averaging steps therefore has an integer coarse side at least  $m \geq 4$ , as required by the clause of Lemma 7.2 on iterated steps. We choose the threshold  $r_{\text{geom}}$  so that  $(10M)^J u \leq u_M$ , where  $u_M$  is the radius of that lemma, and so that the finitely many conditions on logarithms of based loop holonomies hold. Then, by that lemma, we have

$$V_s = T_M^s V_0, \quad \max_e \rho_G((V_s)_e) \leq L_M^s u \quad (0 \leq s \leq J).$$

Gauge covariance can also be read off from (4.3): every based loop holonomy is conjugated at its basepoint, the starting coarse vertex, and so is its logarithm, while the reference transport acquires the gauge factors at its two endpoints. In particular, the branch of based loop holonomies is preserved.

**Step 2: alignment with the given representative.** Let  $h$  be a gauge transformation of the coarse lattice with  $V_J^h = c$ ; if  $U_J = c$ , we take  $h = (k_0|_{\Lambda_m})^{-1}$ . On a coarse bond  $xy$ , we have  $h_x h_y^{-1} = c_{xy} h_y (V_J)_{xy}^{-1} h_y^{-1}$ . Since  $\rho_G$  is conjugation invariant and satisfies the triangle inequality, we obtain

$$\rho_G(h_x h_y^{-1}) \leq \rho_G((V_J)_{xy}) + \rho_G(c_{xy}) < (L_M^J + 1)u.$$

We lift  $h$  to a gauge transformation  $\tilde{h}$  of the fine lattice that is constant on each centered block of  $M^J$  sites per direction around a coarse vertex. These blocks partition the fine torus. They are compatible with the blocks at every intermediate scale. Indeed, in one coordinate, the union of the  $M$  centered intervals of  $M^{s-1}$  sites centered at  $M^{s-1}j$ ,  $-R_M \leq j \leq R_M$ , where  $M = 2R_M + 1$ , is exactly the interval with integer endpoints

$\pm(M^s - 1)/2$ ; taking Cartesian products and inducting in  $s$  proves the nesting. The nesting implies that a link at scale  $s$  either has both endpoints in one block of  $\tilde{h}$  or crosses one boundary between neighboring blocks. In the second case, the increment of  $\tilde{h}$  between its endpoints obeys the preceding bound. Since  $(U^k)^l = U^{lk}$ , the composite gauge transformation is  $\tilde{h}k_0$ , and we obtain

$$\max_e \rho_G((V_s^{\tilde{h}})_e) \leq L_M^s u + (L_M^J + 1)u \leq (2L_M^J + 1)u.$$

At scale  $J$ , the sharper bound  $c \in B_u$  holds. When  $U_J = c$ , the composite gauge transformation is the identity at every coarse vertex. No global logarithm of  $h$  is used.

**Step 3: forest coordinates.** We fix the based block forests successively from the finest scale. Each tree path has at most  $3R_M$  links. On a field with link bound  $v$ , a link outside the forest has, after forest fixing, distance at most  $(6R_M + 1)v$  from the identity. Removing its coarse-field lift increases this bound by at most the bound for the coarse links. Hence all relative logarithms, including those of the eliminated edges, are at most  $C_{M,J}u$ . We reduce  $r_{\text{geom}}$  so that these inequalities, and the finitely many inequalities for intermediate logarithms, hold strictly inside the uniqueness neighborhood of Lemma 7.2 and inside the larger classical charts. Since the field satisfies its block averaging constraint, each eliminated-edge coordinate is the value of  $\Gamma_{c_{i+1}}(z_i)$  on that branch. This excludes a second local solution for any point of the fiber.

**Step 4: bounds on the fluctuation variables.** Since the pairs  $(c_{i+1}, z_i)$  just obtained lie in the larger products of Section 14.2, Lemma 14.1 implies  $\|z_i^*(c_{i+1})\|_\infty \leq C_{M,J}u$  and a uniform row bound on  $K_i^{1/2}$ . By the definition of  $w_i$ , we have

$$g_i \|w_i\|_\infty \leq \|K_i^{1/2}\|_{\infty \rightarrow \infty} (\|z_i\|_\infty + \|z_i^*\|_\infty) \leq C_{\text{TG}} u.$$

Starting at  $c_J = c$  and recovering  $z_i$  and then  $c_i$  in decreasing order of  $i$ , we recover the input configuration.

**Step 5: the based Haar disintegration.** The measure statement in (2) requires the based conclusion of Step 2, not just the existence of a small representative. The union of the nested forests has one connected component for each coarse vertex: a vertex reaches the root of its block at each stage, and reaches its coarse vertex after at most  $J$  stages. Since this union has no cycle, the based gauge transformation that fixes it is unique. The proof of (14.6) applies at every stage. The auxiliary gauge transformation  $k_0$  serves only to verify membership in the local branch after this unique forest fixing; it is not a coordinate and is not summed over. The single change of free coordinates has the Jacobian (14.7), which is the determinant of Theorem 12.13. Finally, the pointwise classical and coarea identities of Section 14.2 hold on the whole branch, before any fluctuation radius is imposed, and the present coordinates lie in this branch. Interaction-dependent identities

retain the additional  $D_\rho$  and reference-value conditions stated there. This proves the measure assertion.

**Step 6: change of representative.** Let  $c, c' \in B_u$  be gauge equivalent to  $U_J$ . The connecting gauge transformation has increments less than  $2u$  on coarse bonds, and its lift that is constant on blocks keeps both coordinate neighborhoods in the same larger classical domain, after a reduction of  $r_{\text{geom}}$ . A gauge transformation fixed at the coarse vertices acts on a relative logarithm by the adjoint action at its source block; reversing the orientation of a link permutes the coordinates and changes their sign, with an adjoint factor. These maps are orthogonal on each color block. By uniqueness of the local inverse, they intertwine the solved coordinates. The classical overlap identity in (14.3) then sends the conditional constrained minimizer and the constrained Hessian to  $Oz_i^*$  and  $OK_iO^\top$ . The positive square root is sent to  $OK_i^{1/2}O^\top$ , since the square of the latter is the transformed positive matrix. This gives  $w'_i = O_i(c_{i+1})w_i$ . Since  $O_i$  depends on the field of the coarser scales, the joint change is triangular, and its diagonal blocks have determinants of absolute value one. This proves the norm preservation and the absence of a multiplicity factor.

**Step 7: the quadratic lower bound.** At each scale, let  $h_i = z_i - z_i^*$ . By stationarity and the classical Hessian bound, we have

$$\begin{aligned} F_i(c_{i+1}, z_i) - F_i(c_{i+1}, z_i^*) &= \int_0^1 (1-s) D_z^2 F_i(c_{i+1}, z_i^* + sh_i) [h_i, h_i] ds \\ &\geq \frac{m_0}{4} \|h_i\|_2^2 \geq \frac{m_0 g_i^2}{4C_K} \|w_i\|_2^2. \end{aligned}$$

In the last inequality, we used  $h_i = g_i K_i^{-1/2} w_i$  and  $K_i^{-1} \geq C_K^{-1} I$ . Both endpoints, and the whole segment between them, lie in the fixed product of free coordinates. Subtracting successive values of the nested minimization identity and using  $g_{i+1}^2 = M g_i^2$ , we see that all intermediate values cancel, as in (14.8). This gives (4), with the coefficient  $m_0/(4C_K)$ .

**Step 8: derivatives of the effective interaction.** Let  $A_s \geq 0$  be the contribution of order  $s$  to the weighted coefficient norm, so that  $\sum_{s \geq 2} A_s \leq E_*$ . On  $\|\xi\|_\infty \leq v\rho$  with  $v = 1 - \eta$ , contracting the undifferentiated indices bounds the first two derivatives by

$$C\rho \sum_{s \geq 2} s v^{s-1} A_s, \quad C \sum_{s \geq 2} s(s-1) v^{s-2} A_s.$$

The multipliers are bounded by  $C/\eta$  and  $C/\eta^2$ : the first is at most  $\sum_{j \geq 0} v^j$ , and applying the same comparison to ordered pairs bounds the second by  $2(\sum_{j \geq 0} v^j)^2$ . This proves the two bounds on  $DR$  and  $D^2R$  in (5), independently of  $R(I)$ .

At a point in the strict interior, we fix a gauge transformation  $k$  sending it into  $B_{(1-\eta)\rho}$  and hold  $k$  fixed. Since the image lies at positive distance from the boundary of  $B_\rho$ , there

exists a neighborhood on which the transformed field stays in  $B_\rho$ . On it, gauge invariance on overlaps identifies  $R_{\text{sat}}$  with this analytic representative. Fixed group multiplications on a link have bounded differentials, and the logarithm has bounded first two differentials on a strictly smaller chart; these operations create no sum over remote links. Combine the minimizer and square-root bounds of Lemma 14.1 with the joint coefficient bounds for  $B$  in Lemma 12.3(3), using Lemma 5.3(1), (3), and (4). These bound the first two differentials of the finite reconstruction in the physical variables  $q_i = g_i w_i$  on the smaller products, with each required input fixed. Substitution of  $q_i = g_i w_i$ , with  $g_i \leq t$ , gives

$$\|D_w \xi\|_{b, \text{root}} \leq C_J t, \quad \|D_w^2 \xi\|_{b, \text{root}} \leq C_J t^2.$$

The two terms of the chain rule are  $D^2 R(\xi)[D_w \xi, D_w \xi]$  and  $DR(\xi)D_w^2 \xi$ . Summing first at the fixed input indices and using a smaller positive rate, we bound their row and column sums by  $C_J E_* t^2 \eta^{-2}$  and  $C_J E_* \rho t^2 \eta^{-1}$ . The gauge transformation  $k$  has not been differentiated. Taking  $\eta = 1 - r/\rho$  gives the stated choice for  $D_r$ .  $\square$

We now fix the parameters. The order matters: first  $J$ , using only constants that are uniform before  $J$ ; then the fixed- $J$  analytic domains; then the small-field radius  $r$ ; then the physical radii of fluctuation products; and finally the coupling bound.

**Definition 14.4.** The parameters of the construction are chosen in the following order.

1. Fix  $M$ , the spatial rates, and the support-enlargement, polymer-summability, and coefficient constants of Theorem 13.5, all of which are uniform before  $J$ . Choose  $J$  once so that

$$C_P(a_{\text{sp}})M^{-J} < 1/4, \quad C_{\text{cen}}M^{-J} < 1/8,$$

and so that the other conditions on  $J$  in Theorem 13.5 hold. Here  $q_P = C_P(a_{\text{sp}})M^{-J}$  is the contraction factor of the pullback of the effective interaction to the background field (Theorems 11.10 and 13.5), and  $C_{\text{cen}}$  is the coefficient constant of the zero-fluctuation background field in Lemma 11.9; both are uniform before  $J$ .

2. Fix  $\rho$ ,  $\rho_{\text{cl}}$ ,  $s_J$ ,  $E_*$ , and the larger construction domains so that the image of the reconstruction majorant and all classical, constraint-coordinate, and coarea conditions of Theorem 13.5 hold. In particular, with the constants  $r_J, C_J$ , the analytic-radius enlargement factor  $K_*$ , and the majorant  $\mathfrak{M}_J$  of Theorem 13.2, we require

$$4K_*(\rho + s_J) < r_J, \quad C_J 4K_*(\rho + s_J) < \frac{1}{2}, \quad \mathfrak{M}_J(4K_*\rho, 4K_*s_J) < \rho/2. \quad (14.11)$$

These domains contain one open branch for an averaging step. Here  $r_{\text{CO}} > 0$  is a fixed radius for the relative logarithms  $X = \Gamma_c(z)$ , including those of the eliminated edges, chosen so that the branch lies strictly inside the uniqueness domain of Lemma 7.2 and the domain of the Haar disintegration of Lemma 6.3. The branch is

$$\begin{aligned} \|\text{Log } c\|_\infty &< r_c, \quad \|z\|_\infty < r_z, \quad \|\Gamma_c(z)\|_\infty < r_{\text{CO}}, \\ \det D_{X_{e_a^*}} C_{c,a}(\Gamma_c(z)) &> 0 \quad (a \text{ a coarse link}), \end{aligned}$$

strictly inside the larger construction domains, with  $r_c, r_z$  as in Lemma 14.1. The square root in the normalized fluctuation variables is the positive square root of the constrained Hessian at the conditional constrained minimizer. All geometric and Hessian constants on these domains are fixed, including  $r_{\text{geom}}$  and  $C_{\text{TG}}$  of Theorem 14.3, which we apply with this branch as its coordinate domain. We decrease  $r_{\text{geom}}$ , before  $r$  is chosen, so that the bounds on all intermediate coarse logarithms, free logarithms, and relative logarithms in the proof of that theorem lie strictly below  $r_c, r_z$ , and  $r_{\text{CO}}$ , respectively, and so that every constraint determinant  $\det D_{X_{c,a}^*} C_{c,a}(\Gamma_c(z))$  is positive. These bounds are linear in  $u$ , and the determinants are continuous and equal to  $\det(M^{-2}I_3) > 0$  at the identity, so this is one fixed reduction. Thus, whenever  $u < r_{\text{geom}}$  and both endpoint fields lie in  $D_u$ , the coordinates of Theorem 14.3 satisfy precisely the branch conditions used in  $\mathcal{A}_{k,J}$  below.

3. Put  $\lambda_* = M^{-(J-1)/2}$  and  $C_{\text{map},J} = (10M)^J$ . Choose  $\mathcal{L}_{\text{en}} > 2C_{\text{TG}}/\lambda_*$ , and choose  $\mathcal{L}'_{\text{en}} \geq \mathcal{L}_{\text{en}}$  larger than  $2C_{\text{TG}}C_{\text{map},J}/\lambda_*$  and than the fixed geometric enlargement constants of the larger chart of coarse fields.
4. Let  $r_{\text{max}} > 0$ , depending only on the data fixed so far, be such that every  $0 < r \leq r_{\text{max}}$  satisfies the following conditions, where  $R_{\text{buf}} = C_{\text{map},J}r$  and  $R_{\text{en}} = \mathcal{L}_{\text{en}}r$ :  $r < \rho/8$ ;  $r < r_{\text{geom}}$  and  $R_{\text{buf}} < r_{\text{geom}}$ ;  $C_{\text{map},J}r < \rho/2$ ; the physical radii  $R_{\text{en}}$  and  $\mathcal{L}'_{\text{en}}r$  lie strictly inside the products of free coordinates fixed in (2);

$$\begin{aligned} R_{\text{buf}} + \mathcal{L}'_{\text{en}}r &< r_J, & C_J(R_{\text{buf}} + \mathcal{L}'_{\text{en}}r) &< \frac{1}{2}, \\ R_{\text{buf}} &< 4K_*\rho, & \mathcal{L}'_{\text{en}}r &< 4K_*s_J; \end{aligned} \tag{14.12}$$

the inequalities

$$C_H \mathcal{L}'_{\text{en}}r < 1, \quad \frac{C_H \mathcal{L}'_{\text{en}}r}{(1 - C_H \mathcal{L}'_{\text{en}}r)^3} < \frac{1}{4},$$

with  $C_H$  fixed to dominate both the constant of (14.10) and the Hessian coefficient constant of Theorem 13.8(2); and  $M^{-J}Z(r) < r/4$ , where  $Z$  is the majorant of the zero-fluctuation reconstruction in Theorem 13.2. Choose  $r \in (0, r_{\text{max}}]$ .

5. Choose  $0 < P_{\text{phys}} < Q_{\text{phys}}$ , fixed multiples of  $r$ , such that  $\mathfrak{M}_J(r, Q_{\text{phys}}) < r/2$  and every point of the product of physical radius  $Q_{\text{phys}}$  satisfies the intermediate coordinate conditions.
6. Choose  $A_w \geq 1$ , put  $p_w(t) = A_w \log(1/t)$ , and choose a coupling bound  $t_* \in (0, 1)$  such that, for all  $0 < t \leq t_*$ , the smallness conditions on  $t$  in Theorems 13.5, 13.8, and 13.10 and in the proofs of Theorems 14.6 and 14.7 hold. These include  $tp_w(t) < P_{\text{phys}}$  and  $p_w(t) \geq p_0$ , where  $p_0$  is the lower bound for the cutoff radius in Theorem 13.5.

The integer  $J$  is not changed after step (1), and  $r$  is not changed after step (4).

Let  $u \in 4\mathbb{N}$ ,  $K \geq 0$ , and  $N = uB^K$ , put  $n_\ell = N/M^\ell$ , and let  $\gamma > 0$  be such that  $t_k^2 = \gamma/(Mn_{(k+1)J}) \leq t_*^2$  for  $0 \leq k < K$ . Put  $V^\ell = T_M^\ell U$ . On the endpoint-admitted set

$\{V^{kJ}, V^{(k+1)J} \in D_r\}$ , define  $\mathcal{A}_{k,J}$  using a representative  $c \in B_r$  of  $V^{(k+1)J}$ : in the based inverse coordinates of this step, every pair  $(c_{i+1}, z_i)$  lies in the branch of (2) at depth  $kJ + i$ , with its conditional constrained minimizer and the normalization by the positive square root of its constrained Hessian, and with the classical and coarea domains fixed above. Set  $\mathcal{A}_{k,J}$  false outside the endpoint-admitted set. By (2) and Theorem 14.3, branch admission is automatic for every such representative on that set; orthogonal transitions compare the fluctuation norms, rather than the bounds on intermediate coarse logarithms. Let  $w_{k,i,e}$  be the normalized fluctuation variables of this step, with couplings  $g_{k,i} = t_k M^{(i-J+1)/2}$ . The small-field event and the conditioned Wilson measure are

$$S_{N,u,\gamma} = \{U \in D_r\} \cap \bigcap_{k=0}^{K-1} \left[ \mathcal{A}_{k,J} \cap \left\{ \max_{i,e} |w_{k,i,e}| \leq A_w \log(1/t_k) \right\} \cap \{V^{(k+1)J} \in D_r\} \right],$$

$$\nu_{N,u,\gamma} = \mu_{N,\gamma}(\cdot \mid S_{N,u,\gamma}).$$

This definition makes precise the event of Definition 4.4. There is no  $D_r$  test at the intermediate scales of an RG step. For  $K = 0$  the event is the initial test  $\{U \in D_r\}$ . The stopped measure after  $k$  RG steps is the normalized pushforward of the Wilson measure restricted by the tests of the first  $k$  steps. The conditioned Wilson measure also contains the tests of the remaining coarser steps, so its marginal at an intermediate scale is in general not the stopped measure.

**Lemma 14.5.** *The choices in Definition 14.4 can be made in the stated order. The integer  $J$  depends only on constants uniform before  $J$ ;  $\rho$  and  $s_J$  as in step (2) exist; there is  $r_{\max} > 0$  as in step (4); in step (5) one can take  $Q_{\text{phys}} = \zeta r$  and  $P_{\text{phys}} = Q_{\text{phys}}/2$ , with  $\zeta > 0$  independent of  $r \in (0, r_{\max}]$ , and then the fluctuation cutoff domain satisfies the same conditions as the product of physical radius  $Q_{\text{phys}}$  whenever  $t \leq t_*$ ; and there is  $t_*$  as in step (6). The event  $S_{N,u,\gamma}$  is a Borel set invariant under lattice gauge transformations, and for  $K = 0$  it equals  $D_r$ .*

*Proof.* Only the constants that are uniform before  $J$  enter the choice of  $J$ . The additional conditions of Theorem 13.5 on the image of the majorant, on support enlargement, and on polymer summability are of this kind, so we can take one  $J$  satisfying all of them and the two displayed contractions. We then fix the larger domains,  $\rho$ ,  $s_J$ ,  $E_*$ , and every coefficient bound at this  $J$ . When choosing the larger domains, we take the strict inclusions of Theorem 13.2 small enough that (14.11) holds. The classical products of Section 14.2 and Theorem 14.3 fix  $C_{\text{TG}}$  before  $r$  is chosen. These choices are made on the larger domains, before  $r$ ; the contraction of the pullback uses only the constants fixed before  $J$ .

Every radius in step (4) is a fixed multiple of  $r$ , and the rational expression in step (4) also tends to zero with  $r$ . Since there are finitely many strict inequalities, their set of solutions contains an interval  $(0, r_{\max}]$  after a strict reduction of its endpoint. The constants used in this reduction have all been fixed already. The majorant condition

already imposed in Theorem 13.5(1) and positivity of its coefficients give  $M^{-J}Z(r) \leq r/(16K_*) < r/4$  for  $r \leq \rho$ : use  $Z(r) \leq rZ(4K_*\rho)/(4K_*\rho)$ .

We choose  $\zeta > 0$  so small that  $C_J\zeta/[1 - C_J(r + \zeta r)] < 1/4$  uniformly for  $0 < r \leq r_{\max}$ , that all intermediate conditions hold on the products of the corresponding physical radii, and that  $\zeta < \mathcal{L}_{\text{en}}$ . We set  $Q_{\text{phys}} = \zeta r$  and  $P_{\text{phys}} = Q_{\text{phys}}/2$ . By (14.12), we have  $r + Q_{\text{phys}} < r_J$ , the denominator of the majorant is positive, and the corresponding product of coarse fields and free coordinates lies inside the products of step (2). Every intermediate classical and coarea condition was included in the choice of  $\zeta$ . The clause of Theorem 13.2 for coarse fields in  $B_r$  therefore applies, and with  $\delta = M^{-J}$  it gives

$$\mathfrak{M}_J(r, Q_{\text{phys}}) = \delta Z(r) + \frac{C_J Q_{\text{phys}}}{1 - C_J(r + Q_{\text{phys}})} < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}.$$

On the fluctuation cutoff domain, we have  $|g_i w_{i,e}| \leq t p_w(t) < P_{\text{phys}}$ , so this domain is contained in the product of physical radius  $Q_{\text{phys}}$  and has the same properties.

For any fixed  $A_w \geq 1$ , all remaining conditions on  $t$  compare a fixed positive quantity with expressions  $t^\alpha \log^h(1/t)$ , where  $\alpha > 0$  and  $h$  is a fixed nonnegative integer. These expressions tend to zero: with  $x = \log(1/t)$ , the expression is  $e^{-\alpha x} x^h$ , which the exponential series bounds by  $C_{\alpha,h}/x$  for large  $x$ . Also,  $p_w(t) \rightarrow \infty$  as  $t \rightarrow 0$ . We conclude that one  $t_* < 1$  enforces every condition, including those of Theorem 13.5. None of them requires increasing  $J$  or changing  $r$ .

On the two endpoint  $D_r$  tests, (2) of Definition 14.4 and Theorem 14.3 imply all branch, determinant, and classical conditions for every chosen  $B_r$  representative. The remaining fluctuation-ball tests are independent of that choice by the orthogonal transitions of the chart theorem. This proves gauge invariance without asserting invariance of intermediate coarse logarithms. For Borelness, cover the endpoint-admitted set by the open sets on which fixed fine and coarse gauge transformations put the endpoints in  $B_r$ . A countable dense set of gauge transformations supplies such a cover because  $B_r$  is open. On each member the based inverse coordinates are continuous and the fluctuation tests are Borel. There are finitely many steps, so their intersections and countable chart unions are Borel. No global continuous gauge choice is used. For  $K = 0$ , the intersection over  $k$  has an empty index set and equals the whole configuration space, so only the initial test remains.  $\square$

## 14.4 The restricted recursion and the enclosing product

This subsection identifies the recursion of Theorem 13.5 with the Wilson measure restricted by the small-field event. It also shows that the fiber of the incoming restricted density over a coarse field lies in a product domain on which the conditional fluctuation potential is uniformly convex.

Fix one RG step from depth  $d_0$ , with coarse side  $m$  and incoming effective interaction  $R$ . The incoming restricted density is

$$P_{\text{in}}(U) = \mathbf{1}_{D_r}(U) e^{-E_{d_0, \text{sat}}(U)/g_0^2 - R_{\text{sat}}(U)}$$

relative to the product Haar measure, as in Theorem 13.4. For a coarse field  $c \in B_{R_{\text{buf}}}$ , let  $S_c$  be the set of normalized fluctuation variables  $w$  such that the pairs  $(c_{i+1}, z_i)$  defined successively by  $c_J = c$ ,  $z_i = z_i^*(c_{i+1}) + g_i K_i(c_{i+1})^{-1/2} w_i$ , and  $c_i = b_i(c_{i+1}, z_i)$  lie in the common branch, and the reconstructed fine configuration  $c_0$ , with the forest variables in their gauge-fixed position, lies in  $D_r$ . For the fixed coarse representative  $c$ , Theorem 14.3 gives these coordinates for every configuration  $U \in D_r$  with  $T_M^J U = c$ , modulo fine gauge transformations equal to the identity at every coarse vertex; the hierarchical forest fixing chooses one representative. We apply the theorem with  $u = r$  for  $c \in B_r$  and with  $u = R_{\text{buf}}$  for  $c \in B_{R_{\text{buf}}}$ . If the coarse image is only gauge equivalent to  $c$ , it is first aligned with  $c$  as in that theorem; we do not assert that this auxiliary alignment is unique. The set  $S_c$  is Borel, since it is the preimage of the open set  $D_r$  under a continuous map on an open domain. Since  $P_{\text{in}}$  is invariant under lattice gauge transformations, the forest variables integrate out with total mass one (Lemma 5.1), and all densities in  $w$  below refer to this reduced integral. They are statements about integrals of gauge-invariant functions, not a pointwise coarea formula for other functions. They do not define a global gauge fixing on the configuration space.

**Theorem 14.6.** *Use the parameters of Definition 14.4 and start with  $R_0 = 0$ .*

1. *For every finite number of RG steps, every coarse side  $m \geq 4$ , and all couplings with  $t \leq t_*$  at each step, there are effective interactions  $R_k$ , holomorphic on  $B_\rho$ , with*

$$DR_k(I) = 0, \quad \mathcal{B}_{\rho, a_{\text{sp}}}(R_k) \leq E_*,$$

*gauge invariant on overlaps and invariant under global color rotations, lattice translations, and spatial inversion.*

2. *Consider one RG step from depth  $d_0$ , with incoming effective interaction  $R = R_k$  and coarse field  $c = e^\theta$ , and set*

$$\begin{aligned} \tilde{F}_+(c) &= A_{\text{coord}} e^{-E_{d_0+J}(c)/g_J^2 - B_{\text{ref}}(c) - R_{\text{ref}}(c)} \int_{\Omega_p^w} e^{-V_c(w)} dw, \\ R_{\text{ref}}(c) &= R(e^{\Phi(\theta, 0)}), \quad V_c(w) = \frac{1}{2}|w|^2 + H_{d_0, J}(\theta, w) + R(e^{\Phi(\theta, (g_i w_i)_i)}) - R_{\text{ref}}(c), \\ A_{\text{coord}} &= \prod_i \mathcal{J}_i(I, 0) g_i^{3f_i}, \quad \Omega_p^w = \prod_{i,e} \{|w_{i,e}| \leq p_w(t)\}. \end{aligned}$$

*For  $c \in B_\rho$  this is an auxiliary integral, positive for real  $c$ , in which  $R$  is evaluated on the image of the reconstruction, which lies in  $B_{\rho/2}$ . With*

$$z_p = \int_{|x| \leq p_w(t)} e^{-|x|^2/2} d^3x, \quad \pi_p^{\text{all}}(dw) = z_p^{-\sum_i f_i} \mathbf{1}_{\Omega_p^w}(w) e^{-|w|^2/2} dw,$$

and  $\Xi_R = \mathbb{E}_{\pi_p^{\text{all}}} e^{-H_{d_0, J} - R(e^{\Phi(\theta, gw)}) + R_{\text{ref}}}$ , we have

$$R_+ = B_{\text{ref}} + R_{\text{ref}} - \left( \sum_i f_i \right) \log z_p - \log \Xi_R, \quad \tilde{F}_+ = A_{\text{coord}} e^{-E_{d_0+J}/g_J^2 - R_+}.$$

For real  $c$ ,  $\log \Xi_R$  is the real logarithm; for complex  $c$  it is the holomorphic logarithm given by Theorem 13.5.

3. For every  $c \in B_r$ , the reconstruction maps the whole fluctuation cutoff domain  $\Omega_p^w$  into  $D_r$ , and all coordinate conditions of the step hold there. Consequently  $\tilde{F}_+(c)$  is the density at  $c$  of the pushforward under  $T_M^I$  of the incoming density restricted by the coordinate conditions and the fluctuation cutoff of this step, and the output density, including the coarse test, is  $\mathbf{1}_{D_r} \tilde{F}_{+, \text{sat}} dH$ . By induction, the stopped density after  $k$  RG steps is proportional to

$$\mathbf{1}_{D_r} e^{-E_{kJ, \text{sat}}/g_{kJ}^2 - R_{k, \text{sat}}} dH,$$

where  $g_{kJ}^2 = \gamma/n_{kJ}$  is the coupling of the lattice of side  $n_{kJ}$ .

4. The field-independent factor of this density is the product of the factors  $A_{\text{coord}}$  of the steps performed, together with the constants removed by centering, which are transferred to the normalization constant. Precisely, if the incoming effective interaction is centered and the incoming density carries the factor  $e^{\Lambda_k}$ , then centering the output of this step means

$$R_{k+1} = R_+ - R_+(I), \quad \Lambda_{k+1} = \Lambda_k + \log A_{\text{coord}} - R_+(I).$$

If  $R_+$  is kept uncentered, its constant stays in the exponent. The normalizing constant is finite and strictly positive. After  $k$  RG steps, normalizing the density in (3) gives the pushforward under  $T_M^{kJ}$  of the Wilson measure conditioned on the initial  $D_r$  test and the tests of the first  $k$  steps. After all  $K$  steps, this measure is  $(T_M^{KJ})_* \nu_{N, u, \gamma}$ , with  $\nu_{N, u, \gamma}$  the conditioned Wilson measure of Definition 14.4.

5. Let  $b_w = \sum_i f_i = 2(B^3 - 1)m^3$ , so that  $\dim_{\mathbb{R}} w = 3b_w$ , and define, for  $a \in \{P_{\text{phys}}, Q_{\text{phys}}\}$ ,

$$\Omega_a^{\text{phys}} = \prod_{i, e} \{|w_{i, e}| \leq a/t\}, \quad \mathcal{K}_t = \prod_{i, e} \{|w_{i, e}| \leq R_{\text{en}}/t\}.$$

Then, for every  $c \in B_r$ ,

$$\Omega_p^w \subset \Omega_{P_{\text{phys}}}^{\text{phys}} \subset \Omega_{Q_{\text{phys}}}^{\text{phys}} \subset S_c \subset \text{int } \mathcal{K}_t.$$

6. The same potential  $V_c$  is smooth on a neighborhood of the product  $\mathcal{K}_t$ , which does not depend on  $c$ , and

$$\|D_w^2(V_c - \frac{1}{2}|w|^2)\|_{b, \text{row} + \text{col}} \leq \frac{C[R_{\text{en}} + t^2(1 + E_*)]}{(1 - CR_{\text{en}})^3} \leq \frac{1}{2}, \quad D_w^2 V_c \geq I/2.$$

7. For  $c \in B_r$ , the fiber of the incoming measure  $P_{\text{in}} dH$  over  $c$ , without fluctuation cutoff, has the density

$$A_{\text{coord}} e^{-E_{d_0+J}(c)/g_J^2 - B_{\text{ref}}(c) - R_{\text{ref}}(c)} \mathbf{1}_{S_c}(w) e^{-V_c(w)} dw.$$

The prefactor is the same for every subset of  $S_c$  that is integrated. The set  $S_c$  need not be convex.

8. On  $\Omega_p^w$  the normalized density is the conditional fluctuation kernel of Theorem 13.8 with the effective interaction  $R$ , and the bounds of Theorem 13.10 on means, derivatives of the potential, and covariances apply to it. The tests of the coarser RG steps cancel when one conditions on the coarse field. A gauge transformation of the reconstructed fine field that depends only on  $c$  and equals the identity at every coarse vertex leaves  $c$ , the normalized fluctuation variables, the conditional fluctuation kernels, and the cutoffs unchanged, provided that the coefficient bounds of Theorem 13.2 continue to hold in the new representative.

All constants depend only on the parameters of Definition 14.4 and the larger domains; they do not depend on the side, the coupling, the depth, or the number of RG steps. The values of  $\tilde{F}_+$  at coarse fields outside  $B_r$  are not identified with pushforwards of the restricted Wilson measure.

*Proof.* Steps 1 to 6 concern the recursion and its identification with the restricted Wilson measure; they prove (1) to (4). Steps 7 to 10 concern the fiber without cutoff and prove (5) to (8).

**Step 1: the auxiliary integral on  $B_\rho$ .** The image of the coefficient majorant on the enlarged product was fixed in Theorem 13.2. By (14.11), the enlarged product of coarse fields and free coordinates lies before the pole of the majorant and within  $u + v < r_J$ , with positive denominator. On the whole auxiliary ball of coarse fields, every reconstructed argument of the effective interaction lies in  $B_{\rho/2}$ , and every intermediate classical and coarea argument satisfies its conditions with strict inequality. Then the formula in (2) defines the same auxiliary operator as Theorem 13.5, with the present parameters and incoming effective interaction, and that theorem supplies its holomorphic logarithm and the common bound on its analytic norm. Positivity of the integral refers to real coarse fields. For complex coarse fields,  $\tilde{F}_+$  is the holomorphic continuation, and the existence of its logarithm is part of Theorem 13.5, not a consequence of positivity.

**Step 2: normalization constants.** Since  $\Omega_p^w$  is a product of  $\sum_i f_i$  balls in  $\mathbb{R}^3$ , each of Gaussian mass  $z_p$ , substituting  $V_c$  gives  $\int_{\Omega_p^w} e^{-V_c} dw = z_p^{\sum_i f_i} \Xi_R$ . Taking the logarithm, with the real branch on real coarse fields, gives the stated  $R_+$  and  $A_{\text{coord}}$ ; this is also the formula of Theorem 13.4. In particular, the normalization  $z_p$  is not absorbed into an unspecified term depending on the coarse field. Subtracting  $R_+(I)$  changes no nonconstant coefficient, no overlap identity, and no symmetry, and the unnormalized densities agree:

$$e^{\Lambda_k} A_{\text{coord}} e^{-E_{d_0+J}/g_J^2 - R_+} = e^{\Lambda_k + \log A_{\text{coord}} - R_+(I)} e^{-E_{d_0+J}/g_J^2 - [R_+ - R_+(I)]}.$$

This proves the sign and every constant in (4).

**Step 3: identification on  $B_r$ .** Let  $c \in B_r$ . On the whole cutoff product, we have  $\|gw\|_\infty \leq tp_w(t) < P_{\text{phys}} < Q_{\text{phys}}$ . By Lemma 14.5, the radius  $Q_{\text{phys}}$  satisfies (14.12),  $r + Q_{\text{phys}} < r_J$ , the positivity of the denominator of the majorant, and all intermediate conditions. The image bound of Theorem 13.2 for coarse fields in  $B_r$  therefore puts the fine field in  $B_{r/2}$ , in the gauge of that theorem, with all intermediate coordinate conditions. The indicator of  $D_r$  for the fine field is identically one there. All hypotheses of Theorem 13.4 now hold:  $R$  is gauge invariant on overlaps, its first derivative at the identity vanishes, its norm is at most  $E_*$ , and it has the required symmetries. This implies that, at coarse fields in  $B_r$ , the auxiliary integral is the fiber integral of the restricted incoming density, including the branch, coarea, cutoff, and coarse tests; we do not use this conclusion outside  $B_r$ .

**Step 4: overlaps, symmetries, and the first derivative.** On an overlap in the common comparison products, the triangular orthogonal change of Theorem 13.4(3), with the domains fixed in Theorem 13.5(1), preserves the fluctuation product, Lebesgue measure, the Haar disintegration densities, and the determinants. The two reconstructed arguments of the effective interaction are related by a gauge transformation and both lie in  $B_{\rho/2}$ , so the values of  $R$  agree. A change of variables shows that the two auxiliary integrals are equal, including all reference factors. Translations and spatial inversion act by the corresponding orthogonal permutations, and constant color rotations also preserve the integral. At the identity, each component of the first derivative of  $R_+$  at a link is a vector in  $\mathfrak{su}(2)$  fixed by all adjoint rotations, and hence zero: rotations by a half-turn about the coordinate axes reverse each component. This verifies the overlap and first-derivative hypotheses at the next step.

**Step 5: the analytic norm.** The induction step for the norm is the inequality

$$\begin{aligned} \mathcal{B}_{\rho, a_{\text{sp}}}(R_+) &\leq C_G + q_P E_* + C_T t^2 E_* (p_w^4 + E_* p_w^2) \\ &\leq C_G + \frac{1}{2}(1 + q_P) E_* \leq E_*. \end{aligned}$$

In the first inequality, we used the bound of Theorem 13.5; in the second, its condition  $C_T t^2 (p_w^4 + E_* p_w^2) \leq (1 - q_P)/2$ ; and in the last,  $C_G \leq (1 - q_P)E_*/2$ . Since  $R_0 = 0$  satisfies all hypotheses, this proves (1) at every finite depth  $kJ$ .

**Step 6: iteration and the normalizing constant.** Iterating these positive operators, and using the last clause of Theorem 13.4, we obtain the pushforward of the Wilson density restricted by exactly the tests of Definition 14.4, and the stopped density after any number of steps. This proves (3). On a finite lattice, the identity configuration and a sufficiently small open neighborhood of it pass all of the finitely many tests strictly. Indeed, the inverse coordinates are continuous on the open branch, the image bound holds on the whole cutoff product, and every normalized fluctuation variable of the

identity configuration is zero. That neighborhood has positive Haar measure. All densities are bounded on smaller chart neighborhoods, and the bound on the oscillation of  $R_{\text{sat}}$  in Theorem 13.4, together with the finiteness of Haar measure on the compact configuration space, gives finiteness. Hence the normalizing constant is finite and strictly positive. When all  $K$  tests are imposed, normalizing the restricted Wilson measure before pushforward gives  $\nu_{N,u,\gamma}$ , and normalizing its pushforward gives the corresponding stopped measure.

**Step 7: the enclosure.** We now consider the fiber without cutoff. The numbers of free variables at the intermediate scales are

$$f_i = 2(M^3 - 1)(mM^{J-i-1})^3, \quad \sum_{i=0}^{J-1} f_i = 2m^3(M^{3J} - 1),$$

where the equality follows by summing the finite geometric series. Since there are three real color components per block, we have  $\dim_{\mathbb{R}} w = 3b_w$ . For a point of  $S_c$ , Theorem 14.3 with  $u = r$  implies that

$$|w_{i,e}| \leq \frac{C_{\text{TGR}}}{g_i} \leq \frac{C_{\text{TGR}}}{\lambda_* t} < \frac{R_{\text{en}}}{2t},$$

where in the last inequality we used  $\mathcal{L}_{\text{en}} > 2C_{\text{TGR}}/\lambda_*$ . This proves that  $S_c$  lies in the interior of  $\mathcal{K}_t$ , before any cutoff. Conversely, if  $|w_{i,e}| \leq Q_{\text{phys}}/t$  for all  $i, e$ , then  $|g_i w_{i,e}| \leq Q_{\text{phys}}$ . The image estimate of Theorem 13.2 puts the fine field in  $B_{r/2}$ , with all intermediate coordinate conditions, on this whole closed product, which is therefore a subset of  $S_c$ . The other two inclusions follow from  $P_{\text{phys}} < Q_{\text{phys}}$  and  $tp_w(t) < P_{\text{phys}}$ . Since all inequalities on the coarse field and the branch hold strictly, the assertions also hold on the boundaries of these closed products.

**Step 8: convexity on  $\mathcal{K}_t$ .** Throughout  $\mathcal{K}_t$ , each input  $g_i w_{i,e}$  has norm at most  $R_{\text{en}} \leq \mathcal{L}_{\text{en}}^l r$ , and together with the coarse-field radius  $r \leq R_{\text{buf}}$  it satisfies (14.12). The image condition of step (2) of Definition 14.4 therefore puts the image of the reconstruction, and an open neighborhood of it, in the analytic domain of  $R$ . The same  $V_c$  is therefore smooth there, also at points not in  $S_c$ . The clause of Theorem 13.8 for products of fixed physical radius applies with  $q_* = R_{\text{en}} < 4K_* s_J$ , strictly inside the reserved product with its image, denominator, and intermediate comparison margins, and  $C_H q_* < 1$  by the gate for  $\mathcal{L}_{\text{en}}^l r$ , and gives the Hessian estimate in (6); equivalently, this is the convergent coefficient calculation of Lemma 14.2 with  $q = R_{\text{en}}$ . The choice of  $r_{\text{max}}$  makes the term with  $R_{\text{en}}$  at most  $1/4$ , and the choice of  $t_*$  then makes the term with  $t^2(1 + E_*)$  at most  $1/4$ . By Schur's test, the weighted row-and-column bound controls the operator norm, which gives  $D_w^2 V_c \geq I/2$  on the whole convex product, whether or not  $S_c$  is convex.

**Step 9: the density on the whole fiber.** By Theorem 14.3(2) and the discussion before the theorem, the uncut fiber has exactly one representation in the based coordinates.

Multiplying (14.6), (14.7), and (14.8) gives the factor (14.9) times  $\mathbf{1}_{S_c} e^{-V_c(w)} dw$ . Gauge invariance on overlaps identifies the value of the effective interaction at every point of  $S_c$ . All other factors are contained in (14.9) or in  $H_{d_0, J}$ , and none of them depends on which subset of the fiber is integrated. This proves (7).

**Step 10: conditional kernels and changes of representative.** On the cutoff product, the normalized density is the kernel of Theorem 13.8 with the effective interaction just constructed; the hypotheses of that theorem, including the fixed domain  $\Omega_p^w$  and the smallness of  $tp_w(t)^3$ , were verified above. The effective interaction is holomorphic on  $B_\rho$  and gauge invariant on overlaps, and it has vanishing first derivative at the identity, the required symmetries, and norm at most  $E_*$ . Further, the product of coarse fields and free coordinates satisfies the image condition of (14.11). The hypotheses of Lemma 14.2 then also hold, and Theorem 13.10 applies to this kernel, including its local  $L^2$  bound along the interpolation. The segment  $s\theta$ ,  $0 \leq s \leq 1$ , stays in  $B_r$ , and the product and the couplings are fixed along it. If the Wilson measure conditioned on the whole small-field event is conditioned further on the coarse field of this step, the tests of the coarser steps are functions of that coarse field and of its deterministic averages. Their common factor therefore cancels in the normalized conditional fluctuation kernel wherever these tests pass.

Finally, a gauge transformation depending only on the coarse field and equal to the identity at the coarse vertices changes the representative of the fine field after reconstruction. The free coordinates, all fluctuation balls, the Haar disintegration density, and the triangular Jacobian are unchanged, and by gauge invariance on overlaps the value of the effective interaction agrees. Hence the potential and the kernel are the same. The proviso on coefficient bounds allows the analytic estimates, including those on derivatives of the potential, to be used in the new representative. Every reduction and every bound used only the fixed parameters, and none involved the length of the preceding recursion.  $\square$

## 14.5 Conditional comparisons and exit probabilities

We now compare three normalized measures on the fluctuation variables over a coarse field  $c \in B_r$ : the conditional fiber measure of the incoming restricted density, its restriction to the fluctuation cutoff domain, and a comparison measure on the enclosing product  $\mathcal{K}_t$ . The last one has a uniformly convex potential on a convex domain, so it satisfies a logarithmic Sobolev inequality and a local covariance bound. Its tail bounds pass to the fiber by conditioning, with the factor  $v_c(S_c)^{-1} \leq (1 - d_t)^{-1}$ ; no logarithmic Sobolev or covariance-decay inequality for the nonconvex fiber is asserted. We also bound the probability that the coarse field leaves  $D_r$  by a ratio of auxiliary integrals.

**Theorem 14.7.** *Use the parameters of Definition 14.4, consider one RG step from depth  $d_0$  with incoming effective interaction  $R$  given by Theorem 14.6, and for every  $c = e^\theta \in B_r$  define the normalized enclosing-product comparison measure  $v_c$  and the conditional fiber measure  $\kappa_c^{\text{old}}$  by*

$$v_c(dw) = \frac{\mathbf{1}_{\mathcal{K}_t}(w)e^{-V_c(w)}dw}{\int_{\mathcal{K}_t} e^{-V_c(w)}dw}, \quad \kappa_c^{\text{old}}(dw) = \frac{\mathbf{1}_{S_c}(w)e^{-V_c(w)}dw}{\int_{S_c} e^{-V_c(w)}dw}.$$

1. *Uniformly in the depth, the side, and  $c$ ,*

$$\sup_i |\mathbb{E}_{v_c} w_i| \leq Ct \|\theta\|_\infty, \quad \kappa_c^{\text{old}} = v_c(\cdot \mid S_c).$$

2. *Put  $m_{\text{cv}} = 1/2$  and*

$$d_t = 6b_w e^{-m_{\text{cv}} Q_{\text{phys}}^2 / (24t^2)}, \quad \delta_t = 6b_w e^{-m_{\text{cv}} p_w(t)^2 / 24}, \quad \epsilon_t = \frac{\delta_t}{1 - d_t}.$$

*If  $d_t < 1$ , then*

$$\kappa_c^{\text{old}}((\Omega_{P_{\text{phys}}}^{\text{phys}})^c) \leq \frac{6b_w e^{-m_{\text{cv}} P_{\text{phys}}^2 / (24t^2)}}{1 - d_t}, \quad \kappa_c^{\text{old}}((\Omega_p^w)^c) \leq \epsilon_t.$$

3. *Let  $\sigma$  be the probability measure  $Z_{\text{in}}^{-1} P_{\text{in}} dH$ , where  $Z_{\text{in}} = \int P_{\text{in}} dH$ ; this is the normalized incoming restricted density of this step. Let  $G(c)$  be the density at  $c$  of the pushforward of the unnormalized measure  $P_{\text{in}} dH$  under  $T_M^J$ , given by the based coarea formula of Theorem 14.6(7), and let  $\tilde{F} = \tilde{F}_+$  be the auxiliary integral of Theorem 14.6(2), both with the same convention for field-independent factors. Then, on  $D_r$ ,*

$$\frac{\tilde{F}(c)}{G(c)} = \kappa_c^{\text{old}}(\Omega_p^w),$$

*without any smallness assumption, and  $1 \leq G(c)/\tilde{F}(c) \leq (1 - \epsilon_t)^{-1}$  if  $d_t < 1$  and  $\epsilon_t < 1$ .*

4. *If  $d_t < 1$ , the normalized measures  $\mathbf{1}_{D_r} G dH / \int_{D_r} G dH$  and  $\mathbf{1}_{D_r} \tilde{F} dH / \int_{D_r} \tilde{F} dH$  differ in total variation  $\sup_A |\mu(A) - \nu(A)|$  by at most  $\epsilon_t$ , and*

$$\sigma(\text{the next RG step fails}) \leq \sigma(T_M^J U \notin D_r) + \epsilon_t,$$

*where the next RG step fails if  $U$  violates at least one of the coordinate conditions of the step, the fluctuation cutoff, and the condition  $T_M^J U \in D_r$ .*

5. *We have  $T_M^J(D_r) \subset D_{R_{\text{buf}}}$  with  $R_{\text{buf}} = (10M)^J r$ . Let  $\mathcal{K}'_t$  be the product of physical radius  $\mathcal{L}'_{\text{en}} r$ , and define*

$$H(c) = C(c) \int_{\mathcal{K}'_t} e^{-V_c(w)} dw, \quad C(c) = A_{\text{coord}} e^{-E_{d_0+J}(c)/g_J^2 - B_{\text{ref}}(c) - R_{\text{ref}}(c)},$$

with the same convention for field-independent factors as  $G$  and  $\tilde{F}$ . If  $\delta_t < 1$ , then

$$G(c) \leq H(c) \leq \frac{\tilde{F}(c)}{1 - \delta_t} \quad (c \in D_{R_{\text{buf}}}),$$

and, with  $Z_{\text{aux}}(A) = \int_A e^{-E_{d_0+J,\text{sat}}/g_f^2 - R_{+,\text{sat}}} dH$ ,

$$\sigma(T_M^J U \notin D_r) \leq \frac{1}{1 - \delta_t} \frac{Z_{\text{aux}}(D_{R_{\text{buf}}} \setminus D_r)}{Z_{\text{aux}}(D_r)}.$$

In (5) no inclusion of the cutoff product in  $S_c$  is required for coarse fields in  $D_{R_{\text{buf}}} \setminus D_r$ . The pointwise statements refer to the explicit density given by the based coarea formula; for an arbitrary disintegration they hold almost everywhere. The constants do not depend on the volume, but the explicit factor  $b_w$  and the conditions  $d_t < 1$ ,  $\epsilon_t < 1$ , and  $\delta_t < 1$  cannot be dropped. At fixed  $L$  and  $\gamma > 0$ , along refinement of the lattice,  $b_w = O_{M,J,\gamma}(t^{-6})$ , so the errors  $d_t$ ,  $\delta_t$ , and  $\epsilon_t$  are smaller than every power of the mesh. The theorem gives no bound on the ratio of auxiliary integrals in (5).

The mean bound in (1) concerns the comparison measure on the enclosing product, not the kernel on the cutoff product, whose means are bounded in Theorem 13.10. Its proof must avoid the large radius  $R_{\text{en}}/t$  of  $\mathcal{K}_t$ . Heuristically, a classical cubic term in the potential has size  $tw^3$ , so its mixed derivative in  $w$  and in the coarse field has size  $tw^2$ , whose  $L^2$  norm is at most  $t\|w\|_4^2$ . The fourth moments are bounded independently of the radius of the product, by concentration. In a term of degree  $k \geq 3$  the remaining  $k - 3$  factors are bounded by  $R_{\text{en}}/t$ , which leaves  $t^{k-2}(R_{\text{en}}/t)^{k-3} = tR_{\text{en}}^{k-3}$ , a summable series. Bounding all factors by the radius of the product, or using a global Cauchy–Schwarz bound for the extensive derivative of the potential, would lose this smallness.

*Proof.* Steps 1 to 4 prove the mean bound in (1): Step 1 recalls the convexity input, Step 2 derives concentration and fourth moments, Step 3 bounds the mixed derivative of the potential, and Step 4 integrates along the segment from 0 to  $\theta$ . Steps 5 and 6 prove the conditioning identity and (2). Steps 7 and 8 prove (3) and (4). Steps 9 and 10 prove (5), and Step 11 proves the remaining statements.

**Step 1: inputs.** We use the coefficient bounds of Lemma 14.2. They apply because the effective interaction, constructed in Theorem 14.6, has the required holomorphy, vanishing first derivative, gauge invariance on overlaps, symmetries, and norm bound. The whole enclosing product and the segment of coarse fields satisfy (14.12). The convexity and the product inequalities are supplied by the clause of Theorem 13.8 for products of fixed physical radius, which rests on Theorem 5.4. Their constants depend on the lower Hessian bound  $m_{\text{cv}} = 1/2$  and on the weighted bounds on the interaction, but not on the radius  $R_{\text{en}}/t$  of the product or on the dimension. These inequalities go back to Gross [103], Bakry and Émery [8], and Helffer and Sjöstrand [110]; the version for reflecting boundary conditions on products of balls is proved in Section 5.

**Step 2: concentration of one coordinate.** Let  $X$  be a scalar coordinate. The logarithmic Sobolev inequality of Step 1 reads  $\text{Ent}(f^2) \leq (2/m_{\text{cv}})\mathbb{E}|\nabla f|^2$ . We apply it to  $f = e^{\lambda X/2}$ , for which  $|\nabla f|^2 = \lambda^2 e^{\lambda X}/4$ , and set  $\psi(\lambda) = \log \mathbb{E}e^{\lambda X}$ . Dividing by  $\mathbb{E}e^{\lambda X}$ , we obtain

$$\lambda\psi'(\lambda) - \psi(\lambda) \leq \frac{\lambda^2}{2m_{\text{cv}}}.$$

The left side is  $\lambda^2$  times the derivative of  $\psi(\lambda)/\lambda$ , whose limit at  $\lambda = 0$  is  $\mathbb{E}X$ . Integrating from 0 to  $\lambda$  gives  $\log \mathbb{E}e^{\lambda(X-\mathbb{E}X)} \leq \lambda^2/(2m_{\text{cv}})$ . This is the argument of Herbst; see [123, Section 2.3]. The exponential Markov inequality with  $\lambda = m_{\text{cv}}v$  gives

$$\mathbb{P}(|X - \mathbb{E}X| \geq v) \leq 2e^{-m_{\text{cv}}v^2/2}. \quad (14.13)$$

The calculation is valid for either sign of  $\lambda$  on the bounded product. Integrating  $4v^3$  times this tail bound gives

$$\|X\|_4 \leq C(1 + |\mathbb{E}X|). \quad (14.14)$$

In particular, the estimates below contain no factor equal to the radius of the product.

**Step 3: the mixed derivative of the potential.** Let  $\theta_s = s\theta$  for  $0 \leq s \leq 1$ , let  $v_s$  be the corresponding comparison measure, and let  $M(s) = \max_\alpha |\mathbb{E}_{v_s} w_\alpha|$ , the maximum over scalar coordinates. In this proof, the running maximum  $M(s)$  of the means should not be confused with the odd integer  $M$ . Since the product  $\mathcal{K}_t$  does not depend on  $s$ , differentiating the finite positive integral by the quotient rule gives

$$\frac{d}{ds} \mathbb{E}_{v_s} w_\alpha = -\text{Cov}_{v_s}(w_\alpha, \partial_s V_{\theta_s}), \quad (14.15)$$

and there is no boundary term.

In a classical monomial of total degree  $k \geq 3$  in  $w$ , the derivative  $D_w \partial_s V$  leaves  $k-1$  fluctuation factors. We keep two of them in  $L^4$  and bound the other  $k-3$  by  $p = R_{\text{en}}/t$ . By Hölder's inequality and (14.14), the  $L^2$  norm of their product is at most  $C(1 + M(s))^2 p^{k-3}$ , and the coupling factor of the monomial becomes

$$t^{k-2} p^{k-3} = t R_{\text{en}}^{k-3}.$$

Since the derivative in the coarse field is taken in the fixed direction  $\theta$ , the summation on a strictly smaller radius in Lemma 14.2 shows that the summed weighted majorant in the coarse field costs  $C\|\theta\|_\infty$ . A nonclassical term of degree one gives  $Ct(1 + E_*)\|\theta\|_\infty$  directly. A term of degree two gives  $Ct^2(1 + E_*)(1 + M(s))\|\theta\|_\infty$ , from its single  $L^2$  factor. For degree  $k \geq 3$  the preceding argument with two factors in  $L^4$  gives the coupling factor  $t^k p^{k-3} = t^3 R_{\text{en}}^{k-3}$ . Summing these convergent geometric series, and the weighted sums over the indices in the coarse field, we obtain the local weighted estimate

$$\sup_\alpha \|D_{w_\alpha} \partial_s V_{\theta_s}\|_{2,\text{loc}} \leq Ct\|\theta\|_\infty(1 + E_* + M(s)^2). \quad (14.16)$$

Here the local norm fixes the index  $\alpha$  of the derivative in  $w$  and sums all coefficient labels and indices in the coarse field with their weights  $e^{bt}$ , and the  $L^2$  norm is taken under  $v_s$ . This definition and the weighted coefficient bounds prevent a volume factor in this estimate.

**Step 4: the mean bound.** We apply the local covariance bound of Step 1 to (14.15). The gradient of  $w_\alpha$  is the unit vector at  $\alpha$ , and the exponentially weighted sum over the other index is finite by the bound on the number of labels in a ball. We obtain

$$\left| \frac{d}{ds} \mathbb{E}_{v_s} w_\alpha \right| \leq Ct \|\theta\|_\infty (1 + E_* + M(s)^2).$$

At  $s = 0$ , every vector mean vanishes and  $M(0) = 0$ , since the potential and each color ball are invariant under simultaneous adjoint rotations. At fixed finite volume, the map  $s \mapsto M(s)$  is continuous. If  $M(s)$  first reached the value one at some  $s_0 \leq 1$ , integration up to  $s_0$  would give  $M(s_0) \leq Ctr(2 + E_*) < 1/2$ , after the corresponding reduction of  $t_*$  in Definition 14.4(6), a contradiction. Hence  $M(s) < 1$  for all  $s \in [0, 1]$ . Inserting this into the last differential inequality, we obtain  $M(1) \leq Ct \|\theta\|_\infty$ . The constants may absorb the fixed  $E_*$ . This proves the mean bound in (1).

**Step 5: the conditioning identity.** By the description of the support and the density of the fiber in Theorem 14.6(5) and (7),

$$\kappa_c^{\text{old}}(A) = \frac{v_c(A \cap S_c)}{v_c(S_c)}$$

for every Borel set  $A$ . The denominator is strictly positive, since  $\Omega_{Q_{\text{phys}}}^{\text{phys}} \subset S_c$  has positive Lebesgue measure and the density of  $v_c$  is positive. This identity requires no convexity of  $S_c$ , and it proves the second statement of (1).

**Step 6: tail bounds.** We reduce  $t_*$  so that every scalar mean under  $v_c$  is at most  $P_{\text{phys}}/(2\sqrt{3}t)$  and at most  $p_w(t)/(2\sqrt{3})$ . This is possible since the means are  $O(tr)$  by Step 4,  $P_{\text{phys}}$  is a fixed positive multiple of  $r$ , and  $p_w(t) \geq p_0$ . Let  $a = P_{\text{phys}}$  or  $a = Q_{\text{phys}}$ . If  $|w_{i,e}| \geq a/t$ , then one of its three scalar coordinates has magnitude at least  $a/(\sqrt{3}t)$ , and hence deviates from its mean by at least  $a/(2\sqrt{3}t)$ . Applying (14.13) to both tails of each scalar coordinate and summing over the  $3b_w$  scalar coordinates, we obtain

$$v_c((\Omega_a^{\text{phys}})^c) \leq 6b_w e^{-m_{cv}a^2/(24t^2)}. \quad (14.17)$$

The same calculation with the threshold  $p_w(t)$  gives

$$v_c((\Omega_p^w)^c) \leq 6b_w e^{-m_{cv}p_w(t)^2/24} = \delta_t. \quad (14.18)$$

Now suppose  $d_t < 1$ . Since  $\Omega_{Q_{\text{phys}}}^{\text{phys}} \subset S_c$ , (14.17) with  $a = Q_{\text{phys}}$  implies  $v_c(S_c) \geq 1 - d_t$ . Dividing the bounds (14.17) with  $a = P_{\text{phys}}$  and (14.18) by this denominator and using Step 5, we obtain both bounds in (2), the second with exactly  $\epsilon_t$ .

**Step 7: density ratios.** Let  $C(c)$  denote the prefactor (14.9), as in (5), with the chosen convention for constants; it depends on  $c$  but not on  $w$ . By Theorem 14.6(2) and (7), on  $B_r$  we have

$$G(c) = C(c) \int_{S_c} e^{-V_c} dw, \quad \tilde{F}(c) = C(c) \int_{\Omega_p^w} e^{-V_c} dw.$$

Since the cutoff product is a subset of  $S_c$ , dividing proves the identity in (3) without any smallness condition. If  $d_t < 1$  and  $\epsilon_t < 1$ , then (2) implies the two-sided bound in (3). Both formulas are compatible with the orthogonal changes of representative of Theorem 14.3(3), and hence hold on  $D_r$  for this explicit density.

**Step 8: total variation and failure of the next step.** For the comparison of normalized measures and the failure bound, we assume only  $d_t < 1$ , so that  $\epsilon_t$  is a nonnegative upper bound; the condition  $\epsilon_t < 1$  is not needed. Let  $\eta$  be the joint law of the coarse field and the fluctuation variables under the uncut fiber densities, conditioned on  $c \in D_r$ , and let  $A$  be the event that  $w \in \Omega_p^w$ . Integrating the second bound of (2) over  $c$ , we obtain  $\eta(A^c) \leq \epsilon_t$ . For any probability measure  $\eta$  and any event  $A$  of positive probability,  $\eta = \eta(A)\eta(\cdot | A) + \eta(A^c)\eta(\cdot | A^c)$ , and the two conditional measures are carried by disjoint events. Hence

$$\sup_B |\eta(B) - \eta(B | A)| = \eta(A^c) \sup_B |\eta(B | A^c) - \eta(B | A)| = \eta(A^c).$$

Taking the image under a measurable map does not increase this distance, since an event for the coarse field pulls back to an event in the joint space. The images of  $\eta$  and  $\eta(\cdot | A)$  under the projection to the coarse field are the two normalized measures in (4), so they differ in total variation by at most  $\epsilon_t$ . For the failure bound, note that  $\sigma$  is carried by  $D_r$ . By Theorem 14.3 with  $u = r$ , and the reduction of  $r_{\text{geom}}$  in step (2) of Definition 14.4, every configuration  $U \in D_r$  with  $T_M^J U \in D_r$  satisfies the coordinate conditions of the step, so the next step can fail only if  $T_M^J U \notin D_r$  or if  $T_M^J U \in D_r$  and the fluctuation cutoff fails. The probability of the second event is  $\sigma(T_M^J U \in D_r) \eta(A^c) \leq \epsilon_t$ . This proves (4).

**Step 9: the larger coarse fields.** We now prove (5), without using any inclusion of the cutoff product in  $S_c$  for coarse fields outside  $D_r$ . Since the fine side is  $M^J m$  and the coarse side is  $m \geq 4$ , each averaging step has coarse side  $M^{J-i-1} m \geq 4$ . Moving an input configuration in  $D_r$  into  $B_r$  and applying Lemma 7.2  $J$  times, we find, by gauge covariance,  $T_M^J(D_r) \subset D_{R_{\text{buf}}}$  with  $R_{\text{buf}} = (10M)^J r$ . For  $c \in B_{R_{\text{buf}}}$ , applying Theorem 14.3 with  $u = R_{\text{buf}}$  and using  $D_r \subset D_{R_{\text{buf}}}$ , we obtain  $|w_{i,e}| \leq C_{\text{TG}} R_{\text{buf}} / (\lambda_* t)$  for every point of  $S_c$ . Since  $\mathcal{L}'_{\text{en}} > 2C_{\text{TG}} C_{\text{map},J} / \lambda_*$ , the set  $S_c$  lies in the interior of  $\mathcal{K}'_t$ .

Both  $\mathcal{K}'_t$  and the fluctuation cutoff domain satisfy the conditions fixed for the larger coarse fields in Definition 14.4. More precisely, by (14.12), we have  $R_{\text{buf}} + \mathcal{L}'_{\text{en}} r < r_J$ , the denominator of the majorant is at least  $1/2$ , and both radii lie in the products

of step (2). Therefore the image condition of Theorem 13.2 also holds on this larger product. Its free-input radius is strictly below  $4K_*s_J$ , and all comparison margins and the Hessian smallness condition of Theorem 13.8(3) are retained. The Hessian argument of Theorem 14.6(6) applies with the physical radius  $\mathcal{L}'_{\text{en}}r$  in place of  $R_{\text{en}}$ , using the inequality for  $C\mathcal{L}'_{\text{en}}r$  in step (4) of Definition 14.4. Steps 2 to 4 apply with the coarse-field radius  $R_{\text{buf}}$  in place of  $r$ , with the fixed product and the weighted sums otherwise unchanged. These are the conditions imposed when choosing  $r_{\text{max}}$  and  $t_*$ . They give scalar means of size at most  $CtR_{\text{buf}}$  and the tail bound (14.18) on the larger product, with the same exponent. With  $H(c) = C(c) \int_{\mathcal{K}'_t} e^{-V_c} dw$  as in (5), the inclusion  $S_c \subset \mathcal{K}'_t$  implies  $G(c) \leq H(c)$ , also when  $S_c$  is empty. The ratio  $\tilde{F}(c)/H(c)$  is the probability of the cutoff product under the normalized measure on  $\mathcal{K}'_t$ , so the concentration bound gives  $\tilde{F}(c)/H(c) \geq 1 - \delta_t$ , that is,  $H(c) \leq \tilde{F}(c)/(1 - \delta_t)$  when  $\delta_t < 1$ . No inclusion of the cutoff product in  $S_c$  has been used outside  $D_r$ .

**Step 10: the exit probability.** Assume  $\delta_t < 1$ . The law of the coarse field under  $\sigma$  has density  $G / \int G dH$ , and by Step 9 it is carried by  $D_{R_{\text{buf}}}$ . Its exit probability satisfies

$$\frac{\int_{D_{R_{\text{buf}}} \setminus D_r} G dH}{\int_{D_{R_{\text{buf}}}} G dH} \leq \frac{1}{1 - \delta_t} \frac{\int_{D_{R_{\text{buf}}} \setminus D_r} \tilde{F} dH}{\int_{D_r} \tilde{F} dH}.$$

In the numerator, we used the upper bound of Step 9, and in the denominator, we used  $G \geq \tilde{F}$  on  $D_r$ , from (3). Since  $\tilde{F} = A_{\text{coord}} e^{-E_{d_0+J}/g_J^2 - R_+}$ , the common constant  $A_{\text{coord}}$  cancels, and we obtain the ratio of the integrals  $Z_{\text{aux}}$  in (5). The same cancellation holds for any convention for centering.

**Step 11: versions of the density and asymptotics.** All pointwise statements used the explicit density given by the based coarea formula. Any other disintegration agrees with it almost everywhere on the set of coarse fields where it is positive, which gives the stated qualification. The explicit factor  $b_w$  cannot be absorbed into a constant independent of the volume. Along a refinement at fixed  $\gamma$ , we have  $t^2 = \gamma/(Mm)$  and hence  $m^3 = (\gamma/M)^3 t^{-6}$ , so that

$$b_w = 2(B^3 - 1)(\gamma/M)^3 t^{-6}.$$

With  $x = \log(1/t)$ , the logarithm of  $\delta_t$  is at most  $C + 6x - cA_w^2 x^2$ , which is below  $-hx$  for every fixed  $h > 0$  once  $x$  is large. The tail with a physical radius has the stronger exponent  $-c/t^2$ . Since the mesh is  $L/m = (LM/\gamma)t^2$ , both errors are smaller than every fixed power of the mesh along such a sequence, and the denominators  $1 - d_t$  tend to one.  $\square$

## 14.6 Partition-function bounds on the stopping lattice

After  $k$  RG steps, the stopped measure lives on the stopping lattice  $\Lambda_n$ ,  $n = n_{kJ}$ , at depth  $d = kJ$ . This subsection bounds its partition function from below and the exponential moments of the Wilson action under it from above. It also compares effective classical actions with the Wilson action at every depth, records an exact identity for normalization constants, and bounds derivatives of the centered effective interactions in fixed charts. The lower bound is a trial integral over the same domain  $D_r$  as the stopped measure, in a periodic tree gauge with all non-tree link variables of size  $a_0g$ . The upper bound enlarges only the numerator, to an integral over an open cube, for which Lemma 6.10 gives an explicit bound. The ranks of the two gauge-fixed integrals are as follows.

| Integral                                 | Group variables   | Power of $g$         |
|------------------------------------------|-------------------|----------------------|
| periodic tree trial inside $D_r$         | $2n^3 + 1$        | $g^{3(2n^3+1)}$      |
| open cube, upper bound for the numerator | $2n^3 - 3n^2 + 1$ | $g^{3(2n^3-3n^2+1)}$ |
| ratio                                    | difference $3n^2$ | $g^{-9n^2}$          |

Since the numerator and the denominator are each estimated once, the boundary term  $9n^2 \log(1/g)$  appears once. These bounds, obtained by elementary gauge integration, are in the tradition of the stability estimates of Faria da Veiga and O'Carroll [64, 65]; here we need the precise periodic correction.

**Theorem 14.8.** *Use the parameters of Definition 14.4. There are  $\kappa_*, a_0, g_* > 0$  and  $C_A, C_R, C_\# > 0$ , independent of the side  $n \geq 4$ , of the depths  $d = kJ$  reached by the recursion, of the number of RG steps, and of  $0 < g \leq g_*$ , with the following properties. The integration domain is exactly  $D_r$  on the stopping lattice  $\Lambda_n$ . Let  $R = R_k$  be the effective interaction of Theorem 14.6 at depth  $d = kJ$ , and put*

$$D = 3n^3, \quad b_n = 2n^3 + 1, \quad c = R(I), \quad \bar{R} = R - c,$$

$$F_g = \frac{E_0}{g^2} = \frac{4}{g^2} \sum_p q(U_p), \quad Z^{\text{raw}} = \int_{D_r} e^{-E_{d,\text{sat}}/g^2 - R_{\text{sat}}} dH_n, \quad \hat{Z}_1 = e^c Z^{\text{raw}}.$$

Here  $D$  is the number of links, and  $c$  is a constant, not a coarse field.

1. For every integer  $d \geq 0$ ,  $E_{d,\text{sat}} \geq \kappa_* E_0$  on the gauge-invariant extension of the larger classical domain, whether or not an effective interaction is attached to the depth  $d$ .
2. For the effective interactions of the stopped measures,  $|\bar{R}| \leq C_R D E_* \rho^2$  on  $D_r$ . On the trial set of configurations which, in a periodic spanning tree gauge based at a root, have every non-tree link logarithm of norm at most  $a_0g < \min(r/2, 1)$ ,

$$E_d \leq C_A D a_0^2 g^2, \quad |\bar{R}| \leq C_R D E_* a_0^2 g^2,$$

$$Z^{\text{raw}} \geq \left( \frac{a_0g - \sin(a_0g)}{2\pi} \right)^{b_n} e^{-c - C_A D a_0^2 - C_R D E_* a_0^2 g^2}.$$

3. Let  $\hat{\nu} = \hat{Z}_1^{-1} \mathbf{1}_{D_r} e^{-E_d/g^2 - \bar{R}} dH_n$  be the stopped measure, and put  $\lambda_{\text{en}} = \kappa_*/2$ . Then

$$\begin{aligned}\hat{\nu}(F_g \geq u) &\leq \exp(C_\# D + 9n^2 \log(1/g) - \lambda_{\text{en}} u), \quad u \geq 0, \\ \mathbb{E}_{\hat{\nu}} e^{sF_g} &\leq \exp(C_\# D + 9n^2 \log(1/g)), \quad 0 \leq s \leq \lambda_{\text{en}}, \\ \mathbb{E}_{\hat{\nu}} F_g &\leq [C_\# D + 9n^2 \log(1/g) + 1]/\lambda_{\text{en}}.\end{aligned}$$

The open cube with  $n$  vertices per side has rank  $b_n^{\text{op}} = 2n^3 - 3n^2 + 1$ . Only an upper bound for a numerator is enlarged to the Haar integral over this open cube; the denominator remains the trial integral over  $D_r$ . The unmatched power of the coupling is exactly  $g^{-3(b_n - b_n^{\text{op}})} = g^{-9n^2}$ .

4. Fix  $0 < \lambda_0 \leq 1$ , let  $\lambda \in [\lambda_0, 1]$ , and put

$$\hat{Z}_\lambda = \int_{D_r} e^{-\lambda E_d/g^2 - \lambda \bar{R}} dH_n, \quad d\tau^{(\lambda)} = \hat{Z}_\lambda^{-1} \mathbf{1}_{D_r} e^{-\lambda E_d/g^2 - \lambda \bar{R}} dH_n.$$

After one reduction of  $g_*$  depending on  $\lambda_0$ , for  $0 \leq s \leq \lambda_0 \kappa_*/2$ ,

$$\log \mathbb{E}_{\tau^{(\lambda)}} e^{sF_g} \leq CD + 9n^2 \log(1/g), \quad \log \left\{ \frac{\hat{Z}_\lambda}{\hat{Z}_1} \mathbb{E}_{\tau^{(\lambda)}} e^{sF_g} \right\} \leq CD + 9n^2 \log(1/g).$$

The second bound is obtained from one numerator and one denominator, not as the sum of bounds for the two factors.

5. For every fixed  $\alpha_0 > 0$  and  $0 < \lambda_0 \leq \lambda \leq \lambda_1 < \infty$ ,

$$\log \frac{\int_{\mathcal{U}_n} e^{-(\lambda\alpha_0 - s)F_g} dH_n}{\hat{Z}_1} \leq C_{\lambda_0, \lambda_1, \alpha_0} D + 9n^2 \log(1/g), \quad 0 \leq s \leq \lambda_0 \alpha_0 / 2,$$

where  $g_*$  is reduced so that the effective inverse coupling  $4(\lambda\alpha_0 - s)/g^2$  is at least one. No condition on charts or small fields is imposed on this numerator.

6. Let an initial probability density with unnormalized normalizing constant  $\mathcal{Z}_{\text{base}}$  be restricted by the small-field tests and pushed forward by finitely many RG steps of Theorem 14.6 to the lattice  $\Lambda_n$ , and suppose that the resulting subprobability measure is written in the centered convention as

$$d\tilde{\zeta}_T = \mathcal{Z}_{\text{base}}^{-1} e^{\Lambda_T^\mathbb{T}} \mathbf{1}_{D_r} e^{-E_d/g^2 - \bar{R}} dH_n, \quad p_T = \tilde{\zeta}_T(\mathcal{U}_n) \leq 1.$$

Then, exactly,

$$\frac{e^{\Lambda_T^\mathbb{T}}}{\mathcal{Z}_{\text{base}}} = \frac{p_T}{\hat{Z}_1} = \frac{p_T e^{-c}}{Z_{\text{raw}}} \leq \hat{Z}_1^{-1}.$$

All normalization factors of integrated Gaussian coordinates, coarea constants, constant terms of cluster expansions, and constants removed from effective interactions belong to this same normalization constant  $\Lambda_T^\mathbb{T}$ .

7. The constants depend only on the parameters of the recursion, the constants of Theorem 11.2 and Lemma 6.10, and the displayed parameters  $\lambda_0, \lambda_1, \alpha_0$ . For every fixed  $\gamma_0 > 0$ , one minimum side makes these estimates hold simultaneously for  $g^2 = \gamma/n$ , all  $0 < \gamma \leq \gamma_0$ , and every larger side. The estimates involve no block regions.
8. The effective interactions  $R_{d,\gamma} = R_k$  satisfy  $\mathcal{B}_{\rho, a_{\text{sp}}}(R_{d,\gamma}) \leq E_*$  and  $DR_{d,\gamma}(I) = 0$ , and they are gauge invariant on overlaps, as in Theorem 14.6. Fix a side  $n$ , a number  $0 < \eta < 1$ , and a finite family of smooth charts  $\Psi_\alpha$  of configurations on  $\Lambda_n$ , defined on parameter domains  $\tilde{W}_\alpha$ , with relatively compact subdomains  $W_\alpha \subseteq \tilde{W}_\alpha$  and smooth based gauge transformations, such that the charts and the gauge transformations extend smoothly to neighborhoods of the closures  $\overline{W}_\alpha$  and the gauge-transformed representatives of  $\Psi_\alpha(y)$ ,  $y \in W_\alpha$ , lie in  $B_{(1-\eta)\rho}$ . Then every derivative of fixed positive order of  $\overline{R}_{d,\gamma,\text{sat}} \circ \Psi_\alpha$  on  $W_\alpha$  is bounded uniformly in  $d = kJ$  and  $0 < \gamma \leq \gamma_0$ . In link logarithms, the bound of order  $j \geq 1$  in the weighted norm is  $C_j E_* \rho^{2-j} \eta^{-j}$ , with  $C_j = j!$ . Passing to a fixed chart costs its fixed derivative bounds and a norm conversion in fixed finite dimension. The charts are composed with fixed smooth gauge transformations, never with a differentiated Borel selection of gauges. The constants may depend on  $n, j, \eta$ , the charts, and the larger domains, but not on the depth or the coupling.
9. The law of the coarsest field  $V^{KJ} = T_M^{KJ} U$ , a configuration on  $\Lambda_u$ , under the conditioned Wilson measure  $\nu_{N,u,\gamma}$  is the stopped measure after all  $K$  RG steps.

Part (1) holds at every integer depth, and part (8) gives bounds on derivatives from the analytic coefficient norm itself; a bound on the oscillation of  $\overline{R}$  alone would not control its derivatives. Any later comparison of normalization constants on block regions must be made with the normalization constant  $\Lambda_T^\mathbb{T}$  of part (6) on the whole torus.

*Proof.* Step 1 relates the stopping lattice to the recursion. Steps 2 and 3 prove (1) and the first bound in (2). Step 4 proves the trial lower bound, and Step 5 the open-cube upper bound. Steps 6 to 8 combine them to prove (3) to (5), Step 9 proves (6), and Step 10 proves (7). Steps 11 and 12 prove (8), and Step 13 proves (9).

**Step 1: depths and couplings.** At a depth  $d = kJ$ , the effective interaction in this proof is the function  $R_{d,\gamma} = R_k$  of Theorem 14.6, in its chosen convention for constants. By that theorem, it is holomorphic on  $B_\rho$  and gauge invariant on overlaps, with  $DR_{d,\gamma}(I) = 0$  and  $\mathcal{B}_{\rho, a_{\text{sp}}}(R_{d,\gamma}) \leq E_*$ , uniformly in  $k$  and in the coupling. Its real density is the stopped density on  $D_r$ . Subtracting the value at the identity changes no derivative of positive order and no analytic norm. We take  $g_*$  so small that  $g_*/\sqrt{M} \leq t_*$ . Since  $g_J^2 = Mt^2$ , at the stopping coupling  $g$  the largest integrated coupling of the last RG step is  $g/\sqrt{M}$ , and all earlier integrated couplings are smaller. This implies that every run of the recursion in the theorem satisfies the conditions on the coupling in Definition 14.4; with zero RG steps there is no such condition.

**Step 2: the classical lower bound.** We apply the comparison of Theorem 11.2 at one stopping field and one mesh, for every integer depth, without reference to an effective interaction. With the larger radius  $r_C$  fixed in Section 14.2, the comparison product of that theorem satisfies

$$P_\infty \leq \exp\left(Cr_C \sum_{j \geq 0} M^{-2(j+1)}\right) < \infty.$$

We conclude that  $E_d \geq \kappa_* E_0$  with  $\kappa_* = P_\infty^{-1}$ . For an argument in the gauge-invariant extension of the classical domain, we fix a gauge transformation taking it into the corresponding classical ball inside  $B_{r_C}$  and use the clause of Theorem 11.2 on transport by a fixed gauge transformation. Theorem 11.6 identifies the transported values with  $E_{d,\text{sat}}$  for every  $d \geq 0$ , while  $E_0$  is gauge invariant. This proves (1) on the whole gauge-invariant domain and at every integer depth. Only the assertions involving  $R_{d,\gamma}$  restrict the depth to the values  $d = kJ$  reached by the recursion. If the larger radius requires a smaller  $\kappa_*$ , it is fixed before the final reductions of the coupling bound.

