

Vertex operator algebra realizations of untwisted Haagerup–Izumi centers

David E. Evans

Andrew Riesen

23 October 2026

Abstract

For every odd integer $N \geq 3$, we construct an untwisted Haagerup–Izumi fusion category $\mathcal{H}_N$ on the negative spherical branch and a simple, self-contragredient, regular, C_2 -cofinite vertex operator algebra E_N of central charge $-16N$, together with an equivalence of modular fusion categories:

$$\mathrm{Rep}(E_N) \simeq Z(\mathcal{H}_N).$$

The VOA E_N is rational, has exactly $N^2 + 3$ irreducible modules, and is lower bounded and integer graded, with least conformal degree $-(N - 1)/2$. The proof begins with a CFT-type cyclic Virasoro extension of central charge $-5N$, whose regularity we establish. Two categorical condensations recover $\mathcal{H}_N$, and a finite extension by shifted lattice modules removes the auxiliary pointed factor. We identify the complete ribbon modular data of $Z(\mathcal{H}_N)$ with $\mathrm{MD}_-(\mathbb{Z}/N; \mathbb{Z}/(N^2 + 4), b \mapsto b^2/(N^2 + 4))$ in the notation of Gannon–Schopieray–Yadav, for every odd N .

1 Introduction and main theorem

The representation category of a rational vertex operator algebra, under the standard finiteness and duality hypotheses, records its fusion, braiding and conformal twists. The converse realization problem asks for a vertex operator algebra with prescribed categorical data. Haagerup–Izumi categories provide a family of test cases: their simple objects consist of a group of invertibles and a single orbit of noninvertibles, while their Drinfeld centers are modular tensor categories.

We construct vertex operator algebra realizations for every odd cyclic order $N \geq 3$, with the negative spherical dimension specified below. The first algebra, W_N , is a CFT-type extension of a tensor product of N irrational Virasoro algebras. Its representation category contains a nondegenerate pointed factor $\mathcal{P}_N$ of rank 2^N . We construct an untwisted cyclic HI category $\mathcal{H}_N$ and prove:

$$\mathrm{Rep}(W_N) \simeq \mathcal{P}_N \boxtimes Z(\mathcal{H}_N), \quad c(W_N) = -5N.$$

An explicit extension using shifted E_7 lattice modules cancels $\mathcal{P}_N$ and gives a second algebra E_N , with central charge $-16N$ and representation category $Z(\mathcal{H}_N)$. Both algebras are simple, self-contragredient, regular and C_2 -cofinite. Their grading conventions and complete invariants are stated in Theorem 1.2.

Two features of the proof are essential. First, the Virasoro factors are not rational, so regularity of their extension requires a separate argument. We prove C_2 -cofiniteness and split extensions of the vacuum before using modular traces. Second, a single conformal-weight residue does not distinguish all exceptional labels when $N^2 + 4$ is composite. Keeping the partial conformal weights resolves these labels and makes the branching calculation sufficient to exhaust the simple modules. The center equivalence then follows from two specified condensation algebras.

Origin of the construction. Evans [33, Section 6.2] proposed the cyclic Virasoro sum V_φ with parameters $\psi_i + \psi_{i-1}^{-1} = k_i$. For $(k_0, k_1, \dots, k_{N-1}) = (N+2, 2, \dots, 2)$, that sum is exactly the underlying frame module of W_N used here. The cited section presents a heuristic cyclic closure of the duality-wall gluing of Creutzig–Gaiotto [34]. The work below supplies the uniform VOA construction, regularity proof, module exhaustion and ribbon center identification for this cyclic family. The earlier physical proposals of Gang–Kim, Gang–Kim–Lee and Jeong–Sohn [37, 38, 39] provide the S-fold and character context; the hyperbolic-word description is developed in [42]. Flohr’s screened one-variable algebras and theta-difference characters [35, 36] are an earlier precedent for these methods. Their character calculations do not supply the N -variable regularity argument proved below.

Relation to categorical existence results. The categorical background includes Izumi’s endomorphism and tube algebras [1], the Evans–Gannon grafting of modular data [2, Section 3.3], and their near-group and nonunitary constructions [40, 3]. Existence for every odd cyclic order is already established in [4] and [5, Theorem A(1)]; the latter includes the pseudo-unitary Galois conjugate. Its Theorem 5.1 identifies the untwisted center modular data. Thus the result here is a VOA and condensation realization, rather than a new categorical existence claim.

We also identify the full modular data of our center with the negative cyclic class $(N^2 + 4)_1^-$ of Gannon–Schopieray–Yadav, including a bijection of simple labels. This is distinct from an equivalence with a prescribed category. At order fifteen, their examples have identical center modular data but are pairwise not categorically Morita equivalent [5, Corollary 5.9]. Our center equivalence retains the condensation algebras and their multiplication; the modular-data comparison is a subsequent theorem. The announced $c = 8$ Haagerup VOA [41] concerns a separate positive-central-charge realization.

1.1 Conventions and statement

Throughout, $N \geq 3$ is odd. All categories are over $\mathbb{C}$. A fusion category is finite, semisimple, rigid, and has simple unit. A modular tensor category carries its specified ribbon structure; dimensions below are spherical dimensions unless marked FPdim . For a fusion category $\mathcal{H}$, $Z(\mathcal{H})$ denotes its Drinfeld center with the ribbon structure induced by the stated spherical structure.

A VOA is lower bounded and integer graded, with finite-dimensional homogeneous spaces. It is of CFT type when its negative degrees vanish and its degree-zero space is $\mathbb{C}\mathbf{1}$. A VOA is *regular* when every weak module is a direct sum of irreducible ordinary modules. Regularity implies rationality, meaning complete reducibility of admissible modules, for both W_N and E_N . The regularity of E_N is proved directly, without assuming CFT type. We use C_2 -cofinite in the sense

$$C_2(V) = \text{span}_{\mathbb{C}}\{a_{-2}b : a, b \in V\}, \quad \dim V/C_2(V) < \infty.$$

This condition depends on the vertex operators, not on the choice of conformal vector. Self-contragredience means an isomorphism $V \cong V^\vee$ with the graded contragredient module for the stated conformal vector. The notation $\text{Rep}(V)$ denotes the category of grading-restricted generalized modules; for the regular VOAs considered here it consists of finite direct sums of ordinary simple modules. For a fusion category, its rank is the number of isomorphism classes of simple objects. We write $\boxtimes$ for Deligne products of categories and, with a VOA subscript when needed, for fusion products of modules.

Definition 1.1. An untwisted cyclic Haagerup–Izumi (HI) fusion category of odd order N has simple objects g and $g\rho$, $g \in G = \mathbb{Z}/N$, with:

$$gh = g + h, \quad g\rho = (-g)\rho, \quad \rho^2 = \mathbf{1} \oplus \bigoplus_{g \in G} g\rho,$$

and its pointed subcategory is tensor equivalent to Vec_G with trivial associator. This condition does not prescribe the remaining associator entries.

Set:

$$\mu = N^2 + 4, \quad t = \frac{\sqrt{\mu} - N}{2}, \quad \delta = t^{-1} = \frac{N + \sqrt{\mu}}{2}, \quad p_N = 2^N.$$

Thus $0 < t < 1$, $t(t + N) = 1$, and $\delta^2 = 1 + N\delta$. The two roots of the HI dimension equation $d^2 = 1 + Nd$ are $-t$ and δ . The term *negative spherical branch* refers to the root $-t$; it does not assert that both roots arise from different pivotal structures on one fixed fusion category. The symbol $\mathcal{P}'_N$ in the theorem denotes the centralizer of $\mathcal{P}_N$ in $\text{Rep}(W_N)$: its objects have trivial double braiding with every object of $\mathcal{P}_N$.

Theorem 1.2. *For every odd integer $N \geq 3$, the constructions in this paper give:*

1. *A simple, self-contragredient, regular, C_2 -cofinite VOA W_N of CFT type, with $c(W_N) = -5N$ and $(W_N)_1 = 0$. Its modular tensor category has rank $2^N(N^2 + 3)$ and contains a nondegenerate pointed ribbon subcategory $\mathcal{P}_N$ of rank 2^N .*
2. *An untwisted cyclic HI spherical fusion category $\mathcal{H}_N$, obtained by the specified two-stage condensation, with $\dim(g) = 1$ and $\dim(gp) = -t$, and a ribbon equivalence:*

$$\mathcal{P}'_N \simeq Z(\mathcal{H}_N).$$

The Drinfeld center $Z(\mathcal{H}_N)$ is a modular tensor category.

3. *A simple, self-contragredient, regular, C_2 -cofinite VOA E_N , given by the explicit finite extension in Theorem 31.1, with:*

$$c(E_N) = -16N, \quad \text{Rep}(E_N) \simeq Z(\mathcal{H}_N), \quad |\text{Irr}(E_N)| = N^2 + 3.$$

Its least conformal degree is exactly $-(N - 1)/2$.

4. *The normalized ribbon modular data of $\mathcal{P}'_N$, $Z(\mathcal{H}_N)$, and $\text{Rep}(E_N)$ are*

$$\mathcal{MD}_-(\mathbb{Z}/N; \mathbb{Z}/\mu, b \mapsto b^2/\mu), \quad \mu = N^2 + 4,$$

in the conventions of Gannon–Schopieray–Yadav [5, Appendix F.1]. The explicit bijection of labels and all entries of S and T^{rib} are given in Theorem 30.2.

Both VOAs are rational. The VOA E_N is lower bounded and integer graded, with finite-dimensional homogeneous spaces; its regularity is in the weak-module sense specified above. Its negative least degree excludes CFT type, but not C_2 -cofiniteness or self-contragredience.

The category $\mathcal{H}_N$ in part (2) is the one constructed by condensation; no uniqueness of an HI category with these fusion rules is assumed. Part (4) identifies its modular data, with the tensor unit fixed and $S_{1,1} = 1/\dim \mathcal{H}_N$. The negative spherical dimensions make these realizations nonunitary. For ordinary VOA characters, $T^{\text{an}} = e^{-2\pi ic/24} T^{\text{rib}}$; this scalar is accounted for in Section 30.

1.2 Frame and lattice notation

We collect the conventions used in the construction. The proof outline in Section 1.3 can be read independently of these formulas.

Indices i, j are cyclic modulo N unless a signed antiperiodic extension is stated. Put:

$$\psi_i = \frac{t + i + 1}{t + i}, \quad c_i = 13 - 6(\psi_i + \psi_i^{-1}), \quad h_i(R, S) = \frac{(\psi_i R - S)^2 - (\psi_i - 1)^2}{4\psi_i}.$$

Each ψ_i is positive and irrational. The key identities are:

$$\psi_i + \psi_{i-1}^{-1} = k_i, \quad k_0 = N + 2, \quad k_i = 2 \ (i > 0).$$

The frame is $\mathcal{B} = \bigotimes_i L(c_i, 0)$, and $L_i(R, S)$ abbreviates the simple Virasoro module of central charge c_i and highest weight $h_i(R, S)$. For positive integral R, S , it is the irreducible quotient at the degenerate Kac weight.

The positive cycle form $K = K_+$ is defined without an ambiguous small-cycle matrix convention by:

$$x^\dagger K_+ x = \sum_i (x_i - x_{i+1})^2 + Nx_0^2.$$

The antiperiodic form K_- replaces the closing difference $(x_{N-1} - x_0)^2$ by $(x_{N-1} + x_0)^2$. Their determinants are N^2 and μ . Use the even positive lattice $L = (\mathbb{Z}^N, 2K_+)$ with basis β_i .

Write $B = \sum_i \beta_i$, $\varepsilon = e^B$, and $D = L(-1)$. In character formulas, $q = e^{2\pi i \tau}$ and $q_i = e^{2\pi i \tau_i}$. Quantum-group parameters denoted by q are introduced locally; the parity rank is always p_N .

For $m \in \mathbb{Z}_{\geq 0}^N$, write:

$$M_m = \bigotimes_i L_i(2m_i + 1, 2m_{i+1} + 1), \quad R_j = M_{e_j}.$$

The state and module called C in the vacuum calculation correspond to $m = \mathbf{1}$, whereas $\mathcal{C} = \text{Rep}(W_N)$ is the full representation category. A proper interval containing site 0 is called *heavy*; an *ordinary* proper interval avoids 0. We also write $W = W_N$ when N is fixed. Bold $\mathbf{1}$ denotes the vacuum or the all-ones coordinate vector according to its ambient space; the distinction is explicit whenever both occur together.

The main categorical notation is collected here for reference.

Symbol	Meaning
$\mathcal{B}$	The tensor product of the N Virasoro frame algebras
A	The deformed lattice VOA, also its algebra object over W_N
$\mathcal{C}$	$\text{Rep}(W_N)$ after regularity is established
$P = (\mathbb{Z}/2)^N, p_N$	The parity-label group and its order 2^N
$\mathcal{P} = \mathcal{P}_N$	The pointed ribbon subcategory on the J_ε
$\mathcal{D} = \mathcal{P}'$	The centralizer of $\mathcal{P}$ in $\mathcal{C}$
$\mathcal{M} = \mathcal{D}_A$	Right A -modules in $\mathcal{D}$
$\mathcal{D}_A^{\text{loc}}$	The full subcategory of local A -modules
$\mathcal{H}_N$	The fusion category obtained by the second condensation

1.3 Structure of the proof

Regularity precedes the module classification and modular traces.

Construction (Part I). Cyclic Virasoro gluing gives W_N (Theorem 2.2). Theorem 3.1 proves C_2 -cofiniteness, and the explicit commuting lattice frame gives a conformal embedding (Theorem 5.1 and Corollary 10.5). Theorem 11.1 constructs the parity simple currents.

Regularity (Part II). Proposition 12.4 reduces vacuum extensions to lowest weights zero and one. Theorems 15.2 and 16.1 exclude these cases, making the vacuum projective. Lemma 17.1 then gives rigidity and semisimplicity, and Theorem 17.2 passes to all weak modules. The parity factorization follows in Section 18.

Modules and traces (Part III). We classify the periodic lattice sectors and prove the needed uniqueness results for rational frame sectors. Formal refined traces retain the joint partial weights and detect every exceptional label, including those not separated by a single conformal-weight residue when $N^2 + 4$ is composite.

Center identification and realization (Part IV). Exact branching and the dimension identity in Section 28 exhaust the simple modules. Two condensations use the lattice algebra A and the regular algebra D_H of a cyclic Lagrangian subgroup:

$$\mathcal{D} = \mathcal{P}'_N \quad \longmapsto \quad \mathcal{M} = \mathcal{D}_A \quad \longmapsto \quad \mathcal{H}_N = \mathcal{M}_{D_H}.$$

The second local category is trivial. Corollary 29.4 identifies the ribbon center; Section 30 computes its full modular data. Theorem 31.1 cancels the parity factor by an explicit E_7 extension and proves its finiteness, duality, weak-module regularity and grading assertions.

The appendices supply the interval and Zhu calculations and the symbolic frame certificate. Section 5 explains the reduction to that finite certificate. No theorem from a companion twisted-case manuscript is assumed.

AI disclosure. Generative AI was used extensively in author-directed proof exploration, symbolic calculations, code development, and drafting, and produced substantial portions of the initial arguments and exposition. The authors formulated the research objective, guided the mathematical strategy, and revised the arguments, notation, and presentation.

The authors take responsibility for the mathematical claims and final text. The accompanying calculations and code make the finite checks inspectable; their role in the proof is explained in the manuscript. Independent human verification of the complete argument remains pending. Corrections and feedback on the mathematics, exposition, attribution, and historical account are welcome.

Acknowledgments. The work of the first author was supported in part by a grant of access to OpenAI models through the ChatGPT for Academic Researchers programme.

Contents

1	Introduction and main theorem	1
I	Construction and lattice realization	
2	Cyclic gluing of generic Virasoro algebras	7
3	Uniform C_2 -cofiniteness	8
4	The ambient lattice and its conformal deformation	12
5	The uniform screening frame and its certificate	13
6	Fock modules and signed Kac labels	17
7	Simple generator modules for the deformed frame	18
8	Closure and classification of the frame socle	21
9	Nonzero self-couplings of all elementary generators	22
10	Recognizing the cyclic algebra from its generator socles	24
11	Parity simple currents	29
II	Vacuum extensions and regularity	
12	Reduction of vacuum extensions to lowest weights zero and one	31
13	Proper intervals and the cyclic first image	37
14	A second ground character from the lattice embedding	40
15	Lowest weight zero and the Zhu algebra	40
16	The closing-root square excludes weight-one extensions	42
17	Vacuum projectivity and regularity	43
18	The parity factor after regularity	44
III	Simple modules and modular traces	
19	Frame decomposition of the nonzero periodic lattice sectors	45
20	BPZ monodromy and simplicity of the periodic sectors	46
21	Frame semisimplicity in generic rational sectors	51
22	Multiplicity one and uniqueness in rational sectors	54
23	Refined modular traces after regularity	59
24	The multivariable theta identity and its Fourier transform	62
25	Exceptional modules for all nonzero discriminant labels	63
IV	Condensation and the Haagerup–Izumi center	
26	The lattice extension and its categorical index	65
27	Branching determines the rank and parity locality	67
28	Modular coefficients, multiplicities, and exhaustion	69
29	Reconstruction of the untwisted ribbon center	71
30	Modular data of the center	74
31	An explicit VOA realization of the parity centralizer	79
	Technical appendices	
A	Ordinary first images of proper cyclic intervals	83
B	Explicit cut polynomials	85
C	Maximum plateaux and the filtered Zhu span	87
D	Uniform invertibility of the all-cut Jacobian	88
E	Uniform elimination of the first-image obstruction	89
F	The weight-zero top has vacuum open-chain action	93
G	Completion of the weight-zero Zhu calculation	96
H	The heavy-root vacuum coefficient excludes a weight-one defect	97
I	The uniform symbolic frame certificate	100
	References	103

Part I

Construction and lattice realization

2 Cyclic gluing of generic Virasoro algebras

Fix an odd integer $N \geq 3$ and retain the parameters and Virasoro modules introduced in Section 1. Thus $\psi_i = (t + i + 1)/(t + i)$ and $t(t + N) = 1$. We construct W_N as an extension of the frame $\mathcal{B}$ by transporting a canonical diagonal algebra.

Lemma 2.1 (Signs of the quantum parameter on integer spins). *Let $q \in \mathbb{C}^\times$ be not a root of unity. The type-I integer-spin ribbon categories for q and $-q$ are ribbon equivalent. The corresponding category for q^{-1} is braid-reversed equivalent to that for q .*

Proof. Use $\Delta E = E \otimes K + 1 \otimes E$ and $\Delta F = F \otimes 1 + K^{-1} \otimes F$. The map from $U_{-q}(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)$ given by $K' \mapsto K$, $E' \mapsto -E$, and $F' \mapsto F$ is a Hopf isomorphism. On even highest weights it preserves the type-I condition. In the standard universal R -matrix its n th nilpotent coefficient is

$$q^{n(n-1)/2} \frac{(q - q^{-1})^n}{[n]_q!} E^n \otimes F^n.$$

Since $[n]_{-q} = (-1)^{n-1} [n]_q$, changing q to $-q$ changes the scalar coefficient by $(-1)^n$, which is cancelled by $E' \mapsto -E$. The diagonal factor $q^{H \otimes H/2}$ is unchanged on even weights: the product of two even weights divided by two is even. The ribbon twist is unchanged as well, since its exponent on highest weight $2m$ is $(2m)(2m + 2)/2 = 2m(m + 1)$. For inversion, the Hopf isomorphism $K' \mapsto K$, $E' \mapsto KF$, $F' \mapsto EK^{-1}$ from $U_{q^{-1}}$ to U_q carries the standard R -matrix to R_{21}^{-1} and the ribbon twist to its inverse; this follows term by term from the same R -matrix formula. These identities give the stated ribbon equivalences. $\square$

Theorem 2.2 (Cyclic gluing). *There is a simple VOA W_N of CFT type and central charge $-5N$, with frame decomposition:*

$$W_N = \bigoplus_{r_0, \dots, r_{N-1} \in 2\mathbb{Z}_{\geq 0} + 1} \bigotimes_{i=0}^{N-1} L_i(r_i, r_{i+1}).$$

It is self-contragredient, $(W_N)_1 = 0$, and $\dim(W_N)_2 = N(N + 1)/2$.

Proof. At an irrational Virasoro parameter, the category of finite-length C_1 -cofinite modules is a semisimple rigid vertex tensor category [7]. Its odd first-row and first-column subcategories are generic integer-spin quantum $\mathfrak{sl}_2$ ribbon categories [6, Theorem 1.1]. They centralize one another: the unique product is $L_i(r, s)$ and

$$h_i(r, s) - h_i(r, 1) - h_i(1, s) = -\frac{(r-1)(s-1)}{2},$$

so balancing gives trivial monodromy for odd r, s . Consequently, the tensor-product functor is braided on these odd-label subcategories. It maps distinct simple pairs bijectively to the odd-label simples, by the fusion rules in [7, Theorem 5.2.5].

The cyclic relations are

$$\psi_0 + \psi_{N-1}^{-1} = N + 2, \quad \psi_i + \psi_{i-1}^{-1} = 2 \quad (1 \leq i < N).$$

For the second equality, substitute the displayed fractions. For the first, use $1/t = t + N$ and $1/(t + N) = t$. The odd first-row category at site i is therefore braid-reversed equivalent to the

odd first-column category at site $i - 1$: their squared quantum parameters are mutual inverses. Lemma 2.1 removes the possible sign of a quantum parameter on integer spins. The twists agree with this reversal because, for odd r , the sum of the two paired conformal weights is an integer.

Let $\mathcal{R}$ be the product of these N odd first-row categories. Its canonical diagonal algebra in $\mathcal{R} \boxtimes \mathcal{R}^{\text{rev}}$ exists in the direct-sum completion and is simple, by [8, Theorem 3.5 and Remark 3.7]. Transport it through the preceding braid-reversed equivalences and reorder factors. This produces exactly the displayed multiplicity-free sum W_N , with nonzero multiplication in every allowed fusion channel. The odd row and column subcategories at each site centralize each other, so this transport is a braided equivalence onto the relevant full subcategory of the tensor-product Virasoro category.