**Step 3: oscillation of the effective interaction.** Let  $A_s \geq 0$  denote the summand of order  $s$  in the weighted coefficient norm of  $R$ , so that  $\sum_{s \geq 2} A_s \leq E_*$ . For  $\|\xi\|_\infty \leq \epsilon \leq \rho$  we expand  $R(e^\xi) - R(I)$ , fix one scalar link and color label in each coefficient, and sum the remaining labels. Since there are  $3D$  choices of the fixed label and the spatial weights are at least one, absolute summation of the Taylor series gives

$$|R(e^\xi) - c| \leq C_R D \epsilon^2 \sum_{s \geq 2} A_s (\epsilon/\rho)^{s-2} \leq C_R D E_* \epsilon^2. \quad (14.19)$$

By gauge invariance on overlaps, this bound holds at every representative. With  $\epsilon = r < \rho$ , we obtain the bound  $|\bar{R}| \leq C_R D E_* \rho^2$  on  $D_r$ . No estimate of  $c$  is involved.

**Step 4: the trial lower bound.** We choose a spanning tree of the periodic lattice graph and a root. By Lemma 5.1, the integral of a gauge-invariant integrand over a gauge-invariant set is an integral over exactly  $b_n = D - (n^3 - 1) = 2n^3 + 1$  independent non-tree link variables, each with the Haar probability measure; the tree variables have total mass one, no root variable is integrated, and no volume factor of a stabilizer appears. We fix  $a_0 > 0$  and reduce  $g_*$  so that  $\epsilon = a_0 g < \min(r/2, 1)$  for all  $0 < g \leq g_*$ . Since all tree-fixed logarithms on the trial set are at most  $\epsilon$ , the trial set lies strictly inside the same domain  $D_r$ . By the upper bound of Theorem 11.2 at radius  $\epsilon$ , we have  $E_d \leq C_A D \epsilon^2$  uniformly in  $d$  and  $n$ , and (14.19) implies  $|\bar{R}| \leq C_R D E_* \epsilon^2$ . By (4.2), the exact Haar mass of the ball for one non-tree link is

$$m(\epsilon) = \frac{1}{16\pi^2} \int_{|X| < \epsilon} \text{sinc}^2(|X|/2) d^3X = \frac{1}{\pi} \int_0^\epsilon \sin^2(s/2) ds = \frac{\epsilon - \sin \epsilon}{2\pi}.$$

Integrating the lower bound for the density over the product of these  $b_n$  balls gives the lower bound for  $Z^{\text{raw}}$  in (2), with the factor  $e^{-c}$  kept.

We will use the weaker form

$$\widehat{Z}_1 \geq e^{-C_L D} g^{3b_n}. \quad (14.20)$$

To prove it, note that  $\text{sinc}(s/2) \geq \text{sinc}(1/2) > 0$  for  $0 \leq s \leq 1$ , since the numerator  $x \cos x - \sin x = -\int_0^x y \sin y dy$  of the derivative of  $\sin x/x$  is nonpositive on  $[0, 1/2]$ . The radial integral therefore gives  $m(\epsilon) \geq c_{\text{ball}} \epsilon^3$ , with  $c_{\text{ball}} = \text{sinc}^2(1/2)/(12\pi)$ . Since  $b_n \leq D$  and  $g_* \leq 1$ , all fixed factors in the trial bound have logarithms bounded in absolute value by a fixed multiple of  $D$ . This proves (14.20).

**Step 5: the open-cube upper bound.** For  $\alpha \geq \alpha_{\min} > 0$ , let

$$I_\alpha(g) = \int_{\mathcal{U}_n} e^{-\alpha F_g} dH_n.$$

Since  $q \geq 0$ , deleting every plaquette that crosses a periodic seam can only increase the integrand. The remaining complex has  $n$  vertices per direction,  $3(n-1)n^2$  links, and  $n^3$  vertices, so its tree rank is

$$b_n^{\text{op}} = 3(n-1)n^2 - n^3 + 1 = 2n^3 - 3n^2 + 1.$$

Each unused seam link contributes a Haar probability integral equal to one. Lemma 6.10, applied to this open cube at inverse coupling  $\beta = 4\alpha/g^2$ , which is legitimate when  $4\alpha_{\min}/g_*^2 \geq 1$ , gives

$$I_\alpha(g) \leq c_G^{b_n^{\text{op}}} (4\alpha/g^2)^{-3b_n^{\text{op}}/2} \leq e^{C_U D} g^{3b_n^{\text{op}}}, \quad (14.21)$$

where  $c_G = \sqrt{2/\pi}$  is the constant of that lemma. The constant  $C_U$  depends on  $\alpha_{\min}$  and  $c_G$ , and is independent of  $n$ ,  $d$ , and  $g$ .

**Step 6: moments under the stopped measure.** We use (1) and the oscillation bound of Step 3 in the numerator. For  $0 \leq s \leq \lambda_{\text{en}} = \kappa_*/2$  we have  $-E_d/g^2 + sF_g \leq -(\kappa_* - s)F_g$ , and hence

$$\int_{D_r} e^{-E_d/g^2 - \bar{R} + sF_g} dH_n \leq e^{C_R D E_* \rho^2} I_{\kappa_* - s}(g) \leq e^{C_U D} g^{3b_n^{\text{op}}},$$

after including the oscillation constant in  $C_U$ . Here  $\alpha_{\min} = \kappa_*/2$  works, including at the endpoint  $s = \lambda_{\text{en}}$ . Dividing by (14.20) leaves exactly

$$3(b_n - b_n^{\text{op}}) = 9n^2$$

unmatched powers of  $g^{-1}$ . This proves the bound on the exponential moment in (3), with  $A = C_\# D + 9n^2 \log(1/g)$ , after enlarging  $C_\#$  if necessary. The exponential Markov inequality at  $s = \lambda_{\text{en}}$  gives the tail bound. Since  $A \geq 0$ , integrating the minimum of the tail bound and one gives

$$\mathbb{E}_{\hat{\nu}} F_g \leq \int_0^\infty \min(1, e^{A - \lambda_{\text{en}} u}) du = \frac{A+1}{\lambda_{\text{en}}}.$$

The exponential moment at the endpoint  $s = \lambda_{\text{en}}$  was proved through its defining integral, not by integrating a tail bound. In the uncentered convention, the same factor  $e^{-c}$  appears in the numerator and in the denominator and cancels.

**Step 7: tempered measures.** Fix  $\lambda_0 \leq \lambda \leq 1$ . On the same tree trial set, the exponent  $\lambda E_d/g^2 + \lambda \bar{R}$  is bounded above by  $CD$ , uniformly in  $\lambda$ , and hence

$$\hat{Z}_\lambda \geq e^{-CD} g^{3b_n}. \quad (14.22)$$

For  $0 \leq s \leq \lambda_0 \kappa_*/2$ , the numerator is bounded above by

$$e^{CD} I_{\lambda \kappa_* - s}(g) \leq e^{CD} g^{3b_n^{\text{op}}}, \quad \lambda \kappa_* - s \geq \lambda_0 \kappa_*/2,$$

and one fixed reduction of  $g_*$  makes the inverse coupling of Lemma 6.10 at least one throughout this interval. Dividing by (14.22) gives the first bound in (4). For the second bound, we keep the exact identity

$$\frac{\hat{Z}_\lambda}{\hat{Z}_1} \mathbb{E}_{\tau(\lambda)} e^{sF_g} = \frac{\int_{D_r} e^{-\lambda E_d/g^2 - \lambda \bar{R} + sF_g} dH_n}{\hat{Z}_1},$$

and use the last upper bound for the numerator and (14.20), once each. The difference of ranks is still  $3n^2$ , so the exponent is again  $9n^2 \log(1/g)$ . Estimating the two factors on the left separately would count this term twice.

**Step 8: the unrestricted numerator.** Let  $\alpha = \lambda \alpha_0 - s \geq \lambda_0 \alpha_0/2$ . Applying (14.21) to the unrestricted Wilson numerator and (14.20) to  $\hat{Z}_1$  proves (5), after the corresponding fixed reduction of  $g_*$ . The constants may depend on  $\lambda_0$ ,  $\lambda_1$ , and  $\alpha_0$ , but not on the depth or the side.

**Step 9: the identity for normalization constants.** Integrating the subprobability measure in (6) gives exactly

$$p_T = \mathcal{Z}_{\text{base}}^{-1} e^{\Lambda_T^\top} \hat{Z}_1.$$

Both  $\mathcal{Z}_{\text{base}}$  and  $\hat{Z}_1$  are positive. The total mass  $p_T$  is at most one, since restricting a probability measure and pushing it forward cannot increase its total mass. Solving this equality, and using  $\hat{Z}_1 = e^c Z^{\text{raw}}$ , proves the identities and the inequality in (6). Removing a Gaussian coordinate, a reference Haar disintegration factor, a constant term, or the constant of an effective interaction changes the representation of the density and the normalization constant together; it cannot change this identity for the total mass.

**Step 10: uniformity in the coupling.** All constants so far were chosen from the parameters of the recursion and the constants of Theorem 11.2 and Lemma 6.10. For a fixed  $\gamma_0 > 0$ , we take a minimum side at least  $\gamma_0/g_*^2$ , rounded up to an integer, and to a multiple of four when the RG steps require it. Then  $g^2 = \gamma/n \leq g_*^2$  for every larger side and every  $0 < \gamma \leq \gamma_0$ . Since the earlier integrated couplings of the recursion are smaller, the same reductions hold at every depth. This proves (7).

**Step 11: derivatives in link logarithms.** We write  $\mathcal{R}_{d,\gamma}(\xi) = R_{d,\gamma}(e^\xi) - R_{d,\gamma}(I)$  and

$$A_{k,d,\gamma} = \frac{\rho^{k-2}}{k!} \|D^k \mathcal{R}_{d,\gamma}(0)\|_{a_{\text{sp}}, \text{root}}, \quad \sum_{k \geq 2} A_{k,d,\gamma} \leq E_*.$$

Let  $j \geq 1$ ,  $v = 1 - \eta$ , and  $\|\xi\|_\infty \leq v\rho$ . We differentiate the absolutely convergent Taylor series on this strictly smaller ball. Contracting its  $k - j$  undifferentiated indices does not increase the weight of the network on the remaining indices, so

$$\|D^j \mathcal{R}_{d,\gamma}(\xi)\|_{a_{\text{sp}}, \text{root}} \leq \rho^{2-j} \sum_{k \geq \max(2,j)} A_{k,d,\gamma} \frac{k!}{(k-j)!} v^{k-j}.$$

Each multiplier satisfies

$$\frac{k!}{(k-j)!} v^{k-j} = j! \binom{k}{j} v^{k-j} \leq j! \eta^{-j},$$

because  $\binom{k}{j} \eta^j v^{k-j}$  is one nonnegative term of  $(\eta + v)^k = 1$ . It follows that

$$\|D^j \mathcal{R}_{d,\gamma}(\xi)\|_{a_{\text{sp}}, \text{root}} \leq j! E_* \rho^{2-j} \eta^{-j} \quad (j \geq 1). \quad (14.23)$$

The uncontrolled constant  $R_{d,\gamma}(I)$  disappears through differentiation; no derivative bound is deduced from the oscillation bound alone.

**Step 12: derivatives in charts.** For  $y \in W_\alpha$ , we write the gauge-transformed representative of  $\Psi_\alpha(y)$  as  $e^{\xi_\alpha(y)} \in B_{(1-\eta)\rho}$ . The chart, the subdomain, and the gauge transformation are fixed independently of  $d$  and  $\gamma$ . By continuity, the extended representatives on  $\overline{W}_\alpha$  lie in the closed ball of radius  $(1 - \eta)\rho$ , strictly inside the domain of the logarithm, and compactness gives a neighborhood of  $\overline{W}_\alpha$  on which the same logarithm is smooth. For  $1 \leq h \leq j$ , let

$$C_{\alpha,h} = \sup_{y \in \overline{W}_\alpha} \|D^h \xi_\alpha(y)\|_{\text{op}, \infty} < \infty;$$

finiteness follows from continuity of these derivatives on the compact closure. The derivatives of the gauge transformation, of the group multiplication, and of the logarithm are included in these constants. By gauge invariance on overlaps,  $\overline{R}_{d,\gamma,\text{sat}} \circ \Psi_\alpha = \mathcal{R}_{d,\gamma} \circ \xi_\alpha$  on  $W_\alpha$ . The chain rule for higher derivatives gives one term for each partition  $\pi$  of  $\{1, \dots, j\}$ , with outer derivative  $D^{|\pi|} \mathcal{R}_{d,\gamma}$  and inner factors  $D^{|B|} \xi_\alpha$ ,  $B \in \pi$ ; this follows by induction, since a new derivative either joins an existing block or forms a new singleton block. Since there are  $9n^3$  scalar link and color labels, fixing the first outer index and summing the others bounds a full contraction by  $9n^3$  times the weighted norm. Using (14.23), we obtain

$$\|D^j(\overline{R}_{d,\gamma,\text{sat}} \circ \Psi_\alpha)\|_{\text{op}} \leq 9n^3 E_* \sum_{\pi} |\pi|! \rho^{2-|\pi|} \eta^{-|\pi|} \prod_{B \in \pi} C_{\alpha,|B|}. \quad (14.24)$$

Every factor on the right is fixed before the depth and the coupling vary, and for a finite family of charts we take the maximum. To cover a compact subset of the gauge-invariant domain with this interior property, we choose at each of its points a local chart and a fixed gauge transformation whose representatives stay in  $B_{(1-\eta)\rho}$ . Since these conditions are open, there exists a smaller parameter neighborhood with compact closure inside the chart on which they still hold. These neighborhoods cover the compact set, and a finite subcover gives subcharts of the required kind, with extensions of the charts and gauge transformations past their closures. We include the derivatives of their logarithmic charts in the same maximum. This proves (8).

**Step 13: the coarsest marginal.** At the coarsest lattice  $\Lambda_u$ , no tests of coarser RG steps remain. The small-field event is then exactly the event whose normalized pushforward defines the stopped density after  $K$  steps, so the law of  $V^{KJ}$  under  $\nu_{N,u,\gamma}$  is this stopped measure.  $\square$

## 14.7 Flat configurations

A configuration  $U$  is flat if  $U_p = I$  for every plaquette  $p$ . Flat configurations are the minima of the Wilson action, and the weak-coupling analysis of Sections 33 to 36 takes place near them. The lemma below first records the classification of flat configurations on a periodic lattice by their holonomies; this is the usual description of flat connections on  $\mathbb{T}_L^3$  by representations of  $\pi_1(\mathbb{T}_L^3) = \mathbb{Z}^3$ , and we include an elementary proof on the lattice. The rest of the lemma is specific to our construction: all flat configurations, including those whose holonomies form one of the eight central triples, satisfy every test of the small-field event, with vanishing fluctuation variables, once the coarse side is large. The reason is that a flat configuration is gauge equivalent to a constant configuration with link variables  $e^{H_i^{\log}/n}$ , which are close to the identity when  $n$  is large, although the holonomies  $e^{H_i^{\log}}$  around the cycles of the torus can be far from it.

**Lemma 14.9.** *1. On every periodic lattice  $\Lambda_n$ , the flat configurations modulo periodic lattice gauge transformations are classified by their commuting holonomies around the three coordinate cycles, up to simultaneous conjugation:*

$$\text{Hom}(\mathbb{Z}^3, \text{SU}(2)) / \text{SU}(2) \simeq (\mathbb{R}/4\pi\mathbb{Z})^3 / \{\boldsymbol{\vartheta} \sim -\boldsymbol{\vartheta}\},$$

where  $\boldsymbol{\vartheta}$  parametrizes the triple  $(e^{\boldsymbol{\vartheta}_1 u}, e^{\boldsymbol{\vartheta}_2 u}, e^{\boldsymbol{\vartheta}_3 u})$  for a unit vector  $u \in \mathfrak{su}(2)$ . There are exactly eight central points, the central triples, at which each coordinate is 0 or  $2\pi$  modulo  $4\pi$ . Every flat configuration is gauge equivalent to a configuration with constant commuting link variables  $e^{H_i^{\log}/n}$  in direction  $i$ , where the  $H_i^{\log} \in \mathfrak{su}(2)$  commute,  $|H_i^{\log}| \leq 2\pi$ , and  $e^{H_i^{\log}}$  are the cycle holonomies. The block averaging map  $T_M$  raises these link variables to

their  $M$ th powers and preserves the three cycle holonomies. Taking commuting  $M$ th roots gives exact flat refinements.

2. Fix  $r_{\text{flat}} > 0$  strictly smaller than  $r$  and than the radii of all real coordinate charts and gauge transformations of one RG step fixed in Definition 14.4. If the coarse side  $u$  satisfies  $2\pi/u < r_{\text{flat}}$ , then, for every  $K \geq 0$ ,  $N = uB^K$ , and every coupling allowed by Definition 14.4, every flat configuration on  $\Lambda_N$  belongs to the small-field event  $S_{N,u,\gamma}$ . In the successive based forests the coarse-field lift is flat, and

$$z_i = 0, \quad z_i^*(c_{i+1}) = 0, \quad w_i = 0, \quad E_{d,\text{sat}}(c_i) = 0$$

at every intermediate scale and every depth  $d$ . The same holds for products of fluctuations of any positive physical radius that fit the fixed domains. All initial and coarse  $D_r$  tests and all intermediate classical and coarea conditions hold. Changes of representative based at the coarse vertices preserve the holonomies and the membership in the event. The gradient of the Wilson action vanishes at flat configurations, so they are stationary for the Wilson flow, and their three cycle holonomies are unchanged at every time.

3. Under the side and parameter hypotheses of (2), at each fixed finite  $N = uB^K$  and each allowed coupling, the compact set of flat configurations has an open neighborhood contained in  $S_{N,u,\gamma}$ ; equivalently, there is  $\delta_{N,u,\gamma} > 0$  with

$$\{\max_p \rho_G(U_p) < \delta_{N,u,\gamma}\} \subset S_{N,u,\gamma}.$$

No bound on  $\delta_{N,u,\gamma}$  uniform in  $N$  or in the coupling is asserted. Separately, on a fixed stopping lattice whose side  $n$  satisfies  $2\pi/n < r_{\text{flat}}$ , the set of flat configurations lies in the open set  $D_r$ , with a neighborhood that depends only on  $n$  and  $r$ , not on the depth or the coupling.

No central triple is removed, and no projectively commuting or twisted flat data are introduced: all gauge transformations are periodic and all cycle holonomies commute.

*Proof.* Steps 1 to 4 prove (1), Steps 5 to 8 prove the membership statement in (2), Step 9 proves the statement on the Wilson flow, and Step 10 proves (3).

**Step 1: path independence.** We lift the periodic configuration to  $\mathbb{Z}^3$ . If all plaquette variables are the identity, the transport along a path is unchanged when an edge followed by its reverse is deleted, and when two adjacent coordinate steps are replaced by the other two sides of their elementary square. These moves connect any two paths with the same endpoints in  $\mathbb{Z}^3$ : one interchanges neighboring steps in different directions until the steps are grouped in coordinate order, and then cancels opposite steps in each group. This implies that a path and its reverse define inverse transports, and the transport  $P_U(x)$  from the origin to a vertex  $x$  of the lift does not depend on the path.

**Step 2: the holonomy triple.** The cycle holonomies are  $P_U(ne_i)$ . By periodicity, the transport from  $ne_i$  to  $ne_i + ne_j$  equals  $P_U(ne_j)$ , so path independence applied to the two orders of going around the  $i$  and  $j$  directions implies  $P_U(ne_i)P_U(ne_j) = P_U(ne_j)P_U(ne_i)$ . A periodic loop whose lift ends at  $n(k_1, k_2, k_3)$  has transport  $P_U(ne_1)^{k_1}P_U(ne_2)^{k_2}P_U(ne_3)^{k_3}$ . A periodic gauge transformation conjugates the triple by its value at the origin.

A commuting triple in  $SU(2)$  lies in one maximal torus. Indeed, suppose that one of the matrices is not central. Then it is unitarily diagonalizable with two distinct eigenvalues, and a matrix commuting with it preserves both eigenlines and is diagonal in this basis. The determinant condition then puts all three diagonal matrices in the same circle. If all three are central, any orthonormal basis works. Hence we can choose commuting logarithms  $H_i^{\log} \in \mathfrak{su}(2)$  with  $e^{H_i^{\log}} = P_U(ne_i)$  and  $|H_i^{\log}| \leq 2\pi$ , and we set  $a_i = e^{H_i^{\log}/n}$ .

**Step 3: an explicit gauge equivalence.** Let  $P_a(x)$  be the transport of the constant configuration with link variables  $a_i$  on  $\mathbb{Z}^3$ , and set  $k_x = P_a(x)^{-1}P_U(x)$ . By flatness and periodicity, we have  $P_U(x + ne_i) = e^{H_i^{\log}}P_U(x)$ , and the same identity holds for  $P_a$ , since  $a_i^n = e^{H_i^{\log}}$ . Since the two factors cancel in  $k_{x+ne_i}$ , the gauge transformation  $k$  is periodic. On an edge  $xy$ , we have  $P_U(y) = P_U(x)U_{xy}$ , so

$$k_x U_{xy} k_y^{-1} = P_a(x)^{-1} P_U(x) U_{xy} P_U(y)^{-1} P_a(y) = P_a(x)^{-1} P_a(y) = a_{xy}.$$

This gauge transformation is the identity at the origin. Conversely, a constant commuting configuration is flat and has the given cycle holonomies. Hence two flat configurations with the same based holonomy triple are equivalent under a gauge transformation based at the origin, and flat configurations modulo all periodic gauge transformations are classified by their triples modulo simultaneous conjugation.

**Step 4: the angle coordinates.** Writing the common element of the Cartan subalgebra as a multiple of a unit vector  $u$  gives the three angles, which have period  $4\pi$  in our normalization. A change of the common eigenbasis either fixes the two eigenlines or interchanges them, and the interchange reverses all three angles. There is no independent change of sign for a single noncentral coordinate. At a central triple each angle is 0 or  $2\pi$  modulo  $4\pi$ , which gives exactly  $2^3 = 8$  points. The sign involution fixes these points and removes none of them. This proves the stated quotient, including the central points.

For constant commuting link variables, every displaced contour in (4.3) and its straight reference path have the same net displacement  $Me_i$ , and hence the same transport  $a_i^M$ . The corresponding based loop holonomy is the identity, so the block averaging map involves no exceptional logarithm and returns  $a_i^M$ . The holonomy around a cycle of the coarse lattice is  $(a_i^M)^{n/M} = a_i^n$ , so the three cycle holonomies are preserved. Taking

$a_i = e^{H_i^{\log}/(Mn')}$  refines a constant configuration on the lattice of side  $n'$  exactly to the lattice of side  $Mn'$ . This completes the proof of (1).

**Step 5: flat lifts.** We now check the minimizer and the membership statements on the larger classical products of Section 14.2. The coarse-field lift  $U_0(c)$  of a flat coarse field  $c$  is flat. Indeed, a fine plaquette inside one block has only identity factors, a plaquette crossing one interface has two equal and opposite interface factors, and a plaquette crossing two interfaces has exactly the four factors of a coarse plaquette. In all three cases, the product is the identity.

**Step 6: zero classical action.** For a constant commuting coarse field on a lattice of side  $n$ , there exists a zero-action configuration at every depth  $d$ , obtained by placing  $e^{H_i^{\log}/(nM^d)}$  on the links of the finer lattice. Its logarithmic potential in units of the coarse lattice is the constant  $H_i^{\log}/n$ , with vanishing differences. Since the coarse link logarithms have norm at most  $2\pi/n < r_{\text{flat}}$ , this configuration, together with all the fixed changes of chart and of gauge, lies strictly inside the domains of the classical minimization. Its Wilson action is zero. Since the effective classical action is nonnegative and is defined as a constrained minimum, its value at this coarse field is zero at every depth. To transport this value to the coarse-field lift, we use Theorem 11.6: by the choice of  $r_{\text{flat}}$ , the constant representative and the forest-fixed representative both lie strictly inside  $B_{r_C}$ .

**Step 7: vanishing fluctuation variables.** Consider one averaging step. The block-tree gauge transformation of the constant configuration is based at every coarse vertex. If  $x = v + r$  lies in the centered block of the coarse vertex  $v$ , the value of this gauge transformation there is  $\prod_j a_j^{r_j}$ . On a link inside a block, the commuting factors cancel to the identity. On an interface link in direction  $i$ , the offsets of the two endpoints differ by  $1 - M$  in the  $i$ th coordinate, and the transformed value is  $a_i^M = c_i$ . Hence the forest-fixed configuration is exactly  $U_0(c)$ , and all its relative logarithms, including those of the eliminated edges, vanish. Its free coordinate is  $z = 0$ , which is a zero of the nonnegative function  $F_d(c, \cdot)$  on the branch. The strict convexity and the uniqueness in Lemma 14.1 force  $z_d^*(c) = 0$ , and the normalization by the positive square root of  $K_d$  then gives  $w = 0$ . This proves the statement on minimizers.

**Step 8: all tests.** At every scale with side  $n_s \geq u$ , the constant link variables have logarithms of norm at most  $2\pi/n_s \leq 2\pi/u < r_{\text{flat}} < r$ . This proves the initial and coarse  $D_r$  tests, with strict inequality. The choice of  $r_{\text{flat}}$  below the radii of all fixed charts and gauge transformations of one RG step gives every intermediate classical, constraint-coordinate, and coarea condition. There are no further  $D_r$  tests at intermediate scales, and every product of fluctuation balls of positive radius contains the zero vector. These facts hold for every number of RG steps, since the smallest coarse side controls the same inequalities at each step. An arbitrary flat configuration is periodically gauge equivalent

to such a constant configuration, and the coordinate transitions on overlaps and the gauge invariance of the event (Lemma 14.5) carry the membership over to it. A change of representative based at the coarse vertices leaves the based cycle holonomies fixed and preserves the norms of the fluctuation variables and the event. A change at the common root only conjugates the triple simultaneously.

**Step 9: the Wilson flow.** We vary one link in a Lie algebra direction. At a flat configuration, each differentiated plaquette variable is the identity, and its derivative is tangent to  $SU(2)$  at the identity, hence traceless. Thus  $Dq(I) = 0$ , every summand of the gradient of the Wilson action vanishes, and the constant trajectory solves the Wilson flow equation. The Wilson flow is the flow of a smooth vector field on the compact manifold  $\mathcal{U}_N$ , so its solutions are unique. Indeed, locally, iterating the integral equation  $h$  times bounds the difference of two solutions with the same initial value by  $C^h t^h / h!$  times a fixed supremum, which tends to zero, and the argument can be repeated on successive time intervals. Hence flat configurations are stationary, and their cycle holonomies are unchanged at every time.

**Step 10: neighborhoods.** Assume the hypotheses of (2), and fix a finite  $N = uB^K$  and an allowed coupling. At each flat configuration, all classical and coarea inequalities hold strictly by Steps 5 to 8, the domains  $D_r$  for the fine and coarse fields are open, and every fluctuation vector is zero, inside a ball of positive radius. The maps defining the event and the inverse coordinates are continuous on their open branches. A finite intersection of their preimages therefore gives an open neighborhood of this configuration contained in the event, after replacing the fluctuation balls by smaller open balls around zero. The union of the gauge images of an open set is open, since each gauge transformation acts by a homeomorphism. Taking the union of these neighborhoods gives an open set  $\mathcal{O}$  with  $\mathcal{F}_N \subset \mathcal{O} \subset S_{N,u,\gamma}$ , where  $\mathcal{F}_N$  is the set of flat configurations.

The configuration space is compact, and  $f(U) = \max_p \rho_G(U_p)$  is continuous with zero set exactly  $\mathcal{F}_N$ . If  $\mathcal{O}^c$  is nonempty,  $f$  attains a strictly positive minimum on this compact set; otherwise any positive threshold works. Choosing  $\delta_{N,u,\gamma}$  below this minimum proves the inclusion in (3). The minimum is taken separately at each  $N$  and each coupling, and the argument gives no uniform lower bound for it.

On a fixed stopping lattice whose side satisfies the same condition, Step 8 shows that the compact set of flat configurations lies in  $D_r$ . The set  $D_r$  is open, being a union of gauge images of the open set  $B_r$ . For any fixed metric on the finite product, the compact set of flat configurations therefore has positive distance from the closed complement of  $D_r$ , unless this complement is empty. The resulting neighborhood depends only on  $n$  and  $r$ ; neither set involves the depth or the coupling. Throughout the argument all gauge transformations were periodic and all cycle holonomies commuted, so no projectively commuting or twisted data have been introduced.  $\square$

## Conditional kernels and reconstruction maps

In this section we prove that the renormalization group construction on the small-field event converges at every fixed resolution. That is, we fix the side  $n_0$  of the coarse lattice and a number  $l$  of RG steps counted from the coarse side, and we let the total number  $k$  of RG steps tend to infinity. At a fixed resolution all objects are finite dimensional. In Theorem 15.6, we show that one RG step contracts effective interactions in the weighted analytic coefficient norm, uniformly in the lattice side and in the preceding depth, and we deduce the convergence of the effective interactions and of the stopped measures at each fixed level. Theorem 15.9 proves the convergence of the conditional fluctuation kernels, and Lemma 15.10 writes the conditioned Wilson measure as an iteration of these kernels. In Theorem 15.13, we construct the reconstruction maps  $\hat{\mathcal{R}}$ , and in Theorem 15.14 we prove that the finite-scale reconstructions converge in law. Lemma 15.16 gives amplitude bounds for the reconstruction that are uniform in the mesh and in the coupling, and Theorem 15.17 makes the convergence uniform for the coupling on compact intervals  $I \subset (0, \infty)$  on which the hypotheses of Theorem 15.6 hold for every  $\gamma \in I$ . The inputs are the classical convergence and the bounds for the zero-fluctuation reconstruction of Theorem 11.2 and Lemma 11.9, and the density and the cluster expansion of one RG step (Theorems 12.7 and 12.13 and Corollary 12.8). We also use the gauge transformations and the invariant ball of Theorems 13.2 and 13.5, and the identification of the auxiliary integral with the conditioned Wilson measure (Theorems 14.3 and 14.6). The later sections approximate the flowed field by the field reconstructed from finitely many RG steps (Sections 3.3 and 3.6). Sections 16, 17, 26 to 29, and 31 to 36 use the reconstruction maps, the conditional kernels, the convergence of finite-scale reconstructions, and the amplitude bounds, and Section 23 uses the deterministic constants and the order in which they are chosen.

The splitting of the fine field into a constrained minimizer and normalized fluctuations, and the integration of the fluctuations scale by scale, are Bałaban's [15, 16, 18]. We combine the standard renormalization group contraction with the cluster expansion estimates of Section 12 to obtain the limits of the conditional kernels and a compatible family of representatives for this construction. These estimates go back to Kotecký and Preiss [120] and use the finite tree-graph bound in Dimock's exposition [68, Appendix B]. The selection of a holomorphic subsequence is the normal-family argument in finitely many variables, and the limits follow from the contraction in the usual way.

## 15.1 Scales, measures, and conventions

Throughout this section  $L > 0$  is fixed,  $M \geq 3$  is odd,  $J \geq 1$ , and  $B = M^J$ . One averaging step is one application of the block averaging map  $T_M$  of Definition 4.3, and one RG step consists of  $J$  averaging steps. We fix the side  $n_0 \in 4\mathbb{N}$  of the coarse lattice, let  $H = L/n_0$ , and for  $j \geq 0$  set

$$n_j = n_0 B^j, \quad a_j = H B^{-j}, \quad g_j^2 = \frac{\gamma}{n_j} = g_3^2 a_j.$$

We call  $\Lambda_{n_j}$  the lattice of level  $j$ . If the finest lattice has level  $k$ , the lattices form the chain

$$\Lambda_{n_0} \xleftarrow{T_M^J} \Lambda_{n_1} \xleftarrow{T_M^J} \cdots \xleftarrow{T_M^J} \Lambda_{n_k}.$$

The fluctuations are integrated from right to left, toward coarser lattices, and the fine field is reconstructed from left to right. The number of averaging steps between the finest lattice and level  $j$  is

$$d_j^k = J(k - j),$$

and we call it the preceding depth at level  $j$ . The RG step from level  $j$  to level  $j - 1$  has  $J$  intermediate scales, indexed by  $0 \leq i < J$ , with couplings

$$g_{j,i} = t_j M^{(i-J+1)/2}, \quad t_j = M^{(J-1)/2} g_j = g_{j-1} / \sqrt{M}.$$

Here  $t_j$  is the largest of these couplings, and the ratios  $g_{j,i}/t_j$  are fixed powers of  $M$ . With  $p(t) = A_w \log(1/t)$  we write  $p_j = p(t_j)$ . The normalized fluctuation variables  $w_j = (w_{j,i,e})$  of this RG step are restricted to the fluctuation cutoff domain  $\Omega_{p_j}$ , the product of the three-dimensional balls  $\{|w_{j,i,e}| \leq p_j\}$ ; it does not depend on  $k$ . We write  $q_{j,i} = g_{j,i} w_{j,i}$  for the coupling-scaled fluctuation inputs of the reconstruction. Since the inverse square roots of the constrained Hessians remain inside the reconstruction maps,  $q_{j,i}$  is not a free logarithmic coordinate of the fiber.

**The three measures.** Let

$$\nu^k = \nu_{n_0 B^k, n_0, \gamma}$$

be the conditioned Wilson measure of Definition 4.4, with the small-field event of (4.6) and Definition 14.4. In the comparison notation, the comparison index  $s$  gives  $n_s = n_* B^s$ ,  $H_s = L/n_s$ , and  $\nu_{K,s} = \nu_{N_K, n_s, \gamma}$  with  $N_K = n_* B^K$ . Then  $n_0 = n_s$  and  $k = K - s$ , the meshes of the reconstruction are  $a_j = H_s B^{-j}$ ,  $0 \leq j \leq K - s$ , and the preceding depths are  $J(K - s - j)$ . At level  $j$  three different measures occur.

1. The marginal of  $\nu^k$  at level  $j$ , that is, the law of  $V_j^{d_j^k}$  under  $\nu^k$ .
2. The stopped measure  $\hat{\nu}_j^k$  of Theorem 15.6. It is obtained by integrating out the finer levels and imposing the small-field restrictions only up to level  $j$ .

3. The conditional fluctuation kernel  $k_j^k(\theta, dw)$  of Theorem 15.9, the conditional law of the fluctuation variables of the RG step from level  $j$  to level  $j - 1$  given the field  $e^\theta$  at level  $j - 1$ .

The marginal in (1) also carries the restrictions at the coarser levels  $j - 1, \dots, 0$ , and in general it differs from the stopped measure. At the coarse level  $j = 0$  no coarser restriction remains, and (1) and (2) coincide (Lemma 15.12). The kernel (3) does not change when coarser restrictions are imposed, once the field at level  $j - 1$  has been fixed (Theorem 15.9).

**Effective interactions.** We write functions of a configuration near the identity in logarithmic coordinates,  $R(e^\zeta)$ , and use the norm

$$\mathcal{B}_{\rho, a_{\text{sp}}}(R) = \sum_{m \geq 2} \frac{\rho^{m-2}}{m!} \|D^m R(0)\|_{a_{\text{sp}}, \text{root}}$$

of Definition 4.6. The tensor norm fixes one index, sums the absolute values of the coefficients over the remaining indices with the weight  $e^{a_{\text{sp}} t}$ , where  $t$  is the length of a shortest connected network through the locations of the indices, and takes the supremum over the fixed index. For maps between lattices of different meshes the input-fixed sums carry the volume factor  $a_{\text{out}}^3 / a_{\text{in}}^3$  of Definition 4.6; we call them volume-normalized column sums. A function  $R$  on  $B_\rho$  is invariant under gauge transformations within  $B_\rho$  if  $R(U^h) = R(U)$  whenever  $U$  and  $U^h$  both lie in  $B_\rho$ , as in Definition 13.3; then  $R_{\text{sat}}(U) = R(U^h)$  for  $U^h \in B_\rho$  defines its gauge-invariant extension to  $D_\rho$ . For a lattice  $\Lambda$  and the parameters  $\rho, a_{\text{sp}}, E$  fixed in Theorem 15.6, let  $\mathcal{A}_E = \mathcal{A}_E(\Lambda)$  be the set of functions  $R$  given by an absolutely convergent Taylor series on  $B_\rho$  that are real on real configurations, invariant under lattice translations, the spatial inversion, global color rotations, and gauge transformations within  $B_\rho$ , and satisfy

$$R(I) = 0, \quad DR(I) = 0, \quad \mathcal{B}_{\rho, a_{\text{sp}}}(R) \leq E.$$

These are the symmetries of the recursion of Theorems 13.5 and 14.6. The norm  $\mathcal{B}_{\rho, a_{\text{sp}}}$  vanishes on constants.

**Constants.** A constant denoted  $C_J, c_J$ , or similar depends only on  $M, J$ , the fixed larger construction domains and their radius ratios, the spatial exponents, the factor  $K_*$ , and the norm ceilings. It is independent of the lattice side and of the preceding depth; this is the dependence described in item (2) of Definition 4.14. A subscript  $\Lambda$  allows, in addition, dependence on a fixed lattice pair. Such constants occur only in convergence moduli and never in the estimates that are used to choose  $J$ . We write  $\|\mu - \nu\|_{\text{TV}} = \sup_A |\mu(A) - \nu(A)|$  for the total variation distance.

## 15.2 Finite-dimensional analytic bounds and product domains

We begin with three elementary facts about total variation, which are used repeatedly below.

**Lemma 15.1.** 1. Let  $\mu$  be a  $\sigma$ -finite measure on a measurable space  $X$ , and let  $V, V'$  be real measurable functions on  $X$  with  $\sup_X |V - V'| \leq \delta$  and  $0 < \int e^{-V} d\mu < \infty$ . Let  $\pi$  and  $\pi'$  be the probability measures proportional to  $e^{-V}\mu$  and  $e^{-V'}\mu$ . Then the ratio of their normalizers lies in  $[e^{-\delta}, e^{\delta}]$ , the density  $d\pi'/d\pi$  takes values in  $[e^{-2\delta}, e^{2\delta}]$ , and

$$\|\pi - \pi'\|_{\text{TV}} \leq e^{2\delta} - 1.$$

2. Let  $\eta, \eta'$  be probability measures on a measurable space  $X_0$ , and for  $1 \leq j \leq l$  let  $K_j, K'_j$  be probability kernels from  $X_0 \times \cdots \times X_{j-1}$  to  $X_j$ . Then  $P = \eta K_1 \cdots K_l$  and  $P' = \eta' K'_1 \cdots K'_l$  satisfy

$$\|P - P'\|_{\text{TV}} \leq \|\eta - \eta'\|_{\text{TV}} + \sum_{j=1}^l \sup_h \|K_j(h, \cdot) - K'_j(h, \cdot)\|_{\text{TV}}.$$

3. Let  $P, P'$  be probability measures on a measurable space  $X$ , and let  $F, F'$  be measurable maps from  $X$  into a set  $\mathcal{K}$  of diameter at most  $D_*$  in a finite-dimensional normed space. Put  $\|F - F'\|_{\infty} = \sup_X \|F - F'\|$ . For every bounded real function  $f$  that is uniformly continuous on  $\mathcal{K}$ , with modulus of continuity  $\omega_f$ ,

$$\left| \int f(F) dP - \int f(F') dP' \right| \leq 2\|f\|_{\infty} \|P - P'\|_{\text{TV}} + \omega_f(\|F - F'\|_{\infty}),$$

and

$$W_1(F_*P, F'_*P') \leq \|F - F'\|_{\infty} + D_* \|P - P'\|_{\text{TV}}.$$

*Proof.* For (1), let  $Z, Z'$  be the two normalizers. Since  $e^{-\delta}e^{-V} \leq e^{-V'} \leq e^{\delta}e^{-V}$ , we have  $e^{-\delta}Z \leq Z' \leq e^{\delta}Z$ . The density  $d\pi'/d\pi = e^{V-V'}Z/Z'$  therefore lies in  $[e^{-2\delta}, e^{2\delta}]$ . For an event  $A$  we have  $|\pi(A) - \pi'(A)| \leq \int |1 - d\pi'/d\pi| d\pi \leq e^{2\delta} - 1$ , where we used  $1 - e^{-2\delta} \leq e^{2\delta} - 1$ .

For (2), first observe that a measurable function  $u$  with values in  $[0, 1]$  satisfies  $|\int u d\mu - \int u d\mu'| \leq \|\mu - \mu'\|_{\text{TV}}$  for probability measures  $\mu, \mu'$ ; indeed  $u = \int_0^1 \mathbf{1}_{\{u > s\}} ds$ . We replace first  $\eta$  by  $\eta'$ , and then the kernels one at a time. Fix an event  $A$  in the product space. Since the function obtained by integrating  $\mathbf{1}_A$  against  $K_1 \cdots K_l$  takes values in  $[0, 1]$ , the first replacement changes the probability of this event by at most  $\|\eta - \eta'\|_{\text{TV}}$ . In the replacement of  $K_j$  by  $K'_j$ , the integral of  $\mathbf{1}_A$  against the later kernels is again a  $[0, 1]$ -valued function of  $(h, x_j)$ , and the change is at most  $\sup_h \|K_j(h, \cdot) - K'_j(h, \cdot)\|_{\text{TV}}$ . Summing these changes proves (2).

For (3), we have

$$\left| \int f(F) dP - \int f(F') dP' \right| \leq \int |f(F) - f(F')| dP + \left| \int f(F') dP - \int f(F') dP' \right|.$$

The first term is at most  $\omega_f(\|F - F'\|_\infty)$ . For the second, the function  $(f(F') + \|f\|_\infty)/(2\|f\|_\infty)$  takes values in  $[0, 1]$ , and the observation in the proof of (2) bounds it by  $2\|f\|_\infty\|P - P'\|_{\text{TV}}$ . For the Wasserstein bound, let  $\lambda = P + P'$ , let  $u, u'$  be the densities of  $P, P'$  with respect to  $\lambda$ , and set  $m = \min(u, u')$  and  $\vartheta_0 = \int m d\lambda = 1 - \|P - P'\|_{\text{TV}}$ . If  $\vartheta_0 = 1$  then  $P = P'$  and we couple  $x$  with itself. Otherwise let  $\Gamma$  be the sum of the image of  $m\lambda$  under  $x \mapsto (x, x)$  and of the product measure  $(1 - \vartheta_0)^{-1}(u - m)\lambda \otimes (u' - m)\lambda$ . Its marginals are  $P$  and  $P'$ . The cost of the pushforward summand is  $\int \|F(x) - F'(x)\| m(x) d\lambda(x)$ , at most  $\vartheta_0\|F - F'\|_\infty$ . The product summand has mass  $1 - \vartheta_0$  and cost at most  $(1 - \vartheta_0)D_*$ . Consequently

$$\int \|F(x) - F'(x')\| d\Gamma(x, x') \leq \|F - F'\|_\infty + (1 - \vartheta_0)D_*,$$

and  $(F \times F')_*\Gamma$  couples  $F_*P$  and  $F'_*P'$ . This completes the proof.  $\square$

In particular, for the bounded-Lipschitz distance, whose test functions are bounded by one and have Lipschitz constant at most one, part (3) gives the bound  $2\|P - P'\|_{\text{TV}} + \|F - F'\|_\infty$ .

**Box supports.** The cluster expansion of Section 12 assigns interactions to connected sets of boxes. We partition each periodic coordinate direction of the coarse lattice into intervals of integer length between  $\mathcal{L}$  and  $2\mathcal{L}$ , or use one interval when the direction is shorter than  $\mathcal{L}$ , as in Definition 12.1. The products of these intervals are the boxes. Each scalar variable has a fixed location in the torus, as in Definition 4.6; the coarse field has nine scalar components (three orientations and three colors) at each vertex. The first lemma bounds the length of a network through the locations of the coarse-field variables in a connected set of boxes.

**Lemma 15.2.** *Fix an integer  $\mathcal{L} \geq 1$  in units of the coarse lattice. There exists a constant  $C_{\text{box}}(\mathcal{L})$ , depending only on  $\mathcal{L}$  and on the fixed metric and location conventions, such that every nonempty connected set  $Y$  of boxes satisfies*

$$\mathfrak{t}(\text{locations of the coarse-field variables in } Y) \leq C_{\text{box}}(\mathcal{L})|Y|. \quad (15.1)$$

*The constant is independent of the side of the torus, the preceding depth,  $J$ , the couplings, the degree of a coefficient, and the density of the fluctuation variables. The bound also holds for every subset or repeated list of these locations, and for every union of connected box sets whose intersection graph is connected.*

*Proof.* Set  $q = 2\mathcal{L} + 2$ , and let  $c_{\text{met}} \geq 1$  bound the length of a coordinate segment of unit length and the distance from the location of a link variable to an endpoint of the link. A lifted box, together with a collar of one cell for the locations of its link variables, lies in an integer cube of side at most  $q$ . The coordinate grid of this cube has at most  $3q(q + 1)^2$

edges. There are at most  $9(q+1)^3$  locations of scalar variables in the cube, and we attach each of them to a grid point by a segment of length at most  $c_{\text{met}}$ . This gives a connected network of length at most  $12c_{\text{met}}(q+1)^3$  containing every such location in the box.

Fix a grid vertex in each box. A traversal of the connected box graph selects  $|Y| - 1$  adjacencies that connect all boxes. For each selected adjacency we join the two chosen grid vertices by a periodic coordinate path. Since the boxes are adjacent, this path has length at most  $c_{\text{met}}(6q+3) \leq 9c_{\text{met}}q$ . These paths and the grids form one connected network containing all these locations in  $Y$ , so we may take

$$C_{\text{box}}(\mathcal{L}) = c_{\text{met}}\{12(q+1)^3 + 9q\}.$$

In a periodic direction shorter than  $\mathcal{L}$  we project the grid of the cube to the torus, which can only decrease its length; for an adjacency across the periodic boundary we use a shortest periodic coordinate path, which obeys the same bound. No lower bound on the side in terms of  $\mathcal{L}$  was used. Repeated labels occupy the same location, and restricting the list of locations requires no new segment. Finally, a union of connected box sets whose intersection graph is connected is itself a connected box set, so the bound applies to it with its own cardinality.  $\square$

The next lemma makes the support assignment of Lemma 12.9 explicit. It assigns a connected box set to a coefficient and bounds the resulting support-counting factor by the length of the network of the coefficient.

**Lemma 15.3.** *Fix deterministic choices of shortest paths. Let  $N \geq 1$  points be given, together with a connected network of length  $\ell$  containing them. Join the points by a minimal spanning tree, replace each edge of the tree by its three coordinate segments along shortest periodic displacements, and let  $Y$  be the set of boxes met by the resulting network, including the boxes of the points. Then  $Y$  is a nonempty connected box set. There exist constants  $c_0, C_{\text{geom}} \geq 1$  depending only on the metric conventions, and a constant  $C_{\mathcal{L}}$  depending only on  $\mathcal{L}$  and the metric conventions, such that*

$$|Y| \leq c_0 \left( N + 1 + \frac{C_{\text{geom}}\ell}{\mathcal{L}} \right), \quad \text{dist}(x, u) \leq C_{\text{geom}}\ell + C_{\mathcal{L}} \quad (15.2)$$

for every box  $x \in Y$  and every given point  $u$ , where the distance is measured from a fixed point of the box. Consequently, for  $h, \mu > 0$  and

$$K \geq e^{hc_0}, \quad \beta \geq C_{\text{geom}} \left( \frac{hc_0}{\mathcal{L}} + \mu \right), \quad C_h = e^{hc_0 + \mu C_{\mathcal{L}}},$$

we have

$$\mathbf{1}_{\{x \in Y\}} e^{h|Y|} \leq C_h K^N e^{\beta\ell} e^{-\mu \text{dist}(x, u)}. \quad (15.3)$$

*Proof.* A minimal spanning tree on the  $N$  points has at most  $N - 1$  edges, and its length is at most twice the length of any connected network containing the points; indeed,

traversing such a network twice and shortcutting between successive points gives a path through all of them. The comparison between coordinate segments and the metric, together with this factor two, is included in  $C_{\text{geom}}$ . A coordinate segment of length  $l'$  meets at most a fixed multiple of  $2 + l'/\mathcal{L}$  boxes. Summing over the at most  $3(N-1)$  segments gives the first bound in (15.2). The second follows by moving along the network from  $u$  to the box  $x$  and then inside the box. Inserting (15.2) into  $e^{h|Y|}$  and using  $\mu \text{dist}(x, u) \leq \mu C_{\text{geom}}\ell + \mu C_{\mathcal{L}}$ , we obtain (15.3).  $\square$

**Order of the spatial choices.** We now fix the spatial parameters, in an order that keeps them independent of  $J$ . Fix a decay rate  $b > 0$  strictly below the common spatial rate of the classical and image bounds of Lemma 11.9 and Theorems 13.2 and 13.5. We choose  $\mathcal{L}$  using this rate alone, so that  $8c_0 C_{\text{geom}}/\mathcal{L} \leq b/4$ . We fix  $\mu > 0$  with  $C_{\text{geom}}\mu \leq b/4$ ; this sets aside a part of  $b$  for summing over the distance between a box and the location of a variable. With  $h = 8$  and  $K = e^{8c_0}$  in Lemma 15.3, the factor  $K^N$  is absorbed by multiplying every input radius by  $K$ ; we take the factor  $K_* \geq 2$  of Theorem 13.5 so large that  $K < 4K_*$ , so that these enlarged radii stay inside the products of Lemma 15.4 below. Next we choose the spatial exponent  $a_{\text{sp}}$  of the effective interactions so that

$$\begin{aligned} 0 < a_{\text{sp}} < \min\{b/4, a_{\text{cen}}/8\}, \quad a_{\text{sp}} C_{\text{box}}(\mathcal{L}) \leq 2, \\ C_P(a_{\text{sp}})/B < 1/4, \quad B^{-1} \leq \delta_0(a_{\text{sp}}), \quad C_{\text{geom}}b/B \leq a_{\text{sp}}, \end{aligned} \quad (15.4)$$

where  $C_P$  is the constant of Theorem 11.10. The condition  $a_{\text{sp}} < a_{\text{cen}}/8$ , where  $a_{\text{cen}}$  is the rate of Lemma 11.9, is the restriction imposed in Theorem 13.5. Indeed, the zero-fluctuation reconstruction used below is the representative of Lemma 11.9, and Theorem 11.10 gives the contraction for this representative only at output rates below  $a_{\text{cen}}/8$ , with the constant  $C_P(a_{\text{sp}})$  at the rate  $a_{\text{sp}}$ . The first two conditions determine  $a_{\text{sp}}$  after  $\mathcal{L}$ . Only then do we choose  $J$ , so that the last three conditions and the condition  $M^{-J}Z(4K_*\rho) \leq \rho/4$  of Theorem 13.5 hold. All constants in these conditions are fixed before  $J$ , and  $M^{-J} \rightarrow 0$ , so they hold for all large  $J$ . The scale  $\mathcal{L}$  is not chosen again after  $a_{\text{sp}}$  or  $J$ , and no exponent is decreased during the iteration. In the common parameter choice of Theorem 23.2, the torus exponent denoted  $a_{\text{sp}}$  in this section is  $a_{\text{T}}$ , distinct from the smaller regional boundary exponent chosen after  $J$ . By (15.1), these choices give

$$a_{\text{sp}} t \leq 2|Y|$$

for the locations of the coarse-field variables in every connected box set  $Y$ , including subsets and repetitions of these locations. This inequality does not involve a radius.

**Nested analytic product domains.** The next lemma fixes the complex products on which the reconstruction is controlled.

**Lemma 15.4.** *Let  $C_{\text{im},J}$  and  $r_J$  be the constants of Theorem 13.2, and let*

$$\mathfrak{M}_J(u, v) = B^{-1}Z(u) + \frac{C_{\text{im},J}v}{1 - C_{\text{im},J}(u + v)}$$

*be its row majorant. Before the radius  $r$  of the small-field domain and the coupling ceiling are chosen, the common product of coarse-field and fluctuation-input variables can be fixed with radii  $u_\circ = 4K_*\rho$  and  $v_\circ = 4K_*s_J$ , such that*

$$\begin{aligned} u_\circ + v_\circ &< r_J, & C_{\text{im},J}(u_\circ + v_\circ) &\leq 1 - \eta_{\text{im}} < 1, \\ B^{-1}Z(u_\circ) &\leq \rho/4, & \mathfrak{M}_J(u_\circ, v_\circ) &< \rho/2 \end{aligned} \quad (15.5)$$

*for a fixed  $\eta_{\text{im}} > 0$ , and every intermediate classical, logarithm, and Haar disintegration domain condition holds with a margin. Every smaller product lies in this domain and has denominator at least  $\eta_{\text{im}}$ . For each  $0 < r \leq \rho$  there exists  $Q_{\text{act}} > 0$ , depending on  $r$ , at most  $s_J$ , and at most any previously chosen positive bound on the inputs  $q$ , such that*

$$r + Q_{\text{act}} < r_J, \quad 1 - C_{\text{im},J}(r + Q_{\text{act}}) > 0, \quad \mathfrak{M}_J(r, Q_{\text{act}}) < r/2. \quad (15.6)$$

*After the coupling ceiling is reduced so that  $tp(t) \leq Q_{\text{act}}$ , these conditions hold on every product  $\{\|\theta\|_\infty < r\} \times \{\max_i \|g_i w_i\|_\infty \leq tp(t)\}$  used in the construction. All these choices are deterministic and independent of the coupling, the lattice side, and the preceding depth.*

*Proof.* The series  $Z$  of Lemma 11.9 is chosen before  $J$ , and the choice of  $J$  in Theorem 13.5 gives  $B^{-1}Z(4K_*\rho) \leq \rho/4$ . After  $J$  has been fixed, we take  $\rho$  strictly below  $r_J/(8K_*)$  and so small that  $4K_*C_{\text{im},J}\rho < 1/4$ , within the common classical and Haar disintegration domains. Then we choose  $s_J > 0$  so small that  $u_\circ + v_\circ < r_J$ , that  $C_{\text{im},J}(u_\circ + v_\circ) < 1/2$ , and that the second term of  $\mathfrak{M}_J(u_\circ, v_\circ)$  is strictly below  $\rho/4$ . This reduction keeps the intermediate margins of Theorem 13.2. No radius is changed during a recursion. We denote the positive margin of the denominator by  $\eta_{\text{im}}$ . Since the coefficients of all majorants are nonnegative, the domain and denominator conditions remain valid on smaller products.

The ratio  $Z(u)/u$  is nondecreasing, because the series of  $Z$  has nonnegative coefficients and no constant term. Hence  $B^{-1}Z(r) \leq r/(16K_*)$  for  $r \leq \rho$ . We take  $Q_{\text{act}}$  strictly below

$$\min\left\{s_J, \frac{r_J - r}{2}, \frac{1 - C_{\text{im},J}r}{2C_{\text{im},J}}, \frac{r(1 - C_{\text{im},J}r)}{16C_{\text{im},J}}\right\},$$

and below any previously chosen bound on the inputs; every entry of the minimum is positive. The denominator is then at least  $(1 - C_{\text{im},J}r)/2$ , and the second term of  $\mathfrak{M}_J(r, Q_{\text{act}})$  is less than  $r/8$ . Thus  $\mathfrak{M}_J(r, Q_{\text{act}}) < r/(16K_*) + r/8 < r/2$ . Since the product lies in the outer product, it has all intermediate margins. Finally,  $tp(t) \rightarrow 0$  as  $t \downarrow 0$ , so one ceiling works on the whole interval of smaller couplings.  $\square$

**A Taylor tail estimate.** The last lemma of this section compares values, derivatives, and coefficient norms at a fixed lattice size.

**Lemma 15.5.** *On a fixed finite lattice  $\Lambda$  with edge set  $\mathcal{E}_\Lambda$ , let  $D_\Lambda = 3|\mathcal{E}_\Lambda|$ . Every function  $F$  given by a Taylor series in logarithmic coordinates with  $F(I) = 0$  and  $DF(I) = 0$  satisfies*

$$\sup_{\|\xi\|_\infty \leq \rho} |F(e^\xi)| \leq D_\Lambda \rho^2 \mathcal{B}_{\rho, a_{\text{sp}}}(F).$$

The same bound holds on  $D_\rho$  if  $F$  is invariant under gauge transformations within  $B_\rho$ . Values on a closed coordinate ball mean the continuous extension of the absolutely convergent Taylor series. For  $0 < \sigma < 1$  and  $s \geq 1$ , the sum over all indices of the absolute values of the coefficients of  $D^s F$  at points of  $B_{\sigma\rho}$  is at most

$$s! D_\Lambda \rho^{2-s} (1 - \sigma)^{-s} \mathcal{B}_{\rho, a_{\text{sp}}}(F).$$

Suppose a family  $F_\alpha$  has a common bound on  $\mathcal{B}_{\rho_1, a_{\text{sp}}}$  for some  $\rho_1 > \rho$ , with the same spatial exponent. If each fixed Taylor coefficient tensor of  $F_\alpha$  is Cauchy, uniformly in an additional parameter, then  $F_\alpha$  is Cauchy in  $\mathcal{B}_{\rho, a_{\text{sp}}}$ , uniformly in that parameter. This also holds for vector-valued maps and for joint products of coarse-field and fluctuation-input variables, including every volume-normalized column sum and every fixed spatial difference, provided the inner and outer radii of each group of variables have a common ratio less than one.

*Proof.* We write  $F(e^\xi) = \sum_{s \geq 2} D^s F(0) [\xi^{\otimes s}] / s!$ . Summing first over one scalar coordinate and then over the remaining ones gives at most  $D_\Lambda$  times the supremum of the sums with one fixed index. Dropping the spatial weights, which are at least one, and using  $|\xi_e| \leq \rho$ , we obtain the value bound. Differentiation produces the falling factorial  $m(m-1) \cdots (m-s+1)$  in degree  $m$ . Its product with  $\sigma^{m-s}$  is at most  $s!(1-\sigma)^{-s}$ , because  $\binom{m}{s} (1-\sigma)^s \sigma^{m-s} \leq 1$  is one term of the binomial expansion of  $((1-\sigma) + \sigma)^m = 1$ . Summing against the coefficient norms gives the derivative bound. Gauge invariance within  $B_\rho$  changes none of these values, since a single gauge transformation is used at the point of evaluation.

Let  $u_m = \|D^m F(0)\|_{a_{\text{sp}}, \text{root}} / m!$ . If  $\mathcal{B}_{\rho_1, a_{\text{sp}}}(F) \leq C$ , then

$$\sum_{m > h} \rho^{m-2} u_m \leq C(\rho / \rho_1)^{h-1}. \quad (15.7)$$

For a difference of two members of the family we use twice the common bound. For fixed  $h$  there are finitely many coefficients, indices, and labels, and their weighted sums converge. We choose  $h$  first in (15.7) and then the indices of the family. This proves the Cauchy property without decreasing  $\rho$  or  $a_{\text{sp}}$ .

For a joint vector series we group the monomials by total degree. If the ratio of each inner radius to the corresponding outer radius is at most  $\zeta < 1$ , the part of degree greater than  $h$  has norm at most  $C\zeta^{h+1}$ . A fixed input contributes a degree factor, which is

absorbed by using an intermediate ratio between  $\zeta$  and one. At a fixed lattice size the volume factors, the spatial weights, and the difference matrices are fixed finite numbers, so the argument applies to all these sums and differences. It also shows that locally uniform holomorphic convergence, together with a common absolute bound on a larger product, gives convergence in these norms.  $\square$

Here and below the Cauchy formula in several variables means the iterated one-variable formula on circles inside the polydisc. If a function is bounded by  $C$  on these circles, the formula bounds the coefficient of  $\zeta^\alpha$  by  $Cr^{-\alpha}$ . In coordinate dimension  $D$  there are  $\binom{m+D-1}{D-1}$  multi-indices of degree  $m$ , a polynomial in  $m$ . On every strictly smaller polydisc, the sums of the absolute values of the coefficients are then finite. This justifies the finite-dimensional uses of locally uniform holomorphic convergence below.

### 15.3 Contraction and limits at fixed resolution

**The auxiliary transformation.** Fix a lattice pair of one RG step: a coarse lattice of side  $m \geq 4$  and the fine lattice of side  $Bm$ , with  $d \geq 0$  preceding averaging steps. Fix  $0 < t \leq g_*$ , and let  $g_i = t\lambda_i$  with  $\lambda_i = M^{(i-J+1)/2}$  for  $0 \leq i < J$ . We write  $c = e^\theta$  for the coarse field and  $q_i = g_i w_i$  for the inputs. Let  $\mathcal{C}_{\text{rawd}}(\theta, q)$  be the reconstruction of the fine field in the block axial gauge, built from the conditional constrained minimizers, the constrained Hessians, and their inverse square roots, as in Lemma 12.3 and Theorem 12.13. Let  $\Omega_d(\theta)$  be the gauge transformation of Theorem 13.2; it depends only on  $\theta$  and equals the identity at the vertices of the coarse lattice. Define

$$\phi_d(\theta, q) = \text{Log}[\mathcal{C}_{\text{rawd}}(\theta, q)^{\Omega_d(\theta)}], \quad P_d R(\theta) = R(e^{\phi_d(\theta, 0)}).$$

Thus  $P_d R$  is the pullback of  $R$  by the zero-fluctuation reconstruction. Let  $H_{d,J}(\theta, w)$  be the fluctuation exponent of Theorem 12.13. It contains the classical remainder, the nonconstant parts of the Haar disintegration densities, and the differences of the determinant terms; the determinant of a finer intermediate scale is evaluated at the argument determined by the coarser fluctuation variables. Let  $\pi_p$  be the normalized product of standard Gaussian measures on  $\mathbb{R}^3$  restricted to balls of radius  $p = p(t)$ , one factor for each three-dimensional block of fluctuation variables. Let

$$z_p = \int_{\{v \in \mathbb{R}^3: |v| \leq p\}} e^{-|v|^2/2} dv,$$

let  $f_i$  be the number of free group variables at the intermediate scale  $i$ , so that the cutoff domain is a product of  $\sum_i f_i$  balls, and let  $B_{\text{ref}}(\theta)$  be the zero-fluctuation Jacobian term of

Theorem 12.13. For an input effective interaction  $R$ , produced by the integrations at finer scales, we set

$$\begin{aligned} A_R(\theta, w) &= R(e^{\phi_d(\theta, (g_i w_i))}) - P_d R(\theta), \\ \Xi_R(\theta) &= \int e^{-H_{d,J}(\theta, w) - A_R(\theta, w)} d\pi_p(w), \\ D_d R &= -\log \Xi_R + \log \Xi_0, \\ Q_d &= B_{\text{ref}} - \left( \sum_i f_i \right) \log z_p - \log \Xi_0. \end{aligned} \tag{15.8}$$

By Theorems 12.13 and 14.6,  $Q_d + P_d R + D_d R$  is the effective interaction produced by one RG step from  $R$ , before normalization at the identity. The three terms are the newly generated effective interaction, the pullback at zero fluctuation, and the fluctuation correction to the pullback. In the gauge of  $\phi_d$ , they are the terms  $Q_{d,J}^w$ ,  $P_\delta R$ , and  $D_J R$  of (13.9), with  $d_0 = d$ , up to field-independent constants. We define the auxiliary transformation

$$\widehat{\mathcal{M}}_{d,t} R = (Q_d + P_d R + D_d R) - (Q_d + P_d R + D_d R)(I).$$

The field-independent factors  $\prod_i \mathcal{J}_i(I, 0) g_i^{3f_i}$ , the Haar disintegration density at the identity, and the powers of the couplings, are kept in the normalization constant of Theorem 14.6, together with the subtracted value at the identity; they are never estimated by  $\mathcal{B}_{\rho, a_{\text{sp}}}$ . The integral  $\Xi_R$  carries no indicator of  $D_r$  on the fine lattice, and it is defined for complex  $\theta$  in the larger analytic product domain of Lemma 15.4. At a coarse field in  $B_r$ , the image condition of the next theorem implies that this indicator would equal one on the whole cutoff domain. There  $\Xi_R$  is the normalized Gaussian fluctuation factor in the fiber formula of Theorem 14.6(2). Multiplying it by the classical, zero-fluctuation, Gaussian, and coordinate prefactors in that formula gives the density of the pushforward under  $T_M^J$ , before global normalization (Theorem 14.6).

**Theorem 15.6.** Fix odd  $M \geq 3$ ,  $J \geq 1$ ,  $B = M^J$ , and the parameters of Theorems 13.5 and 14.6: the radii  $0 < r < \rho$ , the spatial exponent  $a_{\text{sp}} > 0$ , the ceiling  $E$  for effective interactions, the coefficient  $A_w > 0$ , and the coupling ceiling  $g_* > 0$ , with  $p(t) = A_w \log(1/t) > 0$  for  $0 < t \leq g_*$ . Fix a lattice pair of one RG step with coarse side  $m \geq 4$ , a coupling  $0 < t \leq g_*$ , and the couplings  $g_i = t\lambda_i$  of its intermediate scales. Assume that the image under  $\phi_d$  of the complex product of Lemma 15.4, with radii  $4K_*\rho$  and  $4K_*s_J$ , lies strictly inside  $B_{\rho/2}$ ; that for  $\|\theta\|_\infty < r$  the real product  $\{\max_i \|q_i\|_\infty \leq tp(t)\}$  is mapped into  $B_{r/2}$ ; and that all intermediate classical, logarithm, and Haar disintegration domain conditions hold. Then, after one initial reduction of  $g_*$ , all  $R_1, R_2 \in \mathcal{A}_E$  satisfy

$$\begin{aligned} \mathcal{B}_{\rho, a_{\text{sp}}}(\widehat{\mathcal{M}}_{d,t} R_1 - \widehat{\mathcal{M}}_{d,t} R_2) &\leq \{q_{\text{pr}} + C_J t^2 [p(t)^4 + (\epsilon_1 + \epsilon_2) p(t)^2]\} \mathcal{B}_{\rho, a_{\text{sp}}}(R_1 - R_2) \\ &\leq \vartheta \mathcal{B}_{\rho, a_{\text{sp}}}(R_1 - R_2), \end{aligned}$$

where  $\epsilon_i = \mathcal{B}_{\rho, a_{\text{sp}}}(R_i)$ ,  $0 \leq q_{\text{pr}} < 1/4$ , and  $\vartheta = (1 + q_{\text{pr}})/2 < 1$ . The reduction of  $g_*$  imposes  $p(t) \geq 1$ , by taking  $g_* \leq e^{-1/A_w}$ , and

$$t[p(t)^3 + 2Ep(t)] \leq C_J/2, \quad C_J t^2[p(t)^4 + 2Ep(t)^2] \leq (1 - q_{\text{pr}})/2$$

for all  $0 < t \leq g_*$ . At this fixed lattice pair and fixed  $t$ ,

$$\sup_{R \in \mathcal{A}_E} \mathcal{B}_{\rho, a_{\text{sp}}}(\widehat{\mathcal{M}}_{d,t} R - \widehat{\mathcal{M}}_{d',t} R) \longrightarrow 0 \quad (d, d' \rightarrow \infty),$$

at the same radius and exponent. The conditional constrained minimizers, the constrained Hessians, the analytic branches of their square roots and inverse square roots that are positive at real arguments, the Haar disintegration densities, and the reconstructions  $\mathcal{C}_{\text{raw}d}$  converge locally uniformly and holomorphically on smaller product domains, with every fixed derivative.

Now fix  $n_0 \in 4\mathbb{N}$  and  $\gamma > 0$  for which the construction of Theorem 14.6 applies, with  $t_1^2 = \gamma/(Mn_0) \leq g_*^2$  for the reduced ceiling above, and use the hierarchy of Section 15.1. For  $0 \leq j \leq k$  let  $R_j^k$  be the effective interaction at level  $j$  given by Theorem 14.6; thus  $R_k^k = 0$ ,  $R_{h-1}^k = \widehat{\mathcal{M}}_{J(k-h), t_h} R_h^k$  for  $1 \leq h \leq k$ , and  $R_h^k \in \mathcal{A}_E(\Lambda_{n_h})$ . The stopped measure at level  $j$  is

$$d\widehat{v}_j^k = \widehat{Z}_{j,k}^{-1} \mathbf{1}_{D_r} \exp\{-E_{J(k-j), \text{sat}}/g_j^2 - R_{j, \text{sat}}^k\} dH_{n_j}, \quad R_j^k(I) = 0.$$

For every fixed  $j$  there exists  $R_j^\infty \in \mathcal{A}_E(\Lambda_{n_j})$  such that

$$\mathcal{B}_{\rho, a_{\text{sp}}}(R_j^k - R_j^\infty) \longrightarrow 0, \quad \|\widehat{v}_j^k - \widehat{v}_j^\infty\|_{\text{TV}} \longrightarrow 0 \quad (k \rightarrow \infty),$$

where  $\widehat{v}_j^\infty$  has the density of the same form with  $E_{\infty, \text{sat}}$  and  $R_{j, \text{sat}}^\infty$ , the same indicator  $\mathbf{1}_{D_r}$ , and a positive finite normalizer. The contraction constants have the dependence described in item (2) of Definition 4.14; they are uniform in the side and in the preceding depth for fixed analytic parameters. The continuity moduli and the convergence thresholds may in addition depend on the fixed lattices and on  $j$ ,  $t$ ,  $p(t)$ ,  $L$ , and  $\gamma$ . No rate that is uniform in the resolution, in the number of RG steps, or as  $\gamma \rightarrow 0$  is asserted.

Here  $E_\infty$  is the limit of the effective classical actions  $E_d$  at the fixed lattice (Theorem 11.2), and  $E_{\infty, \text{sat}}$  is its gauge-invariant extension.

*Remark 15.7.* The second-order gain in the coupling has the following heuristic origin. Suppose that a reconstruction has the expansion  $F(\theta, tw) = F(\theta, 0) + tL_\theta w + O(t^2)$ . For a difference  $\Delta$  of two effective interactions we then have

$$\Delta(F(\theta, tw)) - \Delta(F(\theta, 0)) = tD\Delta(F(\theta, 0))[L_\theta w] + O(t^2).$$

Since the first term is linear in  $w$ , it has mean zero under the reference measure  $\pi_p$ , which is invariant under  $w \mapsto -w$ . The derivative of the negative logarithm of the fluctuation

integral in the direction  $\Delta$  is the expectation of this difference under the interacting conditional measure. Its leading term is the reference expectation, which vanishes, and the interaction contributes another factor of  $t$ . Lemma 15.8 and the cluster expansion of Theorem 12.7 make this argument uniform in the volume and in the degree of the coefficients, in the norm  $\mathcal{B}_{\rho, a_{\text{sp}}}$ . The cancellation is a property of the reference measure on each support; it does not use any symmetry of the interacting conditional measure.

**Local interaction decompositions.** For a family  $\mathbf{A} = (A_Y)$  indexed by nonempty connected box sets  $Y$ , where  $A_Y(\theta, w) = \sum_v A_{Y,v}(w)\theta^v$  depends only on the coarse-field and fluctuation variables located in  $Y$ , we use the norm

$$\|\mathbf{A}\|_{\mathcal{S}_h} = \sup_x \sum_{Y \ni x} e^{h|Y|} \sum_v \rho^{|v|} \sup_{w \in \Omega_p} |A_{Y,v}(w)| \quad (15.9)$$

of Theorem 12.7, with coarse-field radius  $\rho$  and the cutoff radius  $p \geq 1$  of the RG step; the supremum is over boxes  $x$ . The next lemma gives an explicit decomposition of the fluctuation interactions into such families, with a bound on the difference family that is linear in the norm of the difference.

**Lemma 15.8.** *Assume the hypotheses of Theorem 15.6 at a fixed lattice pair and preceding depth  $d$ , with  $p = p(t) \geq 1$ . Let  $R_1, R_2 \in \mathcal{A}_E$ ,  $\Delta = R_1 - R_2$ ,  $\delta = \mathcal{B}_{\rho, a_{\text{sp}}}(\Delta)$ ,  $\epsilon_i = \mathcal{B}_{\rho, a_{\text{sp}}}(R_i)$ , and  $R_s = R_2 + s\Delta$  for  $0 \leq s \leq 1$ . For  $z \in \mathbb{C}$  replace the couplings  $g_i$  by  $z\lambda_i$ , both in the inputs  $q_i = z\lambda_i w_i$  of  $\phi_d$  and in the classical remainder of  $H_{d,J}$ , as in Definition 12.15, and keep fixed the cutoff radius  $p$  and all parameters on which the input effective interaction depends. There exists  $C_J$  such that, for  $C_J|z|p < 1$ , there exist families  $\mathbf{H}(z)$ ,  $\mathbf{A}_s(z)$ , and  $\mathbf{A}_\Delta(z)$ , holomorphic in  $z$  with values in  $\mathcal{S}_8$ , with the following properties.*

1. *Their sums over  $Y$  are, respectively, the exponent  $H_{d,J}$  with  $g_i = z\lambda_i$ , the function  $R_s(e^{\phi_d(\theta, (z\lambda_i w_i))}) - R_s(e^{\phi_d(\theta, 0)})$ , and the same function with  $R_s$  replaced by  $\Delta$ .*
2.  *$\mathbf{A}_s = \mathbf{A}_0 + s\mathbf{A}_\Delta$ , and  $\mathbf{H}(0) = \mathbf{A}_s(0) = \mathbf{A}_\Delta(0) = 0$ .*
3. *On each  $Y$  the coefficient of  $z$  in  $\mathbf{A}_{\Delta,Y}(z)$  is a linear function of the fluctuation variables.*
4. *They satisfy*

$$\begin{aligned} \|\mathbf{H}(z)\|_{\mathcal{S}_8} &\leq \frac{C_J|z|(p^3 + p)}{1 - C_J|z|p}, \\ \|\mathbf{A}_s(z)\|_{\mathcal{S}_8} &\leq \frac{C_J|z|(\epsilon_1 + \epsilon_2)p}{1 - C_J|z|p}, \\ \|\mathbf{A}_\Delta(z)\|_{\mathcal{S}_8} &\leq \frac{C_J|z|\delta p}{1 - C_J|z|p}. \end{aligned} \quad (15.10)$$

*Supports that wrap around the torus are allowed.*

*Proof.* The family  $\mathbf{H}$  is the local interaction decomposition of the auxiliary fluctuation interaction of Definition 12.15 at the parameter  $\zeta = z$ ; it is built from the decompositions of Lemma 12.9, whose item (4) allows a complex parameter. We use its supported-norm bound at radius  $\rho$  on the pre- $J$  boxes proved in Step 4 of the proof of Theorem 13.5, with the same box scale  $\mathcal{L}$  and enlargement  $K_*$  as here. By stationarity of the conditional constrained minimizers and the normalization of the fluctuation variables by the constrained Hessians, the classical remainder has no terms of degree zero, one, or two in the fluctuation variables, and its coefficients of degree  $m \geq 3$  carry the factor  $z^{m-2}$ . The coefficients of the Haar disintegration and determinant terms of degree  $m \geq 1$  carry the factor  $z^m$ . The coefficient bounds of Lemma 12.9(4), with this support assignment, therefore give the first inequality in (15.10), after enlarging  $C_J$ . In particular  $\mathbf{H}(0) = 0$ .

We now construct  $A_\Delta$ . The construction is linear in the interaction, and we define  $A_s$  by the same construction applied to  $R_s$ ; this proves the first assertion of (2). Since  $\mathcal{B}_{\rho, \text{asp}}(R_s) \leq (1-s)\epsilon_2 + s\epsilon_1 \leq \epsilon_1 + \epsilon_2$ , the bound for  $A_s$  follows from the bound for  $A_\Delta$  with  $\delta$  replaced by  $\epsilon_1 + \epsilon_2$ .

*Coefficients.* We write

$$\Delta(e^{\tilde{\zeta}}) = \sum_{n \geq 2} \frac{1}{n!} \sum_{j_1, \dots, j_n} \Delta_n(j_1, \dots, j_n) \prod_{r=1}^n \tilde{\zeta}_{j_r}, \quad \phi_{d,j}(\theta, q) = \sum_{|\alpha| + |\eta| \geq 1} \phi_{j;\alpha\eta} \theta^\alpha q^\eta,$$

where  $j, j_r$  range over the scalar coordinates of the fine lattice, and  $\alpha$  and  $\eta$  are multi-indices in the coarse-field variables and the inputs. Since  $\phi_d(0, 0) = 0$ , substitution gives, for  $|\alpha| + |\eta| = N$ ,

$$[\theta^\alpha q^\eta] \Delta(e^{\phi_d(\theta, q)}) = \sum_{n=2}^N \frac{1}{n!} \sum_{j_1, \dots, j_n} \sum_{\substack{\alpha_1 + \dots + \alpha_n = \alpha, \\ |\alpha_r| + |\eta_r| \geq 1}} \sum_{\eta_1 + \dots + \eta_n = \eta} \Delta_n(j_1, \dots, j_n) \prod_{r=1}^n \phi_{j_r; \alpha_r \eta_r}. \quad (15.11)$$

No further multinomial factor occurs, since each  $\phi_{j;\alpha\eta}$  is already the coefficient of a monomial. An occurrence  $o$  consists of  $n$ , the indices  $j_1, \dots, j_n$ , and an ordered allocation  $(\alpha_r, \eta_r)_{r \leq n}$  in (15.11). To  $\Delta_n(j_1, \dots, j_n)$  we assign a shortest connected network through the locations of  $j_1, \dots, j_n$ , and to each  $\phi_{j_r; \alpha_r \eta_r}$  a shortest connected network through the location of  $j_r$  and the locations of its inputs. The union  $\mathcal{N}_o$  is a connected network through the locations of all external variables of  $o$ , and its length  $\ell_o$  is at most the sum of the lengths of these networks. Let  $N_o$  be the number of external variables of  $o$ , counted with multiplicity.

*The families.* We keep the occurrences with  $|\eta| \geq 1$ . For such an occurrence, let

$$a_o(z) = \frac{z^{|\eta|}}{n!} \Delta_n(j_1, \dots, j_n) \prod_{r=1}^n \phi_{j_r; \alpha_r \eta_r} \prod_l \lambda_{i(l)}^{\eta_l},$$

where  $l$  runs over the inputs and  $i(l)$  is the intermediate scale of the input  $l$ . Let  $Y_o$  be the box set assigned by Lemma 15.3 to the locations of the external variables of  $o$  and the network  $\mathcal{N}_o$ , and define

$$A_{\Delta,Y}(\theta, w; z) = \sum_{o: Y_o=Y} a_o(z) \theta^\alpha w^\eta.$$

The occurrences with  $\eta = 0$  sum to  $\Delta(e^{\phi_d(\theta,0)})$ . Hence, once absolute convergence is established below,

$$\sum_Y A_{\Delta,Y}(z) = \Delta(e^{\phi_d(\theta, (z\lambda_i w_i))}) - \Delta(e^{\phi_d(\theta,0)}),$$

which is (1) for  $A_\Delta$ . Since every coefficient carries a positive power of  $z$ , we have  $A_\Delta(0) = 0$ , and the coefficient of  $z$  collects the occurrences with  $|\eta| = 1$ , which are linear in  $w$ . This proves (2) and (3).