To check grading restriction, write $r_i = 2m_i + 1$ with $m_i \geq 0$. The lowest weight of its summand is

$$h(\mathbf{m}) = \sum_{i=0}^{N-1} (m_i - m_{i+1})^2 + Nm_0(m_0 + 1). \quad (2.1)$$

It is a nonnegative integer, vanishing only at $\mathbf{m} = 0$. At bounded weight, m_0 is bounded and so are every successive difference and every m_i ; hence only finitely many summands contribute. Each has finite-dimensional graded pieces. The twist is the identity and the unit is injective. Theorem 5.7 of [8] therefore turns this commutative algebra into a conformal vertex algebra, and the grading calculation makes it a VOA. Simplicity follows either from simplicity of the canonical algebra, or by multiplying any nonzero simple summand in an ideal by its dual to reach the vacuum.

Summing the cyclic relations gives $\sum_i (\psi_i + \psi_i^{-1}) = 3N$, hence $\sum_i c_i = -5N$. There is no weight-one state: a nonconstant cyclic integer sequence has squared-difference sum at least two, while a nonzero constant sequence has $Nm_0(m_0 + 1) \geq 2N$. For weight two, $m_0 = 0$ and the sequence consists of ones on a nonempty interval among the $N - 1$ other sites and zeros elsewhere. There are $N(N - 1)/2$ such intervals. Together with the N Virasoro stress tensors this gives $N(N + 1)/2$. Finally a simple CFT-type VOA with zero weight-one space has a nondegenerate invariant bilinear form, so is self-contragredient. $\square$

3 Uniform C_2 -cofiniteness

We prove finiteness in two steps. First, the C_2 quotient is finite over the algebra generated by the stress tensors. Second, the level-three null relations transport the values of these generators around the cycle. The resulting monodromy forces every such value to vanish, so the stress-tensor algebra is itself finite-dimensional.

Theorem 3.1. *For every odd $N \geq 3$, the VOA W_N of Theorem 2.2 is C_2 -cofinite. It admits a finite set of strong generators of conformal weights at most $10N$.*

Finite generation over the stress tensors

Write $M_{\mathbf{m}}$ for the summand in (2.1), and $v_{\mathbf{m}}$ for its highest vector. Put

$$R = W_N/C_2(W_N), \quad x_i = [T_i], \quad R_{\text{st}} = \mathbb{C}[x_0, \dots, x_{N-1}] \subset R, \quad D_i = (T_i)_{(0)} = L_{-1}^{(i)}.$$

Here R_{st} denotes the image of the polynomial algebra. Each D_i preserves C_2 , since zero modes are derivations. In R , $D_i x_j = 0$, the modes $L_{-k}^{(i)}$ for $k \geq 3$ act by zero, and $L_{-2}^{(i)}$ acts by multiplication by x_i .

Lemma 3.2. *The algebra R is finite as a R_{st} -module.*

Proof. For a nonempty binary vector u with $m - u \geq 0$, the top-channel exponent is

$$\begin{aligned}\Delta &= h(m) - h(u) - h(m - u) \\ &= 2 \sum_i (u_i - u_{i+1})((m_i - u_i) - (m_{i+1} - u_{i+1})) + 2Nu_0(m_0 - u_0).\end{aligned}$$

Choose u to be a superlevel set of m . Each boundary contribution in the sum is $|m_i - m_{i+1}| - 1 \geq 0$. It is positive at a boundary gap of at least two. The last term is positive when $u_0 = 1$ and $m_0 \geq 2$. Consequently, whenever

$$m \notin \mathcal{L}_N := \{m : m_0 \leq 1, |m_i - m_{i+1}| \leq 1 \text{ for every } i\},$$

one can choose such a superlevel set with $\Delta > 0$. Both u and $m - u$ are then nonzero.

The coefficient $(v_u)_{(-\Delta-1)v_{m-u}}$ belongs to $C_2(W_N)$. Its projection to M_m is a nonzero multiple of v_m : the canonical multiplication has a nonzero top fusion channel, and its highest-input, highest-output coefficient is nonzero. To see the latter assertion directly, pair its coefficients with all descendants of the highest output vector. Virasoro Ward identities determine these pairings recursively from the leading highest-weight coefficient. If that coefficient vanished, nondegeneracy of the irreducible target’s contravariant form would make the entire intertwiner vanish. Every other summand in this coefficient has labels coordinatewise at most m and strictly smaller total label. Thus $[v_m]$ reduces to lower-label summands. Their images are stable under all D_i and x_i , so the entire image of M_m reduces as well. Induction on $\sum_i m_i$ leaves only $m \in \mathcal{L}_N$.

This is a finite set, since $m_i \leq 1 + \lfloor N/2 \rfloor$. In each surviving module put $r_i = 2m_i + 1$. The singular vector at site i has level $d_i = r_i r_{i+1}$ and can be normalized to have leading term $(L_{-1}^{(i)})^{d_i}$; see [7, Proposition 2.2.2]. Modulo C_2 , it gives a monic relation for D_i with coefficients in $\mathbb{C}[x_i]$. All remaining terms have smaller D_i -degree. Commutators with $L_{-2}^{(i)}$ vanish modulo C_2 . Hence the images

$$\prod_i D_i^{a_i} v_m, \quad m \in \mathcal{L}_N, \quad 0 \leq a_i < d_i,$$

generate R over R_{st} . □

For completeness, a smaller bound for strong generators follows directly from C_1 . If $\max m_i \geq 2$, let u be the indicator of one connected maximal plateau (all sites when m is constant). The same formula gives $\Delta \geq 0$, enough to put the coefficient in C_1 . The zero modes D_i preserve C_1 , while $L_{-k}^{(i)}$, $k \geq 2$, on a positive-weight state belongs to C_1 . Induction leaves binary u and the vacuum summand. The singular-vector bounds give

$$\dim W_N / C_1(W_N) \leq N + \sum_{u \in \{0,1\}^N} 9^{|u|} = N + 10^N.$$

For a binary summand the highest required weight is

$$h(u) + \sum_i (r_i r_{i+1} - 1) = 6|u| + 2 \sum_i u_i u_{i+1} + 2Nu_0 \leq 10N.$$

Lemma 3.3. *For every algebra homomorphism $\ell : R \rightarrow \mathbb{C}$ and every $v \in W_N$,*

$$\ell(D_i v) = 0 \quad \text{for all } i.$$

Proof. In characteristic zero, a derivation of a finite commutative R_{st} -algebra that annihilates R_{st} annihilates its reduced algebra. Indeed derivations preserve the nilradical and each minimal prime. On each integral component, pass to fraction fields; the resulting finite algebraic extension of the fraction field of the image of R_{st} is separable, so a derivation over that base field is zero. Apply this to D_i using Lemma 3.2. □

Thus separate Virasoro Ward identities can be evaluated at any point ℓ of the C_2 algebra, even though the VOA translation operator is the sum of the D_i .

A Laurent-polynomial lemma

For a positive irrational number a , write $h_a = 2a - 1$ and define

$$C_{a,X}(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4ah_a^2z^{-3} & 4aX + 2ah_az^{-2} & -4az^{-1} \end{pmatrix}.$$

This is the companion connection of

$$f''' + 4az^{-1}f'' - (4aX + 2ah_az^{-2})f' - 4ah_a^2z^{-3}f = 0. \quad (3.1)$$

Lemma 3.4. *Let $a > b > 0$ be irrational. If*

$$F' = (C_{a,X} \otimes I + I \otimes C_{b,Y})F$$

has a nonzero vector solution with Laurent-polynomial entries, then $aX = bY$.

Proof. First suppose $X, Y \neq 0$ and put $A = aX$, $B' = bY$. If $A \neq B'$, the constant coefficient matrix has a simple zero eigenvalue: its two constituent spectra are $\{0, \pm 2\sqrt{A}\}$ and $\{0, \pm 2\sqrt{B'}\}$. The right zero eigenvector is $e_0 \otimes e_0$. Comparing the highest Laurent power and projecting the next equation to the left zero eigenspace shows that this power must be zero. The residue matrix annihilates the right zero eigenvector. The formal solution with prescribed constant coefficient is therefore unique: the recursion is invertible off the zero eigenspace, and its scalar equation at order z^{-n-1} has coefficient $-n$ for $n \geq 1$.

The normalized scalar solution with zero exponential part is

$$f_{a,X}(z) = \sum_{n \geq 0} u_n z^{-2n}, \quad u_n = \frac{(a)_n (\frac{1}{2} - a)_n (1 - 2a)_n}{n! A^n}.$$

Substitution into (3.1) proves the recurrence; explicitly the coefficient factorization is

$$-2j(2j+1)(2j+2) + 8aj(2j+1) + 4ah_a j - 4ah_a^2 = -4(a+j)(2a-2j-1)(2a-j-1).$$

Consequently the first component of the unique tensor-product solution is $f_{a,X} f_{b,Y}$. None of its Pochhammer parameters is a nonpositive integer, and the gamma-function asymptotic gives

$$u_n = \frac{(n!)^2 n^{-3/2-2a}}{\Gamma(a)\Gamma(\frac{1}{2}-a)\Gamma(1-2a)A^n} (1 + O(n^{-1})). \quad (3.2)$$

Let v_n be the analogous coefficients for b, Y . Their convolution cannot terminate. The following estimates apply also when X and Y are complex. If $|A| < |B'|$, then

$$\sum_{j=0}^n u_{n-j} v_j = u_n (1 + o(1));$$

if $|A| > |B'|$, the analogous statement holds with v_n . For the first assertion, split the sum at $n/2$. After division by $|u_n|$, the terms with $1 \leq j \leq n/2$ are bounded by a constant times $\binom{n}{j}^{-2} (|A|/|B'|)^j$, because both power exponents in (3.2) are negative. Their sum is $O(n^{-2})$. The other half has the exponentially small factor $(|A|/|B'|)^{n/2}$ times a fixed power of n . Initial finitely many coefficients are absorbed in the constants.

If $|A| = |B'|$, split at $n/2$ and compare each half with its adjacent endpoint. For $1 \leq j \leq n/2$, the first half is bounded by $C|v_n| \binom{n}{j}^{-2}$ after absorbing the fixed negative powers of j and the

bounded ratio $(n-j)/n$; the other half is bounded by $C|u_n|\binom{n}{j}^{-2}$ after reversing j . Since $\sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{j}^{-2} = O(n^{-2})$, we obtain

$$\sum_{j=0}^n u_j v_{n-j} = u_n + v_n + O(n^{-2})(|u_n| + |v_n|).$$

Since $a > b$, $|u_n/v_n| = O(n^{-2(a-b)}) \rightarrow 0$, so again the convolution is nonzero for all sufficiently large n . Thus $f_{a,X}f_{b,Y}$ is not a Laurent polynomial, a contradiction.

If $X = 0$ and $Y \neq 0$, rescale the first-factor coordinates to $(f, zf', z^2 f'')$. Its connection becomes E_a/z , where

$$E_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 4ah_a^2 & 2ah_a & 2-4a \end{pmatrix}, \quad \text{Spec}(E_a) = \{2a, 1-2a, 2-4a\}.$$

The second constant matrix C_0 has a simple zero eigenspace. If the leading coefficient is $z^m v$, the highest-power equation places v in $\mathbb{C}^3 \otimes \ker C_0$. Projecting the next equation to this kernel gives $mv = (E_a \otimes 1)v$: the projected residue of the second factor is zero. Thus the integer m would be an eigenvalue of E_a , impossible because all three are irrational. The case $Y = 0$, $X \neq 0$ is identical with b . This proves the lemma. $\square$

Null vectors transport the stress values around the cycle

Fix an algebra homomorphism $\ell : R \rightarrow \mathbb{C}$ and an index i . Let $v = v_{e_i}$, with $(3, 1)$ label at site i , $(1, 3)$ at site $i-1$, and vacuum labels elsewhere. Set

$$a = \psi_i > 1, \quad b = \psi_{i-1}^{-1} \in (0, 1), \quad X = \ell(T_i), \quad Y = \ell(T_{i-1}),$$

and abbreviate the two translations by D_A, D_B . Define

$$F_{pq}(z) = \ell(Y(D_A^p D_B^q v, z)v), \quad 0 \leq p, q \leq 2.$$

Every entry is a Laurent polynomial with nonpositive powers: all positive-power coefficients belong to C_2 . Moreover $F_{00} \neq 0$, since its most singular coefficient is the nonzero vacuum pairing of v .

The level-three null relation is

$$D_A^3 v - 4aL_{-2}^A D_A v + 2ah_a L_{-3}^A v = 0,$$

and likewise for b . For any D_B -descendant u of v , normal ordering and Lemma 3.3 give

$$\begin{aligned} \ell(Y(L_{-2}^A u, z)v) &= X\ell(Y(u, z)v) - z^{-1}\ell(Y(D_A u, z)v) + h_a z^{-2}\ell(Y(u, z)v), \\ \ell(Y(L_{-3}^A u, z)v) &= z^{-2}\ell(Y(D_A u, z)v) - 2h_a z^{-3}\ell(Y(u, z)v). \end{aligned}$$

Indeed, on the left only L_{-2}^A survives; on the right only L_{-1}^A and L_0^A act nontrivially on the highest vector, and the L_{-1}^A action is moved to the inserted vector using $\ell D_A = 0$. The null relation therefore gives

$$F_{3q} = 4aX F_{1q} - 4az^{-1} F_{2q} + 2ah_a z^{-2} F_{1q} + 4ah_a^2 z^{-3} F_{0q},$$

with the analogous relation for F_{p3} . Total translation gives $F'_{pq} = F_{p+1,q} + F_{p,q+1}$. Thus the nine entries satisfy precisely the tensor-sum system of Lemma 3.4. It follows that

$$\psi_i \ell(T_i) = \psi_{i-1}^{-1} \ell(T_{i-1}) \quad \text{for every } i. \quad (3.3)$$

Finally,

$$\prod_i \psi_i = \frac{t+N}{t} = t^{-2}.$$

Iterating (3.3) gives $\ell(T_0) = t^4 \ell(T_0)$. Since $0 < t < 1$, every $\ell(T_i)$ is zero. By Lemma 3.2, R is a finitely generated commutative algebra; the Nullstellensatz makes each x_i nilpotent. Thus R_{st} is finite-dimensional, and the finite R_{st} -module R is finite-dimensional. This proves C_2 -cofiniteness and completes the proof of Theorem 3.1.

4 The ambient lattice and its conformal deformation

Choose orthogonal real currents A_i with $(A_i, A_j) = 2\psi_i \delta_{ij}$ and put

$$\beta_i = A_i - \psi_{i-1}^{-1} A_{i-1}, \quad L = \bigoplus_{i=0}^{N-1} \mathbb{Z} \beta_i, \quad B = \sum_i \beta_i.$$

The cyclic parameter relations give the Gram matrix $2K$, where

$$m^T K m = \sum_i (m_i - m_{i+1})^2 + N m_0^2. \quad (4.1)$$

It is positive definite. All lattice pairings are even, so we choose the trivial cocycle for exponential fields. Also

$$B = \sum_i \frac{\psi_i - 1}{\psi_i} A_i, \quad (B, B) = 2N.$$

Let D be the lattice translation operator and set

$$\varepsilon = e^B, \quad \omega_0 = \omega_L + \frac{1}{2} B(-2) \mathbf{1} = \sum_i \left(\frac{A_i(-1)^2}{4\psi_i} + \frac{\psi_i - 1}{2\psi_i} A_i(-2) \right) \mathbf{1}.$$

Proposition 4.1 (The ambient conformal vector). *For every $\lambda \in \mathbb{C}$, the vector $\omega_\lambda = \omega_0 + \lambda D^2 \varepsilon$ is a conformal vector on the simple lattice vertex algebra V_L , with central charge $-5N$. Its translation and grading operators agree with those of ω_0 . The grading is nonnegative, has finite-dimensional homogeneous spaces, and has weight-zero space $\mathbb{C} \mathbf{1} \oplus \mathbb{C} \varepsilon$. The resulting vertex operator algebra $A_N = (V_L, \omega_\lambda)$ is regular.*

Proof. The shift by $B/2$ changes the central charge to $N - 12(B/2, B/2) = -5N$. The ground weight of charge $k = \sum_i k_i \beta_i$ is

$$h_0(k) = k^T K k - N k_0 = \sum_i (k_i - k_{i+1})^2 + N k_0 (k_0 - 1). \quad (4.2)$$

For integral k this is a nonnegative even integer and vanishes only for $k = 0$ and $k = (1, \dots, 1)$. Positive definiteness gives finiteness at bounded weight. Thus ε is a primary of weight zero. The self-bracket of ε is zero, since $(B, B) = 2N > 0$. Writing $[\omega_{0z} \varepsilon] = D\varepsilon$, sesquilinearity and skew-symmetry give

$$[\omega_{0z} D^2 \varepsilon] + [D^2 \varepsilon_z \omega_0] = (D+z)^2 D\varepsilon - z^2 D\varepsilon = (D+2z) D^2 \varepsilon.$$

The self-bracket of $D^2 \varepsilon$ also vanishes. Hence ω_λ satisfies the Virasoro relation with the same central charge. The mode formula $(D^2 \varepsilon)_{(n)} = n(n-1) \varepsilon_{(n-2)}$ shows that the translation and grading operators are unchanged.

The positive-definite lattice VOA is regular [9, Theorem 3.16]. Weak modules depend only on the underlying vertex algebra, whose translation operator has not changed. Every weak module is therefore a direct sum of the standard lattice-coset modules. On each such module the new grading is the positive quadratic lattice energy minus the linear term $(B, k)/2$; it is bounded below, diagonalizable, and has finite-dimensional eigenspaces. The $D^2 \varepsilon$ term does not change this grading. Thus these simple modules are ordinary for ω_λ and A_N is regular. $\square$

5 The uniform screening frame and its certificate

We construct commuting Virasoro vectors in the ambient lattice algebra. Each vector has a correction linear in the coupling u ; their sum first changes at order u^N .

Put $r_i = t + i$, $p_i = (r_i + 1)^{-1}$, and

$$P_- = \prod_{i=0}^{N-1} r_i, \quad P_+ = \prod_{i=0}^{N-1} (r_i + 1), \quad s = (-1)^N, \quad C_N = \frac{s}{2}(P_- + P_+).$$

Thus $P_+/P_- = (t + N)/t$. Pairing $r_j r_{N-j} = 1 + j(N - j)$ gives the useful closed formula

$$C_N = -\frac{\sqrt{N^2 + 4}}{2} \prod_{j=1}^{(N-1)/2} (1 + j(N - j)) \neq 0.$$

Use the orthogonal Heisenberg currents $A_i = \sqrt{2\psi_i} H_i$, with $(A_i, A_j) = 2\psi_i \delta_{ij}$. For a charge $m = \sum m_i \beta_i$, set

$$p_i(m) = m_i - \frac{m_{i+1}}{\psi_i}, \quad h_i(m) = \psi_i p_i(m)^2 - (\psi_i - 1) p_i(m).$$

In these coordinates

$$T_i = \frac{A_i(-1)^2 \mathbf{1}}{4\psi_i} + \frac{\psi_i - 1}{2\psi_i} A_i(-2) \mathbf{1}, \quad h(m) = \sum_i h_i(m) = \sum_i (m_i - m_{i+1})^2 + N m_0(m_0 - 1).$$

All lattice pairings are even, so the exponential fields below can use the trivial lattice cocycle.

Theorem 5.1 (Uniform Virasoro frame). *For every odd $N \geq 3$, there are explicitly defined finite sums $\tilde{T}_i(u) \in V_L[u]$ satisfying*

$$[\tilde{T}_i(u)_z \tilde{T}_j(u)] = \delta_{ij}((D + 2z)\tilde{T}_i(u) + c_i z^3 \mathbf{1}/12), \quad \sum_i \tilde{T}_i(u) = \omega_0 + C_N u^N D^2 \varepsilon.$$

They have $\tilde{T}_i(0) = T_i$, and their degree-one corrections are

$$u(e^{\beta_i} - e^{\beta_{i+1}}).$$

Consequently, every nonzero $\lambda \in \mathbb{C}$ admits a conformal embedding

$$\bigotimes_i L(c_i, 0) \hookrightarrow (V_L, \omega_0 + \lambda D^2 \varepsilon).$$

This gives the Virasoro frame embedding. The extension of this map to W_N is established in Corollary 10.5.

Finite formulas for the frame

A proper cyclic interval is an oriented consecutive set $I = [a, \dots, b]$ of length $\ell < N$. Also write I for its binary charge vector. If $k \in I$, define

$$F_{I,k} = (-1)^{\ell-1} \prod_{\substack{j \in I \\ j <_I k}} r_j \prod_{\substack{j \in I \\ k \leq_I j <_I b}} (r_j + 1), \quad c_{i,I} = F_{I,i} - F_{I,i+1},$$

where $F_{I,k} = 0$ for $k \notin I$. Empty products are one and site indices in $c_{i,I}$ are cyclic. In particular $c_{i,\{a\}} = \delta_{ia} - \delta_{i,a-1}$ and $\sum_i c_{i,I} = 0$.