*Spatial weights.* The networks of  $\Delta_n$  have lengths measured in units of the fine lattice; in units of the coarse lattice these lengths are divided by  $B$ . By the last condition in (15.4) and  $C_{\text{geom}} \geq 1$ , a weight  $e^{b\ell}$  on this part, with  $\ell$  in coarse units, is at most the weight  $e^{a_{\text{sp}}t}$  in fine units, which is included in  $\delta$ . The networks attached to the coefficients of  $\phi_d$  are weighted by the spatial rate of the bounds of Theorem 13.2, which exceeds  $b$ . With  $h = 8$ ,  $K = e^{8c_0}$ , and the choices of  $\mathcal{L}$  and  $\mu$  made before (15.4), the exponent  $\beta$  in Lemma 15.3 is at most  $b/2$ , so the weight  $e^{\beta\ell_o}$  is covered by these two weights.

*The bound.* In (15.9) we bound the supremum over  $w \in \Omega_p$  by the sum of the absolute values of the coefficients of  $w^\eta$  times  $p^{|\eta|}$ ; since  $|\lambda_i| \leq 1$ , the input  $l$  then has radius at most  $|z|p$ . For an occurrence  $o$  and a box  $x \in Y_o$  we apply (15.3), with  $u$  the location of an input  $l$  of  $o$ , and we sum over all inputs  $l$  with their multiplicities  $\eta_l \geq 1$ ; this overcounts each occurrence. The factor  $K^{N_o}$  is absorbed by multiplying the coarse-field radius  $\rho$  and the input radius  $|z|p$  by  $K$ . For a fixed input  $l$ , the weighted sum over the occurrences containing  $l$  is bounded by the calculus of nonnegative coefficient bounds of Lemma 5.3. The outer coefficient  $\Delta_n$  is summed with one index fixed, the branch  $r$  containing  $l$  is summed by its volume-normalized column sum at  $l$ , and the remaining branches by their row sums. Let

$$F_\Delta(y) = \sum_{n \geq 2} \frac{\|D^n \Delta(0)\|_{a_{\text{sp}}, \text{root}}}{n!} y^n,$$

let  $c_q(U, V)$  be the majorant of the input-fixed column sums of  $\phi_d$  at a fixed free input, in which every input, including the fixed one, is weighted by its radius, as in Theorem 13.2(1), and let

$$\Gamma_J = \max_l \frac{v_l}{v_{\text{fine}}} \leq M^{3J},$$

where  $v_l$  is the cell volume at the mesh of the input  $l$  and  $v_{\text{fine}}$  the cell volume of the fine lattice. The factor  $\Gamma_J$  converts the volume-normalized column sum of  $\phi_d$  into the counting sum used in the norm of  $\Delta$ . A branch monomial of total degree  $k$  contributes only  $\eta_l/k$

times its absolute coefficient to the normalized column at  $l$ , whereas the occurrence count above uses multiplicity  $\eta_l$ . We therefore mark its total degree and use

$$c_q^{\text{mark}}(U, V) = (U\partial_U + V\partial_V)c_q(U, V).$$

Choosing the outer branch that contains  $l$  separately produces the derivative of  $F_\Delta$ . With  $S_J(\mu) = D_J \sum_{r \geq 0} (r+2)^3 e^{-\mu r}$ , where  $D_J(r+2)^3$  bounds the number of inputs at distance between  $r$  and  $r+1$  from a point, we obtain

$$\|A_\Delta(z)\|_{\mathcal{S}_8} \leq C_8 S_J(\mu) \Gamma_J F'_\Delta(\mathfrak{M}_J(K\rho, K|z|p)) c_q^{\text{mark}}(K\rho, K|z|p). \quad (15.12)$$

Let  $U_0 = 4K_*\rho$  and  $V_0 = 4K_*s_J$  be the radii of Lemma 15.4. Since  $\sum_{n \geq 2} \rho^{n-2} \|D^n \Delta(0)\|_{a_{\text{sp}}, \text{root}}/n! = \delta$ , we have, for  $0 \leq y \leq \rho/2$ ,

$$F'_\Delta(y) \leq \delta \rho \sum_{n \geq 2} n 2^{1-n} = 3\delta \rho.$$

By Theorem 13.2(1),  $c_q \preccurlyeq V\partial_V \widetilde{\mathfrak{M}}_J$  for a majorant  $\widetilde{\mathfrak{M}}_J(U, V) = B^{-1}\widetilde{Z}(U) + C_{\text{im}, J}V/(1 - C_{\text{im}, J}(U + V))$  of the same form. Applying the degree marking coefficientwise and writing  $C = C_{\text{im}, J}$  gives

$$c_q^{\text{mark}}(U, V) \leq \frac{CV(1 - CU + CV)}{[1 - C(U + V)]^3} \leq 2CV\eta_{\text{im}}^{-3}$$

on the reserved product. Thus  $L_J = \sup_{0 < V \leq V_0} c_q^{\text{mark}}(U_0, V)/V \leq 2C_{\text{im}, J}\eta_{\text{im}}^{-3}$ . This changes only a constant chosen after  $J$ . Let

$$B_\Delta = 3\Gamma_J \rho C_8 S_J(\mu) L_J K,$$

and choose  $C_J \geq \max\{1, B_\Delta, K/V_0\}$ . If  $C_J|z|p < 1$ , then  $K|z|p < V_0$ , and  $K\rho < U_0$  because  $K < 4K_*$ ; hence  $\mathfrak{M}_J(K\rho, K|z|p) \leq \mathfrak{M}_J(U_0, V_0) < \rho/2$  by (15.5). Inserting these bounds into (15.12), we find

$$\|A_\Delta(z)\|_{\mathcal{S}_8} \leq B_\Delta \delta |z|p \leq \frac{C_J |z| \delta p}{1 - C_J |z|p}.$$

The factor  $\delta$  enters through the single tensor  $\Delta_n$ , never through the sum of two norms. The bound is uniform in  $|z| \leq r_0$  for every  $r_0 < (C_J p)^{-1}$ . The series defining  $A_\Delta$  therefore converges absolutely in  $\mathcal{S}_8$  and is holomorphic in  $z$ , and all regroupings used above are justified. This proves (4).  $\square$

*Proof of Theorem 15.6.* The proof has seven steps. Step 1 identifies the transformation. Step 2 extracts the contraction of the pullback  $P_d$  from the estimate of Theorem 13.5. Step 3 proves the Lipschitz bound, using Lemma 15.8 and the second-order cancellation. Steps 4 and 5 compare two preceding depths at a fixed lattice pair: Step 4 shows that each fixed coefficient tensor of the output converges, uniformly over  $\mathcal{A}_E$ , and Step 5 deduces convergence in  $\mathcal{B}_{\rho, a_{\text{sp}}}$  from a bound at a larger radius. Step 6 combines the contraction with this continuity to construct  $R_j^\infty$ , and Step 7 passes to the stopped measures.

**Step 1: the transformation.** All products used in this proof are subproducts of the products of Lemma 15.4. In particular, on the cutoff domain  $\max_i \|g_i w_i\|_\infty \leq tp(t) \leq Q_{\text{act}}$ , so that  $r + Q_{\text{act}} < r_J$  and the denominator of the majorant is positive. The complex product of the auxiliary integral satisfies (15.5), with its intermediate domain conditions. Thus the rational bound of Theorem 13.2 is used only inside its domain. As explained before the theorem,  $Q_d + P_d R + D_d R$  is the effective interaction produced from  $R$  before normalization,  $\widehat{\mathcal{M}}_{d,t} R$  subtracts its value at the identity, and at a coarse field in  $B_r$  the prefactors of Theorem 14.6(2) times  $\Xi_R$  give the conditional fiber density before global normalization.

**Step 2: contraction of the pullback.** Let  $F$  have the symmetries of  $\mathcal{A}_E$ , be invariant under gauge transformations within  $B_\rho$ , and satisfy  $F(I) = 0$ ,  $DF(I) = 0$ , and  $\mathcal{B}_{\rho,a_{\text{sp}}}(F) < \infty$ ; we do not require  $\mathcal{B}_{\rho,a_{\text{sp}}}(F) \leq E$ . For each fixed  $h > 0$ , the estimate of Theorem 13.5 applies to the input  $hF$  on a sufficiently small interval of couplings, which may depend on  $h$ . Its constants  $C_G$ ,  $C_T$ , and  $q_P$  are fixed before the ceiling of the invariant ball is chosen. The estimate holds at the rate  $a_{\text{sp}}$ . Indeed,  $a_{\text{sp}} < a_{\text{cen}}/8$  by (15.4), and the pullback  $P_d$  is the evaluation at the representative  $e^{B^{-1}\bar{A}_{d,J}}$  of Lemma 11.9 (Theorem 13.2), for which Theorem 11.10 applies at every rate below  $a_{\text{cen}}/8$ . At the fixed lattice pair, as  $t \downarrow 0$  with  $p = A_w \log(1/t)$ , the fluctuation exponent and the term containing the input tend to zero uniformly on a smaller complex product domain: they are bounded by  $C_{\Lambda,J} t(p^3 + p)$  and  $C_{\Lambda,J} h \mathcal{B}_{\rho,a_{\text{sp}}}(F) tp$ , respectively. Here the factors  $g_i^{-2}$  in the classical remainder have already canceled against its first two orders in the inputs. Since the coarse-field radius of this product exceeds  $\rho$ , Lemma 15.5 implies that the same convergence holds in  $\mathcal{B}_{\rho,a_{\text{sp}}}$ . This gives  $D_d(hF) \rightarrow 0$  in this norm. The estimate with zero input implies  $\mathcal{B}_{\rho,a_{\text{sp}}}(Q_d) \leq C_G$ . Using the estimate with input  $hF$  and the triangle inequality, we find

$$h \mathcal{B}_{\rho,a_{\text{sp}}}(P_d F) \leq 2C_G + q_P h \mathcal{B}_{\rho,a_{\text{sp}}}(F) + C_T t^2 h \mathcal{B}_{\rho,a_{\text{sp}}}(F) (p^4 + h \mathcal{B}_{\rho,a_{\text{sp}}}(F) p^2) + \mathcal{B}_{\rho,a_{\text{sp}}}(D_d(hF)).$$

The pullback  $P_d$  does not depend on  $t$  or  $p$ . Letting  $t \downarrow 0$ , we obtain  $h \mathcal{B}_{\rho,a_{\text{sp}}}(P_d F) \leq 2C_G + q_P h \mathcal{B}_{\rho,a_{\text{sp}}}(F)$ . Dividing by  $h$  and letting  $h \rightarrow \infty$ , we conclude that

$$\mathcal{B}_{\rho,a_{\text{sp}}}(P_d F) \leq q_{\text{pr}} \mathcal{B}_{\rho,a_{\text{sp}}}(F), \quad q_{\text{pr}} = q_P < 1/4. \quad (15.13)$$

All auxiliary limits in this step are taken on one fixed lattice, and the resulting constant  $q_P$  is the constant of Theorem 13.5, which is the same for every lattice and depth. Inputs of arbitrary finite norm were used only in that estimate, before its restriction to  $\epsilon \leq E$ . The order of the two limits matters: the small-coupling limit is taken at fixed  $h$ .

**Step 3: the Lipschitz bound.** Let  $R_1, R_2, \Delta, \delta, \epsilon_i, R_s$  be as in Lemma 15.8. The segment  $R_s$  stays in the class of real, symmetric, gauge-invariant functions with  $R_s(I) = 0$ ,  $DR_s(I) =$

0, and  $\mathcal{B}_{\rho, a_{\text{sp}}}(R_s) \leq \epsilon_1 + \epsilon_2$ . We hold fixed all parameters of the input effective interaction and the cutoff radius, and we vary only the current couplings, through  $g_i = z\lambda_i$ . Set

$$R_z = \frac{c_J}{p^3 + (\epsilon_1 + \epsilon_2)p}, \quad (15.14)$$

where  $c_J \leq \min\{s_J, (2C_J)^{-1}, \eta_0/(8C_J)\}$ , with  $C_J$  from Lemma 15.8 and  $\eta_0$  from Theorem 12.7. For  $|z| \leq R_z$  we have  $|z|p \leq c_J/p^2 \leq c_J$ , so  $C_J|z|p \leq 1/2$ , and the inputs with radii enlarged by  $K$  stay inside (15.5), with all intermediate domain conditions and denominators keeping strict margins. By (15.10) and  $p \leq p^3$ , the family  $\mathbf{U}_s(z) = \mathbf{H}(z) + \mathbf{A}_s(z)$  satisfies

$$\|\mathbf{U}_s(z)\|_{\mathcal{S}_8} \leq 2C_J|z|(p^3 + p + (\epsilon_1 + \epsilon_2)p) \leq 4C_J|z|(p^3 + (\epsilon_1 + \epsilon_2)p) \leq 4C_Jc_J \leq \eta_0/2.$$

This choice depends only on  $J$ . Since the sum of  $\mathbf{U}_s$  at  $z = t$  is  $H_{d,J} + A_{R_s}$ , Theorem 12.7 implies that  $-\log \Xi_{R_s} = \sum_Y \mathcal{L}_Y(\mathbf{U}_s(t))$ . Since  $\partial_s \mathbf{A}_s = \mathbf{A}_\Delta$ , the derivative of the fluctuation correction along the segment is given by the family

$$\mathbf{F}_s(z) = D\mathcal{L}(\mathbf{U}_s(z))[\mathbf{A}_\Delta(z)],$$

and by (12.15) and (15.10), we have

$$\|\mathbf{F}_s(z)\|_{\mathcal{S}_2} \leq C_J|z|\delta p \quad (|z| \leq R_z), \quad (15.15)$$

after enlarging  $C_J$ . At  $z = t$  the sum of  $\mathbf{F}_s(t)$  over  $Y$  is the expectation of  $\mathbf{A}_\Delta$  under the interacting conditional measure, not under the reference measure.

Both  $\mathbf{U}_s$  and  $\mathbf{A}_\Delta$  vanish at  $z = 0$ . By Lemma 15.8(3), the coefficient of  $z$  in  $\mathbf{A}_\Delta$  is linear in the fluctuation variables on each support, so its expectation under the symmetric reference measure  $\pi_p$  vanishes before the sums over supports and over coefficients are taken. The terms containing a further factor of the interaction, including the differences of the Haar disintegration and determinant terms, are of degree at least two in  $z$ . Applying Corollary 12.8 with  $a = 2C_J\delta p$ , which bounds  $\|\mathbf{A}_\Delta(z)\|_{\mathcal{S}_8}/|z|$  on the disc by (15.10), and with  $R = R_z$ , we obtain

$$\|\mathbf{F}_s(t)\|_{\mathcal{S}_2} \leq C_J t^2 \delta [p^4 + (\epsilon_1 + \epsilon_2)p^2], \quad 0 < t \leq R_z/2, \quad (15.16)$$

after enlarging  $C_J$  once more.

The fluctuation correction, its value at zero fluctuation, and its derivative along the segment are invariant under translations and global color rotations, by Theorem 12.13. The coordinate changes map zero fluctuations to zero fluctuations, and the first derivatives with respect to the coarse field vanish at the identity. Every support of the logarithm is a union of supports of the decomposition whose intersection graph is connected. By Lemma 15.2 and (15.4), we have  $a_{\text{sp}}t \leq 2|Y|$  for the locations of the coarse-field

variables in each such support, including subsets and repetitions. The conversion of local interaction decompositions into the norm  $\mathcal{B}_{\rho, a_{\text{sp}}}$  in Theorem 13.5, with nine types of scalar indices, therefore applies to (15.16) at the radius  $\rho$ , with the factor  $9\rho^{-2}$ . We integrate in  $s$  and add (15.13). The common term  $Q_d$  cancels in the difference, and the subtraction of the values at the identity does not change  $\mathcal{B}_{\rho, a_{\text{sp}}}$ . This gives the factor

$$q_{\text{pr}} + C_J t^2 [p^4 + (\epsilon_1 + \epsilon_2) p^2],$$

after enlarging  $C_J$ . One reduction of  $g_*$  imposes  $p(t) \geq 1$ , by taking  $g_* \leq e^{-1/A_w}$ , together with  $t[p(t)^3 + 2Ep(t)] \leq c_J/2$ , which gives  $t \leq R_z/2$  when  $\epsilon_1 + \epsilon_2 \leq 2E$ , and  $C_J t^2 [p(t)^4 + 2Ep(t)^2] \leq (1 - q_{\text{pr}})/2$ , for all  $0 < t \leq g_*$ . The factor is then at most  $\vartheta = (1 + q_{\text{pr}})/2$ .

**Step 4: changing the preceding depth.** By Theorem 11.2, all classical data of the RG step converge on strictly smaller product domains as the preceding depth tends to infinity. These data are the conditional constrained minimizers, the constrained Hessians, the analytic branches of their square roots and inverse square roots that are positive at real arguments, every Haar disintegration density, and the triangular reconstruction. The Hessians here are those of the intermediate scales, with their couplings  $g_i$ ; no inverse of a Hessian in the coarse-field variables is needed. In particular  $H_{d,J}$  and  $B_{\text{ref}}$  converge with every fixed derivative, since their formulas in Theorem 12.13 are finite compositions of these data on products where all domain conditions hold.

It remains to control the term with the input effective interaction, uniformly in  $R \in \mathcal{A}_E$ . For two depths  $d, d'$  we use the same gauge transformation  $\Omega_d$  on both reconstructions  $\mathcal{C}_{\text{raw}d}$  and  $\mathcal{C}_{\text{raw}d'}$ . The arguments stay in the common product (15.5). The product with radii enlarged by  $K$  has radii strictly below  $u_\circ$  and  $v_\circ$ , so their sum is less than  $r_J$  and the denominator of the majorant is at least  $\eta_{\text{im}}$ . Every intermediate map keeps its strict domain margin on this product. The bounds of Theorem 13.2 for the gauge transformation and its inverse, together with the convergence of the reconstructions, put the image  $\text{Log}[\mathcal{C}_{\text{raw}d'}^{\Omega_d}]$  in  $B_{3\rho/4}$  for all sufficiently large pairs of depths, while  $\phi_d$  has image in  $B_{\rho/2}$ . All these bounds hold on a smaller complex product which still contains the output radius  $\rho$  and the cutoff domain. At depth  $d'$  we compare  $R(\exp \text{Log}[\mathcal{C}_{\text{raw}d'}^{\Omega_d}])$  with  $R(e^{\phi_{d'}})$ . On real inputs these are evaluations of the gauge-invariant extension  $R_{\text{sat}}$  at two gauge-equivalent representatives of the same reconstruction, both in  $B_\rho$ , so they agree. Both functions are holomorphic on the comparison product, and their difference vanishes on an open real box. By the one-variable identity theorem applied in each coordinate, it vanishes identically. We may therefore compare the depths  $d$  and  $d'$  in the common gauge  $\Omega_d$ . Since the derivative bound of Lemma 15.5 on  $B_{3\rho/4}$  is uniform over  $\mathcal{A}_E$ , this term and each of its fixed derivatives change by  $o(1)$  as  $d, d' \rightarrow \infty$ , uniformly over  $\mathcal{A}_E$ , without any assumption on the behavior of the sequence  $\Omega_d$ .

Since the cutoff domain and its couplings do not change, the integrands on this fixed finite product, together with their fixed derivatives in the coarse field, are uniformly Cauchy. By Theorem 12.7, the logarithm of each partition function is the sum of a local interaction decomposition over the finitely many boxes of the lattice, with a bound in  $\mathcal{S}_2$ . This sum is bounded on the chosen complex product, and the modulus of each partition function has a positive lower bound there. Its branch agrees with the real logarithm at real coarse fields. Comparing the ratio of two close partition functions on a disc about one, we find that the logarithms, normalized at the identity, converge together with every fixed coefficient tensor of the output, uniformly in  $R \in \mathcal{A}_E$ .

**Step 5: convergence in the norm.** We choose the larger radius uniformly, rather than by continuity separately at each depth. In the nonnegative coefficient compositions of Step 3, bound the degree- $k$  rooted absolute coefficient sums of the input interaction by  $E\rho^{2-k}$ , as supplied by  $\mathcal{B}_{\rho, a_{\text{sp}}}(R) \leq E$ . Together with the depth-uniform classical and reconstruction majorants and the fixed support payments, this gives a common convergent absolute coefficient majorant  $\mathcal{U}(u)$  for the entire supported family  $\mathbf{H} + \mathbf{A}_R$  as a function of the coarse-field radius  $u$ . It converges for  $u$  in a neighborhood of  $\rho$ : the enlarged products have strict margins and the reconstructed input remains strictly inside  $B_\rho$ . At the actual coupling  $t \leq R_z/2$ , the bound of Step 3 is at most  $\eta_0/4$ . For  $u \geq \rho$  the increase in the supported norm is bounded uniformly in  $d$  and  $R \in \mathcal{A}_E$  by  $\mathcal{U}(u) - \mathcal{U}(\rho)$ . Choose one  $\rho < \rho_1 < 2\rho$  so close that this difference is at most  $\eta_0/4$ , while all enlarged-product and input-image margins remain strict. Then

$$\sup_{d, R \in \mathcal{A}_E} \|\mathbf{H} + \mathbf{A}_R\|_{\mathcal{S}_8(\rho_1)} \leq \eta_0/2.$$

Here  $\mathcal{S}_8(u)$  denotes the same supported coefficient norm with coarse-field radius  $u$ . Thus the logarithm expansion applies on one common larger radius; the input interaction is still evaluated below its original radius  $\rho$ . Positive composition and the two expansions of Lemma 12.9 and Theorem 12.6 use the same connected box sets and their unions with connected intersection graph. Hence Lemma 15.2 still gives  $a_{\text{sp}} t \leq 2|Y|$  at this larger output radius. The output is translation invariant and has vanishing first derivative at the identity, as in Step 3. The conversion of Theorem 13.5, now with factor  $9\rho_1^{-2}$ , gives

$$\sup_{d, R \in \mathcal{A}_E} \mathcal{B}_{\rho_1, a_{\text{sp}}}(\widehat{\mathcal{M}}_{d,t} R) \leq C_{\Lambda, J, t, p, E}. \quad (15.17)$$

Only sufficiently large  $d$  are needed here; the finitely many smaller depths have finite bounds and are included after enlarging the constant. At the radius  $\rho_1$  the bound need not be  $E$ . Applying Lemma 15.5 to (15.17) and to the convergent coefficient tensors of Step 4, we obtain the continuity statement at exactly the radius  $\rho$  and the exponent  $a_{\text{sp}}$ . Nothing in this argument identifies the auxiliary integral at coarse fields outside  $B_r$  with a conditional integral of the conditioned measure.

**Step 6: contraction over finitely many scales.** Fix  $j$  and a positive integer  $m$ . If  $k, k' \geq j + m$ , the two hierarchies have the same lattices and couplings at the levels  $j, \dots, j + m$ , and their effective interactions at level  $j + m$  lie in  $\mathcal{A}_E$  by Theorem 14.6. For  $h = j + 1, \dots, j + m$ , define

$$\varepsilon_h(k, k') = \sup_{R \in \mathcal{A}_E} \mathcal{B}_{\rho, a_{\text{sp}}}(\widehat{\mathcal{M}}_{J(k-h), t_h} R - \widehat{\mathcal{M}}_{J(k'-h), t_h} R).$$

For fixed  $m$  and  $h$  this tends to zero as  $k, k' \rightarrow \infty$ , by Step 5. Inserting and subtracting at each level the first operator applied to the second input, and using the contraction and the triangle inequality, we obtain

$$\mathcal{B}_{\rho, a_{\text{sp}}}(R_j^k - R_j^{k'}) \leq 2E\vartheta^m + \sum_{h=j+1}^{j+m} \vartheta^{h-j-1} \varepsilon_h(k, k'). \quad (15.18)$$

We let  $k, k' \rightarrow \infty$  at fixed  $m$ , and then  $m \rightarrow \infty$ . The space of coefficient sequences with finite norm is complete: each coefficient tensor has a limit, Fatou's lemma bounds the weighted sums of the limit, and the Cauchy property makes the tails uniform. The limit series converges absolutely on the same ball and defines  $R_j^\infty$  with  $\mathcal{B}_{\rho, a_{\text{sp}}}(R_j^k - R_j^\infty) \rightarrow 0$ . Its value and first derivative at the identity vanish. Reality, the symmetries, and the identities expressing gauge invariance within  $B_\rho$  pass to the pointwise limit, which defines  $R_{j, \text{sat}}^\infty$  on  $D_\rho$ . Thus  $R_j^\infty \in \mathcal{A}_E$ .

**Step 7: stopped measures.** The stopped density in the statement is the density of Theorem 14.6 after  $k - j$  RG steps, with preceding depth  $J(k - j)$ . Its classical term converges by Theorem 11.2 on this fixed lattice, and the gauge-invariant extension preserves the uniform bound on  $B_r$ . Combining this with Step 6 and the value bound of Lemma 15.5, we obtain

$$\delta_{j,k} = \sup_{D_r} \left| \frac{E_{J(k-j), \text{sat}} - E_{\infty, \text{sat}}}{g_j^2} + R_{j, \text{sat}}^k - R_{j, \text{sat}}^\infty \right| \rightarrow 0. \quad (15.19)$$

Here  $g_j > 0$  is fixed. At this finite volume, the limiting weight is bounded above and below by positive constants on  $D_r$ . Since  $D_r$  contains a real open neighborhood of the identity of positive Haar measure, the limiting normalizer is positive and finite. By Lemma 15.1(1), the total variation distance between  $\widehat{v}_j^k$  and  $\widehat{v}_j^\infty$  is at most  $e^{2\delta_{j,k}} - 1$ , which proves the convergence of the stopped measures. The identification  $n_0 = n_s$ ,  $k = K - s$  follows from Definition 4.4. All the nonuniform constants introduced in (15.17)–(15.19) concern fixed finite resolutions or fixed positive couplings, as asserted.  $\square$

## 15.4 Conditional fluctuation kernels and the coarse-level measure

We now identify the conditional law of the fluctuation variables of one RG step, given the field at the coarser level, and prove its convergence. We then write the conditioned

Wilson measure as an iteration of these kernels, starting from its marginal on the coarse lattice.

**Theorem 15.9.** *In the hierarchy of Theorem 15.6, fix  $j \geq 1$  and  $k \geq j$ , and recall  $d_j^k = J(k - j)$ ,  $t_j = M^{(J-1)/2}g_j$ ,  $p_j = p(t_j)$ , and the cutoff domain  $\Omega_{p_j}$  of the RG step from level  $j$  to level  $j - 1$ , which does not depend on  $k$ . For a field  $c = e^\theta \in B_r$  at level  $j - 1$ , the conditional fluctuation kernel of level  $j$  is*

$$k_j^k(\theta, dw) = \frac{\mathbf{1}_{\Omega_{p_j}}(w) e^{-V_j^k(\theta, w)} dw}{\int_{\Omega_{p_j}} e^{-V_j^k(\theta, v)} dv},$$

with the conditional fluctuation potential

$$\begin{aligned} V_j^k(\theta, w) &= \frac{1}{2}|w|^2 + H_{d_j^k, J}(\theta, w) \\ &\quad + R_{j, \text{sat}}^k(\mathcal{C}_{\text{raw } d_j^k}(\theta, (g_{j,i}w_i)_i)) - R_{j, \text{sat}}^k(\mathcal{C}_{\text{raw } d_j^k}(\theta, 0)). \end{aligned}$$

Here  $H_{d, J}$  is the fluctuation exponent of Theorem 12.13, including every difference of determinant terms that depends on coarser fluctuation variables; only terms independent of  $w$  have been canceled. The reconstruction uses the inputs  $g_{j,i}w_i$ , with their fixed ratios, and contains every inverse square root of the constrained Hessians. There exists a limiting potential  $V_j^\infty$  on the same product such that

$$\Delta_{j,k} = \sup_{e^\theta \in B_r, w \in \Omega_{p_j}} |V_j^k(\theta, w) - V_j^\infty(\theta, w)| \longrightarrow 0, \quad \sup_{e^\theta \in B_r} \|k_j^k(\theta, \cdot) - k_j^\infty(\theta, \cdot)\|_{\text{TV}} \leq e^{2\Delta_{j,k}} - 1.$$

The reconstructions  $\mathcal{C}_{\text{raw } d_j^k}$  converge uniformly on this product. The limiting density and the limiting reconstruction are Borel and continuous in  $\theta$  on the smaller coordinate chart, and all normalizers are positive. The formula defines a version of the conditional law at every field in  $B_r$ , including fields of zero probability. Restrictions at coarser levels are functions of the field at level  $j - 1$ , and they cancel from this conditional law wherever it is used. The constants have the fixed-resolution dependence of Theorem 15.6. No total variation statement is made for the images of these kernels under reconstruction maps that depend on  $k$ .

*Proof.* Here  $t_j = g_{j-1}/\sqrt{M} = M^{(J-1)/2}g_j$  is the largest coupling of the intermediate scales, and  $g_{j,i} = t_j M^{(i-J+1)/2}$ ,  $0 \leq i < J$ . Thus the cutoff domain, its radii, and all ratios of couplings are fixed once  $j$ ,  $L$ ,  $\gamma$ , and the hierarchy are fixed.

We apply Theorem 12.13 with preceding depth  $d_j^k$ . Its telescoping identity for the classical action contributes  $|w|^2/2$  and the classical remainder. Its triangular determinant contributes the factors  $g_{j,i}^{3f_i}$ , the zero-fluctuation terms  $(1/2) \log \det K_i$ , and the differences of these terms that depend on coarser fluctuation variables. Its Haar disintegration

densities contribute their values at the identity and all nonconstant differences. The unnormalized fiber is therefore exactly

$$C_j^k(c) \mathbf{1}_{\Omega_{p_j}}(w) e^{-V_j^k(\theta, w)} dw,$$

where  $C_j^k(c) > 0$  collects the factors that depend only on the field  $c$ , including the value of the input effective interaction at zero fluctuation, and  $V_j^k$  is the displayed potential. By Theorem 14.6, the indicator of  $D_r$  at level  $j$  and all intermediate domain indicators equal one on the whole product, at every field in  $B_r$ . Dividing this fiber by its own integral gives the formula for the kernel.

For the convergence, we first compare the same input effective interaction  $R$  at two preceding depths, using the common gauge transformation of Step 4 in the proof of Theorem 15.6. Both images lie in  $B_{3\rho/4}$  on a common complex product. The difference of the reconstructions tends uniformly to zero, and the derivative bounds for  $R$  on  $B_{3\rho/4}$  are uniform over  $\mathcal{A}_E$ . Next we replace  $R_j^k$  by  $R_j^\infty$  on this image. By the value bound of Lemma 15.5, this changes the potential by at most  $D_{\Lambda_j} \rho^2 \mathcal{B}_{\rho, a_{\text{sp}}}(R_j^k - R_j^\infty)$ , where  $D_{\Lambda_j}$  is the number of scalar link coordinates at level  $j$ . We apply these two comparisons at  $w$  and at  $w = 0$ . The classical and Haar disintegration terms converge by Theorem 11.2. Thus the potentials are uniformly Cauchy on the closed product of fields and fluctuation variables, which gives  $\Delta_{j,k} \rightarrow 0$ .

To define the limiting potential, we fix any sufficiently late depth  $d$  and use its gauge transformation  $\Omega_d$  as a comparison gauge. It puts the image of the limiting reconstruction in  $B_{3\rho/4}$ , so  $R_j^\infty$  can be evaluated holomorphically there. Two such choices agree on real data, by gauge invariance within  $B_\rho$ , and hence as holomorphic functions, by the coordinatewise identity argument of Step 4. The convergence of the reconstructions themselves follows from Theorem 11.2 and does not involve this interaction.

By Lemma 15.1(1), applied to Lebesgue measure on  $\Omega_{p_j}$ , the total variation distance is at most  $e^{2\Delta_{j,k}} - 1$ , uniformly in the field. The integrals are positive, since the product has positive Lebesgue measure. On every closed smaller coordinate chart the limiting potential is continuous and bounded, so its exponential has a positive integral which depends continuously on  $\theta$ . This proves that the normalized density is Borel and continuous in  $\theta$ , and that the limiting reconstruction is continuous. In particular the density defines a version of the conditional law even at fields of zero probability.

Finally, once the field at level  $j - 1$  is fixed, a restriction at a coarser level has the form  $a(c) \in \{0, 1\}$ , because all further averaging maps are deterministic. At a field used by the conditioned measure we have  $a(c) = 1$ , and it cancels between the fiber and its normalizer. For a general nonnegative factor depending only on  $c$  the same cancellation holds wherever the factor is positive. This proves the last assertion.  $\square$

**Lemma 15.10.** Fix  $k \geq 1$  and the conditioned Wilson measure  $\nu^k$ , with meshes  $a_j = HB^{-j}$  and preceding depths  $d_j^k = J(k - j)$ . Fix a Borel map  $S_0$ , independent of  $k$ , from  $D_r$  on the coarse lattice  $\Lambda_{n_0}$  to  $\{\theta : \|\theta\|_\infty < r\}$  such that  $e^{S_0(U)}$  is gauge equivalent to  $U$ ; such a map exists and is constructed in the proof. Let  $\theta_0 = S_0(V^{Jk})$ , where  $V^{Jk}$  is the coarse field of  $U \sim \nu^k$ . Given  $\theta_0, w_1, \dots, w_{j-1}$ , let  $w_j$  have the law  $k_j^k(\theta_{j-1}, \cdot)$ , and put

$$U_0 = e^{\theta_0}, \quad U_j = e^{\theta_j}, \quad \theta_j = \Phi_{d_j^k, j}(\theta_{j-1}, (g_{j,i} w_{j,i})_i), \quad T_M^J U_j = U_{j-1} \quad (1 \leq j \leq k).$$

Here  $\Phi_{d,j}$  is either the map  $\phi_d$  of Section 15.3, built with the gauge transformations of Theorem 13.2, or the logarithmic coordinate representation  $\hat{\Phi}_{d, \Lambda_j}$  of the reconstruction map of Theorem 15.13 below; in both cases the gauge transformation depends only on the field at level  $j - 1$  and equals the identity at the vertices of  $\Lambda_{n_{j-1}}$ . Then the following hold.

1. The gauge orbits  $([U_0], \dots, [U_k])$  have the law of the gauge orbits  $([V^{Jk}], [V^{J(k-1)}], \dots, [V^0])$  under  $\nu^k$ .
2. If the field at level  $j - 1$  is replaced by another gauge-equivalent representative in  $B_r$ , the kernels correspond under a triangular change of fluctuation coordinates with orthogonal diagonal blocks, which preserves the balls of the cutoff domain and the normalized density.
3. Restrictions at coarser levels that are functions of the field at level  $j - 1$  do not change the conditional law at level  $j$ .

The map  $S_0$  need not be constant on gauge orbits.

Suppose in addition that, on the products used in the construction,

$$\|\Phi_{d,j}(\theta, q)\|_\infty \leq a_{\text{cap}} \|\theta\|_\infty + C_{\text{rec}, J} \max_i \|q_i\|_\infty, \quad a_{\text{cap}} = C_{\text{CP}}/B, \quad a_{\text{cap}} \sqrt{B} < 1,$$

where  $C_{\text{CP}} \geq 1$  is fixed before  $J$ . Put  $b_j = \|\theta_j\|_\infty$ . Then on every sample, uniformly in  $k, j$ , and  $d_j^k$ ,

$$b_j \leq a_{\text{cap}} b_{j-1} + C_{\text{rec}, J} t_j p_j, \quad b_j \leq r a_{\text{cap}}^j + C_{\text{cap}, J} g_3 \sqrt{a_j} p_j,$$

for any  $C_{\text{cap}, J} \geq M^{(J-1)/2} C_{\text{rec}, J} / (1 - a_{\text{cap}} \sqrt{B})$ . For  $2 \leq l \leq k$ ,

$$\sum_{j=l}^k b_{j-1} \leq C_J \{ r a_{\text{cap}}^{l-1} + g_3 \sqrt{a_{l-1}} (p_l + 1) \};$$

for  $l = 1$  one adds the term  $b_0 < r$ . These constants have the dependence described in item (2) of Definition 4.14, and they may depend on  $(1 - a_{\text{cap}} \sqrt{B})^{-1}$ ; the physical parameters and the mesh factors are displayed explicitly.

*Proof.* We first construct  $S_0$ . The gauge group of the coarse lattice is a finite product of compact matrix groups and has a countable dense subset  $(h_m)_{m \geq 1}$ . For  $U \in D_r$  the set of gauge transformations carrying  $U$  into the open set  $B_r$  is nonempty and open, so some  $h_m$

does so; we choose the least such  $m$  and set  $S_0(U) = \text{Log}(U^{h_m})$ . The sets  $\{U : U^{h_m} \in B_r\}$  are open, and removing from each of them the finitely many earlier ones gives Borel sets on which the choice is constant. Thus  $S_0$  is Borel and independent of  $k$ , and it carries every field in  $D_r$  into  $B_r$ .

At fixed  $k$ , the density of  $\nu^k$  disintegrates, by the fiber formula of Theorem 14.6, into its coarse marginal and the normalized kernels of Theorem 15.9. To verify this without choosing an abstract version of the conditional law, we multiply the unnormalized fiber by a bounded test function of the coarse field and a bounded test function of the fluctuation variables, and integrate. By Fubini's theorem and the exact fiber normalizer, we obtain the proposed conditional integral. We repeat this finite calculation from the coarse side to the fine side. At each repetition, all remaining restrictions at coarser levels are factors depending only on the coarser field, and they cancel. Integrating the resulting iterated product against the coarse marginal constructs the finite random hierarchy of the statement.

For changes between coarse representatives in  $B_r$ , we use part (3) of Theorem 13.4. The enlarged-domain reconstruction hypothesis inherited from Theorem 15.6 places both intermediate reconstructions in the required comparison domains, with margins. The transition is triangular in the intermediate scales and has orthogonal color rotations and block permutations on its diagonal, so it preserves every ball, the Gaussian term  $|w|^2/2$ , and Lebesgue measure, with absolute Jacobian one. The representative change after reconstruction is given by Theorem 13.2(4). For the initial coarse field, which lies in  $D_r$  but possibly outside  $B_r$ , the passage to the representative  $e^{S_0(U)}$  uses the gauge invariance of the input density and of the restriction event in parts (1)–(3) of Theorem 13.4, not the chart theorem of Theorem 12.13. The Haar disintegration density and the gauge-invariant extension of the input effective interaction transform by the same identity, so the normalized conditional density is preserved. This proves (2). Once the field at level  $j-1$  is fixed, the gauge transformation in  $\Phi$  does not depend on the fluctuation variables and equals the identity at every vertex of  $\Lambda_{n_{j-1}}$ . By gauge covariance of the block averaging map we have

$$T_M^J(\mathcal{C}_{\text{rawd}}(c, q))^{\Omega(c)} = (T_M^J \mathcal{C}_{\text{rawd}}(c, q))^{\Omega|_{\Lambda_{n_{j-1}}}} = c,$$

which proves the relation  $T_M^J U_j = U_{j-1}$  at every step. The same computation applies to the reconstruction maps of Theorem 15.13, which have these properties by construction. We conclude that every function of the tuple that is invariant under gauge transformations at each level has the same expectation as under  $\nu^k$ , which is (1). Assertion (3) is the last part of Theorem 15.9.

On the cutoff domain  $\max_i \|g_{j,i} w_{j,i}\|_\infty \leq t_j p_j$ , and the assumed bound on  $\Phi$  implies the first recurrence. We write  $a = a_{\text{cap}}$ ,  $\xi = a\sqrt{B} < 1$ , and  $c_B = M^{(J-1)/2}$ . Since  $b_0 < r$

and  $t_i = c_B g_3 \sqrt{a_i} = c_B g_3 \sqrt{a_j} B^{(j-i)/2}$ , iteration gives

$$\begin{aligned} b_j &\leq r a^j + C_{\text{rec},J} \sum_{i=1}^j a^{j-i} t_i p_i = r a^j + c_B C_{\text{rec},J} g_3 \sqrt{a_j} \sum_{i=1}^j \xi^{j-i} p_i \\ &\leq r a^j + \frac{c_B C_{\text{rec},J}}{1 - \xi} g_3 \sqrt{a_j} p_j. \end{aligned} \quad (15.20)$$

In the last inequality we used  $p_i \leq p_j$ , which holds since  $p_j = p_1 + (j-1)c_p$  with  $c_p = A_w \log B/2 > 0$ . This gives the second bound on  $b_j$ , on every sample. Only a constant depending on  $J$  is enlarged here;  $C_{\text{CP}}$  and  $a_{\text{cap}}$  stay fixed.

For  $l \geq 2$ , let  $x = B^{-1/2}$ . Using  $\sqrt{a_h} = \sqrt{a_{l-1}} x^{h-l+1}$ ,  $p_h = p_{l-1} + (h-l+1)c_p$ , and the sums  $\sum_{h \geq 0} x^h = (1-x)^{-1}$  and  $\sum_{h \geq 0} h x^h = x(1-x)^{-2}$ , we obtain

$$\begin{aligned} \sum_{j=l}^k b_{j-1} &\leq \frac{r a^{l-1}}{1-a} + C_{\text{cap},J} g_3 \sqrt{a_{l-1}} \left( \frac{p_{l-1}}{1-x} + \frac{c_p x}{(1-x)^2} \right) \\ &\leq C_J \{ r a^{l-1} + g_3 \sqrt{a_{l-1}} (p_l + 1) \}, \end{aligned} \quad (15.21)$$

where one may take  $C_J \geq \max\{(1-a)^{-1}, C_{\text{cap},J}(1-x)^{-2}\}$ : both  $p_{l-1}$  and  $c_p$  are at most  $p_l$ , so the bracket in the first line is at most  $p_l/(1-x)^2$ . Thus no dependence on  $A_w$  is needed in this constant. For  $l = 1$  we separate  $b_0 < r$  and apply this calculation from  $j = 2$ .  $\square$

*Remark 15.11.* Subsequent gauge-invariant comparisons may be made on the probability space of the reconstruction chain in Lemma 15.10. To include a configuration with the full law  $\nu^k$ , adjoin an independent Haar-distributed gauge transformation  $h$  on the finest vertex gauge group and set  $U = U_k^h$ . For any bounded Borel test function, its Haar average over  $h$  depends only on  $[U_k]$ ; the terminal orbit law and gauge invariance of  $\nu^k$  therefore give  $\text{Law}(U) = \nu^k$ . Gauge covariance and  $T_M^J U_j = U_{j-1}$  give, on every sample,

$$(T_M^J)^{k-j} U = U_j^{h|_{\Lambda_{n_j}}} \quad (0 \leq j \leq k).$$

Thus all levels are matched jointly using restrictions of the same gauge transformation. Statements about gauge-invariant observables of the original configuration and a reconstruction prefix are understood in this realization.

**Lemma 15.12.** *In the hierarchy of Theorem 15.6, fix  $n_0, L, \gamma > 0$ , and the Borel map  $S_0$  of Lemma 15.10, independent of  $k$ . Let  $\nu_{0,\text{full}}^k$  be the marginal of  $\nu^k$  on the coarse lattice, that is, the law of  $V^{Jk}$ . Then  $\nu_{0,\text{full}}^k = \hat{\nu}_0^k$ , and the laws  $\eta_0^k = (S_0)_* \nu_{0,\text{full}}^k$  and  $\eta_0^\infty = (S_0)_* \hat{\nu}_0^\infty$  of the coarse logarithm satisfy*

$$\|\eta_0^k - \eta_0^\infty\|_{\text{TV}} \leq \|\hat{\nu}_0^k - \hat{\nu}_0^\infty\|_{\text{TV}} \longrightarrow 0.$$

*The limiting stopped measure has a positive normalizer. The map  $S_0$  need only be Borel. The convergence has the dependence of Theorem 15.6 at the fixed coarse side.*

*Proof.* We integrate the density of  $\nu^k$  down to the coarse lattice, using Theorem 14.6. All  $k$  restrictions on the fluctuation variables and the indicator of  $D_r$  on the coarse lattice have been included, and no restriction at a coarser level remains. The normalized density is exactly that of  $\hat{\nu}_0^k$ . Applying Theorem 15.6 with  $j = 0$ , we obtain its total variation limit and the positive limiting normalizer. For every Borel set  $A$ ,

$$|\eta_0^k(A) - \eta_0^\infty(A)| = |\hat{\nu}_0^k(S_0^{-1}A) - \hat{\nu}_0^\infty(S_0^{-1}A)| \leq \|\hat{\nu}_0^k - \hat{\nu}_0^\infty\|_{\text{TV}}.$$

Taking the supremum over  $A$ , as in Lemma 5.5, completes the proof; no continuity of  $S_0$  is needed.  $\square$

## 15.5 A fixed family of reconstruction maps

The gauge transformations  $\Omega_d$  of Theorem 13.2 are auxiliary: they change only the representative in which the input effective interaction is evaluated, and the conditional measures do not depend on them. However, Theorem 13.2 asserts nothing about their convergence as the preceding depth tends to infinity, so the reconstructed fields need not converge. To fix this, we use that at a fixed lattice pair the gauge transformations form a bounded family of holomorphic functions of finitely many variables, so they have a convergent subsequence. We choose one subsequential limit and use it at every preceding depth beyond a deterministic threshold. The exact averaging identities and the conditional densities are preserved, because the new gauge transformation is the identity at the vertices of the coarse lattice and this interaction is gauge invariant. This is not a gauge fixing on the space of gauge orbits. The threshold must be large enough to preserve certain bounds, and we keep track of the order in which the constants are chosen:

$$\begin{aligned} C_{\text{CP}} \text{ and the spatial parameters} &\longrightarrow J \longrightarrow \text{joint coefficient bounds on larger domains} \\ &\longrightarrow \beta_J, r_{\text{eff}}, r \longrightarrow Q_{\text{act}}, \text{ coupling ceiling, thresholds } d_0(\Lambda). \end{aligned} \tag{15.22}$$

The constants  $\beta_J$  and  $r_{\text{eff}}$  appear in Lemma 15.16. The thresholds can be increased at the last step without changing any earlier constant. The choice of the limiting gauge depends on the lattice pair and on the domains, and not on the coupling.

**Theorem 15.13.** *There exist nested complex domains  $\mathcal{D}_{\text{in}} \subseteq \mathcal{D}_{\text{mid}} \subseteq \mathcal{D}_{\text{out}}$  of coarse-field and input variables, chosen inside the reserved reconstruction and fixed-gauge products before the later working radii, with radius ratios less than one, such that all real products of coarse fields and inputs used in the construction lie in  $\mathcal{D}_{\text{in}}$  with a margin. The domains and the coefficient ceilings depend only on  $M, J$ , and the larger construction domains, and not on the lattice side or the preceding depth.*

1. On each fixed lattice pair  $\Lambda$  of one RG step, choose once a holomorphic subsequential limit  $\Omega_{\infty, \Lambda}$ , on the middle domain of coarse fields, of the gauge transformations  $\Omega_{d, \Lambda}$  of Theorem 13.2 as  $d \rightarrow \infty$ . It is periodic, takes values in  $\text{SU}(2)$  at real arguments, equals the identity at  $\theta = 0$  and at the vertices of the coarse lattice, is equivariant under global color rotations, and has a bounded holomorphic inverse on that domain. Let  $\mathcal{C}_{\text{raw} \infty, \Lambda}$  be the limit of the reconstructions  $\mathcal{C}_{\text{raw} d, \Lambda}$  given by Theorem 15.6, and put

$$\Phi_{\infty, \Lambda}(c, q) = \text{Log}[\mathcal{C}_{\text{raw} \infty, \Lambda}(c, q)^{\Omega_{\infty, \Lambda}(c)}].$$

Then the whole sequence  $\text{Log}[\mathcal{C}_{\text{raw} d, \Lambda}^{\Omega_{\infty, \Lambda}}]$  converges to  $\Phi_{\infty, \Lambda}$  on  $\mathcal{D}_{\text{in}}$  in every fixed derivative and every fixed spatial difference, and in the weighted analytic coefficient norm, including all output-fixed row sums and all volume-normalized column sums. The modulus of convergence may depend on  $\Lambda$ .

2. Fix first tolerances, independent of  $J$ , for the finitely many quantities bounded in (3) below, and a tolerance depending on  $J$  for the joint coefficient norm on  $\mathcal{D}_{\text{in}}$ . There exist finite deterministic thresholds  $d_0(\Lambda)$ , which also enforce the image conditions into  $B_r$  at real coarse fields in  $B_r$  and into  $B_\rho$  on the complex product, such that

$$\hat{\Omega}_{d, \Lambda}(c) = \begin{cases} \Omega_{d, \Lambda}(c), & d < d_0(\Lambda), \\ \Omega_{\infty, \Lambda}(c), & d \geq d_0(\Lambda), \end{cases}$$

and

$$\hat{\mathcal{R}}_{d, \Lambda}(c, q) = \mathcal{C}_{\text{raw} d, \Lambda}(c, q)^{\hat{\Omega}_{d, \Lambda}(c)}, \quad \hat{\Phi}_{d, \Lambda} = \text{Log} \hat{\mathcal{R}}_{d, \Lambda},$$

define maps  $\hat{\Phi}_{d, \Lambda}$  with common joint coefficient bounds on  $\mathcal{D}_{\text{in}}$ , with all row and column sums, uniformly in  $\Lambda$  and  $d$ . The complex image condition is imposed on this inner product, and the real image condition on the actual real cutoff products contained in it; all intermediate classical and Haar disintegration conditions retain their fixed margins. No coefficient bound on a product larger than  $\mathcal{D}_{\text{in}}$  is asserted for the replacement family. The thresholds depend only on the lattice pair, the deterministic domains, and the tolerances; they do not depend on the coupling, on samples, on sources, on an observation scale, on a position, or on the total number of RG steps. We call  $\hat{\mathcal{R}}_{d, \Lambda}$  the reconstruction map and  $\hat{\Phi}_{d, \Lambda}$  its logarithmic coordinate representation.

3. Write  $\hat{\Phi}_{d, \Lambda}(c, 0) = B^{-1} \bar{A}_{d, \Lambda}(c)$ , and let  $\nabla_{1/B}$  denote the first differences at spacing  $1/B$ , in units of the coarse lattice. The field  $\bar{A}_{d, \Lambda}$  is a new representative of the gauge orbit of the classical background field; it is not claimed to equal the field of Lemma 11.9. Set

$$G_{d, \Lambda}(\theta) = (\bar{A}_{d, \Lambda}(\theta), \nabla_{1/B} \bar{A}_{d, \Lambda}(\theta)),$$

and let  $\mathcal{D}_\theta = \{\|\theta\|_\infty \leq r_\theta\}$  be a closed polydisc of coarse fields contained in the coarse-field factor of  $\mathcal{D}_{\text{in}}$  and in the ball  $\|\theta\|_\infty < r_{\text{cen}}$  of Lemma 11.9. There exists  $C_{\text{CP}} \geq 1$ ,

independent of  $J$ ,  $\Lambda$ ,  $d$ , and  $r_\theta$ , with the following properties. For every  $\theta \in \mathcal{D}_\theta$  we have  $\|G_{d,\Lambda}(\theta)\|_\infty \leq C_{\text{CP}}\|\theta\|_\infty$ . The output-fixed row sum and every input-fixed column sum of  $D_\theta G_{d,\Lambda}(0)$  are at most  $C_{\text{CP}}$ . For every  $\theta \in \mathcal{D}_\theta$  and  $1 \leq s \leq 3$ , the output-fixed row sum and every input-fixed column sum of  $D_\theta^s G_{d,\Lambda}(\theta)$  are at most  $s! C_{\text{CP}}^{s+1}$ . These sums use the row and column conventions of Definition 4.6, but their network lengths are measured in coarse/input lattice units, as in Lemma 11.9, with spatial rate  $a_{\text{cen}}$  fixed before  $J$  and column volume factor  $B^{-3}$ . In particular the linear coefficient of  $c \mapsto \hat{\Phi}_{d,\Lambda}(c, 0)$  at  $c = I$  has row and column sums at most  $C_{\text{CP}}/B$ . The full joint constants may depend on  $J$ . The constant  $C_{\text{CP}}$  is enlarged before  $J$  is chosen, and  $J$  is then chosen so that

$$a_{\text{cap}} = C_{\text{CP}}/B, \quad q_{\text{anc}} = C_{\text{CP}}^2/B < 1,$$

with a margin; one may require  $q_{\text{anc}} \leq 1/4$ . The tolerances preserve these conditions.

4. Let  $\hat{L}_{d,\Lambda}$  be the derivative at  $c = I$  of  $c \mapsto \hat{\Phi}_{d,\Lambda}(c, 0)$ . In this item and in Step 4 of the proof, let  $H$  denote the mesh of the coarse lattice of the pair  $\Lambda$  (not the mesh  $L/n_0$  of level zero), and let  $a = H/B$  be the mesh of its fine lattice. Then for every coarse logarithm  $\theta$ , every orientation  $i$ , and every color,

$$a^2 \sum_x (\hat{L}_{d,\Lambda} \theta)_i(x) = H^2 \sum_u \theta_i(u),$$

where  $x$  runs over the vertices of the fine lattice and  $u$  over those of the coarse lattice. In words, for each coarse link and color, the linearized zero-fluctuation reconstruction has the same componentwise integral as the piecewise constant embedding of the coarse potential.

5. The reconstruction maps preserve the exact averaging identity  $T_M^J e^{\hat{\Phi}_{d,\Lambda}(c,q)} = c$ , the gauge orbit of the zero-fluctuation reconstruction, the cutoff domains, the Haar disintegration densities, and the conditional fluctuation kernels.

The constants have the dependence described in item (2) of Definition 4.14; among them, only the finitely many bounds in (3) are chosen before  $J$ . No convergence or uniqueness of the sequence  $\Omega_{d,\Lambda}$  itself is asserted.

*Proof.* The proof has five steps: we choose the limiting gauge transformation, prove convergence of the whole sequence in the fixed limiting gauge, choose the thresholds, prove the identity (4), and verify the preservation statements.

**Step 1: a limiting gauge transformation at a fixed lattice pair.** On a fixed lattice pair  $\Lambda$  there are finitely many coarse-field coordinates and finitely many matrix entries at the vertices. We take the three domains inside the common product of Lemma 15.4, with strict nested margins. Their sums of radii are less than  $r_J$ , the denominator of the rational majorant is at least  $\eta_{\text{im}}$ , and all intermediate classical, logarithm, and Haar disintegration domain conditions hold. By Theorem 13.2, the gauge transformations have holomorphic logarithms  $\kappa_d$  in the identity chart, uniformly bounded on the outer domain. By Cauchy's

formula, every Taylor coefficient is bounded there. We enumerate the coefficients of all matrix entries and choose nested subsequences along which the first coefficient, the second, and so on converge; along the diagonal subsequence, all coefficients converge. Such subsequences exist, since each coefficient lies in a fixed closed disc of  $\mathbb{C}$ .

On every strictly smaller polydisc the Cauchy bounds make the tail of the coefficient series uniformly summable: in coordinate dimension  $D$  there are  $\binom{m+D-1}{D-1}$  monomials of degree  $m$ , and the ratio of radii is bounded by a number less than one. Hence the diagonal series converges uniformly on compact subsets of the middle domain, and so do its fixed derivatives after a further reduction of the domain. Its sum  $\kappa_\infty$  is holomorphic there, and we set  $\Omega_{\infty,\Lambda} = \exp \kappa_\infty$ . The exponential series gives convergence of the corresponding gauge transformations. The bounded inverse entries have the same subsequential limit, and passing to the limit in  $\Omega_d \Omega_d^{-1} = I$ , we see that the limiting inverse is holomorphic and bounded. Periodicity, the identity at zero and at the coarse vertices, equivariance under global color rotations, and the values in  $SU(2)$  at real arguments are closed conditions and pass to the limit; for the last one, a limit of unitary matrices of determinant one is unitary with determinant one. We choose one such subsequence for each  $\Lambda$ .

The identity (15.24) and the common coefficient bounds of Theorem 13.2 make every matrix entry of  $\mathcal{C}_{\text{rawd},\Lambda}$  locally uniformly bounded on the connected product  $\mathcal{D}_{\text{out}}$ . By Theorem 11.2, these entries converge on a real open subset. The analytic compactness statement Lemma 11.4(2) therefore gives convergence of the whole sequence locally uniformly on  $\mathcal{D}_{\text{out}}$ , with each fixed derivative. Along the chosen subsequence both the gauge transformations and the reconstructions converge, so their gauge-transformed logarithms converge to  $\Phi_{\infty,\Lambda}$ . The logarithm stays on its branch: the bound on the images has a strict gap to the boundary of the branch, and the same bound holds in the limit. Passing to the limit coefficient by coefficient in each nonnegative weighted sum and applying Fatou's lemma, we obtain for  $\Phi_{\infty,\Lambda}$  the joint coefficient bounds of the unmodified maps, including its column sums; these bounds may depend on  $J$ . In the same way, we obtain separately the bounds of Lemma 11.9(2) for the pair  $G_{\infty,\Lambda}$  formed by the field  $B\Phi_{\infty,\Lambda}(\cdot, 0)$  and its first differences  $\nabla_{1/B}$  in units of the coarse lattice. These are the value bound and the row and column sums of its first three derivatives in the coarse field at every point of  $\mathcal{D}_\theta$ . Indeed, Lemma 11.9(2) holds at every point of the ball  $\|\theta\|_\infty < r_{\text{cen}}$ , and at a fixed lattice pair these derivatives are finite arrays whose entries converge along the chosen subsequence. Although Lemma 11.9 gives bounds of all orders that are independent of  $J$ , only the finitely many bounds listed in (3) are kept with a constant independent of  $J$  in the construction below.

**Step 2: the whole sequence in the fixed limiting gauge.** We define

$$\Psi_d(c, q) = \text{Log}[\mathcal{C}_{\text{rawd},\Lambda}(c, q)^{\Omega_{\infty,\Lambda}(c)}]. \quad (15.23)$$

For all sufficiently large  $d$  this map is defined on a neighborhood of the inner product contained in the larger analytic domain, and it converges locally uniformly and holomorphically to  $\Phi_{\infty, \Lambda}$ , with every fixed derivative, because the gauge transformation in (15.23) does not depend on  $d$ . We now show that it also converges in the coefficient norm.

All operations needed below preserve nonnegative coefficient bounds with fixed indices; this is the calculus of nonnegative coefficient bounds of Lemma 5.3. For a product of coefficient tensors, the union of their networks connects the external indices, so the exponential weight of the product is at most the product of the weights. An output-fixed sum uses the row sums of all factors; an input-fixed sum uses the column sums of the factors on the path from that input to the output, and the row sums of the others. The intermediate volume factors cancel, and only the ratio of the volumes of the external output and of the fixed input remains. Expanding derivatives of compositions, with factorials divided out, sums the partitions as in the composition of nonnegative power series. Matrix products contribute only a factor depending on the fixed matrix dimension. The series

$$e^X = \sum_{m \geq 0} X^m / m!, \quad \log(I + Y) = \sum_{m \geq 1} (-1)^{m+1} Y^m / m$$

give the same conclusion for the exponential and the logarithm on products strictly inside their domains of convergence; the second series is used only where  $\|Y\| < 1$  with a fixed margin.

To apply these facts uniformly before any threshold is imposed, we recover a link of the reconstruction from the gauge-transformed link by

$$(\mathcal{C}_{\text{raw}_d})_e = \Omega_d(x)^{-1} e^{\phi_{d,e}} \Omega_d(y), \quad e = (x, y). \quad (15.24)$$

The gauge transformation  $\Omega_d$ , its inverse, and the map  $\phi_d$  have the common bounds of Theorem 13.2 on a larger product domain. The input product is strictly inside (15.5), so its denominator is positive and its intermediate margins hold. The limiting gauge transformation has the same bounds, by Fatou's lemma. Using (15.24), then multiplying by the limiting gauge transformation and applying the logarithm series, we obtain a joint bound, depending on  $J$ , for (15.23) on an intermediate product domain. The construction domains are chosen inside this product at the outset, with these margins. This preliminary bound need not carry the small factor  $B^{-1}$  of the zero-fluctuation reconstruction. Its constants and radii are independent of  $\Lambda$ , of  $d$ , and of the working radius chosen later. At fixed  $\Lambda$  each fixed coefficient tensor of (15.23) converges, and Lemma 15.5 controls the tail in the degree on  $\mathcal{D}_{\text{in}}$ . Then the whole sequence converges in the coefficient norm, including every volume-normalized column sum. Each fixed spatial difference is a fixed finite difference matrix at this lattice size, so the same argument applies to it. The modulus may depend on  $\Lambda$ . This proves (1).

**Step 3: finitely many derivatives, then thresholds.** Let  $C_0 \geq 1$  be a common constant for the quantities in (3) of the unmodified maps, that is, for the pair  $G_d$  formed by  $B\phi_d(\cdot, 0)$  and its first differences  $\nabla_{1/B}$ . These quantities are the value bound on  $\mathcal{D}_\theta$ , the row and column sums of  $D_\theta G_d(0)$ , and the row and column sums of  $D_\theta^s G_d(\theta)$  for  $1 \leq s \leq 3$  and every  $\theta \in \mathcal{D}_\theta$ . Since the unmodified zero-fluctuation map is  $B^{-1}\bar{A}_{d,J}$  of Lemma 11.9 (Theorem 13.2), and Lemma 11.9(2) holds at every point of the ball  $\|\theta\|_\infty < r_{\text{cen}}$ , the constant  $C_0$  is independent of  $J$ ,  $\Lambda$ , and  $d$ . For the difference  $H_d$  between the fixed-gauge and limiting zero-fluctuation pairs, including their spatial-difference components,  $H_d(0) = 0$ . The radial identity  $H_d(\theta) = \int_0^1 DH_d(s\theta)[\theta] ds$  therefore turns derivative convergence into a value error proportional to  $\|\theta\|_\infty$ . We prescribe errors at most one in this finite list of scaled quantities, smaller if necessary to keep a chosen strict condition. We enlarge  $C_{\text{CP}}$  to cover  $C_0$  and these errors. Then  $C_{\text{CP}}$  serves directly as the constant in the value bound and as the bound for the row and column sums of the linear coefficient tensor, and it can be chosen so that the bounds of orders  $1 \leq s \leq 3$  have the form  $s! C_{\text{CP}}^{s+1}$ . This enlargement uses no analytic radius depending on  $J$  and no modulus of convergence. We choose  $J$  using this enlarged constant, with  $C_{\text{CP}}^2/B \leq 1/4$  and the earlier conditions of Theorem 13.5 and (15.4). The construction of Steps 1 and 2 is then carried out for this  $J$ .

We also prescribe one error, depending on  $J$ , in the full joint coefficient norm on the inner product. By the convergence just proved, for each fixed  $\Lambda$  there exists a finite  $d_0(\Lambda)$  such that, for every  $d \geq d_0(\Lambda)$ , the map (15.23) differs from  $\Phi_{\infty, \Lambda}$  by at most this error, and the quantities in (3) for  $B\Psi_d(\cdot, 0)$  and its first differences differ from those of  $G_{\infty, \Lambda}$  by at most the prescribed errors, uniformly in  $\theta \in \mathcal{D}_\theta$ . At a fixed lattice pair these are finitely many entries of finitely many arrays, and they converge uniformly on the closed set  $\mathcal{D}_\theta$  by Step 2, since  $\mathcal{D}_\theta$  lies in the inner domain. We include in this finite list of requirements the image conditions on the real and complex products and every intermediate logarithm, classical, and Haar disintegration margin. The limiting images have strict margins, inherited from the common bounds of the unmodified maps, and the convergence on these product domains allows us to keep half of each margin. All image requirements concern the inputs  $q_i$ , on fixed products that contain every product  $\{\max_i \|g_i w_i\|_\infty \leq tp(t)\}$  used in the construction, so they do not involve the coupling. For the image into  $B_r$  we take the whole product  $\{\|\theta\|_\infty \leq r\} \times \{\max_i \|q_i\|_\infty \leq Q_{\text{act}}\}$  of (15.6), including  $r + Q_{\text{act}} < r_J$  and the positive denominator. On this product, the limiting image satisfies the strict bound inherited from the majorant of the unmodified maps, and the convergence in the fixed gauge keeps half of its margin above the threshold. For the complex image condition use the closure of  $\mathcal{D}_{\text{in}}$ , which lies strictly inside the larger product: the limit has the uniform  $B_{\rho/2}$  margin inherited from the unmodified majorant, and uniform convergence there places the replacement image in  $B_\rho$ . The intermediate domain conditions belong to the reconstruction  $\mathcal{C}_{\text{rawd}, \Lambda}$ , which is unchanged. These

requirements use  $Q_{\text{act}}$  itself, not the varying radius  $tp(t)$ .

The piecewise family of (2) now has the asserted bounds: below the threshold we use the bounds of the unmodified maps, and above it the bounds of the limit plus the prescribed errors. The maximum of the two is independent of  $\Lambda$  and  $d$ , even if the thresholds diverge with the lattice size. Each threshold is a deterministic property of finitely many classical maps, domains, and tolerances.

In (3),  $\bar{A}_{d,\Lambda} = B\hat{\Phi}_{d,\Lambda}(\cdot, 0)$  is a representative of the gauge orbit of the classical background field. Its preserved bounds give  $a_{\text{cap}} = C_{\text{CP}}/B$  and, under the choice of  $J$  in (3),  $a_{\text{cap}}\sqrt{B} \leq 1/2$ . The full joint constants may depend on  $J$ .

**Step 4: preservation of componentwise integrals.** We prove the identity in (4) directly, by differentiating one averaging step at  $I$ . For an array  $X$  of logarithms on the links of the fine lattice, we write  $X(\sigma)$  for the signed sum of  $X$  along a path  $\sigma$ , and  $X_i(x)$  for the value on the link from  $x$  to  $x + e_i$ . Differentiating the exponent and the reference path in Definition 4.3, we obtain

$$\begin{aligned} (DT_M(I)X)_i(v) &= M^{-3} \sum_{x \in B_M(v)} \sum_{s=0}^{M-1} X_i(x + se_i) + h_X(v) - h_X(v + Me_i), \\ h_X(v) &= M^{-3} \sum_{x \in B_M(v)} X(\sigma_{v,x}). \end{aligned} \quad (15.25)$$

Here the derivative of the reference transport cancels the derivative of the reference transport subtracted inside the logarithm, and the paths  $\sigma_{v+Me_i, x+Me_i}$  are translates of  $\sigma_{v,x}$ . Summing over the periodic coarse vertices  $v$  removes the last two terms. For each  $s$  the blocks  $B_M(v)$  partition the fine vertices, and translation by  $se_i$  permutes them. Thus every oriented fine link is counted exactly  $M$  times in the double sum, and

$$\sum_v (DT_M(I)X)_i(v) = M^{-2} \sum_x X_i(x). \quad (15.26)$$

Multiplying by the square of the coarse mesh converts this into the equality of the integrals of the two piecewise constant potentials, logarithm divided by mesh. Applying (15.26)  $J$  times gives the factor  $B^{-2}$  for  $T_M^J$ .

Every reconstruction map satisfies the exact averaging identity, since its gauge transformation is the identity at the coarse vertices. At zero fluctuation this reads  $T_M^J e^{\hat{\Phi}_{d,\Lambda}(\theta, 0)} = e^\theta$ , and its derivative at  $\theta = 0$  gives  $D(T_M^J(I))\hat{L}_{d,\Lambda}\theta = \theta$ . With the coarse mesh  $H$  and the fine mesh  $a = H/B$ , (15.26) yields, for each orientation and color,

$$a^2 \sum_x (\hat{L}_{d,\Lambda}\theta)_i(x) = H^2 \sum_u \theta_i(u). \quad (15.27)$$

If  $\theta = Hv$  for a coarse potential  $v$  in physical units, the two sides are exactly the integrals of the fine linearized reconstruction  $a^{-1}\hat{L}_{d,\Lambda}\theta$  and of the piecewise constant embedding of  $v$ . Taking  $v$  supported on one oriented coarse link and one color proves (4).

**Step 5: preservation of the conditional measures.** At a fixed coarse field, the reconstruction map is a fixed fine gauge transformation applied after the unchanged reconstruction  $\mathcal{C}_{\text{rawd},\Lambda}$ . It fixes the coarse vertices, preserves the product Haar measure and the gauge orbit of the classical background field, and changes neither the fluctuation variables nor the cutoff domain. The Haar disintegration density, the triangular determinant, and the fluctuation exponent of Theorem 12.13 are those of  $\mathcal{C}_{\text{rawd},\Lambda}$ . The input effective interaction takes the same value at the two representatives, by gauge invariance within  $B_\rho$ , since both representatives lie in its domain. Thus the fiber and its normalizer are unchanged, and so is the conditional fluctuation kernel. This proves (5).  $\square$

## 15.6 Convergence of finite-scale reconstructions

We now use the reconstruction maps of Theorem 15.13 in every hierarchy. For the RG step from level  $j$  to level  $j - 1$  let  $\Lambda_j$  be its lattice pair, and use the preceding depth  $d_j^k = J(k - j)$ . With the coarse logarithm  $\theta_0$  and the fluctuation variables  $w_1, \dots, w_l$  of Lemma 15.10, the finite-scale reconstruction is defined by

$$F_0^k = \theta_0, \quad F_j^k = \widehat{\Phi}_{d_j^k, \Lambda_j}(F_{j-1}^k, (g_{j,i} w_{j,i})_i) \quad (1 \leq j \leq l).$$

**Theorem 15.14.** *Use the reconstruction maps of Theorem 15.13 at the preceding depths  $d_j^k = J(k - j)$  in every hierarchy.*

1. *For every finite  $k$ , the replacement of  $\Omega_d$  by  $\widehat{\Omega}_d$  is a gauge transformation that depends only on the coarse field and equals the identity at the coarse vertices, applied after the same reconstruction  $\mathcal{C}_{\text{rawd}}$ . At each coarse field in  $B_r$  it preserves the small-field event, every ball of the cutoff domain, the coordinates of the reconstruction, the effective classical action, the Haar disintegration density, the triangular determinant, the gauge-invariant extension of the input effective interaction, the normalizer, and the conditional fluctuation kernel. It preserves the exact averaging identities at every level, and the gauge invariance within  $B_\rho$  after the triangular change of fluctuation coordinates that accompanies a change of the coarse representative.*
2. *Fix  $l < \infty$ ,  $L, \gamma > 0$ , and the Borel map  $S_0$  of Lemma 15.10. As  $k \rightarrow \infty$ , the joint law  $P_l^k$  of  $(\theta_0, w_1, \dots, w_l)$  converges in total variation. The law of the finite-scale reconstruction  $(F_0^k, \dots, F_l^k)$  converges weakly and in bounded-Lipschitz distance. Since these reconstructions have a common compact range, it also converges in  $W_1$  for any norm on the finite-dimensional space of logarithmic fields. The limiting laws are compatible: they are the finite iterations of one limiting coarse-level law, the kernels  $k_j^\infty$ , and the maps  $\Phi_{\infty, \Lambda_j}$ , and the limit for  $l$  is a marginal of the limit for  $l + 1$ . All these objects are Borel.*

No total variation statement is made for the laws of the reconstructed fields. The constants and moduli have the dependence of Theorem 15.6 at fixed  $l$ . The construction requires neither a limit of fields in a space of distributions nor a modulus that is uniform in  $l$ .

*Remark 15.15.* The restriction to weak convergence for the reconstructed fields is necessary. The reconstruction maps depend on  $k$ , and pushforwards by maps that move with  $k$  do not preserve total variation convergence. For example, if  $x_k \rightarrow x$  with  $x_k \neq x$ , then  $\delta_{x_k} \rightarrow \delta_x$  weakly, while  $\|\delta_{x_k} - \delta_x\|_{\text{TV}} = 1$ . The coordinates  $(\theta_0, w_1, \dots, w_l)$ , on the other hand, live on fixed products, and their total variation convergence holds.

*Proof.* At a fixed coarse field, the replacement changes no coordinate of the reconstruction, no fluctuation variable, no ball, and no Jacobian. The indicators of  $D_r$  are gauge invariant, and the new representative satisfies the strict domain conditions of Theorem 15.13. Gauge covariance preserves the averaging identity, and gauge invariance within  $B_\rho$  preserves the effective classical action and the input effective interaction. By Theorem 14.6, every factor of the fiber, the normalizer, and the conditional fluctuation kernel are therefore preserved.

For a change of the coarse representative, we write  $C' = C^{\tilde{h}}$  for the reconstructions, after the triangular change of fluctuation coordinates. The convention (4.1) gives

$$(C')^{\hat{\Omega}(c')} = (C^{\hat{\Omega}(c)})^{\hat{\Omega}(c')\tilde{h}\hat{\Omega}(c)^{-1}}. \quad (15.28)$$

The change of coordinates preserves every ball and has absolute Jacobian one, by Theorem 13.4(3), on the comparison domains of the enlarged-domain reconstruction hypothesis. Both representatives lie in the domain of this interaction, so its values at them agree. This proves the gauge invariance within  $B_\rho$  without any equivariance of the limiting gauge transformation under local gauge transformations, and completes the proof of (1).

By Lemma 15.12,  $\eta_0^k \rightarrow \eta_0^\infty$  in total variation. Fix  $l < \infty$  and define

$$X_l = \{\|\theta_0\|_\infty \leq r\} \times \prod_{j=1}^l \Omega_{p_j}.$$

The domain of each reconstruction map strictly contains its closed product. By Lemma 15.10 and Theorem 15.13, the reconstructions  $F_j^k$  are defined on  $X_l$  and have a common compact range. For  $k$  large, each of the finitely many depths  $d_j^k$ ,  $j \leq l$ , exceeds its threshold  $d_0(\Lambda_j)$ . The gauge transformation is then the fixed limiting one, and Theorem 15.13(1) implies uniform convergence of each map. Let  $u_{j,k} = \|F_j^k - F_j^\infty\|_\infty$ , let  $v_{j,k}$  be the supremum distance between  $\hat{\Phi}_{d_j^k, \Lambda_j}$  and  $\Phi_{\infty, \Lambda_j}$  on the smaller product domain, and let  $\omega_{\Phi_j}$  be the modulus of continuity of  $\Phi_{\infty, \Lambda_j}$  in its first argument there. Adding and subtracting  $\Phi_{\infty, \Lambda_j}(F_{j-1}^k, \cdot)$ , we obtain

$$u_{j,k} \leq v_{j,k} + \omega_{\Phi_j}(u_{j-1,k}), \quad u_{0,k} = 0, \quad v_{j,k} \rightarrow 0. \quad (15.29)$$

By induction on  $j$ ,  $F_j^k \rightarrow F_j^\infty$  uniformly on  $X_l$  for every  $j \leq l$ .

By Theorem 15.9, the limiting density is continuous on the compact product of fields and fluctuation variables, and its continuous positive normalizer has a positive minimum. Hence the limiting kernel is uniformly continuous in the field, in  $L^1(dw)$  and therefore in total variation; let  $\omega_{k_j}$  be its modulus. For points  $h = (\theta_0, w_1, \dots, w_{j-1})$  of the coordinate space, define

$$K_j^k(h, dw) = k_j^k(F_{j-1}^k(h), dw), \quad K_j^\infty(h, dw) = k_j^\infty(F_{j-1}^\infty(h), dw).$$

The two kernels are evaluated at different reconstructed fields, so the comparison contains a term for this discrepancy. By Theorem 15.9,

$$\sup_h \|K_j^k(h, \cdot) - K_j^\infty(h, \cdot)\|_{\text{TV}} \leq e^{2\Delta_{j,k}} - 1 + \omega_{k_j}(u_{j-1,k}) \longrightarrow 0. \quad (15.30)$$

Every term of the conditional fluctuation potential is kept here, including the input effective interaction, the Haar disintegration terms, and the differences of the determinant terms.

By Lemma 15.10,  $P_l^k = \eta_0^k K_1^k \cdots K_l^k$  is the law of the coordinates. We define  $P_l^\infty = \eta_0^\infty K_1^\infty \cdots K_l^\infty$ . Lemma 15.1(2) implies that

$$\|P_l^k - P_l^\infty\|_{\text{TV}} \leq \|\eta_0^k - \eta_0^\infty\|_{\text{TV}} + \sum_{j=1}^l \sup_h \|K_j^k(h, \cdot) - K_j^\infty(h, \cdot)\|_{\text{TV}} \longrightarrow 0. \quad (15.31)$$

We write  $F^k = (F_0^k, \dots, F_l^k)$ . The ranges lie in a common compact subset of the finite-dimensional space of logarithmic fields, of diameter  $D_l$ . For every continuous  $f$  on this set, Lemma 15.1(3) gives

$$\left| \int f(F^k) dP_l^k - \int f(F^\infty) dP_l^\infty \right| \leq 2\|f\|_\infty \|P_l^k - P_l^\infty\|_{\text{TV}} + \omega_f(\|F^k - F^\infty\|_\infty), \quad (15.32)$$

which proves weak and bounded-Lipschitz convergence. For a fixed norm, Lemma 15.1(3) also gives

$$W_1((F^k)_* P_l^k, (F^\infty)_* P_l^\infty) \leq \|F^k - F^\infty\|_\infty + D_l \|P_l^k - P_l^\infty\|_{\text{TV}} \longrightarrow 0.$$

The limiting coarse-level law is a Borel pushforward, and the limiting maps and kernels are continuous on their product domains. Integrating the last probability kernel to one proves the compatibility. Only finitely many lattices, thresholds, and moduli were used.  $\square$

## 15.7 Deterministic amplitude bounds for the reconstruction

The bounds of Lemma 15.10 assume a linear bound on the reconstruction maps. We now derive such bounds, in a nonlinear form, from the joint coefficient bounds of Theorem 15.13. The constants are uniform in the mesh of the coarse lattice and in the coupling; this uniformity is used in Sections 17, 23, 31, and 34.

**Lemma 15.16.** *For the reconstruction maps of Theorem 15.13 write  $\widehat{P}_{d,\Lambda}(c) = \widehat{\Phi}_{d,\Lambda}(c, 0)$  and  $\widehat{L}_{d,\Lambda} = D\widehat{P}_{d,\Lambda}(0)$ . Fix the larger joint domains and one common joint coefficient bound, with all row sums and volume-normalized column sums of Definition 4.6, using the coarse/input units for network lengths specified in Theorem 15.13(3), before the working radius and the radius  $r$  are chosen. Let  $a_m$  be the larger of the row and column norms of the homogeneous part of degree  $m$  of  $\widehat{P}_{d,\Lambda}$ . There exist  $A_J, R_J > 0$ , uniform in  $\Lambda$  and  $d$ , such that*

$$a_1 \leq \alpha_* = C_{CP}/B, \quad a_m \leq A_J R_J^{-m} \quad (m \geq 2).$$

After a fixed reduction of the radius, which absorbs the degree factors produced by fixing an input, choose

$$\beta_J \geq \max\{1, R_J^{-1}, A_J/(\alpha_* R_J^2)\}.$$

Then, uniformly in  $\Lambda$  and  $d$ ,

$$\widehat{P} \prec \varphi(z) = \frac{\alpha_* z}{1 - \beta_J z}, \quad \widehat{P} - \widehat{L} \prec \frac{\alpha_* \beta_J z^2}{1 - \beta_J z}.$$

Here  $\prec$  means that each  $a_m$  (with  $m \geq 2$  in the second relation) is at most the coefficient of  $z^m$  of the series on the right, and that a fixed input is bounded by the corresponding derivative of the same series. The full joint series in the coarse field and the inputs, not only these coefficients, has a common domain depending on  $J$ , with its column sums, and the bounds (3) and the identity (4) of Theorem 15.13 hold at the same time.

Choose  $J$ , using the constant  $C_{CP}$ , so that  $B\alpha_*^2 = q_{\text{anc}} \leq 1/4$ , and only afterwards choose

$$0 < r_{\text{eff}} < \min\{\rho, r_\theta\}, \quad \frac{\beta_J r_{\text{eff}}}{1 - \alpha_*} < \frac{1}{8}, \quad 0 < r < r_{\text{eff}}/4.$$

On the working ball  $\{\|\theta\|_\infty \leq r_{\text{eff}}\}$ ,

$$\|D\widehat{P}\| \leq a_{\text{map}} = \frac{\alpha_*}{(1 - \beta_J r_{\text{eff}})^2} \leq \frac{64}{49} \alpha_*, \quad a_{\text{map}} \sqrt{B} \leq \frac{32}{49} < 1.$$

The bounds on the inputs and the coupling ceiling can then be chosen so that every image of the reconstruction, and every repeated zero-fluctuation reconstruction, lies in the working domain. Increasing the thresholds  $d_0(\Lambda)$  to obtain these smaller image margins changes none of  $C_{CP}$ ,  $A_J$ ,  $R_J$ ,  $\beta_J$ , or the joint bound. These constants have the dependence described in item (2) of Definition 4.14; they are independent of  $r$ , of the thresholds, of the coupling, of the side, of the depth, and of the lattice offset, and among them only  $C_{CP}$  is independent of  $J$ .

The same joint bounds give the following amplitude bound. Let  $H > 0$ ,  $a_j = HB^{-j}$ ,  $t_j = c_B g_3 \sqrt{a_j}$  with  $c_B = M^{(J-1)/2}$ , and  $p_j = A_w \log(1/t_j) \geq 1$ . Fix the free-input radius  $v_0 > 0$  used in (15.34), whose doubled product lies in the joint domain. Consider any finite chain  $\theta_j = \widehat{\Phi}_{d_j, \Lambda_j}(\theta_{j-1}, q_j)$ ,  $1 \leq j \leq k$ , of logarithmic representations of reconstruction maps in the

working ball, with  $b_0 = \|\theta_0\|_\infty \leq r$  and inputs  $\max_i \|q_{j,i}\|_\infty \leq t_j p_j \leq v_0$ , and put  $b_j = \|\theta_j\|_\infty$ . There exists a constant  $C_{\text{free},J}$ , determined by the joint bounds, such that

$$b_j \leq \frac{\alpha_* b_{j-1}}{1 - \beta_J b_{j-1}} + C_{\text{free},J} t_j p_j.$$

Choose a coupling ceiling  $\tau$ , after the working radii, so that these chains stay in the working ball. Then, whenever  $t_1 \leq \tau$ , uniformly in  $H$ ,  $g_3$ ,  $k$ , the preceding depths, the sides, and the lattice offset,

$$b_j \leq C_{\text{cap},J} r \alpha_*^j + C_{\text{cap},J} g_3 \sqrt{a_j} p_j, \quad \sum_{j=1}^k b_{j-1} \leq C_{\text{cap},J} \left[ \frac{r}{1 - \alpha_*} + g_3 \sqrt{H} (p_1 + 1) \right].$$

The constant  $C_{\text{cap},J}$  may depend on the joint bounds,  $A_w$ ,  $M$ ,  $J$ ,  $r_{\text{eff}}$ ,  $\beta_J$ , and  $\tau$ , but not on  $H$  or on the coupling. These bounds hold on every sample; they are not moment bounds.

*Proof.* We first build the majorant  $\varphi$  from the joint bounds and choose the radii. We then prove the amplitude bounds in two passes. The first pass uses the ratio  $a_{\text{map}}$  and bounds the sum of the amplitudes uniformly; the second uses this sum to recover the smaller ratio  $\alpha_*$ , which is what makes the forcing summable.

**Step 1: the majorant.** We fix the common joint coefficient bound of Theorem 15.13 on a larger product domain, before any working radius is chosen, and restrict it to  $q = 0$ . If its coarse-field radius is  $R_0$  and its bound is  $A_0$ , then the norm of degree  $m$  is at most  $A_0 R_0^{-m}$ . The row and every volume-normalized column use the same bound. Fixing an input produces at most a degree factor  $m$ , which is absorbed by replacing  $R_0$  with  $R_0/2$ , since  $m2^{-m}$  is bounded; a fixed finite list of such factors is absorbed by a further fixed reduction. This gives  $A_J, R_J > 0$  as in the statement for every  $m \geq 2$ . The linear coefficient has the separate bound  $a_1 \leq C_{\text{CP}}/B = \alpha_*$  from Theorem 15.13(3). Only this last bound is needed uniformly before  $J$  is fixed.

We choose  $\beta_J$  at least the displayed maximum; it may be increased to cover the inverse radius of the fixed joint product as well. For  $m \geq 2$ ,

$$\alpha_* \beta_J^{m-1} \geq \frac{A_J}{R_J^2} \beta_J^{m-2} = A_J R_J^{-m} (\beta_J R_J)^{m-2} \geq A_J R_J^{-m}. \quad (15.33)$$

These are the coefficients of  $\varphi(z) = \alpha_* z / (1 - \beta_J z)$ , whose first coefficient is treated separately. Subtracting the linear term gives the second majorant. In a nonnegative coefficient series, a derivative with respect to a fixed input marks that input and differentiates the same series. The same calculation therefore proves the statement for fixed inputs, with the volume normalization of the column of that input. The full joint series and the identities of Theorem 15.13 are those already proved there; restricting to  $q = 0$  does not replace them.

**Step 2: the working radii.** We first choose  $J$  with  $C_{\text{CP}}^2/B \leq 1/4$ . In particular  $\alpha_* < 1$  and  $\alpha_*\sqrt{B} \leq 1/2$ . All constants defining  $\beta_J$  are now fixed. We choose  $r_{\text{eff}}$  as small as necessary inside the coarse-field radius of the common product domain, with the displayed inequality, and then  $0 < r < r_{\text{eff}}/4$ . Such choices exist since  $\beta_J$  is finite. Differentiating the nonnegative series, we have on this working ball

$$\|D\hat{P}\| \leq \varphi'(r_{\text{eff}}) = \frac{\alpha_*}{(1 - \beta_J r_{\text{eff}})^2}.$$

The working inequality implies  $\beta_J r_{\text{eff}} < (1 - \alpha_*)/8 < 1/8$ . The last display is therefore at most  $(64/49)\alpha_*$ , and its product with  $\sqrt{B}$  is at most  $32/49$ . Since  $\hat{P}(0) = 0$ , the zero-fluctuation image of the working ball lies in the strictly smaller ball of radius  $a_{\text{map}} r_{\text{eff}}$ .

For the remaining choices, we note that the common joint series gives a continuous nonnegative majorant  $G_J(u, v)$ , with  $G_J(u, 0) = 0$ , for  $\hat{\Phi}(\theta, q) - \hat{P}(\theta)$  on the working ball  $\|\theta\|_\infty \leq r_{\text{eff}}$ . We choose a fixed  $v_* > 0$  so small that

$$G_J(r_{\text{eff}}, v_*) < (1 - a_{\text{map}})r_{\text{eff}}/4,$$

and, after a further reduction, the image condition into  $B_r$  and every intermediate domain margin hold. For example, at a coarse field in  $B_r$  the zero-fluctuation reconstruction is at most  $a_{\text{map}}r < r/2$ , and we choose the bound on the inputs so that their contribution is less than  $r/2 - a_{\text{map}}r$ . This number is positive, since  $a_{\text{map}} \leq 32/(49\sqrt{B}) < 1/2$  for  $B \geq 3$ . The coupling ceiling is then reduced so that  $tp(t)$  is below that bound for every smaller coupling. These choices of fixed domains and ceilings do not depend on the coupling.

If necessary, we increase each threshold  $d_0(\Lambda)$  to impose these more restrictive image conditions, by the same convergence in the fixed gauge. For depths below the enlarged threshold we use the unmodified family, whose bounds satisfy the chosen requirements. For depths above it we keep the prescribed tolerances for the joint norm and for the quantities in Theorem 15.13(3), and add only these image requirements. Thus increasing a threshold changes none of the earlier coefficient constants. The order of the choices is the joint bound, then  $\beta_J$ , then  $r_{\text{eff}}$  and  $r$ , and then the bounds on the inputs and the coupling ceiling, as in (15.22). This proves the first part of the lemma.

**Step 3: a uniform bound for the inputs and a first recursion.** All products in the rest of the proof are closed subproducts of the common product of Lemma 15.4; in particular their sums of radii are less than  $r_J$  and the denominators of the majorants are positive. Fix  $v_0 > 0$  such that the product with coarse-field radius  $r_{\text{eff}}$  and input radius  $2v_0$  lies in the common joint coefficient domain, with a margin. Differentiating the nonnegative joint series in one input, and using the factor two in the radius, we find that every output-fixed absolute row sum of the derivative with respect to all fluctuation inputs is bounded by

one constant  $C_{\text{free},J} < \infty$ , uniformly in the lattice pair, the preceding depth, and the lattice offset. Integrating  $D_q \widehat{\Phi}(\theta, sq)q$  over  $0 \leq s \leq 1$  and using its absolute row sum, we obtain

$$\|\widehat{\Phi}_{d,\Lambda}(\theta, q) - \widehat{\Phi}_{d,\Lambda}(\theta, 0)\|_\infty \leq C_{\text{free},J} \max_i \|q_i\|_\infty, \quad \|\theta\|_\infty \leq r_{\text{eff}}, \quad \max_i \|q_i\|_\infty \leq v_0. \quad (15.34)$$

The number of output links is never summed in this estimate. The inverse square roots of the constrained Hessians remain inside  $\widehat{\Phi}$ , whose inputs are exactly  $q_i = g_i w_i$ . Combining (15.34) with the majorant of  $\widehat{P}$ , which gives  $\|\widehat{P}(\theta)\|_\infty \leq \varphi(\|\theta\|_\infty)$ , we obtain

$$b_j \leq \varphi(b_{j-1}) + f_j, \quad f_j = C_{\text{free},J} t_j p_j, \quad \varphi(b) = \frac{\alpha_* b}{1 - \beta_j b}. \quad (15.35)$$

Let  $a = a_{\text{map}}$ ,  $x = B^{-1/2}$ , and  $\kappa_w = A_w \log B/2$ . We choose the common ceiling  $\tau > 0$ , after the working radii, so that

$$\tau \leq e^{-1}, \quad A_w \log(1/\tau) \geq 1, \quad \tau p(\tau) \leq v_0, \quad C_{\text{free},J} \tau p(\tau) \leq (1-a)r_{\text{eff}}/4. \quad (15.36)$$

We also keep every previously imposed image and domain condition, by decreasing  $\tau$  if necessary. For  $0 < t \leq \tau$  both  $tp(t)$  and  $t(p(t) + 1)$  are increasing functions of  $t$ , since their derivatives are  $A_w(\log(1/t) - 1)$  and  $A_w(\log(1/t) - 1) + 1$ . Thus these conditions hold for every  $t_j$  whenever  $t_1 \leq \tau$ . Since  $\varphi(b) \leq ab$  on the working ball, (15.35) implies

$$b_j \leq ab_{j-1} + f_j.$$

It also keeps the chain in the working ball. Starting from  $b_0 \leq r < r_{\text{eff}}/4$ , if  $b_{j-1} \leq r_{\text{eff}}/2$ , then (15.36) gives  $b_j \leq ar_{\text{eff}}/2 + (1-a)r_{\text{eff}}/4 < r_{\text{eff}}/2$ . By induction, every evaluation in (15.35) takes place in the same working ball.

The identities  $t_j = t_1 x^{j-1}$  and  $p_j = p_1 + \kappa_w(j-1)$  give

$$\sum_{j \geq 1} t_j p_j = t_1 \left\{ \frac{p_1}{1-x} + \frac{\kappa_w x}{(1-x)^2} \right\} \leq D_J t_1 (p_1 + 1), \quad D_J = \frac{1}{1-x} + \frac{\kappa_w x}{(1-x)^2}. \quad (15.37)$$

For a finite chain, the infinite sum is an upper bound for the forcing. Iterating the linear recursion and summing its nonnegative terms, we obtain

$$\sum_{h=0}^{k-1} b_h \leq \frac{r + C_{\text{free},J} D_J t_1 (p_1 + 1)}{1-a} \leq S_* = \frac{r_{\text{eff}}/4 + C_{\text{free},J} D_J \tau (p(\tau) + 1)}{1-a}. \quad (15.38)$$

This bound is independent of  $H$ ,  $g_3$ ,  $k$ , the lattice pairs, and their preceding depths. In particular  $S_*$  contains neither  $p_1$  nor  $H$  as a varying parameter.

**Step 4: the sharper ratio.** Set

$$K_\beta = \frac{1}{1 - \beta_J r_{\text{eff}}} \leq \frac{8}{7}, \quad \zeta_h = \frac{\alpha_*}{1 - \beta_J b_h}, \quad C_{\text{prod}} = \exp(K_\beta \beta_J S_*).$$

For  $0 \leq y \leq \beta_J r_{\text{eff}}$  we have  $-\log(1 - y) = \int_0^y (1 - s)^{-1} ds \leq K_\beta y$ . Since  $b_h \leq r_{\text{eff}}/2$ , every finite product satisfies

$$\prod_{h=u}^v \zeta_h \leq \alpha_*^{v-u+1} \exp\left(K_\beta \beta_J \sum_{h=u}^v b_h\right) \leq C_{\text{prod}} \alpha_*^{v-u+1}, \quad (15.39)$$

where we used (15.38); for an empty product we use  $1 \leq C_{\text{prod}}$ . Since  $\varphi(b_h) = \zeta_h b_h$ , successive substitution in  $b_j \leq \zeta_{j-1} b_{j-1} + f_j$  gives

$$\begin{aligned} b_j &\leq b_0 \prod_{h=0}^{j-1} \zeta_h + \sum_{i=1}^j f_i \prod_{h=i}^{j-1} \zeta_h \leq C_{\text{prod}} r \alpha_*^j + C_{\text{prod}} C_{\text{free},J} \sum_{i=1}^j \alpha_*^{j-i} t_i p_i \\ &\leq C_{\text{prod}} r \alpha_*^j + \frac{C_{\text{prod}} C_{\text{free},J}}{1 - \alpha_* \sqrt{B}} t_j p_j. \end{aligned} \quad (15.40)$$

In the last inequality we used  $t_i = t_j B^{(j-i)/2}$ ,  $p_i \leq p_j$ , and  $\alpha_* \sqrt{B} \leq 1/2$ . Substituting  $t_j = c_B g_3 \sqrt{a_j}$ , we obtain the pointwise bound of the statement, with the ratio  $\alpha_*$  rather than  $a$ .

Summing (15.40) over  $h = 1, \dots, k-1$ , treating  $b_0 \leq r$  separately, and using (15.37), we obtain

$$\begin{aligned} \sum_{h=0}^{k-1} b_h &\leq \frac{C_{\text{prod}} r}{1 - \alpha_*} + \frac{C_{\text{prod}} C_{\text{free},J} D_J}{1 - \alpha_* \sqrt{B}} t_1 (p_1 + 1) \\ &= \frac{C_{\text{prod}} r}{1 - \alpha_*} + \frac{C_{\text{prod}} C_{\text{free},J} D_J c_{B^X}}{1 - \alpha_* \sqrt{B}} g_3 \sqrt{H} (p_1 + 1). \end{aligned} \quad (15.41)$$

For the last two displays, we may take any

$$C_{\text{cap},J} \geq C_{\text{prod}} \max\left\{1, \frac{C_{\text{free},J} c_B}{1 - \alpha_* \sqrt{B}}, \frac{C_{\text{free},J} D_J c_{B^X}}{1 - \alpha_* \sqrt{B}}\right\}. \quad (15.42)$$

It may also be enlarged to cover the constant  $C_{\text{cap},J}$  of Lemma 15.10. This constant depends only on the joint bounds, the working radii, and the ceiling. Thus it is uniform in  $H$  and  $g_3$  subject to the single condition  $t_1 \leq \tau$ , and in the length of the chain, the sides, the lattice offset, and the preceding depths. The constant  $C_{\text{CP}}$  is not enlarged in this proof.  $\square$

## 15.8 Uniformity on compact intervals of couplings

In this section the coupling varies in a compact interval  $I = [\gamma_-, \gamma_+]$  with  $0 < \gamma_- \leq \gamma_+ < \infty$ . The lattice of level  $j$  has side  $n_j = nB^j$  for a fixed coarse side  $n \in 4\mathbb{N}$ , and  $a_j = (L/n)B^{-j}$ ,  $g_j^2 = \gamma/n_j$ ,  $t_j = M^{(J-1)/2}g_j$ , and  $d_j^k = J(k-j)$ . On a fixed lattice pair  $\Lambda$  the largest coupling of the intermediate scales is  $t = c_\Lambda \sqrt{\gamma}$  for a constant  $c_\Lambda > 0$ , with the fixed ratios  $\lambda_i$ . Constants with a subscript  $I$  may depend on both endpoints of  $I$ , on  $n, L, M, J$ , the fixed parameter collection  $\Theta$  of Theorem 23.2, the analytic parameters and exponents, moment orders, and fixed lists of times and observables; constants for a fixed lattice pair may also depend on that pair. None of them depends on the value of  $\gamma \in I$ , on the preceding depth, or on a finer mesh. These are compact-coupling constants with the explicit dependences just listed, not constants chosen before  $J$ . Since the cutoff radius  $p(t_j)$  depends on  $\gamma$ , we use the rescaled fluctuation variables  $v_j = w_j/p(t_j)$ , which live on the fixed product  $\Omega_1$  of unit balls.

**Theorem 15.17.** *Fix the parameters and products of Theorem 15.6, with its coupling ceiling  $g_*$ , a coarse side  $n \in 4\mathbb{N}$ , and  $I = [\gamma_-, \gamma_+]$  with  $0 < \gamma_- \leq \gamma_+ < \infty$ . Assume that the image, product, and coupling conditions of Theorem 15.6 hold for every  $\gamma \in I$ . Then the following hold.*

1. *On every fixed lattice pair  $\Lambda$ , at the same radius and exponent,*

$$\lim_{d, d' \rightarrow \infty} \sup_{\gamma \in I} \sup_{R \in \mathcal{A}_E} \mathcal{B}_{\rho, a_{\text{sp}}}(\widehat{\mathcal{M}}_{d,t}R - \widehat{\mathcal{M}}_{d',t}R) = 0. \quad (15.43)$$

*The contraction factor  $\vartheta < 1$  of Theorem 15.6 is the same for all  $\gamma \in I$ . At each finite  $d$  the transformation is jointly continuous in  $(\gamma, R)$  in this norm. After the substitution  $w = p(t)v$ , the classical data and the reconstruction maps converge uniformly in  $\gamma \in I$  on every fixed smaller product domain, with every fixed derivative and every fixed spatial difference.*

2. *For every fixed  $j$ , the effective interactions  $R_j^k$  and the stopped measures  $\widehat{v}_j^k$  converge uniformly in  $\gamma \in I$ , in  $\mathcal{B}_{\rho, a_{\text{sp}}}$  and in total variation respectively, and the limits are continuous in  $\gamma$  in these norms.*
3. *For every fixed  $l$ , the joint law of the coarse logarithm and of the rescaled fluctuation variables  $(v_1, \dots, v_l)$  converges in total variation, uniformly in  $\gamma \in I$ . The laws of the finite-scale reconstructions converge in bounded-Lipschitz distance and in  $W_1$ , for any norm on the finite-dimensional space of logarithmic fields, uniformly in  $\gamma \in I$ . The limiting laws are continuous in  $\gamma$  in these distances, and they are compatible.*

*All moduli have the dependence described above, at fixed  $l$ . No rate that is uniform in  $l$ , and no statement as  $\gamma \rightarrow 0$ , is asserted.*

The hypothesis holds in particular when  $I \subset (0, \gamma_0]$  and the hierarchy satisfies the hypotheses of Theorem 15.6 for every  $\gamma \leq \gamma_0$ . The theorem does not by itself guarantee that these hypotheses hold for a given coarse side and an arbitrary upper endpoint  $\gamma_+$ .

*Proof.* The proof follows the proofs of Theorems 15.6, 15.9, and 15.14. The new point is that every estimate is proved uniformly in  $\gamma \in I$  on fixed products; uniform convergence is never deduced from pointwise convergence and compactness of  $I$ .

**Step 1: rescaling.** On a fixed lattice pair, let  $t = c_\Lambda \sqrt{\gamma}$ ,  $t_\pm = c_\Lambda \sqrt{\gamma_\pm}$ ,  $p_- = p(t_+) > 0$ , and  $p_+ = p(t_-) < \infty$ . Let  $D$  be the finite number of scalar fluctuation variables, and we change variables  $w = p(t)v$ ,  $v \in \Omega_1$ . The Jacobian  $p(t)^D$  cancels from normalized integrals and from logarithms normalized at the identity. For the Gaussian reference the remaining normalizer is

$$Z_p^{\text{unit}} = \int_{\Omega_1} e^{-p^2|v|^2/2} dv, \quad |\Omega_1| e^{-p_+^2 \sup_{\Omega_1} |v|^2/2} \leq Z_p^{\text{unit}} \leq |\Omega_1|. \quad (15.44)$$

The scalar factor  $z_p^{D/3}$  of (15.8) satisfies the exact relation  $z_p^{D/3} = p^D Z_p^{\text{unit}}$ , so no normalization is discarded. Every input is  $q_i = \lambda_i t p(t) v_i$ , with fixed  $\lambda_i > 0$  and  $t \in [t_-, t_+]$ . By Lemma 15.4 and the hypothesis, these inputs lie in one closed subproduct of the larger analytic product domain, and we also use its complex enlargement in the coarse field and a slightly larger input product. Their margins were chosen before the coupling.

**Step 2: classical data.** In Lemma 12.3, the conditional constrained minimizer minimizes  $E_d \circ b$ , the constrained Hessian is its second derivative in the free variables, and its square root is the analytic branch that is positive at real arguments. These objects do not depend on  $\gamma$ ; the coupling enters only through  $q_i = g_i w_i$  and the explicit divisors  $g_i^2$  of the action. The finitely many ratios of couplings are fixed powers of  $M$ . By Theorem 11.2, the minimizers, the constrained Hessians, their square roots and inverses, the Haar disintegration densities, and the reconstructions converge with every fixed derivative on this one product of inputs. The formulas of Theorem 12.13 are finite compositions of these data, with factors  $g_i^{-1} \leq (\lambda_i t_-)^{-1}$  and bounded  $p$  and  $p^{-1}$ . Thus their substitutions converge uniformly in  $\gamma \in I$ , including every fixed derivative in the coarse field of the classical and Haar disintegration exponent, with all dependences of the determinant terms on coarser fluctuation variables. This proves the last assertion of (1).

**Step 3: the input effective interaction.** We use the same gauge transformation  $\Omega_d$  of Theorem 13.2 on the two reconstructions being compared; its derivatives in the coarse field and its inverse are bounded on the common product domain. One image lies in  $B_{\rho/2}$ , and the uniform convergence of the reconstructions puts the other in  $B_{3\rho/4}$  for all sufficiently large pairs of depths, uniformly over the product of Step 1 and over  $\gamma \in I$ . By Lemma 15.5, every  $R \in \mathcal{A}_E$  satisfies

$$\sup_{B_{3\rho/4}} |D^s R| \leq C_{s,\Lambda} \rho^{2-s} E. \quad (15.45)$$

Using the chain rule and the integral of  $DR$  along the segment between the two images, we find that every fixed derivative of the term with the input effective interaction converges,

uniformly in  $R$  and in  $\gamma$ . Gauge invariance within  $B_\rho$  identifies these evaluations with the values of  $R_{\text{sat}}$ , and the coordinatewise identity argument of Step 4 in the proof of Theorem 15.6 identifies the holomorphic extensions. No convergence of  $\Omega_d$  is needed.

**Step 4: the partition functions.** The integrands on  $\Omega_1$  are now uniformly Cauchy, together with each fixed derivative in the coarse field. On the finite set  $\mathcal{Q}_\Lambda$  of boxes of the lattice, let  $\mathbf{U}_{d,t,R}$  be the local interaction decomposition of Lemma 15.8, after the zero-fluctuation terms and the Gaussian factor have been removed. Its reference measure is the product probability

$$d\pi_p^{\text{unit}}(v) = (Z_p^{\text{unit}})^{-1} e^{-p^2|v|^2/2} \mathbf{1}_{\Omega_1}(v) dv.$$

The estimates of Lemma 15.8 for the supports, the support-counting factors, and the radii give  $\|\mathbf{U}_{d,t,R}\|_{\mathcal{S}_8} \leq \eta_0/4$  at coarse-field radius  $\rho$ , uniformly in  $\gamma \in I$ ,  $d$ , and  $R$ . These smallness conditions hold throughout  $I$  by hypothesis; the bounds involve only the fixed domains, their radius ratios, and bounded  $p$ ,  $t$ ,  $t^{-1}$ , and  $E$ . Theorem 12.7 applies to this family of coefficients in the coarse field, allows scalar coefficients, and has constants independent of the reference probability and of the number of fluctuation variables. It gives a family  $\mathcal{L}$  with

$$-\log \Xi_{d,t,R} = \sum_Y \mathcal{L}_Y, \quad \|\mathcal{L}\|_{\mathcal{S}_2} \leq C \|\mathbf{U}_{d,t,R}\|_{\mathcal{S}_8}.$$

Every support is nonempty, so on the complex product

$$\left| \sum_Y \mathcal{L}_Y \right| \leq \sum_Y \sum_v \rho^{|v|} |\mathcal{L}_{Y,v}| \leq |\mathcal{Q}_\Lambda| \|\mathcal{L}\|_{\mathcal{S}_2} \leq C |\mathcal{Q}_\Lambda| \eta_0/2.$$

The branch of the logarithm vanishes at zero interaction and agrees with the real logarithm on real data. This proves that the normalized complex partition functions do not vanish and satisfy

$$|\log \Xi_{d,t,R}| \leq C_{\Lambda,I,E}, \quad |\Xi_{d,t,R}| \geq e^{-C_{\Lambda,I,E}}. \quad (15.46)$$

Comparing the ratio of two close partition functions on a disc about one, we find that their logarithms normalized at the identity, and their fixed coefficient tensors, are uniformly Cauchy. The uniform lower bound (15.46) is not inferred from positivity of the normalizers at real arguments.

**Step 5: the transformations.** The common absolute coefficient majorants used for (15.17) are uniform also in  $\gamma \in I$ : the rescaled inputs have fixed product margins,  $t$  and  $p(t)$  range over compact intervals, and every denominator has one common positive margin. A common convergent majorant controls the increase of the supported norm when the coarse-field radius increases from  $\rho$  to  $u$ . At  $u = \rho$  the actual-coupling reserve gives  $\|\mathbf{H} + \mathbf{A}_R\|_{\mathcal{S}_8} \leq \eta_0/4$  uniformly in  $d, R, \gamma$ . Choose one  $\rho < \rho_1 < 2\rho$  so that the common

majorant's increment is at most  $\eta_0/4$  and all product and input-image margins remain strict. The whole supported family is then bounded by  $\eta_0/2$  at  $\rho_1$ , uniformly over these parameters. The logarithm expansion and the conversion used for (15.17) give

$$\sup_{\gamma \in I, d, R \in \mathcal{A}_E} \mathcal{B}_{\rho_1, a_{\text{sp}}}(\widehat{\mathcal{M}}_{d,t} R) \leq C_{\Lambda, I, E}. \quad (15.47)$$

Indeed, the argument of the input effective interaction stays below its input radius  $\rho$ , the fixed connected box sets still satisfy  $a_{\text{sp}} t \leq 2|Y|$ , and the conversion factor is  $9\rho_1^{-2}$ ; none of these facts depends on  $t$ . The finitely many small depths are included in the bound, on their own product domains. We split the difference of two outputs at a Taylor degree  $h$ . Its tail is at most  $2C_{\Lambda, I, E}(\rho/\rho_1)^{h-1}$ , by Lemma 15.5. We choose  $h$  first, and then  $d, d'$  so large that the finitely many remaining tensors are close, by Step 4. This proves (15.43) in the coefficient norm at the unchanged  $\rho$  and  $a_{\text{sp}}$ .

At a finite depth  $d$  the rescaled integrand and  $Z_p^{\text{unit}}$  depend continuously on  $t > 0$ . The same logarithm and tail argument proves continuity in the norm for fixed  $R$ . The contraction of Theorem 15.6, applied at a fixed parameter to two inputs, implies continuity in  $R$ . Its conditions on  $t$ ,  $p(t)$ , and  $E$  hold for every  $\gamma \in I$  by hypothesis, so the factor  $\vartheta$  is unchanged. This proves (1).

**Step 6: effective interactions and stopped measures.** For fixed  $j$  and  $m$  we take the supremum over  $I$  in (15.18). By (1),

$$\sup_{\gamma \in I} \mathcal{B}_{\rho, a_{\text{sp}}}(R_j^k - R_j^{k'}) \leq 2E\vartheta^m + \sum_{h=j+1}^{j+m} \vartheta^{h-j-1} \varepsilon_{h,I}(k, k'), \quad \varepsilon_{h,I}(k, k') \longrightarrow 0, \quad (15.48)$$

where  $\varepsilon_{h,I}$  is the supremum over  $\gamma \in I$  of the quantity  $\varepsilon_h$  of (15.18). We choose  $m$  first and then  $k, k'$ . By the completeness of the coefficient space, proved in Step 6 of the proof of Theorem 15.6, we obtain the uniform limit  $R_j^\infty(\gamma) \in \mathcal{A}_E$ . Each finite effective interaction is continuous in  $\gamma$ , since it is a finite composition of transformations that are continuous by (1), starting from the zero interaction on the finest lattice. The uniform limit is then continuous as well.

On the fixed domain  $D_r$  at level  $j$ , the difference of the classical terms in (15.19) is divided by  $g_j^2 \geq \gamma_-/n_j$ . Using the value bound of Lemma 15.5, we obtain

$$\delta_{j,k,I} = \sup_{\gamma \in I} \sup_{D_r} \left| \frac{E_{J(k-j), \text{sat}} - E_{\infty, \text{sat}}}{g_j^2} + R_{j, \text{sat}}^k - R_{j, \text{sat}}^\infty \right| \longrightarrow 0. \quad (15.49)$$

At this fixed volume all these exponents are bounded uniformly in  $\gamma \in I$ . This common domain contains an open neighborhood of the identity of positive Haar measure, so the normalizers have a positive lower bound uniform in  $\gamma$ . By Lemma 15.1(1), the total variation distance is at most  $e^{2\delta_{j,k,I}} - 1$ , which proves the uniform convergence of the

stopped measures. The same bounds and the fixed domain prove continuity in  $\gamma$  in total variation, both at finite  $k$  and in the limit. In particular, with the fixed Borel map  $S_0$  of Lemmas 15.10 and 15.12, the coarse-level laws  $\eta_0^k$  and  $\eta_0^\infty$  converge in total variation uniformly in  $\gamma$  and depend continuously on  $\gamma$ . No continuity of  $S_0$  is needed. This proves (2).

**Step 7: kernels and finite-scale reconstructions.** For the potential of Theorem 15.9, with all its terms, we write

$$\tilde{V}_j^k(\gamma, \theta, v) = V_j^k(\gamma, \theta, p_j(\gamma)v), \quad v \in \Omega_1.$$

By Steps 2 and 3 and (15.49), we have

$$\Delta_{j,k,I} = \sup_{\gamma, \theta, v} |\tilde{V}_j^k - \tilde{V}_j^\infty| \longrightarrow 0, \quad (15.50)$$

where all suprema are over closed smaller product domains. Each finite potential is jointly continuous in  $(\gamma, \theta, v)$ , and so is the uniform limit, which is bounded there. The integral of  $e^{-\tilde{V}_j^\infty}$  over  $\Omega_1$  is at least  $|\Omega_1| \exp(-\sup |\tilde{V}_j^\infty|) > 0$ . Thus the limiting normalized kernel on the unit product is jointly continuous in  $(\gamma, \theta)$  in total variation, with one modulus  $\omega_{j,I}$  in the field. By Lemma 15.1(1), the finite kernel differs from it in total variation by at most  $e^{2\Delta_{j,k,I}} - 1$ , uniformly in the field and in  $\gamma$ .

The thresholds  $d_0(\Lambda_j)$  of Theorem 15.13 do not depend on the coupling. For a fixed  $l$ , all gauge transformations are therefore the fixed limiting ones once  $k$  is large. After the substitution  $q_i = g_{j,i}(\gamma)p_j(\gamma)v_i$ , the reconstruction maps converge uniformly on the common compact product of  $\gamma$  and the coordinates, and their first derivatives in the coarse field are uniformly bounded. Let  $F_j^k$  be the finite-scale reconstruction, and let  $u_{j,k,I} = \sup_{\gamma, h} |F_j^k(\gamma, h) - F_j^\infty(\gamma, h)|$ . Then

$$u_{0,k,I} = 0, \quad u_{j,k,I} \leq v_{j,k,I} + C_{j,I}u_{j-1,k,I}, \quad v_{j,k,I} \longrightarrow 0, \quad (15.51)$$

where  $v_{j,k,I}$  is the uniform distance between the maps and  $C_{j,I}$  bounds the first derivative of the limiting map. Since there are finitely many  $j \leq l$ , all  $u_{j,k,I}$  tend to zero. Evaluated along the reconstruction, the kernels differ by at most  $e^{2\Delta_{j,k,I}} - 1 + \omega_{j,I}(u_{j-1,k,I})$ . Using the disintegration of Lemma 15.10 and Lemma 15.1(2), we obtain

$$\sup_{\gamma \in I} \|P_l^k(\gamma) - P_l^\infty(\gamma)\|_{\text{TV}} \leq \sup_{\gamma \in I} \|\eta_0^k - \eta_0^\infty\|_{\text{TV}} + \sum_{j=1}^l \{e^{2\Delta_{j,k,I}} - 1 + \omega_{j,I}(u_{j-1,k,I})\} \longrightarrow 0. \quad (15.52)$$

This bound accounts for the whole finite-scale reconstruction, including the dependence of later fields on earlier fluctuation variables. The logarithmic fields have a common compact range, by the amplitude bounds. Lemma 15.1(3), as in (15.32), implies uniform

convergence in bounded-Lipschitz distance, and its coupling bound gives uniform convergence in  $W_1$ , with error at most  $\|F^k - F^\infty\|_\infty + D_l \|P_l^k - P_l^\infty\|_{\text{TV}}$ . The continuity in  $\gamma$  of the coarse-level laws and of the kernels in total variation, and the continuity of the limiting reconstruction maps, prove the continuity of the limiting laws. Compatibility is the disintegration identity. This proves (3).  $\square$

## PART V

# Heat estimates and backward reconstruction

## Heat-kernel estimates, interpolation, and concentration

This section proves the analytic and probabilistic estimates through which the conditional fluctuation kernels of Sections 13 and 15 control the reconstructed field in norms of negative regularity. The main result is Theorem 16.6. It concerns one renormalization group step, from a coarse field  $c$  to the lattice of mesh  $a$ , and the lattice potential  $A = a^{-1}\Phi$ , where  $\Phi$  is the logarithm of the reconstructed fine field in a gauge that depends only on  $c$ . Conditionally on  $c$ , the fluctuation  $F = A - \mathbb{E}[A \mid c]$  of  $A$  about its conditional expectation satisfies, for  $0 < \alpha < 3/2$  and every  $p \geq 1$ ,

$$\left( \mathbb{E} \left[ \left( \sup_{0 < s \leq T_0} s^{\alpha/2} \|P_s F\|_{\infty} \right)^p \mid c \right] \right)^{1/p} \leq C_p g_3 a^{\alpha-1/2} \sqrt{\log(2 + L/a)},$$

where  $P_s$  is the continuum heat semigroup applied to  $F$ , viewed as a function that is constant on lattice cells. The bound is uniform in  $c$  and in the number of averaging steps that precede the step. For  $\alpha > 1/2$  it tends to zero with the mesh and is summable over a geometric sequence of meshes. It controls only the fluctuation about the conditional expectation; the conditional expectation itself is estimated in Lemma 17.1.

The auxiliary results are deterministic heat and interpolation estimates and probabilistic maximal inequalities. We prove bounds for the heat kernel on the periodic lattice and on the torus (Lemma 16.1), and we construct a smooth cardinal interpolation between lattices whose meshes have an integer ratio, which preserves the integral of every column exactly (Lemma 16.3). We also prove an elementary chaining lemma (Lemma 16.4), estimates for time convolutions (Lemma 16.8), and the convergence of the discrete heat evolution of a fixed bounded finite-dimensional family of smooth fields to the continuum heat evolution (Lemma 16.9). All heat estimates are stated on a fixed finite time interval and keep the constant mode.

**The scale in the main estimate.** The factor  $g_3 a^{\alpha-1/2}$  has a simple origin. Let  $g^2 = g_3^2 a$  be the coupling at mesh  $a$ . The Jacobian of the map from the normalized fluctuation variables  $w$  to the lattice potential  $A$  has counting  $\ell^2$  operator norm at most  $C_J g/a = C_J g_3 a^{-1/2}$ . At a time  $s \geq a^2$ , the heat kernel averages over about  $(\sqrt{s}/a)^3$  cells, and a linear function of  $w$  with this averaging has Lipschitz constant smaller by the factor  $a^{3/2} s^{-3/4}$ . When  $\alpha < 3/2$ , the product  $s^{\alpha/2} a^{3/2} s^{-3/4}$  is largest at  $s \asymp a^2$ , where it equals  $a^{\alpha}$ . By the concentration

inequality for the conditional fluctuation kernel, the fluctuations at each point have size  $g_3 a^{\alpha-1/2}$ , and the maximum over space and time costs the factor  $\sqrt{\log(2 + L/a)}$ .

**Three linear operations.** Three linear operations on lattice arrays occur in this and the following sections.

| Operation                                                    | Input and output                                                                                                           | Use                                                                                                                                             |
|--------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|
| Continuum heat semigroup $P_s$ applied to the cell embedding | An array on the lattice of mesh $a$ , extended as a constant on each half-open cell; a smooth function on $\mathbb{T}_L^3$ | Negative-regularity norms of reconstructed fields (Theorem 16.6, Section 17)                                                                    |
| Discrete heat semigroup $P_t^a$                              | An array on the lattice of mesh $a$ ; an array on the same lattice                                                         | Heat bounds for the lattice flow (Lemmas 16.1 and 16.9)                                                                                         |
| Cardinal interpolation $I_{a_0 \rightarrow a}$               | An array on the lattice of mesh $a_0$ ; an array on the finer lattice of mesh $a = a_0/m$                                  | Comparison of fields from different scales on a common lattice; values and difference arrays are interpolated separately (Lemmas 16.3 and 16.9) |

**Two points that need care.** First, the conditional fluctuation kernel is supported on a product of balls, but it is not a product measure: the normalized fluctuation variables interact, and they are not Gaussian. We use only that its potential  $V$  satisfies  $D_w^2 V \geq I/2$  on the convex support. This implies a logarithmic Sobolev inequality and Gaussian concentration for Lipschitz functions, with constants independent of the dimension. Second, the column sums of the reconstruction Jacobian must be normalized by the cell volume of the lattice on which each fluctuation variable lives. A uniform bound on the row sums alone does not imply the  $\ell^2$  operator bound used in the concentration argument, and the column bound with the correct input volume produces the factor  $a^{3/2}$  above.

**Inputs, uses, and attribution.** The conventions for meshes, heat semigroups, analytic norms, and flow norms are those of Section 4. The probabilistic input is the concentration theorem on convex products of balls, Theorem 5.4. It rests on the variance inequality of Brascamp and Lieb [33], the logarithmic Sobolev inequality of Gross [103], and the curvature criterion of Bakry and Émery [8], with the boundary of the product treated in Section 5. The conditional fluctuation kernel and the bounds on the reconstruction and its derivatives come from Theorems 13.2 and 13.8; the reconstruction maps and the preservation of the kernels under them come from Theorems 15.13 and 15.14. The concentration theorem is used in Section 17; the heat bounds, the interpolation, and

the chaining lemma are used in Sections 17, 24, 26, 27, 29, 31, and 34; the convolution estimates in Section 24; and the consistency lemma in Section 25.

The heat-kernel bounds are standard; we prove them by Fourier analysis because the normalizations matter. The maximal inequalities use chaining in the sense of Dudley [79]. For general tail bounds by generic chaining, see Dirksen [74, Theorems 3.2 and 3.5]; the second of these theorems treats increments with mixed sub-Gaussian and exponential tails. Those results are stated for semimetrics satisfying the triangle inequality. Our version assumes only the covering and continuity properties that we use, and we include its short proof. Only the verification that the conditional fluctuation kernel and the reconstruction satisfy the hypotheses of these general estimates, uniformly in the coarse field and in the depth, depends on the construction.

**Notation.** Throughout this section,  $L > 0$  is the side length of the torus  $\mathbb{T}_L^3$ . For a mesh  $a = L/n$  with an integer  $n \geq 4$ , we write  $\Gamma_a = (a\mathbb{Z}/L\mathbb{Z})^3$  for the vertex set of  $\Lambda_n$  in physical units, and we use the measure  $a^3 \sum_{x \in \Gamma_a}$  in all norms  $L_a^r$ . Arrays on positive links are identified with arrays on vertices by attaching the link  $(x, x + ae_i)$  to its starting vertex  $x$ ; the direction  $i$  and the color form a fixed finite set of components, and all operators below act componentwise. For a direction  $i$ , we write  $D_{a,i}^+ f(x) = (f(x + ae_i) - f(x))/a$  and  $D_{a,i}^- f(x) = (f(x) - f(x - ae_i))/a$  for the physical forward and backward differences,  $\Delta_a = \sum_i D_{a,i}^- D_{a,i}^+$ , and  $P_t^a = e^{t\Delta_a}$ ; this semigroup keeps the constant mode. For an integer  $q \geq 0$ , we write  $D_a^q$  for any ordered product of  $q$  physical forward or backward differences (the identity if  $q = 0$ ). When the particular product matters, we write  $D_a^\epsilon$ , and  $\partial^\epsilon$  for the corresponding product of partial derivatives; bounds stated for  $D_a^q$  hold uniformly over the finitely many products of order  $q$ . The continuum heat semigroup is  $P_t = e^{t\Delta}$  on  $\mathbb{T}_L^3$ , with kernel  $p_t(x - y)$ . The cell embedding of an array  $f$  on  $\Gamma_a$  is the function on  $\mathbb{T}_L^3$  equal to  $f_y$  on the half-open cell  $Q_y = y + [0, a)^3$ . In this section, the letter  $q$  denotes an order of differentiation or an exponent; it is unrelated to the plaquette energy. Constants  $C$  may change from line to line, and depend only on the parameters listed in the statement being proved.

## 16.1 Heat-kernel bounds

The following lemma collects the bounds on the discrete and continuum heat kernels used in the rest of the paper. Its last assertion is a maximal inequality for sub-Gaussian processes. We state it here because it is applied to processes built from the heat kernel; it is a special case of Lemma 16.4. For a set  $I$ , a nonnegative symmetric function  $d$  on  $I \times I$ , and  $v > 0$ , we write  $N(I, d, v)$  for the smallest cardinality of a set  $\mathcal{N} \subset I$  such that every  $i \in I$  has some  $i' \in \mathcal{N}$  with  $d(i, i') \leq v$ .

**Lemma 16.1.** Fix  $L, T > 0$ . Let  $a = L/n$  with an integer  $n \geq 4$ , and let  $p_t^a(x, y)$  be the density of  $P_t^a$  with respect to the measure  $a^3 \sum_y$ . Then  $p_0^a(x, y) = a^{-3} \mathbf{1}_{x=y}$ ,  $p_t^a \geq 0$ , and  $a^3 \sum_y p_t^a(x, y) = 1$ . For every integer  $q \geq 0$ , every product  $D_a^\epsilon$  of  $q$  physical forward or backward differences acting on the variable  $y$ , every  $x \in \Gamma_a$ , and every  $0 \leq t \leq T$ ,

$$\begin{aligned} \|D_a^\epsilon p_t^a(x, \cdot)\|_{L_a^1} &\leq C_q (t + a^2)^{-q/2}, \\ \|D_a^\epsilon p_t^a(x, \cdot)\|_{L_a^2} &\leq C_q (t + a^2)^{-3/4 - q/2}, \\ \|D_a^\epsilon p_t^a(x, \cdot)\|_\infty &\leq C_q (t + a^2)^{-(3+q)/2}. \end{aligned} \quad (16.1)$$

The continuum heat kernel  $p_t$  is smooth for  $t > 0$  and satisfies the same three bounds with  $a = 0$ , for  $0 < t \leq T$ , with  $D_a^\epsilon$  replaced by a product of  $q$  partial derivatives. The constants depend only on  $q$ , the fixed sets of directions and colors, and  $T/L^2$ .

For  $y \in \Gamma_a$ ,  $0 < t \leq T$ , and  $x \in \mathbb{T}_L^3$  let

$$\omega_y(t, x) = \int_{Q_y} p_t(x - z) dz.$$

Then

$$\|\omega(t, x)\|_{\ell^2} \leq C \min\{1, a^{3/2} t^{-3/4}\} \quad (0 < t \leq T),$$

and

$$\|\omega(t, x) - \omega(t, x')\|_{\ell^2} \leq C a^{3/2} t^{-3/4} \min\{1, \text{dist}(x, x')/\sqrt{t}\} \quad (a^2 \leq t \leq T).$$

More generally, for every multi-index  $v$  and every integer  $r \geq 0$ , the array of cell integrals of  $\partial^v \partial_t^r p_t(x - \cdot)$  has  $\ell^2$  norm at most  $C_{v,r} a^{3/2} t^{-3/4 - |v|/2 - r}$  for  $0 < t \leq T$ . Thus every additional spatial derivative costs a factor  $t^{-1/2}$  and every time derivative a factor  $t^{-1}$ , and the corresponding increment bounds follow; for example, with  $v = \min\{s, t\}$ ,

$$\|\omega(t, x) - \omega(s, x)\|_{\ell^2} \leq C a^{3/2} v^{-3/4} \min\{1, |t - s|/v\} \quad (0 < s, t \leq T).$$

For  $0 < t \leq a^2$  we use instead that  $\|P_t f\|_\infty \leq \max_y |f_y|$  for the cell embedding of an array  $f$ .

Finally, let  $(X_i)_{i \in I}$  be a centered real process and let  $d$  be a nonnegative symmetric function on  $I \times I$  with  $\sup d \leq \sigma$ . Suppose that

$$\mathbb{E} e^{z X_i} \leq e^{z^2 \sigma^2 / 2}, \quad \mathbb{E} e^{z(X_i - X_j)} \leq e^{z^2 d(i, j) / 2} \quad (z \in \mathbb{R}, i, j \in I),$$

that  $N(I, d, \eta \sigma) \leq A \eta^{-v}$  for every  $0 < \eta < 1$ , where  $A \geq 1$  and  $v \geq 0$ , and that the process has a version which is continuous with respect to  $d$  and whose supremum equals its supremum over a countable  $d$ -dense subset of  $I$  (this holds automatically if  $I$  is finite). Choose such a continuous version and denote it again by  $X$ ; the following suprema are taken for this version. Then there is a universal constant  $C$  such that, for  $u \geq 0$ ,

$$\begin{aligned} \mathbb{P} \left\{ \sup_{i \in I} |X_i| > C \sigma [\sqrt{\log(2A)} + \sqrt{v} + \sqrt{u}] \right\} &\leq 2e^{-u}, \\ \mathbb{E} \sup_{i \in I} |X_i| &\leq C \sigma [\sqrt{\log(2A)} + \sqrt{v}]. \end{aligned}$$

No Gaussian or independence assumption is made.

*Remark 16.2.* The finite horizon  $T$  cannot be removed, because the semigroups keep the constant mode. For instance, as  $t \rightarrow \infty$  the density  $p_t^a(x, \cdot)$  tends to the constant  $L^{-3}$ , whose  $L_a^2$  norm is  $L^{-3/2}$ , while  $(t + a^2)^{-3/4} \rightarrow 0$ . In the same way, for  $E_\theta = I$  and  $a = a_0$  in Lemma 16.3 below, the undifferentiated row of  $W_t$  tends to the constant row  $n^{-3}$ , whose counting  $\ell^2$  norm is  $n^{-3/2}$ . The bound  $Ca_0^{3/2}(t + a_0^2)^{-3/4}$  therefore cannot hold with a fixed constant on an unbounded time interval. For this reason, all row and covering estimates in this section are stated for times in  $[0, T]$ , and their constants depend on  $T/L^2$ .

*Proof of Lemma 16.1.* We use Euclidean norms on the fixed spaces of directions and colors; changing between equivalent norms costs a fixed factor. All heat times below lie in  $[0, T]$ , and they are positive for the continuum kernel and the cell integrals.

**The discrete kernel and its  $L^1$  bounds.** We begin with unit spacing on  $\mathbb{Z}$ . Let  $S_+$  and  $S_-$  be the two shifts. The kernel  $r_s$  of

$$e^{s(S_+ + S_- - 2I)} = e^{-2s} \sum_{k, l \geq 0} \frac{s^{k+l}}{k! l!} S_+^k S_-^l$$

is nonnegative and has sum one; the series converges in operator norm on bounded arrays. Its Fourier multiplier on  $[-\pi, \pi]$  is  $e^{-4s \sin^2(\theta/2)}$ . Let  $\nabla^\epsilon$  be a product of  $r$  unit forward or backward differences in this coordinate, and set  $h_s = \nabla^\epsilon r_s$ . Up to a sign, the multiplier of  $h_s$  is

$$m_s(\theta) = e^{-il\theta} (e^{i\theta} - 1)^r e^{-4s \sin^2(\theta/2)}, \quad 0 \leq l \leq r,$$

where  $l$  is the number of backward differences. For  $s \leq 1$ , we have  $\sum_j |h_s(j)| \leq 2^r$  by the difference stencil. For  $s \geq 1$ , the inequalities  $|e^{i\theta} - 1| \leq |\theta|$  and  $\sin^2(\theta/2) \geq \theta^2/\pi^2$  on  $[-\pi, \pi]$  imply

$$\int_{-\pi}^{\pi} |m_s|^2 d\theta \leq C_r s^{-r-1/2}, \quad \int_{-\pi}^{\pi} |m'_s|^2 d\theta \leq C_r s^{-r+1/2}. \quad (16.2)$$

Here and below, the finitely many sign patterns are included in  $C_r$ . To verify the second bound, note that  $m'_s$  is a sum of terms bounded by

$$C_r (|\theta|^r + \mathbf{1}_{r \geq 1} |\theta|^{r-1} + s |\theta|^{r+1}) e^{-cs\theta^2};$$

the first term is absent when  $l = 0$ , and only the last term is present when  $r = 0$ . Squaring, integrating, and substituting  $v = \sqrt{s} \theta$ , we obtain (16.2), since  $\int_{\mathbb{R}} |v|^m e^{-cv^2} dv < \infty$  for every  $m \geq 0$ . For this finiteness, we split  $[0, \infty)$  into the intervals  $[k, k+1]$ ,  $k \geq 0$ ; on each

of them,  $|v|^m e^{-cv^2} \leq (k+1)^m e^{-ck^2}$ , and these bounds form a summable sequence. The negative half-line is treated by symmetry. The Fourier coefficients of  $m'_s$  are  $ijh_s(j)$ . By Parseval's identity, we have

$$\sum_j |h_s(j)|^2 \leq C_r s^{-r-1/2}, \quad \sum_j j^2 |h_s(j)|^2 \leq C_r s^{-r+1/2}.$$

These identities may be applied first to the absolutely convergent kernel series, or obtained by approximation with trigonometric polynomials in  $L^2$ . For  $s \geq 1$ , we have  $\sum_{j \in \mathbb{Z}} (1 + j^2/s)^{-1} \leq C\sqrt{s}$ ; to see this, we separate  $|j| \leq \sqrt{s}$  from  $|j| > \sqrt{s}$  and use  $s \sum_{|j| > \sqrt{s}} j^{-2} \leq C\sqrt{s}$ . Then the weighted Cauchy–Schwarz inequality implies

$$\sum_j |h_s(j)| \leq \left( \sum_j (1 + j^2/s)^{-1} \right)^{1/2} \left( \sum_j (1 + j^2/s) |h_s(j)|^2 \right)^{1/2} \leq C_r s^{-r/2}. \quad (16.3)$$

The kernel on  $\mathbb{Z}^3$  is the product of three copies of  $r_s$ . A product of  $q$  differences puts  $r_i$  differences on coordinate  $i$ , with  $r_1 + r_2 + r_3 = q$ , and the absolute counting  $L^1$  norm of the differentiated kernel is the product of the three one-dimensional norms. By (16.3), it is at most  $C_q(1+s)^{-q/2}$  for every  $s \geq 0$ . Periodizing with period  $n$  does not increase this norm, by the triangle inequality, and it preserves positivity and mass. At spacing  $a$ , the time is  $s = t/a^2$ , the density carries the factor  $a^{-3}$ , and a product of  $q$  physical differences carries the factor  $a^{-q}$ . We obtain

$$a^3 \sum_y |D_a^\epsilon p_t^a(x, y)| \leq C_q a^{-q} (1 + t/a^2)^{-q/2} = C_q (t + a^2)^{-q/2}.$$

The periodized kernel solves the finite heat equation with initial density  $a^{-3} \mathbf{1}_{x=y}$ , and by uniqueness for this finite linear system it equals  $p_t^a$ . This proves the  $L^1$  bound and the assertions on positivity and mass, including at  $t = 0$ .

**The  $L^2$  and supremum bounds.** Fix one representative of each  $k \in (\mathbb{Z}/n\mathbb{Z})^3$  with  $|k_i| \leq n/2$ . Then

$$p_t^a(x, y) = L^{-3} \sum_k e^{-t\lambda_a(k)} e^{2\pi i k \cdot (x-y)/L}, \quad \lambda_a(k) = 4a^{-2} \sum_i \sin^2(\pi k_i/n). \quad (16.4)$$

Indeed, these exponentials are eigenfunctions of  $\Delta_a$ , and their orthogonality on  $\Gamma_a$  yields the required initial density at  $t = 0$ . For these representatives,

$$c|k|^2/L^2 \leq \lambda_a(k) \leq C|k|^2/L^2,$$

and the multiplier of  $D_a^\epsilon$  has absolute value at most  $C_q(|k|/L)^q$ . If  $0 \leq t \leq a^2$ , we obtain the supremum bound  $2^q a^{-3-q}$  and the  $L_a^2$  bound  $2^q a^{-3/2-q}$  from the difference stencil

and the mass identity; since  $t + a^2 \leq 2a^2$ , these are the asserted bounds. If  $t \geq a^2$ , we extend the sum over nonzero frequencies to  $\mathbb{Z}^3 \setminus \{0\}$ . For every  $r \geq 0$  and  $c > 0$ ,

$$L^{-3} \sum_{k \neq 0} (|k|/L)^r e^{-ct|k|^2/L^2} \leq C_{r,c} t^{-(3+r)/2}. \quad (16.5)$$

To check this estimate, we set  $u = t/L^2$  and group the frequencies with  $j \leq |k| < j+1$ . There are at most  $C(j+1)^2$  of them, since the disjoint unit cubes centered at these points lie in an annulus of fixed extra width. For  $u \leq 1$ , the sum of  $(j+1)^{r+2} e^{-cu j^2}$  over  $j \geq 1$  is at most  $C_{r,c} u^{-(r+3)/2}$ : on the interval  $[j, j+1]$ , we may replace  $j^2$  by  $x^2/4$  and  $(j+1)^{r+2}$  by  $C_r x^{r+2}$ , and then integrate. For  $u \geq 1$ , we have  $e^{-cu j^2} \leq e^{-cu/2} e^{-c j^2/2}$  for  $j \geq 1$ , and  $e^{-cu/2} \leq C_{r,c} u^{-(r+3)/2}$ . Rescaling gives (16.5).

The triangle inequality in (16.4), together with (16.5) for  $r = q$ , implies the supremum bound. By orthogonality on  $\Gamma_a$ , the square of the  $L_a^2$  norm equals  $L^{-3}$  times the sum over  $k$  of the squared multipliers times  $e^{-2t\lambda_a(k)}$ , and (16.5) with  $r = 2q$  bounds it. For  $q = 0$ , the frequency  $k = 0$  contributes  $L^{-3}$  both to the supremum and to the square of the  $L_a^2$  norm. On the horizon, we have  $L^{-3} \leq (T/L^2)^{3/2} t^{-3/2}$ , and this contribution is at most  $C(T/L^2) t^{-3/2}$  in both cases; every difference of positive order annihilates it. Since  $t + a^2 \leq 2t$  in this range, these are the asserted bounds, with constants depending only on  $q$  and  $T/L^2$ .

**The continuum kernel and the cell integrals.** The continuum kernel is the periodization of

$$G_t(z) = (4\pi t)^{-3/2} e^{-|z|^2/(4t)},$$

which has mass one. For each multi-index  $\nu$ , the derivative  $\partial^\nu G_t$  equals  $t^{-(3+|\nu|)/2}$  times a fixed polynomial in  $z/\sqrt{t}$  times  $e^{-|z|^2/(4t)}$ . A change of variables yields  $\|\partial^\nu G_t\|_1 \leq C_\nu t^{-|\nu|/2}$ , and periodization does not increase the absolute  $L^1$  norm. The periodic Fourier coefficients are  $L^{-3} e^{-4\pi^2 t |k|^2/L^2}$ , obtained by integrating the periodized Gaussian over one period. The other two bounds follow from (16.5), with the term  $k = 0$  treated as before. For  $t > 0$ , all these series and their derivatives converge absolutely; multiplying Fourier coefficients then verifies the heat equation and the semigroup property.

By positivity and mass, we have  $\sum_y \omega_y(t, x) = 1$ , and hence  $\|\omega(t, x)\|_2 \leq 1$ . Using the Cauchy–Schwarz inequality on each cell, we find

$$\sum_y \left| \int_{Q_y} \partial^\nu \partial_t^r p_t(x - z) dz \right|^2 \leq a^3 \|\partial^\nu \Delta^r p_t\|_2^2 \leq C_{\nu,r} a^3 t^{-3/2 - |\nu| - 2r}, \quad (16.6)$$

where we used the heat equation  $\partial_t p_t = \Delta p_t$  and, in the last inequality, the continuum  $L^2$  bound with  $q = |\nu| + 2r$ . The case  $\nu = 0, r = 0$  is the second bound on  $\|\omega(t, x)\|_2$ . Integrating one spatial derivative along a shortest segment in  $\mathbb{T}_L^3$  from  $x$  to  $x'$ , we bound the difference of the two cell arrays by  $C a^{3/2} t^{-5/4} \text{dist}(x, x')$ . The value bound, used

twice, bounds it by  $Ca^{3/2}t^{-3/4}$ , and the minimum of the two is the increment bound. In the same way, (16.6) controls every fixed spatial and time derivative. For the time increment, we integrate the time derivative over  $[v, v + |t - s|]$  when  $|t - s| \leq v$ , and we use the two value bounds otherwise. The same reasoning, applied to a differentiated array of cell integrals, yields its increment bounds, with the additional factors displayed in (16.6). Finally, for  $t \leq a^2$ , positivity and unit mass imply  $\|P_t f\|_\infty \leq \max_y |f_y|$  for every cell embedding  $f$ ; this needs no regularity across cell boundaries.

The chaining assertion is a special case of Lemma 16.4; we deduce it at the end of Section 16.3.  $\square$

## 16.2 Cardinal interpolation and volume-normalized columns

We next define the cardinal interpolation used to compare arrays on different lattices.

Fix an even nonnegative function  $b \in C_c^\infty([-1, 1])$  that is positive on  $(-1, 1)$ ; all its derivatives vanish at  $\pm 1$ . Set

$$\zeta(v) = \frac{b(v)}{\sum_{m \in \mathbb{Z}} b(v - m)} \quad (v \in \mathbb{R}), \quad \zeta_3(u) = \prod_{i=1}^3 \zeta(u_i) \quad (u \in \mathbb{R}^3).$$

Let  $a_0 = L/n$  with  $n \geq 4$ , let  $m \geq 1$  be an integer, and let  $a = a_0/m$ . For  $x \in \Gamma_{a_0}$  let

$$\psi_x(z) = \sum_{l \in \mathbb{Z}^3} \zeta_3((z - x + Ll)/a_0) \quad (z \in \mathbb{T}_L^3)$$

be the periodization of the translated function  $\zeta_3((\cdot - x)/a_0)$ . The cardinal interpolation of an array  $f$  on  $\Gamma_{a_0}$  is the array on  $\Gamma_a$  defined componentwise by

$$(I_{a_0 \rightarrow a} f)_i(y) = \sum_{x \in \Gamma_{a_0}} \psi_x(y) f_i(x) \quad (y \in \Gamma_a), \quad (16.7)$$

where, as above, each positive link is attached to its starting vertex. The continuous interpolant is  $I_{a_0} f(z) = \sum_x \psi_x(z) f(x)$  for  $z \in \mathbb{T}_L^3$ . In (16.7) we also write  $\zeta_3((y - x)/a_0)$  for  $\psi_x(y)$ .

For a matrix  $K$  from scalar variables  $\xi$  attached to the cells of the lattice of mesh  $a_0$  to arrays on the lattice of mesh  $a$ , we use the row sums  $\sup_y \sum_{\xi} |K(y, \xi)|$  and the volume-normalized column sums  $\sup_{\xi} (a^3/a_0^3) \sum_y |K(y, \xi)|$ , as in Definition 4.6. Throughout, a column is normalized by the cell volume  $a_0^3$  of its input.

**Lemma 16.3.** *Fix  $L, T > 0$  and the function  $b$ . Let  $a_0 = L/n$  with  $n \geq 4$ , let  $m \geq 1$  be an integer, and let  $a = a_0/m$ . All times below lie in  $[0, T]$ .*

1. The map  $I_{a_0 \rightarrow a}$  is local within distance  $\sqrt{3}a_0$ , commutes with constant color rotations, preserves constants, satisfies  $(I_{a_0 \rightarrow a}f)(x) = f(x)$  for  $x \in \Gamma_{a_0}$ , and  $I_{a \rightarrow a}$  is the identity. For every integer  $q \geq 0$ ,

$$\|D_a^q I_{a_0 \rightarrow a}f\|_\infty \leq C_q a_0^{-q} \|f\|_\infty, \quad a^3 \sum_y \psi_x(y) = a_0^3,$$

$$a^3 \sum_y \text{dist}(y, x) \psi_x(y) \leq C a_0^4, \quad \|D_a^q P_t^a I_{a_0 \rightarrow a}f\|_\infty \leq C_q (t + a_0^2)^{-q/2} \|f\|_\infty.$$

The continuous interpolant  $I_{a_0}f$  satisfies the corresponding derivative bounds, and  $\int_{\mathbb{T}_L^3} \psi_x(z) dz = a_0^3$ .

2. Let  $E_\theta$  be a deterministic matrix, depending on a parameter  $\theta$ , from scalar variables  $\xi$  attached to the cells of the lattice of mesh  $a_0$ , at most  $v_J$  of them per cell, to arrays on that lattice. Suppose that, uniformly in  $\theta$ , its row sums and volume-normalized column sums are at most  $C_J$ , that is,

$$\sup_x \sum_\xi |E_\theta(x, \xi)| \leq C_J, \quad \sup_\xi a_0^3 \sum_x |E_\theta(x, \xi)| \leq C_J a_0^3, \quad (16.8)$$

and that its columns decay exponentially at scale  $a_0$ , with fixed rate and amplitude. Set  $W_t = P_t^a I_{a_0 \rightarrow a} E_\theta$ . Then for every integer  $q \geq 0$ ,

$$\sup_{x, \theta} \|D_a^q W_t(x, \cdot)\|_{\ell^2} \leq C_{J,q} a_0^{3/2} (t + a_0^2)^{-3/4 - q/2},$$

$$\|D_a^q W_t\|_{\ell^2 \rightarrow L_a^2} \leq C_{J,q} a_0^{3/2} (t + a_0^2)^{-q/2},$$

where  $\ell^2$  refers to the variables  $\xi$ . For  $q \in \{0, 1\}$ , put  $A_q(t) = a_0^{3/2} (t + a_0^2)^{-3/4 - q/2}$  and  $v = \min\{s, t\}$ . Then

$$\|D_a^q W_t(x, \cdot) - D_a^q W_s(x, \cdot)\|_{\ell^2} \leq C_{J,q} A_q(v) \min\{1, |t - s|/(v + a_0^2)\},$$

$$\|D_a^q W_t(x, \cdot) - D_a^q W_t(y, \cdot)\|_{\ell^2} \leq C_{J,q} A_q(t) \min\{1, \text{dist}(x, y)/\sqrt{t + a_0^2}\}.$$

At fixed  $\theta$ , for  $q \in \{0, 1\}$  and  $0 < \eta < 1$ , the set  $\Gamma_a \times [0, T]$  with the distance  $a_0^q \|D_a^q W_t(x, \cdot) - D_a^q W_s(y, \cdot)\|_{\ell^2}$  between  $(x, t)$  and  $(y, s)$  has an  $\eta$ -net with at most

$$C_{J,q} (1 + L/(\eta a_0))^3 (1 + T/(\eta a_0^2))$$

points. For the normalized rows  $D_a^q W_t(x, \cdot)/A_q(t)$  the same holds, up to a factor which is a fixed power of  $1 + T/a_0^2$ .

3. Let  $u$  be an array on the lattice of mesh  $a_1$  with  $a_1/a \in \mathbb{N}$ . Then, componentwise,

$$a^3 \sum_{y \in \Gamma_a} I_{a_1 \rightarrow a} u(y) = a_1^3 \sum_{x \in \Gamma_{a_1}} u(x).$$

Hence, if arrays originating on lattices of different meshes are interpolated separately from their own lattices to a common finer lattice before they are added or subtracted, every componentwise integral is preserved exactly; in particular every prescribed zero integral is preserved.

All the estimates are uniform in the ratio  $m = a_0/a$ , in  $n$ , and in  $\theta$ , at fixed  $J$  and fixed coefficient constants. The constants depend only on  $b$ ,  $q$ ,  $T/L^2$ ,  $v_J$ ,  $C_J$ , and the decay constants of the columns. The hypotheses of the second part hold, with a constant  $C_J$  depending only on  $J$ , the fixed construction domains, and the order of differentiation, for the normalized Jacobian of the reconstruction and for each of its mixed derivatives of fixed order in the coarse field and the fluctuation variables, uniformly in the coarse field and in the depth; see Lemma 16.5.

*Proof.* All matrices in this proof are written in counting coordinates, and all times lie in  $[0, T]$ ; we apply Lemma 16.1 with the same  $L$  and  $T$ . The parameter  $\theta$  is fixed during the construction of each net, and the constants are uniform in  $\theta$ ; the covering assertion is not an entropy estimate for the family indexed by  $\theta$ .

**The basis functions and their masses.** Let  $d_b(u) = \sum_{m \in \mathbb{Z}} b(u - m)$ . Since only finitely many terms are nonzero near any point,  $d_b$  is smooth and one-periodic. It is strictly positive, since some integer has distance at most  $1/2$  from  $u$ , and its minimum over one period is positive. The function  $\zeta = b/d_b$  is smooth, nonnegative, even, and supported in  $[-1, 1]$ ; all its derivatives vanish at  $\pm 1$ , and

$$\zeta(k) = \mathbf{1}_{k=0} \quad (k \in \mathbb{Z}), \quad \sum_{k \in \mathbb{Z}} \zeta(u - k) = 1. \quad (16.9)$$

The first identity holds because  $b(k) = 0$  for integers  $k \neq 0$  and  $d_b(0) = b(0)$ ; the second follows by replacing every  $d_b(u - k)$  by  $d_b(u)$ . The periodization in  $\psi_x$  accounts for every periodic seam. These functions sum to one as  $x$  ranges over  $\Gamma_{a_0}$ , they are supported within torus distance  $\sqrt{3}a_0$  of  $x$ , and  $\psi_x(x') = \mathbf{1}_{x=x'}$  for  $x' \in \Gamma_{a_0}$ . The map preserves constants and samples, and it is the identity when  $a = a_0$ . Since it acts by the same scalar coefficients on each direction and color, it commutes with every constant color rotation.

Since at any point of  $\mathbb{T}_L^3$  at most a fixed number of the functions  $\psi_x$  are nonzero, we have

$$\|\partial^\epsilon I_{a_0} f\|_\infty \leq C_q a_0^{-q} \|f\|_\infty$$

for every product  $\partial^\epsilon$  of  $q$  partial derivatives. For a smooth periodic function  $h$ , each physical forward or backward difference is an average of the corresponding derivative,

$$D_{a,i}^\pm h(z) = \int_0^1 \partial_i h(z \pm rae_i) dr. \quad (16.10)$$

Applying this identity repeatedly, we write a product of  $q$  differences as an average of the corresponding derivative of order  $q$ . The preceding bound therefore implies the bound on  $D_a^q I_{a_0 \rightarrow a} f$ , uniformly in  $m$ .

For the mass, we write an integer  $l$  as  $l = mk + r$  with  $0 \leq r < m$ . By (16.9), we have

$$\sum_{l \in \mathbb{Z}} \zeta(l/m) = \sum_{r=0}^{m-1} \sum_{k \in \mathbb{Z}} \zeta(k + r/m) = m.$$

Taking the product over the three coordinates and regrouping the periodization, we obtain

$$a^3 \sum_{y \in \Gamma_a} \psi_x(y) = a^3 m^3 = a_0^3. \quad (16.11)$$

Since  $\psi_x \geq 0$  is supported within distance  $\sqrt{3}a_0$  of  $x$ , the first moment is at most  $\sqrt{3}a_0^4$ . Integrating the second identity in (16.9) over  $[0, 1]$ , we find  $\int_{\mathbb{R}} \zeta(u) du = 1$ , and the product over the coordinates gives  $\int_{\mathbb{T}_L^3} \psi_x(z) dz = a_0^3$ . These arguments also prove the assertions on the continuous interpolant.

**Heat smoothing.** If  $t \leq a_0^2$ , we commute the differences with the discrete heat semigroup and use that  $P_t^a$  contracts the supremum norm, together with the bound on  $D_a^q I_{a_0 \rightarrow a} f$ ; this gives  $C_q a_0^{-q} \|f\|_\infty$ . If  $t \geq a_0^2$ , we move the differences onto the heat kernel and use its  $L_a^1$  bound from Lemma 16.1, together with  $\|I_{a_0 \rightarrow a} f\|_\infty \leq \|f\|_\infty$ ; since  $a \leq a_0$ , we obtain  $C_q t^{-q/2} \|f\|_\infty$ . Both bounds are at most  $C_q (t + a_0^2)^{-q/2} \|f\|_\infty$ . The same argument, with continuous derivatives and the continuum heat kernel, proves the corresponding continuous estimate.

**Row sums and volume-normalized column sums.** We write  $\xi$  for the scalar input variables. The finitely many variables per cell and the color components are included in  $C_J$ . The estimates below use only (16.8), and they apply to every matrix obeying (16.8). The interpolation, whose range is  $\sqrt{3}a_0$ , preserves the exponential decay of the columns, but this decay is not needed for the numerical bounds.

Let  $K = I_{a_0 \rightarrow a} E_\theta$ . For every product of  $q$  differences,

$$\sup_y \sum_{\xi} |D_a^q K(y, \xi)| \leq C_{J,q} a_0^{-q}, \quad \sup_{\xi} a^3 \sum_y |D_a^q K(y, \xi)| \leq C_{J,q} a_0^{3-q}. \quad (16.12)$$

For the first bound, we expand  $K$  over the coarse vertices. By (16.10) and the bounded overlap of the differentiated continuous basis functions, we have  $\sum_x |D_a^q \psi_x(y)| \leq C_q a_0^{-q}$ , and the row bound in (16.8) implies the claim. For the second bound, each differentiated basis function has magnitude at most  $C_q a_0^{-q}$  and support within distance  $C_q a_0$  of its vertex, which contains at most  $C_q (a_0/a)^3$  fine vertices. This also holds if the support wraps around the torus, by summing over its lifts. We conclude that  $a^3 \sum_y |D_a^q \psi_x(y)| \leq C_q a_0^{3-q}$ , and summing over  $x$  with the column bound in (16.8) gives the second bound.

For  $t \leq a_0^2$ , the heat semigroup is a contraction on absolute row sums and on volume-normalized column sums, and (16.12) holds for  $D_a^q W_t$ . In particular, the  $\ell^2$  norm of a

row of  $D_a^q W_t$  is at most its absolute row sum  $C_{J,q} a_0^{-q} = C_{J,q} A_q(0)$ . For  $t \geq a_0^2$ , we use the undifferentiated  $K$  in the heat convolution and move all differences to the heat kernel. The  $L_a^1$  bound and the supremum bound of Lemma 16.1 yield, respectively,

$$\sup_x \sum_{\xi} |D_a^q W_t(x, \xi)| \leq C_{J,q} t^{-q/2}, \quad \sup_{x, \xi} |D_a^q W_t(x, \xi)| \leq C_{J,q} a_0^3 t^{-(3+q)/2}. \quad (16.13)$$

The square of the  $\ell^2$  norm of a row is at most the product of these two quantities, that is, at most  $C_{J,q} a_0^3 t^{-3/2-q}$ . Together with the bound for  $t \leq a_0^2$ , this proves the row bound with  $t + a_0^2$  on the whole interval.

For the operator norm, we use Schur's test with the volume weights of the output lattice,

$$a^3 \sum_x \left| \sum_{\xi} H(x, \xi) u_{\xi} \right|^2 \leq \left( \sup_x \sum_{\xi} |H(x, \xi)| \right) \left( \sup_{\xi} a^3 \sum_x |H(x, \xi)| \right) \sum_{\xi} |u_{\xi}|^2, \quad (16.14)$$

which follows from the Cauchy–Schwarz inequality in each row and an interchange of two finite sums. For  $t \leq a_0^2$ , we use (16.12). For  $t \geq a_0^2$ , the first factor is bounded by (16.13), and the second is at most  $C_{J,q} a_0^3 t^{-q/2}$ , by the  $L_a^1$  bound on the heat kernel and the undifferentiated column bound for  $K$ . Taking the square root gives  $C_{J,q} a_0^{3/2} (t + a_0^2)^{-q/2}$ . The volume in this computation is the input cell volume  $a_0^3$ ; refining the output lattice does not change it.

**Increments and nets.** Let  $R_q(t, x)$  denote the row  $D_a^q W_t(x, \cdot)$ . The bounds already proved hold for every order of differentiation. Since  $\partial_t W_t = \Delta_a W_t$ , they imply

$$\|\partial_t R_q(t, x)\|_{\ell^2} \leq C_{J,q} \frac{A_q(t)}{t + a_0^2}, \quad \|D_{a,i}^{\pm} R_q(t, x)\|_{\ell^2} \leq C_{J,q} \frac{A_q(t)}{\sqrt{t + a_0^2}}.$$

Since  $A_q$  is decreasing, integrating the first bound between  $s$  and  $t$  gives  $C_{J,q} |t - s| A_q(v) / (v + a_0^2)$ ; the sum of the two row norms is at most  $C_{J,q} A_q(v)$ , and the minimum of the two bounds is the time increment. For the spatial increment, we join two fine vertices by a shortest periodic path along coordinate directions, of total physical length at most  $\sqrt{3} \operatorname{dist}(x, y)$ . Summing the second bound along this path gives the distance times the displayed derivative bound; the sum of the two value bounds is the alternative without the distance.

For the distance  $d_q((x, t), (y, s)) = a_0^q \|R_q(t, x) - R_q(s, y)\|_{\ell^2}$ , these estimates imply

$$d_q((x, t), (y, s)) \leq C_{J,q} \left( \frac{\operatorname{dist}(x, y)}{a_0} + \frac{|t - s|}{a_0^2} \right).$$

We choose spatial points at physical separation at most  $c_{J,q} \eta a_0$  and times at separation at most  $c_{J,q} \eta a_0^2$ . If  $a$  exceeds the proposed spatial separation, we use all vertices, whose

number is then smaller than the bound below; otherwise, we round a continuum cubic covering to fine vertices, which at most doubles its radius. The resulting net has at most  $C_{J,q}(1 + L/(\eta a_0))^3(1 + T/(\eta a_0^2))$  points, independently of the number of fine vertices. For the normalized rows  $R_q(t, x)/A_q(t)$ , differentiation in time produces the additional term  $A'_q(t)/A_q(t) = -(3/4 + q/2)/(t + a_0^2)$ . The same estimates give a spatial Lipschitz bound  $C_{J,q}/a_0$  and a time Lipschitz bound  $C_{J,q}/a_0^2$  for the normalized rows, and the same net works. Alternatively, using only the global bounds costs powers of  $a_0^{-q}/A_q(T) = (1 + T/a_0^2)^{3/4+q/2}$ .