For the charge- B term write $a_j = A_j(-1)$, $b_j = A_j(-2)$ and define the quadratic oscillator polynomial S_i by its coefficients:

monomial	coefficient in S_i
b_i	$C_N p_i$
$b_j \ (j \neq i)$	0
a_i^2	$\frac{s}{2}(P_+ p_i - P_- r_i p_i^2)$
$a_j^2 \ (j < i)$	$-\frac{s P_+ r_j p_j}{2 r_i (r_i + 1)}$
$a_j^2 \ (j > i)$	$-\frac{s P_- r_j p_j}{2 r_i (r_i + 1)}$
$a_i a_j \ (j < i)$	$s P_+ p_i p_j$
$a_i a_j \ (j > i)$	$s P_- p_i p_j$
$a_j a_k \ (j \neq k, i \notin \{j, k\})$	0

The order $<$ in this table is the ordinary order $0 < 1 < \dots < N - 1$. Telescoping $1/[r_i(r_i + 1)] = 1/r_i - 1/(r_i + 1)$ gives

$$\sum_i S_i \varepsilon = C_N D^2 \varepsilon. \quad (5.1)$$

There are further scalar terms $d_{i,I} e^{B+I}$ for intervals $I \subseteq \{1, \dots, N - 1\}$. Define them in increasing order of $|I|$. In the already constructed partial frame at $u = 1$, let $q_{i,B+I}$ be the coefficient of $z e^{B+I}$ in

$$[\tilde{T}_{iz} \tilde{T}_i] - (D + 2z) \tilde{T}_i - c_i z^3 \mathbf{1}/12.$$

Set

$$d_{i,I} = -\frac{q_{i,B+I}}{2(h_i(B + I) - 1)}. \quad (5.2)$$

This is a finite, explicit recursion in lattice OPE coefficients. Its denominators never vanish, as checked below. The complete formula is

$$\tilde{T}_i(u) = T_i + \sum_{\text{proper cyclic } I} u^{|I|} c_{i,I} e^I + u^N S_i \varepsilon + \sum_{I \subseteq \{1, \dots, N-1\} \text{ interval}} u^{N+|I|} d_{i,I} e^{B+I}. \quad (5.3)$$

Proof and the complete charge bound

If a nonnegative charge has weight at most three, its weight is zero or two: the displayed formula for $h(m)$ is an even nonnegative integer. If $m_0 \geq 2$, then $h(m) \geq 2N \geq 6$. If $m_0 = 0$, squared differences totaling two force a binary interval avoiding zero. If $m_0 = 1$, the possibilities are B , a proper binary cyclic interval containing zero, or $B + I$ with I avoiding zero. Weight zero gives precisely $0, B$.

For the singular OPE of two weight-two fields, the coefficient of z^k has total weight $3 - k \leq 3$. The preceding list therefore exhausts every positive charge at which a frame equation can fail. Hence the finite ansatz (5.3) contains every possible correction.

For proper intervals, the only nonlinear contributions come from decompositions $I = J \sqcup K$ into adjacent intervals. For v, w in the oriented interval, direct product telescoping gives, when $v <_I w$,

$$\sum_{J|K} F_{J,v} F_{K,w} = F_{I,w} - F_{I,v}.$$

The sum is over cuts separating v and w . It is the elementary identity for $\prod(r_j + 1) - \prod r_j$, expanded by the place at which the factor is changed. The version with an extra momentum

factor is

$$\sum_{J|K} F_{J,v} F_{K,w} p_l(J) = \begin{cases} p_l(I)(F_{I,w} - F_{I,v}), & l <_I v, \\ -F_{I,v}/(r_l + 1), & v \leq_I l <_I w, \\ 0, & l \geq_I w. \end{cases}$$

Here sites are ordered along the interval with its left boundary; outside its support and boundary the expression is zero. For the reversed ordered pair, use $p_l(K) = p_l(I) - p_l(J)$.

Taking discrete differences in v, w produces exactly the two required coefficient identities

$$\begin{aligned} \sum_{J|K} (c_{i,J} c_{j,K} + c_{i,K} c_{j,J}) &= -h_i(I) c_{j,I} - h_j(I) c_{i,I} + 2\delta_{ij} c_{i,I}, \\ \sum_{J|K} (c_{i,J} c_{j,K} p_l(J) + c_{i,K} c_{j,J} p_l(K)) &= -c_{j,I} p_i(I) \delta_{li} + c_{i,I} p_j(I) \delta_{lj} \\ &\quad - c_{i,I} h_j(I) p_l(I) + \delta_{ij} c_{i,I} p_l(I). \end{aligned}$$

They are respectively the coefficients of ze^I and $A_l(-1)e^I$ in the frame equations. The singleton equations hold directly, and induction proves every proper-interval equation.

At charge B , the possible residuals are cubic oscillator polynomials and their z -multiples. Verification that the displayed S_i cancel these residuals is the computer-assisted step in the proof. The complete program in Appendix I checks the resulting rational identities with N, t and the site positions left indeterminate. The finite reduction, described below, has 172 coefficient types. Equation (5.1) follows separately by telescoping.

It remains to prove that the recursion (5.2) solves every equation at the remaining charges. For $\mu = B + [a, b]$, with $1 \leq a \leq b < N$, its nonzero individual weights are

$$h_{a-1}(\mu) = \frac{r_{a-1} - 1}{r_{a-1} + 1}, \quad h_i(\mu) = \frac{2}{r_i(r_i + 1)} \quad (a \leq i < b), \quad h_b(\mu) = 1 + \frac{2}{r_b}.$$

All other weights are zero. None equals one, since t is irrational and positive. The momenta are also nonzero: outside the interval they equal $1/(r_i + 1)$, in its interior they equal $2/(r_i + 1)$, and at its left and right boundaries they equal $(1 - r_i)/(r_i + 1)$ and $(r_i + 2)/(r_i + 1)$, respectively. Irrationality excludes $r_i = 1$.

Lemma 5.2 (Automatic mixed equations). *At a first unresolved charge μ of minimum weight two, suppose $h_i(\mu) \neq 1$ and $p_i(\mu) \neq 0$ for all i . If the self-Virasoro equations have been repaired at that charge, all mixed equations vanish there for $N \geq 3$.*

Proof. For $i \neq j$, the mixed residual has the form

$$[\tilde{T}_i \tilde{T}_j]_\mu = \left(\sum_k v_k A_k(-1) + zw \right) e^\mu.$$

Apply $[L_0^i, L_{-1}^i] = L_{-1}^i$ to $\tilde{T}_j$ and take charge μ . All contributions involving earlier mixed residuals vanish. The original Fock action then gives

$$(h_i(\mu) + \delta_{ki} - 1)v_k = wp_i(\mu)\delta_{ki}.$$

Thus v is supported at i . Skew-symmetry says that the reversed constant coefficient is $wp(\mu) - v$; the same argument shows that it is supported at j . A third site k has $p_k(\mu) \neq 0$, forcing $w = 0$, then $v = 0$. $\square$

Adding de^μ changes a self-equation's ze^μ coefficient by $2(h_i(\mu) - 1)d$. Therefore (5.2) kills that coefficient. Skew-symmetry kills the constant coefficient too: any self-bracket at this charge

has constant term equal to one half the derivative of its z coefficient. Higher powers of z are impossible by weight. The lemma now kills every mixed equation, completing induction over the tail charges.

Finally the sum of the repaired frame is Virasoro. Its proper-interval corrections sum to zero, and its charge- B correction is (5.1). Compare it with the Virasoro vector $\omega_0 + C_N D^2 \varepsilon$ from Proposition 4.1. A first nonzero tail difference would be a scalar de^μ of total weight two. Its self-equation has linear coefficient $2dze^\mu$, so $d = 0$. Hence all tail coefficients sum to zero. Charge homogeneity reinstates the powers of u in (5.3) and proves the theorem. Each parameter ψ_i is irrational, so the corresponding Virasoro vacuum algebra is simple; the nonzero homomorphism from their simple tensor product is injective.

Reduction to finitely many symbolic identities

Every complementary pair of proper intervals consists uniquely of $I = [a, b] \subseteq \{1, \dots, N-1\}$ and its cyclic complement J . Put

$$m_k = \mathbf{1}_{a \leq k \leq b}, \quad H_k = \mathbf{1}_{b < k}, \quad R = \frac{t+N}{t}, \quad G = sP_-.$$

To see precisely which factors are extracted, put

$$\alpha_I = (-1)^{|I|-1} \prod_{j \in I} r_j, \quad \alpha_J = (-1)^{|J|-1} \prod_{j \in J} r_j.$$

The identities $r_j + 1 = r_{j+1}$ give $F_{I,k} = \alpha_I m_k / r_k$ and $F_{J,k} = \alpha_J [1 - m_k + (R-1)H_k] / r_k$. For the second formula the factor R appears exactly when the product defining $F_{J,k}$ crosses the closing edge. In particular $\alpha_I \alpha_J = (-1)^{N-2} P_- = G$. Thus $c_{i,I} = \alpha_I u_i$, $c_{i,J} = \alpha_J v_i$, and the normalized coefficients and momenta are

$$\begin{aligned} u_i &= \frac{m_i}{r_i} - \frac{m_{i+1}}{r_i + 1}, & v_i &= \frac{g_i}{r_i} - \frac{g_{i+1}}{r_i + 1}, & g_i &= 1 - m_i + (R-1)H_i, \\ p_i(I) &= m_i - \frac{r_i}{r_i + 1} m_{i+1}, & p_i(J) &= p_i - p_i(I). \end{aligned}$$

In these formulas use $m_0 = m_N = 0$, $H_0 = 0$, $H_N = 1$. For a Boolean monomial $\prod_{k \in M} m_k \prod_{l \in H} H_l$, summation over all $1 \leq a \leq b < N$ gives exactly

$$\begin{cases} (\min M) \max(0, U - \max M), & M \neq \emptyset, \\ U(U-1)/2, & M = \emptyset, \end{cases} \quad U = \min(H \cup \{N\}).$$

Indeed, for $M \neq \emptyset$ the allowed choices are $1 \leq a \leq \min M$ and $\max M \leq b < U$; for $M = \emptyset$ there are b choices of a for each $1 \leq b < U$. Boolean products are reduced using $m_k^2 = m_k$, $H_k^2 = H_k$. Successor sites are represented as $j+1$ before counting. When $j+1$ coincides with another site, the same minimum and maximum values give the same count; an empty endpoint range gives zero. At the closing edge $g_N = R$, so $g_N/(t+N) = g_0/t = 1/t$. Thus the successor convention gives the cyclic coefficient at site zero exactly. The factors at m_0, m_N and H_0, H_N are also enforced by the endpoint count. This replaces every rank-dependent interval sum by a polynomial on each equality and order pattern. The charge- B identities hold with N, t independent; the relation $t(t+N) = 1$ is still used in the preceding lattice and charge-bound arguments.

The complementary exponential bracket is

$$[e^I z e^J] = \left(\frac{P_1^3}{6} + \frac{P_1 P_2}{2} + \frac{P_3}{3} + \frac{z}{2} (P_1^2 + P_2) + \frac{z^2}{2} P_1 + \frac{z^3}{6} \right) \varepsilon, \quad P_k = \sum_l p_l(I) A_l(-k).$$

Thus a coefficient involves at most the two frame sites and three oscillator sites. The script enumerates every equality and order type of at most five distinct sites, keeping their positions

$j_0 < \cdots < j_{m-1}$ symbolic. The seven shapes are z^3 , $z^2A(-1)$, $zA(-2)$, $zA(-1)^2$, $A(-3)$, $A(-2)A(-1)$ and $A(-1)^3$. There are 7, 40, 69, 46, 10 types with one through five distinct sites, respectively, for a total of 172. For each type, the program computes the source and the linear correction, clears denominators, and expands the numerator of their sum. The claimed identity is precisely that this numerator is zero. The complete Python/SymPy implementation is printed in Appendix I; it makes no numerical substitution for N , t , or the site positions. This certifies the finite algebraic calculation, subject to the reduction proved above; it is not a proof-assistant verification of the theorem.

For clarity, the linear correction is computed with the undeformed Fock modes

$$L_i(n) = \frac{1}{4\psi_i} \sum_{k \in \mathbb{Z}} : A_i(k) A_i(n-k) : - \frac{(n+1)(\psi_i-1)}{2\psi_i} A_i(n),$$

where $A_i(0)e^B = 2\psi_i p_i e^B$ and $A_i(k) = 2\psi_i k \partial/\partial A_i(-k)$ for $k > 0$. The first linear bracket is $\sum_{k=0}^3 z^k L_i(k-1)(S_j e^B)/k!$; the reversed one is obtained by skew-symmetry, replacing z by $-z - D$ and negating. These formulas explain the mode action in the certificate.

A nondegenerate invariant form on the ambient algebra

Proposition 5.3. *For $\lambda \neq 0$, (V_L, ω_λ) is regular and has a unique normalized nondegenerate invariant bilinear form. Its weight-zero space is two-dimensional, so it is not of CFT type.*

Proof. The total grading is unchanged by the $D^2\varepsilon$ correction. It has

$$(V_L)_0 = \mathbb{C}\mathbf{1} \oplus \mathbb{C}\varepsilon, \quad (V_L)_1 = \bigoplus_i (\mathbb{C}A_i(-1)\mathbf{1} \oplus \mathbb{C}A_i(-1)\varepsilon).$$

The degree-one-to-zero action is

$$L_1^{(\lambda)} A_i(-1)\mathbf{1} = -2(\psi_i - 1)(\mathbf{1} + 2\lambda\varepsilon), \quad L_1^{(\lambda)} A_i(-1)\varepsilon = 0.$$

Therefore $(V_L)_0/L_1(V_L)_1$ is one-dimensional, and the vacuum has nonzero image exactly when $\lambda \neq 0$. The invariant-form theorem [10] gives a unique form with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Its radical is an ideal of the underlying simple lattice vertex algebra, hence is zero. On weight zero its matrix is

$$\begin{pmatrix} 1 & -1/(2\lambda) \\ -1/(2\lambda) & 0 \end{pmatrix}.$$

Regularity follows from Proposition 4.1. □

6 Fock modules and signed Kac labels

We fix the free-field conventions used below, since the distinction between a Verma module and its contragredient is essential. Let $[a_m, a_n] = m\delta_{m+n,0}$ and

$$Q = \frac{\psi-1}{\sqrt{2\psi}}, \quad T = \frac{1}{2}a(-1)^2\mathbf{1} + Qa(-2)\mathbf{1}, \quad c = 1 - 12Q^2,$$

where $\psi > 0$ is irrational. The Fock space F_ξ has ground momentum $a_0 = \xi$, and its highest weight is $h(\xi) = \xi(\xi - 2Q)/2$. Put

$$\alpha_+ = \sqrt{2\psi}, \quad \alpha_- = -\sqrt{2/\psi}, \quad \xi_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-.$$

The classical free-field determinant formula [11, Proposition 2.7] for the ground-vector map $M(c, h(\xi)) \rightarrow F_\xi$ is, at level n and up to a nonzero basis constant,

$$\det \Phi_{\xi, n} = C_n \prod_{\substack{r, s \geq 1 \\ rs \leq n}} (\xi - \xi_{r, s})^{p(n-rs)}, \quad C_n \neq 0. \quad (6.1)$$

Here $p(n)$ is the partition number. To check conventions against [11], set $w = \xi - Q$, $\lambda = -Q$, and $s_\pm = \alpha_\pm$. Their raising operator L_n is our L_{-n} ; their determinant factor $w + rs_+/2 + ss_-/2$ is exactly $\xi - \xi_{r, s}$. At level one this gives $L_{-1}|\xi\rangle = \xi a_{-1}|\xi\rangle$. Fock contragredience is $F_\xi^\vee \simeq F_{2Q-\xi}$; the pairing is obtained by transposing nonzero oscillator modes as $a_n \mapsto -a_{-n}$ and the zero mode as $a_0 \mapsto 2Q - a_0$.

Lemma 6.1 (Signed Kac coordinates). *For nonzero integers r, s and irrational ψ the following hold. If $r, s > 0$, then $F_{\xi_{r, s}}$ is the contragredient Verma module at $h_{r, s}$, with unique simple socle $L(r, s)$. If $r, s < 0$, it is the Verma module at $h_{-r, -s}$; its simple socle is the irreducible partner Verma module and is not C_1 -cofinite. If $rs < 0$, it is an irreducible Verma module and is not C_1 -cofinite.*

Proof. Irrationality implies that $\xi_{r, s} = \xi_{u, v}$ only when $(r, s) = (u, v)$. Thus (6.1) identifies $F_{\xi_{r, s}}$ with its Verma module whenever at least one of r, s is negative. Reflection sends (r, s) to $(-r, -s)$, so contragredience proves the assertion when both are positive. The generic Verma-module classification gives one primitive singular vector at level rs for a positive pair and an irreducible partner Verma module below it. A mixed-sign pair is not a positive Kac weight, even after reflection, so its Verma module is irreducible. The C_1 -cofinite simple modules are exactly the positive Kac modules [7, Theorems 2.2.4 and 3.1.4], which proves the remaining assertions. $\square$

At site i of our lattice, the momentum of charge k is $\xi = \alpha_+ k_i + \alpha_- k_{i+1}$, hence its signed Kac coordinates are $(1 - 2k_i, 1 - 2k_{i+1})$. In particular the charges $-m$ and $1 + m$ have, respectively, contragredient-Verma and Verma orientation for the positive pair $(2m_i + 1, 2m_{i+1} + 1)$.

7 Simple generator modules for the deformed frame

Retain the lattice L , the vector B , and the exponential $\varepsilon = e^B$ from Section 4. Fix a nonzero coupling in the frame of Theorem 5.1, so that:

$$\sum_i \tilde{T}_i = \omega_0 + \lambda D^2 \varepsilon, \quad \lambda \neq 0.$$

Each correction to T_i has a strictly positive charge in the cone $\sum_i \mathbb{Z}_{\geq 0} \beta_i$. The ambient normalized invariant form satisfies $\langle 1, \varepsilon \rangle = -1/(2\lambda)$. We write $L_i(n)$ for the modes of $\tilde{T}_i$.

For $m \in \mathbb{Z}_{\geq 0}^N$, set

$$r_i = 2m_i + 1, \quad \eta_i = h_i(r_i, r_{i+1}), \quad \ell_i = r_i r_{i+1}, \quad H = \sum_i \eta_i = \sum_i (m_i - m_{i+1})^2 + N m_0 (m_0 + 1).$$

The two distinguished charges are $\mu_- = -\sum_i m_i \beta_i$ and $\mu_+ = B + \sum_i m_i \beta_i$. Their undeformed Fock modules are, respectively, the tensor product of the contragredient Verma modules $M_i^\vee$ and the tensor product of the Verma modules M_i , by Lemma 6.1. Here $M_i = M(c_i, \eta_i)$ has one primitive singular vector, at level ℓ_i , and its simple quotient is $L_i(r_i, r_{i+1})$.

Proposition 7.1 (Simple generator socles). *For each $m = e_j$ with $0 \leq j < N$, the ambient lattice contains exactly one simple frame submodule of type*

$$M_m = \bigotimes_i L_i(2m_i + 1, 2m_{i+1} + 1).$$

Its highest vector can be normalized to have lowest charge μ_- , with coefficient one at e^{μ_-} .

The finite spectral calculation

For a lattice charge $k = (k_i)$, its undeformed partial highest weights are

$$h_i(k) = \psi_i k_i (k_i - 1) + \psi_i^{-1} k_{i+1} (k_{i+1} - 1) - 2k_i k_{i+1} + k_i + k_{i+1}.$$

Put $a_i = 1 + i(N - i)$, including $a_0 = a_N = 1$. Since

$$\psi_i = 1 + \frac{N - i + t}{a_i}, \quad \psi_i^{-1} = 1 - \frac{N - i - 1 + t}{a_{i+1}},$$

the condition that every $h_i(k) - \eta_i$ is an integer implies

$$\frac{k_i(k_i - 1) - m_i(m_i + 1)}{a_i} = \kappa \quad \text{for every } i. \quad (7.1)$$

This follows by comparing coefficients of the irrational number t . The total Fock ground weight is

$$h(k) = \sum_i (k_i - k_{i+1})^2 + N k_0 (k_0 - 1).$$

For $m = e_j$, $j > 0$, all grades needed below are at most $H + 3 = 5$. Equation (7.1) gives $\kappa = k_0(k_0 - 1) \geq 0$; if $\kappa > 0$, then $h(k) \geq 2N > 5$. Hence $\kappa = 0$.

For the cyclic elementary generator $m = e_0$, the required grades are at most $H + 3 = 2N + 5 < 6N$. Here $\kappa = k_0(k_0 - 1) - 2$. If $\kappa = -2$, every site $i > 0$ would satisfy $k_i(k_i - 1) = -2a_i < 0$, which is impossible. If $\kappa > 0$, then $k_0(k_0 - 1) \geq 6$ contradicts the grade bound. Thus $\kappa = 0$ for the closing generator as well.

Thus throughout these finite ranges $k_i \in \{-m_i, m_i + 1\}$. Write $k_i = (1 + \epsilon_i r_i)/2$, $\epsilon_i \in \{\pm 1\}$. Direct substitution gives

$$h_i(k) = \eta_i + \begin{cases} 0, & \epsilon_i = \epsilon_{i+1}, \\ \ell_i, & \epsilon_i \neq \epsilon_{i+1}. \end{cases} \quad (7.2)$$

A nonconstant cyclic sign sequence has at least two domain walls. Consequently, at the joint weight $\eta = (\eta_i)$, and at each joint weight $\eta + \ell_i e_i$, the only Fock charge quotients are μ_- and μ_+ . Below η in any one coordinate there are no quotients. Oscillators add nonnegative integers separately in each coordinate, so they do not change these conclusions.

Spectral projections, without a semisimplicity assumption

All arguments take place in finite-dimensional total-weight spaces. The commuting operators $L_i(0)$ are triangular for the positive-charge filtration; their associated graded operators are the undeformed Fock $L_i(0)$ operators. Their joint generalized spectral projections are polynomials in these commuting operators and preserve the filtration. One can equivalently choose a linear combination separating all the finitely many joint eigenvalue tuples and use its ordinary spectral projections. This construction does not require the individual $L_i(0)$ to be semisimple.

The joint generalized eigenspace at η has dimension two. Its intersection with the cone-filtered subspace supported at charges $\mu_+ + \sum_i \mathbb{Z}_{\geq 0} \beta_i$ has dimension one. To justify the dimension count, restrict the polynomial spectral projector to this invariant filtered subspace. Its associated graded image is the sum of the Fock eigenspaces with tuple η in that cone. Of the two possible charges, only μ_+ belongs to the cone, and only its ground vector has that tuple. Thus the image has dimension one. The commuting nilpotent parts vanish on a one-dimensional space. It is therefore spanned by a genuine joint eigenvector e with lowest charge μ_+ and coefficient one at e^{μ_+} . Projecting e^{μ_-} gives a vector f with lowest charge μ_- and coefficient one there. Thus

$$L_i(0)e = \eta_i e, \quad L_i(0)f = \eta_i f + b_i e.$$

The absence of spectral targets below η implies $L_i(n)e = L_i(n)f = 0$ for all $n > 0$. The submodule E generated by e is the tensor product of the Verma modules M_i : its map from that tensor product is injective, since its lowest-charge projection is exactly the isomorphism to the positive Fock module at μ_+ .