**Column integrals.** Let  $u$  be an array on the lattice of mesh  $a_1$  with  $a_1/a \in \mathbb{N}$ . Interchanging finite sums and using (16.11) with  $a_1$  in place of  $a_0$ , we obtain

$$a^3 \sum_y I_{a_1 \rightarrow a} u(y) = \sum_x u(x) a^3 \sum_y \psi_x(y) = a_1^3 \sum_x u(x).$$

Applying this identity to each array on its own lattice before adding or subtracting arrays from different lattices preserves every signed componentwise integral exactly. This is an identity for arrays of components, not for lattice cochains or holonomies. All the constants above depend only on  $b$ , the order of differentiation,  $T/L^2$ , and the coefficient constants.  $\square$

### 16.3 Supremum bounds from covering numbers

The maximal inequalities of this paper are obtained by elementary chaining. We consider a process indexed by a set  $\mathcal{T}$  with a nonnegative symmetric function  $d$  that controls the tails of the increments. We do not assume that  $d$  satisfies the triangle inequality; the proof uses only the covering numbers of  $d$  and the continuity of the process with respect to  $d$ . As in Section 16.1, for  $v > 0$ , a  $v$ -net is a set  $\mathcal{N} \subset \mathcal{T}$  such that every  $z \in \mathcal{T}$  has some  $z' \in \mathcal{N}$  with  $d(z, z') \leq v$ , and  $N(\mathcal{T}, d, v)$  is the smallest cardinality of a  $v$ -net. A process  $(F_z)_{z \in \mathcal{T}}$  is continuous with respect to  $d$  if  $F_{z_k} \rightarrow F_z$  whenever  $d(z_k, z) \rightarrow 0$ .

**Lemma 16.4.** *Let  $\mathcal{T}$  be a nonempty set and  $d : \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$  a symmetric function. Let  $(F_z)_{z \in \mathcal{T}}$  be a real process which has a version that is continuous with respect to  $d$  and whose supremum equals its supremum over a countable  $d$ -dense subset of  $\mathcal{T}$ . Choose such a continuous version and denote it again by  $F$ ; all suprema below are taken for this chosen version. Let  $z_0 \in \mathcal{T}$ ,  $s, b \geq 0$ ,  $D < \infty$ ,  $A \geq 1$ , and  $q_{\text{ent}} \geq 0$ , and assume that*

$$\sup_{z, z' \in \mathcal{T}} d(z, z') \leq D, \quad N(\mathcal{T}, d, v) \leq Av^{-q_{\text{ent}}} \quad (0 < v \leq 1),$$

*and that for every  $u > 0$  and all  $z, z' \in \mathcal{T}$ ,*

$$\mathbb{P}\{|F_{z_0}| > s\sqrt{u} + bu\} \leq 2e^{-u}, \quad \mathbb{P}\{|F_z - F_{z'}| > d(z, z')(s\sqrt{u} + bu)\} \leq 2e^{-u}.$$

Put  $D_1 = \max\{1, D\}$  and  $H = \log(2A) + q_{\text{ent}}$ . There is a universal constant  $C$  such that, for  $u \geq 0$  and  $1 \leq p < \infty$ ,

$$\mathbb{P}\left\{\sup_z |F_z| > CD_1[s(\sqrt{H} + \sqrt{u}) + b(H + u)]\right\} \leq 2e^{-u},$$

$$\left\|\sup_z |F_z|\right\|_{L^p} \leq CD_1[s(\sqrt{H} + \sqrt{p}) + b(H + p)].$$

No Gaussian or independence assumption is made; in particular the lemma applies to centered processes.

Further, let  $Y_1, \dots, Y_N$ ,  $N \geq 1$ , be real random variables with  $\mathbb{P}\{|Y_i| > s\sqrt{u} + bu\} \leq 2e^{-u}$  for every  $u > 0$ , and let  $M \geq 0$  be a random variable with  $M \leq \max_{i \leq N} |Y_i| + e$  almost surely, for a deterministic  $e \geq 0$ . Then, for  $u \geq 0$  and  $1 \leq p < \infty$ ,

$$\mathbb{P}\{M > e + s\sqrt{\log(2N) + u} + b(\log(2N) + u)\} \leq e^{-u},$$

$$\|M\|_{L^p} \leq e + C\{s[\sqrt{\log(2N)} + \sqrt{p}] + b[\log(2N) + p]\}.$$

Every assertion holds under a fixed conditional probability kernel, with the same constants, if all the hypotheses hold for each fixed value of the conditioning variable. The constant  $C$  does not depend on the dimension, the meshes, the radii, the couplings, or the conditioning value. A bound which is uniform in the conditioning value requires that the parameters  $s, b, D, A, q_{\text{ent}}$  (and  $e$ ) be bounded uniformly in that value.

*Proof.* We use the chosen continuous version; equality of finite-dimensional distributions preserves every fixed-point and increment hypothesis, and countably many net points can be compared on one event of probability one. We first normalize. Dividing  $d$  by  $D_1$  gives a function of supremum at most one. At the radii  $2^{-k}$ ,  $k \geq 1$ , it has nets  $\mathcal{T}_k$  with at most  $A2^{kq_{\text{ent}}}$  points: if  $D_1 2^{-k} \leq 1$ , we use the given net for  $d$  at radius  $D_1 2^{-k}$ , and otherwise we use the net at radius one. In the normalized function, the increment parameters become at most  $D_1 s$  and  $D_1 b$ , while the parameters of the point  $z_0$  are unchanged and are at most  $D_1 s$  and  $D_1 b$ . If a net of minimal cardinality does not exist at some radius, we enlarge the radius by a factor two; the universal constant absorbs this factor. It therefore suffices to consider  $D_1 = 1$  and nets  $\mathcal{T}_k$  of radius at most  $2^{1-k}$ .

We choose the nets and the connections deterministically. We connect every point of  $\mathcal{T}_1$  to  $z_0$ , and each point of  $\mathcal{T}_k$ ,  $k \geq 2$ , to one point of  $\mathcal{T}_{k-1}$  within distance  $2^{2-k}$ ; such a point exists because  $\mathcal{T}_{k-1}$  is a net of radius  $2^{2-k}$  for all of  $\mathcal{T}$ . There are at most  $A2^{kq_{\text{ent}}}$  connections at level  $k$ . Set

$$h_0 = u + \log 8, \quad h_k = u + \log(2A) + kq_{\text{ent}} \log 2 + (k+3) \log 2 \quad (k \geq 1).$$

The point  $z_0$  violates the bound  $s\sqrt{h_0} + bh_0$  with probability at most  $e^{-u}/4$ . By a union bound over the connections at level  $k$ , the probability that one of them violates its bound is at most  $e^{-u}2^{-k-3}$ . The sum of these probabilities is less than  $2e^{-u}$ .

On the complementary event, we follow the connections from a point of  $\mathcal{T}_k$  back to  $z_0$ . The value of the process at every net point is then bounded by the bound at  $z_0$ , plus the bound for its connection at level one, plus

$$\sum_{k \geq 2} 2^{2-k} (s\sqrt{h_k} + bh_k).$$

Since the sums of  $2^{-k}$ ,  $k2^{-k}$ , and  $\sqrt{k}2^{-k}$  converge, and  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ , this expression is at most

$$C\{s[\sqrt{\log(2A) + q_{\text{ent}} + \sqrt{u}}] + b[\log(2A) + q_{\text{ent}} + u]\}.$$

Every  $z \in \mathcal{T}$  is the limit, with respect to  $d$ , of net points  $z_k \in \mathcal{T}_k$ , since  $d(z, z_k) \leq 2^{1-k}$ . On the event where the chosen version is continuous, we have  $F_{z_k} \rightarrow F_z$ , and the bound extends to every point. The supremum equals a countable supremum and is therefore measurable. Restoring  $D_1$  proves the tail bound. If  $s = b = 0$ , the same countable argument applies: the value at  $z_0$  and every increment along a connection vanish almost surely. This also covers points at distance zero.

For the moments, we use that a nonnegative random variable  $X$  with  $\mathbb{P}\{X > \alpha_0 + c\sqrt{u} + d_0u\} \leq 2e^{-u}$  for all  $u \geq 0$  satisfies  $\|X\|_{L^p} \leq \alpha_0 + C(c\sqrt{p} + d_0p) + C(c + d_0)$ . To see this, let  $E$  be an exponential random variable with  $\mathbb{P}(E > v) = e^{-v}$ . Since  $\min\{1, 2e^{-u}\} = \mathbb{P}(E + \log 2 > u)$ , the variable  $X$  is stochastically dominated by  $\alpha_0 + c\sqrt{E + \log 2} + d_0(E + \log 2)$ . Minkowski's inequality and

$$\mathbb{E}E^r = \int_0^\infty t^r e^{-t} dt \leq 2(2r/e)^r \quad (r > 0)$$

imply the claim; the last inequality follows by maximizing  $t^r e^{-t/2}$  and integrating the remaining factor  $e^{-t/2}$ , and we apply it with  $r = p/2$  and  $r = p$ . In our case,  $\alpha_0 = CD_1(s\sqrt{H} + bH)$ ,  $c = CD_1s$ , and  $d_0 = CD_1b$ , and  $H \geq \log 2$  absorbs the term  $C(c + d_0)$ . This proves the  $L^p$  bound, including the bound on the expectation at  $p = 1$ .

For the finite family, we apply the tail bound for each  $Y_i$  with  $u + \log(2N)$  in place of  $u$  and sum the  $N$  probabilities of failure; then we add the deterministic error  $e$ . The same comparison with an exponential random variable yields the moment bound.

Every step above is an inequality under one fixed probability measure, and we may therefore apply it separately under each conditional kernel. In the applications, the supremum is over rational times and a finite spatial grid, or over a set on which the process is continuous; its measurability is part of the hypotheses. The nets are constructed for the fixed value of the conditioning variable and require no random choice. Bounds on the parameters that are uniform in the conditioning value give bounds that are uniform in that value.  $\square$

*Proof of the chaining assertion in Lemma 16.1.* We use the chosen continuous version of  $X$ , which has the same fixed-point and increment laws. If  $\sigma > 0$ , we apply Lemma 16.4 to

the function  $d/\sigma$ , with  $s = \sqrt{2}\sigma$ ,  $b = 0$ ,  $D = 1$ ,  $q_{\text{ent}} = v$ , the given  $A$ , and an arbitrary point  $i_0 \in I$ . The moment generating function bounds imply the required tails. Indeed, for a variable  $Y$  with  $\mathbb{E}e^{zY} \leq e^{z^2 r^2/2}$  for all real  $z$ , minimizing  $e^{-zx} \mathbb{E}e^{zY}$  at  $z = \pm x/r^2$  gives  $\mathbb{P}\{|Y| > x\} \leq 2e^{-x^2/(2r^2)}$ , which is the required tail with  $u = x^2/(2r^2)$ . Since the one-point variance proxy is at most  $\sigma$  at every point, the choice of  $i_0$  is arbitrary. The hypothesis on nets concerns radii  $\eta < 1$ ; at radius one, a single point is a net, since  $d/\sigma \leq 1$ . The lemma then gives the displayed tail and expectation bounds, where we use  $\sqrt{\log(2A) + v} \leq \sqrt{\log(2A)} + \sqrt{v}$ . If  $\sigma = 0$ , every value of the process on the countable dense set vanishes almost surely by its moment generating function bound, and the claim follows from the continuity of the chosen version.  $\square$

## 16.4 Concentration of conditionally centered reconstructions

**One renormalization group step.** Fix an odd integer  $M \geq 3$ , an integer  $J \geq 1$ ,  $B = M^J$ , and the parameters of Theorems 13.8 and 15.14: the analytic radii, the cutoff amplitude  $A_w$ , the bound  $E_*$  on the effective interaction, and the larger construction domains. Fix  $L > 0$  and the continuum coupling  $g_3 > 0$ . Let  $a = L/n$  be a mesh such that  $n/B \geq 4$  is an integer. We consider the renormalization group step from the lattice of mesh  $Ba$ , which we call the coarse lattice, to the lattice of mesh  $a$ , preceded by  $d \geq 0$  averaging steps; we call  $d$  the depth. Its intermediate scales are indexed by  $0 \leq i \leq J-1$ . The intermediate scale  $i$  has mesh  $a_i = M^i a$  and coupling  $g_i = g_3 \sqrt{a_i} = M^{i/2} g$ , where  $g = g_3 \sqrt{a}$ , so that  $g^2 = g_3^2 a$ . The largest coupling of the step is  $t = g_{J-1} = M^{(J-1)/2} g$ . The normalized fluctuation variables  $w = (w_{i,e})$  take values in the fluctuation cutoff domain  $\Omega_p = \prod_{i,e} \{w_{i,e} \in \mathbb{R}^3 : |w_{i,e}| \leq p_i\}$ , whose radii satisfy  $p_i \leq p(t) = A_w \log(1/t)$ . We write  $q_i = g_i w_i$  for the corresponding coupling-scaled fluctuation inputs, which range over a product of balls of radius at most  $Q = \max_i g_i p_i \leq tp(t)$ .

**The conditional fluctuation kernel.** Let  $R_{\text{supp}}$  be a support radius, either  $R_{\text{supp}} = r$  or  $R_{\text{supp}} = \rho$  as described below, and let  $c \in B_{R_{\text{supp}}}$  be a coarse field, with  $\theta = \text{Log } c$ . Its conditional fluctuation kernel is

$$\begin{aligned} V_{d,c}^{[R_{\text{supp}}]}(w) &= \frac{1}{2}|w|^2 + H_{d,J}(\theta, w) + R_{\text{in}}^{[R_{\text{supp}}]}(e^{\varphi_\theta(w)}) - R_{\text{in}}^{[R_{\text{supp}}]}(e^{\varphi_\theta(0)}), \\ Z_{d,c}^{[R_{\text{supp}}]} &= \int_{\Omega_p^{[R_{\text{supp}}]}} e^{-V_{d,c}^{[R_{\text{supp}}]}(w)} dw, \\ d\mu_{d,c}^{[R_{\text{supp}}]}(w) &= \frac{\mathbf{1}_{\Omega_p^{[R_{\text{supp}}]}}(w) e^{-V_{d,c}^{[R_{\text{supp}}]}(w)}}{Z_{d,c}^{[R_{\text{supp}}]}} dw. \end{aligned} \tag{16.15}$$

Here  $\varphi_\theta(w) = \Phi_{d,J}(\theta, (g_i w_i)_i)$  is the reconstruction of Theorem 13.2, at which the effective interaction is evaluated, and  $R_{\text{in}}^{[R_{\text{supp}}]}$  is the effective interaction produced by the

$d$  preceding averaging steps. The function  $H_{d,J}(\theta, w)$  collects the remaining terms of the conditional fluctuation potential: the part of the effective classical action beyond its quadratic approximation, the Haar disintegration densities, and the logarithms of the Jacobian determinants of the conditional Hessian normalizations. The density in (16.15) is exactly the kernel  $k(c, dw)$  of Theorem 13.8 (compare (4.8)). The subtraction of  $R_{\text{in}}^{[R_{\text{supp}}]}(e^{\varphi_\theta(0)})$  depends only on  $c$  and cancels in the normalized kernel. The domain  $\Omega_p^{[R_{\text{supp}}]}$  does not depend on  $c$ . Since the potential is smooth on the compact closed product and the exponential is positive,  $0 < Z_{d,c}^{[R_{\text{supp}}]} < \infty$ . When no confusion can arise, we write  $\mu_{d,c}$ ,  $V_c$ , and  $\Omega_p$ .

For  $R_{\text{supp}} = r$ , the coarse field ranges over the small-link domain  $B_r$ , and  $\mu_{d,c}$  is the conditional law, given the coarse field, of the fluctuation variables of the step under the conditioned Wilson measure. Indeed, once the coarse field is fixed, the restrictions at coarser scales are fixed factors and cancel in the normalized kernel (Theorem 13.8). For  $R_{\text{supp}} = \rho$ , we assume that a separate instance of Theorem 13.8 with support radius  $\rho$  is given. It has its own incoming effective interaction  $R_{\text{in}}^{[\rho]}$ , its own fluctuation cutoff domain, normalized kernel, and change-of-gauge identity, and the image conditions of Theorem 13.2 hold for all coarse fields in  $B_\rho$ , including the margins of Lemma 16.5 below. This instance is a different conditional measure; it is not obtained by evaluating the auxiliary integral of Section 14 at coarse fields in  $B_\rho \setminus B_r$ , and we make no statement about that integral there. In Section 17, only the case  $R_{\text{supp}} = r$  is used.

**The reconstruction.** Let  $\Phi(c, q)$  be the logarithm of a reconstruction of the fine field, in a gauge transformation that depends only on the coarse field  $c$ . We require the following properties on the whole product of coarse fields  $c \in B_{R_{\text{supp}}}$  and coupling-scaled fluctuation inputs of radius at most  $Q$ . The image of the reconstruction lies in the domain prescribed by Theorem 13.2. Its derivatives in  $q$  satisfy the absolute row bounds and the volume-normalized column bounds of that theorem, with each variable of the intermediate scale  $i$  normalized by its own cell volume  $a_i^3$ . The map  $\Phi_{d,J}$  of Theorem 13.2 and the logarithmic form  $\hat{\Phi}_d$  of the reconstruction maps  $\hat{\mathcal{R}}$  of Theorem 15.13 have these properties; see Lemma 16.5. We set

$$A(w) = a^{-1} \Phi(c, (g_i w_i)_i),$$

the lattice potential of the reconstructed field, and we view it as a cell function: the component of  $A$  on a positive link  $(y, y + ae_i)$  is extended as a constant on the cell  $Q_y$ .

**The domain conditions.** The next lemma collects the conditions under which the whole product of fluctuation variables lies in the domain where the coefficient bounds of Theorem 13.2 and the Hessian bound of Theorem 13.8 hold. It fixes the order of choices: the larger construction domains and coefficient constants first, then the support radius, and the coupling bound  $g_*$  last;  $J$  is fixed throughout.

**Lemma 16.5.** *Let  $K_* \geq 2$ ,  $\rho$ ,  $s_J$ ,  $r_J$ ,  $C_J$ , and the coefficientwise majorant series*

$$\mathfrak{M}_J(u, v) = B^{-1}Z(u) + \frac{C_J v}{1 - C_J(u + v)}$$

*for the absolute row sums of the reconstruction, be as in Theorem 13.2. Thus  $Z$  has nonnegative coefficients and no constant term, and*

$$4K_*\rho < r_J, \quad C_J 4K_*(\rho + s_J) < 1, \quad \mathfrak{M}_J(4K_*\rho, 4K_*s_J) < \rho/2.$$

*For the reconstruction and conditional-potential assertions, fix a support radius  $0 < R_{\text{supp}} \leq \rho$  and a conditional-kernel instance on  $B_{R_{\text{supp}}}$ . Assume that this ball of coarse fields satisfies the coordinate-domain conditions of that instance, and that its product with the coupling-scaled fluctuation inputs lies in the domain of the chosen reconstruction, with the fixed radius ratios; for the reconstruction maps of Theorem 15.13, assume that this product lies in their common inner domain. In the application below  $R_{\text{supp}} = r$ , or  $R_{\text{supp}} = \rho$  with the separately hypothesized instance and its own domains. For this fixed ball and instance, choose  $Q_\circ$  with  $0 < Q_\circ \leq s_J$  and  $Q_\circ < (r_J - \rho)/2$ , inside every further domain of coarse fields and coupling-scaled fluctuation inputs used by the reconstruction, and set  $\eta_\circ = 1 - C_J(\rho + s_J)$ . Then the following statements hold.*

1. *We have  $\eta_\circ > 0$  and*

$$0 < R \leq \rho, \quad 0 \leq Q \leq Q_\circ \implies R + Q < r_J, \quad 1 - C_J(R + Q) \geq \eta_\circ. \quad (16.16)$$

2. *For  $0 \leq u \leq \rho$  and  $0 \leq v \leq Q_\circ$ ,*

$$\partial_v \mathfrak{M}_J(u, v) = \frac{C_J(1 - C_J u)}{[1 - C_J(u + v)]^2} \leq C_J \eta_\circ^{-2}.$$

*On the fixed product of coarse fields  $c \in B_{R_{\text{supp}}}$  and coupling-scaled fluctuation inputs of radius at most  $Q_\circ$ , the normalized Jacobian  $g^{-1}D_w \Phi(c, (g_i w_i)_i)$ , with each fluctuation variable attached to a cell of the lattice of mesh  $a$ , and each of its mixed derivatives of fixed order in the coarse field and the fluctuation variables, satisfy (16.8) with  $a_0 = a$  and a constant depending only on  $J$ , the fixed construction domains, the order of differentiation, and  $\eta_\circ$ .*

3. *For  $0 < R \leq \rho$  put  $Q_{\text{gate}}(R) = \min\{Q_\circ, R\eta_\circ/(8C_J)\}$ . If  $0 \leq Q \leq Q_{\text{gate}}(R)$ , then  $\mathfrak{M}_J(R, Q) < 3R/16 < R/2$ .*
4. *Fix  $0 < R \leq \rho$ . There is  $0 < g_* \leq t_*$ , where  $t_*$  is the threshold of the selected instance of Theorem 13.8, depending only on the fixed construction data and on  $R$ , such that for all  $0 < t \leq g_*$ ,*

$$\begin{aligned} tp(t) &\leq Q_{\text{gate}}(R), \quad 1 - Ctp(t) > 0, \\ \frac{C[tp(t) + t^2(1 + E_*)]}{[1 - Ctp(t)]^3} &\leq \frac{1}{2}, \end{aligned} \quad (16.17)$$

where  $p(t) = A_w \log(1/t)$  and  $C$  is the constant in the Hessian bound of Theorem 13.8 (not the row constant  $C_J$ ).

In particular, for the fixed support radius and instance specified above, let  $g_*$  be as in the fourth part with  $R = R_{\text{supp}}$ , let  $0 < t \leq g_*$ , and let the radii of the fluctuation cutoff domain satisfy  $p_i \leq p(t)$ . Then for every coarse field in  $B_{R_{\text{supp}}}$  the whole product of fluctuation variables, and not only the point  $w = 0$ , is mapped into the product of the first part; the image condition  $\mathfrak{M}_J(R, Q) < R/2$  of Theorem 13.2 holds with  $Q = \max_i g_i p_i$ ; and the conditional fluctuation potential satisfies  $D_w^2 V_c \geq I/2$  on the closed product.

*Proof.* Since  $K_* \geq 2$ , we have  $C_J(\rho + s_J) < 1/(4K_*) < 1$ , and hence  $\eta_o > 0$ . If  $0 < R \leq \rho$  and  $0 \leq Q \leq Q_o$ , then  $R + Q \leq \rho + Q_o < \rho + (r_J - \rho)/2 < r_J$  and  $1 - C_J(R + Q) \geq 1 - C_J(\rho + s_J) = \eta_o$ . This proves (16.16). The choice of  $Q_o$  is possible because  $r_J - \rho > 0$ , and it can be made strictly inside every further domain used by the map, since these domains are open and have fixed radius ratios. For the reconstruction maps of Theorem 15.13, the required room comes from the hypothesis that the fixed product of coarse fields and coupling-scaled fluctuation inputs lies in their common inner domain with fixed radius ratios. We use their coefficient bound for mixed derivatives at fixed  $J$  and their uniform interior-domain conditions, rather than a majorant uniform in  $J$ .

Every product in (16.16) is contained in the product with radii  $4K_*\rho$  and  $4K_*s_J$ . By Theorem 13.2, the image of this larger product lies in  $B_{\rho/2}$ , with interior margins for the intermediate classical, logarithm, and Haar disintegration conditions; subsets inherit these margins. On this product, the formula for  $\partial_v \mathfrak{M}_J$  follows by differentiation, and the bound follows from  $1 - C_J u \leq 1$  and  $1 - C_J(u + v) \geq \eta_o$ . The derivatives of the column majorants with respect to one fixed input give the volume-normalized column sums, with the same lower bound  $\eta_o$  on the denominator. Each further derivative of fixed order costs only a factorial, the order, and inverse powers of  $\eta_o$ ; it does not depend on the number of lattice points. These are derivatives of the absolute coefficient series, not consequences of a bound on the values of a holomorphic function. It remains to convert the normalizations. A fluctuation variable of the intermediate scale  $i$  lives on the lattice of mesh  $a_i = M^i a$ , and its column is normalized by  $a_i^3$ . Attaching it to a cell of the lattice of mesh  $a$  changes the normalization by the factor  $M^{3i} \leq B^3$ , and replacing  $q_i$  by  $g_i w_i$  and dividing by  $g$  multiplies the derivative by  $g_i/g = M^{i/2}$ . At most a number of variables depending only on  $J$  is attached to each cell. These conversions only enlarge  $C_J$  by a factor depending on  $J$ , and they yield (16.8) for the normalized Jacobian and its mixed derivatives of fixed order. This proves the second part.

For the third part, the coefficients of  $Z$  are nonnegative and  $Z(0) = 0$ . Comparing term by term and using  $(R/(4K_*\rho))^k \leq R/(4K_*\rho)$  for every degree  $k \geq 1$ , we find  $Z(R) \leq (R/(4K_*\rho))Z(4K_*\rho)$  for  $0 < R \leq \rho$ . Since  $B^{-1}Z(4K_*\rho) \leq \mathfrak{M}_J(4K_*\rho, 4K_*s_J) < \rho/2$ , we

obtain

$$B^{-1}Z(R) \leq \frac{R}{4K_*\rho} B^{-1}Z(4K_*\rho) < \frac{R}{8K_*} \leq \frac{R}{16}.$$

If  $Q \leq Q_{\text{gate}}(R)$ , then by (16.16),

$$\mathfrak{M}_J(R, Q) \leq B^{-1}Z(R) + C_J Q / \eta_\circ < \frac{R}{16} + \frac{R}{8} = \frac{3R}{16} < \frac{R}{2}.$$

For the fourth part, let  $x = \log(1/t)$ . Then  $tp(t) = A_w x e^{-x} \rightarrow 0$  as  $t \downarrow 0$ , since  $e^x \geq x^3/6$  for  $x > 0$ , and  $t^2(1 + E_*) \rightarrow 0$ . By the definition of a limit, the three conditions in (16.17) hold for all  $t$  in an interval  $(0, g_*]$ , not only for one value of  $t$ . This is a single reduction of the upper bound on the coupling, which uses only the fixed construction data and  $R$ ; it introduces no dependence on the lattice side, the depth, or the coarse field.

Finally, let  $R = R_{\text{supp}}$ , let  $t \leq g_*$ , and let  $p_i \leq p(t)$ . Since  $g_i \leq t$ , we have  $Q = \max_i g_i p_i \leq tp(t) \leq Q_{\text{gate}}(R) \leq Q_\circ$ . Hence, for every coarse field in  $B_{R_{\text{supp}}}$ , the whole product of fluctuation variables lies in the fixed product on which the reconstruction and the conditional potential of the chosen instance are defined, and in the product of the first part. The third part gives  $\mathfrak{M}_J(R, Q) < R/2$ . Shrinking  $Q$  does not enlarge the domain of coarse fields; the ball  $B_{R_{\text{supp}}}$  and its conditional kernel are given by the hypothesis. In particular, the majorant of Theorem 13.2 is never evaluated beyond this product or across its pole, and both the image condition and the domain conditions under which it is stated hold. The last condition in (16.17) is its numerical sufficient bound on the Hessian perturbation; since also  $t \leq g_* \leq t_*$  for the chosen instance of Theorem 13.8, that theorem implies  $D_w^2 V_c \geq I/2$  on the closed product.  $\square$

**Theorem 16.6.** Fix  $M, J$ , and the parameters of Theorems 13.8 and 15.14, and fix  $L, T_0 > 0$ . Let  $R_{\text{supp}} = r$ , or  $R_{\text{supp}} = \rho$  under the additional hypothesis on the separate instance described above, and let  $g_*$  be as in Lemma 16.5 with  $R = R_{\text{supp}}$ . Let  $a = L/n$  be a mesh with  $n/B \geq 4$ , let  $d \geq 0$ , let  $g^2 = g_3^2 a$ , and assume that  $t = M^{(J-1)/2} g \leq g_*$ . Let  $c \in B_{R_{\text{supp}}}$ , let  $\mu_{d,c}$  be the conditional fluctuation kernel (16.15), and let  $\Phi$  be a reconstruction with the properties described above; the reconstruction maps of Theorem 15.13 are an example. Put  $A(w) = a^{-1} \Phi(c, (g_i w_i)_i)$ , viewed as a cell function, and

$$F = A - \mathbb{E}_{\mu_{d,c}} A, \quad \|F\|_{-\alpha; T_0} = \sup_{0 < s \leq T_0} s^{\alpha/2} \|P_s F\|_\infty,$$

where  $P_s$  is the continuum heat semigroup acting componentwise. Then for every  $0 < \alpha < 3/2$  and  $u \geq 0$ ,

$$\mu_{d,c} \left\{ \|F\|_{-\alpha; T_0} > C g_3 a^{\alpha-1/2} \sqrt{\log(2 + L/a) + u} \right\} \leq C e^{-u}.$$

For every  $p \geq 1$  and  $\lambda \geq 0$ , with  $\tau_a = C g_3 a^{\alpha-1/2}$ ,

$$\begin{aligned} (\mathbb{E}_{\mu_{d,c}} \|F\|_{-\alpha; T_0}^p)^{1/p} &\leq C_p g_3 a^{\alpha-1/2} \sqrt{\log(2 + L/a)}, \\ \mathbb{E}_{\mu_{d,c}} e^{\lambda \|F\|_{-\alpha; T_0}} &\leq \exp \left\{ C \lambda \tau_a \sqrt{\log(2 + L/a)} + C \lambda^2 \tau_a^2 \right\}. \end{aligned}$$

In particular, if  $1/2 < \alpha < 3/2$  and  $a \downarrow 0$  with  $g_3$  fixed (equivalently, with  $L$  and  $\gamma = g_3^2 L$  fixed), then for each fixed  $p$  the conditional  $L^p$  norms of  $\|F\|_{-\alpha; T_0}$  tend to zero, for each fixed  $\lambda$  the exponential moments  $\mathbb{E}_{\mu_{d,c}} e^{\lambda \|F\|_{-\alpha; T_0}}$  tend to one, and the bounds on  $\mathbb{E}_{\mu_{d,c}} \|F\|_{-\alpha; T_0}$  are summable along every geometric sequence of meshes; these statements are uniform in  $c$  and  $d$ .

For every deterministic real  $\varphi \in L^2(\mathbb{T}_L^3)$ , paired componentwise with  $A$ ,

$$\text{Var}_{\mu_{d,c}} \langle A, \varphi \rangle \leq C_J g_3^2 a^2 \|\varphi\|_2^2, \quad \log \mathbb{E}_{\mu_{d,c}} e^{z \langle F, \varphi \rangle} \leq C_J z^2 g_3^2 a^2 \|\varphi\|_2^2 \quad (z \in \mathbb{R}).$$

For finitely many real  $\varphi_1, \dots, \varphi_m \in L^2(\mathbb{T}_L^3)$  and every compact set  $\mathcal{K} \subset \mathbb{C}^m$ ,

$$\sup_{z \in \mathcal{K}} \left| \mathbb{E}_{\mu_{d,c}} \exp \left( \sum_{j=1}^m z_j \langle F, \varphi_j \rangle \right) - 1 \right| \longrightarrow 0$$

as  $a \downarrow 0$  with  $g_3$  fixed.

If  $c \in D_{R_{\text{supp}}}$ , choose a coarse-lattice gauge transformation  $k$  with  $c' = c^k \in B_{R_{\text{supp}}}$ . Use the kernel  $\mu_{d,c'}^{[R_{\text{supp}}]}$  and evaluate the reconstruction at  $c'$ . All the preceding estimates hold uniformly in  $c$  and in this choice of representative.

The constants depend only on  $M, J$ , the fixed construction domains and their margins, the Hessian bound, the row and column constants of the derivatives of  $\Phi$ ,  $\alpha$ ,  $L$ ,  $T_0$ , and, where displayed,  $p$ . They do not depend on  $n, d$ , or  $c$ , and they do not depend on the coupling below  $g_*$ , except through the displayed factors of  $g_3$ .

*Remark 16.7.* Theorem 16.6 bounds the fluctuation  $F$  about the conditional expectation under the conditional fluctuation kernel. It gives no bound on  $\mathbb{E}_{\mu_{d,c}} A$ , which is estimated in Lemma 17.1. The norm  $\|\cdot\|_{-\alpha; T_0}$  is not invariant under gauge transformations that vary in space, and the estimates refer to the chosen gauge representative of the reconstruction.

*Proof of Theorem 16.6.* We fix  $c$  and the kernel  $\mu = \mu_{d,c}$  for the whole proof; all expectations and probabilities are taken with respect to  $\mu$ , and nothing in the proof differentiates in  $c$ . The constants have the dependence stated in the theorem, and  $C_J$  may increase by factors depending on  $M, J$ , and the fixed construction domains. The proof has seven steps. We verify the hypotheses of the concentration theorem and bound the Jacobian of  $w \mapsto A(w)$ . We then bound  $P_s F$  at a fixed time by chaining over space, using the cell integrals of the heat kernel, take the supremum over times by a dyadic decomposition, and derive the moments, the exponential moments, and the limits. Finally, we estimate deterministic source pairings and their generating functions, and we transport the conclusions to coarse fields in  $D_{R_{\text{supp}}}$ .

**Step 1: the kernel and its convexity.** Let  $R_{\text{supp}}$  and  $g_*$  be as in the statement. We apply Lemma 16.5 to this support radius and to the corresponding instance of the conditional kernel: for  $R_{\text{supp}} = r$  the kernel of Theorem 13.8 on  $B_r$ , and for  $R_{\text{supp}} = \rho$  the

separately hypothesized instance on  $B_\rho$ , each with its own domains. By that lemma, for every  $c \in B_{R_{\text{supp}}}$ , the whole product  $\Omega_p$  is mapped into the fixed product on which the reconstruction and the potential are defined, and the image condition of Theorem 13.2 holds. For the reconstruction of Theorem 13.2, each intermediate classical, logarithm, and Haar disintegration input lies in the enlarged product of that theorem, and its interior margin remains valid. For the reconstruction maps of Theorem 15.13, the same inputs lie in their inner domain, where that theorem preserves the uniform interior-domain conditions and the normalized conditional kernels. We use only the joint coefficient bound of that family at fixed  $J$ . In the case  $R_{\text{supp}} = \rho$ , the image condition is required on all of  $B_\rho$ , including the margins of Lemma 16.5, and not only on  $B_r$ ; the analytic radius and the coefficient constants of that instance cover  $B_\rho$  by hypothesis.

By the last condition in (16.17), Theorem 13.8 gives  $D_w^2 V_c \geq I/2$  on the closed convex product, with a  $C^2$  extension to an open neighborhood. Then  $\mu$  satisfies the hypotheses of Theorem 5.4 with  $m = 1/2$ ; that theorem includes the faces and corners of the product of balls. We use only the following two of its conclusions: for every function  $H$  of  $w$  in  $H^1(\mu)$  and every real Euclidean-Lipschitz function  $H$ ,

$$\text{Var}_\mu H \leq 2\mathbb{E}_\mu |\nabla_w H|^2, \quad \log \mathbb{E}_\mu e^{z(H - \mathbb{E}_\mu H)} \leq z^2 \text{Lip}_w(H)^2 \quad (z \in \mathbb{R}). \quad (16.18)$$

These inequalities require no independence under  $\mu$ .

The kernel does not depend on the gauge in which the observable  $A$  is computed: another gauge transformation depending only on  $c$  changes  $A$  but not  $V_c$ . The reconstruction maps of Theorem 15.13 are obtained from the same reconstruction before gauge transformation by a gauge transformation depending only on the coarse field. By Theorems 15.13 and 15.14, they preserve the fluctuation cutoff domain, the free logarithmic coordinates, the effective classical action, the Haar disintegration density, the Jacobian determinants of the conditional Hessian normalizations, the incoming effective interaction, the coarse field, and the normalization constant. Hence (16.15) defines the same conditional measure for the reconstruction maps, and its Hessian is that of the same potential.

**Step 2: the Jacobian.** We write  $\xi$  for a scalar fluctuation coordinate,  $e$  for a scalar component of the output on a positive link, and  $\varphi(w) = \Phi(c, (g_i w_i)_i)$ . By the derivative bounds of Theorem 13.2 on the whole product, converted as in the second part of Lemma 16.5, we have

$$\sup_{w,e} \sum_{\xi} |\partial_{w_\xi} \varphi_e(w)| \leq C_J g, \quad \sup_{w,\xi} \sum_e |\partial_{w_\xi} \varphi_e(w)| \leq C_J g. \quad (16.19)$$

We convert the normalizations as in the proof of Lemma 16.5. The column of a variable of the intermediate scale  $i$  is normalized by  $a_i^3$ , and the column bound of Theorem 13.2 reads  $a^3 \sum_e |\partial_{q_\xi} \Phi_e| \leq C_J a_i^3$ . The substitution  $q_\xi = g_i w_\xi$  contributes the factor  $g_i =$

$M^{i/2}g$ , and the passage to counting coordinates costs the factor  $M^{3i}$ ; the row bound is converted with the same bounded ratios of couplings. These factors, and the number of variables per cell, depend only on  $J$ , and (16.19) holds with no factor depending on the number of cells or on  $d$ . The gauge transformation depends only on  $c$  and has no derivative in  $w$ , and the derivatives of the conditional Hessian normalizations and of the reconstruction before gauge transformation are included in (16.19). For the reconstruction maps of Theorem 15.13, that theorem provides the joint coefficient bound, every volume-normalized column, and the image conditions on their common domains, uniformly in the depth and with deterministic thresholds that do not depend on the coupling or on the sample. The same computation applies, with the constant of that family at fixed  $J$ .

By Schur's test (16.14) in counting coordinates, we have

$$\sup_w \|D_w A(w)\|_{\ell^2 \rightarrow \ell^2} \leq K_a, \quad K_a = C_J g/a = C_J g_3 a^{-1/2}, \quad (16.20)$$

where the norm is from the fluctuation variables to the components of  $A$  on the links. Since the product is convex, integrating the derivative along the segment from  $w$  to  $w'$  gives

$$\|A(w) - A(w')\|_{\ell^2} \leq K_a \|w - w'\|_2. \quad (16.21)$$

**Step 3: a fixed time.** Fix one scalar component, given by a direction and a color, and let  $X_s(x) = P_s F(x)$ . Since  $F$  is constant on the half-open cells, we have

$$X_s(x) = \sum_y \omega_y(s, x) (A_y(w) - \mathbb{E}_\mu A_y), \quad (16.22)$$

by integration of the cell function against the continuum heat kernel. As a function of  $w$ , the right side is a linear combination of the components of  $A$ . By (16.20), its Lipschitz constant is at most  $K_a \|\omega(s, x)\|_2$ , and the Lipschitz constant of  $X_s(x) - X_s(y)$  is at most  $K_a \|\omega(s, x) - \omega(s, y)\|_2$ . Both have mean zero, because the subtracted mean is the conditional expectation under  $\mu$ . By (16.18) and Lemma 16.1 with horizon  $T_0$ , their sub-Gaussian parameters, in the normalization  $e^{z^2 \sigma^2 / 2}$ , can be taken as

$$\sigma_s = C_J K_a \min\{1, a^{3/2} s^{-3/4}\} \quad (16.23)$$

and, when  $s \geq a^2$ ,

$$d_s(x, y) = C_J K_a a^{3/2} s^{-3/4} \min\{1, \text{dist}(x, y) / \sqrt{s}\}. \quad (16.24)$$

The constants include the factor  $\sqrt{2}$  that converts (16.18) to this normalization. The function  $d_s$  is nonnegative and symmetric, and after enlarging the constants we have  $\sup d_s \leq \sigma_s$  for  $s \geq a^2$ .

For  $s \geq a^2$ , a cubic grid on  $\mathbb{T}_L^3$  of spacing at most  $c\eta\sqrt{s}$  is an  $\eta\sigma_s$ -net for  $d_s$ , which gives

$$N(\mathbb{T}_L^3, d_s, \eta\sigma_s) \leq C(1 + L/\sqrt{s})^3 \eta^{-3} \quad (0 < \eta < 1).$$

For each  $w$ , the function  $x \mapsto P_s F(x)$  is smooth for  $s > 0$ . Its supremum is determined by a countable dense set, and the process is continuous with respect to  $d_s$ . We set  $\ell_a = \log(2 + L/a)$ . Applying the chaining assertion of Lemma 16.1 with  $A = C(1 + L/\sqrt{s})^3$  and  $v = 3$ , we obtain

$$\mathbb{P}_\mu \left\{ \|P_s F\|_\infty > CK_a a^{3/2} s^{-3/4} \sqrt{\ell_a + u} \right\} \leq Ce^{-u}, \quad \mathbb{E}_\mu \|P_s F\|_\infty \leq CK_a a^{3/2} s^{-3/4} \sqrt{\ell_a}. \quad (16.25)$$

Here we used  $\log(2 + C(1 + L/\sqrt{s})^3) \leq C\ell_a$ , which holds because  $s \geq a^2$ . There are nine scalar components; a union bound over them and the equivalence of norms on the components change only  $C$ .

**Step 4: the supremum over times.** Each scalar cell variable  $A_y - \mathbb{E}_\mu A_y$  is centered and  $K_a$ -Lipschitz. The exponential bound in (16.18), applied to it and to its negative, yields

$$\mathbb{P}_\mu \{|A_y - \mathbb{E}_\mu A_y| > 2K_a \sqrt{h}\} \leq 2e^{-h} \quad (h \geq 0).$$

There are  $9n^3$  scalar cell variables. Choosing  $h = u + \log(18n^3)$  and summing, we obtain

$$\mathbb{P}_\mu \left\{ \|F\|_\infty > CK_a \sqrt{\ell_a + u} \right\} \leq Ce^{-u}. \quad (16.26)$$

Since the heat semigroup contracts the supremum norm, we have, for  $0 < s \leq \min\{a^2, T_0\}$ ,

$$s^{\alpha/2} \|P_s F\|_\infty \leq a^\alpha \|F\|_\infty.$$

Together with (16.26), this implies the required bound at all these times, including the case  $T_0 < a^2$ .

Suppose now that  $T_0 \geq a^2$ . Let  $s_j = 2^j a^2$  for  $0 \leq j \leq r$ , where  $r$  is the largest integer with  $s_r \leq T_0$ . By the semigroup property and the contraction of the supremum norm, we have, for  $s_j \leq s \leq \min\{2s_j, T_0\}$ ,

$$s^{\alpha/2} \|P_s F\|_\infty \leq 2^{\alpha/2} s_j^{\alpha/2} \|P_{s_j} F\|_\infty. \quad (16.27)$$

The weighted scale at the time  $s_j$  satisfies

$$K_a a^{3/2} s_j^{\alpha/2-3/4} = K_a a^\alpha 2^{-j(3/4-\alpha/2)} \leq K_a a^\alpha, \quad (16.28)$$

where we used  $\alpha < 3/2$ . Further,  $r+1 \leq 1 + \log_2(T_0/a^2) \leq C_{T_0/L^2} \ell_a$ , since  $n = L/a \geq 4$ . We use (16.25) at every  $s_j$  with  $u + \log(r+2)$  in place of  $u$ , and (16.26) with the same parameter. The sum of the probabilities of failure is at most  $Ce^{-u}$ . Since  $\log(r+2) \leq C_{T_0/L^2} \ell_a$ , all the thresholds in (16.27) and (16.28) are bounded by

$$CK_a a^\alpha \sqrt{\ell_a + u} = Cg_3 a^{\alpha-1/2} \sqrt{\ell_a + u}.$$

This proves the tail bound for all meshes in the statement. The time supremum is measurable, because the heat field is continuous in  $(s, x)$  for  $s > 0$  and a countable dense subset of  $(0, T_0] \times \mathbb{T}_L^3$  determines it. For each finite lattice, the field is bounded, and the norm is finite.

**Step 5: moments, exponential moments, and limits.** Let  $\tau_a = Cg_3a^{\alpha-1/2}$ , with  $C$  large enough for the preceding bounds, and let  $Z = \|F\|_{-\alpha;T_0}$ . Since  $\sqrt{\ell_a + u} \leq \sqrt{\ell_a} + \sqrt{u}$ , the positive part of  $Z - C\tau_a\sqrt{\ell_a}$  has tail at most  $C \exp[-x^2/(C\tau_a^2)]$  at height  $x$ . Multiplying this tail by  $px^{p-1}$ , integrating over  $x > 0$ , and substituting  $x = \tau_av$ , we obtain  $C_p\tau_a^p$ . By the triangle inequality in  $L^p$  and  $\ell_a \geq \log 2$ , we have

$$\mathbb{E}_\mu Z \leq C\tau_a\sqrt{\ell_a}, \quad (\mathbb{E}_\mu Z^p)^{1/p} \leq C_p\tau_a\sqrt{\ell_a}.$$

The norm  $Z$  is itself a Lipschitz function of  $w$ , with constant at most  $C\tau_a$ . Indeed, by (16.21) and the bound on the cell integrals of the heat kernel, the difference of its values at  $w$  and  $w'$  is at most

$$CK_a\|w - w'\|_2 \sup_{0 < s \leq T_0} s^{\alpha/2} \min\{1, a^{3/2}s^{-3/4}\} \leq CK_a a^\alpha \|w - w'\|_2,$$

where in the last inequality we used  $s^{\alpha/2} \leq a^\alpha$  for  $s \leq a^2$ , and that  $s^{\alpha/2-3/4}$  is decreasing for  $s \geq a^2$ . The subtracted conditional mean does not depend on  $w$ . Applying (16.18) to  $Z$ , we obtain

$$\mathbb{E}_\mu e^{\lambda Z} \leq \exp\{\lambda \mathbb{E}_\mu Z + C\lambda^2 \tau_a^2\} \leq \exp\{C\lambda \tau_a \sqrt{\ell_a} + C\lambda^2 \tau_a^2\}.$$

Since the concentration theorem applies to every real Lipschitz function, no smoothness of  $Z$  is needed.

All limits and summability statements are at fixed  $L$  and  $g_3$ , equivalently at fixed  $L$  and  $\gamma = g_3^2 L$ . If  $\beta = \alpha - 1/2 > 0$ , then  $a^\beta \sqrt{\log(2 + L/a)} \rightarrow 0$  as  $a \downarrow 0$ , and every fixed conditional  $L^p$  norm therefore tends to zero, uniformly in  $c$  and  $d$ . The exponential moment is at least one because  $Z \geq 0$ , and its upper bound tends to one for fixed  $\lambda$ . For a geometric sequence of meshes  $a_k = a_* \mathfrak{b}^{-k}$  with a fixed  $\mathfrak{b} > 1$ , we have

$$a_k^\beta \sqrt{\log(2 + L/a_k)} \leq a_*^\beta \mathfrak{b}^{-k\beta} \left\{ \sqrt{\log(2 + L/a_*)} + \sqrt{k \log \mathfrak{b}} \right\},$$

and both resulting series converge; the ratio of consecutive terms in the second tends to  $\mathfrak{b}^{-\beta} < 1$ . This proves the summability of the bounds on the expected norms, without any independence between different meshes.

**Step 6: deterministic sources and generating functions.** For a real test field  $\varphi$ , let  $q_y = \int_{Q_y} \varphi(x) dx$ , with the pairing of directions and colors included. By the Cauchy–Schwarz inequality on each cell, we have

$$\sum_y |q_y|^2 \leq a^3 \|\varphi\|_2^2. \quad (16.29)$$

Combining this with (16.20), we find that the pairing  $H_\varphi = \langle A, \varphi \rangle$  has Lipschitz constant at most

$$K_a a^{3/2} \|\varphi\|_2 = C_J g_3 a \|\varphi\|_2,$$

and its gradient satisfies the same bound pointwise. The two inequalities in (16.18) then imply

$$\mathrm{Var}_\mu H_\varphi \leq C_J g_3^2 a^2 \|\varphi\|_2^2, \quad \log \mathbb{E}_\mu e^{z\langle F, \varphi \rangle} \leq C_J z^2 g_3^2 a^2 \|\varphi\|_2^2 \quad (z \in \mathbb{R}).$$

Next, fix finitely many real sources  $\varphi_1, \dots, \varphi_m$  and a compact set  $\mathcal{K} \subset \mathbb{C}^m$ , and let  $Y_z = \sum_j z_j \langle F, \varphi_j \rangle$ . By the variance bound and the triangle inequality in  $L^2$ , we have

$$(\mathbb{E}_\mu |Y_z|^2)^{1/2} \leq C_J g_3 a \sum_j |z_j| \|\varphi_j\|_2.$$

Since the real part of  $Y_z$  is the pairing of  $F$  with the real source  $\sum_j \Re(z_j) \varphi_j$ , the exponential bound gives

$$\mathbb{E}_\mu e^{2|\Re Y_z|} \leq \mathbb{E}_\mu e^{2\Re Y_z} + \mathbb{E}_\mu e^{-2\Re Y_z} \leq 2 \exp \left\{ C_J g_3^2 a^2 \left( \sum_j |z_j| \|\varphi_j\|_2 \right)^2 \right\}.$$

By the identity  $e^y - 1 = y \int_0^1 e^{\varrho y} d\varrho$  and the Cauchy–Schwarz inequality, uniformly for  $z \in \mathcal{K}$ ,

$$|\mathbb{E}_\mu e^{Y_z} - 1| \leq (\mathbb{E}_\mu |Y_z|^2)^{1/2} (\mathbb{E}_\mu e^{2|\Re Y_z|})^{1/2} \longrightarrow 0$$

as  $a \downarrow 0$  with  $g_3$  fixed. For each finite lattice, these functions of  $z$  are entire. Indeed, the domain of the fluctuation variables is compact and the reconstruction is continuous on a neighborhood of it; hence the exponential of the source pairing has a uniformly convergent power series on every compact set of sources. Complex test fields are handled by including their real and imaginary parts among the real sources.

**Step 7: coarse fields in  $D_{R_{\mathrm{supp}}}$ .** Let  $c \in D_{R_{\mathrm{supp}}}$ , and let  $c' = c^k$  and  $c'' = c^{k'}$  both lie in  $B_{R_{\mathrm{supp}}}$ . For support  $r$ , the change-of-gauge identity of Theorem 13.4(3), the reconstruction identity of Theorem 13.2(4), and Theorem 15.13(5) when the replaced reconstruction is used give a triangular fluctuation-coordinate map  $\mathcal{T}$  between the two small representatives. Its diagonal blocks are orthogonal; it preserves the cutoff product and has absolute Jacobian one. Thus

$$\mathcal{T}_* \mu_{d, c'}^{[R_{\mathrm{supp}}]} = \mu_{d, c''}^{[R_{\mathrm{supp}}]}.$$

For  $R_{\mathrm{supp}} = \rho$ , we use the change-of-gauge identity assumed for the separate instance in (16.15). At each small representative separately, the same radius  $Q$  and reconstruction-domain margins apply, so all preceding estimates hold with the same constants. This proves uniformity in  $c$  and the chosen small representative; it does not assert gauge invariance of the heat norms.  $\square$

## 16.5 Weighted heat convolutions

The Duhamel integrals for the Wilson flow in Section 24 lead to time convolutions of two powers, each regularized by the same positive parameter  $\epsilon$ ; in the applications,  $\epsilon = a_0^2$  for a mesh  $a_0$ . The following lemma estimates these scalar integrals.

**Lemma 16.8.** *Let  $\epsilon > 0$ ,  $t \geq 0$ ,  $0 \leq p < 1$ , and  $q \geq 0$ , and define*

$$I_{p,q}^\epsilon(t) = \int_0^t (s + \epsilon)^{-p} (t - s + \epsilon)^{-q} ds.$$

*For fixed  $p$  and  $q$ , uniformly in  $\epsilon$  and  $t$ ,*

$$I_{p,q}^\epsilon(t) \leq C_{p,q} \begin{cases} (t + \epsilon)^{1-p-q}, & 0 \leq q < 1, \\ (t + \epsilon)^{-p} [1 + \log(1 + t/\epsilon)], & q = 1, \\ \epsilon^{1-q} (t + \epsilon)^{-p}, & q > 1. \end{cases}$$

*For  $0 \leq t \leq \epsilon$  we also have  $I_{p,q}^\epsilon(t) \leq t\epsilon^{-p-q}$ , and for  $p = q = 1/2$  we have  $I_{1/2,1/2}^\epsilon(t) \leq \pi$ .*

*In particular, let  $1/2 < \alpha_{\text{fl}} < 2/3$ ,  $p = (\alpha_{\text{fl}} + 1)/2$ , and  $\epsilon = a_0^2$  with  $a_0 > 0$ . Then*

$$I_{p,3/4}^{a_0^2}(t) \leq C(t + a_0^2)^{-\alpha_{\text{fl}}/2 - 1/4}, \quad I_{p,5/4}^{a_0^2}(t) \leq C a_0^{-1/2} (t + a_0^2)^{-p},$$

$$\sup_{t \geq 0} \left\{ (t + a_0^2)^{\alpha_{\text{fl}} - 1/2} I_{p,3/4}^{a_0^2}(t) + (t + a_0^2)^{\alpha_{\text{fl}}} I_{p,5/4}^{a_0^2}(t) \right\} \leq C_{\alpha_{\text{fl}}} a_0^{\alpha_{\text{fl}} - 3/2}.$$

*The same bound holds if either of the two weights  $(t + a_0^2)^{\alpha_{\text{fl}} - 1/2}$  and  $(t + a_0^2)^{\alpha_{\text{fl}}}$  is replaced by the same power of  $t + a^2$ , for some  $0 < a \leq a_0$ . For  $p = 3\alpha_{\text{fl}}/2$ , the integrals with  $q = 0$  and  $q = 1/2$ , which correspond to the value and to one spatial difference of a heat convolution, are bounded by  $C(t + \epsilon)^{1-3\alpha_{\text{fl}}/2}$  and  $C(t + \epsilon)^{1/2-3\alpha_{\text{fl}}/2}$ , respectively. The constants depend only on the displayed exponents.*

*Proof.* All integrands are nonnegative. If  $t = 0$ , the integral is zero. If  $0 < t \leq \epsilon$ , each factor of the integrand is at most its value with  $s$  or  $t - s$  replaced by zero, which gives

$$I_{p,q}^\epsilon(t) \leq t\epsilon^{-p-q}. \quad (16.30)$$

Since  $\epsilon \leq t + \epsilon \leq 2\epsilon$  in this range, this implies the three general bounds, with a constant depending only on  $p$  and  $q$ ; in the case  $q = 1$ , we use  $1 + \log(1 + t/\epsilon) \geq 1$ .

For  $t \geq \epsilon$ , we split the integral at  $t/2$ . On the first half,  $t - s + \epsilon \geq (t + \epsilon)/2$ , which gives

$$\int_0^{t/2} (s + \epsilon)^{-p} (t - s + \epsilon)^{-q} ds \leq 2^q (t + \epsilon)^{-q} \int_0^{t/2} (s + \epsilon)^{-p} ds \leq \frac{2^q}{1-p} (t + \epsilon)^{1-p-q}.$$

On the second half, we set  $r = t - s$  and use  $s + \epsilon \geq (t + \epsilon)/2$ ; this half is at most

$$2^p (t + \epsilon)^{-p} \int_0^{t/2} (r + \epsilon)^{-q} dr.$$

The last integral is at most  $(t + \epsilon)^{1-q}/(1-q)$  if  $q < 1$ , it equals  $\log(1 + t/(2\epsilon))$  if  $q = 1$ , and it is at most  $\epsilon^{1-q}/(q-1)$  if  $q > 1$ . The first half is absorbed in the bounds for the

second half: for  $q = 1$ , it is of order  $(t + \epsilon)^{-p}$ , and for  $q > 1$  we use  $(t + \epsilon)^{1-q} \leq \epsilon^{1-q}$ . This proves the general estimate for all  $t$  and  $\epsilon$ .

For  $p = q = 1/2$  and  $t > 0$ , we decrease the two positive shifts to zero and obtain

$$I_{1/2,1/2}^\epsilon(t) \leq \int_0^t s^{-1/2}(t-s)^{-1/2} ds = \int_0^{\pi/2} 2 d\vartheta = \pi,$$

where we substituted  $s = t \sin^2 \vartheta$ . At  $t = 0$ , the bound is trivial.

Now let  $\alpha = \alpha_{\mathbb{H}}$  and  $p = (\alpha + 1)/2$ . Then  $p < 1$ , and the cases  $q = 3/4$  and  $q = 5/4$  of the general estimate yield

$$I_{p,3/4}^\epsilon(t) \leq C_\alpha(t + \epsilon)^{-\alpha/2-1/4}, \quad I_{p,5/4}^\epsilon(t) \leq C_\alpha \epsilon^{-1/4}(t + \epsilon)^{-(\alpha+1)/2}.$$

Multiplying by the two weights, we obtain

$$\begin{aligned} (t + \epsilon)^{\alpha-1/2} I_{p,3/4}^\epsilon(t) &\leq C_\alpha(t + \epsilon)^{\alpha/2-3/4}, \\ (t + \epsilon)^\alpha I_{p,5/4}^\epsilon(t) &\leq C_\alpha \epsilon^{-1/4}(t + \epsilon)^{(\alpha-1)/2}. \end{aligned} \tag{16.31}$$

Since both powers of  $t + \epsilon$  on the right are negative for  $\alpha$  in the stated range, both right sides are at most  $C_\alpha \epsilon^{\alpha/2-3/4}$ , which equals  $C_\alpha a_0^{\alpha-3/2}$  for  $\epsilon = a_0^2$ . Replacing either weight by the corresponding power of  $t + a^2$  decreases it, since  $a \leq a_0$  and both powers are positive. The short-time bound (16.30) remains available; in particular, none of these bounds replaces an integral that vanishes at  $t = 0$  by a singular expression.

Finally,  $p = 3\alpha/2 < 1$ , and the cases  $q = 0$  and  $q = 1/2$  of the general estimate give  $(t + \epsilon)^{1-3\alpha/2}$  and  $(t + \epsilon)^{1/2-3\alpha/2}$ , respectively. All constants in this proof are built from  $2^p$ ,  $2^q$ ,  $(1-p)^{-1}$ ,  $|1-q|^{-1}$  (when  $q \neq 1$ ), and fixed powers of two; they do not depend on any mesh ratio, coupling, lattice side, or depth.  $\square$

## 16.6 Heat consistency for finite-scale reconstructions

The last lemma of this section compares the discrete heat evolution of a sampled smooth field with the continuum heat evolution. It is applied in Section 25 to the cardinal interpolation of a finite-scale reconstruction, that is, of a field reconstructed from a fixed finite number of renormalization group steps. The family of fields is fixed before the mesh tends to zero, and no uniformity is claimed as the number of scales grows. The comparison concerns the linear heat evolution only; the corresponding statement for the quadratic Picard term is proved in Section 25.

For a smooth periodic function  $f$  we write  $\|f\|_{C^r}$  for the maximum of the supremum norms of its derivatives of order at most  $r$ . For a mesh  $a = L/n$  we write  $S_a f = f|_{\Gamma_a}$  for the restriction of  $f$  to the vertices.

**Lemma 16.9.** Fix  $L, T, H > 0$  with  $L/H \in \mathbb{N}$ , a finite collection of meshes  $a_1, \dots, a_l$  commensurate with  $H$ , and a deterministic bound  $M_H$ . Let  $\mathcal{F}_H$  be the set of functions  $f = \sum_{j=1}^l I_{a_j} u^{(j)}$ , where each  $u^{(j)}$  is an array of link variables on the lattice of mesh  $a_j$  with  $\|u^{(j)}\|_\infty \leq M_H$ ; it is a bounded finite-dimensional family of smooth periodic functions. Let  $a \downarrow 0$  run through meshes  $a = L/n$  such that  $a_j/a \in \mathbb{N}$  for every  $j$ . Then, uniformly in  $f \in \mathcal{F}_H$  and  $0 \leq t \leq T$ ,

$$\|P_t^a S_a f - S_a P_t f\|_\infty \leq C_H a^2,$$

and for every ordered product  $D_a^\epsilon$  of  $q \geq 1$  forward or backward differences, with corresponding product of partial derivatives  $\partial^\epsilon$ ,

$$\|D_a^\epsilon P_t^a S_a f - S_a \partial^\epsilon P_t f\|_\infty \leq C_{H,q} a.$$

For a general smooth periodic function  $f$  the two right sides may be replaced by  $CTa^2\|f\|_{C^4}$  and  $C_q[a\|f\|_{C^{q+1}} + Ta^2\|f\|_{C^{q+4}}]$ , respectively. These bounds include  $t = 0$ , the periodic seams, and the constant mode.

Let  $u_a(t) = P_t^a S_a f$  and  $u_a^\epsilon(t) = D_a^\epsilon P_t^a S_a f$ , and let  $I_a u_a(t)$  and  $I_a u_a^\epsilon(t)$  be their cardinal interpolations on  $\mathbb{T}_L^3$ , each array being interpolated separately. Then  $I_a u_a \rightarrow P_t f$  and  $I_a u_a^\epsilon \rightarrow \partial^\epsilon P_t f$  uniformly on  $[0, T] \times \mathbb{T}_L^3$ , and the same convergence holds after multiplying the discrete and the continuum quantities by the weights  $(t + a^2)^\delta$  and  $t^\delta$ , respectively, for any  $\delta > 0$ . In particular, the separately interpolated value and difference arrays converge in the positive-time weighted heat norms of Definition 4.8.

The constants depend on  $L, T, q$ , the meshes  $a_1, \dots, a_l$ , the function  $b$ , and  $M_H$ , but not on the finer mesh  $a$ , the coupling, the depth, or the number of steps of the whole reconstruction.

*Remark 16.10.* The convergence in Lemma 16.9 concerns the value array and each difference array, interpolated separately. It is not a statement about the derivatives of the interpolated value field, and such a statement would be false. Indeed,  $\zeta'(k) = 0$  for every integer  $k$ : at  $k = 0$  because  $\zeta$  is even, at  $k = \pm 1$  because all derivatives of  $\zeta$  vanish there, and elsewhere because  $\zeta$  vanishes near  $k$ . It follows that  $\partial_i(I_a u)$  vanishes at every vertex of  $\Gamma_a$  for every array  $u$ , and it cannot converge uniformly to  $\partial_i f$  even when  $u = S_a f$ . The proof below uses no identity between  $D_a^\epsilon$  and derivatives of  $I_a$ .

*Proof of Lemma 16.9.* The fields are deterministic. By Lemma 16.3, for every integer  $r \geq 0$ , the functions in  $\mathcal{F}_H$  satisfy

$$\sup_{f \in \mathcal{F}_H} \|f\|_{C^r} \leq C_{b,r} M_H \sum_{j=1}^l (1 + a_j^{-r}), \quad (16.32)$$

and we denote the right side by  $K_{H,r}$ . Since there are only finitely many link variables on the meshes  $a_j$ , the family  $\mathcal{F}_H$  is bounded and finite-dimensional; no compactness argument is needed for the uniform estimates below.

Taylor's formula with integral remainder in the coordinate  $i$  gives, for a smooth periodic function  $f$ ,

$$\|D_{a,i}^- D_{a,i}^+ S_a f - S_a \partial_i^2 f\|_\infty \leq C a^2 \|\partial_i^4 f\|_\infty.$$

To see the cancellation, we expand  $f(x + ae_i)$  and  $f(x - ae_i)$  to order three at  $x$ ; the odd terms cancel, and the two remainders of order four, divided by  $a^2$ , are at most  $C a^2 \|\partial_i^4 f\|_\infty$ . Summing over  $i$ , we obtain

$$\|\Delta_a S_a f - S_a \Delta f\|_\infty \leq C a^2 \|f\|_{C^4}. \quad (16.33)$$

The Taylor expansions are carried out on the periodic lift of  $f$  to  $\mathbb{R}^3$ , so they apply at every seam with the same constants.

The discrete and the continuum semigroups both commute with their own constant-coefficient derivatives, and both contract the supremum norm; for the continuum semigroup, this follows from its convolution kernel in Lemma 16.1. Differentiating  $P_{t-s}^a S_a P_s f$  in  $s$  and integrating, we obtain the identity

$$P_t^a S_a f - S_a P_t f = \int_0^t P_{t-s}^a (\Delta_a S_a - S_a \Delta) P_s f \, ds. \quad (16.34)$$

The differentiation is legitimate because  $f$  is smooth and the discrete space is finite. Since  $\|P_s f\|_{C^4} \leq \|f\|_{C^4}$ , inserting (16.33) into (16.34), we obtain the bound  $C T a^2 \|f\|_{C^4}$ , uniformly on  $[0, T]$ .

Fix an ordered product of  $q$  differences  $D_a^\epsilon$  and the corresponding product of derivatives  $\partial^\epsilon$ . Using (16.10) repeatedly, we write  $D_a^\epsilon S_a f(x)$  as an average of  $\partial^\epsilon f$  over shifts of  $x$  of total length at most  $qa$ . The fundamental theorem of calculus along these shifts then gives

$$\|D_a^\epsilon S_a f - S_a \partial^\epsilon f\|_\infty \leq C_q a \|f\|_{C^{q+1}}. \quad (16.35)$$

We commute the differences with  $P_t^a$  and compare the differentiated initial data:

$$D_a^\epsilon P_t^a S_a f - S_a \partial^\epsilon P_t f = P_t^a (D_a^\epsilon S_a f - S_a \partial^\epsilon f) + (P_t^a S_a \partial^\epsilon f - S_a P_t \partial^\epsilon f).$$

The first term is bounded by (16.35), and the second by the value estimate already proved, applied to  $\partial^\epsilon f$ , and we obtain  $C_q \{a \|f\|_{C^{q+1}} + T a^2 \|f\|_{C^{q+4}}\}$ . This argument does not divide an error in the values by  $a$ . It includes  $t = 0$ , and both errors vanish identically for constant  $f$ . Together with (16.32), this proves the two estimates for the family  $\mathcal{F}_H$ ; since  $a \leq L$ , the term of order  $a^2$  is absorbed into  $C_{H,q} a$ .

For the interpolation statement, we use the nonnegative partition of unity and its support radius: for every smooth periodic  $g$  and every array  $u$ ,

$$\|I_a S_a g - g\|_\infty \leq C a \|\nabla g\|_\infty, \quad \|I_a u\|_\infty \leq \|u\|_\infty. \quad (16.36)$$

We apply the first bound to  $g = P_t f$  and  $g = \partial^\epsilon P_t f$ , and the second to the differences between the discrete arrays and the sampled continuum functions. The smooth norms of

these continuum functions are bounded uniformly by (16.32) and heat contraction. Then the interpolated value array and each separately interpolated difference array converge uniformly on  $[0, T] \times \mathbb{T}_L^3$  to the corresponding continuum functions.

For a positive power  $\delta$ , the weights  $(t + a^2)^\delta$  are bounded by  $(T + L^2)^\delta$  and converge to  $t^\delta$  uniformly on  $[0, T]$  as  $a \downarrow 0$ , by the uniform continuity of  $v \mapsto v^\delta$  on  $[0, T + L^2]$ . Multiplying the uniform errors in the values and the differences by the weights of Definition 4.8 therefore preserves the convergence, also when one compares the weighted discrete and the weighted continuum functions themselves. All constants depend only on the meshes  $a_j$ ,  $M_H$ ,  $b$ ,  $L$ ,  $T$ , and  $q$ .  $\square$

## Backward reconstruction and conditional quadratic moments

This section proves three estimates on the coarse-to-fine reconstruction of the fine field under the conditioned Wilson measure. They are used in Sections 26 and 27 to control the heat evolution of the reconstructed field and its quadratic Picard term, and also in Sections 28 and 31. Throughout, the torus, the coarsest mesh  $H$ , the continuum coupling  $g_3 > 0$ , and a time horizon  $T_0$  are fixed, and all estimates are uniform in the number  $k$  of renormalization group steps between the coarsest and the finest lattice.

**The purpose of the three estimates.** Recall from Section 3.3 that the fine field is reconstructed from the coarsest field one renormalization group step at a time, from coarse to fine. At the  $j$ th step, the logarithm  $\theta_j$  of the reconstructed field on the lattice of mesh  $a_j$  is obtained from the logarithm  $\theta_{j-1}$  on the lattice of mesh  $a_{j-1} = Ba_j$  and from the normalized fluctuation variables  $w_j$  of that step. Let  $A_j = a_j^{-1}\theta_j$  be the lattice potential, and let  $C_j$  be the zero-fluctuation reconstruction in the same units, that is, the reconstruction with  $w_j = 0$ . We write  $\bar{P}$ ,  $\bar{Q}$ , and  $\bar{R}$  for the linear, quadratic, and cubic parts of the zero-fluctuation reconstruction defined in Lemma 17.2, so that  $C_j = a_{j-1}^{-1}[\bar{P}\theta_{j-1} + \bar{Q}(\theta_{j-1}, \theta_{j-1}) + \bar{R}(\theta_{j-1})]$ . Viewing all terms as functions that are constant on lattice cells, we obtain the exact decomposition

$$A_j - E_{a_{j-1}}A_{j-1} = (A_j - \mathbb{E}[A_j \mid \mathcal{F}_{j-1}]) + m_j + D_d A_{j-1} + a_{j-1}^{-1}\bar{Q}(\theta_{j-1}, \theta_{j-1}) + a_{j-1}^{-1}\bar{R}(\theta_{j-1}). \quad (17.1)$$

Here  $E_{a_{j-1}}$  is the cell embedding on the coarser lattice, and  $\mathcal{F}_{j-1}$  is generated by the coarsest field and the fluctuation variables of the first  $j-1$  steps. The term  $m_j = \mathbb{E}[A_j \mid \mathcal{F}_{j-1}] - C_j$  is the conditional mean correction (in the notation of Sections 26 and 34,  $m_j = m_j^{\text{ph}}$ ), and  $D_d A_{j-1}$  is the difference between the linearized reconstruction of  $A_{j-1}$  and its cell embedding; the precise definitions are given below. The first term is controlled by Theorem 16.6. Lemma 17.1 gives a deterministic bound on  $\theta_j$ , the reconstruction amplitude bound, which tends to zero with the mesh. It also bounds  $m_j$  and shows that the sum over  $j$  of  $A_j - C_j$ , which is the sum of the first two terms, is small in a negative-regularity norm. Lemma 17.2 shows that each column of the linear operator  $D_d$  has zero integral and is localized at the scale  $a_{j-1}$ , so that heat smoothing at time  $t$  gains the factor  $a_{j-1}/\sqrt{t}$ . Theorem 17.4 controls the quadratic term. The cubic term is

controlled by the reconstruction amplitude bound on  $\theta_{j-1}$ .

**Inputs and attribution.** The deterministic inputs are the reconstruction maps of Theorem 15.13, with the representation of the conditioned Wilson measure in Lemma 15.10, the preservation of the conditional kernels in Theorem 15.14, and the derivative bounds of Lemma 15.16. The zero-fluctuation reconstruction is built from the classical background fields of Bałaban [16], and the averaging is a version of Bałaban's averaging operation [12]. The probabilistic inputs are the bounds on conditional means and covariances of Theorem 13.10, the concentration theorem Theorem 5.4, and the results of Section 16. The concentration theorem rests on the convexity methods of Brascamp and Lieb [33], Gross [103], and Bakry and Émery [8]. The gain from a column of zero integral is the standard cancellation of a localized kernel with vanishing mass, and the maximal inequalities are elementary chaining in the sense of Dudley [79] (see also [74]), in the form of Lemma 16.4.

## 17.1 Setting

**Order of the parameters.** We use the constructions of Section 15 with the following order of choices. The constant  $C_{\text{CP}} \geq 1$  of Theorem 15.13, which bounds the first three derivatives in the coarse field and the first spatial difference of the zero-fluctuation reconstruction, is fixed before  $J$ . Before  $J$  is chosen, it may be enlarged to include finitely many further bounds of the same kind, as in the proofs below. Then  $J$  is chosen so that  $q_{\text{anc}} = C_{\text{CP}}^2/B < 1$ , where  $B = M^J$ ; Theorem 15.13 allows  $q_{\text{anc}} \leq 1/4$ . The larger analytic domains and their coefficient bounds, which may depend on  $J$ , are fixed next. Then the radii are chosen as in Lemma 15.16,

$$0 < r < r_{\text{eff}}/4, \quad \frac{\beta_J r_{\text{eff}}}{1 - \alpha_*} < \frac{1}{8}, \quad \alpha_* = C_{\text{CP}}/B,$$

and the bounds on the coupling-scaled fluctuation inputs and on the coupling are chosen so that every reconstructed field lies in the working domain of that lemma. Finally, the upper bound  $g_*$  on the couplings is fixed. It includes the bound of Theorem 16.6 with  $R_{\text{supp}} = r$ , it ensures that  $p_j \geq 1$ , and it makes  $t_j p_j^3$  as small as required in Theorem 13.10, in the notation introduced next.

**The coarse-to-fine reconstruction.** Fix  $L > 0$ ,  $T_0 > 0$ , a coarsest mesh  $H = L/n_0$  with  $n_0 \in 4\mathbb{N}$ , and  $g_3 > 0$  such that  $t_1 \leq g_*$ , in the notation introduced below; here  $\gamma = g_3^2 L$ . We also fix the Borel choice of gauge of Lemma 15.10, which carries every stopping field in  $D_r$  into  $B_r$  and does not depend on  $k$ , and the reconstruction maps  $\hat{\mathcal{R}}$  of Theorem 15.13. We write  $\hat{\Phi}_{d,j}$  for their logarithmic form at depth  $d$ , at the  $j$ th step. For  $k \geq 1$ , let  $a_j = HB^{-j}$  and  $d_j^k = J(k-j)$  for  $0 \leq j \leq k$ . For  $1 \leq j \leq k$ , we define

$$g_j = g_3 \sqrt{a_j}, \quad g_{j,s} = M^{s/2} g_j \quad (0 \leq s < J), \quad t_j = M^{(J-1)/2} g_j, \quad p_j = A_w \log(1/t_j).$$

Since  $t_j = t_1 B^{-(j-1)/2}$ , we have  $t_j \leq t_1 \leq g_*$  at every fluctuation step. We take  $g_*$  no larger than the working-domain ceiling  $\tau$  of Lemma 15.16. Here  $g_j^2$  is the lattice coupling at mesh  $a_j$ ,  $g_{j,s}$  is the coupling at the intermediate scale  $s$  of the  $j$ th step, and  $t_j$  is the largest of these couplings. Further,  $p_j$  is the radius of the fluctuation cutoff domain of the  $j$ th step, and  $d_j^k$  is the number of averaging steps that precede it, counted from the finest lattice. The reconstruction is

$$\theta_j = \widehat{\Phi}_{d_j^k, j}(\theta_{j-1}, (g_{j,s} w_{j,s})_{s=0}^{J-1}), \quad \mathcal{F}_j = \sigma(\theta_0, w_1, \dots, w_j) \quad (1 \leq j \leq k), \quad (17.2)$$

where  $\theta_0$  is the logarithm of the stopping field in the chosen gauge, and  $w_j = (w_{j,s})_s$  are the normalized fluctuation variables of the  $j$ th step. The meshes and depths are

$$(\theta_0; H; Jk) \longrightarrow (\theta_1; H/B; J(k-1)) \longrightarrow \dots \longrightarrow (\theta_k; H/B^k; 0).$$

**The measure.** Let  $N = n_0 B^k$ , and let  $\nu = \nu_{N, n_0, \gamma}$  be the conditioned Wilson measure. By Lemma 15.10, the gauge orbits of the fields  $(e^{\theta_0}, \dots, e^{\theta_k})$  have exactly the law of the averaged configurations under  $\nu$ . The conditional law of  $w_j$  given  $\mathcal{F}_{j-1}$  is the conditional fluctuation kernel  $k(e^{\theta_{j-1}}, dw)$  of Theorem 13.8 at the coarse field  $e^{\theta_{j-1}}$ , and the reconstruction maps preserve it (Theorem 15.14). This kernel has a density  $Z^{-1} \mathbf{1}_\Omega e^{-V}$  on a fixed product  $\Omega$  of three-dimensional balls of radius  $p_j$ , where  $V$  is smooth on a neighborhood of the closed product and  $D_w^2 V \geq I/2$  there. It is not a product of independent Gaussian measures. All conditional expectations below are taken under  $\nu$ . The marginal law of  $\theta_{j-1}$  under  $\nu$  includes the restrictions at all coarser steps, and it is in general not a stopped measure. Once  $\mathcal{F}_{j-1}$  is fixed, these restrictions are fixed factors, and they cancel in the conditional law of  $w_j$ .

At each step, the coarse field  $e^{\theta_{j-1}}$  lies in  $B_r$ , after the change of gauge of Lemma 15.10 if it is given in  $D_r$ . By the strict domain inclusions established in Lemma 15.16 and Theorem 15.14, the reconstruction of Theorem 13.2 and the reconstruction maps act on the same product. All segments of coarse fields and of coupling-scaled fluctuation inputs used below lie in the larger domains on which the derivative bounds hold. We use Theorem 16.6 with  $R_{\text{supp}} = r$ , together with its change-of-gauge assertion.

**Norms.** Every array is embedded as a componentwise constant function on the half-open cells of its own lattice, each positive link being attached to its starting vertex. We use the heat norm

$$\|f\|_{-\alpha; T_0} = \sup_{0 < s \leq T_0} s^{\alpha/2} \|P_s f\|_\infty, \quad (17.3)$$

where  $P_s$  is the continuum heat semigroup acting componentwise; it keeps the constant mode. Since the heat kernel is positive and has unit mass, we have

$$\|f\|_{-\alpha; T_0} \leq T_0^{\alpha/2} \|f\|_\infty \quad (\alpha > 0). \quad (17.4)$$

Discrete heat semigroups carry their mesh as a superscript, as in Section 16. For linear and quadratic coefficient arrays, we use the row sums and the volume-normalized column sums of Definition 4.6: a column of an input on a lattice of mesh  $a_{\text{in}}$  is multiplied by  $a_{\text{out}}^3/a_{\text{in}}^3$ .

**Constants.** We vary the depth at fixed  $L$ ,  $T_0$ ,  $H$ , and  $g_3$ , and all limits are taken at these fixed values. Constants have the dependence specified in each result: the lemma on conditional means allows dependence on the upper bound on the coupling, the lemma on column cancellation has the independence stated there, and the theorem on quadratic moments allows dependence on the fixed positive  $g_3$  and the fixed  $H$ . No stronger statement, uniform as  $H$  varies or as  $g_3 \downarrow 0$ , is claimed; the bounds of Lemma 15.16 that are uniform in  $H$  are not used to improve this dependence. A subscript  $J$  indicates a constant that may depend on  $J$ ; such constants are chosen after  $J$ . The applications of Lemmas 16.1 and 16.3 use the horizon  $T = T_0$ , and their constants therefore depend on  $T_0/L^2$ .

## 17.2 Conditional means along the reconstruction

The first lemma bounds the reconstructed fields and their conditional mean corrections. The stopping-field contribution to the reconstruction amplitude bound is  $r(a_j/H)^{\kappa_{\text{term}}}$ , where  $\kappa_{\text{term}}$  is determined by the contraction factor  $a_{\text{cap}} = C_{\text{CP}}/B$  of the linearized reconstruction; the fluctuation contribution is bounded by  $C_J g_3 \sqrt{a_j} p_j$ . The conditional mean correction is at most  $C_J g_3^2$  times the size of the coarser field. Indeed, by Theorem 13.10, the conditional mean of the reconstructed link variables differs from the zero-fluctuation reconstruction by  $O(t_j^2 \|\theta_{j-1}\|_\infty)$ , and  $t_j^2/a_j$  is a fixed multiple of  $g_3^2$ .