Let S_i be the primitive singular operator in M_i , with coefficients fixed at η_i . The leading μ_- part of $S_i f$ vanishes, since that Fock module is contragredient Verma. The spectral calculation at $\eta + \ell_i e_i$ therefore places $S_i f$ in E . In more detail, inside the cone-filtered subspace above μ_- , the kernel of projection to the μ_- Fock quotient has generalized-block dimension $p(\ell_i)$, supplied entirely by the μ_+ Fock quotient; the corresponding ordinary weight space of E has that same dimension. They are equal. This remains valid before the coefficients b_i have been shown to vanish.

Eliminating the possible Jordan coupling

Each frame vector is quasi-primary for the total conformal vector, so the ambient invariant form satisfies $\langle L_i(n)x, y \rangle = \langle x, L_i(-n)y \rangle$. The module E is isotropic. Indeed e is a total primary of weight H , and every charge in it is at least $\mu_+ \geq B$. Therefore the weight-zero product $e_{(2H-1)}e$ cannot have charge 0 or B , the only weight-zero lattice charges. This proves $\langle e, e \rangle = 0$, and contravariance proves $\langle E, E \rangle = 0$.

On the other hand,

$$\langle e, f \rangle = (-1)^H \langle 1, e_{(2H-1)}f \rangle = -\frac{1}{2\lambda} \neq 0. \quad (7.3)$$

Here H is even, the lowest-charge product is $e_{(2H-1)}^{\mu_+} e^{\mu_-} = \varepsilon$, and every correction has strictly larger charge and hence cannot contribute to weight zero.

Let $P_i(z)$ be the Shapovalov norm polynomial of the fixed vector S_i : on a generalized highest vector, $S_i^\dagger S_i$ acts as $P_i(L_i(0))$. The Kac determinant has a simple zero at η_i in level ℓ_i , because ψ_i is irrational and the only positive Kac pair with this highest weight is (r_i, r_{i+1}) . Thus the level- ℓ_i Gram matrix has a one-dimensional kernel and

$$P_i(\eta_i) = 0, \quad P_i'(\eta_i) \neq 0.$$

The last assertion follows directly by differentiating the determinant: for a symmetric matrix with a one-dimensional kernel, its adjugate is a nonzero scalar times the outer product of a kernel vector with itself. A simple determinant zero is therefore equivalent to a nonzero derivative of that vector's norm.

Since $S_i f \in E$, isotropy and contravariance now give

$$0 = \langle S_i f, S_i f \rangle = \langle f, P_i(L_i(0))f \rangle = b_i P_i'(\eta_i) \langle f, e \rangle.$$

Equation (7.3) proves $b_i = 0$ for every site.

The common singular-vector kernel

Now f is a genuine joint highest vector. Thus $S_i f$ is a joint singular vector in E at weight $\eta + \ell_i e_i$, and

$$S_i f = c_i S_i e$$

for some scalar c_i . Operators belonging to different Virasoro factors commute, and E is a tensor product of Verma modules. Hence

$$(c_i - c_j) S_i S_j e = 0, \quad S_i S_j e \neq 0,$$

so all c_i have a common value c . The vector $v = f - ce$ is killed by every primitive singular operator. It therefore generates the simple tensor product M_m . Its leading negative exponential

has coefficient one. No other such simple submodule can occur in this two-dimensional bottom space, since a kernel vector in E would contradict its Verma structure. This proves the proposition.

8 Closure and classification of the frame socle

We first show that the sum of simple C_1 -cofinite frame submodules is closed under vertex products. The category of lower-bounded C_1 -cofinite modules for $\mathcal{B} = \bigotimes_i L(c_i, 0)$ is the Deligne product of the generic Virasoro C_1 categories, and is therefore semisimple. This follows by applying McRae’s tensor-product theorem inductively [12, Theorems 3.7 and 4.13]. The required closure under contragredients holds in each generic Virasoro factor by [7].

For two simple C_1 -cofinite frame submodules M, N of the ambient algebra, let P be the span of the coefficients of $Y(M, z)N$. The intertwining-operator Jacobi identity makes P a frame submodule. It inherits a lower-bounded, finite-dimensional total-weight grading from the lattice. The intertwining operator with target P is surjective by this definition, so Miyamoto’s image theorem, in the form of [13, Corollary 2.14], makes P C_1 -cofinite. The preceding tensor-product theorem then makes P a finite direct sum of simple C_1 frame modules. Thus the sum of all such simple submodules is closed under vertex products. This argument does not assume that the full ambient lattice module is locally C_1 -cofinite.

Lemma 8.1 (Aligned socle and multiplicity). *Every simple C_1 -cofinite frame submodule of the ambient lattice is of type M_m , for a unique $m \in \mathbb{Z}_{\geq 0}^N$, and its highest vector has a nonzero component at $e^{-\sum_i m_i \beta_i}$. Each such type occurs with multiplicity at most one.*

Proof. Let v be the highest vector of a simple C_1 frame submodule M . Filter by the scalar charge height $d(k) = \sum_i k_i$. The vector v has finite charge support; let d_0 be its minimum height. Every deformed frame mode preserves or strictly increases this height, since every correction charge lies in the positive cone. The same lower bound therefore holds on the module generated by v . For any charge k at height d_0 with nonzero component in v , projection to that charge is a nonzero module homomorphism

$$M \longrightarrow F(k) = \bigotimes_i F_i(k).$$

Indeed a correction could contribute at k only from a charge of height strictly less than d_0 , which is absent. Simplicity of M makes this map injective.

Here the two signed Kac coordinates of $F_i(k)$ are

$$R_i = 1 - 2k_i, \quad S_i = 1 - 2k_{i+1}.$$

They are nonzero odd integers. Lemma 6.1 gives the following three possibilities. If $R_i, S_i > 0$, then $F_i(k)$ is a contragredient Verma module, whose unique simple socle is $L_i(R_i, S_i)$ and is C_1 -cofinite. If $R_i, S_i < 0$, it is the Verma module at the reflected positive Kac label, whose simple socle is the irreducible partner Verma module; that partner is not C_1 -cofinite. If the signs are opposite, the Fock module itself is an irreducible Verma module and is not C_1 -cofinite. Irrationality of ψ_i excludes any additional positive Kac label in the last two cases.

By the tensor-product description above, M is an external tensor product of simple C_1 Virasoro modules. Its nonzero embedding in $F(k)$ forces such a simple socle in every factor. This can also be seen directly on its highest vector: the highest-weight and primitive-null equations act separately on the finite-dimensional site weight spaces, so their joint solution space is the tensor product of the corresponding site solution spaces. Consequently $R_i, S_i > 0$ at every site, equivalently $k_i \leq 0$ for every i . Writing $m = -k$, the embedded socle is exactly M_m . Its highest vector in $F(k)$ is the Fock ground exponential, so the coefficient at e^{-m} is nonzero.

The label uniquely determines m : equality of two positive Kac weights at irrational ψ_i forces equality of their positive Kac pairs. Evaluation of the highest-vector coefficient at this one ground exponential is therefore an injective linear map

$$\mathrm{Hom}_{\mathcal{B}}(M_m, V_L) \longrightarrow \mathbb{C}.$$

If a nonzero linear combination had coefficient zero, applying the preceding argument to its nonzero simple image would require that same coefficient to be nonzero. This contradiction proves the multiplicity assertion. $\square$

9 Nonzero self-couplings of all elementary generators

The elementary generators have an explicit nonzero self-channel. The calculation is uniform for all edges j , including $j = 0$.

Set

$$a = \psi_j, \quad b = \psi_{j-1}^{-1}, \quad k = a + b, \quad A = 2a, \quad B' = 2b, \quad H = 2k - 2.$$

For an ordinary edge $k = 2$; for the cyclic edge $k = N + 2$. The symbol B' in this section is a scalar and is not the lattice vector B . Use the two orthogonal currents $X = A_j$ and $Z = -bA_{j-1}$; their norms are A and B' and $\beta_j = X + Z$. The corresponding exponential $E = e^{\beta_j}$ has individual frame weights $(1, 1)$. The charge- β_j frame corrections are exactly $+uE$ and $-uE$.

Define the Schur states by

$$\sum_{p \geq 0} S_p(X) z^p = \exp \left(\sum_{n \geq 1} \frac{X(-n)}{n} z^n \right),$$

and similarly for Z . The normalized simple generator v_j has lowest term $e^{-\beta_j}$. At charge zero its state is exactly

$$uP, \quad P = \sum_{p=0}^H \frac{S_p(X) S_{H-p}(Z)}{A - 1 - p}. \quad (9.1)$$

Indeed, $E_{(1)} e^{-\beta_j}$ is the Schur state of total degree $H = \beta_j^2 - 2$, and the original partial $L_j(0)$ acts by p on its summand of X -degree p . The eigenvalue equation at charge zero therefore gives (9.1). Its denominators are nonzero because a is irrational. No other charge in the global frame can enter this equation: both v_j and the frame corrections are supported in the nonnegative cone above their respective lowest charges.

Let σ_j denote the coefficient of the normalized highest vector v_j in $v_{j(H-1)} v_j$. The leading charge $-\beta_j$ can come only from the ordered charge pairs $(-\beta_j, 0)$ and $(0, -\beta_j)$. Contracting $X(-n)$ against $e^{-\beta_j}$ replaces it by $(-1)^n A$, and similarly replaces $Z(-n)$ by $(-1)^n B'$. Thus the Schur states evaluate to $(-1)^p (A)_p / p!$ and $(-1)^{H-p} (B')_{H-p} / (H-p)!$. Since H is even, their total sign is positive. Skew-symmetry gives the same coefficient for the reversed charge pair; derivative terms cannot change the leading highest vector at its minimum weight. Consequently

$$\frac{\sigma_j}{u} = 2 \sum_{p=0}^H \frac{(A)_p (B')_{H-p}}{p! (H-p)! (A-1-p)}, \quad A + B' = H + 2. \quad (9.2)$$

Vacuum descendants have only nonnegative charge and cannot contribute to this leading negative exponential. The next possible diagonal channel, with edge spin two, has weight $6k - 4 > H$, so the coefficient in (9.2) is indeed the self-channel scalar.

Lemma 9.1 (An elementary finite-sum identity). *For every integer $H \geq 0$, as an identity of rational functions of A ,*

$$2 \sum_{p=0}^H \frac{(A)_p (H+2-A)_{H-p}}{p! (H-p)! (A-1-p)} = 2^{H+1} \frac{\prod_{j=1}^H (2A-2j-1)}{\prod_{j=1}^{H+1} (A-j)}. \quad (9.3)$$

Proof. The only possible finite poles on the left are simple poles at $A = p+1$, $0 \leq p \leq H$. The residue there is

$$2 \binom{2p}{p} \binom{2H-2p}{H-p}.$$

The right side has the same residue: evaluating its numerator at $A = p+1$ gives the product of the odd integers up to $2p-1$ and, with sign $(-1)^{H-p}$, the odd integers up to $2(H-p)-1$; the derivative of its denominator is $(-1)^{H-p} p! (H-p)!$. The double-factorial formula gives precisely the displayed central-binomial residue.

It remains to check that the left side is proper. For $0 \leq p \leq H$, the finite Newton expansion is

$$\frac{1}{A-1-p} = \sum_{r=0}^p \frac{p(p-1) \cdots (p-r+1)}{(A-1)(A-2) \cdots (A-r-1)}.$$

Vandermonde convolution gives, for $A+B' = H+2$,

$$\sum_{p=0}^H p(p-1) \cdots (p-r+1) \frac{(A)_p (B')_{H-p}}{p! (H-p)!} = \frac{(A)_r (A+B'+r)_{H-r}}{(H-r)!}.$$

After substitution, every term has numerator of degree r in A and denominator of degree $r+1$, since $A+B'$ is constant. The left side is therefore $O(A^{-1})$. The right side is also proper. Their difference is a polynomial vanishing at infinity, hence zero. $\square$

All zeros and poles of (9.3) occur at rational A . Therefore $\sigma_j \neq 0$ for every edge in the cyclic frame and every nonzero common coupling u .

9.1 Exact ordinary-root coefficients

Proposition 9.2. *Let $1 \leq j < N$, put $h = 2\psi_j - 1$, and normalize the elementary highest vector v_j to have coefficient one at $e^{-\beta_j}$. For the common screening coupling $u \neq 0$, its vacuum and self-channel coefficients are*

$$\nu_j = u^2 \frac{3(3h-5)(3h-1)}{h(h-2)(h-1)^2}, \quad (9.4)$$

$$\sigma_j = u \frac{8(2h-1)(2h-3)}{h(h-1)(h-2)}. \quad (9.5)$$

Both coefficients are nonzero.

Proof. We compute the invariant norm of the ordinary generator v_j . Set $a = \psi_j$, $b = \psi_{j-1}^{-1} = 2-a$, $X = A_j$, $Y = A_{j-1}$, and $E_j = e^{\beta_j}$. Use the common screening coupling u , so the charge- β_j parts of the two frame vectors are $+uE_j$ and $-uE_j$. Only the charges $-\beta_j, 0, \beta_j$ in the normalized highest vector can contribute to the charge-zero vacuum coefficient in its self-product. Joint primarity fixes those terms as

$$v_j = e^{-\beta_j} + uP_j + u^2 c_j E_j + \text{terms at other charges},$$

where

$$P_j = \alpha X(-1)^2 + \beta X(-1)Y(-1) + \gamma Y(-1)^2 + \delta X(-2) + \epsilon Y(-2),$$

$$\alpha = \delta = -\frac{1}{2(3-2a)}, \quad \beta = \frac{b}{2(1-a)}, \quad \gamma = \frac{b^2}{2(2a-1)}, \quad \epsilon = -\frac{b}{2(2a-1)},$$

and

$$c_j = \frac{10a^2 - 20a + 9}{(a-1)^2(2a-3)(2a-1)}.$$

For example, the $L_j(0)$ equation at charge zero gives these five coefficients directly, since $L_j(0)$ has eigenvalues 2, 1, 0, 2, 0 on the five displayed oscillator monomials, and its source is $\frac{1}{2}((X - bY)(-1)^2 + (X - bY)(-2))$. The $L_j(1)$ and $L_j(2)$ equations then hold identically.

The contractions are $(X, X) = 2a$, $(Y, Y) = 2/b$, $(X, Y) = 0$. The coefficient $E_{j(1)}P_j$ is

$$4a^2\alpha - 4a\beta + 4\gamma - 2a\delta + 2\epsilon,$$

giving the stated c_j . Finally the charge-zero coefficient in $v_{j(3)}v_j$ is

$$\begin{aligned} \nu_j &= u^2 \left(2c_j + 8a^2\alpha^2 + \frac{4a}{b}\beta^2 + \frac{8}{b^2}\gamma^2 - 12a\delta^2 - \frac{12}{b}\epsilon^2 \right) \\ &= u^2 \frac{3(3a-4)(3a-2)}{(a-1)^2(2a-3)(2a-1)} \neq 0. \end{aligned}$$

Every possible zero or pole in the last expression would require a rational value of a , whereas $a = \psi_j$ is irrational. The C_1 image theorem puts this self-product in the frame vacuum submodule, whose weight-zero space is $\mathbb{C}1$; there is therefore no ϵ term. Thus ν_j is also the invariant norm. Substituting $a = (h+1)/2$ gives (9.4). For an ordinary edge, $H = 2$ and $A = 2a = h+1$ in (9.3). Hence

$$\frac{\sigma_j}{u} = 8 \frac{(2A-3)(2A-5)}{(A-1)(A-2)(A-3)} = \frac{8(2h-1)(2h-3)}{h(h-1)(h-2)},$$

which is (9.5). Irrationality again excludes all zeros and poles. $\square$

10 Recognizing the cyclic algebra from its generator socles

We distinguish three questions: closure of the simple socle, survival of its generators in a simple quotient, and uniqueness of the resulting diagonal multiplication. The following lemmas address the last two questions without assuming that every fusion-allowed operator product is already nonzero.

10.1 Object-fixing automorphisms of the generic integer-spin category

Write $\mathcal{R}_q$ for the semisimple category of integer-spin type-I $U_q(\mathfrak{sl}_2)$ modules, with its usual ribbon structure, where q is not a root of unity. Its simple objects are X_m , $m \geq 0$, with $X_0 = \mathbf{1}$ and X_1 the three-dimensional representation.

Lemma 10.1. *Every $\mathbb{C}$ -linear braided tensor autoequivalence of $\mathcal{R} = \boxtimes_{i=0}^{N-1} \mathcal{R}_{q_i}$ that fixes the isomorphism class of every simple object is monoidally naturally isomorphic to the identity. The parameters q_i need not be equal.*

Proof. First consider one factor. The generic $SO(3)_q$ category is the additive idempotent completion of the pivotal category generated by a symmetrically self-dual object $X = X_1$, its cup and cap, and a rotationally invariant trivalent vertex $v : \mathbf{1} \rightarrow X^{\otimes 3}$. All morphisms are linear combinations of trivalent diagrams. This is the standard even Temperley–Lieb realization;

see [14, Proposition 4.3]. In their normalization the bigon coefficient is one, so the closed theta evaluation is $\dim X = q^2 + 1 + q^{-2} \neq 0$.

Let F be the autoequivalence. Since every simple object is fixed and its twist is scalar, F preserves the twist. Being braided, it consequently preserves the associated pivotal structure. Choose an isomorphism $F(X) \simeq X$. After multiplying this isomorphism by a nonzero scalar, the transported cup equals the original cup; this is possible over $\mathbb{C}$ since the cup scales quadratically. The transported cap then also agrees, by the snake identities. The transported trivalent vertex is αv for some $\alpha \neq 0$. Preservation of the nonzero theta evaluation gives $\alpha^2 = 1$. Replacing the chosen isomorphism $F(X) \simeq X$ by its negative leaves cup and cap unchanged and changes the sign of the trivalent vertex. Thus it can also be arranged that $F(v) = v$.

Tensor the chosen isomorphism on words in X . Because all morphisms between these words are generated by cup, cap, and v , the resulting maps are natural; by construction they are monoidal. They extend to finite direct sums and idempotent summands, proving the claim for one factor. For a Deligne product, perform the same normalization in each factor. All remaining interchanges between generators from distinct factors are braidings, which F preserves. The same argument therefore proves naturality and monoidality for the product. $\square$

The lemma controls the tensor structure, including its coherence. In particular, an object-fixing braided autoequivalence cannot produce a different diagonal algebra through a residual tensor 2-cocycle.

10.2 Recovering an algebra from the mirror functor

Lemma 10.2. *Let $\mathcal{U}, \mathcal{V}$ be semisimple ribbon categories with simple units, and let*

$$A = \bigoplus_{i \in I} U_i \boxtimes V_i$$

be a simple commutative algebra in their direct-sum completion, with trivial twist. Suppose the U_i and the V_i are complete sets of distinct simple objects, the unit summand occurs once, and $U_0 = \mathbf{1}_{\mathcal{U}}$, $V_0 = \mathbf{1}_{\mathcal{V}}$ for the unit index 0. Then A , including its multiplication, is the canonical diagonal algebra associated to the braid-reversed equivalence constructed from A in [8, Proposition 4.4 and Theorem 4.5].

Proof. We reconstruct the multiplication from the mirror equivalence. Let F be free A -module induction and let $F_{\mathcal{U}}$ and $F_{\mathcal{V}}$ denote its restrictions to the two factors. The proof of the cited theorem constructs a tensor category $\mathcal{M} \subset \text{Rep } A$ in which both restrictions are tensor equivalences onto $\mathcal{M}$. Its unit is A . The central lifts of the two functors identify their braidings oppositely. If $\tau = F_{\mathcal{V}}^{-1} F_{\mathcal{U}}$ is the resulting mirror functor, then, on transporting $\mathcal{M}$ to $\mathcal{U}$, the action of $\mathcal{U} \boxtimes \mathcal{V}$ on its unit is the tensor functor

$$(U, V) \mapsto U \otimes \tau^{-1}(V).$$

The coherence here is inherited from the central lifts, so it is the same coherence used for the canonical diagonal algebra.

For finite objects Z of $\mathcal{U} \boxtimes \mathcal{V}$, Frobenius reciprocity gives the natural isomorphism

$$\text{Hom}(Z, A) \simeq \text{Hom}_{\text{Rep } A}(F(Z), A).$$

Thus A represents the internal endomorphisms of the unit for that action. Moreover this identification preserves multiplication. Indeed, if $f : Z \rightarrow A$ and $g : T \rightarrow A$, their free A -linear extensions $\hat{f} : F(Z) \rightarrow A$ and $\hat{g} : F(T) \rightarrow A$ have tensor product

$$F(Z \otimes T) \simeq F(Z) \otimes_A F(T) \xrightarrow{\hat{f} \otimes_A \hat{g}} A \otimes_A A \simeq A.$$

Under Frobenius reciprocity this is exactly $\mu_A \circ (f \otimes g)$. The unit is preserved as well. The representing algebra is therefore the canonical internal-End algebra of the action, as in Creutzig–Kanade–McRae, Theorems 3.2–3.5. Uniqueness of representing objects proves the algebra assertion. For infinitely many simple labels, test the representing property on finite objects Z . The argument then identifies every component of the multiplication in the direct-sum completion. $\square$

10.3 Recognition from a generating family with nonzero vacuum channels

Fix an edgewise braid-reversed equivalence between $\mathcal{R}$ and a ribbon category $\mathcal{D}$. Write D_m for its image of X_m^* , so the canonical algebra is

$$A_{\text{can}} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}^N} X_m \boxtimes D_m.$$

All these labels are self-dual. We write $M_m = X_m \boxtimes D_m$.

Theorem 10.3 (Diagonal recognition). *Let S be a connected commutative unital algebra of trivial twist in $\text{Ind}(\mathcal{R} \boxtimes \mathcal{D})$. Suppose its underlying object is a subobject of A_{can} : only the indicated diagonal simples occur, each with multiplicity at most one. Suppose a collection $\{M_g : g \in G\}$ occurs in S , its $\mathcal{R}$ -labels tensor-generate $\mathcal{R}$, and each vacuum channel*

$$M_g \otimes M_g \xrightarrow{\mu_S} S \xrightarrow{\pi_0} \mathbf{1}$$

is nonzero. Then S is simple, has every diagonal summand, and is isomorphic as an algebra to A_{can} . Moreover, the unital subalgebra generated by the summands M_g , $g \in G$, is all of S .