**Lemma 17.1.** *Assume the setting of Section 17.1, and let the constant  $C_{\text{CP}} \geq 1$ , fixed before  $J$ , satisfy*

$$a_{\text{cap}} = C_{\text{CP}}/B, \quad a_{\text{cap}} \sqrt{B} < 1, \quad \kappa_{\text{term}} = -\frac{\log a_{\text{cap}}}{\log B} > \frac{1}{2}.$$

For  $1 \leq j \leq k$  set

$$A_j = a_j^{-1} \theta_j, \quad C_j = a_j^{-1} \hat{\Phi}_{a_j^k, j}(\theta_{j-1}, 0), \quad m_j = \mathbb{E}[A_j \mid \mathcal{F}_{j-1}] - C_j,$$

*each embedded as a componentwise constant function on the cells of its own lattice. Then, on every sample of the conditioned Wilson measure,*

$$b_j = \|\theta_j\|_\infty \leq \bar{b}_j, \quad \bar{b}_j = r(a_j/H)^{\kappa_{\text{term}}} + C_J g_3 \sqrt{a_j} p_j, \quad \|m_j\|_\infty \leq C_J g_3^2 b_{j-1}.$$

For  $2 \leq l \leq k$  let

$$M_l = C_J g_3^2 \{ r a_{\text{cap}}^{l-1} + g_3 \sqrt{a_{l-1}} (p_l + 1) \}.$$

Then  $\sum_{j=l}^k \|m_j\|_\infty \leq M_l$ , and  $M_l \rightarrow 0$  as  $l \rightarrow \infty$ , uniformly in  $k \geq l$ . For every  $1/2 < \alpha < 3/2$  and  $u \geq 0$ ,

$$\sum_{j=l}^k \mathbb{E} \|A_j - C_j\|_{-\alpha; T_0} \leq C_{J,\alpha} g_3 a_l^{\alpha-1/2} \sqrt{\log(2 + L/a_l)} + C_{\alpha, T_0} M_l,$$

and, with probability at least  $1 - Ce^{-u}$ , simultaneously for all terms of this finite sum,

$$\sum_{j=l}^k \|A_j - C_j\|_{-\alpha; T_0} \leq C_{J,\alpha} g_3 a_l^{\alpha-1/2} [\sqrt{\log(2 + L/a_l)} + \sqrt{u}] + C_{\alpha, T_0} M_l.$$

These estimates also hold after integration over the law of the stopping field under the conditioned Wilson measure, uniformly in  $k$ . The constants depend on the fixed construction parameters, on  $L$ ,  $T_0$ ,  $\alpha$ ,  $H$ , and on the upper bound on the coupling, but not on  $j$ ,  $k$ , the depth, or the choice of the representative of the stopping field.

The factors  $g_3$  and the powers of the mesh are displayed explicitly. The exponent  $\kappa_{\text{term}}$  is defined by the contraction of the linearized reconstruction and is unrelated to the exponent  $\kappa_{\text{fl}} = 1 - 3\alpha_{\text{fl}}/2$  of the flow norms. The lemma controls  $A_j - C_j$ , the deviation of the reconstructed field from its zero-fluctuation reconstruction; the adjacent difference  $A_j - A_{j-1}$  contains in addition the last three terms of (17.1), which are treated in Sections 17.3 and 17.4.

*Proof.* All constants have the dependence allowed in the statement. A constant with subscript  $J$  may include the finitely many ratios of couplings within a step and the factor  $(1 - a_{\text{cap}}\sqrt{B})^{-1}$ . The proof has four steps: the reconstruction amplitude bound, its summation, the bound on the conditional mean in the gauge of the reconstruction maps, and the bounds on the centered part.

**Step 1: the reconstruction amplitude bound.** The recurrence and its summation in Steps 1 and 2 are also stated in Lemma 15.10. We repeat the short computation because the distinction between the constant  $C_{\text{rec}, J}$  of the recurrence and the constant  $C_{\text{cap}, J}$  of its solution is used again in the proof of Theorem 17.4. We first explain the choice of  $C_{\text{CP}}$ . Let  $K_1$ ,  $K_2$ , and  $K_3$  be the bounds, independent of  $J$ , from Theorem 15.13(3), retained in Lemma 15.16, for the linear term, the quadratic term, and the cubic remainder of the normalized zero-fluctuation reconstruction  $\bar{A}_d$  (see Lemma 17.2) on its fixed inner ball. For a coarse field of norm  $R \leq 1$ , the value of  $\bar{A}_d$  is at most  $K_1 R + K_2 R^2 + K_3 R^3 \leq (K_1 + K_2 + K_3)R$ . We include this sum in  $C_{\text{CP}}$ ; the working ball is chosen inside the inner ball, with  $R \leq 1$ , after  $J$  is fixed. No constant that depends on all orders of differentiation at fixed  $J$  is used to choose  $J$ . With this choice, the bound

$$\|\hat{\Phi}_{d,j}(\theta, q)\|_\infty \leq a_{\text{cap}} \|\theta\|_\infty + C_{\text{rec}, J} \max_s \|q_s\|_\infty$$

holds with the same  $a_{\text{cap}}$ : the zero-input term has the bound just established, and integrating the free-input derivative bound (15.34) of Lemma 15.16 gives  $C_{\text{rec},J} = C_{\text{free},J}$ . The summation is also recorded in Lemma 15.10.

Set  $\varrho = a_{\text{cap}}\sqrt{B} < 1$ . Since  $|g_{j,s}w_{j,s}| \leq t_j p_j$  on the fluctuation cutoff domain, the last bound implies that, at every point of the support of the conditioned measure,

$$b_j \leq a_{\text{cap}} b_{j-1} + C_{\text{rec},J} t_j p_j, \quad b_0 \leq r.$$

Let  $c_B = M^{(J-1)/2}$ , so that  $t_j = c_B g_3 \sqrt{a_j}$ , and fix

$$C_{\text{cap},J} \geq \frac{c_B C_{\text{rec},J}}{1 - \varrho}. \quad (17.5)$$

Both  $C_{\text{rec},J}$  and  $C_{\text{cap},J}$  are chosen after  $J$ , and neither changes  $C_{\text{CP}}$ . Since  $t_j = t_1 B^{-(j-1)/2}$ , we have

$$p_j = p_1 + c_w(j-1), \quad c_w = \frac{1}{2} A_w \log B > 0; \quad (17.6)$$

in particular  $p_j$  is increasing in  $j$ . Iterating the recurrence, with no exceptional set, and using  $a_{\text{cap}}^{j-i} \sqrt{a_i} = \varrho^{j-i} \sqrt{a_j}$  and  $p_i \leq p_j$  for  $i \leq j$ , we obtain

$$b_j \leq r a_{\text{cap}}^j + c_B C_{\text{rec},J} g_3 \sqrt{a_j} \sum_{i=1}^j \varrho^{j-i} p_i \leq r a_{\text{cap}}^j + C_{\text{cap},J} g_3 \sqrt{a_j} p_j.$$

Since  $a_{\text{cap}}^j = B^{-\kappa_{\text{term}} j} = (a_j/H)^{\kappa_{\text{term}}}$ , this is the asserted bound, with a constant  $C_J \geq C_{\text{cap},J}$  that may also dominate the constants of the summation and of the mean below. The condition  $\kappa_{\text{term}} > 1/2$  is equivalent to  $\varrho < 1$ .

**Step 2: summation of the amplitude bound.** Let  $q_B = B^{-1/2}$  and  $j = l + h$ . By the amplitude bound, (17.6), and  $a_{j-1} = a_{l-1} B^{-h}$ , we have

$$\sum_{j=l}^k b_{j-1} \leq \frac{r a_{\text{cap}}^{l-1}}{1 - a_{\text{cap}}} + C_{\text{cap},J} g_3 \sqrt{a_{l-1}} \sum_{h \geq 0} q_B^h (p_l + c_w h) \leq C_J \{ r a_{\text{cap}}^{l-1} + g_3 \sqrt{a_{l-1}} (p_l + 1) \}.$$

The two series equal  $(1 - q_B)^{-1}$  and  $q_B(1 - q_B)^{-2}$ . The constant does not depend on  $k$  or on the sample.

**Step 3: the conditional mean in the gauge of the reconstruction maps.** Conditionally on  $\mathcal{F}_{j-1}$ , the coarse field is fixed, the gauge transformation of the reconstruction maps depends only on it, and the conditional fluctuation kernel is the same as for the reconstruction of Theorem 13.2. Equality of the kernels does not by itself identify the two reconstructed means, because the observable changes; we show that the bound on the mean survives the change of gauge. We write  $\phi = \phi_{j-1}$  and  $R = \|\phi\|_\infty$ . Let  $X_e(w)$  be the

logarithm of the reconstruction of Theorem 13.2 on one output link  $e$ , and let  $x_e = X_e(0)$ . By the bound on the reconstructed mean in Theorem 13.10, we have

$$|\mathbb{E}X_e - x_e| \leq C_J g_j^2 R,$$

where we used  $t_j^2 = c_B^2 g_j^2$ . The variance of  $X_e$  is at most  $C_J g_j^2$ , by Theorem 16.6, applied under the conditional fluctuation kernel at the coarse field  $e^\phi \in B_r$ , with fine mesh  $a_j$  and horizon  $T_0$ . Indeed, we apply its variance bound to the test function  $a_j^{-3} \mathbf{1}_{Q_e}$  in one color component of  $X/a_j$ , whose squared  $L^2$  norm is  $a_j^{-3}$ , and multiply by  $a_j^2$ . Summing over the three colors, we obtain

$$\mathbb{E}|X_e - x_e|^2 \leq C_J g_j^2 + C_J g_j^4 R^2 \leq C'_J g_j^2. \quad (17.7)$$

The reconstruction of Theorem 13.2 satisfies the hypotheses of both theorems used here.

At the fixed coarse field, the reconstruction maps differ from this reconstruction by a gauge transformation given by matrices  $h_x$  at the vertices. On the link  $e = (x, y)$ , the logarithm of the reconstruction map is  $H_e(X_e)$ , where  $H_e(X) = \text{Log}(h_x e^X h_y^{-1})$ . By the bounds of Theorems 13.2 and 15.13, the two zero-fluctuation reconstructions  $x_e$  and  $\hat{x}_e = H_e(x_e)$  have norm at most  $CR$ . The identity  $e^{\hat{x}_e} = h_x e^{x_e} h_y^{-1}$  therefore implies  $|\text{Log}(h_y^{-1} h_x)| \leq CR$ . Let  $z_e = \text{Log}(h_y^{-1} h_x)$ . By conjugation, we have

$$H_e(X) = \text{Ad}(h_x) \text{Log}(e^X e^{z_e}).$$

On the fixed small ball of logarithms, the analytic series of the exponential and the logarithm give bounded derivatives of  $(X, z) \mapsto \text{Log}(e^X e^z)$  through order three. More precisely, we enlarge this ball within the fixed chart so that  $\|e^X e^z - I\| \leq q < 1$  in matrix norm. Termwise differentiation of  $\text{Log}(I + Z) = \sum_{m \geq 1} (-1)^{m+1} Z^m / m$  is justified by the convergent majorants  $\sum_{m \geq r} m^{r-1} q^{m-r}$ ,  $r = 1, 2, 3$ , and the derivatives of the exponential are bounded by its absolutely convergent series on this ball. These bounds depend only on the chart. At  $z = 0$ , the function equals  $X$ , and its second derivative in  $X$  vanishes there. Integrating the mixed derivative  $D_z D_X^2$  along the segment  $sz_e$ ,  $0 \leq s \leq 1$ , we therefore find  $\|D_X^2 H_e\| \leq CR$ , while  $\|D_X H_e\| \leq C$ . All segments used here lie in the fixed chart. Since the matrices  $\text{Ad}(h_x)$  are orthogonal, these constants do not depend on the matrices  $h_x$  or on the number of vertices. By Taylor's formula at  $x_e$ , the bound on the mean, and (17.7), we obtain

$$|\mathbb{E}H_e(X_e) - H_e(x_e)| \leq C |\mathbb{E}(X_e - x_e)| + CR \mathbb{E}|X_e - x_e|^2 \leq C_J g_j^2 R.$$

This is the bound on the mean for the reconstruction maps, on the same product of balls. Dividing by the fine mesh, we obtain

$$\|m_j\|_\infty \leq C_J \frac{t_j^2}{a_j} b_{j-1} \leq C_J g_3^2 b_{j-1}.$$

Together with Step 2, this implies  $\sum_{j=l}^k \|m_j\|_\infty \leq M_l$ , after one enlargement of  $C_J$ . Both terms of  $M_l$  tend to zero as  $l \rightarrow \infty$ , because  $a_{\text{cap}} < 1$  and  $p_l$  grows linearly in  $l$  while  $\sqrt{a_{l-1}}$  decays geometrically.

**Step 4: the centered part.** We define

$$F_j = A_j - \mathbb{E}[A_j \mid \mathcal{F}_{j-1}].$$

Theorem 16.6 applies to  $F_j$  conditionally on  $\mathcal{F}_{j-1}$ . Indeed, the fine mesh is  $a_j$ , the side of the coarse lattice of the step is  $n_0 B^{j-1} \geq 4$ , the depth is  $d_j^k$ , the largest coupling is  $t_j \leq g_*$ , the coarse field is in  $B_r$ , the horizon is  $T_0$ , and the reconstruction maps satisfy the hypotheses of that theorem on the whole product. The same holds for the reconstruction of Theorem 13.2, which we used in Step 3. Let  $\beta = \alpha - 1/2 > 0$  and  $\ell_j = \log(2 + L/a_j)$ . Taking the conditional expectation and then the expectation, we obtain

$$\mathbb{E}\|F_j\|_{-\alpha; T_0} \leq C_{J,\alpha} g_3 a_j^\beta \sqrt{\ell_j}.$$

For  $h \geq 0$ , we have  $\ell_{l+h} \leq \ell_l + h \log B$  and  $a_{l+h} = a_l B^{-h}$ . It follows that

$$\sum_{h \geq 0} a_{l+h}^\beta \sqrt{\ell_{l+h}} \leq a_l^\beta \left\{ \frac{\sqrt{\ell_l}}{1 - B^{-\beta}} + \sqrt{\log B} \sum_{h \geq 0} B^{-\beta h} \sqrt{h} \right\} \leq C_{B,\beta} a_l^\beta \sqrt{\ell_l}. \quad (17.8)$$

The last series converges, for instance because  $\sqrt{h} \leq h + 1$ , and  $\ell_l \geq \log 2$  absorbs its contribution. Since  $A_j - C_j = F_j + m_j$ , the bound on the expectation follows from (17.4) applied to  $m_j$  and from Step 3.

For the bound in probability, we use the tail bound of Theorem 16.6 at level  $j = l + h$  with  $u + 2h \log 2$  in place of  $u$ . Integrating the conditional probability does not change its upper bound. By a union bound over the finitely many levels, the probability of failure is at most

$$C e^{-u} \sum_{h \geq 0} 4^{-h} \leq C' e^{-u}.$$

On the complementary event, we sum the thresholds, using  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ , (17.8), and the convergent series  $\sum_h B^{-\beta h}$  and  $\sum_h B^{-\beta h} \sqrt{h}$ , and we add the deterministic sum of the means by (17.4). This proves the bound in probability. No independence between levels has been used.

Since all conditional estimates are uniform in the coarse fields that occur and in the stopping field, the same inequalities hold after integration over the law of the stopping field under the conditioned Wilson measure. The argument concerns  $A_j - C_j$  throughout, and  $C_j$  is never replaced by  $A_{j-1}$ .  $\square$

### 17.3 Column cancellation and heat-kernel estimates

The normalization of an input column is essential here. For an operator  $K$  from arrays on a lattice of mesh  $A$  to componentwise functions on the torus, the volume-normalized absolute column of an input  $x$  is  $A^{-3} \int |K_x(y)| dy$ . A zero integral of a column is a separate identity, which no absolute bound implies. On a lattice of mesh  $\delta$ , the integral is replaced by  $\delta^3 \sum_y$ , still divided by the input volume  $A^3$ .

The lemma below concerns the difference  $D_d$  between the linearized zero-fluctuation reconstruction and the cell embedding on the coarse lattice. Each column of  $D_d$  is concentrated within distance of order  $A$  of its input, and it has zero integral. At times  $t \gg A^2$ , the heat kernel is nearly constant on the scale  $A$ , and the zero integral allows us to replace  $p_t(z - y)$  by  $p_t(z - y) - p_t(z - x)$ , which costs the distance from  $y$  to  $x$  times the gradient of the kernel. This yields the factor  $A/\sqrt{t}$ .

**Lemma 17.2.** *Assume the setting of Section 17.1. Let  $d \geq 0$ , let  $A > 0$  be a mesh with  $L/A \in \mathbb{N}$  and  $L/A \geq 4$ , let  $a = A/B$ , and let  $\hat{\Phi}_d$  be the logarithmic form of the reconstruction map of one renormalization group step at depth  $d$ , from the lattice of mesh  $A$  to the lattice of mesh  $a$ . In units of the coarse lattice, its zero-fluctuation reconstruction has the expansion*

$$\hat{\Phi}_d(\theta, 0) = B^{-1} \bar{A}_d(\theta), \quad \bar{A}_d(\theta) = \bar{P}_d \theta + \bar{Q}_d(\theta, \theta) + \bar{R}_d(\theta),$$

where  $\bar{P}_d = D \bar{A}_d(0)$ ,  $\bar{Q}_d = D^2 \bar{A}_d(0)/2$ , and  $\|\bar{R}_d(\theta)\|_\infty \leq C_{CP} \|\theta\|_\infty^3$  on a fixed inner ball of coarse fields. The first three derivatives in  $\theta$ , together with their first spatial differences  $\nabla_{1/B}$  in units of the coarse lattice, have weighted row sums and volume-normalized column sums bounded by one constant, uniformly in  $d$ , the lattice side, and  $J$ , as in Lemma 15.16. Here  $\nabla_{1/B} = A \nabla_a$ , where  $\nabla_a$  denotes the first differences at the physical mesh  $a$ ; thus a physical first difference of  $\bar{A}_d$  carries the factor  $A^{-1}$ , and a physical first difference of the potential  $A^{-1} \bar{A}_d$  carries the factor  $A^{-2}$ . The cubic remainder and these finitely many bounds are included in  $C_{CP}$  before  $J$  is fixed. This pre- $J$  enlargement also includes the row, column, and first-moment bounds below after subtracting the coarse cell embedding; their additional constants depend only on the fixed spatial rate and the cell geometry. In physical units the zero-fluctuation reconstruction is  $A^{-1} \bar{A}_d(\theta)$ .

Let  $E_A v$  be the cell embedding on the lattice of mesh  $A$  of a coarse lattice potential  $v$ , let  $P_d v$  be the cell embedding on the lattice of mesh  $a$  of the array  $A^{-1} \bar{P}_d(Av)$ , and let  $D_d = P_d - E_A$ . For each input link  $x$  and input color  $b$ , and each output direction and color  $(i, c)$ , the corresponding column of  $D_d$  satisfies

$$\int_{\mathbb{T}_L^3} (D_d)_{(i,c),(x,b)}(y) dy = 0, \quad A^{-3} \int_{\mathbb{T}_L^3} \text{dist}(y, x) |(D_d)_{(i,c),(x,b)}(y)| dy \leq C_{CP} A.$$

Its volume-normalized column sums and its row sums are bounded by  $C_{CP}$ , with exponential localization at the scale  $A$ . In particular, for  $0 < t \leq T_0$ ,

$$\|P_t D_d v\|_\infty \leq C \min\{1, A/\sqrt{t}\} \|v\|_\infty, \quad \|D_d v\|_{-\alpha; T_0} \leq C_\alpha A^\alpha \|v\|_\infty \quad (0 < \alpha < 1).$$

Let  $m \in \mathbb{N}$  and  $\delta = a/m$ . Let  $P_d^{\text{arr}} v$  be the array on the lattice of mesh  $a$  whose cell embedding is  $P_d v$ , and let

$$D_d^\delta = I_{a \rightarrow \delta} P_d^{\text{arr}} - I_{A \rightarrow \delta},$$

where the two cardinal interpolations of Lemma 16.3 start from the lattices of meshes  $a$  and  $A$ , respectively. Then every componentwise column of  $D_d^\delta$  has zero integral  $\delta^3 \sum_y$ , its absolute column sums and first moments satisfy the same bounds, and for every integer  $q \geq 0$ , every product  $D_\delta^q$  of  $q$  physical forward or backward differences, and every  $0 \leq t \leq T_0$ ,

$$\|D_\delta^q P_t^\delta D_d^\delta v\|_\infty \leq C_{J,q} (t + A^2)^{-q/2} \min\{1, A/\sqrt{t + A^2}\} \|v\|_\infty.$$

In the continuum estimates,  $C$  depends only on the column and moment bounds, on  $\alpha$  where it appears, and on  $T_0/L^2$ . In the estimates on the lattice of mesh  $\delta$  the constant may also depend on  $J$ ,  $q$ , and the function  $b$  of the cardinal interpolation. No constant depends on  $\delta/a$ , the lattice side, the depth, the coupling, or the coarse field. The zero integral is a linear identity for each input column; no zero integral of  $\bar{Q}_d$  is asserted.

*Proof.* The proof has five steps: the expansion, the zero integrals, the localization, the continuum heat estimates, and the estimates on the finer lattice.

**Step 1: the expansion and its units.** By Lemma 15.16, the reconstruction maps keep the bounds on the first three derivatives of the zero-fluctuation reconstruction in the coarse field and on its first spatial difference in units of the coarse lattice, the volume-normalized columns, and the zero-integral identity of Theorem 15.13. Their common domain contains a fixed inner ball of coarse fields. Taylor's formula on this ball gives

$$\begin{aligned} \bar{A}_d(\theta) &= D \bar{A}_d(0) \theta + \frac{1}{2} D^2 \bar{A}_d(0) [\theta, \theta] + \bar{R}_d(\theta), \\ \bar{R}_d(\theta) &= \frac{1}{2} \int_0^1 (1-s)^2 D^3 \bar{A}_d(s\theta) [\theta, \theta, \theta] ds. \end{aligned}$$

The value at  $\theta = 0$  is zero by global color equivariance. The bound on the third derivative implies  $\|\bar{R}_d(\theta)\|_\infty \leq (1/6) \sup \|D^3 \bar{A}_d\| \|\theta\|_\infty^3$ . We enlarge  $C_{\text{CP}}$  to include this bound, the finitely many derivative bounds, and their first spatial differences in units of the coarse lattice, before  $J$  is fixed. This uses the finitely many constants that are uniform in  $J$ , not the bounds on all derivatives at fixed  $J$ . The conversion to physical units is

$$a^{-1} \hat{\Phi}_d(\theta, 0) = (A/B)^{-1} B^{-1} \bar{A}_d(\theta) = A^{-1} \bar{A}_d(\theta). \quad (17.9)$$

**Step 2: zero integrals.** We first restate the zero-integral identity of Theorem 15.13 in these units. The reconstruction satisfies the averaging constraint exactly. Differentiating this identity at  $\theta = 0$ , we find that the linearized reconstruction preserves the componentwise physical integral of each input column. For each physical input basis vector, at an input link  $x$ , with a direction and a color, this gives

$$\int_{\mathbb{T}_L^3} (P_d)_{(i,c),(x,b)}(y) dy = \int_{\mathbb{T}_L^3} (E_A)_{(i,c),(x,b)}(y) dy.$$

The direction of  $x$  is suppressed in the notation, as in the statement. The matching component on the right has integral  $A^3$ , and the other components have integral zero. Subtracting, we obtain the zero integral of each column of  $D_d$ . This is the exact identity of Theorem 15.13, not a consequence of its absolute bounds. In that identity, the periodic gradient part of the differentiated reconstruction has zero sum, including across the seams of the torus; hence no constant mode has been discarded.

**Step 3: localization and the first moment.** After a fixed reduction of the exponent of the spatial weight, there exist constants  $C, c > 0$ , independent of  $J$ , such that for every column and every integer  $h \geq 0$ ,

$$A^{-3} \int_{hA \leq \text{dist}(y, x) < (h+1)A} |(D_d)_x(y)| dy \leq Ce^{-ch}, \quad (17.10)$$

and the row sums are bounded. These statements follow by subtracting the coarse cell column from the volume-normalized column of  $P_d$ , weighted exponentially; moving a point of a cell to the vertex of the cell costs at most  $\sqrt{3}A$ . The same argument, applied to each weighted row, yields the localization of the rows. Multiplying (17.10) by  $(h+1)A$  and summing over  $h$ , we obtain

$$A^{-3} \int \text{dist}(y, x) |(D_d)_x(y)| dy \leq CA \sum_{h \geq 0} (h+1)e^{-ch} \leq C'A.$$

All sums over directions and colors are finite and included in  $C$ .

**Step 4: heat-kernel cancellation in the continuum.** In this step, all heat estimates use the horizon  $T = T_0$  in Lemma 16.1. To pass from the moments of individual columns to a bound on the supremum over outputs, we use the following elementary inequality. For a smooth periodic function  $f$  and a cube  $Q$  of side  $A$ ,

$$A^3 \sup_Q |f| \leq C \sum_{S \subset \{1,2,3\}} A^{|S|} \int_Q |\partial_S f(y)| dy. \quad (17.11)$$

Indeed, on an interval of length  $A$ , we have  $|f(x)| \leq A^{-1} \int |f| + \int |f'|$ , and applying this successively in the three variables yields (17.11); the endpoints are handled by continuity. If  $Q$  is enlarged to the set of points within distance  $(h+2)A$ , the enlargement is covered by at most  $C(h+3)^3$  cubes of side  $A$ . Summing the resulting bounds on the supremum over the lattice of inputs costs the same overlap factor  $C(h+3)^3$ .

Let  $A^2 \leq t \leq T_0$  and  $f_z(y) = p_t(z - y)$ . Since every column has zero integral, we have

$$(P_t D_d v)(z) = \sum_x v_x \int [p_t(z - y) - p_t(z - x)] (D_d)_x(y) dy.$$

On the  $h$ th annulus around  $x$ , the bracket is at most  $(h+1)A$  times the supremum of  $|\nabla p_t(z - \cdot)|$  on the enlarged cube around  $x$ ; a shortest periodic path proves this also

when the annulus crosses a seam. Using (17.10), then (17.11) with the overlap bound, and finally the  $L^1$  bounds on the derivatives of the heat kernel in Lemma 16.1, we obtain

$$\|P_t D_d v\|_\infty \leq CA \|v\|_\infty \sum_{h \geq 0} e^{-ch} (h+3)^4 \sum_{r=0}^3 A^r \|\nabla^{r+1} p_t\|_1 \leq C \frac{A}{\sqrt{t}} \|v\|_\infty \quad (A^2 \leq t \leq T_0),$$

where in the last inequality we used that each factor  $(A/\sqrt{t})^r$  is at most one. For  $0 < t \leq \min\{A^2, T_0\}$ , the contraction of the heat semigroup and the bound on the row sums imply  $\|P_t D_d v\|_\infty \leq C \|v\|_\infty$ . This proves the value estimate, with constants depending only on the column bounds and on  $T_0/L^2$ . When  $0 < \alpha < 1$ , multiplying by  $t^{\alpha/2}$  gives at most  $CA^\alpha$  in both ranges of time; if  $A^2 > T_0$ , the bound for small times suffices. This proves the bound on the heat norm.

**Step 5: the finer lattice.** We now keep the lattices of meshes  $a$  and  $A$  distinct. Let  $m \in \mathbb{N}$  and  $\delta = a/m$ , so that

$$\frac{a}{\delta} = m \in \mathbb{N}, \quad \frac{A}{\delta} = Bm \in \mathbb{N}. \quad (17.12)$$

Since  $L/A \in \mathbb{N}$  and  $L/A \geq 4$ , the numbers  $L/a = B(L/A)$  and  $L/\delta = Bm(L/A)$  are integers at least four. We therefore apply Lemma 16.3 to the pairs of meshes  $(a, \delta)$  and  $(A, \delta)$ , with integer ratios  $m$  and  $Bm$ , the same function  $b$ , and the horizon  $T = T_0$ . For a fixed integer  $q \geq 0$ ,  $D_\delta^q$  denotes any ordered product of  $q$  physical forward or backward differences, and the identity if  $q = 0$ ; these are exactly the products allowed in both lemmas of Section 16. Let  $p_{z,x}$  be the coefficients of the array  $P_d^{\text{arr}}$ . By the mass identity of Lemma 16.3, we have, for each componentwise column,

$$\begin{aligned} \delta^3 \sum_y (D_a^\delta)_{y,x} &= \delta^3 \sum_y \sum_{z \in \Lambda_{L/a}} \zeta_3((y-z)/a) p_{z,x} - \delta^3 \sum_y \zeta_3((y-x)/A) I \\ &= a^3 \sum_z p_{z,x} - A^3 I = 0, \end{aligned} \quad (17.13)$$

where the last equality is the discrete form of the zero integral in Step 2. The identity matrices in this formula involve the matching direction and color only. Since the function  $b$  is nonnegative, the mass and moment bounds of the interpolation imply

$$\begin{aligned} \delta^3 \sum_y \left| \sum_z \zeta_3((y-z)/a) p_{z,x} \right| &\leq a^3 \sum_z |p_{z,x}|, \\ \delta^3 \sum_y \text{dist}(y,x) \left| \sum_z \zeta_3((y-z)/a) p_{z,x} \right| &\leq a^3 \sum_z (\text{dist}(z,x) + Ca) |p_{z,x}|. \end{aligned}$$

The interpolated coarse column has mass  $A^3$  and first moment at most  $CA^4$ . Both terms of  $D_d^\delta$  therefore satisfy the bounds of the columns before interpolation, with a fixed enlargement of the constants. Since their supports are displaced by at most  $Ca \leq CA$ ,

(17.10) holds with the integral replaced by  $\delta^3 \sum_y$ . The row sums are bounded as well, by the positive partition of unity of each interpolation.

The argument of Step 4 applies on the lattice of mesh  $\delta$ , and we only describe its new ingredient. On an interval of  $m_A = A/\delta = Bm$  lattice points, averaging the inequality

$$|f(r)| \leq |f(s)| + \delta \sum_{u \text{ between } r,s} |D_\delta f(u)|$$

over  $s$  yields the one-dimensional analog of (17.11). Taking the product over the coordinates, with an enlargement by one lattice step to include the differences at the boundary, we obtain (17.11) with sums in place of integrals and products of forward differences in place of derivatives. The overlap bound for the enlarged cubes is unchanged. To subtract the kernel, we use (17.13) and a shortest lattice path, whose length is at most a fixed multiple of the torus distance. Then the  $L^1$  bounds on discrete differences of the heat kernel in Lemma 16.1 imply

$$\|P_t^\delta D_d^\delta v\|_\infty \leq C \min\{1, A/\sqrt{t}\} \|v\|_\infty \quad (0 < t \leq T_0). \quad (17.14)$$

No constant here depends on  $a/\delta$ .

For  $0 \leq t \leq \min\{2A^2, T_0\}$ , we commute the product of  $q$  differences with the heat semigroup and use the derivative bounds of the two interpolations:

$$\|D_\delta^q P_t^\delta D_d^\delta v\|_\infty \leq \|D_\delta^q I_{a \rightarrow \delta} P_d^{\text{arr}} v\|_\infty + \|D_\delta^q I_{A \rightarrow \delta} v\|_\infty \leq C_q (a^{-q} + A^{-q}) \|v\|_\infty \leq C_{J,q} A^{-q} \|v\|_\infty.$$

For  $2A^2 \leq t \leq T_0$ , we split the heat semigroup into two equal halves, use the  $L^1$  bound on the differentiated heat kernel for the first half and (17.14) for the second:

$$\|D_\delta^q P_t^\delta D_d^\delta v\|_\infty \leq C_q (t/2 + \delta^2)^{-q/2} \frac{A}{\sqrt{t/2}} \|v\|_\infty \leq C_q A t^{-(q+1)/2} \|v\|_\infty.$$

Up to fixed constants, the two ranges of time give the asserted bound with  $t + A^2$ ; they include  $t = 0$  and every fixed order of differences. Only the estimate for small times introduces the factor  $B^q$ , through  $a^{-q} = B^q A^{-q}$ , and this factor is allowed in  $C_{J,q}$ . All the cancellations used in this proof are identities for the columns of linear operators; neither Taylor's formula nor the heat-kernel cancellation yields a zero integral of  $\bar{Q}_d$ .  $\square$

## 17.4 Conditional quadratic moments

**Quadratic maps and their norms.** Let  $Q$  be a quadratic map from arrays on a lattice of mesh  $a_{\text{in}}$  to arrays on a lattice of mesh  $a_{\text{out}} \leq a_{\text{in}}$ . We write a scalar output component as

$$Q_y(v, v) = \sum_{x,z} T_{y;x,z} v_x v_z,$$

where the indices include directions and colors. We use the symmetric coefficient tensor of the quadratic map: replacing  $T_{y;x,z}$  by  $(T_{y;x,z} + T_{y;z,x})/2$  leaves  $Q(v, v)$  unchanged, and all bounds below are assumed or proved for this symmetric tensor. We write  $Q(u, v)$  for the corresponding symmetric bilinear map, and  $Q : \mathcal{M}$  for the array with components  $\sum_{x,z} T_{y;x,z} \mathcal{M}_{xz}$ , for a matrix  $\mathcal{M}$  indexed by the inputs. Let  $t(y, x, z)$  be the length of a shortest connected network in  $\mathbb{T}_L^3$  containing the locations of  $y$ ,  $x$ , and  $z$ , with internal vertices allowed, as in Definition 4.6. We fix a spatial exponent  $a'_{\text{sp}} > 0$ , smaller than the exponents of the coefficient bounds of Section 15 by a fixed amount, and set

$$\|T\|_{a_{\text{in}}, a_{\text{out}}} = \max \left\{ \sup_y \sum_{x,z} e^{a'_{\text{sp}} t(y,x,z)/a_{\text{in}}} |T_{y;x,z}|, \sup_x \frac{a_{\text{out}}^3}{a_{\text{in}}^3} \sum_{y,z} e^{a'_{\text{sp}} t(y,x,z)/a_{\text{in}}} |T_{y;x,z}| \right\}. \quad (17.15)$$

By symmetry, the column with the second input fixed has the same bound. In particular, if  $a_{\text{in}} = a_i$ ,  $a_{\text{out}} = \delta$ , and  $\|T\|_{a_i, \delta} \leq K_Q$ , then

$$\sup_y \sum_{x,z} |T_{y;x,z}| \leq K_Q, \quad \sup_x \frac{\delta^3}{a_i^3} \sum_{y,z} |T_{y;x,z}| \leq K_Q. \quad (17.16)$$

The same exponent  $a'_{\text{sp}}$  is used at every level below; it is not reduced after each composition. At the coarsest level, we set  $\bar{b}_0 = r$ . For  $i \geq 1$ ,  $\bar{b}_i$  always denotes the reconstruction amplitude bound of Lemma 17.1, never a random bound left outside a conditional expectation.

**Why a conditional decomposition is needed.** For  $j \geq 2$ , by the reconstruction amplitude bound, the quadratic term  $a_{j-1}^{-1} \bar{Q}(\theta_{j-1}, \theta_{j-1})$  in (17.1) is at most  $C a_{j-1}^{-1} \bar{b}_{j-1}^2$ , which is in general of order  $g_3^2 p_{j-1}^2$  and is not summable in  $j$ . We therefore regard  $\theta_{j-1}$  as reconstructed from  $\theta_{j-2}$ , condition on  $\mathcal{F}_{j-2}$ , and separate a conditionally centered part, which concentrates, from a conditional expectation error, which is smaller. The conditional expectation is close to the quadratic term evaluated on the linearized reconstruction  $L_{j-1} \theta_{j-2}$ , and repeating the argument pulls the quadratic coefficient back through the linearized reconstructions. Each pullback contracts the coefficient by the factor  $a_{\text{cap}}^2$ , one factor for each argument, while dividing by the coarser mesh costs a factor  $B$ ; the resulting ratio  $q_{\text{anc}} = B a_{\text{cap}}^2 < 1$  makes the final convolution summable. Volume-normalized columns are essential here: with counting column sums, each pullback would lose an additional factor  $B^3$ .

Three errors remain in the conditional expectation, and they explain the three terms in the first part of Theorem 17.4 below. Replacing the nonlinear zero-fluctuation reconstruction by its linearization costs  $a_{\text{cap}}^2 \|\phi\|_\infty^3$ . The conditional means of the fluctuation variables and the change of their covariance with the coarse field cost  $g_i^2 \|\phi\|_\infty$ . Here we use that the conditional kernel at the zero coarse field is invariant under constant

color rotations, although it is spatially correlated; an expected adjoint vector built from it therefore vanishes. The terms of higher order in the fluctuations cost  $g_i^3 p_i^3$ .

The deterministic part of the pullback is the following lemma.

**Lemma 17.3.** *Let  $1 \leq i \leq j$ , let  $T$  be the symmetric coefficient array of a quadratic map from arrays on the lattice of mesh  $a_i$  to arrays on the lattice of mesh  $a_j$ , and let  $L$  be the coefficient array of a linear map from arrays on the lattice of mesh  $a_{i-1} = Ba_i$  to arrays on the lattice of mesh  $a_i$ . Suppose that  $\|T\|_{a_i, a_j} \leq K$  and that*

$$\sup_x \sum_u e^{a'_{\text{sp}} \text{dist}(x,u)/a_{i-1}} |L_{x,u}| \leq a_{\text{cap}}, \quad \sup_u \frac{a_i^3}{a_{i-1}^3} \sum_x e^{a'_{\text{sp}} \text{dist}(x,u)/a_{i-1}} |L_{x,u}| \leq a_{\text{cap}}. \quad (17.17)$$

Then the coefficient array

$$T'_{y;u,v} = \sum_{x,z} T_{y;x,z} L_{x,u} L_{z,v}$$

of the map  $v \mapsto Q(Lv, Lv)$  is symmetric and satisfies  $\|T'\|_{a_{i-1}, a_j} \leq Ka_{\text{cap}}^2$ . If the quadratic map is equivariant under the adjoint action of constant color rotations and  $L$  commutes with them, then the map  $v \mapsto Q(Lv, Lv)$  is equivariant.

*Proof.* We first prove the unweighted bounds. The output row of  $T'$  is at most  $Ka_{\text{cap}}^2$ , by the row bounds of  $T$  and of  $L$ . For the column with the first input  $u$  fixed, we use the volume-normalized column of  $L$  in the first argument and its row in the second:

$$\frac{a_j^3}{a_{i-1}^3} \sum_{y,v} |T'_{y;u,v}| \leq \frac{a_i^3}{a_{i-1}^3} \sum_x |L_{x,u}| a_{\text{cap}} \left( \frac{a_j^3}{a_i^3} \sum_{y,z} |T_{y;x,z}| \right) \leq Ka_{\text{cap}}^2.$$

Fixing the second input instead repeats this calculation with  $x, z$  and  $u, v$  interchanged. The factor  $a_i^3/a_{i-1}^3 = B^{-3}$  cancels the factor  $B^3$  in the counting column sum of  $L$ .

The weights cause no additional loss. A shortest network for  $(y, x, z)$ , joined with shortest paths from  $x$  to  $u$  and from  $z$  to  $v$ , is a connected network containing  $(y, u, v)$ , which implies  $t(y, u, v) \leq t(y, x, z) + \text{dist}(x, u) + \text{dist}(z, v)$ ; periodic shortest networks satisfy the same inequality. In the units  $a_{i-1} > a_i$  of the new input mesh, the weight of the old network is at most its weight in the units  $a_i$ , and the weights in (17.17) are already in the units  $a_{i-1}$ . The new weight is therefore at most the product of the three old weights, and the preceding calculation applies with these weights inserted and the same exponent. Symmetry of  $T'$  follows from that of  $T$ . Equivariance is preserved because  $L$  commutes with the color rotations.  $\square$

**Theorem 17.4.** *Assume the setting of Section 17.1 and fix  $k \geq 1$ . Let  $\bar{b}_i$ ,  $1 \leq i \leq k$ , be the reconstruction amplitude bounds of Lemma 17.1, and  $\bar{b}_0 = r$ . Let  $a_{\text{cap}} = C_{\text{CP}}/B$  and  $q_{\text{anc}} = Ba_{\text{cap}}^2 = C_{\text{CP}}^2/B < 1$ , and for  $1 \leq i \leq k$  let  $L_i = D_{\theta} \hat{\Phi}_{a_i^k, i}(0, 0)$  be the linearized zero-fluctuation reconstruction of the  $i$ th step.*

Let  $1 \leq i \leq k$ , and let  $Q$  be a deterministic quadratic map from arrays on the lattice of mesh  $a_i$  to  $\mathfrak{su}(2)$ -valued arrays on a lattice of mesh  $\delta \leq a_i$ , which is equivariant under the adjoint action of constant color rotations and satisfies  $\|T\|_{a_i, \delta} \leq K_Q$  for its coefficient array  $T$ , with all input and output directions and colors included. Then the following statements hold.

1. For every coarse field  $e^\phi \in B_r$ , under the conditional fluctuation kernel of the  $i$ th step at the coarse field  $e^\phi$ ,

$$\|\mathbb{E}[Q(\theta_i, \theta_i) \mid \theta_{i-1} = \phi] - Q(L_i \phi, L_i \phi)\|_\infty \leq C_J K_Q \{a_{\text{cap}}^2 \|\phi\|_\infty^3 + g_i^2 \|\phi\|_\infty + g_i^3 p_i^3\}.$$

Let  $\mu_\phi = \mathbb{E}_\phi w$  and  $\Sigma_\phi = \text{Cov}_\phi(w, w)$  be the conditional mean and covariance of the fluctuation variables under this kernel (in the notation of Section 26,  $\mu_\phi = m_\phi$ ), and let  $E_\phi$  be the derivative at  $w = \mu_\phi$  of the map  $w \mapsto g_i^{-1} \hat{\Phi}_{d_i^k, i}(\phi, (g_{i,s} w_s)_s)$ , which includes the ratios  $g_{i,s}/g_i$ . Then the expectation of  $Q(g_i E_\phi(w - \mu_\phi), g_i E_\phi(w - \mu_\phi))$  equals  $g_i^2 Q : (E_\phi \Sigma_\phi E_\phi^\top)$ . This contraction vanishes at  $\phi = 0$ , and

$$\|E_\phi \Sigma_\phi E_\phi^\top - E_0 \Sigma_0 E_0^\top\|_{\text{op}} \leq C_J \|\phi\|_\infty.$$

The first estimate accounts for the value of the reconstruction at  $\mu_\phi$ , its displacement from the zero-fluctuation reconstruction, and all remainders of the means and of the covariance; it does not assume that the conditional expectation of a quadratic term vanishes when  $\phi \neq 0$ .

2. For every  $0 < \alpha < 3/2$ , on every sample,

$$\mathbb{E} \left[ \left\| Q(\theta_i, \theta_i) - \mathbb{E}[Q(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}] \right\|_{-\alpha; T_0} \mid \mathcal{F}_{i-1} \right] \leq C_{J, \alpha} K_Q g_i \bar{b}_i a_i^\alpha \sqrt{\log(2 + L/a_i)}.$$

3. For  $2 \leq j \leq k$  define quadratic maps  $Q_{j,i}$  from arrays on the lattice of mesh  $a_i$  to arrays on the lattice of mesh  $a_j$ , for  $0 \leq i \leq j-1$ , by

$$Q_{j,j-1} = \bar{Q}_{d_j^k, j}, \quad Q_{j,i-1}(v, v) = Q_{j,i}(L_i v, L_i v) \quad (1 \leq i < j),$$

where  $\bar{Q}_{d_j^k, j}$  is the quadratic part of the zero-fluctuation reconstruction of the  $j$ th step, as in Lemma 17.2. These maps are equivariant. Let  $T_{j,i}$  be the symmetric coefficient array of  $Q_{j,i}$ , and set  $K_{j,i} = \|T_{j,i}\|_{a_i, a_j}$ . Then

$$K_{j,i} \leq C a_{\text{cap}}^{2(j-1-i)}, \quad \frac{K_{j,i}}{a_{j-1}} \leq C \frac{q_{\text{anc}}^{j-1-i}}{a_i}.$$

4. For  $2 \leq j \leq k$  and  $1 \leq i \leq j-1$  put

$$\begin{aligned} Z_{j,i} &= Q_{j,i}(\theta_i, \theta_i) - \mathbb{E}[Q_{j,i}(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}], \\ e_{j,i} &= \mathbb{E}[Q_{j,i}(\theta_i, \theta_i) \mid \mathcal{F}_{i-1}] - Q_{j,i-1}(\theta_{i-1}, \theta_{i-1}). \end{aligned}$$

Then, on every sample,

$$\overline{Q}_{d_j^k,j}(\theta_{j-1}, \theta_{j-1}) = Q_{j,0}(\theta_0, \theta_0) + \sum_{i=1}^{j-1} (Z_{j,i} + e_{j,i}).$$

Further, with

$$v_i = g_i \bar{b}_i a_i^{\alpha-1}, \quad u_i = \frac{q_{\text{anc}} \bar{b}_{i-1}^3}{a_{i-1}} + g_3^2 \bar{b}_{i-1} + g_3^3 \sqrt{a_i} p_i^3,$$

we have, for  $0 < \alpha < 3/2$ ,

$$\mathbb{E} \|a_{j-1}^{-1} \overline{Q}_{d_j^k,j}(\theta_{j-1}, \theta_{j-1})\|_{-\alpha; T_0} \leq \frac{Cr^2}{H} q_{\text{anc}}^{j-1} + C \sum_{i=1}^{j-1} q_{\text{anc}}^{j-1-i} \left\{ v_i \sqrt{\log(2 + L/a_i)} + u_i \right\}.$$

5. For  $1/2 < \alpha < 3/2$  and every  $0 < \tau < \min\{1/2, -\log q_{\text{anc}} / \log B\}$ ,

$$\sum_{j=l}^k \mathbb{E} \|a_{j-1}^{-1} \overline{Q}_{d_j^k,j}(\theta_{j-1}, \theta_{j-1})\|_{-\alpha; T_0} \leq C_\tau a_l^\tau,$$

uniformly in  $k \geq l \geq 2$ .

The constants depend on the fixed construction parameters and coefficient bounds, on  $L$ ,  $H$ ,  $T_0$ ,  $\alpha$ , and  $\tau$ , and on the fixed coupling  $g_3$ , but not on  $k$ , the depth, the ratio  $a_i/\delta$ , or the sample. The identity in the fourth part does not identify compositions of the nonlinear zero-fluctuation reconstructions with products of their linearizations; their difference is contained in  $e_{j,i}$ .

*Proof.* We fix  $k$  for the whole proof, so that all coefficients, zero-fluctuation reconstructions, and conditional kernels have the depths  $d_i^k = J(k-i)$  of one hierarchy. Intermediate constants in the estimates for a single kernel may depend on  $J$ ; all estimates are uniform in the depth. The proof has five steps. Steps 1 and 2 prove the first two parts, Step 3 proves the third, Step 4 the fourth, and Step 5 the last.

**Step 1: the conditional expectation of a quadratic term.** Fix a level  $i$  and a coarse field  $e^\phi \in B_r$ , and write  $R = \|\phi\|_\infty$ ,  $g = g_i$ ,  $t = t_i$ , and  $p = p_i$ . In this step, all expectations are under the conditional fluctuation kernel at  $e^\phi$ . Let

$$\mu_\phi = \mathbb{E}_\phi w, \quad \xi = w - \mu_\phi, \quad \Sigma_\phi = \mathbb{E}_\phi \xi \xi^\top, \quad \psi_\phi(w) = \widehat{\Phi}_{d_i^k,i}(\phi, (g_{i,s} w_s)_s).$$

The segment of coarse fields  $u\phi$ ,  $0 \leq u \leq 1$ , stays in  $B_r$ . The fluctuation cutoff domain, the free logarithmic coordinates, and the conditional fluctuation potential are those preserved by Theorem 15.14. The domains and bounds on the coupling-scaled fluctuation inputs are those fixed in Section 17.1; hence the whole segment satisfies the domain conditions.

The bounds on means and on the dependence on the coarse field in Theorem 13.10 apply to this kernel, and give

$$\|\mu_\phi\|_\infty \leq C_J t R, \quad \|\Sigma_\phi\|_{\text{op}} \leq 2, \quad \|\Sigma_\phi - \Sigma_0\|_{\text{row+col}} \leq C_J t p^3 R, \quad (17.18)$$

where the last norm is the larger of the maximal row and column sums, possibly with a fixed smaller exponential weight; since it is at most their sum, the cited bound holds with either convention. The product of balls does not depend on  $\phi$ . To make precise which normalization enters, let  $V_\phi$  be the conditional fluctuation potential and  $S_u = \partial_u V_{u\phi} - \mathbb{E}_{u\phi} \partial_u V_{u\phi}$  its centered potential derivative along the segment. In the notation of Lemma 13.6,  $S_u = Q_u$  for this path, and  $-S_u$  is the normalized log-density derivative. By differentiation, we have

$$\partial_u \mathbb{E}_{u\phi} F = \mathbb{E}_{u\phi} \partial_u F - \mathbb{E}_{u\phi} (F S_u), \quad \partial_u \Sigma_{u\phi}(r, s) = -\mathbb{E}_{u\phi} (\xi_r \xi_s S_u). \quad (17.19)$$

Indeed, we differentiate the numerator and the denominator; this is legitimate because the support is compact and fixed and the integrands are smooth. In the second identity, the terms containing the derivative of  $\mu_{u\phi}$  multiply  $\mathbb{E} \xi = 0$  and cancel. Thus (17.18) refers to the covariance of the interacting truncated kernel and to its centered derivative; there is no boundary term and no reference Gaussian covariance.

By the joint coefficient bounds kept by Lemma 15.16, we may expand the reconstruction map by Taylor's formula at the conditional mean  $\mu_\phi$ . We define

$$v_\phi = \psi_\phi(\mu_\phi), \quad E_\phi = g^{-1} D_w \psi_\phi(\mu_\phi), \quad B_\phi = (2g^2)^{-1} D_w^2 \psi_\phi(\mu_\phi);$$

the ratios  $g_{i,s}/g$  are included in these definitions. The mean lies in the convex product, and so does the segment from it to each  $w$ . Since  $\|\xi\|_\infty \leq 2p$ , Taylor's formula with integral remainder and the bound on the third derivative in  $w$  imply

$$\theta_i = v_\phi + g E_\phi \xi + g^2 B_\phi(\xi, \xi) + \mathcal{R}_\phi(\xi), \quad \|\mathcal{R}_\phi(\xi)\|_\infty \leq C_J g^3 p^3. \quad (17.20)$$

Each derivative in  $w$  contributes its coupling factor, and the joint absolute bound sums the remaining coefficients. These bounds are at fixed  $J$  and impose no further condition on  $J$ .

By the bounds on the row sums and volume-normalized column sums kept by the reconstruction maps, converted between the finitely many types of fluctuation variables of a step, we have

$$\|E_\phi\|_{\text{op}} + \|B_\phi\|_{\text{abs}} \leq C_J,$$

where  $\|B_\phi\|_{\text{abs}}$  is the larger of its absolute row and column sums. The joint bounds in the coarse field and the coupling-scaled fluctuation inputs also give

$$\|E_\phi - E_0\|_{\text{op}} + \|B_\phi - B_0\|_{\text{abs}} \leq C_J (R + g \|\mu_\phi\|_\infty) \leq C'_J R. \quad (17.21)$$

For this bound, we integrate the derivative in the coarse field along  $s\phi$  with the coupling-scaled fluctuation inputs equal to zero, and then the derivative in these inputs from zero to  $(g_{i,s}\mu_{\phi,s})_s$ . Both segments lie in the fixed inner domain, where the normalized coefficients have bounded row and column sums. At the zero coarse field, we have  $\mu_0 = 0$  by the global color symmetry, and the endpoint is exactly  $E_0, B_0$ . The row and column bounds imply the displayed operator bounds by Schur's test,

$$\sum_x \left| \sum_y K_{xy} u_y \right|^2 \leq \left( \sup_x \sum_y |K_{xy}| \right) \left( \sup_y \sum_x |K_{xy}| \right) \sum_y |u_y|^2,$$

and the conversion from volume-normalized to counting column sums costs only a factor depending on  $J$ .

Let  $\mathcal{Z}_\phi = E_\phi \Sigma_\phi E_\phi^\top$ . Adding and subtracting  $E_0 \Sigma_\phi E_\phi^\top$  and  $E_0 \Sigma_0 E_\phi^\top$ , and using (17.18) and (17.21), we obtain

$$\|\mathcal{Z}_\phi - \mathcal{Z}_0\|_{\text{op}} \leq C_J R(1 + tp^3) \leq C'_J R. \quad (17.22)$$

This is the covariance comparison of the first part, for the coefficients of the reconstruction maps themselves.

At the zero coarse field, the product of balls and the potential are invariant under every constant adjoint color rotation. This implies  $\Sigma_0 = \Sigma_{\text{sp}} \otimes I_3$  for a spatial matrix  $\Sigma_{\text{sp}}$ , and the linear coefficient  $E_0$  acts as a scalar on the adjoint color representation. To check this, note that rotations by  $\pi$  about the three coordinate axes force each  $3 \times 3$  color block commuting with them to be diagonal, and rotations exchanging the axes force its three diagonal entries to agree. The same invariance for the equivariant map  $Q$  implies

$$Q : \mathcal{Z}_0 = 0. \quad (17.23)$$

Indeed, each output component of the contraction is an adjoint vector fixed by every rotation, and rotations by  $\pi$  about two different axes force it to vanish. This argument is componentwise in the output and keeps every spatial entry of the covariance; it needs no translation invariance of  $Q$ .

The same reasoning shows that  $B_0 : \Sigma_0 = 0$ . Together with (17.21) and (17.18), this implies

$$\|\mathbb{E}_\phi B_\phi(\xi, \xi)\|_\infty \leq C_J R(1 + tp^3) \leq C'_J R. \quad (17.24)$$

Here we used that each scalar coefficient matrix of  $B_\phi$  has absolute entry sum at most  $C_J$ , and that a contraction against a matrix of operator norm  $D$  has absolute value at most  $C_J D$ , since every entry of that matrix is bounded by  $D$ . The same remark justifies contractions against (17.22) without any dependence on the dimension.

The displacement of the reconstruction at the mean satisfies

$$\|v_\phi - \psi_\phi(0)\|_\infty \leq C_J g \|\mu_\phi\|_\infty \leq C_J g^2 R, \quad (17.25)$$

where we integrated the derivative in  $w$ , whose absolute row sums are at most  $C_J g$ , along the segment from 0 to  $\mu_\phi$ , and used that  $t/g$  depends only on  $J$ . By the bounds on the first three derivatives of the zero-fluctuation reconstruction and the majorant of Lemma 15.16, we have

$$\|\psi_\phi(0) - L_i\phi\|_\infty \leq C_J a_{\text{cap}} R^2, \quad \|L_i\phi\|_\infty \leq a_{\text{cap}} R. \quad (17.26)$$

The constant  $C_{\text{CP}}$  used to choose  $J$  has already been enlarged by the uniform nonlinear bounds, as explained in Step 1 of Lemma 17.1. The present  $J$ -dependent quadratic-error constant is chosen afterwards and does not change  $a_{\text{cap}}$ . By (17.25), (17.26), and  $\|Q(u, v)\|_\infty \leq K_Q \|u\|_\infty \|v\|_\infty$ , we have

$$\|Q(v_\phi, v_\phi) - Q(L_i\phi, L_i\phi)\|_\infty \leq C_J K_Q (a_{\text{cap}}^2 R^3 + g^2 R). \quad (17.27)$$

To see every power, let  $\epsilon = v_\phi - L_i\phi$ ; then the left side is at most  $2K_Q a_{\text{cap}} R \|\epsilon\|_\infty + K_Q \|\epsilon\|_\infty^2$ , with  $\|\epsilon\|_\infty \leq C_J (a_{\text{cap}} R^2 + g^2 R)$ . Since  $R$  and  $g$  are bounded by fixed constants, the term  $R^4$  is absorbed into  $R^3$  and the remaining terms into  $g^2 R$ .

We now expand  $Q(\theta_i, \theta_i)$  using (17.20). The term linear in  $gE_\phi \xi$  has mean zero. The term quadratic in it has mean

$$\mathbb{E}_\phi Q(gE_\phi \xi, gE_\phi \xi) = g^2 Q : \mathcal{Z}_\phi,$$

which is the identity of the first part; by (17.23) and (17.22), it is at most  $C_J K_Q g^2 R$ . The cross term of  $v_\phi$  with  $g^2 B_\phi(\xi, \xi)$  has mean at most  $C_J K_Q g^2 R$ , by (17.24) and the boundedness of  $v_\phi$ . Every remaining product contains either one linear and one quadratic term in  $\xi$ , or a remainder, or two quadratic terms. Their pointwise bounds are  $C_J K_Q g^3 p^3$ ,  $C_J K_Q g^3 p^3$ , and  $C_J K_Q g^4 p^4$ , respectively. Products of a remainder with a term of positive degree in  $\xi$  satisfy the same bound  $C_J K_Q g^3 p^3$ , since  $gp$  is bounded by a fixed constant, and so does the square of the remainder. In the term  $Q(v_\phi, \mathcal{R}_\phi)$ , only the boundedness of  $v_\phi$  is needed. Further,  $g^4 p^4 \leq C_J g^3 p^3$ . Combining these bounds with (17.27), we obtain the first estimate.

**Step 2: the centered heat norm of a quadratic term.** We prove the second part without assuming smoothness of the output. Fix a sample, so that  $\phi = \theta_{i-1}$  is fixed and  $\|\phi\|_\infty \leq \bar{b}_{i-1}$ . Let  $Y(w) = Q(\theta_i(w), \theta_i(w))$  and  $F = Y - \mathbb{E}_\phi Y$ . On the whole product of balls, differentiating the polarized map, we find

$$D_w Y = Q(D_w \theta_i, \theta_i) + Q(\theta_i, D_w \theta_i).$$

The absolute row sums of the derivative of  $\theta_i$  in  $w$  are at most  $C_J g_i$ , and its volume-normalized columns satisfy the same bound. At most a number of fluctuation variables depending only on  $J$  is attached to each cell of the lattice of mesh  $a_i$ . Further,  $\|\theta_i(w)\|_\infty \leq \bar{b}_i$  at every point  $w$  of the product. Indeed, the recurrence of Step 1 in the proof of

Lemma 17.1 gives  $\|\theta_i(w)\|_\infty \leq a_{\text{cap}}\bar{b}_{i-1} + C_{\text{rec},J}t_i p_i$ , and this is at most  $\bar{b}_i$  by (17.5), since  $C_J \geq C_{\text{cap},J}$ . Let  $\mathcal{A} = C_J K_Q g_i \bar{b}_i$ . Then the derivative matrix  $G = D_w Y$  satisfies

$$\sup_y \sum_r |G_{y,r}| \leq \mathcal{A}, \quad \sup_r \frac{\delta^3}{a_i^3} \sum_y |G_{y,r}| \leq \mathcal{A}, \quad (17.28)$$

where  $r$  runs over the scalar fluctuation variables. For the column bound, we fix the differentiated fluctuation variable, use the column of  $Q$  for its differentiated argument, bound its other argument by  $\bar{b}_i$ , and then use the volume-normalized column of  $D_w \theta_i$ ; the volume factors multiply to  $\delta^3/a_i^3$ . Since both differentiated arguments satisfy the same bound, their sum is included in  $C_J$ . This holds for all directions and colors.

By Schur's test, the counting  $\ell^2$  operator norm of  $G$  is at most  $\mathcal{A}(a_i/\delta)^{3/2}$ . For  $0 < s \leq T_0$ , let  $\omega_y(s, x) = \int_{Q_y} p_s(x - z) dz$  for the output cells  $Q_y$  of side  $\delta$ . For each scalar component of  $P_s Y$  at a point  $x$ , the Lipschitz constant in  $w$  is at most

$$C\mathcal{A} \min\{1, a_i^{3/2} s^{-3/4}\}. \quad (17.29)$$

The first bound follows from the absolute row sums and  $\sum_y \omega_y = 1$ . For the second, we compose the operator bound with  $\|\omega(s, x)\|_2 \leq C\delta^{3/2}s^{-3/4}$ , from Lemma 16.1; the factor  $(a_i/\delta)^{3/2}$  cancels the factor  $\delta^{3/2}$  of the output cells. By the derivative and time-increment bounds of the same lemma, one spatial derivative of the cell integrals costs  $s^{-1/2}$  and one time derivative costs  $s^{-1}$  when  $a_i^2 \leq s \leq T_0$ , in which case  $s \geq \delta^2$ . These bounds also hold for differences of two evaluations, by integration along a shortest spatial path or a time segment.

We now take the supremum over times, beginning with times below  $a_i^2$ . Let  $M_i = \mathbb{E}_\phi \theta_i$ . By the amplitude bound, we have  $\|M_i\|_\infty \leq \bar{b}_i$  as well as  $\|\theta_i\|_\infty \leq \bar{b}_i$ . The output row bound in (17.16) then gives

$$\|Q(\theta_i, \theta_i) - Q(M_i, M_i)\|_\infty \leq 2K_Q \bar{b}_i \|\theta_i - M_i\|_\infty,$$

and subtracting the conditional expectation, we obtain

$$\|F\|_\infty \leq 2K_Q \bar{b}_i (\|\theta_i - M_i\|_\infty + \mathbb{E}_\phi \|\theta_i - M_i\|_\infty). \quad (17.30)$$

This uses the output row; a bound on the input columns alone would not imply it. There are  $N_i = 9(L/a_i)^3$  scalar coordinates of  $\theta_i$  on its own lattice. By the bounds on the derivative rows of the reconstruction maps in Lemma 15.16, each of them is a Lipschitz function of  $w$  with Euclidean constant at most  $C_J g_i$ , since the  $\ell^2$  norm of a row of the derivative is at most its absolute sum. The conditional kernel is the one on a convex product described in Section 17.1 (Lemma 15.10 and Theorem 15.14), with Hessian at least  $I/2$ . By Theorem 5.4 with  $m = 1/2$  and a union bound, we have

$$\mathbb{P}_\phi \{\|\theta_i - M_i\|_\infty > x\} \leq \min\{1, 2N_i \exp[-x^2/(C_J g_i^2)]\}. \quad (17.31)$$

To integrate this bound, we take  $x_0 = g_i \sqrt{C_J \log(2N_i)}$ , and for this integral we let  $A^2 = C_J g_i^2$ . We use  $\int_{x_0}^{\infty} e^{-x^2/A^2} dx \leq A^2 e^{-x_0^2/A^2} / (2x_0)$ , which follows by inserting the factor  $x/x_0 \geq 1$  in the integrand. After enlarging the constant, we obtain

$$\mathbb{E}_\phi \|\theta_i - M_i\|_\infty \leq C_J g_i \sqrt{\log(2 + L/a_i)}, \quad \mathbb{E}_\phi \|F\|_\infty \leq C_J K_Q g_i \bar{b}_i \sqrt{\log(2 + L/a_i)},$$

where we used  $\log(2N_i) \leq C \log(2 + L/a_i)$ , which holds because  $L/a_i \geq 4$ . Since the heat semigroup contracts the supremum norm, the expected supremum over  $0 < s \leq \min\{a_i^2, T_0\}$  of  $s^{\alpha/2} \|P_s F\|_\infty$  is at most  $C A a_i^\alpha \sqrt{\log(2 + L/a_i)}$ . Only the coordinates on the lattice of mesh  $a_i$  have been counted; there is no union bound over the arbitrarily fine output lattice. The mean  $M_i$  is the conditional mean of the reconstruction, which need not vanish.

For the remaining times, let  $S_h = 2^h a_i^2$ , and consider each nonempty interval  $[S_h, \min\{2S_h, T_0\}]$ ,  $h \geq 0$ . The centered process  $s^{\alpha/2} (P_s F)(x)$ , with its nine scalar components, has one-point sub-Gaussian parameter at most

$$\sigma_h = C_{J,\alpha} A a_i^{3/2} S_h^{\alpha/2-3/4},$$

by (17.29) and (16.18). Its increments have parameter at most  $C\sigma_h$  times

$$\min\{1, \text{dist}(x, x') / \sqrt{S_h}\} + \min\{1, |s - s'| / S_h\};$$

this includes the derivative of  $s^{\alpha/2}$ , whose relative size on the interval is at most  $C_\alpha / S_h$ . Between distinct components, we take the distance to be  $C\sigma_h$ ; adding the Lipschitz constants of the two components justifies this choice. Using spatial nets of spacing  $c\eta \sqrt{S_h}$  and temporal nets of spacing  $c\eta S_h$ , together with the nine components, we obtain covering numbers at most

$$C(1 + L/\sqrt{S_h})^3 \eta^{-4} \quad (0 < \eta < 1).$$

Since the process is continuous for  $s > 0$  and smooth in  $x$  on these intervals, a countable dense set of parameters determines its supremum. We apply Lemma 16.4 to the increment function divided by  $\sigma_h$ , with  $s = C\sigma_h$ ,  $b = 0$ ,  $D = C$ ,  $A = C(1 + L/\sqrt{S_h})^3$ , and  $q_{\text{ent}} = 4$ ; the point  $z_0$  of that lemma has the same one-point bound. At  $p = 1$ , the lemma bounds the expected supremum over the interval by

$$C\sigma_h \sqrt{\log(2 + L/a_i)},$$

because  $\sqrt{S_h} \geq a_i$ , so that  $A \leq C(1 + L/a_i)^3$ ; no factor depending on the output mesh enters. Since  $\sigma_h = C_{J,\alpha} A a_i^\alpha 2^{-h(3/4-\alpha/2)}$ , the sum over the intervals converges with ratio  $2^{\alpha/2-3/4} < 1$  when  $\alpha < 3/2$ . Adding the bound for small times, we obtain

$$\mathbb{E}_\phi \|F\|_{-\alpha; T_0} \leq C_{J,\alpha} K_Q g_i \bar{b}_i a_i^\alpha \sqrt{\log(2 + L/a_i)}.$$

Conditionally on  $\mathcal{F}_{i-1}$ , all constants here are deterministic and uniform over the coarse fields that occur. This is the second part.

**Step 3: the pulled-back coefficients.** The linearized reconstruction  $L_i$  has output mesh  $a_i$  and input mesh  $a_{i-1} = Ba_i$ , and it satisfies (17.17). These are the weighted bounds on the first derivative of the zero-fluctuation reconstruction, with the constant  $C_{CP}$ , uniform in  $J$ , enlarged as in Section 17.1. By the bounds on the second derivative of the zero-fluctuation reconstruction, the coefficient  $Q_{j,j-1} = \overline{Q}_{a_j^k, j}$  has norm at most a fixed constant  $C$ . By induction down to the input level  $i$ , using Lemma 17.3 at each step, we obtain

$$K_{j,i} \leq Ca_{\text{cap}}^{2(j-1-i)}.$$

The global color equivariance of the zero-fluctuation reconstruction implies the equivariance of its quadratic part, and each  $L_i$  commutes with constant color rotations; hence every  $Q_{j,i}$  is equivariant. With  $m = j - 1 - i$ , the identity  $a_i = B^m a_{j-1}$  gives

$$\frac{K_{j,i}}{a_{j-1}} \leq \frac{CB^m a_{\text{cap}}^{2m}}{a_i} = \frac{Cq_{\text{anc}}^m}{a_i}. \quad (17.32)$$

**Step 4: the telescoping identity and the convolution.** Let  $Y_i = Q_{j,i}(\theta_i, \theta_i)$  for  $0 \leq i \leq j - 1$ . By the definitions in the fourth part,

$$Y_i - Y_{i-1} = \{Y_i - \mathbb{E}[Y_i \mid \mathcal{F}_{i-1}]\} + \{\mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] - Y_{i-1}\} = Z_{j,i} + e_{j,i}.$$

Summing this finite identity over  $1 \leq i \leq j - 1$ , we obtain the telescoping identity on every sample. Since every conditional expectation is taken under the kernel of the same hierarchy that generated  $\theta_i$ , Steps 1 and 2 apply with  $\phi = \theta_{i-1}$  and the amplitude bound  $\bar{b}_{i-1}$  on  $\|\theta_{i-1}\|_\infty$ .

The term from the coarsest level is bounded deterministically by

$$\left\| \frac{Q_{j,0}(\theta_0, \theta_0)}{a_{j-1}} \right\|_{-\alpha; T_0} \leq \frac{Cr^2}{H} q_{\text{anc}}^{j-1},$$

by (17.4), (17.32) with  $i = 0$ ,  $a_0 = H$ , and  $\|\theta_0\|_\infty \leq r$ . For the centered terms, the second part and (17.32) give

$$\frac{\mathbb{E}\|Z_{j,i}\|_{-\alpha; T_0}}{a_{j-1}} \leq Cq_{\text{anc}}^{j-1-i} g_i \bar{b}_i a_i^{\alpha-1} \sqrt{\log(2 + L/a_i)}.$$

For the conditional expectation errors, we use the first part, (17.4), and the three identities

$$\frac{a_{\text{cap}}^2}{a_i} = \frac{q_{\text{anc}}}{a_{i-1}}, \quad \frac{g_i^2}{a_i} = g_3^2, \quad \frac{g_i^3}{a_i} = g_3^3 \sqrt{a_i},$$

and obtain

$$\frac{\mathbb{E}\|e_{j,i}\|_{-\alpha; T_0}}{a_{j-1}} \leq Cq_{\text{anc}}^{j-1-i} u_i.$$

By the triangle inequality, we obtain the convolution estimate of the fourth part, with  $v_i$  as defined there. No composition of the nonlinear zero-fluctuation reconstructions has been identified with a product of linearizations; the difference remains in  $e_{j,i}$ .

**Step 5: summation.** Let  $1/2 < \alpha < 3/2$ . By the reconstruction amplitude bound of Lemma 17.1 and (17.6), we have, with constants depending on the fixed  $H$ ,  $g_3$ , and  $r$ ,

$$\bar{b}_i \leq C(1+i)B^{-i/2} \quad (i \geq 0), \quad (17.33)$$

since  $a_{\text{cap}}^i = B^{-\kappa_{\text{term}} i}$  with  $\kappa_{\text{term}} > 1/2$ ; the case  $i = 0$  is included by enlarging  $C$ . More precisely, inserting the two terms of  $\bar{b}_i$ , we obtain

$$v_i \leq C\{B^{-(\kappa_{\text{term}} + \alpha - 1/2)i} + (1+i)B^{-\alpha i}\},$$

and both exponents exceed  $1/2$ . In  $u_i$ , the part of  $\bar{b}_{i-1}^3/a_{i-1}$  coming from the coarsest field decays with the exponent  $3\kappa_{\text{term}} - 1 > 1/2$ , and the part coming from the fluctuations is at most  $C(1+i)^3 B^{-i/2}$ ; the inequality  $(x+y)^3 \leq 4(x^3 + y^3)$  handles the mixed terms. The other two terms of  $u_i$  are at most  $C(1+i)^3 B^{-i/2}$  directly. It follows that

$$v_i + u_i \leq C(1+i)^3 B^{-i/2}. \quad (17.34)$$

For  $i < j$ , we use the cruder bound  $\sqrt{\log(2 + L/a_i)} \leq \sqrt{\log(2 + L/a_j)} \leq C\sqrt{1+j}$ .

Let  $0 < \tau < \min\{1/2, -\log q_{\text{anc}}/\log B\}$ , and set  $r_1 = q_{\text{anc}} B^\tau < 1$  and  $r_2 = B^{\tau-1/2} < 1$ . Let  $\mathcal{X}_j$  be the deterministic upper bound obtained from the convolution estimate by inserting (17.34) and  $\sqrt{\log(2 + L/a_i)} \leq C\sqrt{1+j}$ ; it is defined for every  $j \geq 2$  and does not depend on  $k$ . Writing  $j = i + m + 1$ , we have

$$\sum_{j \geq 2} B^{\tau j} \mathcal{X}_j \leq C \sum_{j \geq 2} B^{\tau j} q_{\text{anc}}^{j-1} + C B^\tau \sum_{i \geq 1} \sum_{m \geq 0} (2+i+m)^{1/2} (1+i)^3 r_1^m r_2^i < \infty.$$

All terms are nonnegative, and  $(2+i+m)^{1/2} \leq (2+i)(1+m)$  reduces the double series to the product of  $\sum_i (2+i)^4 r_2^i$  and  $\sum_m (1+m) r_1^m$ , which converge by comparison with geometric series. The bound depends only on the fixed data and on the gaps in the strict inequalities for  $\tau$ . The finite sequence for a given  $k$  is bounded by this deterministic majorant; no coefficients from different values of  $k$  are identified. We conclude that

$$\sum_{j=l}^k \mathbb{E} \|a_{j-1}^{-1} \bar{Q}_{d_j^k, j}(\theta_{j-1}, \theta_{j-1})\|_{-\alpha; T_0} \leq B^{-\tau l} \sum_{j \geq 2} B^{\tau j} \mathcal{X}_j \leq C_\tau H^{-\tau} a_l^\tau,$$

and the factor  $H^{-\tau}$  is absorbed into  $C_\tau$ . This is the last part. The proof used only  $q_{\text{anc}} < 1$  and the stated choice of  $\tau$ ; no condition of the form  $q_{\text{anc}} B^\alpha < 1$  is needed.  $\square$

## PART VI

# Large-field control and the small-field event

## Local coordinates and constrained minimizers on block regions

This section begins the large-field analysis. Its subject is a block region: a union of final blocks of  $s \leq J$  consecutive averaging steps, possibly disconnected and with holes, on which the configuration is integrated while the configuration on the fixed complement is held fixed. On a block region, we construct coordinates adapted to the block averaging constraints, compute Haar measure in these coordinates, and construct the constrained minimizers of the effective classical action, with weighted bounds on all their Taylor coefficients. All estimates are uniform in the shape, the topology, and the number of components of the block region.

We need these objects for the following reason. As explained in Section 3.4, the exponential-moment bound for blocked Wilson actions (Theorem 22.1) runs the renormalization group in a localized form. The configurations are partitioned according to the regions where some averaged plaquette has large curvature (Theorem 21.5). Outside these regions, background fields, fluctuation integrals, and effective interactions are constructed on arbitrary unions of blocks, and the link variables in the excluded regions are held fixed as frozen coordinates. This section supplies the finite-dimensional part of that construction. Section 19 compares the constructions on a block region with those on the whole torus, and Section 20 integrates the fluctuations on block regions and tracks the normalization constants. Further, Section 21 uses the change of variables and the coercivity estimate at the first block integration to write exact densities, and Section 23 uses the substitution estimates in its choice of parameters.

Definition 18.1 introduces coefficient norms with exponential weights in the distance to the boundary of the block region, and Lemma 18.3 shows that these norms are stable under contractions, inverses, inverse square roots, and Schur complements. Lemma 18.4 proves that the iterated block averaging map is local for every output side  $m \geq 4$ . Lemma 18.7 computes the pushforward of Haar measure in the local constraint coordinates of a block region, under the hypothesis that the fiber is covered by the chosen chart. Lemma 18.8 records the derivatives of constrained minimum values, and Lemma 18.9 states how coefficient norms, normalization constants, and determinants change when variables are frozen. The main result is Theorem 18.12: the constrained Hessian of the effective classical action on a block region is bounded below by a constant that does

not depend on the region, and the constrained minimizer is analytic, with weighted coefficient bounds. Its coercivity estimate rests on the patch inequality of Lemma 18.11. Lemma 18.14 identifies the objective of the first block integration with the Wilson action on the block region and gives the exact form and range of its Hessian, as needed in Section 21. Finally, Lemma 18.15 shows that connected expressions containing a perturbation in the fixed complement decay exponentially into the block region, without a factor proportional to the volume of the complement.

The inputs are the coefficient calculus of Lemma 5.3, the local constraint coordinates and Haar disintegration of Lemma 6.3, the nested minimization identity and the weighted coefficient bounds for the effective classical actions in Theorems 10.10 and 10.11, the triangular change of variables of Theorem 12.13, the domains of Definition 14.4, and, from Theorem 14.8, only the inequality  $E_{d,\text{sat}} \geq \kappa_* E_0$  for the effective classical action at every depth  $d \geq 0$ . The geometry of Lemma 7.4 is used only for a comparison.

Two observations organize the argument. The first concerns coercivity. Let  $u$  be a linearized variation that satisfies the tree gauge conditions and the linearized block averaging constraints of the block region. If  $u$  has zero curl on a single final block, then  $u$  is a gradient there, the tree conditions make its potential constant on the block, and hence  $u$  vanishes on the block. On two adjacent final blocks, the constraint on the coarse bond between them equates the two constants. Coercivity on these two kinds of patches, a covering with bounded overlap, and an explicit extraction of the free coordinates from  $u$  give a lower bound on the constrained Hessian that does not see the topology of the region. The second observation concerns locality. A connected contraction of coefficient tensors that contains a perturbation in the fixed complement must connect each of its external indices in the block region to the complement. Its length therefore bounds the distance of these indices to the boundary, and summing over the positions in the complement with exponential weights gives decay without a volume factor.

The constructions are analytic near the identity. Whether every point of a fiber of the block averaging map lies in the chart is a hypothesis of Lemma 18.7; it is verified in Section 21. Frozen coordinates from earlier block integrations are fixed real parameters: we never differentiate in them or continue them to complex values. Constant terms are tracked separately from the nonconstant Taylor series, and the coefficient norms never bound them.

Charts adapted to the averaging constraints and constrained minimizers of the action (background fields) are central ingredients of Balaban's renormalization group. Configurations with prescribed averages and the axial gauge conditions on them are introduced in [14, §§A–B]; the minimizers, their analytic dependence on the coarse field, and the decay of their derivatives are constructed in [16, Theorem 1, p. 279, and §G, Proposition 9, p. 309]. We follow that method, but the block regions, the centered averaging map, the constraint coordinates, and the coefficient norms are different, and we prove the estimates

needed here directly. The weighted inverse bounds adapt the exponential conjugation of Combes and Thomas [61]. Gauge fixing along trees by successive Haar translations, the analytic implicit function theorem, the formulas for derivatives of minimum values, Schur complements, the resolvent identity, and the integral formula for inverse square roots are standard.

### 18.1 Block regions, coordinates, and constants

Fix an odd integer  $M = 2R_M + 1 \geq 3$  and an integer  $J \geq 1$ . Throughout this section  $1 \leq s \leq J$  and  $m \geq 4$  are integers, and we consider  $s$  consecutive averaging steps  $T_M^s$ , from the lattice of side  $M^s m$  (the input lattice) to the lattice of side  $m$  (the output lattice). The lattices of sides  $M^{s-i} m$ ,  $0 \leq i \leq s$ , are the levels, indexed by  $i$ ; the integrated intermediate scales have  $i < s$ . The level  $i = 0$  is the input lattice and  $i = s$  the output lattice. Distances are measured in units of the output mesh. The final blocks are the centered blocks of  $M^s$  input sites per direction at the vertices of the output lattice; they are described precisely in Section 18.3. A block region is a union  $\Omega$  of final blocks. The fixed complement consists of the input vertices and links not in  $\Omega$ , and the boundary  $W$  of  $\Omega$  is the set of positions of the input links with one endpoint in  $\Omega$  and the other in the fixed complement. A block region may be empty, may be disconnected, may have holes, and may contain isolated final blocks.

**Labels and positions.** A scalar coordinate is one component of the logarithm of a link variable. It is specified by its link, its color, and the level on which the link lives; a frozen coordinate also records the scale at which it was introduced. Coordinates that differ in direction, color, scale, or scale of introduction are different labels, even if their positions coincide. Each label is assigned the position of its link, embedded in output units. Inner products on lists of coordinates are counting inner products. Volume-normalized column sums carry the factors  $a_{\text{out}}^3/a_{\text{in}}^3$  of Definition 4.6. Changing the counting unit across at most  $J$  scales changes a counting sum by a factor depending only on  $M$  and  $J$ , and it introduces no volume factor. In analytic formulas we write  $E_d(A)$  for  $E_d(e^A)$  on the identity chart.

**Coordinates.** We use the coordinates listed in Table 18.1. The coarse field, the free coordinates, and the tree coordinates are defined precisely in Definition 18.6 and Lemma 18.7, and the normalized fluctuation variables in Lemma 18.9. The logarithms  $c$  of the coarse links, the free coordinates  $z$ , and the logarithms  $\zeta$  of the frozen coordinates introduced at the current block integration are analytic variables, and they may be complexified on the construction chart. The frozen coordinates  $b_{\text{rec}}$  introduced at earlier block integrations are fixed real parameters. Figure 18.1 shows the different kinds of links in a two-dimensional sketch.

| Symbol           | Meaning                                                                                                 | Reference measure                                      | Analytic role                           |
|------------------|---------------------------------------------------------------------------------------------------------|--------------------------------------------------------|-----------------------------------------|
| $c$              | logarithms of the coarse links of $\Omega$ , that is, of the bonds joining two final blocks of $\Omega$ | Haar measure<br>$dH(e^c)$ of the group values          | variable;<br>complexified               |
| $z_i$            | free logarithmic coordinates at intermediate scale $i$ , on the free edges $\mathcal{F}_i$              | Lebesgue measure, with the density $J_i$ of Lemma 18.7 | variable;<br>complexified               |
| $w_i$            | normalized fluctuation variables, $z_i = a_i + g_i K_i^{-1/2} w_i$                                      | Lebesgue measure                                       | variable                                |
| $\mathfrak{f}$   | tree coordinates (gauge transformations at the nonroot vertices of the trees)                           | product of Haar probability measures                   | integrated; never differentiated        |
| $\zeta$          | logarithms of the frozen coordinates introduced at the current block integration                        | Haar measure on the group values                       | small; complexified                     |
| $b_{\text{rec}}$ | frozen coordinates introduced at earlier block integrations, at their scales of introduction            | product Haar measure                                   | fixed and real;<br>never differentiated |

Table 18.1. The coordinates used on a block region.

**Uniform constants.** In this section a constant is called uniform if it depends only on the following data:  $M$  and  $J$ ; the radius  $r_E$  of the identity chart on which the effective classical actions  $E_d$  are controlled, the bound  $C_E$  on their weighted Taylor coefficients with each index fixed, and their spatial rate  $a_E > 0$ , all from Theorem 10.11; the constant  $\kappa_* > 0$  in the lower bound (18.20) below; the radii of the local constraint coordinates of Lemma 6.3, strictly smaller than their radii of definition, and the coefficient bounds of the finite group products that define them; the label density  $v_J$  of (18.1); and fixed ratios of radii and fixed differences of spatial rates, stated where they are used. Uniform constants are independent of  $L$ ,  $\gamma$ ,  $m$ ,  $N$ ,  $K$ , the depth  $d_0$ , the number of preceding block integrations, the number of components and the shape of the block region, the values of the frozen coordinates  $b_{\text{rec}}$ , and every width chosen later (boundary strips, interior restriction distances, protected neighborhoods, and cells). A bound for a general analytic input may depend in addition on the coefficient bound and the inverse bound assumed for that input. No uniformity as  $J \rightarrow \infty$  is claimed. This is the dependence described in item (3) of Definition 4.14. Frozen coordinates are fixed before Taylor coefficients are taken or degrees are summed.

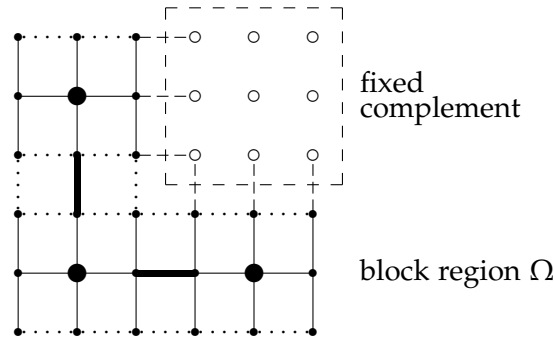


Figure 18.1. A two-dimensional sketch of a block region with  $M = 3$  and  $s = 1$ : three final blocks (filled sites, with enlarged roots) and one final block of the fixed complement (open sites, dashed square). Solid lines are tree edges. The two thick edges are the eliminated edges of the two coarse bonds of  $\Omega$ . Dotted lines are free edges. Dashed edges are links across the boundary; they are not coordinates of the block region, and they carry frozen coordinates.

## 18.2 Coefficient norms with boundary decay

In the regional estimates, distances between label positions are periodic shortest-path distances on the common refined position graph: the periodic cubic lattice of mesh  $(2M^s)^{-1} \leq 1$  in output units, which contains the endpoints and the midpoints of the links of all levels. Its vertices need not carry a coordinate. In this section  $\text{dist}$  denotes this graph distance, and the boundary distance uses it. The connecting-network length  $t$  uses the associated periodic  $\ell^1$  metric, with internal vertices allowed. If  $d_2$  and  $t_2$  denote the Euclidean distance and network length of Definition 4.6, then  $d_2 \leq \text{dist} \leq \sqrt{3}d_2$  and  $t_2 \leq t \leq \sqrt{3}t_2$ . A coefficient bound imported at Euclidean rate  $b$  is therefore used at graph rate  $b/\sqrt{3}$ . The regional norms below are the graph-metric variants of the norms of Definition 4.6 and (4.11), written out for a block region.

**Definition 18.1.** Fix the data of Section 18.1 and a block region  $\Omega$ . Assume that the positions of the labels satisfy the density bound

$$\#\{y : R \leq \text{dist}(x, y) < R + 1\} \leq v_J(R + 2)^3 \quad (R \geq 0) \quad (18.1)$$

for every label position  $x$ , where the count is over labels  $y$ . Each region carries its geometric position domain, a subset of the vertices of the refined position graph, as well as its list of labels. A vertex in that domain with no coordinate is not a hole. For a label  $i$ , let  $d_\Omega(i)$  be the graph distance from its position to the geometric complement of this domain. When a geometric position is removed from a block region, all labels at that position are removed together; the distances for a retained subregion use its retained geometric position domain. A different convention for the positions of link endpoints

changes these distances by at most a fixed amount, which is absorbed in the widths below.

Let  $\rho > 0$ ,  $a_{\text{sp}} > 0$ , and  $\alpha_{\text{bd}} > 0$ , and let  $\phi$  be a spatially varying analytic radius on  $\Omega$ , with  $0 < \phi(i) \leq \rho$ . Let  $f(v; b_{\text{rec}}) = \sum_{\nu} f_{\nu}(b_{\text{rec}}) v^{\nu}$  be a function of the variables  $v = (v_i)_{i \in \Omega}$  of the block region, given by its Taylor series in  $v$ , which depends on the frozen coordinates  $b_{\text{rec}}$ . Define

$$\mathcal{X}_{\Omega}^{\phi}(f) = \sup_{b_{\text{rec}}} \rho^{-2} \sum_{k \geq 1} \sup_{i \in \Omega} e^{\alpha_{\text{bd}} d_{\Omega}(i)} \sum_{|\nu|=k} \frac{\nu_i}{k} |f_{\nu}(b_{\text{rec}})| e^{a_{\text{sp}} t(\nu)} \phi^{\nu}.$$

Here  $t(\nu)$  is the length of a shortest connecting network for the positions of the labels in  $\text{supp } \nu$ , with internal vertices allowed, measured in the periodic  $\ell^1$  metric of the refined position graph (the graph-metric variant of the length in Definition 4.6), and  $\phi^{\nu} = \prod_i \phi(i)^{\nu_i}$ . The supremum over  $i$  is taken inside the sum over degrees, and the supremum over  $b_{\text{rec}}$  outside it;  $b_{\text{rec}}$  ranges over the given compact set of real values of the frozen coordinates, each at the scale at which it was introduced. The norm involves no derivative in  $b_{\text{rec}}$  and no complex value of  $b_{\text{rec}}$ . The constant term  $f(0; b_{\text{rec}})$  is not part of the norm: it is kept in the effective classical action or in the normalization constant  $\Lambda$ . All nonconstant degrees, including degree one, enter the norm. The norm of a function on the empty block region is zero. If the fixed complement is empty, a boundary perturbation is required to have zero nonconstant part, and its norm is zero. For the effective classical action we use the same definition with its own radius  $\rho_{\text{cl}}$ , its own radius function, and its own rates.

For  $K_* \geq 2$ ,  $w_{\text{bd}} \geq 1$ , and  $0 < \rho_{\partial} \leq \rho$  we use the radius

$$\phi(i) = \begin{cases} \rho_{\partial}, & d_{\Omega}(i) < w_{\text{bd}}, \\ \rho, & d_{\Omega}(i) \geq w_{\text{bd}}, \end{cases} \quad \psi_{\text{prof}} = \sqrt{K_*} \phi, \quad \hat{\phi} = K_* \phi,$$

which is smaller in the boundary strip of width  $w_{\text{bd}}$ . The enlarged radii  $\psi_{\text{prof}}$  and  $\hat{\phi}$  are used in Sections 19 and 20, on correspondingly larger analytic domains.

For a coefficient tensor with a set  $A$  of external indices and Taylor multi-index  $\nu$ , put  $S = A \cup \text{supp } \nu$  and

$$t_W(S) = \inf_{w \in W} t(S \cup \{w\}).$$

The bulk and boundary coefficient norms of such a tensor at rate  $a$  are the sums of the absolute values of its coefficients at the given radii, with weights  $e^{at(S)}$  and  $e^{at_W(S)}$  respectively, in which one external index is fixed and all other indices are summed; the norm is the maximum over the choice of the fixed index. Differentiated analytic inputs count as external indices. Column sums at input indices are volume-normalized as in Definition 4.6. A tensor has a bulk bound or a boundary bound if the corresponding norm is bounded by a stated constant. The algebraic properties of these norms are collected in Lemma 18.3.

**Remark 18.2.** The weight  $e^{at_W(S)}$  of the boundary coefficient norm and the weight  $e^{a_{\text{sp}}t(v)+\alpha_{\text{bd}}d_\Omega(i)}$  in  $\mathcal{X}_\Omega^\phi$  are different, and a boundary bound at rate  $a$  does not by itself bound  $\mathcal{X}_\Omega^\phi$  with  $a_{\text{sp}} = \alpha_{\text{bd}} = a$ . The two are related by the length inequalities (18.51) and (18.52) in the proof of Lemma 18.15, at the cost of a fixed fraction of the rate.

For example, freezing the variable  $y$  at the value  $c$  in the function  $vy + y^2$  produces the linear term  $cv$  and the constant  $c^2$ . This is why the norm  $\mathcal{X}_\Omega^\phi$  includes degree one, and why constants are tracked separately (see Lemma 18.9).

**Lemma 18.3.** *Use the norms of Definition 18.1. Fix radii and rates, together with strictly smaller radii and rates. For every analytic substitution  $F \circ G$ , let  $R$  be the coefficient-radius vector of the outer function and  $r$  the input-radius vector of the substitution, and require the absolute coefficient image margin*

$$\widehat{G}(r) \leq (1 - \eta)R$$

*componentwise, for one fixed  $\eta > 0$ . Fix common convergent majorants of the factors on these larger products, including their bounds with each permitted input distinguished; all constants may depend on these majorants and on  $\eta$ . The bounds below hold at the smaller radii and rates, with constants depending only on these choices, on  $v_J$ , and on the stated bounds of the factors. For inverse differences and each Schur elimination, the output rate is chosen strictly below the rate supplied by part (3) for every inverse factor used in that operation. In an iterated elimination this requirement is imposed successively, at most  $J$  times, retaining all real-gap, bulk-coefficient, and complex-perturbation hypotheses of part (3) at each stage. The final output rate is below the minimum of these finitely many rates.*

1. For a scalar function  $f$ , a label  $i$ , and  $k \geq 1$ ,

$$\sum_{|v|=k} \frac{v_i}{k} |f_v| e^{at(v)} = \frac{1}{k!} \sum_{i_2, \dots, i_k} |D^k f(0)_{i, i_2, \dots, i_k}| e^{at(i, i_2, \dots, i_k)}. \quad (18.2)$$

*In particular, a bound on the Taylor tensors of  $f$  with each index fixed in turn is the same as a bound on the absolute values of its coefficients.*

2. A connected product, contraction, or analytic composition of tensors, in which one factor has a boundary bound and the other factors have bulk bounds, has a boundary bound at every remaining external index. This includes at most  $J$  Schur complements of constrained Hessians in the free coordinates.
3. For  $u > 0$  put  $S_J(u) = 8v_J \sum_{r \geq 0} (r+1)^3 e^{-ur}$ . For a matrix  $K$  on a list of labels let  $\|K\|_b$  be the maximum of its absolute row and column sums with weight  $e^{b \text{dist}(x,y)}$ . Let  $K_0 = K_0^\top \geq \lambda I$  be real with  $\|K_0\|_b \leq Q$ , and put

$$\mu = \frac{b\lambda}{8(Q + \lambda)}, \quad C_R = 2S_J(\mu/2).$$

Let  $L$  be a complex symmetric matrix, analytic in further variables, with bulk coefficient bounds on a larger radius such that  $C_R \|L\|_{\mu/2} \leq \lambda/4$ , where the norm includes the absolute values of all coefficients at that radius. Then  $(K_0 + L)^{-1}$  and the inverse square root  $(K_0 + L)^{-1/2}$ , obtained by holomorphic continuation from the positive inverse square root of  $K_0$ , have bulk coefficient bounds at rate  $\mu/2$ , depending only on  $b, \lambda, Q, v_J$ , and the ratio of the radii.

4. For two matrices  $A$  and  $B$  as in (3) on a common list, a boundary bound on  $A - B$  gives boundary bounds on  $A^{-1} - B^{-1}$  and  $A^{-1/2} - B^{-1/2}$ , through the ordered identities

$$A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$$

and

$$A^{-1/2} - B^{-1/2} = -\frac{1}{\pi} \int_0^\infty t^{-1/2} (A + t)^{-1} (A - B) (B + t)^{-1} dt.$$

The lower bound  $\lambda$  is a hypothesis on the free coordinates only, not on the coarse field. All constants are uniform.

*Proof.* We begin with the identity (18.2). Fix  $\nu$  with  $|\nu| = k$  and  $\nu_i > 0$ . An ordered list  $(i_2, \dots, i_k)$  whose multiplicities are  $\nu - e_i$  occurs  $(k-1)! / \prod_j (\nu - e_i)_j!$  times in the sum on the right, and for each such list the derivative is  $\nu! f_\nu$ . Multiplying these two numbers and dividing by  $k!$ , we obtain  $\nu_i/k$ , which proves (18.2). In particular, a bound on Taylor tensors with each index fixed is an absolute coefficient bound, not a bound on a derivative in a single direction.

Next we prove (2). When two factors are contracted at a label, we join their connecting networks at that label. The union spans every remaining external label, and if the first factor carries a boundary bound, it also reaches the point of  $W$  of that factor. Hence

$$t_W(S_{\text{ext}}) \leq t_W(S_1) + \sum_{r \geq 2} t(S_r), \quad (18.3)$$

where  $S_{\text{ext}}$  is the set of external labels of the contraction and  $S_r$  the label set of the  $r$ -th factor. We fix any remaining external index. On the branch leading to that label we use the corresponding input-column bound, and on the other branches we use output-row bounds; these are the contractions of Lemma 5.3, with the same counting measures and with the attachment and external-normalization mesh-volume ratios retained in the majorant for general contractions. A connected matrix cycle with an external insertion can be cut at that insertion, and successive absolute row and column sums give the same bound. (This does not apply to a disconnected scalar trace.) Multiplication of majorant series with nonnegative coefficients sums over the partitions of the Taylor labels, and division by the Taylor factorials turns the multinomial factors into ordinary products of power series. The same reasoning, applied to the convergent composed majorant  $\hat{F}(\hat{G}(r))$ , proves (2) for compositions. The fixed absolute image margin justifies the outer

derivative series, and the distinguished-input majorants bound a specified analytic input through the chain rule. A pointwise inclusion of the image alone would not justify these absolute coefficient sums. Fixing a specified analytic input only changes the fixed index. An external index within distance  $R_0$  of  $W$  may instead be joined to  $W$  at the price of a factor  $e^{aR_0}$ . No step involves a sum over the positions of  $W$ .

For (3) we first record a density sum. By the density bound (18.1), we have, for every  $u > 0$  and every subset of a periodic list of labels,

$$\sup_x \sum_y e^{-u \operatorname{dist}(x,y)} \leq 8v_J \sum_{r \geq 0} (r+1)^3 e^{-ur} = S_J(u). \quad (18.4)$$

Here we divided the displacements according to  $r \leq \operatorname{dist}(x,y) < r+1$  and used  $(r+2)^3 \leq 8(r+1)^3$ . The series is finite, since for large  $r$  the ratio of successive summands is at most  $e^{-u/2} < 1$ .

We now adapt the exponential conjugation of Combes and Thomas [61] to the matrix  $K_0$ . Let  $q$  be a real function that is one-Lipschitz for the distance on the labels, and let  $D_q = \operatorname{diag}(e^{\mu q(x)})$ . The entries of  $D_q K_0 D_q^{-1} - K_0$  are  $(K_0)_{xy}(e^{\mu(q(x)-q(y))} - 1)$ , so the maximum of its unweighted absolute row and column sums is at most

$$Q \sup_{r \geq 0} e^{-br} (e^{\mu r} - 1) \leq \frac{Q\mu}{b-\mu} \leq \frac{\lambda}{8}.$$

For the first inequality we used  $e^{\mu r} - 1 \leq \mu r e^{\mu r}$  and  $r e^{-(b-\mu)r} \leq (b-\mu)^{-1}$ ; the second follows from the definition of  $\mu$ . The Euclidean operator norm is bounded by the row and column sums: by the Cauchy–Schwarz inequality,

$$\|Av\|_2^2 \leq \left( \sup_x \sum_y |A_{xy}| \right) \left( \sup_y \sum_x |A_{xy}| \right) \|v\|_2^2.$$

For  $t \geq 0$  the matrix  $(K_0 + t)^{-1}$  has operator norm at most  $(\lambda + t)^{-1}$ . The Neumann series for the conjugated resolvent therefore converges, and we obtain  $\|D_q(K_0 + t)^{-1}D_q^{-1}\|_2 \leq 2/(\lambda + t)$ . Taking  $q(x) = \operatorname{dist}(x, y)$  for a fixed  $y$  and reading off the  $(x, y)$  entry, we find

$$|(K_0 + t)^{-1}_{xy}| \leq \frac{2e^{-\mu \operatorname{dist}(x,y)}}{\lambda + t}, \quad \|(K_0 + t)^{-1}\|_{\mu/2} \leq \frac{C_R}{\lambda + t},$$

where the second bound follows from the first and (18.4).

Next let  $L$  be the perturbation in (3). Since  $C_R\|L\|_{\mu/2}/\lambda \leq 1/4$ , we have

$$(K_0 + L + t)^{-1} = \sum_{n \geq 0} [-(K_0 + t)^{-1}L]^n (K_0 + t)^{-1}, \quad \|(K_0 + L + t)^{-1}\|_{\mu/2} \leq \frac{4C_R}{3(\lambda + t)}. \quad (18.5)$$

The series converges absolutely in the coefficient norms, uniformly for  $t \geq 0$ . Bounds with every index fixed, and on the derivatives in the further variables, follow by inserting

the differentiated perturbation and its connecting network into this series, or by differentiating its scalar geometric majorant on a strictly smaller radius. Here we use that  $L$  has bulk coefficient bounds with every index fixed; a pointwise bound on a matrix-valued function would not suffice.

For the inverse square root we use the scalar identity  $\pi^{-1} \int_0^\infty t^{-1/2}(x+t)^{-1} dt = x^{-1/2}$  for  $x > 0$ , which follows from the substitution  $t = xu^2$ . Diagonalization gives the same formula for a real positive matrix. Expanding the resolvents as in (18.5) proves the formula for the complex perturbations of (3), and shows that it defines the branch obtained by holomorphic continuation from the positive inverse square root. Multiplying out the two resolvents, we have

$$(A+t)^{-1} - (B+t)^{-1} = -(A+t)^{-1}(A-B)(B+t)^{-1}.$$

At  $t = 0$  this is the first identity in (4), and integrating it against  $\pi^{-1}t^{-1/2} dt$  gives the second. The order of the factors matters, since they need not commute. By (2) and (18.5), the integrand of the second identity has a boundary bound by a constant times  $t^{-1/2}(\lambda+t)^{-2}$  times the boundary norm of  $A-B$ , and  $\int_0^\infty t^{-1/2}(\lambda+t)^{-2} dt = \pi/(2\lambda^{3/2})$ . The integral therefore preserves absolute coefficient bounds and the boundary weight. This proves (3) and (4).

Finally we treat Schur complements, completing the proof of (2). If  $K = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \geq \lambda I$ , then  $A \geq \lambda I$  and

$$v^\top(C - B^\top A^{-1}B)v = \min_u (u, v)^\top K(u, v) \geq \lambda|v|^2, \quad (18.6)$$

where the equality follows by substituting the minimizer  $u = -A^{-1}Bv$ . In the coefficient norms the Schur complement is bounded by  $\|C\| + \|B^\top\| \|A^{-1}\| \|B\|$ . To compare two Schur complements, we replace the three factors of the product one at a time; each of the resulting terms contains one difference, possibly the inverse difference of (4), and therefore has a boundary bound by (2). For at most  $J$  successive eliminations we obtain a finite recursion: starting from a bound  $Q_0$ , the next bound can be taken to be  $Q_{k+1} = Q_k + Q_k^2 I_k$ , where  $I_k$  is the preceding inverse bound from (18.5). At each stage we first choose an inverse rate supplied by (3), and then choose the next coefficient rate strictly below this rate. We make these finitely many choices before subsequent applications and take their minimum, together with the corresponding smaller radii; there is no loss depending on the Taylor order or on the number of sites. The lower bounds used here are lower bounds for matrices in the free coordinates, and all constants depend only on the quantities listed in the statement.  $\square$

### 18.3 Local coordinates and Haar disintegration on block regions

The coordinates of a block region are obtained by solving, scale by scale, only the block averaging constraints of the coarse bonds that join two blocks of the region, with the local constraint coordinates of Lemma 6.3.

We set  $R^{(t)} = (M^t - 1)/2$  for  $t \geq 0$ , so that  $R^{(0)} = 0$  and  $R^{(1)} = R_M$ . In input lattice units, the final block with anchor  $v$  has the vertices  $v + [-R^{(s)}, R^{(s)}]^3 \cap \mathbb{Z}^3$ . Balanced base- $M$  digits give the disjoint decomposition

$$[-R^{(s)}, R^{(s)}]^3 \cap \mathbb{Z}^3 = \bigsqcup_{x \in [-R^{(s-1)}, R^{(s-1)}]^3 \cap \mathbb{Z}^3} (Mx + [-R_M, R_M]^3 \cap \mathbb{Z}^3). \quad (18.7)$$

To see this in one coordinate, we write an integer uniquely as  $Mq + r$  with  $-R_M \leq r \leq R_M$ ; since  $MR^{(s-1)} + R_M = R^{(s)}$ , the integers in  $[-R^{(s)}, R^{(s)}]$  are exactly those with  $|q| \leq R^{(s-1)}$ . We apply this in each of the three coordinates and iterate. The internal links of a block region have both endpoints in  $\Omega$ , and the links across the boundary of the block region have exactly one endpoint in  $\Omega$ . At every intermediate scale we use exactly the blocks of this nested partition that lie in  $\Omega$ .

**Lemma 18.4.** *Let  $M \geq 3$  be odd, and let  $s \geq 1$  and  $m \geq 4$  be integers. On the input lattice of side  $M^s m$ , the value of  $T_M^s$  at a final link with endpoints  $v$  and  $v + M^s e_i$  depends only on the input links with both endpoints in the union of the two centered  $M^s$ -blocks at  $v$  and  $v + M^s e_i$ . The tree paths of all intermediate scales inside a final block stay in that block. The same holds for the Borel extension of  $T_M$  by the reference-path transport of Definition 4.3. No divisibility of  $m$  by four is needed.*

*Proof.* We begin with one averaging step and write  $x = v + \delta$  with  $\delta \in [-R_M, R_M]^3 \cap \mathbb{Z}^3$ . The coordinate-ordered path  $\sigma_{v,x}$  stays in the block at  $v$ , since each coordinate of each intermediate vertex lies between its initial and final value. The path  $\sigma_{v+Me_i, x+Me_i}$  stays in the translated block for the same reason. Along the straight segment from  $x$  to  $x + Me_i$ , the transverse coordinates stay equal to  $\delta_j$ , and the longitudinal coordinate lies in  $[-R_M, M + R_M]$ . Since  $M = 2R_M + 1$ , we have

$$[-R_M, M + R_M] \cap \mathbb{Z} = ([-R_M, R_M] \cap \mathbb{Z}) \sqcup ([M - R_M, M + R_M] \cap \mathbb{Z}).$$

Every vertex of the segment lies in one of the two endpoint blocks, and its only link between the blocks joins the two adjacent boundary vertices. The reference path from  $v$  to  $v + Me_i$  satisfies the same containment. Therefore every contour, based loop holonomy, and tree path in the definition (4.3) of the output uses only links with both endpoints in the union of the two blocks. The test whether some based loop holonomy equals  $-I$  reads these same holonomies, and the exceptional output is the reference transport, so the Borel extension has no additional dependence.

For the induction, suppose that the assertion holds for  $r$  averaging steps. A one-step formula for a link at the next scale reads links whose endpoints lie in its two endpoint  $M$ -blocks on the current lattice. Each such link, pulled back through  $r$  steps, reads only the two centered  $M^r$ -blocks at its endpoints, by the induction hypothesis. By (18.7), these blocks partition the two centered  $M^{r+1}$ -blocks at the endpoints. All links that are read, including links between the smaller blocks, therefore have their endpoints in the union of the two large blocks. A tree path inside one block of the current lattice pulls back to tree paths in its smaller blocks and to the links joining these, all inside the same final block. This proves both assertions for  $r + 1$  steps, including the exceptional outputs. When  $m \geq 4$  the two final blocks lift injectively to a rectangular box in  $\mathbb{Z}^3$ ; alternatively, the same coordinate containment can be read periodically. No step used a tiling, a reflection, or a congruence condition on the side.  $\square$

*Remark 18.5.* When  $m \in 4\mathbb{N}$ , the dependence in Lemma 18.4 agrees with the geometry of Lemma 7.4, applied with  $D = s$  and  $n = m$ . The reflection and tiling conclusions of that lemma are not needed here. For the other sides  $m \geq 4$ , and for all shapes of block regions, Lemma 18.4 is the dependence statement used below; it requires no filling across a hole.

When we solve only the block averaging constraints of the internal coarse bonds, their eliminated-edge equations keep their independence and their derivatives after restriction to  $\Omega$ : by Lemma 18.4, an internal equation reads only links in its two endpoint blocks, and it contains no eliminated edge other than its own. The following definition makes the construction explicit and fixes the linear maps used in the coercivity estimates of Sections 18.6 and 18.7.

**Definition 18.6.** Fix  $1 \leq s \leq J$  and  $m \geq 4$ , put  $n = M^s m$ , and work in input lattice units on  $(\mathbb{Z}/n\mathbb{Z})^3$ . For an anchor  $a \in (M^s \mathbb{Z}/n\mathbb{Z})^3$ , let  $B_s(a) = a + \{-R^{(s)}, \dots, R^{(s)}\}^3$  be the final block at  $a$ . Let  $\mathcal{K}$  be the set of anchors of the final blocks of  $\Omega$ , let  $\Omega_0 = \bigcup_{a \in \mathcal{K}} B_s(a)$  be the vertex set of  $\Omega$ , and let  $\Omega_i = \Omega_0 \cap (M^i \mathbb{Z}/n\mathbb{Z})^3$  for  $0 \leq i \leq s$ , so that  $\Omega_s = \mathcal{K}$ .

1. *Edges.* For  $0 \leq i \leq s$ , let  $\mathcal{L}_i$  be the set of positive edges of length  $M^i$  with both endpoints in  $\Omega_i$ . Thus  $\mathcal{L}_0 = L_\Omega$  is the set of internal links of  $\Omega$ , including the links that join two final blocks of  $\Omega$ , and  $\mathcal{L}_s$  is the set of coarse bonds. An input link with exactly one endpoint in  $\Omega_0$  is a link across the boundary of the block region; such links are not coordinates of the block region. A reversed edge carries the opposite cochain value and the inverse group value.
2. *Blocks and trees.* For  $i < s$  the blocks  $B_i(a) = a + M^i \{-R_M, \dots, R_M\}^3$ ,  $a \in \Omega_{i+1}$ , partition  $\Omega_i$ , by (18.7). Write  $\pi_i(x) = a$  for  $x \in B_i(a)$ . For  $x \in B_i(a)$ , let  $\sigma_{a,x}$  be the path from  $a$  to  $x$  with steps of length  $M^i$  that changes the coordinates in the order

1, 2, 3. In each block these paths form a tree rooted at  $a$ , and  $\mathcal{T}_i \subset \mathcal{L}_i$  is the union of these trees.

3. *Eliminated and free edges.* For a bond  $c' = (a, a + M^{i+1}e_k) \in \mathcal{L}_{i+1}$ , its eliminated edge is

$$e_i^*(c') = (a + R_M M^i e_k, a + (R_M + 1) M^i e_k) \in \mathcal{L}_i.$$

Let  $\mathcal{D}_i = \{e_i^*(c') : c' \in \mathcal{L}_{i+1}\}$ , and let  $\mathcal{F}_i = \mathcal{L}_i \setminus (\mathcal{T}_i \cup \mathcal{D}_i)$  be the set of free edges at scale  $i$ . No constraint and no free coordinate is attached to a link across the boundary.

4. *Linearized maps.* For  $c' = (a, a + M^{i+1}e_k) \in \mathcal{L}_{i+1}$  and  $x \in B_i(a)$ , let

$$F_{i,c',x} = \sigma_{a,x}[x, x + M^{i+1}e_k] \sigma_{a+M^{i+1}e_k, x+M^{i+1}e_k}^{-1}$$

a path with steps of length  $M^i$ . With  $[\omega : e]$  the signed number of traversals of the edge  $e$  by the path  $\omega$ , define the scalar matrices

$$(Q_i)_{c'e} = M^{-3} \sum_{x \in B_i(a)} [F_{i,c',x} : e], \quad \Pi_i^T u = u|_{\mathcal{T}_i},$$

which act on three colors by tensoring with  $I_3$ . Put  $Q_{[0,0]} = I$  and  $Q_{[0,i]} = Q_{i-1} \cdots Q_0$ , and define

$$G_\Omega u = ((\Pi_i^T Q_{[0,i]} u)_{0 \leq i < s}, Q_{[0,s]} u) \quad (18.8)$$

for edge cochains  $u$  on  $\mathcal{L}_0$ . The last component prescribes zero variation of every coarse bond of  $\Omega$ ; no bond to a final block outside  $\Omega$  is constrained. The kernel  $\ker G_\Omega$  is the space of linearized variations of the block region.

5. *Reconstruction.* For a group field  $b$  on  $\mathcal{L}_{i+1}$ , its coarse-field lift is

$$(\mathcal{H}_i b)_e = \begin{cases} I, & \pi_i(x) = \pi_i(y), \\ b_{(\pi_i(x), \pi_i(y))}, & \pi_i(x) \neq \pi_i(y), \end{cases} \quad e = (x, y) \in \mathcal{L}_i, \quad (18.9)$$

with the orientation convention in the second line. Start with  $U^{(s)} = \exp(c)$  on  $\mathcal{L}_s$  and descend by

$$U_e^{(i)} = \exp(\Gamma_{U^{(i+1)}, e}(z_i)) (\mathcal{H}_i U^{(i+1)})_e, \quad e \in \mathcal{L}_i. \quad (18.10)$$

Here  $\Gamma_b$  is the map of local constraint coordinates of Lemma 6.3 at the target  $b = U^{(i+1)}$ , solving only the constraints indexed by  $\mathcal{L}_{i+1}$  for the logarithms on the eliminated edges  $\mathcal{D}_i$ , with free entries  $z_i \in (\mathbb{R}^3)^{\mathcal{F}_i}$  and zero entries on  $\mathcal{T}_i$ . All products are taken on a common identity chart, smaller than the chart of definition, on which these finitely many inverses exist. Write  $\Phi_\Omega(c, z) = U^{(0)}$  for the resulting field on  $\mathcal{L}_0$ . Thus  $\Phi_\Omega$  receives the logarithm  $c$  of the coarse field on  $\mathcal{L}_s$ , not its group value. The logarithms  $c$  of the coarse links, the free coordinates  $z = (z_0, \dots, z_{s-1})$ , and the logarithms  $\zeta$  of the current frozen coordinates are independent parameters of this chart.

The reference measure of the coarse links is Haar measure of their group values, written  $dH(c)$  in Lemma 18.7; in these coordinates it is  $dH(e^c) = \prod_e c_{\text{Haar}} j_H(c_e) d^3 c_e$ , with the product over the coarse links, not Lebesgue measure.

6. *Tree-path transports to the basepoints.* For  $x \in \Omega_0$  let  $x_0 = x$  and  $x_{i+1} = \pi_i(x_i)$ . For  $1 \leq k \leq s$  and a configuration  $U$  on  $\mathcal{L}_0$ , let

$$\Theta_{x,k}(U) = U^{(k-1)}(\sigma_{x_k, x_{k-1}}) \cdots U^{(0)}(\sigma_{x_1, x_0}), \quad (18.11)$$

where  $U^{(i)} = T_M^i U$  on  $\mathcal{L}_i$ ; this is a transport with endpoints  $x_k$  and  $x$ . A frozen coordinate on a link across the boundary at  $x$  that is based at the level- $k$  ancestor  $x_k$  is represented by its based transport, obtained by multiplying the link value from the left by  $\Theta_{x,k}$ ; a link across the boundary frozen at the first block integration is based at the root of its final block,  $k = s$ . We write  $\Theta_x = \Theta_{x,s}$ .

On the reconstruction we have  $T_M^i \Phi_\Omega(c, z) = U^{(i)}$  on  $\mathcal{L}_i$ , since each  $\Gamma$  solves the constraints of the internal bonds, and these read only internal links by Lemma 18.4. Each factor in (18.11) is a transport along a tree path, and the tree links of (18.10) are equal to  $I$ . Hence  $\Theta_{x,k}(\Phi_\Omega(c, z)) = I$  identically, also at nonzero  $(c, z)$  in the chart. Under a gauge transformation  $f$ , the transport has the covariance  $\Theta_{x,k}^f = f(x_k) \Theta_{x,k} f(x)^{-1}$ . Thus the based transport of a boundary link from  $x$  to  $v$  transforms by  $f(x_k)$  at the left endpoint and  $f(v)^{-1}$  at the right endpoint. It is invariant when both these endpoints are fixed. This applies to a coordinate frozen at the current integration, where  $k = s$  and  $x_s$  is a fixed final root, and to older frozen coordinates whose endpoints are protected from the active trees. An intermediate ancestor  $x_k$  with  $k < s$  need not be fixed; those intermediate transports are covariant and are not additional frozen coordinates. The coarse output on a bond across the boundary of the block region is evaluated from the reconstructed field after this construction; it is not a prescribed target. The frozen coordinates supply the based transports of frozen links across the boundary and the links of the fixed complement, each at the scale at which it was introduced. Arguments of an analytic torus action outside the block region that were outside earlier block regions are set to zero (logarithm zero), as described in Theorem 18.12. Arbitrary data deep in the fixed complement are never arguments of an analytic function.

**Lemma 18.7.** *Fix  $1 \leq s \leq J$ , a block region  $\Omega$ , and Borel sets  $\mathcal{S}_{c, b_{\text{rec}}}$  of free coordinates  $z$  in the chart of Definition 18.6, depending measurably on the coarse links  $c$  and on the frozen coordinates  $b_{\text{rec}}$ . For tree coordinates  $\mathfrak{f}$ , that is, a gauge transformation at the nonroot vertices of the trees of all scales, let  $U(c, z, \mathfrak{f}; b_{\text{rec}})$  be the configuration obtained from  $\Phi_\Omega(c, z)$  by the based gauge transformation  $\mathfrak{f}$  and completed outside  $\Omega$  by the frozen coordinates. Let the incoming finite positive measure have a Borel density  $p(U, b_{\text{rec}})$  with respect to the product of Haar probability measures on the links of  $\Omega$  and Haar measures on the independently introduced frozen coordinates, each at the scale at which it was introduced; any normalizing constant of the incoming measure*

is included in  $p$ . Let  $\chi(U, b_{\text{rec}})$  be the product of all indicators carried by the incoming measure: those of the large-field decomposition, of the coordinate domains, of the small-field conditions of Definition 14.4, and of the fluctuation cutoffs. Assume coverage of the fiber by the chart: every configuration  $U$  with  $p(U, b_{\text{rec}})\chi(U, b_{\text{rec}}) > 0$  has the form  $U(c, z, \mathfrak{f}; b_{\text{rec}})$  with  $z \in \mathcal{S}_{c, b_{\text{rec}}}$ , after the based tree gauge transformation that fixes its coarse field and the based transports of its frozen coordinates.

Then the pushforward of the measure with density  $p\chi$  to  $(c, b_{\text{rec}})$  has the density

$$p^{\text{out}}(c, b_{\text{rec}}) = \int dH(\mathfrak{f}) \int_{\mathcal{S}_{c, b_{\text{rec}}}} \chi(U(c, z, \mathfrak{f}; b_{\text{rec}}), b_{\text{rec}}) p(U(c, z, \mathfrak{f}; b_{\text{rec}}), b_{\text{rec}}) \prod_{i=0}^{s-1} J_i \prod_{i=0}^{s-1} dz_i$$

with respect to  $dH(c) \prod_e dH(b_{\text{rec}, e})$ , where the product is over the independently introduced frozen coordinates,  $dH(\mathfrak{f})$  is the product of Haar probability measures, and we use the letter  $c$  both for the coarse links and for their logarithms. If  $f_i$  is the number of free group variables at scale  $i$ , the Haar disintegration density at scale  $i$  is

$$J_i = c_{\text{Haar}}^{f_i} \frac{\prod_{e \in \mathcal{L}_i \setminus \mathcal{T}_i} j_H(\Gamma_{i, e})}{\prod_{c' \in \mathcal{L}_{i+1}} \det D_{X_{e'_i}(c')} C_{b, c'}(\Gamma_i)}, \quad j_H(X) = \text{sinc}^2(|X|/2), \quad c_{\text{Haar}} = \frac{1}{16\pi^2},$$

where  $\Gamma_i = \Gamma_{U^{(i+1)}}(z_i)$  and  $C_{b, c'}$  is the block averaging constraint of  $c'$  at the target  $b = U^{(i+1)}$ . All Haar factors of the eliminated edges appear in the numerator. The determinants are positive on the real chart and continue holomorphically, without zeros, to its complex neighborhood. There is no further Haar density of the coarse field, no Gram determinant, no gauge-orbit volume, and no Jacobian of a field-dependent choice of gauge. The formula is an identity tested against nonnegative Borel functions, and the displayed integral defines the density at every covered target.

When a frozen coordinate is introduced on a link  $e$  across the boundary at a fixed field  $v$  of the block region, the change from the link value  $a_e$  to the based transport  $\widehat{b}_e = U_v(\sigma_e)a_e$ , where  $U_v(\sigma_e)$  is the tree-path transport of  $e$  to its basepoint in the field  $v$ , is a Haar translation that is triangular in the coordinates. Frozen coordinates introduced earlier are unchanged, and a blocked image of an earlier frozen coordinate is a function of the existing coordinates, not an additional Haar coordinate. If  $p\chi$  is invariant under the based tree gauge transformations that fix the coarse field and the frozen coordinates, the integral over  $\mathfrak{f}$  equals one and may be omitted; otherwise it remains in the formula. The identities are finite-dimensional, and the analytic bounds of the chart are uniform. Coverage is a hypothesis of this lemma.

*Proof.* We first fix one scale and the frozen coordinates. On each block we can remove a leaf of its rooted tree: we replace the variable on that edge by the transport from the root to the leaf, and translate the variables on the other edges at the leaf accordingly. At fixed values of the remaining variables this is a left or right Haar translation. We remove the leaves successively. The inverse map reconstructs each edge variable from the transports to its endpoints and its based value, so this is a bijective triangular change of

group coordinates. It gives Haar probability measure in each tree variable at a nonroot vertex and in each based link outside the trees. No division by a stabilizer or by an orbit volume occurs, also for a single isolated block.

For an internal bond its eliminated-edge equation is the equation of Lemma 6.3 on its two endpoint blocks. Its determinant equals  $\det(M^{-2}I_3) > 0$  at zero and stays positive on the real chart. We compute the normalization when only internal equations are solved. Let  $n_i$  be the number of internal group links outside  $\mathcal{T}_i$  and  $q_i$  the number of internal constraints, so that  $f_i = n_i - q_i$ . At a fixed target  $c_{i+1}$  we write  $U_e = e^{X_e} U_{0,e}(c_{i+1})$ , where  $U_0$  is the coarse-field lift. Right translation preserves Haar measure, and these  $n_i$  factors have the density  $c_{\text{Haar}}^{n_i} \prod_e j_H(X_e)$  with respect to  $dX$ , by the coordinate formula for Haar measure in (4.2).

The tested pushforward can be computed by inserting a unit point mass at  $T_\Omega U = c_{i+1}$  with respect to Haar measure on the targets, where  $T_\Omega$  denotes the internal outputs of  $T_M$ . More explicitly, if  $\delta_H$  is the density of the unit point mass at the identity, then

$$\int dH(c_{i+1}) \delta_H((T_\Omega U) c_{i+1}^{-1}) G(c_{i+1}, U) = G(T_\Omega U, U)$$

for nonnegative tests  $G$ . This notation only expresses a change of variables. In logarithmic coordinates  $Y$  at the identity it reads  $\delta_H(e^Y) = c_{\text{Haar}}^{-1} \delta_0(Y)$ , since  $j_H(0) = 1$ . The logarithms of the constraints are  $Y_{c'} = C_{b,c'}(X)$ . The local substitution  $(X_{e_i^*(c')})_{c'} \mapsto (Y_{c'})_{c'}$ , with the free logarithms fixed, has Jacobian equal to the product of the eliminated-edge determinants, since each equation contains no other eliminated variable. We obtain

$$c_{\text{Haar}}^{n_i - q_i} \frac{\prod_{e \in \mathcal{L}_i \setminus \mathcal{T}_i} j_H(\Gamma_{i,e})}{\prod_{c'} \det D_{X_{e_i^*(c')}} C_{b,c'}(\Gamma_i)} dz_i = J_i dz_i,$$

with the Haar density of every solved eliminated edge still in the numerator. The point-mass calculation can equivalently be carried out by first making this invertible local substitution and then integrating the unit mass in  $Y$ ; no global disintegration theorem is needed. This proves the identity for continuous tests on compact subsets of the chart. Exhaustion of the open chart and monotone approximation extend it to nonnegative Borel tests. The reference measure  $dH(c_{i+1})$  is already present in this calculation; expressing it once more in logarithms and keeping its density as another factor would count it twice.

We apply this one-scale identity to a nonnegative Borel test of the coarse links and frozen coordinates, multiplied by the Borel function  $p\chi$ , keeping the tree variables in the integrand. The internal output Haar variables of scale  $i$  are the internal input Haar variables of scale  $i + 1$ . When the based tree gauge at a coarser scale is restored, its block-constant extension is absorbed into the finer free coordinates by Lemma 6.3(5). At fixed coarser data it acts by orthogonal adjoint maps, preserves the constraint reconstruction and its coarea density, and preserves the three-dimensional fluctuation balls. Ordering

the variables from coarse to fine, the joint coordinate change is triangular with orthogonal diagonal blocks, so its determinant has absolute value one. We retain the transformed  $p\chi$  and the forest variables; the tree Haar integrals are removed only under the invariance condition in the statement. Repeated nonnegative integration therefore gives the product of the  $J_i$ , Lebesgue measure in all free logarithms, Haar probability measure in each tree variable of each scale, and Haar measure on the coarse links. This proves the displayed formula against every nonnegative Borel test. Restricting each free integral to  $\mathcal{S}_{c,b_{\text{rec}}}$  and keeping  $\chi$  records the domain of the chart. The coverage hypothesis says that the image of this domain under the based tree gauge transformations contains every point that is pushed forward; without it the argument proves only the identity restricted to the chart.

The local coordinate formulas are Borel, their domains are Borel, and their Jacobians are continuous on the chart. After extension of the integrand by zero outside these domains, the integral is a Borel function of  $(c, b_{\text{rec}})$ : this holds first for indicators of rectangles, then for nonnegative simple functions, and then by increasing approximation. At an exceptional target its value may be  $+\infty$ ; since the incoming measure is finite, it is finite almost everywhere. No unspecified almost-everywhere density enters the pointwise formula.

When a frozen coordinate is introduced, we hold the field  $v$  of the block region fixed. For every nonnegative Borel function  $G$ ,

$$\int G(v, U_v(\sigma_e)a_e) dH(a_e) = \int G(v, \widehat{b}_e) dH(\widehat{b}_e),$$

by left invariance of Haar measure. Iterating over the newly frozen links and then integrating over  $v$  proves the assertion on Haar translations, although the translations depend on  $v$ . If the orientation gives a right translation instead, right invariance gives the same identity. An earlier frozen coordinate is not changed by this operation, and a later blocked value computed from it is a function on the existing coordinate space, not a new independent coordinate. This proves the statement on the reference measure of the frozen coordinates; it makes no claim of probabilistic independence under the density.

If  $p\chi$  is invariant under the based gauge transformations in the statement, its value in these coordinates does not depend on  $\mathfrak{f}$ , since these gauge transformations fix both the target and the endpoints of the based transports of the frozen coordinates. Each Haar probability factor of the tree variables then integrates to one. Finally, the determinants do not vanish on the complex neighborhood of the chart, by the local inverse bounds of Lemma 6.3. Since there are at most  $J$  scales, each with bounded stencils and fixed radii, the analytic constants of the chart are uniform. The measure identity itself needs no analyticity of  $p$ .  $\square$

## 18.4 Derivatives of constrained minima

The constrained minimizers below are stationary points of an objective in the free coordinates, with the coarse field and the current frozen coordinates as parameters. The identities below hold at complex arguments; positivity of the Hessian, and hence minimization, enters only at real data.

**Lemma 18.8.** *Let  $F(u, y; b_{\text{rec}})$  be holomorphic on a finite-dimensional complex product and real at real arguments, where  $u$  denotes free coordinates and  $y$  parameters. Let  $a(y)$  be a holomorphic branch of solutions of  $F_u(a(y), y; b_{\text{rec}}) = 0$  along which  $A = F_{uu}$  is invertible. All derivatives below are evaluated at  $(a(y), y; b_{\text{rec}})$ , with the frozen coordinates  $b_{\text{rec}}$  fixed. For a direction  $\eta$  of the parameters (the coarse field or the current frozen coordinates), set  $V_\eta = (-A^{-1}F_{uy}\eta, \eta)$ , and let  $F^{\min}(y) = F(a(y), y)$ . Then*

$$Da\eta = -A^{-1}F_{uy}\eta, \quad DF^{\min} = F_y, \quad D^2F^{\min} = F_{yy} - F_{yu}A^{-1}F_{uy},$$

and

$$D^3F^{\min}[\eta_1, \eta_2, \eta_3] = D^3F[V_{\eta_1}, V_{\eta_2}, V_{\eta_3}].$$

At intermediate scale  $i$  we define the regional intermediate objective to be  $M^i F^{\min}$ , and each of its derivatives carries the same factor  $M^i$ . In the torus case, the Bellman identity identifies this objective with the named effective classical action. Successive and joint constrained minimizers agree provided that every conditional branch is unique in its stated solution domain and the corresponding coordinates of the joint stationary point belong, at each successive stage, to both that stage's parameter domain and its solution domain, including the outer solution domain.

If  $F = E \circ \Psi$ , then for any directions  $u, v$  of the free coordinates or of the parameters,

$$F_{uv} = \Psi_u^\top E''(\Psi) \Psi_v + E'(\Psi) \Psi_{uv}.$$

The second term is present also at a constrained minimizer. In particular, differentiation in a current frozen coordinate  $\xi$  gives  $D_\xi a = -F_{uu}^{-1}F_{u\xi}$ , where  $F_{u\xi}$  contains both terms. At complex arguments, transposes are bilinear transposes. Higher derivatives of  $a$  require only the inverse of  $F_{uu}$ . Weighted coefficient bounds on these derivatives hold in the coefficient norms on smaller fixed products, with constants depending on the coefficient bounds of  $F$ , on the bound for  $F_{uu}^{-1}$ , and on uniform data. No inverse of a Hessian in the coarse field or in the frozen coordinates is used, and no derivative in a frozen coordinate  $b_{\text{rec}}$  is taken.

*Proof.* We begin with the existence and uniqueness of the branch near a given point of it. At that point the matrix  $A_* = F_{uu}$  is invertible, so the map  $u \mapsto u - A_*^{-1}F_u(u, y)$  has derivative zero there. By continuity, there exist a ball of free coordinates on which this derivative has norm at most  $1/2$  and a smaller neighborhood of parameters on which the map moves the base point by less than a quarter of the radius of that ball. The iteration is then a contraction of the ball into itself, and the analytic implicit-series statement of

Lemma 5.3 gives its holomorphic solution. Local uniqueness identifies this solution with the given branch, and these local constructions agree on their common neighborhoods.

Let  $L(y) = (a(y), y)$ , with  $b_{\text{rec}}$  fixed. Differentiating  $F_u \circ L = 0$ , we obtain

$$A Da \eta + F_{uy} \eta = 0.$$

This proves the formula for  $Da$  and shows that  $DL \eta = V_\eta$ . The first derivative of  $F \circ L$  is  $F_y$ , since  $F_u = 0$ . Differentiating this equality once more and inserting the formula for  $Da$ , we find the stated Schur complement formula for  $D^2 F^{\min}$ .

For the third derivative it is convenient to keep the tangent vectors. For every vector  $h$  of free coordinates and every parameter direction  $\eta$ , we have

$$D^2 F[(h, 0), V_\eta] = h^\top (A Da \eta + F_{uy} \eta) = 0. \quad (18.12)$$

The second derivative can also be written  $D^2(F \circ L)[\eta_1, \eta_2] = D^2 F[V_{\eta_1}, V_{\eta_2}]$ . Its derivative in the direction  $\eta_3$  consists of  $D^3 F[V_{\eta_1}, V_{\eta_2}, V_{\eta_3}]$  and two further terms, in each of which one of the vectors  $V$  is differentiated. A differentiated  $V$  has the form  $(D^2 a[\eta_j, \eta_3], 0)$ , so both further terms vanish by (18.12). This proves the polarized identity for the third derivative.

By the chain rule, before any stationarity is imposed, we have

$$D_u D_v (E \circ \Psi) = (D_u \Psi)^\top E''(\Psi) D_v \Psi + E'(\Psi) D_u D_v \Psi.$$

Stationarity of the constrained objective says only that  $E'(\Psi) \Psi_u = 0$  for the free directions  $u$ ; it does not remove  $E'(\Psi) \Psi_{uv}$ . Taking  $v$  to be the direction of a current frozen coordinate, we obtain the formula for  $D_{\bar{c}} a$ . Since these are polynomial identities between derivatives, their holomorphic versions use the same bilinear transpose, without complex conjugation of the parameters.

We now turn to higher derivatives. Differentiating  $F_u \circ L = 0$  in  $r$  given directions and using the chain rule indexed by the partitions  $\pi$  of these directions, we obtain

$$0 = \sum_{\pi} D^{|\pi|} F_u(L) [D^{|B|} L[\eta_j : j \in B] : B \in \pi].$$

For  $r \geq 2$  the partition with a single block contributes  $A D^r a$ , since the higher derivatives of the second component of  $L$  vanish. Every other partition contains only derivatives of  $a$  of order less than  $r$ . The new derivative is therefore obtained by solving one linear equation with the matrix  $A$ . For  $r = 1$  this is the formula for  $Da$  proved above.

A quantitative bound uses given coefficient majorants for  $F$  and for the inverse  $A^{-1}$ . On a product where these bounds hold, we divide the partition formula by  $r!$ , use the contraction bounds of Lemma 5.3, and pass to a strictly smaller product of parameters. After the derivative condition for the preconditioned nonlinear map has been imposed,

the solution is bounded by the majorant series  $Y$  with nonnegative coefficients determined by

$$Y = Ct + \kappa Y + \frac{C(t + Y)^2}{1 - C(t + Y)}, \quad \kappa < 1.$$

By the implicit-series statement of Lemma 5.3, we obtain the factorial bounds and their versions with a specified input index. For the boundary-weighted version, mark one boundary factor in the differentiated implicit series. If  $Z$  denotes its absolute marked majorant, the unmarked branches are bounded by  $Y$ , and the recursion has the coefficientwise form  $Z \preceq H(t, Y) + A(t, Y)Z$ . Here  $H$  contains the marked input and  $A$  the derivative of the unmarked fixed-point map. On a fixed smaller product the derivative condition gives  $A \leq a_0 < 1$ , so  $Z \preceq \sum_{n \geq 0} A^n H$ . Every marked term connects its external index to the marked factor. The rooted contraction and composition bounds of Lemma 18.3(2) therefore bound this series in the boundary coefficient norm, including any specified analytic input. This proves the weighted implicit bound directly, without using the later result on general boundary networks. The constants depend only on these coefficient bounds, the bound for  $A^{-1}$ , and uniform data. The hypotheses of this quantitative application are verified for the objective of a block region in Theorem 18.12.

To compare successive and joint minimizers, split the free coordinates as  $(u, v)$  and let  $(u_*, v_*)$  be the joint stationary point. By the domain compatibility in the statement,  $v_*$  is an admissible parameter for the conditional branch and  $u_*$  belongs to its solution domain. Conditional stationarity and uniqueness imply  $a(v_*) = u_*$ . The envelope identity then gives  $D_v F^{\min}(v_*) = F_v(u_*, v_*) = 0$ . Since  $v_*$  belongs to the outer solution domain, uniqueness of the outer branch identifies it with the successive solution. Repeating this argument identifies all stages. Conversely, a successive stationary point is jointly stationary, but this observation alone does not identify a joint branch in a different domain. Finally, the regional rescaling convention multiplies the partial minimum value by  $M$  at each intermediate scale, so after  $i$  scales the factor is  $M^i$ . On the torus, Theorem 10.10 additionally identifies the result with the effective classical action. Multiplication by a positive constant does not change the minimizer, and every derivative of the value carries the same factor.  $\square$

## 18.5 Freezing variables

When a block region shrinks, the variables at the removed labels are frozen: they are replaced by fixed values, which become frozen coordinates. The next lemma describes how the coefficient norm, a torus interaction, the effective density, and the triangular determinant behave under this substitution.

**Lemma 18.9.** *Use the norms and the position convention of Definition 18.1. Let  $D$  be the label list of a block region with geometric position domain  $V_D$ , with the radius  $\phi$  of Definition 18.1*

for  $D$ , let  $s_{\text{er}} \geq \max\{2, w_{\text{bd}}\}$ , and choose a retained geometric position domain contained in  $\{x \in V_D : \text{dist}(x, V_D^c) \geq s_{\text{er}}\}$ . Let  $I$  be the labels of  $D$  at the retained positions and let  $F = D \setminus I$ . The labels in  $F$  are frozen. Since  $s_{\text{er}} \geq w_{\text{bd}}$ , every label of  $D$  at which  $\phi < \rho$  lies in  $F$ ; additional geometric positions may be removed as holes. The domains here are domains of graph vertices, including vertices without labels, as in Definition 18.1. Let  $c = (c_j)_{j \in F}$  satisfy  $|c_j| \leq \eta\phi(j)$  with  $0 \leq \eta \leq 1/4$ . For a function  $v$  on  $I$ , write  $v \oplus c$  for the variables on  $D$  equal to  $v$  on  $I$  and to  $c$  on  $F$ , and for an analytic function  $f$  set  $T_c f(v) = f(v \oplus c) - f(0 \oplus c)$ . The following statements hold.

1. For  $i \in I$  we have  $d_D(i) \geq d_I(i) + s_{\text{er}} - 1$ , and

$$\mathcal{X}_I^\rho(T_c f) \leq (1 - \eta)^{-2} e^{-\alpha_{\text{bd}}(s_{\text{er}} - 1)} \mathcal{X}_D^\phi(f) \leq 2e^{-\alpha_{\text{bd}}(s_{\text{er}} - 1)} \mathcal{X}_D^\phi(f).$$

The norm on the left includes every remaining nonconstant degree, including degree one. The same statement holds with the separate classical radius and rates.

2. Let  $r_{\mathbb{T}}$  be a function on the torus with no linear term and  $\mathcal{B}_{\rho, \bar{a}}(r_{\mathbb{T}}) \leq E$ , where  $\mathcal{B}_{\rho, \bar{a}}$  is the torus norm of Definition 4.6, with the Euclidean network length, and  $\bar{a} \geq \sqrt{3}(a_{\text{sp}} + \alpha_{\text{bd}})$ , and assume  $|c_j| \leq \beta = \eta\rho$  for  $j \in F$ . The boundary-completion difference

$$G_c r_{\mathbb{T}}(v) = r_{\mathbb{T}}(v, c) - r_{\mathbb{T}}(0, c) - r_{\mathbb{T}}(v, 0) + r_{\mathbb{T}}(0),$$

that is, the difference between the substitution of the frozen values  $c$  and the substitution of zero, satisfies  $\mathcal{X}_I^\rho(G_c r_{\mathbb{T}}) \leq 8\eta E$ . All torus labels outside  $D$  are set to zero in these evaluations. The analogous difference for the effective classical action has its own radius and bound.

3. Consider a joint density, relative to its given reference measure, of the form

$$Z^{-1} e^{\Lambda(b_{\text{rec}})} W(b_{\text{rec}}) e^{-E(v, c; b_{\text{rec}})/g^2 - R(v, c; b_{\text{rec}})} \chi(v, c, b_{\text{rec}}),$$

where  $W(b_{\text{rec}}) \geq 0$  is a weight depending only on the frozen coordinates (not to be confused with the boundary  $W$ ). Freezing  $c$  gives the same joint measure, with the same  $Z$ , the same indicators, and the same effective classical action  $E$ , with the enlarged frozen coordinates  $(b_{\text{rec}}, c)$ , and with

$$R_{\text{new}}(v) = R(v, c) - R(0, c), \quad \Lambda_{\text{new}} = \Lambda - R(0, c).$$

If the effective classical action is centered as well, its factor  $e^{-E(0, c; b_{\text{rec}})/g^2}$  remains as an external factor. No partition function at fixed frozen coordinates is introduced. At uniform radius, write  $R = r_{\mathbb{T}}|_D + f$  with  $r_{\mathbb{T}}$  as in (2), and set

$$\epsilon_D(f) = \sum_{k \geq 2} \rho^{k-2} \sup_{i \in D} e^{\alpha_{\text{bd}} d_D(i)} \sum_{|v|=k} \frac{v_i}{k} |f_v| e^{a_{\text{sp}} t(v)}, \quad \ell_D(f) = \sup_{i \in D} e^{\alpha_{\text{bd}} d_D(i)} |f_{e_i}|.$$

If  $\|c\|_\infty \leq \beta \leq \rho/4$ , then

$$|R(0; c; b_{\text{rec}}) - R(0; b_{\text{rec}})| \leq |F| \{ \beta \ell_D(f) + \beta^2 [\epsilon_D(f) + E] \}.$$

This bounds the change of the constant term; it does not bound  $R(0; b_{\text{rec}})$  itself.

4. Consider a sequential normalization by the constrained Hessians,

$$z_i = a_i(c_{i+1}; b_{\text{rec}}) + g_i K_i(c_{i+1}; b_{\text{rec}})^{-1/2} w_i, \quad c_{i+1} = c_{i+1}(w_{>i}),$$

where  $c_{i+1}$  denotes the coarser fields as functions of the coarser normalized fluctuation variables, and let  $d_i = 3f_i$  be the number of scalar free coordinates at scale  $i$ . The Jacobian determinant of  $w \mapsto z$  is

$$\prod_i g_i^{d_i} \det K_i(c_{i+1}(w_{>i}); b_{\text{rec}})^{-1/2}.$$

Adding and subtracting half of  $\log \det K_i(\hat{c}_{i+1}; b_{\text{rec}})$ , where  $\hat{c}_{i+1} = c_{i+1}(0)$ , splits this single determinant into its value at zero fluctuation and the logarithmic difference containing all the dependence on the coarser normalized fluctuation variables. Missing Gaussian coordinates and all factors at zero fluctuation remain explicit.

At frozen coordinates represented by their based transports, we have the identity  $f(v \oplus c) = T_c f(v) + f(0 \oplus c)$ . Every algebraic identity above is finite-dimensional. All bounds are uniform and independent of additional holes, of the number of preceding block integrations, of the widths chosen later, and of the number of frozen labels except through the displayed factor  $|F|$ .

*Proof.* We begin with the distance inequality in (1). If  $I$  is empty there is no nonconstant coefficient to estimate. If  $D$  has no boundary, the boundary perturbation is zero by the convention of Definition 18.1. Otherwise let  $i \in I$ , and let  $h = (2M^s)^{-1} \leq 1$  be the mesh of the refined position graph. We choose a shortest path  $i = x_0, \dots, x_k$  of vertices of this graph with  $x_k$  outside  $D$ , so that  $kh = d_D(i)$ , and we set  $s' = \lceil s_{\text{er}}/h \rceil$ . Since  $kh = d_D(i) \geq s_{\text{er}}$  and  $k$  is an integer, we have  $k \geq s'$ . The vertex  $x_{k-s'+1}$  is at distance at most  $(s' - 1)h < s_{\text{er}}$  from the complement of  $D$ , so it is outside the retained geometric position domain, whether or not it carries a label. We conclude that

$$d_I(i) \leq (k - s' + 1)h \leq d_D(i) - s_{\text{er}} + h \leq d_D(i) - s_{\text{er}} + 1.$$

Additional holes can only decrease  $d_I$ . This proves the distance inequality, with the exact term  $-1$ . Removing all labels at a position together is what makes this a statement on the graph of positions. The fixed buffer of Definition 18.1 transfers the same argument to another convention for link endpoints.

Next we prove the coefficient bound in (1). We fix the earlier frozen coordinates and an allowed  $c$ , and for each input degree  $m$  we set

$$A_m = \rho^{-2} \sup_{i \in D} e^{\alpha_{\text{bd}} d_D(i)} \sum_{|v|=m} \frac{v_i}{m} |f_v| e^{a_{\text{sp}} \mathbf{t}(v)} \phi^v.$$

A monomial of degree  $m$  splits uniquely as  $(v_I, v_F)$ , of degrees  $k \geq 1$  and  $j \geq 0$  with  $m = k + j$ . Every remaining radius is  $\rho$ , since all labels with smaller radius are frozen. For a fixed index  $i \in I$ , the new factor  $v_i/k$  is at most  $m/k$  times the old factor  $v_i/m$ , and  $m/k \leq 1 + j$ . Substituting the frozen values costs at most a factor  $\eta^j$ , and  $t(v_I) \leq t(v)$ . The distance inequality contributes the factor  $q = \exp[-\alpha_{\text{bd}}(s_{\text{er}} - 1)]$ . Combining these factors, we find that the outgoing summand of degree  $k$  is at most  $q \sum_{j \geq 0} (1 + j) \eta^j A_{k+j}$ ; coinciding outgoing monomials are bounded by the triangle inequality. We sum over  $k$  after taking the suprema over the fixed index. Since all terms are nonnegative, we obtain

$$\sum_{k \geq 1} q \sum_{j \geq 0} (1 + j) \eta^j A_{k+j} \leq q \left( \sum_{j \geq 0} (1 + j) \eta^j \right) \sum_{m \geq 1} A_m = q(1 - \eta)^{-2} \sum_{m \geq 1} A_m.$$

This includes  $k = 1$ , even when the linear term comes from a monomial of high degree. No binomial factor for the choice of frozen factors appears, since the two label sets are fixed. The suprema over the earlier frozen coordinates and over the allowed real values  $c$  are taken after this inequality and the degree sums. Since  $(1 - \eta)^{-2} \leq 16/9 < 2$ , this proves the two inequalities in (1). For the classical statement we replace the three parameters by the classical radius and rates in the same calculation; the partition of labels and the ratios of the factors  $v_i/k$  are unchanged.

We now explain why the outgoing norm has uniform radius. We write  $\phi_{\text{cl}}$  for the classical radius function. When all labels with smaller radius are frozen, we have

$$\phi|_I \equiv \rho, \quad \phi_{\text{cl}}|_I \equiv \rho_{\text{cl}}. \quad (18.13)$$

Indeed,  $d_D(i) \geq s_{\text{er}} \geq w_{\text{bd}}$  excludes every remaining position from the boundary strip, and additional holes remove labels without changing the radius at the remaining ones. This means that  $\mathcal{X}_I^\rho(T_c f)$  is the norm with the uniform radius  $\rho$  at every remaining variable, and its classical counterpart has uniform radius  $\rho_{\text{cl}}$ . These are the incoming norms at uniform radius for the next calculation on the same list of labels, with their existing scales of introduction. The new values  $c$  are fixed real frozen coordinates, not additional variables with the radius  $\rho$ , and no extension beyond an earlier analytic domain is asserted, since each remaining variable already had the radius  $\rho$ .

For (2), only monomials that meet both  $I$  and  $F$  contribute to  $G_c r_{\mathbb{T}}$ . For any fixed index  $i \in I$ , the network of such a monomial reaches the complement of  $I$ , so its  $\ell^1$  length is at least  $d_I(i)$ , as well as at least  $t(v_I)$ . Since the torus norm uses the Euclidean network length  $t_2$  and  $t \leq \sqrt{3}t_2$ , we obtain

$$a_{\text{sp}} t(v_I) + \alpha_{\text{bd}} d_I(i) \leq (a_{\text{sp}} + \alpha_{\text{bd}}) t(v) \leq \sqrt{3} (a_{\text{sp}} + \alpha_{\text{bd}}) t_2(v) \leq \bar{a} t_2(v).$$

We use (18.2) for the torus norm. The degree calculation above now has  $j \geq 1$ , and its incoming sum is at most  $E$ . We obtain the stronger estimate

$$\mathcal{X}_I^\rho(G_c r_{\mathbb{T}}) \leq [(1 - \eta)^{-2} - 1]E \leq 4\eta E \leq 8\eta E.$$

For the middle inequality we wrote the bracket as  $\eta(2 - \eta)/(1 - \eta)^2$  and used  $\eta \leq 1/4$ . This calculation counts the linear term produced by the substitution. Setting the other torus labels to zero only deletes coefficients. For the classical difference we replace the radius, the rates, and the bound by their classical counterparts in the same calculation, with the same factor  $\sqrt{3}$  in the rate hypothesis.

For (3), the statement on the measure is an algebraic identity with the given reference measure. Indeed,

$$e^{\Lambda - R(0,c)} e^{-[R(v,c) - R(0,c)]} = e^{\Lambda - R(v,c)}.$$

The weight  $W(b_{\text{rec}})$ , every indicator, and the term  $E(v, c; b_{\text{rec}})/g^2$  are unchanged. Splitting  $E$  into  $E(v, c) - E(0, c)$  and  $E(0, c)$  gives the external classical factor in the statement. Fubini's theorem with the given reference measure proves the identity of joint measures, with the same  $Z$ ; no normalization at fixed frozen coordinates has taken place. If the frozen variables are based transports, the identity is evaluated at those same values; it is a substitution in existing variables.

Only monomials supported in  $F$  contribute to  $R(0, c; b_{\text{rec}}) - R(0; b_{\text{rec}})$ . At each degree  $m$ , we assign such a monomial to its labels with weights  $v_i/m$ , whose sum is one, and drop the network and distance weights, which are at least one. In degree one the sum is at most  $|F|\beta\ell_D(f)$ . For  $m \geq 2$  we use  $\beta^m \leq \beta^2 \rho^{m-2}$ . Summing over  $i \in F$  and then over  $m$  bounds the contribution of  $f$  by  $|F|\beta^2 \epsilon_D(f)$  and that of the torus part by  $|F|\beta^2 E$ . The constant terms cancel in this difference, which is why the bound says nothing about an arbitrary incoming  $R(0; b_{\text{rec}})$ . This proves the bound on the change of the constant term.

Finally we prove (4). We order the blocks of free coordinates from the coarser scales to the finer ones. The sequential map has the blocks

$$z_i = a_i(c_{i+1}; b_{\text{rec}}) + g_i K_i(c_{i+1}; b_{\text{rec}})^{-1/2} w_i, \quad c_{i+1} = c_{i+1}(w_{>i}).$$

Its derivative is block triangular, with diagonal blocks  $g_i K_i(c_{i+1}(w_{>i}); b_{\text{rec}})^{-1/2}$ , so the determinant is the product in the statement. This is the triangular argument of Theorem 12.13, applied to the list of free coordinates of the block region; the dimensions  $d_i = 3f_i$  are those of this list, not of the torus. With  $L_i(w) = \log \det K_i(c_{i+1}(w_{>i}); b_{\text{rec}})$  and  $\hat{L}_i = \log \det K_i(\hat{c}_{i+1}; b_{\text{rec}})$ , the splitting reads

$$\det D_w Z = \left( \prod_i g_i^{d_i} \right) e^{-\frac{1}{2} \sum_i \hat{L}_i} e^{-\frac{1}{2} \sum_i [L_i(w) - \hat{L}_i]}.$$

The second exponential contains all the dependence on the coarser normalized fluctuation variables. This is one determinant, with its real positive branch and its analytic continuation. If some free labels are absent, their Gaussian coordinates are absent from this product and remain explicit in any comparison, and the factors at zero fluctuation remain constant factors. The only estimates were sums of coefficients with a fixed index and graph distances, and neither introduces a dependence on the number of preceding

block integrations, the number of holes, or the later widths, beyond the displayed factor  $|F|$ .  $\square$

## 18.6 Coercivity and analytic constrained minimizers

We first prove the patch inequality, which bounds a linearized variation of a block region by its curl on single final blocks and on pairs of adjacent final blocks. The patches and the restriction maps are defined explicitly, so that the inequality applies both to the effective classical action (Theorem 18.12) and to the Wilson action at the first block integration (Lemma 18.14).

**Definition 18.10.** Use the notation of Definition 18.6. The set  $\mathfrak{P}_\Omega$  of patches of  $\Omega$  consists of every final block  $B_s(a)$  with  $a \in \mathcal{K}$  and of every union  $B_s(a) \cup B_s(a + M^s e_k)$  of two adjacent final blocks with both anchors in  $\mathcal{K}$ . For  $P \in \mathfrak{P}_\Omega$ , let  $\mathcal{E}(P)$  be the set of positive input edges with both endpoints in  $P$ , and  $\mathcal{P}(P)$  the set of elementary positive plaquettes with all four vertices in  $P$ ; thus  $P$  is a complete cuboid, with its boundary edges and faces. Construct  $\mathcal{L}_i(P)$ ,  $Q_i^P$ , and  $\Pi_i^{\mathcal{T},P}$  by the rules of Definition 18.6, using only the final block or blocks of  $P$ , and let

$$G_P h = ((\Pi_i^{\mathcal{T},P} Q_{[0,i]}^P h)_{0 \leq i < s}, Q_{[0,s]}^P h), \quad \mathcal{V}_P = \ker G_P. \quad (18.14)$$

For a single block  $\mathcal{L}_s(P)$  is empty, and for a pair it consists of its one coarse bond. The tree conditions are imposed at every scale, not only at the finest one. Let  $d_P$  be the signed plaquette incidence matrix (the curl) of the cuboid  $P$ . For an edge cochain  $u$  on  $\mathcal{L}_0$ , the restriction to the patch is  $(R_P u)_e = u_e$  for  $e \in \mathcal{E}(P)$ . For a patch that crosses a periodic seam we use the orientations induced by its rectangular lift.

**Lemma 18.11.** Use Definitions 18.6 and 18.10. Let  $\dot{\mathcal{H}}_i$  be the derivative of the coarse-field lift  $\mathcal{H}_i$  at the identity, let  $I_{\mathcal{F}_i}$  and  $I_{\mathcal{D}_i}$  be the inclusions of coordinates on  $\mathcal{F}_i$  and  $\mathcal{D}_i$  into cochains on  $\mathcal{L}_i$ , and let  $\Pi_{\mathcal{F}_i}$  be the restriction to  $\mathcal{F}_i$ . Define

$$\Gamma'_i = I_{\mathcal{F}_i} - M^2 I_{\mathcal{D}_i} Q_i I_{\mathcal{F}_i}, \quad A_\Omega z = \sum_{i=0}^{s-1} \dot{\mathcal{H}}_0 \cdots \dot{\mathcal{H}}_{i-1} \Gamma'_i z_i, \quad (18.15)$$

where the product preceding  $\Gamma'_0$  is the identity, and

$$(B_{\text{ext}} v)_i = \Pi_{\mathcal{F}_i} (I - \dot{\mathcal{H}}_i Q_i) Q_{[0,i]} v. \quad (18.16)$$

For a matrix  $B$  write  $\|B\|_\infty = \sup_x \sum_y |B_{xy}|$  and  $\|B\|_1 = \sup_y \sum_x |B_{xy}|$ . The following statements hold.

1. The map  $A_\Omega$  is the derivative of  $\text{Log } \Phi_\Omega(0, \cdot)$  at  $z = 0$ . It maps the free coordinates bijectively onto  $\ker G_\Omega$ , and  $B_{\text{ext}}$  is its inverse on  $\ker G_\Omega$ .

2. We have

$$\begin{aligned} \|A_\Omega\|_\infty &\leq J(1 + 4M^3), & \|A_\Omega\|_1 &\leq M^{2(J-1)}(1 + 24M^3), \\ \|B_{\text{ext}}\|_\infty &\leq b_\infty = (1 + 4M)(4M)^{J-1}, & \|B_{\text{ext}}\|_1 &\leq b_1 = (1 + 24M^3) \sum_{i=0}^{J-1} (24M)^i. \end{aligned}$$

Consequently  $\|z\|_2 \leq C_{\text{ext},J} \|A_\Omega z\|_{\mathcal{L}_0}$  with  $C_{\text{ext},J} = \max(1, \sqrt{b_\infty b_1})$ . Every nonzero entry of  $A_\Omega$  or  $B_{\text{ext}}$  joins two positions at periodic maximum distance at most  $R_{M,J} = 4 \sum_{i=0}^{J-1} M^{i+1}$  in input units, hence also in output units.

3. Let  $u \in \ker G_\Omega$ , and let  $du$  be the curl of any edge cochain on all input edges that agrees with  $u$  on  $\mathcal{L}_0$ . For every patch  $P$ , we have  $R_P u \in \mathcal{V}_P$  and  $(d_P R_P u)_p = (du)_p$  for  $p \in \mathcal{P}(P)$ .
4. There is  $\sigma_J > 0$ , depending only on  $M$  and  $J$ , such that  $\|h\|^2 \leq \sigma_J^{-1} \|d_P h\|^2$  for every patch  $P$  and every  $h \in \mathcal{V}_P$ . For every  $u \in \ker G_\Omega$ ,

$$\|u\|_{\mathcal{L}_0}^2 \leq \sum_{P \in \mathfrak{P}_\Omega} \|R_P u\|^2 \leq \sigma_J^{-1} \sum_{P \in \mathfrak{P}_\Omega} \|d_P R_P u\|^2 \leq \frac{7}{\sigma_J} \sum_{p \in \mathcal{P}_\Omega} |(du)_p|^2, \quad (18.17)$$

where  $\mathcal{P}_\Omega = \bigcup_{P \in \mathfrak{P}_\Omega} \mathcal{P}(P)$ .

All constants are independent of  $m$ , of the block region, of its number of components and holes, and of its topology. For an empty list of free coordinates all statements hold with the conventions for zero-dimensional spaces.

*Proof.* We begin with the derivatives of the block averaging map at the identity. Differentiating the averaging formula (4.3) at the identity gives  $Q_i$ : the derivative of the reference transport in the averaged logarithm cancels the derivative of the final reference factor. The exceptional rule plays no role, since all based loop holonomies are close to  $I$  in a neighborhood of the identity. For a scalar vertex function  $\varphi$  and a bond  $c' = (a, a + M^{i+1}e_k)$ , each contour  $F_{i,c',x}$  telescopes, and hence

$$(Q_i \nabla \varphi)_{c'} = \varphi(a + M^{i+1}e_k) - \varphi(a). \quad (18.18)$$

This holds at every scale, and it uses only the contours of internal bonds; no gradient is extended across a hole.

The derivative  $\mathcal{H}_i$  copies a coarse value to each edge of the interface of its bond and vanishes on the interiors of the blocks. We have

$$Q_i \mathcal{H}_i = I, \quad \Pi_i^\mathcal{T} \mathcal{H}_i = 0, \quad Q_i I_{\mathcal{D}_i} = M^{-2} I. \quad (18.19)$$

For the first identity, each contour of  $c'$  crosses the interface of  $c'$  exactly once, with positive orientation, and its tree parts lie in block interiors. The second holds since tree edges lie in block interiors. For the third, a contour of a different bond never uses the eliminated edge of  $c'$ , and among the  $M^3$  contours of  $c'$  exactly  $M$  use  $e_i^*(c')$ , each once

with positive orientation. Since tree paths stay in their blocks, these identities hold also for an internal bond next to a final block outside  $\Omega$ .