For example, either of the following generating collections suffices:

$$G = \{e_1, \dots, e_{N-1}, \mathbf{1}\}, \quad G = \{e_0, \dots, e_{N-1}\},$$

where $\mathbf{1} = (1, \dots, 1)$.

Proof. The ind-completion of a semisimple category is semisimple as an abelian category. A proper ideal of S has no unit summand: an ideal containing the unit is all of S . Because the unit has multiplicity one, the sum of all proper ideals still has no unit summand. This sum J is therefore a proper ideal and is the unique maximal ideal. Put $Q = S/J$.

No M_g is contained in J . If it were, multiplying it by itself and using its nonzero vacuum channel would put the unit in J . Hence all the chosen generators survive in the simple algebra Q . The diagonal decomposition gives the commuting-pair hypothesis: $\text{Hom}_{\mathcal{R}}(\mathbf{1}, X_m)$ and $\text{Hom}_{\mathcal{D}}(\mathbf{1}, D_m)$ vanish for $m \neq 0$. Apply [8, Proposition 4.4] to Q : its simple-label support is closed under tensor products and taking summands. Since it contains the generating family, it is all of $\mathcal{R}$. Thus Q has every diagonal summand. But S has no off-diagonal summands and has multiplicity at most one on each diagonal summand. The quotient $S \twoheadrightarrow Q$ can consequently have no kernel, and $S = Q$ is simple with full support.

Theorem 4.5 of the same paper supplies a mirror equivalence for S . Relative to the fixed edgewise equivalence, it is a braided autoequivalence of $\mathcal{R}$ fixing every simple label. Lemma 10.1 makes it monoidally naturally isomorphic to the identity. By Lemma 10.2, this identifies the entire algebra structure of S with the canonical diagonal one.

For the stated generating set, $X_{\mathbf{1}} \otimes \bigotimes_{j=1}^{N-1} X_{e_j}$ has X_{e_0} as a summand: choose the vacuum channel in each of the factors $j > 0$. Then the spin-one objects X_{e_i} generate every integer-spin factor, hence their product category. This argument takes place in the support of the already simple quotient; it does not presume that those operator-product channels were nonzero in S . Finally the unital subalgebra generated by the M_g satisfies the same hypotheses: it is again a

multiplicity-free diagonal subobject, and it has the same nonzero generator vacuum channels. Applying the proof to it shows that it too has every diagonal summand, so its inclusion in S is an isomorphism. $\square$

Necessity of the two-point hypothesis. Multiplicity-free diagonal support and associativity alone are not enough. For example, a Cartan-product degeneration can retain the same diagonal object while killing lower channels, including a nontrivial generator’s vacuum channel. The proof above uses precisely the vacuum channels to rule out such degenerations before applying the mirror theorem.

10.4 A nonzero self-channel forces the vacuum channel

Lemma 10.4 (The spin-one associativity test). *In the setup of Theorem 10.3, let $M = M_{e_i}$ occur in S . If the projection of $\mu_S : M \otimes M \rightarrow S$ to M is nonzero, then its projection to the unit is nonzero. No simplicity assumption on S and no prior occurrence of M_{2e_i} are needed.*

Proof. Only one generic $SO(3)_q$ factor is involved. Write its spin-one object as X and its three tensor-square summands as $X_0 = \mathbf{1}, X_1 = X, X_2$. Use the nonzero canonical diagonal multiplication to fix bases for every allowed diagonal multiplication channel. Express the multiplication of S in these bases as scalars a_{ij}^ℓ , assigning scalar zero if the output summand is absent. Unit normalization gives $a_{01}^1 = a_{10}^1 = 1$. Because the reference multiplication is itself commutative, commutativity of S gives $a_{1j}^1 = a_{j1}^1$; this statement includes any braiding scalars and does not assume that a naked fusion basis has trivial braiding.

Put $T = \text{Hom}(X^{\otimes 3}, X)$, which has dimension three. Dual bases in the second factor identify the corresponding nine-dimensional space of diagonal-category morphisms with $\text{End}(T)$. The left-associated multiplication of $M^{\otimes 3}$ into M is the diagonal endomorphism

$$D = \text{diag}(a_{11}^0, (a_{11}^1)^2, a_{11}^2 a_{21}^1)$$

in the left fusion-tree basis. The right-associated multiplication is the same diagonal endomorphism in the right fusion-tree basis. If F is the spin-one Racah change-of-basis matrix, the second factor changes by its inverse transpose: the evaluation pairing of the two fusion spaces must remain the identity. Explicitly, if $v'_a = \sum_b F_{ba} v_b$, then $(v'^a) = \sum_b (F^{-1})_{ab} v^b$. Consequently a diagonal channel tensor $\sum_a d_a v'_a \otimes v'^a$ has matrix FDF^{-1} in the unprimed bases. Associativity therefore says

$$D = FDF^{-1}, \quad FD = DF.$$

The vacuum-to-spin-one entry F_{01} is nonzero. For completeness, in the usual normalized pivotal fusion-tree bases the vacuum row is, up to nonzero signs,

$$F_{0j} = \frac{\sqrt{\dim X_j}}{\dim X}, \quad j = 0, 1, 2.$$

This follows by closing the vacuum cup-cap pair against the two trivalent vertices; the remaining closed diagram is the nonzero theta evaluation. All quantum dimensions in this row are nonzero for generic q . In particular $F_{01} = \pm 1/\sqrt{\dim X} \neq 0$. The $(0, 1)$ entry of $FD = DF$ now yields $a_{11}^0 = (a_{11}^1)^2$. The implication in the statement is independent of all the nonzero choices of channel bases. $\square$

10.5 Application to the cyclic frame

Let $\mathcal{B} = \bigotimes_i L(c_i, 0)$ denote the Virasoro frame subalgebra in the deformed lattice construction, and let

$$\mathcal{S}(V_L) = \sum \{M \subset V_L : M \text{ is a simple } C_1\text{-cofinite } \mathcal{B}\text{-module}\}.$$

Miyamoto’s image theorem (in the form of Creutzig–McRae–Yang, *Direct limit completions*, Corollary 2.14), together with generic Virasoro semisimplicity, makes this a vertex subalgebra. By Lemma 8.1, its $\mathcal{B}$ -module decomposition is a subobject of

$$\bigoplus_{m \geq 0} \bigotimes_i L_i(2m_i + 1, 2m_{i+1} + 1).$$

On odd labels the row/column interchange has trivial double braiding, so the edgewise rearrangement identifies this with the diagonal categorical setup of Theorem 10.3.

Corollary 10.5 (Uniform cyclic embedding). *For every odd $N \geq 3$ and every nonzero coupling, the cyclic VOA W_N embeds conformally in the deformed lattice VOA. More precisely:*

$$\mathcal{S}(V_L) \simeq W_N$$

as an extension of $\mathcal{B}$, with canonical diagonal multiplication.

Proof. The uniform calculation (9.3) gives a nonzero self-channel for every M_{e_i} . Lemma 10.4 therefore gives a nonzero vacuum channel for every elementary generator. These objects tensor-generate the product of integer-spin categories. Theorem 10.3 applies and identifies the full socle, with its multiplication, with W_N . In particular the frame and the elementary highest vectors v_{e_i} vertex-generate W_N , and all fusion-allowed diagonal channels are nonzero. $\square$

10.6 The orthogonal splitting and a second weight-zero simple module

Proposition 10.6. *For $\lambda \neq 0$ the conformal embedding just constructed gives a W_N -module decomposition*

$$A_N = W_N \oplus X, \quad X = W_N^\perp.$$

The line X_0 is spanned by $\zeta = \mathbf{1} + 2\lambda\varepsilon$. There is a simple W_N -module $T \not\cong W_N$ of lowest weight zero, obtained as a quotient of the cyclic submodule generated by ζ . If C denotes the highest vector of the all-ones frame summand, normalized to have leading charge e^{-B} , then $o(C)\zeta = 2\lambda\zeta$.

Proof. The normalized ambient invariant form restricts to a nonzero invariant form on the simple CFT-type subalgebra W_N . Its restriction is therefore nondegenerate. Both algebras have finite-dimensional homogeneous spaces, so grade-by-grade orthogonal projection gives the asserted module splitting. The weight-zero Gram matrix in Proposition 5.3 identifies X_0 as stated.

The all-ones summand is present by the diagonal recognition theorem. Lemma 8.1 supplies its normalized lowest charge $-B$. Its total highest weight is $2N$. Consequently the charge-zero coefficient of $o(C)\varepsilon$ is one, while weight considerations give $o(C)\varepsilon = \mathbf{1} + b\varepsilon$ for some b . Since C is primary of even weight and $o(C)\mathbf{1} = 0$, invariance gives

$$0 = \langle o(C)\varepsilon, \mathbf{1} \rangle = 1 - \frac{b}{2\lambda}.$$

Thus $b = 2\lambda$ and $o(C)\zeta = 2\lambda\zeta$. The cyclic submodule generated by ζ has finite length: it is grading restricted over the CFT-type C_2 -cofinite algebra W_N , so the finite-category theorem applies [15]. Any simple quotient has nonzero image of its cyclic generator and therefore has lowest weight zero. Its C zero mode acts there by $2\lambda \neq 0$, whereas it acts by zero on the vacuum top. The simple quotient is consequently different from W_N . $\square$

11 Parity simple currents

We extend the odd-label gluing to all parity sectors. Continue to write $\mathcal{B} = \bigotimes_i L(c_i, 0)$ and let $K = K_+$ be the cycle form from Section 1. Its diagonal entries are $k_0 = N + 2$ and $k_i = 2$ for $i \neq 0$, and its cycle off-diagonal entries are -1 . For $N = 3$, each distinct off-diagonal entry is -1 . The identities needed below are:

$$x^T K x = \sum_i (x_i - x_{i+1})^2 + N x_0^2, \quad K \mathbf{1} = N e_0, \quad \det K = N^2.$$

The canonical odd-label gluing is:

$$W = \bigoplus_{r_i \geq 1, r_i \text{ odd}} \bigotimes_i L_i(r_i, r_{i+1}).$$

Here $W = W_N$ is the extension constructed in Theorem 2.2.

The ambient category used throughout is $\mathcal{V}_{\mathcal{B}}$, the category of C_1 -cofinite grading-restricted generalized $\mathcal{B}$ -modules. The generic Virasoro results [7, Theorems 3.1.4, 4.2.6, 5.1.2, 5.5.3], followed by [12, Theorem 1.2, equivalently Theorem 4.13], show that $\mathcal{V}_{\mathcal{B}}$ is exactly the semisimple rigid Deligne product of the corresponding generic Virasoro categories; it is closed under contragredients. Thus [13, Theorems 7.1, 7.5, 7.7] applies to its direct-limit completion, to algebra extensions there, and to their local modules. These theorems do not require W to be rational or C_2 -cofinite.

11.1 Construction and unrestricted fusion

For $\epsilon \in P = (\mathbb{Z}/2\mathbb{Z})^N$, define

$$J_{\epsilon} = \bigoplus_{\substack{r_i \geq 1 \\ r_i \equiv 1 + \epsilon_i \pmod{2}}} \bigotimes_i L_i(r_i, r_{i+1}).$$

Represent the entries of ϵ by 0, 1 when evaluating integral quadratic expressions.

Theorem 11.1. *Each J_{ϵ} is a simple local grading-restricted W -module, and*

$$J_{\epsilon} \boxtimes_W J_{\eta} \simeq J_{\epsilon + \eta}.$$

These fusion products have the intertwining-operator universal property against every grading-restricted generalized W -module, not only against modules whose $\mathcal{B}$ -restriction belongs to $\text{Ind}(\mathcal{V}_{\mathcal{B}})$.

The full finite pointed ribbon subcategory $\mathcal{P}$ on these modules has self-braiding quadratic form, conformal twist, and categorical dimension

$$\begin{aligned} Q(\epsilon) &= \exp\left(\frac{\pi i}{2} \epsilon^T K \epsilon\right), \\ \theta_{\epsilon} &= \exp\left(\frac{\pi i}{2} (\epsilon^T K \epsilon + 2N\epsilon_0)\right), \\ d_{\epsilon} &= (-1)^{\epsilon_0}. \end{aligned}$$

Its monodromy is $(-1)^{\epsilon^T K \eta}$, hence is nondegenerate.

Proof. We first work in $\text{Ind}(\mathcal{V}_{\mathcal{B}})$. The generic first-row category is braided equivalent to type-I $U_q(\mathfrak{sl}_2)$ modules with $q = e^{\pi i \psi_i}$; see [6, Theorem 1.1, Theorems 5.10–5.11]. The first-column category uses $e^{\pi i \psi_i^{-1}}$. At edge i , the parameters of the row at i and column at $i - 1$ sum to k_i . Moriwaki’s integer parameter shift theorem for type A [16, Theorem 2.16] and quantum-coordinate-ring functor [16, Lemma 3.4, Proposition 3.5] therefore transport the $\mathbb{Z}/2$ -graded

pieces of $\mathcal{O}_q(SL_2)$ to a lax braided functor whose source quadratic form has value i^{k_i} on the odd grade.

Reordering the edge factors and multiplying the row and column factors at each site gives a strong monoidal equivalence onto $\mathcal{V}_{\mathcal{B}}$; the monoidal structure is the usual braiding interchange, and $L_i(r, 1) \boxtimes L_i(1, s) \simeq L_i(r, s)$ by [7, Theorem 5.2.5]. This functor is not braided for the naive product braiding. Indeed, balancing gives

$$h_i(r, s) - h_i(r, 1) - h_i(1, s) = -\frac{(r-1)(s-1)}{2},$$

so the row/column double braiding is $(-1)^{(r-1)(s-1)}$. One must multiply the source braiding by the bicharacter

$$\beta(\epsilon, \eta) = (-1)^{\sum_i \epsilon_{i+1} \eta_i}.$$

The exponent is bilinear over $\mathbb{F}_2$, so $\beta(\epsilon + \eta, \zeta) = \beta(\epsilon, \zeta)\beta(\eta, \zeta)$ and likewise in the second argument. Multiplying a braiding by β therefore preserves both hexagons with the same associator. Its polarization is $\beta(\epsilon, \eta)\beta(\eta, \epsilon) = (-1)^{\sum_i (\epsilon_{i+1} \eta_i + \eta_{i+1} \epsilon_i)}$, exactly the row–column monodromy introduced by the interchange. This is the correction in the braided compatibility diagram for the multiplication functor. Transposing the convention corresponds to reversing that interchange. The resulting source self-braiding is

$$\prod_i i^{k_i \epsilon_i} (-1)^{\sum_i \epsilon_i \epsilon_{i+1}} = \exp\left(\frac{\pi i}{2} \epsilon^T K \epsilon\right).$$

The zero-grade image is the canonical odd-label algebra W , and the ϵ -grade image is J_ϵ . Normalized abelian cocycles have trivial components whenever one entry is zero. The lax braided functor therefore supplies ordinary local W -module structures on every J_ϵ and balanced multiplication maps between them.

These balanced maps are isomorphisms, not merely nonzero maps. Before the transports, $\mathcal{O}_q(SL_2)$ is strongly $\mathbb{Z}/2$ -graded. In its standard presentation a, b, c, d are odd and $ad - qbc = 1$, so $1 \in \mathcal{O}_1 \mathcal{O}_1$. Strong grading implies

$$\mathcal{O}_a \otimes_{\mathcal{O}_0} \mathcal{O}_b \xrightarrow{\sim} \mathcal{O}_{a+b}.$$

For example a finite decomposition $1 = \sum_j x_j y_j$ in odd elements supplies the inverse to multiplication. The map is equivariant, hence so is its inverse. Taking N factors and transporting through the grading twists and strong monoidal equivalences preserves these relative tensor isomorphisms. Thus $J_\epsilon \boxtimes_W J_\eta \simeq J_{\epsilon+\eta}$ in the local direct-limit category. Since W is simple, invertibility also implies simplicity of all these modules.

The total top weight of a $\mathcal{B}$ -summand is

$$\sum_i h_i(r_i, r_{i+1}) = \frac{r^T K r - N}{4}.$$

For $r = \mathbf{1} + v$, with $v_i \geq 0$, this equals

$$\frac{1}{4} \sum_i (v_i - v_{i+1})^2 + \frac{N}{4} v_0 (v_0 + 2).$$

Positive definiteness gives a lower bound and finite-dimensional weight spaces for each parity sum. Reduction modulo integers gives the displayed conformal twist. Dividing the twist by the self-braiding quadratic form gives the displayed dimension character. The polarization of Q is $(-1)^{\epsilon^T K \eta}$; $\det K$ is odd, so it is nondegenerate.

It remains to verify that the calculated fusion products are universal outside the chosen ambient category. Let $\mathcal{Y}$ be any W -intertwining operator of type $(\begin{smallmatrix} X \\ J_\epsilon \ J_\eta \end{smallmatrix})$, where X is a grading-restricted

generalized W -module. Restrict to finite $\mathcal{V}_{\mathcal{B}}$ -subobjects $U \subset J_{\epsilon}$ and $V \subset J_{\eta}$. By [13, Corollary 2.14], the image of this $\mathcal{B}$ -intertwining operator is C_1 -cofinite and grading restricted. That corollary permits an arbitrary generalized target; it is the direct-limit version of Miyamoto’s image theorem [17]. Consequently the full image $\text{Im } \mathcal{Y}$ is a union of $\mathcal{V}_{\mathcal{B}}$ -submodules and belongs to $\text{Ind}(\mathcal{V}_{\mathcal{B}})$. The intertwining Jacobi identity makes it a W -submodule. Factor $\mathcal{Y}$ through that image and use the already-established local-category tensor universal property. Composing with the inclusion into X proves the claimed unrestricted universal property. $\square$

Remark 11.2 (Comparison of braiding conventions). The braided identification above uses the first-row equivalence of [6]. With the Kac weights fixed here, balancing gives row–column monodromy $(-1)^{(r-1)(s-1)}$. In particular, it equals -1 when both highest-weight parities are odd. Thus the trivial row–column braiding asserted in [6, Proposition 6.3] does not agree with these weights. The monoidal multiplication functor of its Proposition 6.2 is sufficient for the construction; the displayed bicharacter supplies the required braiding correction.

Part II

Vacuum extensions and regularity

12 Reduction of vacuum extensions to lowest weights zero and one

We prove regularity by splitting every extension of the vacuum by a simple module. The first step reduces this problem to simple modules of lowest weight zero or one.

Let $W = W_N$ be the vertex operator algebra of Theorem 2.2. We use its conformal frame $\mathcal{B} = \bigotimes_{i=0}^{N-1} L(c_i, 0)$, its frame decomposition, and generation by the frame and the elementary summands $R_j = M_{e_j}$; see Corollary 10.5. Thus

$$h_i(r, s) = \frac{(r\psi_i - s)^2 - (\psi_i - 1)^2}{4\psi_i}, \quad \psi_i = \frac{t + i + 1}{t + i}, \quad t(t + N) = 1,$$

and, with cyclic indices,

$$\psi_j + \psi_{j-1}^{-1} = k_j, \quad k_0 = N + 2, \quad k_j = 2 \quad (1 \leq j < N).$$

Each ψ_i is irrational. Throughout this section, modules are grading-restricted generalized W -modules. Their restrictions to $\mathcal{B}$ need not lie in the C_1 -cofinite Virasoro category; the block argument below establishes the finiteness properties that we need.

12.1 Exact frame-block projections

For an irrational parameter ψ , the Feigin–Fuchs structure theorem says that a Virasoro Verma module is simple unless its highest weight is $h(r, s)$ for positive integers r, s . In that case its unique nonzero proper submodule is the simple Verma module of highest weight $h(r, -s) = h(r, s) + rs$; see [7, Proposition 2.2.1 and Theorem 2.2.4]. Hence every Verma module has length at most two. The corresponding blocks are singletons and the disjoint pairs

$$\{h(r, s), h(r, -s)\}, \quad r, s > 0.$$

The pairs are disjoint because equality of two signed Kac weights at irrational ψ forces equality of their signed labels up to simultaneous reversal.

Lemma 12.1. *The restriction of a grading-restricted generalized W -module E to $\mathcal{B}$ is the direct sum of its joint Virasoro block components. The projections onto these components are canonical $\mathcal{B}$ -module maps and are exact on short exact sequences. Every fixed joint block component has finite length.*

Proof. The subspace generated by finitely many vectors under the joint positive Borel algebra is finite-dimensional. Indeed, positive modes lower total conformal weight, partial zero modes preserve total weight, the total grading is bounded below, and each total generalized weight space is finite-dimensional. The image of this Borel algebra is solvable: the positive modes strictly lower the finite set of relevant total weights and the zero modes commute. Lie’s theorem therefore gives a finite flag with one-dimensional quotients, on which positive modes vanish. PBW induction is exact. Consequently the frame module generated by the chosen vectors is a quotient of a module with a finite filtration by tensor products of ordinary Virasoro Verma modules. It has finite length.

We justify block separation also for generalized partial zero modes. In a finite generalized highest-weight module for one Virasoro factor, the nilpotent part of $L(0)$ commutes with all modes. In an extension of two nonisomorphic ordinary simple modules, that nilpotent part therefore factors through a homomorphism from the quotient simple to the submodule simple, and is zero. Such an extension is ordinary.

An ordinary extension between two simples in different displayed blocks splits. If their highest weights are incongruent modulo $\mathbb{Z}$, weight cosets split it. Otherwise, dualize if necessary so that the quotient has the smaller highest weight. Its highest vector lifts to a highest vector of the extension, since the submodule has no weights below its larger highest weight. The resulting Verma-module map has zero intersection with the submodule: the Verma structure theorem lists no constituent of that highest weight in a different block. Its image is thus a section. Restricted contragredience is exact, so this also proves the other orientation. Finite-length induction now gives the canonical block decomposition for finite generalized highest-weight modules.

The restriction of all of E to one factor can have infinite partial weight multiplicities. For each vector, however, its one-factor positive-Borel orbit is finite by the total-weight argument above. The induced module has finite partial weight spaces and finite Verma length. Thus this restriction is a union of finite generalized highest-weight submodules. Their canonical block decompositions agree under inclusions. They define canonical projections on the union. The remaining factors commute with the chosen one and preserve its block components. Iterating gives the joint projections. Their functoriality and the direct-sum decompositions prove exactness.

A fixed joint block contains at most 2^N simple frame types. Let D be their finite set of total lowest weights. Every finite submodule of this component has length at most

$$\sum_{d \in D} \dim E_{[d]}.$$

This follows by additivity of generalized weight-space dimensions along a composition series: each constituent contributes at least one dimension at its own total lowest weight. The lengths of the finitely generated submodules are therefore bounded integers. Choose one of maximal length. Adjoining any further vector gives a finite-length submodule of the same length, so the inclusion is an equality. Thus the chosen submodule is the entire joint block component. $\square$

12.2 The aligned odd collection

Call a joint frame block *aligned and odd* if it has a simple constituent

$$\bigotimes_i L_i(r_i, s_i), \quad r_i, s_i \in 2\mathbb{Z} + 1, \quad r_i^2 = s_{i-1}^2.$$

Linkage preserves this condition because it changes only signs of the Kac labels. Let $\mathcal{P}$ be this collection of blocks, and let $E_{\mathcal{P}}$ be the sum of their components in E .

Lemma 12.2. *If the total generalized conformal weights of E are integers, then $E_{\mathcal{P}}$ is a W -submodule and has no negative total weights. The restriction of W to the frame belongs to $\mathcal{P}$.*

Proof. We first record the necessary local null constraint, including the case of signed Kac labels. For a nonzero intertwining operator with first input $L(3, 1)$, second input $L(r, s)$, and a simple highest-weight output of weight x , one has

$$(x - y)[(x - y - 2\psi + 1)(x - y - 1) - 4\psi y] = 0, \quad y = h(r, s).$$

Its three roots are $h(r - 2, s)$, $h(r, s)$, $h(r + 2, s)$. The reciprocal formula for first input $L(1, 3)$ changes s by $-2, 0, 2$. The polynomial is the image of the elementary level-three singular vector in the Virasoro Zhu bimodule [7, Proposition 5.2.3]. Necessity for arbitrary simple highest-weight second and output modules follows from the corrected injectivity theorem [18, Lemma 5.5 and Propositions 5.6–5.8]: both the second module and the output contragredient are generated by their lowest spaces, and the first Zhu bimodule is cyclic on its highest class. Thus a nonzero intertwiner cannot have zero highest coefficient. No sufficiency assertion or fusion decomposition for signed-label modules is being used. For the joint frame these same necessary relations hold separately at each factor.

Project the action of an elementary summand to a fixed source and target frame block. Both blocks have finite length by Lemma 12.1. If this projected intertwiner is nonzero, restrict it to the first nonzero source-filtration term; it factors through that term's simple quotient. The span of all coefficients is a frame submodule, by the intertwiner commutator formula. Choose the smallest target-filtration term containing this coefficient span and project to its simple quotient. Minimality makes the resulting intertwiner between simple subquotients nonzero. The preceding necessary constraints therefore determine the possible target blocks even when the original blocks are not semisimple.

The summand R_j changes only r_j and s_{j-1} , each by $0, \pm 2$. Let Δ_r and Δ_s be the respective changes of their squares. The possibly irrational part of the change of total highest weight is

$$\frac{1}{4}(\psi_j \Delta_r + \psi_{j-1}^{-1} \Delta_s) = \frac{1}{4}(\psi_j(\Delta_r - \Delta_s) + k_j \Delta_s).$$

All other terms are rational. Both highest weights occur in the integer-graded module E , so irrationality of ψ_j forces $\Delta_r = \Delta_s$. Alignment and odd parity are preserved. The frame and every R_j therefore preserve $E_{\mathcal{P}}$; the vertex-generation theorem proves W -stability.

For nonnegativity, reverse both labels at each site if necessary to write $r_i = \rho_i > 0$ and $s_i = \epsilon_i \rho_{i+1}$, where every ρ_i is odd and $\epsilon_i \in \{1, -1\}$. Direct substitution gives

$$4 \sum_i h_i(r_i, s_i) = \sum_i (\rho_i - \rho_{i+1})^2 + N(\rho_0^2 - 1) + 4 \sum_{i: \epsilon_i = -1} \rho_i \rho_{i+1} \geq 0. \quad (12.1)$$

Frame descendants only increase total weight. Finally, the constructed summands M_m of W have $r_i = s_{i-1} = 2m_i + 1 > 0$, so belong to this collection. $\square$

Lemma 12.3 (Vacuum-like vectors). *If a vector v of a weak W -module is killed by the total translation operator $D = L(-1)$, then $a_n v = 0$ for every $a \in W$ and every $n \geq 0$. The map $a \mapsto a_{-1} v$ is a W -module homomorphism from W to the module containing v .*

Proof. Lower truncation gives $a_n v = 0$ for all sufficiently large integers n . Descending induction using $[D, a_n] = -n a_{n-1}$ proves the vanishing for $n \geq 0$. The module Jacobi identity then gives $b_m(a_{-1} v) = (b_m a)_{-1} v$, which is the asserted homomorphism. $\square$

Proposition 12.4. *Let S be a simple grading-restricted W -module. Every nonsplit grading-restricted generalized extension*

$$0 \longrightarrow S \longrightarrow E \longrightarrow W \longrightarrow 0$$

has $E = E_{\mathcal{D}}$, has $S \not\cong W$, and has lowest weight $h(S) \in \{0, 1\}$. On its lowest space, the possible joint generalized partial spectra are $\{0\}$ if $h(S) = 0$, and $\{e_0, \dots, e_{N-1}\}$ if $h(S) = 1$. In the weight-zero case the joint generalized spectrum on all of E_0 is also zero and $L(0) = 0$ there exactly.

Proof. An irreducible lower-bounded generalized module is ordinary: the kernel of the nilpotent part of $L(0)$ is a nonzero submodule, hence the entire module. Its weights lie in one coset modulo $\mathbb{Z}$. If the coset differs from the vacuum coset, the two total weight cosets split the extension.

In the remaining case E is integer graded. The map $E \rightarrow W$ restricts to a surjection $E_{\mathcal{D}} \rightarrow W$, by exactness of the frame-block projections. Its kernel is $S \cap E_{\mathcal{D}}$, which is either zero or all of S by simplicity and Lemma 12.2. If the kernel is zero, this restriction is an isomorphism onto W and gives a section. If the kernel is S , then $E_{\mathcal{D}}$ contains S and surjects onto E/S , so $E = E_{\mathcal{D}}$. Lemma 12.2 then excludes negative weights.

If $h(S) > 1$, the unique vacuum lift $v \in E_0$ has $Dv = 0$ because $E_1 = 0$. Lemma 12.3 splits the extension. The same argument handles $S \cong W$: here $E_1 = 0$ because $W_1 = 0$, and any lift of the vacuum is killed by D . Hence only the two stated lowest weights remain, with $S \not\cong W$.

To determine the partial spectra, apply (12.1). If the positive odd sequence (ρ_i) is nonconstant, its cyclic differences include at least two nonzero even integers, so its total lowest weight is at least two. If it is constant and greater than one, that weight is at least $2N$. At weights zero and one, therefore, all $\rho_i = 1$. No negative sign gives the vacuum tuple $(0, \dots, 0)$; exactly one negative sign at site j gives the tuple e_j . The simple vacuum frame module has no total weight-one descendants. Composition series of the finite frame blocks consequently give exactly the stated possible generalized spectra, with no assertion of partial semisimplicity.

Finally, the nilpotent part of total $L(0)$ on E vanishes. It is a module endomorphism, kills S , is zero on the quotient, and so factors through a homomorphism $W \rightarrow S$. That homomorphism is zero because $S \not\cong W$. In particular $L(0) = 0$ on E_0 in the weight-zero case. $\square$

12.3 Exact identities for a weight-one vacuum defect

Suppose now $h(S) = 1$, choose the unique vacuum lift $v \in E_0$, and set

$$H_i = L_i(0), \quad w_i = L_i(-1)v, \quad w = \sum_i w_i.$$

Every positive-weight zero mode and every degree-lowering mode kills v . The w_i are jointly highest and satisfy $H_i w_j = \delta_{ij} w_j$. A nonsplit extension has $w \neq 0$ by Lemma 12.3.

For a positive-weight joint frame eigenvector $a \in W$, of total weight p and partial weights h_i , put $r_a = a_{p-2}v$. The following identities hold without a semisimplicity assumption on the partial zero modes acting on S_1 :

$$o(a)w_i = (h_i - H_i)r_a, \quad o(a)w = (p-1)r_a, \quad (12.2)$$

$$(L_i(-n)a)_{p+n-2}v = (-1)^n o(a)w_i \quad (n \geq 1). \quad (12.3)$$

Indeed, the iterate formula for $(\omega_{i1-n}a)_{p+n-2}v$ leaves only the term containing $\omega_{i0}v = w_i$: all a -modes of degree at most zero on v vanish and $\omega_{ik}v = 0$ for $k \geq 1$. This gives (12.3). The commutator with H_i at mode $p-2$ gives the first identity in (12.2); summing it uses $\sum_i H_i = 1$ on S_1 .

Lemma 12.5. *If the local factor of a joint primary a is $(3, 1)$ or $(1, 3)$, with partial weight h , then its first image satisfies*

$$H_i(H_i - 1)(H_i - h)r_a = 0.$$

In particular, its components at partial eigenvalues zero and one are ordinary at that site.

Proof. The local singular vector is

$$(L_i(-1)^3 - 2(h+1)L_i(-2)L_i(-1) + h(h+1)L_i(-3))a = 0.$$

Write $X = H_i$. Equations (12.2)–(12.3) give, respectively, the following polynomials on r_a for its three terms:

$$(X - h)(X - h - 1)(X - h - 2), \quad -(X - h)(X - h - 1), \quad X - h.$$

Their stated linear combination is $(X - h)X(X - 1)$. Here h is irrational, so these roots are distinct. The assertion follows on each finite generalized eigenspace. This is a direct first-image mode computation, rather than an ordinary Zhu calculation applied to a degree-raising mode. $\square$

For homogeneous states a, b of weights $p, q \geq 2$, the usual mode commutator and (12.2) give

$$o(a)o(b)w = (q-1) \sum_{\ell=0}^{p-1} \binom{p-1}{\ell} \frac{o(a_\ell b)w}{p+q-\ell-2}. \quad (12.4)$$

To see this, replace $o(b)w$ by $(q-1)b_{q-2}v$, commute $o(a)$ past b_{q-2} , and use $o(a)v = 0$. The summand is the first image of $a_\ell b$, whose weight is $p+q-\ell-1$. For two weight-two states, skew symmetry and $W_1 = 0$ reduce this to

$$\{o(a), o(b)\}w = o(a_1 b)w.$$

Together with the ordinary-root identities proved below, this anticommutator relation shows that $U = \text{span}\{w_i\}$ is invariant under every ordinary interval zero mode: one adjoins adjacent disjoint intervals inductively. We will use this invariant subspace without requiring a description of all of S_1 .

We also use the exact partial-conformal Zhu identity

$$[L_i(-n)a] = (-1)^n(-[\omega_i] * [a] + n[a] * [\omega_i] + h_i[a]), \quad n \geq 1. \quad (12.5)$$

For $n = 1$ this is the Zhu commutator formula; $n = 2$ follows by expanding $[\omega_i] * [a]$. The higher circle relations give the recurrence $[L_i(-n-2)a] + 2[L_i(-n-1)a] + [L_i(-n)a] = 0$, proving all remaining cases. When it is iterated, h_i is the weight of the descendant currently being reduced, not the weight of its original primary.

12.4 An algebraic criterion for second-order nilpotence

The following form of Nakayama’s lemma controls the partial nilpotent operators on the first cyclic image.

Lemma 12.6. *Let V be finite-dimensional, let commuting operators H_i have joint generalized spectrum e_j on a component, and suppose $\sum_i H_i = 1$. Let r be a vector in that component. Suppose polynomials $K_0, \dots, K_{N-2}$ vanish at e_j , their gradients are independent modulo $\sum_i (H_i - \delta_{ij}) = 0$, and every $K_a(H)r$ is an ordinary simultaneous eigenvector. Then*

$$(H_i - \delta_{ij})(H_k - \delta_{kj})r = 0 \quad \text{for all } i, k.$$

Proof. On the cyclic space generated by r , let $\mathcal{A}$ be the finite commutative algebra generated by $n_i = H_i - \delta_{ij}$. It is local with nilpotent maximal ideal $\mathfrak{m} = (n_i)$. Invertibility of the gradients says that the K_a span $\mathfrak{m}/\mathfrak{m}^2$; Nakayama’s lemma makes them generators of $\mathfrak{m}$. Ordinarity of their values gives $\mathfrak{m}K_a(H)r = 0$, so $\mathfrak{m}^2r = 0$. The maximal ideal is nilpotent because $\mathcal{A}$ is a finite local algebra. The argument therefore requires no additional bound from a $(3, 3)$ singular vector. $\square$

12.5 The ordinary-root defect calculation

The ordinary roots determine the initial defect space without any classification of its ambient module. We give the calculation because it is used in every interval equation below.

Lemma 12.7. *For an ordinary root $R = R_j$, $1 \leq j < N$, write*

$$a = 1 + \frac{2}{t+j}, \quad b = 2 - a, \quad r = R_0v, \quad O = R_1, \quad w_a = w_j, \quad w_b = w_{j-1}.$$

In the root normalization with nonzero scale u , one has the formulas below. For the root with leading charge coefficient one, u equals the screening parameter. Rescaling R rescales u by the same factor, so the formulas apply to independently normalized ordinary roots.

$$\begin{aligned} r &= u \left(\frac{a+1}{(a-2)(a-1)} w_a + \frac{a-3}{a(a-1)} w_b \right), \\ Ow_a &= u \left(\frac{a+1}{a-2} w_a + \frac{a-3}{a-1} w_b \right), \\ Ow_b &= u \left(-\frac{a+1}{a-1} w_a - \frac{a-3}{a} w_b \right), \quad Ow_i = 0 \quad (i \notin \{j-1, j\}). \end{aligned}$$

If the vacuum extension is nonsplit, all w_i are nonzero and $U = \text{span}\{w_i\}$ has dimension N .

Proof. Off the two active sites $L_i(-1)R = 0$, so (12.3) gives $H_i r = 0$ there. At the active sites Lemma 12.5 and Proposition 12.4 make r an ordinary sum $r_a + r_b$ with spectra $(1, 0)$ and $(0, 1)$. The first-image commutator gives

$$Ow_a = (a-1)r_a + ar_b, \quad Ow_b = br_a + (b-1)r_b.$$

The explicit ordinary-root OPE, in the stated normalization, has vacuum and self-channel coefficients

$$\nu = u^2 \frac{3(3a-5)(3a-1)}{a(a-2)(a-1)^2}, \quad \sigma = u \frac{8(2a-1)(2a-3)}{a(a-1)(a-2)}. \quad (12.6)$$

These are the ordinary-edge specializations of the root product calculation in Proposition 9.2. The next aligned channel has lowest weight eight, so only these two channels occur through weight four.

For a local degenerate weight h , put

$$\begin{aligned} c_h &= \frac{-3h^2 + 7h - 2}{h+1}, \quad A_h = \frac{h(h+1)}{(h+2)(2h-1)}, \quad B_h = \frac{(h-2)(h+1)}{4(h+2)(2h-1)}, \\ \alpha_h &= \frac{3h(c_h - 2h + 4)}{c_h(5c_h + 22)}, \quad \beta_h = \frac{2h(5h+1)}{c_h(5c_h + 22)}. \end{aligned}$$

The level-two self-channel is $A_h L(-2)R + B_h L(-1)^2 R$, and the level-four vacuum channel is $\alpha_h L(-4)\mathbf{1} + \beta_h L(-2)^2 \mathbf{1}$. Applying $L(1), L(2)$ to the former and $L(2), L(4)$ to the latter gives the displayed coefficients.

The identity $R_2R = 0$, together with skew symmetry, gives

$$Or = \nu \left(\frac{a}{c_a} w_a + \frac{b}{c_b} w_b \right) + \frac{\sigma}{2} r.$$

The iterate formula also gives $(R_{-1}R)_2v = Or$. Its vacuum channel contributes $\nu((\alpha_a + \beta_a)w_a + (\alpha_b + \beta_b)w_b)$. Its self-channel contributes $\sigma F(H_j, H_{j-1})r$, where

$$\begin{aligned} F(x, y) = & A_a(a - x) + B_a(x - a - 1)(x - a) \\ & + A_b(b - y) + B_b(y - b - 1)(y - b) + \frac{1}{4}(x - a)(y - b). \end{aligned}$$

Here the mixed vacuum descendant contributes zero, and the mixed self descendant has coefficient $1/4$. These statements follow directly from (12.2)–(12.3). The two scalar denominators in comparing the formulas are

$$F(1, 0) - \frac{1}{2} = -\frac{a(a-2)(3a-5)}{4(a-4)(2a-3)(2a-1)}, \quad F(0, 1) - \frac{1}{2} = -\frac{a(a-2)(3a-1)}{4(a+2)(2a-3)(2a-1)}.$$

They are nonzero because a is irrational. Substituting (12.6) yields the asserted r_a, r_b , and hence the two matrices.

Every off-diagonal coefficient is nonzero. If one w_i vanished, the equations on each adjacent ordinary edge would make its neighbor vanish. Connectedness of the ordinary path would give $w = 0$, splitting the extension. Thus all w_i are nonzero. Their distinct joint spectra make them linearly independent. $\square$

The anticommutator identity now gives invariance of U under every ordinary interval. If adjacent disjoint intervals J, K are shorter than their union, then $(P_J)_1 P_K = \kappa P_{J \cup K}$ with $\kappa \neq 0$, so $o(P_{J \cup K})w \in U$. Equation (12.2) then puts $o(P_{J \cup K})w_i$ in U for every i .

13 Proper intervals and the cyclic first image

Let $H_* = 2N + 2$. Every connected proper interval containing root zero has primary P_J of weight H_* ; ordinary intervals have weight two. Write

$$r_J = (P_J)_{H_*-2}v.$$

For the full cyclic primary C , set

$$H = 2N, \quad h_i = \text{wt}_i(C), \quad r_C = C_{H-2}v, \quad H_i = L_i(0).$$

The universal frame commutators give

$$o(C)w_i = (h_i - H_i)r_C, \quad o(C)w = (H - 1)r_C. \quad (13.1)$$

13.1 Semisimplicity of proper-interval first images

Appendix A proves that every heavy proper-interval first image r_J is a sum of simultaneous eigenvectors of the H_i , with eigenvalue tuples among the e_i . Thus the partial nilpotent operators vanish on these images. The proof uses growth by adjacent weight-two intervals. For a prefix $J = [0, k]$, cutting at $a < k$ gives the operator

$$D_a = \sum_{i=a+1}^k H_i - (t+a)H_a - 1$$

on $r_{[0,k]}$, with right-hand side a linear combination of $(h_i - H_i)r_{[0,a]}$. On the old eigencomponents and the fixed endpoint e_{N-1} , D_a has nonzero scalar part; induction forces ordinariness. On the new endpoint e_k , the right-hand sides vanish for all cuts. The endpoint $(3, 1)$ null relation makes H_k semisimple, and the cut equations then successively kill the internal nilpotents.

The reversed growth proves the suffix statement. For a wrapping proper interval with growth on both sides of zero, a cut on the side opposite a given eigencomponent has nonzero scalar part and reduces to a shorter interval. These arguments prove the claim for every proper heavy interval. The full operator identities and both directions of growth are given in Appendix A.

13.2 The closing-root product through level three

For the ordinary complementary interval $P = P_{[1, N-1]}$ and the closing root R_0 , canonical fusion gives

$$(R_0)_3 P = \kappa C, \quad (R_0)_j P = 0 \quad (j > 3), \quad \kappa \neq 0.$$

The other singular coefficients are level-one, level-two, and level-three descendants of C . For a rank-one parameter p , the $(3, 1)$ followed by $(1, 3)$ descendant coefficients are

$$\begin{aligned} v_1 &= \frac{p}{p-1} L_{-1} C, \\ v_2 &= -\frac{2p}{(p-2)(2p-1)} L_{-2} C + \frac{p^2(2p-3)}{2(p-2)(p-1)(2p-1)} L_{-1}^2 C, \\ v_3 &= -\frac{p+2}{3(p-2)} L_{-3} C - \frac{2p}{3(p-2)(p-1)} L_{-2} L_{-1} C + \frac{p^2}{6(p-2)(p-1)} L_{-1}^3 C. \end{aligned}$$

Reversing the inputs gives

$$\begin{aligned} \tilde{v}_1 &= -\frac{1}{p-1} L_{-1} C, \\ \tilde{v}_2 &= -\frac{2p}{(p-2)(2p-1)} L_{-2} C + \frac{3p-2}{2(p-2)(p-1)(2p-1)} L_{-1}^2 C, \\ \tilde{v}_3 &= \frac{2p+1}{3(2p-1)} L_{-3} C - \frac{2p}{3(p-1)(2p-1)} L_{-2} L_{-1} C + \frac{1}{6(p-1)(2p-1)} L_{-1}^3 C. \end{aligned}$$

The Ward equations for L_1, L_2, L_3 determine these coefficients uniquely, since the output has no singular vector below level nine.

13.3 The all-cut polynomial system

For each prefix cut $a = 0, \dots, N-2$, let $B_a = P_{[0,a]}$, $A_a = P_{[a+1, N-1]}$, and normalize

$$(B_a)_3 A_a = \kappa_a C, \quad v_a = \kappa_a^{-1} o(B_a) o(A_a) w, \quad \kappa_a \neq 0.$$

Put

$$r = t + a, \quad Z = \sum_{i < a} H_i, \quad X = H_a, \quad Y = H_{N-1}.$$

Tensoring the preceding local coefficients with the vacuum descendants at the prefix sites and applying (12.4) gives

$$K_a(H) r_C = v_a. \tag{13.2}$$

Each v_a is a sum of simultaneous eigenvectors. Indeed, $o(A_a)w \in U$, and the analogue of (13.1) for B_a expresses its image under $o(B_a)$ in terms of the semisimple proper-interval first image. Appendix B gives a finite explicit polynomial prescription for every K_a .