The linearized constraint at the identity is  $Q_i X = 0$  for a cochain  $X$  on  $\mathcal{L}_i$  with zero tree entries, and (18.19) gives its unique solution with free entries  $z_i$ :

$$\Gamma'_i z_i = I_{\mathcal{F}_i} z_i - M^2 I_{\mathcal{D}_i} Q_i I_{\mathcal{F}_i} z_i, \quad \Pi_i^T \Gamma'_i = 0, \quad Q_i \Gamma'_i = 0, \quad \Pi_{\mathcal{F}_i} \Gamma'_i z_i = z_i.$$

No inversion of a matrix of unbounded range occurs, since the coefficient matrix of the eliminated edges is  $M^{-2}I$ . Thus  $\Gamma'_i$  is the derivative at zero of the local constraint coordinates at the identity target. Since  $\Gamma_b(0) = 0$  for every target  $b$ , differentiating (18.10) at  $(c, z) = (0, 0)$ , we obtain

$$u_s = 0, \quad u_i = \dot{\mathcal{H}}_i u_{i+1} + \Gamma'_i z_i, \quad u_0 = A_\Omega z,$$

where  $u_i$  is the derivative of  $\text{Log } U^{(i)}$  in the direction  $z$ . By (18.19), we have  $u_i = Q_{[0,i]} u_0$ ,  $\Pi_i^T u_i = 0$ , and  $u_s = 0$ , so  $u_0 \in \ker G_\Omega$ . Conversely, let  $v \in \ker G_\Omega$  and set

$$v_i = Q_{[0,i]} v, \quad z_i = \Pi_{\mathcal{F}_i} (v_i - \dot{\mathcal{H}}_i v_{i+1}).$$

The cochain  $v_i - \dot{\mathcal{H}}_i v_{i+1}$  has zero tree entries and satisfies  $Q_i(v_i - \dot{\mathcal{H}}_i v_{i+1}) = v_{i+1} - v_{i+1} = 0$ . By (18.19) its eliminated entries are determined by its free entries, so it equals  $\Gamma'_i z_i$ . Since  $v_s = 0$ , descending induction shows that  $v = A_\Omega z$ . We conclude that  $A_\Omega$  maps the free coordinates bijectively onto  $\ker G_\Omega$ , with inverse (18.16). This is the successive extraction of the averaged fields and of their free coordinates, written as a matrix, and it proves (1).

Next we prove (2). A contour has at most  $4M - 3$  steps, so

$$\|Q_i\|_\infty \leq 4M, \quad \|Q_i\|_1 \leq 24M, \quad \|\dot{\mathcal{H}}_i\|_\infty \leq 1, \quad \|\dot{\mathcal{H}}_i\|_1 \leq M^2.$$

For the column bound for  $Q_i$ , an interior edge of a block occurs in the contours of at most the six bonds incident to its block, and an interface edge only in the contours of its own bond; each such row has absolute sum at most  $4M$ . A bond has  $M^2$  interface edges, which gives the column bound for  $\dot{\mathcal{H}}_i$ . We obtain  $\|\Gamma'_i\|_\infty \leq 1 + 4M^3$  and  $\|\Gamma'_i\|_1 \leq 1 + 24M^3$ , and the bounds for  $A_\Omega$  follow from (18.15), since a column of  $A_\Omega$  belongs to a single scale  $i \leq J - 1$ . For  $B_{\text{ext}}$  we multiply the row and column bounds of the factors in (18.16); stacking over  $i$  takes the maximum of the row bounds and the sum of the column bounds. By the Cauchy–Schwarz inequality, we have  $\|Bv\|_2^2 \leq \|B\|_\infty \|B\|_1 \|v\|_2^2$ , and therefore  $\|z\|_2 = \|B_{\text{ext}} A_\Omega z\|_2 \leq C_{\text{ext},J} \|A_\Omega z\|_{\mathcal{L}_0}$ . These estimates apply to each color and to the Euclidean norms over the colors.

We now prove the range bound. We assign to each edge its midpoint. Each nonzero entry of  $Q_i$ ,  $\dot{\mathcal{H}}_i$ , or  $\Gamma'_i$  joins positions at periodic maximum distance at most  $2M^{i+1}$ . A term of (18.15) uses at most one such factor at each scale. In (18.16), the term with  $\dot{\mathcal{H}}_i Q_i$

can use two factors at scale  $i$ , in addition to the factors at finer scales. Adding these ranges bounds every term of both formulas by  $4 \sum_{i=0}^{J-1} M^{i+1} = R_{M,J}$ . Dividing coordinates by  $M^s \geq 1$  only decreases this range. All these estimates hold after restriction to a block region: only equations of internal bonds were used, and deleting other equations does not increase a row sum, a column sum, or a range.

We now prove (3). For a patch  $P$  and each  $i$ , let  $r_i^P$  select the edges of  $\mathcal{L}_i(P)$  among those of  $\mathcal{L}_i$ . Since all contours of a bond stay in its two complete endpoint blocks (Lemma 18.4), we have the matrix identity

$$Q_i^P r_i^P = r_{i+1}^P Q_i,$$

where the right side has zero coefficients on the input edges outside  $P$ ; these zeros come from the supports of the contours, not from an extension of  $u$  by zero. The same identity holds for the rows of the trees. Iterating it and using  $G_\Omega u = 0$ , we obtain  $G_P R_P u = 0$ , that is,  $R_P u \in \mathcal{V}_P$ . This holds for every patch, including the patches next to the boundary of  $\Omega$ . Every plaquette in  $\mathcal{P}(P)$  has its four edges in  $\mathcal{E}(P) \subset \mathcal{L}_0$ , so  $(d_P R_P u)_p = (du)_p$  for  $p \in \mathcal{P}(P)$ .

Next we show that the curl is injective on each space  $\mathcal{V}_P$ . A single block or an adjacent pair lifts injectively to a rectangular cuboid in  $\mathbb{Z}^3$ : its longest side has at most  $2M^s - 1$  links, which is less than  $n/2$  since  $m \geq 4$ . Let  $h \in \mathcal{V}_P$  with  $d_P h = 0$ . A closed cochain on a rectangular cuboid is a gradient, as we now recall. Fix a corner  $o$ , and for a vertex  $x$  let  $\tau_x$  be the coordinate-ordered path from  $o$  to  $x$ . For an edge  $(x, y)$ , the closed chain  $\tau_x + (x, y) - \tau_y$  is an integer combination of elementary squares in the cuboid: we move the last coordinate step through the portions of  $\tau_x$  in later coordinates, and all squares used lie between the relevant coordinate values, hence in the cuboid. Pairing with  $h$  shows that  $\varphi(x) = h(\tau_x)$  satisfies  $h_{xy} = \varphi(y) - \varphi(x)$ . By (18.18),  $Q_{[0,i]}^P h$  is the gradient of the values of  $\varphi$  at the anchors of scale  $i$ . The tree equations at the finest scale make  $\varphi$  constant on each block of the first scale. If  $\varphi$  is constant on the blocks of scale  $i$ , the tree equations at scale  $i$  equate these constants along a connected tree in each block of scale  $i + 1$ . After  $s$  steps  $\varphi$  is constant on each final block. On a single block this implies  $h = 0$ . On a pair, the remaining equation of  $G_P$  for its coarse bond says, by (18.18), that the two final constants agree, so  $h = 0$  on the pair as well. No equation with an endpoint in a final block outside  $\Omega$  was used.

For  $1 \leq s \leq J$  there are finitely many cuboids of these types, including the directions of the pairs. For each one with  $\mathcal{V}_P \neq \{0\}$ , we take an orthonormal basis matrix  $B_P$  of  $\mathcal{V}_P$  and let  $\sigma_P$  be the least eigenvalue of  $B_P^\top d_P^\top d_P B_P$ , which is positive by the injectivity just proved; if  $\mathcal{V}_P = \{0\}$  we set  $\sigma_P = 1$ . Let  $\sigma_J$  be the minimum of these finitely many numbers. It depends only on  $M$  and  $J$ , and we have  $\|h\|^2 \leq \sigma_J^{-1} \|d_P h\|^2$  for  $h \in \mathcal{V}_P$ .

It remains to prove the covering and overlap bounds in (18.17). Every edge of  $\mathcal{L}_0$  has its endpoints either in one final block or in two adjacent final blocks of  $\Omega$ ; the first kind

is covered by the single block and the second by the pair. This gives the first inequality, including for isolated blocks. The second inequality follows from (3) and the bound on each  $\mathcal{V}_p$ . For the third, fix a plaquette  $p$  and one of its vertices  $x$ . Every patch containing  $p$  contains  $x$ , hence the final block of  $x$ . There are at most seven such patches: that block and the pairs formed with its six possible neighbors. So each plaquette occurs in at most seven of the sets  $\mathcal{P}(P)$ ; a plaquette that meets three or four final blocks may occur in none. Using (3), we obtain (18.17).  $\square$

We now state the main result. Its objective is the effective classical action at depth  $d_0$ , evaluated on the reconstruction of a block region and completed outside  $\Omega$  as described in the statement.

**Theorem 18.12.** *Fix  $M$  and  $J$ , and let  $\kappa_* > 0$  be the constant of Theorem 14.8. Then*

$$E''_{d_0}(0) \geq \kappa_* \operatorname{curl}^T \operatorname{curl} \quad \text{for every depth } d_0 \geq 0, \quad (18.20)$$

*as quadratic forms on real link variations. Let  $1 \leq s \leq J$ ,  $m \geq 4$ , let  $\Omega$  be a block region and  $d_0 \geq 0$  a depth, and set*

$$F_R(c, z; \xi) = E_{d_0}(\Psi_R(c, z; \xi)).$$

*Here  $\Psi_R(c, z; \xi)$  is the logarithm, on all links of the torus, of the reconstruction  $\Phi_\Omega(c, z)$  of Definition 18.6, completed outside  $\Omega$  by the small values with logarithms  $\xi$  prescribed by the current frozen coordinates (at the first block integration, by the completion of Definition 18.13), and with logarithm zero at the torus labels that lay outside earlier block regions. The parameters  $\xi$  may be complexified on the construction chart. Frozen coordinates from earlier block integrations and arbitrary data in the fixed complement are not arguments of this analytic action. The following statements hold with uniform constants and radii; in particular they are independent of the widths chosen later, of the shape of  $\Omega$  (which may contain isolated final blocks), of  $d_0$ , and of the values of the frozen coordinates.*

1. *On a common list of coordinates, the linearizations at zero  $E_1$  and  $E_0$  of the reconstruction on  $\Omega$  and of the reconstruction on the whole torus with the complementary coordinates set to zero, defined in (18.22), satisfy  $E_1 = E_0 + D_W$ ; here  $E_0$  and  $E_1$  denote linear embeddings, not effective classical actions. Both indices of every nonzero entry of  $D_W$  lie in a band of fixed width  $L_J$  around the boundary  $W$ , the weighted rows and columns of  $D_W$  are at most  $C_J$ , and its rank is at most  $C_J|W|$ . On matched coordinates in the common chart, the nonlinear formulas for the outputs in  $\Omega$  of the two reconstructions agree away from this band. Prescribed values in the fixed complement may differ; their effect is estimated as a bounded source in Lemma 18.15. Coarse outputs on bonds across the boundary of the block region that are not prescribed are holomorphic functions of  $(c, z; \xi)$ , with all their derivatives. This compares two embeddings; it does not identify their fibers.*

2. There are  $\nu_J, C_{\text{ext},J}, \lambda_J, Q_J > 0$  with the following properties. Let  $u = D_z \Psi_R(0)z$ ; its restriction to  $L_\Omega = \mathcal{L}_0$  lies in  $\ker G_\Omega$ . Then

$$\|\text{curl } u\|_2^2 \geq \nu_J \|u\|_{L_\Omega}^2, \quad \|z\|_2 \leq C_{\text{ext},J} \|u\|_{L_\Omega},$$

$$K_R^0 = F_{R,zz}(0) \geq \lambda_J I, \quad \lambda_J = \kappa_* \nu_J / C_{\text{ext},J}^2, \quad \|K_R^0\|_{a_J} \leq Q_J.$$

One may take  $\nu_J = \sigma_J/7$  and  $C_{\text{ext},J}$  as in Lemma 18.11. Every principal block of  $K_R^0$  and every real Schur complement of  $K_R^0$  is bounded below by  $\lambda_J I$ . Let  $z_H$  be the free coordinates of the torus outside  $\Omega$ . On the common list  $(z_L, z_H)$ , the Hessian at  $(c, z, \xi) = 0$  of the interpolation between the joint objective on the torus and  $F_R + \|z_H\|_2^2/2$  is bounded below by  $\lambda_{U,J} I$ , where  $\lambda_{U,J} = \min(\lambda_J, \lambda_{T,J}, 1) > 0$  and  $\lambda_{T,J}$  is the corresponding flat bound on the torus.

3. There are radii  $r_{a,J}, r_{z,J}, \rho_J, \beta_{\text{rec},J} > 0$ , a rate  $a_J > 0$ , and an inverse bound  $I_J$  such that, on an open product containing

$$\|c\|_\infty \leq 2\rho_J, \quad \|z\|_\infty \leq r_{z,J}, \quad \|\xi\|_\infty \leq 2\beta_{\text{rec},J},$$

the reconstruction lies in  $B_{r_{a,J}}$ , inside the identity chart on which  $E_{d_0}$  is controlled, and

$$\|F_{R,zz} - K_R^0\|_{a_J} \leq Q_J (\|c\|_\infty + \|z\|_\infty + \|\xi\|_\infty) \leq \lambda_J/8.$$

Mixed derivative matrices, and tensors of order  $l \geq 2$  with each index fixed in turn, are bounded by  $l! C_J^{l+1}$ . The inverse of the complex free Hessian has weighted row-column norm at most  $I_J$  at a fixed smaller rate.

4. Choose the parameter radii below the solution radius so that

$$I_J Q_J (2\rho_J + 2\beta_{\text{rec},J}) < r_{z,J}/16, \quad 8G_J (2\rho_J + r_{z,J} + 2\beta_{\text{rec},J}) < \min(r_{\Gamma,J}, r_E),$$

where  $G_J$  and  $r_{\Gamma,J}$  are the coefficient bound and the radius of the reconstruction in (18.21). Then there exists a unique holomorphic stationary point  $z_R = z_R(c; \xi)$  in the free box,

$$F_{R,z}(c, z_R; \xi) = 0, \quad \|z_R\|_\infty \leq C_J (\|c\|_\infty + \|\xi\|_\infty) < r_{z,J}/8.$$

It has factorial coefficient bounds on output rows and on volume-normalized input columns on a smaller fixed product. For real data it is the unique minimizer of  $F_R(c, \cdot; \xi)$  in the convex free box.

5. Conditional minimizers over initial segments of the scales exist on smaller products of the coarser free coordinates around this joint minimizer, chosen so that the bounds (18.31) on the Hessian and on the inverse times the gradient hold. No claim is made on other slices of the joint box.

*Proof of Theorem 18.12.* The proof has six steps. In the first, we construct the reconstruction with coefficient bounds that do not depend on the size of the lattice, and compare the embeddings on  $\Omega$  and on the torus. In the second, the patch inequality gives the coercivity of the curl. In the third, we derive (18.20) from Theorem 14.8 and obtain the lower bound on  $K_R^0$ . In the fourth, we bound the Hessian on a complex product and choose the radii. In the fifth, we construct the minimizer by a contraction argument, and in the sixth, the conditional minimizers. All constants are constructed on larger products before any of the later widths is chosen. If the list of free coordinates is empty, the minimizer is empty and the matrices are zero-dimensional, so we assume that it is nonempty.

**Step 1: local charts and comparison of embeddings.** For one averaging step the reconstruction is the eliminated-edge solution of Lemma 6.3, restricted to internal bonds as in Definition 18.6. Let  $L_M$  bound its stencil radius in output units of one step. Its local scalar majorants can be chosen of the form  $G_M t / (1 - G_M t)$  on a fixed radius. This absolute coefficient bound does not depend on the size of the lattice, for the following reason. Each defining group product reads a bounded number  $n_M$  of variables. Matrix exponentials are bounded coefficientwise by  $e^t$ , and on  $\|V - I\| < 1$  the logarithm has the convergent series  $\sum_{k \geq 1} (-1)^{k+1} (V - I)^k / k$ . The inverse of the linear coefficient of an eliminated edge is  $M^2 I_3$ , and the nonlinear equation for that edge has zero constant term and a quadratic remainder after its linear part is removed. On a sufficiently small fixed radius, the implicit-series statement of Lemma 5.3 gives  $G_M$  and factorial coefficients for the local inverse. Since only  $n_M$  variables occur in each local equation, summing over its finite stencil, also with any input fixed, we obtain absolute row and column bounds.

By composing at most  $J$  such maps, with the embeddings of positions, we obtain numbers  $G_J, L_J, r_{T,J} > 0$  such that the logarithm  $\Psi_R$  of the reconstruction is holomorphic, vanishes at zero, and has a majorant series with nonnegative coefficients on that radius, with the bounds

$$\|\Psi_R\|_{\text{coeff}} \leq \frac{G_J T}{1 - G_J T}, \quad \|D^l \Psi_R\|_{\text{all indices}} \leq l! G_J^{l+1} \quad (l \geq 1) \quad (18.21)$$

on a fixed smaller product. Here  $T = r_c + r_z + r_\xi$ , where  $r_c$ ,  $r_z$ , and  $r_\xi$  are the maximal radii of the coarse logarithms  $c$ , of the free coordinates  $z$ , and of the current frozen coordinates  $\xi$ , respectively; the second bound absorbs the fixed ratios of radii. The first norm is a row majorant for the image; the derivative bounds include every input column. One may take  $L_J \geq \sum_{i=0}^{J-1} M^i L_M$ , and enlarge  $G_J$  for the finitely many compositions. There are at most a fixed multiple of  $J M^{3J}$  labels per unit cube of the output lattice, and  $v_J$  includes this multiplicity. Volume-normalized input columns acquire their mesh-volume ratios under composition, by Lemma 5.3, and these factors are bounded by fixed powers of  $M^J$ .

We now compare the embeddings, before any minimization. Let  $\Psi_T(c_T, z_T)$  be the logarithm of the reconstruction on the whole torus, and let  $P_L$  insert the free coordinates

of  $\Omega$  into the list of free coordinates of the torus, with zero in every other slot. At matched data we use  $c_T = c \oplus 0$  and  $z_T = P_L z$ . The two linearized embeddings on this common list are

$$E_1 z = D_z \Psi_R(0, 0; 0) z, \quad E_0 z = \frac{d}{dt} \Psi_T(0, t P_L z) \Big|_{t=0}. \quad (18.22)$$

The symbols  $E_0$  and  $E_1$  in this formula and in (18.23) denote linear embeddings, not effective classical actions. All current frozen coordinates and logarithms of the coarse links vanish in these derivatives, and both reconstructions vanish there. At one scale, the equations on  $\Omega$  and on the torus for an internal bond have the same two endpoint blocks, the same internal target, and the same free values, so their eliminated-edge solutions agree by local uniqueness. The additional equations of the torus concern bonds meeting the complement. Descending through the scales, a change in such an equation can propagate only through the accumulated stencil of width  $L_J$  of Lemma 18.4. Therefore, with matching free values in  $\Omega$  and zero complementary torus values, the nonlinear formulas for the outputs in  $\Omega$  agree outside a band of fixed width around  $W$ . Prescribed values in the fixed complement are separate formulas and may differ there. An input in  $\Omega$  farther than one further stencil from the band cannot affect a changed formula, and we enlarge  $L_J$  once to include that stencil. Differentiating with the conventions of (18.22), we obtain

$$E_1 = E_0 + D_W, \quad (D_W)_{ex} = 0 \text{ unless } e, x \text{ lie in the } L_J\text{-band around } W. \quad (18.23)$$

Here the  $L_J$ -band is the set of positions within distance  $L_J$  of  $W$ . The weighted row and column bounds follow from the finite range, (18.21), and the density bound. We cover the band by balls of radius  $L_J + 1$  around the positions of  $W$ . Each ball contains at most  $C v_J (L_J + 2)^3$  labels, so the number of nonzero output rows of  $D_W$ , and hence its rank, is at most  $C_J |W|$ .

Every coarse output on a bond across the boundary of the block region that is not prescribed is obtained by applying the finite group products of  $T_M^s$  to the reconstructed field. In the chart it is a holomorphic function of  $(c, z, \xi)$ , with all its derivatives given by these products and (18.21). No equation sets it to a prescribed value, so (18.23) compares two embeddings of a common list and asserts no equality of their fibers.

**Step 2: coercivity of the curl.** Let  $u = D_z \Psi_R(0) z$ . On  $\mathcal{L}_0$  it equals  $A_\Omega z$ , by Lemma 18.11(1), so its restriction lies in  $\ker G_\Omega$ . Then (18.17) implies

$$\|u\|_{\mathcal{L}_0}^2 \leq \frac{7}{\sigma_J} \sum_{p \in \mathcal{P}_\Omega} |(du)_p|^2 \leq \frac{7}{\sigma_J} \|\operatorname{curl} u\|_2^2,$$

where  $\operatorname{curl} u$  is the curl of the completed variation on all plaquettes of the torus. In the last inequality we used that restricting a sum of squares to the plaquettes of the patches can only decrease it; since these plaquettes have all their edges in  $\mathcal{L}_0$ , variations of the

completed boundary values induced by the tree-path transports to the basepoints do not enter. This proves the first inequality of (2) with  $\nu_J = \sigma_J/7$ . The second inequality is Lemma 18.11(2); it shows in particular that a nonzero vector of free coordinates is visible on the internal links, also for an isolated final block.

**Step 3: the lower bound on the Hessian at the identity.** The quadratic form  $E''_{d_0}(0)$  is the Hessian at the identity, that is, the effective Maxwell quadratic form of Theorem 8.3 (see Theorem 10.10). We derive (18.20) from Theorem 14.8, which gives  $E_{d,\text{sat}} \geq \kappa_* E_0$  for every integer  $d \geq 0$  on the domain of the gauge-invariant extension  $E_{d,\text{sat}}$  of  $E_d$ , whether or not an effective interaction is attached to that depth. We choose a ball of link logarithms at the identity that is contained both in the construction chart of that extension and in the domain used here; on it,  $E_{d,\text{sat}}$  equals the analytic function  $E_d$ . For a fixed real variation  $u$ , we choose  $|t|$  so small that  $tu$  lies in this ball. Then for every  $d_0 \geq 0$  we have  $E_{d_0}(tu) \geq \kappa_* E_0(tu)$ , with the same  $\kappa_*$ . Both functions vanish at zero and are stationary there: for  $E_{d_0}$  this follows from Theorems 10.10 and 10.11 by induction from the Wilson action. Further,  $E_0(tu) = t^2 \|\text{curl } u\|_2^2/2 + O(t^3)$ , by expanding the plaquette product and using  $4q(e^X) = |X|^2/2 + O(|X|^4)$ . Dividing the inequality by  $t^2/2$  and letting  $t \rightarrow 0$  proves (18.20) for every integer depth. Only this inequality from Theorem 14.8 has been used, and this step gives no positivity at a nonflat coarse field.

By Lemma 18.8, the Hessian of  $E_{d_0} \circ \Psi_R$  at zero has no term involving the gradient of the action, since  $DE_{d_0}(0) = 0$ . Using (18.20) and Step 2, we obtain

$$z^T K_R^0 z \geq \kappa_* \|\text{curl } D_z \Psi_R(0) z\|_2^2 \geq \frac{\kappa_* \nu_J}{C_{\text{ext},J}^2} \|z\|_2^2, \quad (18.24)$$

which is the claimed value of  $\lambda_J$ . The same patch construction with every final block of the torus gives a lower bound  $\lambda_{T,J} > 0$  for the joint free Hessian on the torus. Every principal block of either joint matrix has the same lower bound, and (18.6) implies the same lower bound for its real Schur complements. On the common list  $(z_L, z_H)$ , where  $z_L$  are the free coordinates of  $\Omega$ , the interpolating Hessian at the identity is

$$H_t^0 = (1-t)H_T^0 + t(K_R^0 \oplus I_H), \quad 0 \leq t \leq 1,$$

where  $H_T^0$  is the joint Hessian at the identity on the torus and  $I_H$  the identity on the  $z_H$  coordinates. It is bounded below by  $\lambda_{U,J} = \min(\lambda_J, \lambda_{T,J}, 1) > 0$ . This is a convex combination of two objectives on a common list of coordinates, not an identification of fibers. All the flat and nonflat estimates below can be chosen uniformly in  $t$ , using this common lower bound and the maximum of the two coefficient bounds.

**Step 4: analytic products and the inverse of the free Hessian.** Let  $b_J > 0$  be smaller than the rate  $a_E$  of the torus tensors after the transfer of positions to output units and the conversion to the graph metric; this conversion divides a Euclidean rate by

$\sqrt{3}$ . We include the finite-stencil weights of (18.21). We compose the torus tensors of Theorem 10.11 with the majorant series of  $\Psi_R$ , using Lemma 5.3. The scalar composition converges on a fixed product, and after enlarging  $C_J$  and  $Q_J$  we have

$$\|D^l F_R\|_{b_J, \text{all indices}} \leq l! C_J^{l+1} \quad (l \geq 2), \quad \|K_R^0\|_{b_J} \leq Q_J. \quad (18.25)$$

The mixed matrices in  $z$ ,  $c$ , and  $\xi$  satisfy the same bound after enlargement. The labels of  $\xi$  have their positions on the input lattice and their own density, not the density of the output lattice. The volume-normalized column bounds follow from the contraction rule, with the volume factors kept in each factor.

At a general point of the product, the Hessian is

$$F_{R,zz} = \Psi_{R,z}^\top E''_{d_0}(\Psi_R) \Psi_{R,z} + E'_{d_0}(\Psi_R) \Psi_{R,zz}, \quad (18.26)$$

and the last term, which contains the gradient of the action, is kept. The gradient bound  $\|E'_{d_0}(A)\|_\infty \leq C_E \|A\|_\infty$  is part of Theorem 10.11. We integrate the third-derivative bound of the composed objective along the segment from zero to  $(c, z, \xi)$ , which lies in its product domain. In the weighted row-column norm, we obtain

$$\|F_{R,zz}(c, z; \xi) - K_R^0\|_{b_J} \leq Q_J(\|c\|_\infty + \|z\|_\infty + \|\xi\|_\infty), \quad (18.27)$$

$$\|F_{R,z}(c, 0; \xi)\|_\infty \leq Q_J(\|c\|_\infty + \|\xi\|_\infty). \quad (18.28)$$

These estimates also follow by expanding all the terms of (18.26); integrating the complete third derivative ensures that none of the terms involving the gradient of the action or the second derivative of the coordinates is omitted.

We apply Lemma 18.3(3) with the lower bound  $\lambda_{U,J}$ , the bound  $Q_J$ , and the rate  $b_J$  to every flat principal block of free coordinates and to every interpolating Hessian. We obtain a common smaller rate  $a_J > 0$  and a flat inverse bound; we take  $I_J$  so large that it bounds these inverses and their perturbations. For example, the constants  $\mu$  and  $C_R$  of that lemma give the flat bound  $C_R/\lambda_{U,J}$  and the perturbed bound  $4C_R/(3\lambda_{U,J})$  on a sufficiently small coefficient product. We enlarge  $I_J$  and  $Q_J$  if necessary; these enlargements precede every choice of radius. Bounds at the rate  $b_J$  imply the same bounds at the rate  $a_J$ .

We now choose the radii, on a product larger than the one in the statement. Let  $r_0 = \min(r_{\Gamma,J}, r_E)$ , where  $r_E$  is replaced by a strictly smaller radius of the torus chart if necessary. We first choose  $r_{z,J} > 0$ , and then  $\rho_J, \beta_{\text{rec},J} > 0$ , so that, with  $T_+ = 2r_{z,J} + 4\rho_J + 4\beta_{\text{rec},J}$ ,

$$\begin{aligned} Q_J T_+ &< \min\{\lambda_{U,J}/8, (8I_J)^{-1}\}, & 8G_J T_+ &< r_0, \\ I_J Q_J (4\rho_J + 4\beta_{\text{rec},J}) &< r_{z,J}/16. \end{aligned} \quad (18.29)$$

We also require that the nonconstant part of the Hessian on this product satisfies the condition of Lemma 18.3(3) with the factor  $1/4$  in (18.5). By (18.25) this perturbation

tends to zero with  $T_+$ , so this is one more upper bound on the radii. All these inequalities can be satisfied: we first choose  $r_{z,J}$  to satisfy the two inequalities that involve it alone, with strict inequality, and then make the two parameter radii sufficiently small. We choose  $r_{a,J} < r_E$  larger than the bound on the image in (18.21) at  $T_+$ , which is possible by the inequality  $8G_J T_+ < r_0$ .

These choices hold on the open product defined by  $\|z\|_\infty < 2r_{z,J}$ ,  $\|c\|_\infty < 4\rho_J$ , and  $\|\xi\|_\infty < 4\beta_{\text{rec},J}$ , which contains the closed product of the statement. In particular the right side of (18.27) is at most  $\lambda_{U,J}/8 \leq \lambda_J/8$  there, the image lies in  $B_{r_{a,J}}$ , and (18.29) implies both inequalities on the parameter radii in (4). The inverse of the complex free Hessian is bounded by  $I_J$  at its smaller rate by the resolvent series (18.5), not by a positivity argument at complex data. This proves (3).

**Step 5: the constrained minimizer and its coefficients.** On the free box we use the map

$$\mathcal{T}_{c,\xi}(z) = z - (K_R^0)^{-1} F_{R,z}(c, z; \xi).$$

By (18.27) and (18.29), its derivative has maximal row norm at most  $1/8$ . By (18.28), its value at zero is at most  $I_J Q_J(\|c\|_\infty + \|\xi\|_\infty) < r_{z,J}/16$  on the larger parameter product. Thus it maps the closed box of radius  $r_{z,J}$  into itself, strictly. Starting at zero, its iterates satisfy  $\|z^{(n+1)} - z^{(n)}\|_\infty \leq 8^{-n} r_{z,J}/16$ . Their geometric sum converges to a fixed point in the box, and it is unique, since the distance between two fixed points would be at most one eighth of itself. Since  $\|z_R\|_\infty \leq \|\mathcal{T}_{c,\xi}(0)\|_\infty + \|z_R\|_\infty/8$ , we have

$$\|z_R\|_\infty \leq \frac{8}{7} I_J Q_J(\|c\|_\infty + \|\xi\|_\infty) < r_{z,J}/8,$$

which gives the stated  $C_J$ . The iterates are holomorphic, and their convergence is geometric and uniform on compact subsets of the product. By the Cauchy integral formula in each coordinate, the limit is holomorphic: the integral of the uniform tail tends to zero, also on each smaller contour used for derivatives.

For the absolute coefficient bounds with each index fixed, we use the majorant version of this iteration. We subtract the flat linear part. By (18.25), the nonlinear remainder is quadratic and has the majorant  $C_J(t+Y)^2/[1 - C_J(t+Y)]$ . The coefficients of the solution, and their versions with a specified input index, are therefore bounded by the majorant series  $Y$  with nonnegative coefficients determined by

$$Y = C_J t + \frac{1}{2} Y + \frac{C_J(t+Y)^2}{1 - C_J(t+Y)}. \quad (18.30)$$

All hypotheses of the implicit-series statement of Lemma 5.3 have now been checked: zero constant term, a bounded inverse of the linear part, tensor bounds with each index fixed, the bound  $1/8$  on the derivative, and a strictly larger product. The lemma therefore

gives factorial coefficient bounds on a smaller fixed product. Fixing a chosen current input index in the same majorant series, we obtain the bound on volume-normalized columns. One spatial rate is used throughout, with at most the single initial reduction for external trees described in Lemma 5.3.

At real data the objective and the chart are real. Complex conjugation maps a stationary point in the box to a stationary point in the same box, and uniqueness shows that  $z_R$  is real. Throughout the box the real Hessian is at least  $7\lambda_{U,J}I/8$ , by (18.27) and (18.29). For a real  $z$  in the convex box, by Taylor's formula at  $z_R$  with  $h = z - z_R$ , we have

$$F_R(c, z; \xi) - F_R(c, z_R; \xi) = \int_0^1 (1-t) h^\top F_{R,zz}(c, z_R + th; \xi) h dt \geq \frac{7\lambda_{U,J}}{16} \|h\|_2^2.$$

We conclude that  $z_R$  is the unique minimizer in the box. This proves (4).

**Step 6: domains for the conditional minimizers.** It remains to specify the domains of the conditional minimizers. Let  $I$  be an initial segment of the scales, write  $z = (u, y)$  with  $u = z_I$  and  $y = z_{I^c}$ , and let  $z^0 = z_R(c; \xi)$ . We use the parameters  $v = y - z_{I^c}^0$  and solve for  $h = u - z_I^0$ . The matrix  $A_I = F_{R,uu}(c, z^0; \xi)$  has a common inverse bound  $I'_I$ , by the estimates for principal blocks proved above. We choose a solution radius  $r_I > 0$  inside the larger domain, and then a smaller radius  $\eta_I > 0$  for the coarser parameters  $v$ , such that throughout the corresponding product

$$\begin{aligned} \|A_I^{-1}(F_{R,uu} - A_I)\|_{\text{coeff}} &\leq 1/4, \\ \|A_I^{-1}F_{R,u}(c, z_I^0, z_{I^c}^0 + v; \xi)\|_{\text{coeff}} &\leq r_I/8. \end{aligned} \tag{18.31}$$

The first condition holds after shrinking the product, by the convergent majorant of the third derivative. In the second, the gradient vanishes at  $v = 0$ , and its change is

$$\int_0^1 F_{R,uy}(c, z_I^0, z_{I^c}^0 + tv; \xi) v dt.$$

Its coefficient norm is at most  $Q'_J\|v\|$ , by the bounds on mixed derivatives, so  $I'_JQ'_J\eta_I \leq r_I/8$  suffices. The solution radius is chosen first and the parameter radius second; they are not equal.

The map  $h \mapsto h - A_I^{-1}F_{R,u}$  now has derivative at most  $1/4$  and value at most  $r_I/8$  at  $h = 0$ . It maps the box of radius  $r_I/2$  into itself, and the convergence and majorant arguments of Step 5 apply. The conditional minimizer  $a_I$  is holomorphic, real at real data, and satisfies

$$a_I(c, z_{I^c}^0; \xi) = z_I^0, \quad D_y a_I = -F_{R,uu}^{-1} F_{R,uy}.$$

Since there are at most  $J$  initial segments, their solution radii and parameter radii have a common positive minimum. For a sequential application we also choose a smaller radius for the fluctuation inputs. The finite composition of these conditional minimizers

and of their inverse square roots equals the joint minimizer when all the inputs  $g_i w_i$  vanish, and its deviation has a majorant  $C_{\text{seq},J} t / (1 - C_{\text{seq},J} t)$ . We choose  $r_q > 0$  so that this expression is less than a quarter of the least of the preceding solution and parameter radii. All sequential images then lie in the domains of the conditional minimizers. This is a choice for finitely many scales, independent of the number of sites. Lemma 18.8 identifies successive and joint minimizers there and gives the factors  $M^i$ . The real lower bounds for principal blocks and Schur complements persist there by (18.6), and their complex coefficient bounds follow from the Schur calculus of Lemma 18.3.

These choices also give the coefficient-image reserves required by Lemma 18.3: the raw reconstruction majorant in (18.21) stays inside the action radius, the implicit majorant (18.30) stays inside its solution product, and the sequential deviation majorant stays below a quarter of every stage margin. Choose fixed smaller coefficient products inside these larger products. The same common majorants and their distinguished-input derivatives converge there, and the finite minimum over the at most  $J$  stages gives a common positive image margin. This controls absolute coefficient substitutions, rather than just pointwise images. The conditions (18.31) are part of the smaller products just constructed. Every constant used to obtain them comes from the finite local formulas, the tensor bounds on the torus, the constants of the patch inequality, and the density bound. Taking maxima and minima over  $s \leq J$  and over initial segments therefore gives uniform constants, independent in particular of isolated final blocks, of the later widths, and of the values of the frozen coordinates. This completes the proof.  $\square$

## 18.7 The first block integration

At depth zero, we identify the completed objective with the part of the Wilson action that depends on  $(c, z)$ , and we give the exact form and range of its Hessian, uniformly in the arbitrary frozen group values deep in the fixed complement, as used in Section 21. The restriction and extraction maps are those of Lemma 18.11.

**Definition 18.13.** Use Definition 18.6 at the first block integration, and let  $\mathcal{E}(\Lambda)$  be the set of all positive input links.

1. *Dependencies.* Collect in a finite list all contours  $F_{i,c',x}$  with  $c' \in \mathcal{L}_{i+1}$ , their reference paths and tree paths, and the tree-path transports (18.11) to the basepoints of frozen coordinates on links across the boundary, with the edges of the coarser scales expanded recursively through the definition of  $T_M$ . The formulas of the local constraint coordinates and of the Haar disintegration densities are included in this list. Let  $\mathcal{E}_\Omega^{\text{dep}}$  be the union of the input links on which these formulas depend. If a further formula at the boundary is to be used analytically, its input links are added to this list before the near links are fixed.

2. *Near and deep links.* Let  $\mathcal{P}_{\text{inc}}$  be the set of plaquettes  $p$  with at least one edge in  $\mathcal{L}_0$ , and set

$$\mathcal{E}_{\Omega}^{\text{near}} = \left( \mathcal{E}_{\Omega}^{\text{dep}} \cup \bigcup_{p \in \mathcal{P}_{\text{inc}}} \partial p \right) \setminus \mathcal{L}_0, \quad \mathcal{E}_{\Omega}^{\text{deep}} = \mathcal{E}(\Lambda) \setminus (\mathcal{L}_0 \cup \mathcal{E}_{\Omega}^{\text{near}}). \quad (18.32)$$

Further links may be added to  $\mathcal{E}_{\Omega}^{\text{near}}$ ; none is needed below.

3. *Near values.* Assume that every near link carries a value in the identity chart. The near values  $b^{\text{near}}$  are independent small group values on  $\mathcal{E}_{\Omega}^{\text{near}}$ : on a link with both endpoints outside  $\Omega_0$  the value itself, and on a link across the boundary its based transport. If a link  $(x, y)$  across the boundary is first given by its value  $a_{xy}$ , we set  $b_e = \Theta_x(U)a_{xy}$  at fixed field of the block region; this is the Haar translation of Lemma 18.7, and all indicators are kept under this substitution. The logarithms of the near values are denoted  $\xi$ .
4. *Completion.* For a link  $e = (x, y)$  across the boundary, with  $x \in \Omega_0$ ,  $y \notin \Omega_0$ , and based transport  $b_e$ , set

$$U_{xy}^{\text{act}} = \Theta_x(\Phi_{\Omega})^{-1}b_e; \quad (18.33)$$

for a positive link whose endpoint in  $\Omega_0$  is its terminal endpoint, we apply this rule to the reversed link and invert. On the remaining links we set

$$U_e^{\text{act}}(c, z; \xi, \eta) = \begin{cases} \Phi_{\Omega, e}(c, z), & e \in \mathcal{L}_0, \\ b_e, & e \in \mathcal{E}_{\Omega}^{\text{near}} \text{ with both endpoints outside } \Omega_0, \\ \eta_e, & e \in \mathcal{E}_{\Omega}^{\text{deep}}, \end{cases} \quad (18.34)$$

where  $\eta \in \text{SU}(2)^{\mathcal{E}_{\Omega}^{\text{deep}}}$  are the frozen group variables in the fixed complement on the deep links (the deep frozen group variables); they are arbitrary real group values, not small analytic logarithms. The analytic completion  $U^{\text{an}}(c, z; \xi)$  is defined by the same rules with  $\eta_e = I$  on the deep links, and we write  $\Psi_R = \text{Log } U^{\text{an}}$ , componentwise on the chart.

5. *Varying plaquettes.* Let  $f_p(c, z; \xi) = 4q(U_p^{\text{an}}(c, z; \xi))$ , and

$$\mathcal{P}_{\text{min}} = \{p \in \mathcal{P}_{\text{inc}} : f_p(\cdot, \cdot; \xi) \text{ is nonconstant in } (c, z) \text{ for some small real } \xi\}. \quad (18.35)$$

Fix a list  $\mathcal{P}_{\text{var}}$  with  $\mathcal{P}_{\text{min}} \subseteq \mathcal{P}_{\text{var}} \subseteq \mathcal{P}_{\text{inc}}$ , and define the objective of the first block integration

$$F_{\text{bare}}(c, z; \xi) = \sum_{p \in \mathcal{P}_{\text{var}}} f_p(c, z; \xi). \quad (18.36)$$

Every link across the boundary is a near link: if  $x \in \Omega_0$  is the endpoint of such a link in direction  $k$ , a step from  $x$  in a transverse direction toward the interior of the final block of  $x$  gives a plaquette that contains the link and an edge of  $\mathcal{L}_0$ , since a final block has

at least three vertices per side. This holds also at corners and at periodic seams. The requirement that the near values lie in the identity chart is the condition imposed on the protected neighborhoods in Section 21; the conversion between the value  $a_{xy}$  and the based transport  $b_e$  is a finite group product, whose smallness is also controlled there. Since  $\Theta_x(\Phi_\Omega(c, z)) = I$ , the value (18.33) of such a link equals  $b_e$  on the reconstruction. Hence, at fixed frozen coordinates, the derivatives in  $(c, z)$  of all links outside  $\Omega$  vanish. Completion by zero means logarithm zero, that is, the group value  $I$ ; it does not mean solving the block averaging constraints of the torus with the missing targets set to zero. Coarse outputs on bonds across the boundary of the block region are evaluated by  $T_M$  on the completion, and they are not imposed as additional equations. Exceptional values of the Borel extension of  $T_M$  at the deep frozen group variables are kept in the measure, and no logarithm of such a value enters  $U^{\text{an}}$ .

The list  $\mathcal{P}_{\min}$  is fixed on the chart, not chosen anew at each value of the frozen coordinates. A plaquette function outside  $\mathcal{P}_{\min}$  is also independent of the complex variables: its derivatives in  $(c, z)$  vanish on the real product, hence, by the identity theorem applied successively in each scalar coordinate, on the complex product. If the split is made before some internal constraint coordinates are eliminated, the list of varying plaquettes may be larger than  $\mathcal{P}_{\min}$ . All conclusions below hold for every fixed list between  $\mathcal{P}_{\min}$  and  $\mathcal{P}_{\text{inc}}$ .

For the statement of the next lemma, let  $d_{\text{var}}u = ((du)_p)_{p \in \mathcal{P}_{\text{var}}}$ , let  $\iota_\Omega$  be the extension by zero from edge cochains on  $\mathcal{L}_0$  to all input links, and let  $\hat{A}_\Omega = \iota_\Omega A_\Omega$ , with  $A_\Omega$  as in (18.15). For a patch  $P$  define a linear map from cochains on the varying plaquettes to cochains on the plaquettes of  $P$  by

$$(R_P^{(2)}v)_p = \begin{cases} v_p, & p \in \mathcal{P}(P) \cap \mathcal{P}_{\text{var}}, \\ 0, & p \in \mathcal{P}(P) \setminus \mathcal{P}_{\text{var}}. \end{cases} \quad (18.37)$$

**Lemma 18.14.** *Use Definition 18.13. The following statements hold.*

1. *For  $p \in \mathcal{P}_{\text{inc}}$ , the plaquette variables of  $U^{\text{act}}$  and  $U^{\text{an}}$  agree for every  $\eta$ . The sum of the Wilson terms of the plaquettes not in  $\mathcal{P}_{\text{var}}$ , evaluated at  $U^{\text{act}}$ , is independent of  $(c, z)$ ; denote it by  $E_{\text{hole}}(\xi, \eta)$ . Then*

$$S_W(U^{\text{act}}) = F_{\text{bare}}(c, z; \xi) + E_{\text{hole}}(\xi, \eta) \quad (18.38)$$

and

$$E_0(\Psi_R(c, z; \xi)) = F_{\text{bare}}(c, z; \xi) + C_\Omega(\xi), \quad C_\Omega(\xi) = \sum_{p \notin \mathcal{P}_{\text{var}}} f_p(0, 0; \xi), \quad (18.39)$$

where  $E_0 = S_W$  is the Wilson action. The function  $C_\Omega$  is not the energy of the deep frozen group variables. Every derivative that involves  $(c, z)$  is the same on the two sides of (18.39).

2. Let  $u = D_z \text{Log } U^{\text{an}}(0, 0; 0)z$  and  $K_\Omega = \partial_z^2 F_{\text{bare}}(0, 0; 0)$ . Only the variables of the block region and the near values vanish here; the deep frozen group variables  $\eta$  are arbitrary, and the same  $u$  on the varying plaquettes and the same  $K_\Omega$  are obtained from  $U^{\text{act}}$ , independently of  $\eta$ . Then  $u = \hat{A}_\Omega z$ , and for every patch  $P$ ,

$$R_P u \in \mathcal{V}_P, \quad d_P R_P u = R_P^{(2)} d_{\text{var}} u. \quad (18.40)$$

Further,

$$\|u\|_{\mathcal{L}_0}^2 \leq \sum_{P \in \mathfrak{P}_\Omega} \|R_P u\|^2, \quad \sum_{P \in \mathfrak{P}_\Omega} \|R_P^{(2)} d_{\text{var}} u\|^2 \leq 7 \|d_{\text{var}} u\|^2, \quad (18.41)$$

$$\|d_{\text{var}} u\|^2 \geq \frac{\sigma_J}{7} \|u\|_{\mathcal{L}_0}^2, \quad z = B_{\text{ext}} u|_{\mathcal{L}_0}, \quad \|z\|_2 \leq C_{\text{ext}, J} \|u\|_{\mathcal{L}_0}, \quad (18.42)$$

with  $\sigma_J$ ,  $B_{\text{ext}}$ , and  $C_{\text{ext}, J}$  as in Lemma 18.11.

3. We have

$$K_\Omega = \hat{A}_\Omega^T d_{\text{var}}^T d_{\text{var}} \hat{A}_\Omega, \quad \lambda_{M, J} I \leq K_\Omega \leq Q_{M, J} I, \quad \lambda_{M, J} = \frac{\sigma_J}{7 C_{\text{ext}, J}^2}, \quad (18.43)$$

where  $Q_{M, J} = 16a_1 a_\infty$  with  $a_\infty = J(1 + 4M^3)$  and  $a_1 = M^{2(J-1)}(1 + 24M^3)$ . In positions measured in output units,

$$(K_\Omega)_{ab} = 0 \quad \text{if } \text{dist}_\infty(a, b) > D_{M, J} = 2R_{M, J} + 2. \quad (18.44)$$

Principal submatrices of  $K_\Omega$  satisfy the same lower bound, the same upper bound, and the same range condition. Real Schur complements of  $K_\Omega$  are bounded below by  $\lambda_{M, J}$  and have uniform exponentially weighted row and column bounds; no finite range is asserted for them.

4. There is a chart radius  $\epsilon_{M, J} > 0$ , depending in addition only on a fixed choice of the local analytic chart, such that for real  $(c, z, \xi)$  with  $\max(\|c\|_\infty, \|z\|_\infty, \|\xi\|_\infty) \leq \epsilon_{M, J}$ ,

$$\partial_z^2 F_{\text{bare}}(c, z; \xi) \geq \frac{1}{2} \lambda_{M, J} I. \quad (18.45)$$

All constants depend only on  $M$ ,  $J$ , and the coordinate conventions, and are independent of  $m$ , of the block region, of its number of components and holes, of its topology, of the widths of the protected neighborhoods, and of the deep frozen group variables. No uniformity as  $J \rightarrow \infty$  is claimed. For an empty list of free coordinates the statements on matrices and on the extraction hold with the conventions for zero-dimensional spaces. By (18.39),  $K_\Omega$  is the matrix  $K_R^0$  of Theorem 18.12 at depth zero, where (18.20) holds with  $\kappa_* = 1$ , and (18.43) is the corresponding case of that theorem, with  $v_J = \sigma_J/7$ .

*Proof.* We begin with (1). Let  $p \in \mathcal{P}_{\text{inc}}$ . By (18.32), each edge of  $p$  lies in  $\mathcal{L}_0$  or in  $\mathcal{E}_\Omega^{\text{near}}$ , so the deep values do not enter  $U_p^{\text{act}}$ , and  $U_p^{\text{act}} = U_p^{\text{an}}$  for every  $\eta$ . A plaquette outside  $\mathcal{P}_{\text{inc}}$  has no edge in  $\mathcal{L}_0$ . On the reconstruction its edges are constant in  $(c, z)$ : links across

the boundary are equal to  $b_e$ , other near links to  $b_e$ , and deep links to  $\eta_e$ . Its Wilson term is therefore independent of  $(c, z)$ . A plaquette in  $\mathcal{P}_{\text{inc}} \setminus \mathcal{P}_{\text{var}}$  is not in  $\mathcal{P}_{\text{min}}$ , so its term is independent of  $(c, z)$  as well. This proves that  $E_{\text{hole}}$  is independent of  $(c, z)$ , and (18.38) follows by splitting the sum over all plaquettes. The same argument for  $U^{\text{an}}$  shows that  $E_0(\Psi_R) = F_{\text{bare}} + \sum_{p \notin \mathcal{P}_{\text{var}}} f_p(c, z; \xi)$  with each summand independent of  $(c, z)$  and hence equal to  $f_p(0, 0; \xi)$ . This proves (18.39). If the list  $\mathcal{P}_{\text{var}}$  contains additional plaquettes whose terms are constant in  $(c, z)$ , only the complementary constants change.

Next we prove (2). On  $\mathcal{L}_0$  the derivative  $u$  equals  $A_\Omega z$ , by Lemma 18.11(1). The links outside  $\Omega$  are constant in  $z$  at zero near-link logarithms and fixed deep frozen group variables, as noted after Definition 18.13, so  $u = \hat{A}_\Omega z$ . The first assertion of (18.40) is Lemma 18.11(3). Let  $p \in \mathcal{P}(P)$ . All four edges of  $p$  belong to  $\mathcal{L}_0$ , so  $p \in \mathcal{P}_{\text{inc}}$ , and the link variables of  $U^{\text{act}}$  and  $U^{\text{an}}$  on  $p$  are equal, independently of the deep frozen group variables. At  $(c, z, \xi) = 0$  all these links are equal to  $I$ . The Wilson expansion of a single term is

$$f_p(0, tz; 0) = \frac{1}{2} t^2 |(du)_p|^2 + O(t^3). \quad (18.46)$$

Indeed, the first derivative of the plaquette product is the signed sum of the derivatives of its four edge logarithms, and  $4q(e^X) = |X|^2/2 + O(|X|^4)$ . If  $p \notin \mathcal{P}_{\text{var}}$ , then  $f_p(0, tz; 0)$  is constant, equal to its value zero at  $t = 0$ , and (18.46) implies  $(du)_p = 0$ . This holds for every  $u \in \ker G_\Omega$ , since  $A_\Omega$  maps onto  $\ker G_\Omega$  by Lemma 18.11(1); it does not assert that every plaquette of a patch is a varying plaquette. On a varying plaquette of  $P$  the entries of  $d_p R_p u$  are those of  $du$ , and on the other plaquettes of  $P$  both sides vanish. This proves the second assertion of (18.40).

The first inequality of (18.41) is the first inequality of (18.17). The second follows since every plaquette lies in at most seven of the sets  $\mathcal{P}(P)$ , as shown in the proof of Lemma 18.11. Using the bound  $\|h\|^2 \leq \sigma_J^{-1} \|d_p h\|^2$  on  $\mathcal{V}_P$  and (18.40), we obtain

$$\|u\|_{\mathcal{L}_0}^2 \leq \sum_P \|R_P u\|^2 \leq \sigma_J^{-1} \sum_P \|R_P^{(2)} d_{\text{var}} u\|^2 \leq \frac{7}{\sigma_J} \|d_{\text{var}} u\|^2.$$

Together with Lemma 18.11(1) and (2), this proves (18.42).

For (3), all varying plaquettes have only internal or near links, by (18.32). At their common zero they are identity plaquettes, even when the deep links are arbitrary. Summing (18.46) over the varying plaquettes proves the formula for  $K_\Omega$  in (18.43). In particular, the term in the Hessian of a composition that involves the gradient of the action vanishes here, since each plaquette term has zero first derivative at the identity; it is not being dropped at a nonflat field. The lower bound follows from (18.42): we have  $z^\top K_\Omega z = \|d_{\text{var}} u\|^2 \geq (\sigma_J/7) \|u\|_{\mathcal{L}_0}^2 \geq \lambda_{M,J} \|z\|_2^2$ .

For the upper bound and the range, note that each row of the curl matrix has four entries, and each positive edge lies in four plaquettes, so its row and column norms are at most 4. Using Lemma 18.11(2) for  $A_\Omega$ , we find that  $16a_1 a_\infty$  bounds both the operator

norm and the unweighted absolute row and column norms of  $K_\Omega$ . If  $(K_\Omega)_{ab} \neq 0$ , then two edges in the supports of the columns  $a$  and  $b$  of  $A_\Omega$  belong to a common plaquette, so their midpoints are within maximum distance 2 in input units. Since each column of  $A_\Omega$  is supported within maximum distance  $R_{M,J}$  of its label, this proves (18.44). This is a finite range after the elimination of the trees and of the eliminated edges, before any minimization over a whole scale; no orthonormal basis of  $\ker G_\Omega$  was used as coordinates.

We record the cases at the boundary. If a final block of  $\Omega$  is next to the fixed complement, all trees of that block and all equations used for its restriction still lie in the complete block, and a pair is used only when both of its blocks lie in  $\Omega$ . No missing target of a link across the boundary is set to zero: such a link is fixed through (18.33), and its tree-path transport to the basepoint is identically  $I$  on the reconstruction. Plaquettes that contain an internal link and a near link belong to  $\mathcal{P}_{\text{inc}}$ , have only small links, and contribute nonnegative squares of curls at zero; the estimate above is already obtained from the plaquettes of the patches. If a plaquette of a patch has become constant in  $(c, z)$  after the elimination of the constraints, (18.46) shows that its curl vanishes, so no missing energy term is needed, and no boundary condition is imposed on a face of a hole.

At the boundary between near and deep links, a plaquette containing a deep link cannot contain a link of  $\mathcal{L}_0$ , by (18.32). All its links are constant on the reconstruction, and it belongs to  $E_{\text{hole}}$  even when some of its other links are near links. Likewise, a deep link occurs in none of the formulas for the coordinates and the Haar disintegration densities. Changing  $\eta$  therefore changes neither  $A_\Omega$ ,  $B_{\text{ext}}$ ,  $F_{\text{bare}}$ , nor any of their derivatives in  $(c, z)$ . There is no smallness or branch condition on  $\eta$ , and no derivative in  $\eta$  is taken. In particular, an exceptional value of the Borel averaging map at the deep frozen group variables cannot affect an internal formula or the matrices above.

Next we treat principal submatrices and Schur complements. Principal submatrices keep the lower bound, the upper bound, and the range. For a decomposition  $K = \begin{pmatrix} K_{II} & K_{IR} \\ K_{RI} & K_{RR} \end{pmatrix}$  of a matrix with  $\lambda I \leq K \leq QI$ , the real Schur complement satisfies

$$\lambda \|y\|^2 \leq y^\top (K_{RR} - K_{RI} K_{II}^{-1} K_{IR}) y = \min_x (x, y)^\top K (x, y) \leq Q \|y\|^2. \quad (18.47)$$

The lower inequality uses  $\lambda(\|x\|^2 + \|y\|^2) \geq \lambda\|y\|^2$ , and the upper one the choice  $x = 0$ . This proves the lower bound for the Schur complement of every initial segment of the scales. The factors  $M^i$  of the effective classical actions multiply whole quadratic forms and lie between 1 and  $M^J$ .

We now prove locality without asserting a finite range for Schur complements. In output units, the number of scalar labels in a ball of radius  $r$  is at most  $v_{M,J}(r+2)^3$  for a fixed  $v_{M,J}$ . Indeed, there are at most  $J$  scales and at most  $9M^{3J}$  scalar edge labels per scale and per unit cube of the output lattice, with a fixed enlargement for the boundaries of the cubes. For a principal matrix  $K$  of  $K_\Omega$  let  $a = 2Q_{M,J}$  and  $T = I - K/a$ . Its operator norm is at most  $q = 1 - \lambda_{M,J}/a < 1$ . The matrix  $T^k$  has range  $kD_{M,J}$  in the maximum

distance, hence range at most  $3kD_{M,J}$  in the graph distance  $\text{dist}$ , its entries are bounded by  $q^k$  in absolute value, and its rows and columns are supported on at most  $C_{M,J}(k+1)^3$  labels. For  $b > 0$  so small that  $qe^{3bD_{M,J}} < 1$ , the series

$$K^{-1} = a^{-1} \sum_{k \geq 0} T^k, \quad K^{-1/2} = a^{-1/2} \sum_{k \geq 0} \binom{2k}{k} 4^{-k} T^k \quad (18.48)$$

converge in the norm  $\|\cdot\|_b$  of Lemma 18.3(3), uniformly, since their scalar majorants are constant multiples of  $\sum_{k \geq 0} (k+1)^3 (qe^{3bD_{M,J}})^k$ ; the second series is the binomial series of  $(1-x)^{-1/2}$ , whose coefficients are at most one. The norm  $\|\cdot\|_b$  is submultiplicative by the triangle inequality for the distance, so a Schur complement  $S$  satisfies

$$\|S\|_b \leq \|K\|_b + \|K\|_b^2 \|K_I^{-1}\|_b, \quad S \geq \lambda_{M,J} I,$$

and these bounds can be repeated through the at most  $J$  eliminations of initial segments.

Finally we bound the inverse and the inverse square root of such an exponentially localized matrix directly. Suppose  $S \geq \lambda I$  and  $\|S\|_b \leq H$ . For a one-Lipschitz real function  $f$  and  $W_f = \text{diag}(e^{\mu f})$  with  $0 < \mu < b$ , we have

$$\|W_f S W_f^{-1} - S\|_2 \leq \frac{H\mu}{b-\mu},$$

by  $|e^t - 1| \leq |t|e^{|t|}$ , the inequality  $re^{-(b-\mu)r} \leq 1/(b-\mu)$ , the row and column bounds, and the Cauchy–Schwarz inequality. We choose  $\mu$  so that the right side is at most  $\lambda/2$ . The Neumann series about  $S + tI$  then bounds the conjugated resolvent by  $2/(\lambda + t)$  for every  $t \geq 0$ . Taking  $f = \text{dist}(\cdot, y)$ , we obtain

$$|(S + tI)_{xy}^{-1}| \leq \frac{2e^{-\mu \text{dist}(x,y)}}{\lambda + t}.$$

The density of labels sums this bound in the weighted norm at rate  $\mu/2$ . At  $t = 0$  it bounds  $S^{-1}$ , and the formula

$$S^{-1/2} = \frac{1}{\pi} \int_0^\infty t^{-1/2} (S + tI)^{-1} dt$$

bounds  $S^{-1/2}$ , since  $\int_0^\infty t^{-1/2} (\lambda + t)^{-1} dt = \pi/\sqrt{\lambda}$ . The integral formula follows by orthogonal diagonalization of the positive matrix and the substitution  $t = \lambda x^2$  in the scalar integral. This proves the assertions on Schur complements in (3), without a finite range for them.

It remains to prove (4). We fix local constraint coordinates of Lemma 6.3 on strictly smaller charts. Their finite-dimensional analytic bounds and (18.10) give a common complex product  $\max(\|c\|_\infty, \|z\|_\infty, \|\xi\|_\infty) < 2\delta$ , for some  $\delta > 0$  depending only on  $M, J$ ,

and the radii of these charts. To check that  $\delta$  does not depend on the region, we start at scale  $s$ , reduce its target and free radii so that its image lies strictly inside the target chart of the next scale, and repeat at most  $J$  times. Since the inverse at one scale depends on at most  $6M^3$  links, this finite sequence of choices does not depend on the number of blocks. The completion outside  $\Omega$  is by constants on the reconstruction.

On this product each  $f_p$  depends on at most  $N_*$  scalar labels, and each label occurs in the dependency lists of at most  $N_{\text{inc}}$  plaquettes; both numbers depend only on the finite stencils and on  $M$  and  $J$ . There are finitely many patterns of final blocks of  $\Omega$  in one stencil. Let  $C_3$  be the maximum, over these patterns and all scalar indices, of the absolute values of the third derivatives of a single  $f_p$  on the closed product of radius  $\delta$ . It is finite, since each local function is holomorphic on the larger product and the list of functions and of their arguments is finite; this also describes exactly how  $C_3$  depends on the chart. Integrating the third derivative from zero to  $x = (c, z, \xi)$  and summing over the second free index, we obtain

$$\|\partial_z^2 F_{\text{bare}}(x) - K_\Omega\|_\infty \leq C_{\text{lip}} \|x\|_\infty, \quad C_{\text{lip}} = N_{\text{inc}} N_*^2 C_3.$$

The column bound is the same by symmetry, so this is also a bound on the operator norm. The calculation differentiates the whole composition, including the term with the gradient of the action at nonzero  $x$ . We take  $\epsilon_{M,J} \leq \min(\delta, \lambda_{M,J} / (2 \max(1, C_{\text{lip}})))$ . For real data the Hessian is real symmetric, and (18.45) follows. All plaquette functions in  $F_{\text{bare}}$  are independent of  $\eta$  before differentiation, so the uniformity in the deep frozen group variables persists.  $\square$

## 18.8 Decay of connected boundary perturbations

We finally consider expressions, built from the objects of Theorem 18.12, that contain one perturbation located in the fixed complement or near the boundary: for example the difference between the reconstructions on  $\Omega$  and on the torus, or a bounded source in the fixed complement. A contraction network is a finite contraction, or a convergent sum of contractions, of the following tensors: the tensors of the chart and of the effective classical action, with coefficient bounds with each index fixed; inverses of constrained Hessians; at most  $J$  Schur complements; and the convergent Neumann and implicit-function series of these operations. Its coefficients are summed with the convergent coefficients of these operations, so a network may contain arbitrarily many inverse or implicit vertices. One constant cannot, however, bound expressions of unbounded algebraic complexity; multiplying an allowed expression by a large scalar already destroys any common bound. We therefore consider families of networks with one common scalar majorant. A perturbation factor has a stated bound on its amplitude. If it depends analytically on current variables, these variables occur among its external indices and

satisfy the coefficient bounds of the family; a fixed real source needs only its amplitude bound. The perturbation factor must lie in the same connected component of the network as the external index in the block region.

**Lemma 18.15.** *Use the products of Theorem 18.12. Let  $H$  be the fixed complement, let  $H_{L_J}$  be its enlargement by the fixed width  $L_J$ , and for  $b > 0$  put*

$$S_J(b) = Cv_J \sum_{r \geq 0} (r+1)^3 e^{-br} < \infty.$$

1. *On every periodic subset of labels we have  $\sup_x \sum_y e^{-b \text{dist}(x,y)} \leq S_J(b)$ . For  $0 < a < b$  and every position  $u$  in  $\Omega$ ,*

$$e^{a \text{dist}(u,W)} \sum_{h \in H_{L_J}} e^{-b \text{dist}(u,h)} \leq e^{aC_J} S_J(b-a).$$

2. *Let  $\mathfrak{N}$  be a family of connected contraction networks, each with at least one external index in  $\Omega$  and with a perturbation factor at a position of  $H_{L_J}$  in the same connected component. The perturbation may be a bounded source  $s_h$  at any position  $h$  of  $H_{L_J}$ , with  $|s_h| \leq s_*$ . Assume that for one fixed  $b_0 > 0$  the diagram sums of all members of  $\mathfrak{N}$  are bounded, on one common coefficient product and including all mesh-volume factors, by one common scalar majorant  $\mathcal{P}$  with nonnegative coefficients after insertion of the weight  $e^{b_0 \ell}$ . Here  $\ell$  is the sum of the lengths of the constituent connecting networks and propagator paths, including a route to the perturbation factor; the bound holds with every external index fixed in turn. Thus the majorant is a spatially weighted majorant of the full resolved contractions, not just an unweighted amplitude bound. It is required to converge on the common larger product, together with the distinguished-input majorants, and every analytic substitution retains the fixed absolute coefficient image margins of Lemma 18.3. Then there are  $c_J, C_J > 0$ , depending only on uniform data, on  $b_0$ , on  $\mathcal{P}$ , and on  $s_*$ , such that every tensor  $T_l$  of order  $l \geq 1$  in  $\mathfrak{N}$  with a specified external index  $i_1$  in  $\Omega$  satisfies*

$$\sup_{i_1 \in \Omega} e^{c_J \text{dist}(i_1, W)} \sum_{i_2, \dots, i_l} e^{c_J t(i_1, \dots, i_l)} |T_l(i_1, \dots, i_l)| \leq l! C_J^{l+1}.$$

*A single fixed reduction of the spatial rate suffices, independently of the Taylor order and of the number of contractions.*

3. *For a member of  $\mathfrak{N}$  with output index  $e$  and Taylor series  $T_e(v) = \sum_v f_{e;v} v^v$ , and an input index  $x$ , let*

$$\mathcal{C}_{ex} = \sum_{v: v_x > 0} v_x |f_{e;v}| r^{v-e_x}$$

*be the entrywise absolute coefficient envelope of  $\partial_x T_e$  at the radii  $r$  of a strictly smaller coefficient product. Let  $\mathcal{I}_{\text{an}}$  be the union of the unfixed analytic input labels and the current*

frozen-coordinate labels on their construction chart, and let  $v_e$  and  $v_x$  be the mesh volumes of the lattices on which  $e$  and  $x$  live. Then

$$\sup_{x \in \mathcal{I}_{\text{an}}} \sum_{e \in \Omega} \frac{v_e}{v_x} e^{c_J [\text{dist}(e, W) + \text{dist}(e, x)]} r_x \mathcal{C}_{ex} \leq C_J. \quad (18.49)$$

In particular, if  $x$  and  $e$  lie in  $\Omega$ , then  $\mathcal{C}_{ex} \leq C_J e^{-c_J [\text{dist}(e, W) + \text{dist}(e, x)]}$ , where the fixed conversions of radii and mesh volumes are absorbed in  $C_J$ . For two Hessian indices  $e$  and  $f$  in  $\Omega$ ,

$$|T_{ef}| \leq C_J e^{-c_J [\text{dist}(e, f) + \text{dist}(e, W) + \text{dist}(f, W)]},$$

with factorial joint versions that include every differentiated background and free index in the external network. These statements also apply to derivatives in the current frozen coordinates on their construction chart, at the density of their labels.

Traces confined to  $H$ , which have no external index in  $\Omega$ , remain sums over  $H$ ; no bound on them independent of the size of  $H$  is asserted. Differences of coordinates deep in  $H$  need only be bounded on their construction domain; they need not be exponentially small. The constants do not depend on the volume of  $H$ , on the area of  $W$ , on the shape or number of components of  $\Omega$ , on the number of preceding block integrations, or on the widths chosen later.

*Proof.* We begin with (1). The first bound is (18.4), with  $C = 8$ , possibly enlarged to account for a fixed change between positions of links and of their endpoints. It holds on every periodic subset, since restriction deletes terms from a sum of nonnegative terms. A shortest path from a position  $u$  in  $\Omega$  to a position in  $H$  crosses  $W$ . If  $h \in H_{L_J}$  lies in  $\Omega$ , we add a path of length at most  $L_J$  to  $H$  and then its crossing of  $W$ . In both cases

$$\text{dist}(u, W) \leq \text{dist}(u, h) + C_{\text{band}, J}, \quad (18.50)$$

where the fixed buffer for link endpoints is included in  $C_{\text{band}, J}$ . Multiplying termwise by  $e^{a \text{dist}(u, W)}$  and using (18.4), we obtain

$$e^{a \text{dist}(u, W)} \sum_{h \in H_{L_J}} e^{-b \text{dist}(u, h)} \leq e^{a C_{\text{band}, J}} \sum_h e^{-(b-a) \text{dist}(u, h)} \leq e^{a C_{\text{band}, J}} S_J(b-a).$$

This proves (1), after enlarging  $C_J$ . The bound sums over all of  $H$ , with no separate constant for each face of the boundary.

Next we record how a network acquires its distance to the boundary. For the coefficient of each factor we choose a connected network spanning its external indices and Taylor labels, with length as close as desired to its shortest connecting length, and for each matrix propagator a shortest path between its endpoints. Joining these networks at the contracted labels gives a connected network  $\mathcal{N}$  of total length  $\ell$ . If  $S$  is the set of its external positions, then

$$t(S) \leq \ell, \quad \text{dist}(i, W) \leq \ell + C_{\text{band}, J} \quad (i \in S \cap \Omega). \quad (18.51)$$

For the second inequality, we follow a route in  $\mathcal{N}$  from  $i$  to its perturbation vertex  $h \in H_{L_J}$  and use (18.50). The same route reaches a point of  $W$ , after a segment of length at most  $C_{\text{band},J}$  if necessary, so also  $t_W(S) \leq \ell + C_{\text{band},J}$ . Adding these inequalities, we obtain for all rates  $a, \alpha \geq 0$  and every external index  $i$  in  $\Omega$

$$e^{at(S) + \alpha \text{dist}(i,W)} \leq e^{\alpha C_{\text{band},J}} e^{(a+\alpha)\ell}, \quad e^{at_W(S)} \leq e^{a C_{\text{band},J}} e^{a\ell}. \quad (18.52)$$

This is how the two weights of Remark 18.2 are related: both are bounded by the weight of the total length of the joined network, at a fixed rate. For two external indices  $e$  and  $f$  in  $\Omega$  we have the three inequalities

$$\text{dist}(e, f) \leq \ell, \quad \text{dist}(e, W) \leq \ell + C_{\text{band},J}, \quad \text{dist}(f, W) \leq \ell + C_{\text{band},J} \quad (18.53)$$

simultaneously; they do not require a separate perturbation for each index. Summing them, we find, for every  $c \geq 0$ ,

$$e^{c[\text{dist}(e,f) + \text{dist}(e,W) + \text{dist}(f,W)]} \leq e^{2c C_{\text{band},J}} e^{3c\ell}. \quad (18.54)$$

We now divide the spatial rate. Use the rate  $b_0$  of the weighted majorant, decreased if necessary to a common rate for the coefficients of the factors, including the smaller rate of the inverse of the free Hessian. We take all contraction and implicit-series estimates first at the rate  $b_0/2$ , and we choose  $0 < c_J \leq b_0/32$ . A factor  $e^{b_0\ell}$  is available in the absolute product bounds at the rate  $b_0$ . We use at most  $e^{b_0\ell/4}$  of it for the desired external network and the distances in (18.51) and (18.53). For example, the network length plus one distance to the boundary is at most  $2\ell + C_{\text{band},J}$ , and the network length plus the three distances in (18.53) is at most  $4\ell + 2C_{\text{band},J}$ . The choice of  $c_J$  covers either sum and leaves a positive fixed fraction of  $b_0\ell$  for the internal summations. The fixed added lengths contribute at most the factor  $e^{2c_J C_{\text{band},J}}$ .

Next we give the summation. We orient a finite tree of contractions toward the specified external index and sum the labels on the outermost branches first, using the corresponding row bounds. On the unique branch containing a specified input index we use the input-column bound instead. The network weights multiply by (18.3), and Lemma 5.3 gives the product of the bounds for the factors, with the normalization of the columns unchanged. A redundant edge or a matrix cycle with a fixed external index is bounded by a further absolute row sum after cutting it at an attached vertex; alternatively, its propagators can be expanded into their convergent chains. All internal sums then use the same row and column bounds. In particular, a final sum over a bounded source  $s_h$ ,  $h \in H_{L_J}$ , uses

$$\sum_{h \in H_{L_J}} |s_h| e^{-u \text{dist}(i,h)} \leq \|s\|_\infty S_J(u),$$

or the weighted version proved in (1), rather than the bound  $|H| \|s\|_\infty$ . Only one source insertion is needed; further perturbation vertices already present in a coefficient do not require independent sums of this kind.

A fixed product has a finite scalar majorant obtained by multiplying the majorants of its factors, keeping the attachment and external-normalization mesh-volume ratios of Lemma 5.3. For a composition, finiteness follows from the stipulated common convergent majorants and absolute image margins of Lemma 18.3; separate factor bounds or a pointwise image inclusion would not suffice. Each inverse or resolvent series is an infinite sum of chains with geometric ratio at most  $1/4$  on the larger product, as in (18.5). The perturbation in an inverse difference stays in its single difference factor. A difference of Schur complements is the finite telescoping sum of Lemma 18.3, and at most  $J$  such operations have a fixed complexity. The common majorant  $\mathcal{P}$  bounds the number of summed chains and contractions. The division of the spatial rate is applied once to each complete joined network, not anew at each vertex of an infinite expansion.

For an implicit series we make this precise. Let  $Y(t)$  be the solution with nonnegative coefficients of (18.30). A derivative in a parameter that inserts the perturbation once gives an equation whose majorant satisfies

$$Z \preceq H(t, Y(t)) + A(t)Z, \quad A(t) = \frac{1}{2} + \partial_Y \frac{C_J(t+Y)^2}{1 - C_J(t+Y)} \Big|_{Y=Y(t)}. \quad (18.55)$$

Here  $\preceq$  denotes domination of coefficients, and  $H(t, Y)$  is the series with the source inserted (not to be confused with the fixed complement  $H$ ); its constant and higher coefficients have the bounds with a fixed index just proved. The implicit series is obtained by repeated substitution, so all its coefficients are bounded by those of

$$\sum_{n \geq 0} A(t)^n H(t, Y(t)) = \frac{H(t, Y(t))}{1 - A(t)}.$$

We choose a radius  $t_0 > 0$  strictly inside the radius of the solution such that  $A(t_0) \leq 3/4$ ; this is possible since  $Y(0) = 0$  and  $A(0) = 1/2$ . The series converges absolutely at  $t_0$ , and its value is at most  $4H(t_0, Y(t_0))$ . This is a bound on the value of a series with nonnegative coefficients; it is not the false coefficientwise inequality  $Z_l \leq 4H_l$ . A derivative in another analytic input differentiates this convergent series on a strictly smaller radius, where the additional geometric denominators are bounded; this gives the estimate with a specified input index. All indices are allowed, since all factor tensors and all inserted inverses were controlled with every index fixed.

For example, consider the interpolation from Step 3 of the proof of Theorem 18.12, and let  $d_a = \Psi_R - \Psi_T$  on the common list of link variables. Its changed formulas in  $\Omega$  are confined to the band of width  $L_J$ , and its values in the fixed complement are bounded on the construction domain. The difference of the two objectives is

$$\Delta F = \int_0^1 DE_{d_0}(\Psi_T + td_a)[d_a] dt + \frac{1}{2} \|z_H\|_2^2. \quad (18.56)$$

The segment of integration lies in the torus chart, by the choice of radii in that proof. If a derivative in a current or free variable acts on  $d_a$ , the changed formula keeps its perturbation. If it acts on the action or on another argument of the chart, the tensor of the action connects that derivative to the remaining position of  $d_a$ . The auxiliary quadratic form has only indices in the complement, and it reaches an index in  $\Omega$  only through a connected contraction. A derivative at a remote background position therefore cannot disconnect the perturbation. Differentiating the stationarity equation in the interpolation parameter, we obtain  $\dot{z} = -F_{zz}^{-1}(\Delta F)_z$ , which is an instance of (18.55). This verifies that the implicit networks used here have a perturbation connected to every external index, rather than an unrelated scalar factor.

To pass from convergence to the factorial bounds in (2), we fix a common radius  $t_1$  with  $0 < t_1 < t_0$  and let  $D_J$  bound the preceding majorants with a fixed index there, after the division of the spatial rate. If  $a_l$  denotes their Taylor coefficients of order  $l$ , normalized as in (18.2), nonnegativity implies  $a_l t_1^l \leq D_J$ . Hence the derivative tensors of order  $l$  are bounded by  $l! D_J t_1^{-l}$ . Taking  $C_J \geq \max(D_J, t_1^{-1}, 1)$  proves the bound in (2). Every degree sum and every supremum over the fixed index in this argument has been taken at fixed frozen coordinates, and the uniform bound permits the final supremum over their real values.

For (3), we keep both the input index  $x$  and the output index  $e$  in the complete network before summing the other leaves. The length then covers  $\text{dist}(e, W)$  and  $\text{dist}(e, x)$  at the rate  $C_J$ . For each differentiated occurrence we have  $\text{dist}(e, W) + \text{dist}(e, x) \leq 2\ell + C_{\text{band}, J}$ , by (18.51). Summing the differentiated majorant series on a smaller product over  $e$ , with the input column of  $x$  and its volume normalization, we obtain (18.49). Each entry is then at most  $(v_x/v_e)r_x^{-1}C_J e^{-C_J[\text{dist}(e, W) + \text{dist}(e, x)]}$ ; since at most  $J$  scales are involved and  $r_x$  is a fixed radius of the smaller product, this gives the entrywise bound after enlarging  $C_J$ . This bound includes the multiplicity of the input in the differentiated series; it is not only a bound on a first derivative at zero. For two Hessian indices we use (18.53). A background or free coefficient of order  $k$  is bounded by

$$\frac{(k+2)!}{k!} C_J^{k+3} R^k = C_J^3 (k+1)(k+2)(C_J R)^k,$$

whose sum over  $k$  is  $2C_J^3/(1 - C_J R)^3$  for  $C_J R < 1$ . The external network at this order includes both Hessian indices and all  $k$  further leaves. Reducing  $R$  by a fixed factor gives the bound on  $T_{ef}$  and its joint factorial versions. Derivatives in the current frozen coordinates use the same tensors and the same counting, at the density of the labels on the input lattice.

No step of the proof estimates a trace confined to the complement by fixing an index in  $\Omega$ , and no step requires decay of a coordinate difference at a position deep in the complement. For example, a prescribed small link  $\exp(\varepsilon\tau_a)$  in the complement can differ from the corresponding link  $I$  of the torus completion by the logarithm  $\varepsilon\tau_a$ ,

whatever its distance from  $W$ ; its effect at an index in  $\Omega$  is bounded instead by (18.50) and the convergent sum over sources. All choices used only fixed coefficient bounds, the density of labels, and fixed rates, so no volume, area, number of components, number of preceding block integrations, or later width enters  $c_J$  and  $C_J$ . The empty block region and the convention for a block region without boundary give the remaining degenerate cases.  $\square$

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