The polynomial K_a vanishes at each allowed joint eigenvalue. Its gradient with respect to (Z, X, Y) has four forms:

$$\begin{aligned} G_{\text{suffix}} &= \left(-\frac{1}{6}, -\frac{r+1}{6}, -\frac{t-1}{6t} \right), \\ G_{\text{prefix}} &= \left(\frac{1}{3}, \frac{(r+1)(r^2+4r-2)}{3(r-1)(r+2)}, \frac{(t-1)(2t^2+4t-1)}{3t(t+1)(2t-1)} \right), \\ G_{\text{cut}} &= \left(\frac{3r+2}{6}, -\frac{r+1}{3(r-1)}, \frac{(t-1)(6rt^2+9rt-3r+4t^2+8t-2)}{6t(t+1)(2t-1)} \right), \\ G_{\text{last}} &= \left(\frac{2t-3}{6t}, \frac{(r+1)(2r^2t-3r^2+8rt-9r-4t+6)}{6t(r-1)(r+2)}, \frac{t-1}{3(t+1)} \right). \end{aligned}$$

For the component e_j , use prefix if $j < a$, cut if $j = a$, suffix if $a < j < N-1$, and last if $j = N-1$. All denominators are nonzero for $r = t + a$.

Theorem 13.1 (Uniform second-order nilpotent bound). *In the hypothetical weight-one extension above, decompose the first cyclic image as $r_C = \sum_j r_{C,j}$, with joint generalized eigenvalue e_j . Put $N_i = H_i - \delta_{ij}$ on that component. Then, for every odd $N \geq 3$,*

$$N_i N_{\ell} r_{C,j} = 0 \quad \text{for all } i, \ell, j.$$

The statement does not assert $N_i r_{C,j} = 0$.

Proof. The operators N_i commute and are nilpotent on the finite joint generalized eigenspace, and $\sum_i N_i = 0$ because total $L(0)$ is the scalar one on S_1 . Appendix D proves that the Jacobian formed from all the displayed gradients is invertible after imposing this sum.

Lemma 12.6, applied to the ordinary right-hand sides of (13.2), now gives the assertion. $\square$

The theorem kills products of two partial nilpotents, but may leave first-order nilpotent images. To eliminate these, we next prove that the proper heavy interval images on the right-hand sides of (13.2) vanish.

13.4 All first partial nilpotents vanish

Theorem 13.2 (Uniform first-image elimination). *For every odd $N \geq 3$, in the hypothetical weight-one extension,*

$$r_B = 0 \quad \text{for every proper heavy cyclic interval } B, \quad (H_i - \delta_{ij})r_{C,j} = 0 \quad \text{for all } i, j.$$

Consequently every proper heavy interval zero mode annihilates U , and r_C is a sum of simultaneous partial eigenvectors. The final contradiction will use the consequence $o(R_0)U = 0$.

Proof. Appendix E proves that the two possible components of the closing-root first image r_{R_0} vanish. For $N \geq 5$, triangle differences of complementary-interval equations give the result uniformly; for $N = 3$, two explicit matrices have nonzero determinants in $\mathbb{Q}[t]/(t^2 + 3t - 1)$.

An ordinary interval zero mode preserves U . If a heavy interval B annihilates U , its commutator with an adjoining ordinary interval therefore annihilates U . The unique order-two OPE identifies this commutator with $\kappa[Q, o(B')]$, where B' is their proper union and Q is an explicit linear combination of the H_i . The first image of B' is ordinary, and Q has at least two distinct coefficients on its active support. Evaluating on suitable source vectors w_i forces every component of $r_{B'}$ to vanish. Starting at R_0 proves this for every proper heavy interval.

All right sides of (13.2) consequently vanish. The same invertible Jacobian now gives $\mathfrak{m}r_{C,j} = 0$, proving the claim. The detailed scalar coefficients and propagation identity are given in Appendix E. $\square$

14 A second ground character from the lattice embedding

Here a *ground character* means a one-dimensional representation of the Zhu algebra on the lowest conformal-weight space. We construct one that differs from the vacuum character; it will determine the second root of the cyclic Zhu relation in the next section.

The construction gives a conformal embedding $W \subset A$ into the deformed lattice vertex operator algebra, with

$$A_0 = \mathbb{C}\mathbf{1} \oplus \mathbb{C}\varepsilon, \quad \varepsilon = e^B, \quad \langle \mathbf{1}, \mathbf{1} \rangle = 1, \quad \langle \varepsilon, \mathbf{1} \rangle = -\frac{1}{2\lambda}, \quad \lambda \neq 0.$$

The frame zero modes kill both ground vectors. For $\mathbf{1}$ this is the vacuum property. For ε , the undeformed partial weights are zero, while every correction to a frame vector has strictly positive charge. Such a correction has no degree-zero target above charge B and hence also kills ε . Normalize the full cyclic primary C by its leading charge term e^{-B} .

Each elementary highest vector was constructed in the cone above its charge $-\beta_j$. For disjoint binary root supports, diagonal fusion has only their union as output. Taking the highest coefficient of successive disjoint products therefore places every binary highest vector in the cone above $-\sum_{j \in J} \beta_j$. In root coordinates, a primary with binary support J thus has charges componentwise at least $-\mathbf{1}_J$.

Lemma 14.1. *There is a simple grading-restricted W -module T of lowest weight zero whose ground character has*

$$H_i = 0, \quad o(P_J) = 0 \quad (J \text{ a proper interval}), \quad o(C) = 2\lambda.$$

Its existence does not require semisimplicity of the restriction of A to W .

Proof. For a proper interval J , choose a coordinate absent from its support. Applied to ε , its primary zero mode can only have charge B in degree zero: charge zero is excluded by that coordinate. Thus $o(P_J)\varepsilon$ is a scalar multiple of ε . This scalar is zero. Indeed, P_J is a primary of even total weight, so its zero mode is self-adjoint for the invariant form, and

$$\langle o(P_J)\varepsilon, \mathbf{1} \rangle = \langle \varepsilon, o(P_J)\mathbf{1} \rangle = 0, \quad \langle \varepsilon, \mathbf{1} \rangle \neq 0.$$

For C , extraction of its leading charge gives $o(C)\varepsilon = \mathbf{1} + d\varepsilon$. The same pairing gives $0 = 1 - d/(2\lambda)$, and hence

$$o(C)\varepsilon = \mathbf{1} + 2\lambda\varepsilon.$$

Thus the nonzero one-dimensional line $(A/W)_0$ has the stated Zhu character. Let M be the W -submodule of A/W generated by that line. Any proper submodule misses its degree-zero line, because that line generates M . The sum of all proper submodules still misses degree zero and is therefore proper. The quotient by this sum is simple, grading restricted, and retains the same ground character; call it T . Since its C -eigenvalue differs from the vacuum value zero, $T \not\cong W$. $\square$

15 Lowest weight zero and the Zhu algebra

We now treat lowest weight zero. A spanning result for the Zhu algebra controls the number of factors involving the distinguished root. Appendix C gives the full maximum-plateau argument; we record its filtered consequence below.

15.1 A filtered spanning lemma

The partial frame classes, all proper cyclic interval highest classes, and the full-cycle class $[C]$ generate $A(W)$. Disconnected binary supports factor by regular normal products; higher labels reduce by the maximum-plateau induction in Appendix C. Let $A_0 \subset A(W)$ be the ordinary open-chain image, including the partial frame classes, and put

$$F_p W = \bigoplus_{m_0 \leq p} M_m.$$

Aligned fusion gives $(F_p W)_n (F_q W) \subset F_{p+q} W$. If the first input has label zero at the distinguished root, fusion preserves the second input's label at that root.

Lemma 15.1 (Filtered Zhu span). *One has*

$$[F_1 W] \subseteq A_0 + \sum_{P \in \{C, \text{ proper heavy intervals}\}} A_0[P]A_0. \quad (15.1)$$

Proof. If $m_0 \leq 1$ and $\max_i m_i \geq 2$, peel a connected maximal plateau with indicator u . It avoids root zero. The top-channel weight formula gives $\Delta = h(m) - h(u) - h(m - u) \geq 0$, and $(v_u)_{-\Delta-1} v_{m-u}$ has a nonzero v_m component. Every other aligned output has strictly smaller total label and the same root-zero label. Induction on total label therefore retains at most one root-zero-bearing binary factor in a normal-product expression. Partial Virasoro descendants retain this count: each $L_i(-1)$ is a derivation, and higher negative Virasoro modes add only a frame factor. A terminal binary support has at most one connected component containing zero, which is heavy or the full cycle. Its other components lie in the open chain.

Pass to Zhu classes by a second induction on total conformal weight. Replacing a normal product by its Zhu product leaves only lower-weight singular terms, still in $F_1 W$, so the count is preserved. Higher negative modes are handled by total derivatives. Partial descendants reduce using

$$[L_i(-n)a] = (-1)^n (-H_i * [a] + n[a] * H_i + h_i[a]), \quad n \geq 1,$$

which adds only left and right frame classes. This proves (15.1) without commuting its noncommutative factors. $\square$

15.2 The two-character quotient

Theorem 15.2 (Closure of the weight-zero branch). *Let M be a finite-dimensional Zhu representation with generalized partial joint spectrum zero and exact total conformal operator $\sum_i H_i = 0$. Then $H_i = 0$ for every i , every proper interval primary acts by zero, and*

$$C(C - 2\lambda) = 0, \quad \lambda \neq 0.$$

In particular,

$$A(W)/\langle H_0, \dots, H_{N-1} \rangle \cong \mathbb{C}[X]/(X(X - 2\lambda)) \cong \mathbb{C} \oplus \mathbb{C}.$$

The relevant aligned lowest-weight-zero simples are exactly W, T , and every vacuum extension in this branch splits.

Proof. Appendix F first proves that ordinary roots commute with all the partial nilpotents. Comparing a root square with its derivative commutator gives

$$(t + j + 1)H_j - (t + j - 1)H_{j-1} \in \mathfrak{m}^2,$$

where $\mathfrak{m}$ is the nilpotent ideal generated by the H_i . Together with $\sum_i H_i = 0$, the positive recurrence coefficients give $\mathfrak{m} = \mathfrak{m}^2$, hence $\mathfrak{m} = 0$. A difference of endpoint commutators then kills every ordinary interval; the possible height-two output channels cancel explicitly.

Appendix G eliminates maximal proper heavy intervals using two independent coefficients in an ordinary-root product with C , calibrated on the ground character constructed in Lemma 14.1. A nonzero level-two coefficient descends to every heavy interval. Lemma 15.1 now implies $C^2 = \alpha + \beta C$: the square has weight at most $4N$, below the $6N$ threshold for root-zero label two. The actual vacuum and T_0 characters give $\alpha = 0$, $\beta = 2\lambda$, and both distinct characters occur. To make this an identity of the quotient algebra, apply the preceding operator relations to its left regular module; it is finite-dimensional because $A(W)$ is finite-dimensional. This proves the exact quotient. Zhu’s lowest-space correspondence [19, Theorem 2.2.2] identifies its two simple characters with the already constructed simple modules W and T .

Proposition 12.4 verifies these hypotheses for the aligned lowest-weight-zero tops and the vacuum extensions under consideration. For an extension of W by a distinct simple of weight zero, the nilpotent part of total $L(0)$ factors through $\text{Hom}(W, S) = 0$, so the total operator on its top is also zero. Lemma 15.3 below lifts the splitting of the lowest space to a splitting of the module extension. $\square$

15.3 Why a split top splits a vacuum extension

Lemma 15.3 (Lifting a splitting of the lowest space). *For a simple $S \not\cong W$ of lowest weight zero, restriction to degree zero induces an injection*

$$\text{Ext}_W^1(W, S) \hookrightarrow \text{Ext}_{A(W)}^1(\mathbb{C}_{\text{vac}}, S_0).$$

The left-hand side is computed in grading-restricted generalized modules, and $\mathbb{C}_{\text{vac}}$ is the vacuum Zhu character.

Proof. Suppose the induced extension of lowest spaces splits. Choose a lift $v \in E_0$ with the vacuum Zhu character. Every positive-weight zero mode kills v . The commutator

$$a_n L(-1)v = L(-1)a_n v + n a_{n-1} v$$

shows that all degree-lowering modes kill $L(-1)v$. PBW degree ordering puts its generated submodule in weights at least one. It lies in the simple S , whose weight-zero space is nonzero, so $L(-1)v = 0$. The lift is vacuum-like and splits the original extension. Thus the map on extension classes has zero kernel. $\square$

The semisimple quotient in Theorem 15.2 supplies the splitting of the lowest space. Vacuum self-extensions were already handled in Proposition 12.4.

The exact-total-operator hypothesis matters: the theorem is not an assertion that every generalized zero-eigenvalue block of an arbitrary module is semisimple. The argument needs precisely the simple tops and vacuum extensions specified above.

16 The closing-root square excludes weight-one extensions

Theorem 16.1 (Closure of the weight-one branch). *There is no nonsplit extension of W by a simple module of lowest weight one.*

Proof. Assume such an extension exists. Its nonzero defect space belongs to the aligned odd collection by Proposition 12.4. That space $U = \text{span}\{w_j\}$ has $H_i w_j = \delta_{ij} w_j$, with every $w_j \neq 0$. Theorem 13.2 gives $o(R_0)U = 0$. The partial Zhu descendant formula then shows that every partial Virasoro descendant of R_0 also annihilates U . This uses only $H_i U \subset U$; it does not require a classification of the full target top.

Put

$$K = \text{wt}(R_0) = 2N + 2, \quad a = 1 + 2/t, \quad b = 1 - 2t, \quad a + b = K.$$

The only aligned channels of $R_0 \times R_0$ are the vacuum, R_0 , and $2e_0$. The latter has lowest weight $6N + 8 > 2K = 4N + 4$, so it cannot occur in the Zhu square. Every contribution of the self-channel annihilates U .

At a local degenerate weight h , evaluation of the normalized vacuum channel on Virasoro highest weight one has exact series

$$V_h(z) = {}_3F_2\left(\begin{matrix} -h-1, -h, 2 \\ 1-h/2, (1-3h)/2 \end{matrix}; -\frac{z^2}{4(1+z)}\right).$$

Appendix H derives this formula from the level-three null relation and proves its uniqueness as the normalized vacuum solution. On w_0 , the active partial weights are $(1, 0)$, so the other endpoint vacuum series evaluates to 1. The resulting Zhu coefficient is

$$\begin{aligned} P_K(a) &= [z^{2K}](1+z)^K V_a(z) \\ &= \left(-\frac{1}{4}\right)^K \frac{(-a-1)_K (-a)_K (2)_K}{(1-a/2)_K ((1-3a)/2)_K K!} \neq 0. \end{aligned} \tag{16.1}$$

To see the extraction directly, write $V_a(z) = \sum_n d_n z^{2n} (1+z)^{-n}$. For $n < K$, the corresponding term after multiplication by $(1+z)^K$ is a polynomial of degree $K+n < 2K$. For $n > K$, its lowest degree is $2n > 2K$. Only d_K remains. Irrationality of a excludes all zeros and poles of the finite Pochhammer products.

Let $\nu_0 \neq 0$ be the vacuum-channel coefficient. Its nonvanishing follows from the nondegenerate invariant form on the multiplicity-one self-contragredient frame summand M_{e_0} . Applying $R_0 * R_0$ to w_0 gives

$$0 = \nu_0 P_K(a) w_0 \neq 0,$$

a contradiction. This proves the theorem. $\square$

Only the closing-root first-image vanishing is used in this proof; no description of repeated cyclic action on the full target top is required.

17 Vacuum projectivity and regularity

The passage from vacuum projectivity to semisimplicity is categorical. We isolate it from the extension calculations, using the rigidity criterion of Etingof–Penneys [21, Theorem 1.1].

Lemma 17.1 (A projective-unit criterion). *Let $\mathcal{C}$ be a finite $\mathbb{C}$ -linear abelian braided monoidal category with simple unit $\mathbf{1}$ and right-exact tensor product. Suppose it has a contravariant duality $X \mapsto X'$ with natural isomorphisms*

$$\text{Hom}(Y \boxtimes X, \mathbf{1}) \cong \text{Hom}(Y, X').$$

If $\mathbf{1}$ is projective, then $\mathcal{C}$ is rigid and semisimple.

Proof. For every nonzero X , the identity of X' corresponds to a nonzero map $e_X : X' \boxtimes X \rightarrow \mathbf{1}$. Since the unit is simple, e_X is an epimorphism; projectivity supplies a section. Thus $\mathbf{1}$ is a direct summand of $X' \boxtimes X$. Every indecomposable X is therefore non-negligible in the sense of Etingof–Penneys. This summand argument does not assume the snake identities.

We verify moderate growth without using tensor exactness. Write ℓ for composition length, let S_i run through the finitely many simple objects, and set $c_X = \max_i \ell(S_i \boxtimes X)$. Right exactness and induction on a composition series give:

$$\ell(Y \boxtimes X) \leq c_X \ell(Y), \quad \ell(X^{\boxtimes n}) \leq \ell(X) c_X^{n-1}.$$

Every simple endomorphism algebra is $\mathbb{C}$. Induction on length in the two variables, using left exactness of Hom , gives $\dim \text{Hom}(A, B) \leq \ell(A) \ell(B)$. Consequently:

$$\dim \text{End}(X^{\boxtimes n}) \leq \ell(X)^2 c_X^{2n-2} < n!$$

for sufficiently large n . The category is idempotent complete, has finite-dimensional Hom spaces and finite direct sums, and $\text{End}(\mathbf{1}) = \mathbb{C}$. Etingof–Penneys therefore makes each indecomposable object rigid. Finite direct sums of rigid objects are rigid, so $\mathcal{C}$ is rigid.

The tensor functors now have both adjoints and are exact. For every X , rigidity gives:

$$\text{Hom}(X, -) \cong \text{Hom}(\mathbf{1}, X^* \boxtimes -).$$

The right-hand side is exact because the unit is projective. Every object is therefore projective, and the category is semisimple. $\square$

Theorem 17.2 (Regularity). *The vertex operator algebra W_N is regular.*

Proof. Let $\mathcal{C}$ be the category of grading-restricted generalized W_N -modules. The CFT-type, C_2 -cofiniteness and self-contragredience properties established in Part I allow us to apply Huang’s vertex tensor-category theorems [15] and the duality theorem [20, Theorem 2.12]. Thus $\mathcal{C}$ is a finite abelian braided Grothendieck–Verdier category, with right-exact tensor product and dualizing object $W'_N \cong W_N$.

Proposition 12.4 reduces a nonsplit extension of the vacuum by a simple module to lowest weight zero or one. Theorems 15.2 and 16.1 exclude these two cases. Hence:

$$\text{Ext}_{\mathcal{C}}^1(W_N, S) = 0 \quad \text{for every simple } S. \quad (17.1)$$

Induction on length and the long exact sequence for Ext extend this vanishing to every object. The vacuum is projective, and Lemma 17.1 makes $\mathcal{C}$ rigid and semisimple.

It remains to pass from grading-restricted modules to all weak modules. By [22, Proposition 2.11], every finitely generated weak module for an $\mathbb{N}$ -graded C_2 -cofinite VOA is grading-restricted generalized. Every cyclic weak W_N -module is therefore a finite sum of ordinary simple modules. An arbitrary weak module is the sum of its cyclic submodules, hence a sum of simple submodules. Such a module is semisimple. Thus every weak W_N -module is a direct sum of irreducible ordinary modules, as required. In particular W_N is rational; compare [23]. $\square$

18 The parity factor after regularity

By Theorem 17.2, the category $\mathcal{C} = \text{Rep}(W_N)$ is semisimple and rigid. McRae’s factorizability theorem [22, Theorem 5.3], applied to the simple, self-contragredient, $\mathbb{N}$ -graded, C_2 -cofinite VOA W_N , makes it a modular tensor category. The parity modules constructed earlier generate a pointed braided subcategory $\mathcal{P}$. Its self-braiding form is:

$$Q(\epsilon) = i^{\epsilon K_+ \epsilon}, \quad \epsilon \in (\mathbb{Z}/2)^N.$$

The monodromy pairing is $(-1)^{\epsilon K_+ \eta}$. Since $\det K_+ = N^2$ is odd, this pairing is nondegenerate. The centralizer factorization for a nondegenerate fusion subcategory [24] consequently gives a ribbon equivalence:

$$\mathcal{C} \simeq \mathcal{P} \boxtimes \mathcal{D}, \quad \mathcal{D} = \mathcal{P}'.$$

Every simple has a unique expression $J_\epsilon \boxtimes Z$ with $Z \in \text{Irr } \mathcal{D}$. In particular the parity action on simples is free, every orbit has 2^N members, and it has a unique representative in $\mathcal{D}$. This use of finite semisimple category theory occurs after regularity, and is not an input to its proof.

Part III

Simple modules and modular traces

19 Frame decomposition of the nonzero periodic lattice sectors

Let $L = (\mathbb{Z}^N, 2K_+)$ and $B = \sum_i \beta_i$. Choose a class $\xi \in L^*/L$ with nonzero odd-order component γ . In the coordinate basis β_i , write:

$$\gamma_i = \frac{a + ib}{N} \pmod{\mathbb{Z}}, \quad \xi = \gamma + \frac{\eta}{2}, \quad \eta \in \{0, 1\}^N.$$

Here $a, b \in \mathbb{Z}/N\mathbb{Z}$ are not both zero. We write $M_\xi = V_{L+\xi} \downarrow W_N$ for restriction to the conformally embedded algebra. A charge vector $k = (k_i)$ denotes the lattice vector $\sum_i k_i \beta_i$; thus e^k is its lattice exponential.

The frame decomposition follows from the block projections of Lemma 12.1 and the triangularity of the screening deformation: every correction strictly raises the total coordinate height $\sum_i k_i$. We first show that the undeformed charge Focks occupy distinct singleton blocks.

Lemma 19.1 (Nondegeneracy and separation of charge Focks). *For every charge $k \in \xi + \mathbb{Z}^N$, the undeformed tensor Fock space $F(k)$ is an irreducible tensor product of nondegenerate Virasoro Verma modules. Its joint frame linkage block is a singleton, and no other charge in the same coset has the same joint block.*

Proof. A nonzero affine progression modulo N cannot have two consecutive zero entries: their difference would give $b = 0$, and then $a = 0$. The Kac labels at site i are $R_i = 1 - 2k_i$ and $S_i = 1 - 2k_{i+1}$. At least one of these is nonintegral, because their integrality would make $\gamma_i = \gamma_{i+1} = 0$. If its weight were a degenerate Kac weight $h_i(r, s)$ with positive integers r, s , the difference-of-squares identity and irrationality of ψ_i would force $(R_i, S_i) = \pm(r, s)$, a contradiction. The Verma module at that weight is therefore irreducible. The charge Fock space has exactly its character, so the highest-vector homomorphism from the irreducible Verma is an isomorphism. The irrational Virasoro linkage theorem makes its block a singleton.

If $F(k)$ and $F(k')$ have the same joint block, then at every site $(R'_i, S'_i) = \sigma_i(R_i, S_i)$ with $\sigma_i = \pm 1$. A negative sign would force $2\xi_i, 2\xi_{i+1} \in \mathbb{Z}$, hence $\gamma_i = \gamma_{i+1} = 0$, because k, k' lie in the same lattice coset. This is impossible, so every sign is positive and $k' = k$. $\square$

Proposition 19.2 (Deformed frame decomposition). *There are canonical simple $\mathcal{B}$ -submodules $E_k \subset M_\xi$ indexed by $k \in \xi + \mathbb{Z}^N$, and*

$$M_\xi \downarrow \mathcal{B} = \bigoplus_{k \in \xi + \mathbb{Z}^N} E_k, \quad E_k \cong F(k).$$

Projection to the lowest charge identifies E_k with $F(k)$. In particular its highest vector f_k can be normalized to have lowest charge k and leading term e^k .

Proof. Filter M_ξ by scalar charge height $d(k) = \sum_i k_i$, using the descending subspaces $F_{\geq d}$ of vectors supported in heights at least d . These are $\mathcal{B}$ -submodules because every frame correction strictly raises height. The associated quotient at height d is the direct sum of the undeformed charge Focks at that height.

Fix the singleton block of $F(k)$. Any nonzero vector of the corresponding deformed block has a minimum charge height. Projecting its generated finite-length frame module to that minimum height gives a nonzero map to the undeformed Focks at that height. Since the only such Fock in this block is $F(k)$, that minimum must be $d(k)$, and the image lies in $F(k)$. Consequently projection to $F(k)$ is injective on the entire deformed block: a vector in its kernel would have strictly larger minimum height. It is surjective by exact block projection applied to $F_{\geq d(k)} \twoheadrightarrow F_{\geq d(k)}/F_{> d(k)}$. Thus the deformed block is $F(k)$. The same minimum-height argument excludes all other blocks. $\square$

Proposition 19.3 (Downward charge transitions). *For every elementary generator R_j and every k , the component of its vertex action from E_k to $E_{k-\beta_j}$ is nonzero.*

Proof. The normalized highest vector $v_j \in R_j$ has lowest charge $-\beta_j$ and leading term $e^{-\beta_j}$, including for the closing generator $j = 0$. All its other charges have strictly larger scalar height. The normalized f_k has lowest charge k . Therefore the minimum-charge part of $Y(v_j, z)f_k$ is exactly $Y(e^{-\beta_j}, z)e^k$, which is nonzero. Projecting to the canonical frame block of $F(k - \beta_j)$ retains this leading term. Hence the asserted block component is nonzero. $\square$

A nonzero submodule could a priori be supported on a proper subset closed under downward charge transitions. To prove simplicity, we also need the upward transitions. They are obtained from the monodromy of the level-three BPZ equation in the next section.

20 BPZ monodromy and simplicity of the periodic sectors

We prove that every elementary generator acts nontrivially in both charge directions. The argument combines the frame decomposition in Proposition 19.2 with Mimachi’s irreducibility criterion for the Dotsenko–Fateev equation [25, Theorem 11]. At this stage we use only the frame restrictions of the lattice modules; the corresponding statement for arbitrary modules is proved in Section 21.

20.1 Frame labels and vacuum coefficients

Write $\mathcal{B} = \bigotimes_i L(c_i, 0)$ and $M_\xi = V_{L+\xi} \downarrow W_N$, where the odd-order component of $\xi \in L^*/L$ is nonzero. The exact frame decomposition is

$$M_\xi = \bigoplus_{k \in \xi + \mathbb{Z}^N} E_k.$$

Every E_k is an irreducible tensor product of generic Virasoro Verma modules, occurs once, and has highest vector f_k with rational Kac labels

$$R_i = 1 - 2k_i, \quad S_i = 1 - 2k_{i+1}.$$

All these labels have odd denominator. At every site at least one of R_i, S_i is nonintegral. The summands E_k are pairwise nonisomorphic. The elementary generator $v_j \in R_j$ has $(3, 1)$ at site j , $(1, 3)$ at site $j - 1$, and the vacuum elsewhere.

Put

$$a = \psi_j, \quad b = \psi_{j-1}^{-1}, \quad \kappa_j = a + b = \begin{cases} 2, & j \neq 0, \\ N + 2, & j = 0. \end{cases}$$

Both a, b are positive irrational numbers. At site $j - 1$ use the equivalent parametrization $b = \psi_{j-1}^{-1}$, interchanging the two Kac labels. Thus both active factors of v_j are of type $(3, 1)$, at parameters a and b respectively. Their source momenta are

$$p = aR_j - S_j, \quad q = bS_{j-1} - R_{j-1}.$$

In particular

$$p + q = \kappa_j - 2 - 2(K_+k)_j \in \mathbb{Z}.$$

The generator has a nonzero vacuum two-point coefficient:

$$(v_j)_{2\text{wt}(v_j)-1}v_j = \nu_j \mathbf{1}, \quad \nu_j \neq 0.$$

This holds also for $j = 0$. Indeed the cyclic construction is a multiplicity-free sum of pairwise distinct self-dual simple frame modules. Its nondegenerate invariant form restricts nondegenerately to each of them; distinct summands are orthogonal. Its restriction to the one-dimensional highest-weight line of R_j is therefore nonzero. Equivalently, one may use directly the nonzero vacuum channel of the canonical diagonal algebra in the cyclic-gluing theorem.

20.2 Exact identification of the third-order local system

For an irrational parameter a , write

$$d = 2a - 1, \quad H = \frac{p^2 - (a - 1)^2}{4a}.$$

The level-three singular relation for the $(3, 1)$ primary X is

$$(L_{-1}^3 - 4aL_{-2}L_{-1} + 2a(2a - 1)L_{-3})X = 0.$$

Consider the Virasoro four-point blocks $F(z) = \langle H|X(1)X(z)|H \rangle$. Ward identities give

$$F''' - 4aUF'' + (-4aV + 2adT)F' + 2adWF = 0, \quad (\text{BPZ})$$

where

$$\begin{aligned} U &= \frac{1 - 2z}{z(z - 1)}, \\ V &= \frac{H}{z^2} + \frac{d}{(z - 1)^2} - \frac{2d + 1}{z(z - 1)}, \\ T &= \frac{3z^2 - 3z + 1}{z^2(z - 1)^2}, \\ W &= -\frac{2H}{z^3} - \frac{2d}{(z - 1)^3} - \frac{2d}{z^2} + \frac{2d}{(z - 1)^2}. \end{aligned}$$

For example, translation and dilation give $\partial_0 F = (z - 1)F' + 2dF$ and $\partial_1 F = -zF' - 2dF$. Applying these identities to the $L_{-1}X$ insertion gives the displayed V coefficient. Thus the operator includes its accessory parameter, rather than being inferred only from its local exponents.

To compare this equation with [25, equation (1.1)], denote the Dotsenko–Fateev parameters by A, B, C, G and set:

$$A = a - 1 - p, \quad B = C = -2a, \quad G = 2a.$$

His equation (1.1) is

$$\begin{aligned} &z^2(z - 1)^2 I''' + (K_1 z + K_2(z - 1))z(z - 1)I'' \\ &+ (L_1 z^2 + L_2(z - 1)^2 + L_3 z(z - 1))I' + (M_1 z + M_2(z - 1))I = 0, \end{aligned}$$

with

$$\begin{aligned}
 K_1 &= -G - 3B - 3C, & K_2 &= -G - 3A - 3C, \\
 L_1 &= (B + C)(2B + 2C + G + 1), \\
 L_2 &= (A + C)(2A + 2C + G + 1), \\
 L_3 &= (B + C)(2A + 2C + G + 1) + (A + C)(2B + 2C + G + 1) \\
 &\quad + (C - 1)(A + B + C) + (3C + G)(A + B + C + G + 1), \\
 M_1 &= -C(2B + 2C + G + 1)(2A + 2B + 2C + G + 2), \\
 M_2 &= -C(2A + 2C + G + 1)(2A + 2B + 2C + G + 2).
 \end{aligned}$$

The substitution

$$F(z) = z^{-A}(1 - z)^{2a}I(z)$$

turns this equation exactly into (BPZ). One checks this by setting $w = -A/z + 2a/(z - 1)$ and replacing

$$\begin{aligned}
 I' &= z^A(1 - z)^{-2a}(F' - wF), \\
 I'' &= z^A(1 - z)^{-2a}(F'' - 2wF' + (w^2 - w')F), \\
 I''' &= z^A(1 - z)^{-2a}(F''' - 3wF'' + (3w^2 - 3w')F' + (-w^3 + 3ww' - w'')F).
 \end{aligned}$$

Substitution recovers each of the three monic coefficients in (BPZ). The two local systems therefore differ by a rank-one twist, which preserves irreducibility.

Lemma 20.1 (A sufficient generic irreducibility criterion). *The rank-three local system (BPZ) is irreducible if*

$$p \notin (a + \mathbb{Z}) \cup (-a + \mathbb{Z}) \cup (2a + \mathbb{Z}) \cup (-2a + \mathbb{Z}) \cup (\tfrac{1}{2} + \mathbb{Z}).$$

If $p = aR - S$, with R, S rational of odd denominator and not both integral, this condition holds. Under this additional rational-label hypothesis, its three exponents at zero

$$\lambda_+ = 1 - a + p, \quad \lambda_0 = 1 - 2a, \quad \lambda_- = 1 - a - p$$

are pairwise distinct modulo $\mathbb{Z}$.

Proof. Mimachi’s Theorem 11 requires the following eleven quantities to be nonintegral:

$$\begin{aligned}
 &A, \ A + G/2, \ B, \ B + G/2, \ C, \ C + G/2, \\
 &A + B + C + G/2, \ A + B + C + G, \ (G + 1)/2, \\
 &A + C + (G + 1)/2, \ B + C + (G + 1)/2.
 \end{aligned}$$

After substitution the nonautomatic conditions are exactly $a - 1 - p, 2a - 1 - p, -2a - 1 - p, -a - 1 - p, -p - 1/2 \notin \mathbb{Z}$. All other conditions follow from irrationality of a .

If $p = ma + n$ with $m, n \in \mathbb{Z}$, irrationality gives $R = m$ and $S = -n$, contrary to the hypothesis. If $2p \in \mathbb{Z}$, irrationality gives $R = 0$ and $2S \in \mathbb{Z}$; the odd-denominator condition then makes S integral, again a contradiction. These observations prove that the rational-label hypothesis implies the criterion and also gives nonresonance of the three exponents, whose differences are $a + p, a - p, 2p$. Nonresonance is not asserted from the irreducibility criterion alone: that criterion need not exclude $2p \in \mathbb{Z}$. $\square$

20.3 Factorization and a finite-jet invariance lemma

Fix k, j and let f_k^* be the grade-zero functional on the frame summand E_k , normalized by $f_k^*(f_k) = 1$ and extended by zero on the other summands. This is a homogeneous vector in the restricted dual of M_ξ , and is a highest vector for the contragredient frame action. Consider

$$\mathcal{F}(z) = f_k^* Y(v_j, 1) Y(v_j, z) f_k.$$

It is a rational function of z by the ordinary VOA module Jacobi identity. In particular, it is single-valued about both zero and one.

The exact frame decomposition justifies the following factorization without a rationality assumption on the frame or on W_N . Project the inner action onto an intermediate E_l . The singular relation for each active $(3, 1)$ factor restricts its intermediate highest weight to the three values with momentum $p, p \pm 2a$ at site j , and $q, q \pm 2b$ at site $j - 1$. This is also obtained directly from the indicial polynomial of (BPZ). Every other site is unchanged. All these intermediate Verma modules are generic: changing a rational Kac label by an even integer does not make a nonintegral pair integral. Their nondegenerate Shapovalov forms and the Virasoro Ward identities uniquely determine each projected four-point block from its leading three-point constants. The PBW bases and the inverse Shapovalov forms are tensor products, so these projected blocks factor into the product of the two rank-three Virasoro blocks. This is a direct formal-series factorization; regular-singular ODE convergence makes it analytic near zero.

Let $\mathbf{f}(z) = (f_+(z), f_0(z), f_-(z))^T$ and $\mathbf{g}(z) = (g_+(z), g_0(z), g_-(z))^T$ be the normalized Frobenius bases of the two equations. There is therefore a constant matrix Q such that

$$\mathcal{F}(z) = \mathbf{f}(z)^T Q \mathbf{g}(z).$$

More generally the two-variable expression

$$\mathcal{H}(z_1, z_2) = \mathbf{f}(z_1)^T Q \mathbf{g}(z_2)$$

has the property

$$\partial_{z_1}^r \partial_{z_2}^s \mathcal{H}(z_1, z_2) \big|_{z_1=z_2=z} = f_k^* Y(v_j, 1) Y(L_j(-1)^r L_{j-1}(-1)^s v_j, z) f_k. \quad (20.1)$$

Here the componentwise $L(-1)$ derivative identity is applied to the factorized frame intertwiners. Its constants are the same Q : the descendants are obtained by differentiating the same moving insertion. Each right-hand side is again a rational VOA module matrix coefficient.

Lemma 20.2 (Nine rational derivatives force tensor invariance). *The matrix Q defines a monodromy-invariant tensor in the tensor product of the two rank-three local systems.*

Proof. After continuation along a loop about zero or one, write the new coefficient matrix, in the original local bases, as Q' . Rationality of (20.1) for $0 \leq r, s \leq 2$ implies

$$\mathbf{f}^{(r)}(z)^T (Q' - Q) \mathbf{g}^{(s)}(z) = 0 \quad (0 \leq r, s \leq 2).$$

At any common ordinary point of the two differential equations, their two Wronskian matrices are invertible. Stacking these nine equations therefore gives $Q' = Q$. The two loops generate the fundamental group.

The nine descendant coefficients are essential here. The scalar identity for $\mathcal{F}$ alone need not determine Q , because multiplication of the two solution spaces can have a kernel. $\square$

20.4 All elementary charge channels are nonzero

Proposition 20.3. *For every $k \in \xi + \mathbb{Z}^N$ and every elementary index j , both components*

$$R_j \times E_k \longrightarrow E_{k-\beta_j}, \quad R_j \times E_k \longrightarrow E_{k+\beta_j}$$

of the W_N vertex action are nonzero. This includes the closing index $j = 0$.

Proof. The two source Kac pairs satisfy Lemma 20.1, so both rank-three monodromy representations are irreducible. The invariant tensor Q is nonzero. Indeed the vacuum channel of $Y(v_j, 1)Y(v_j, z)$ gives

$$\mathcal{F}(z) = \nu_j(1-z)^{-2\text{wt}(v_j)} + \text{terms with smaller pole order}, \quad \nu_j \neq 0.$$

The remaining W_N states have positive conformal weight, and $f_k^*(f_k) = 1$. Thus $\mathcal{F}$ is not zero.

An invariant tensor between two irreducible three-dimensional representations is a homomorphism from the dual of one to the other. Since it is nonzero, it is an isomorphism. Hence Q is invertible.

The intermediate frame summands in the fixed lattice coset are limited to:

$$E_{k-\beta_j}, \quad E_k, \quad E_{k+\beta_j}.$$

Indeed, equality of rational Kac weights at an irrational parameter identifies the two Kac pairs up to simultaneous sign. A negative sign between two pairs whose charges are in the same coset would force both adjacent odd-discriminant coordinates to be zero, which is excluded. Thus at all inactive sites the labels agree literally, and at the two active sites their changes must be the same 0, 2, or -2 . In the two Frobenius bases, Q is consequently diagonal, with entries belonging to these three intermediate summands.

Equivalently, this diagonality follows from local monodromy. The exponents for the second factor are $1-b+q, 1-2b, 1-b-q$. Since $p+q \in \mathbb{Z}$, the matched pairs have integral sums. The unmatched corner pairs would require $2p \in \mathbb{Z}$; a pair involving just one middle exponent would require $p \in \pm a + \mathbb{Z}$. Both are impossible by the arithmetic in Lemma 20.1.

Invertibility makes each diagonal entry nonzero. The entry for $E_{k \pm \beta_j}$ is the product of the two corresponding three-point coefficients, for the passage from E_k to that intermediate summand and back. Its nonvanishing implies that each of those two actions is nonzero, proving the claim. $\square$

Theorem 20.4 (Simplicity of every nonzero odd lattice restriction). *For every odd $N \geq 3$ and every $\xi \in L^*/L$ whose odd-order component is nonzero, the restriction*

$$V_{L+\xi} \downarrow W_N$$

is simple. Its character is therefore the genuine simple-module character

$$\text{ch } M_\xi(\tau) = \frac{1}{\eta(\tau)^N} \sum_{k \in \xi + \mathbb{Z}^N} q^{(k-1/2)^T K_+(k-1/2)}.$$

Here k is the coordinate column in the basis (β_i) , so the lattice vector $B/2$ has coordinate column $\mathbf{1}/2$.

Proof. A nonzero submodule is a direct sum of some of the pairwise nonisomorphic simple frame summands E_k . Proposition 20.3 makes its charge support closed under both $k \mapsto k + \beta_j$ and $k \mapsto k - \beta_j$, for every j . The lattice graph is connected, so its support is the entire coset. The submodule is all of M_ξ . The character is the ambient lattice character computed above. $\square$

21 Frame semisimplicity in generic rational sectors

We next pass from the lattice modules to simple W_N -modules in nonzero rational sectors. The main issue is that semisimplicity of the total conformal operator does not imply semisimplicity of its commuting partial operators. The level-three singular vectors give polynomial relations for those partial operators separately; their simple roots eliminate the possible nilpotent parts.

Lemma 21.1 (Structure of a generic generalized frame block). *Let Λ be a joint frame block whose unique simple object is the irreducible Verma tensor F_λ of joint highest weight λ . Every finite-length module Y in this block has the form*

$$Y \cong U(\mathfrak{n}_-) \otimes T, \quad T = Y_{[\lambda]},$$

where all positive modes annihilate T and $L_i(0)|_T = \lambda_i + N_i$ for commuting nilpotent N_i . Its ordinary frame socle is

$$\text{soc}_B(Y) = U(\mathfrak{n}_-) \otimes \bigcap_i \ker N_i. \quad (21.1)$$

Both Y and its restricted contragredient are generated by their joint highest spaces. Moreover the Zhu annihilation space is exactly $\Omega(Y) = T$: there are no additional vectors annihilated by all positive Virasoro modes at a strictly higher joint degree.

Proof. All simple constituents have weights in $\lambda + \mathbb{Z}_{\geq 0}^N$, so the same is true of Y , and positive modes kill T . Induction from this highest space gives a generalized Verma tensor $\mathcal{V}(T)$ and a map $\mathcal{V}(T) \rightarrow Y$. The commuting nilpotents on T admit a flag with one-dimensional trivial nilpotent quotients. Induction is exact by PBW. Hence $\mathcal{V}(T)$ has a finite flag of irreducible ordinary Verma tensors F_λ . The cokernel of the displayed map has zero joint highest space; since every nonzero object in the block has a constituent F_λ , exact highest-space extraction makes that cokernel zero. The kernel has zero joint highest space by construction and vanishes for the same reason. This proves the asserted description.

A vector in $\bigcap_i \ker N_i$ generates an ordinary irreducible Verma tensor, and every simple submodule must start at joint weight λ . This proves (21.1). Restricted contragredience preserves finite length and the single simple type, so the same argument applies to Y' .

For the last assertion, the Shapovalov matrix at every positive joint degree is invertible at the ordinary generic highest weight λ . Replacing its entries by their values at the commuting operators $\lambda_i + N_i$ changes this invertible matrix by a nilpotent matrix over the finite local algebra generated by the N_i . It remains invertible. Therefore a positive-degree vector killed by all positive Virasoro modes must be zero. The highest space is killed by every degree-lowering mode of the frame VOA, proving $\Omega(Y) = T$. $\square$

Lemma 21.2 (The elementary generalized top constraint). *Let F be an ordinary irreducible Verma tensor with rational Kac pairs of odd denominator, with no Kac pair consisting of two integers. Let Y be a finite-length generic joint block, ordinary for the total conformal zero mode. Every frame intertwining operator obtained by restricting the W_N vertex action,*

$$\mathcal{I} : R_j \times F \longrightarrow Y,$$

has image contained in $\text{soc}_B(Y)$.

Proof. Write $T = Y_{[\lambda]}$ and $L_i(0)|_T = \lambda_i + N_i$ as in Lemma 21.1. Since total $L(0)$ is semisimple, $\sum_i N_i = 0$, but individual N_i need not vanish a priori. Let f be the highest vector of F with partial weights h_i . For a homogeneous $u \in R_j$, project $\mathcal{I}(u, z)f$ to T and take its unique total-weight-compatible coefficient. Denote it by $t(u) \in T$.

The standard necessary leading-coefficient Zhu map, obtained directly by the intertwining Jacobi identity, sends the left and right actions of the i th conformal vector to $\lambda_i + N_i$ and h_i , respectively. This assertion is valid for a nonsemisimple partial $L_i(0)$: the residue proof uses only the zero-mode operators and the annihilation of the source and target highest spaces by positive modes. In particular, with x_i, y_i denoting the two Zhu actions, the identities for a state of partial weight d_i are

$$\begin{aligned} [L_i(-1)u] &= (x_i - y_i - d_i)[u], \\ [L_i(-2)u] &= (2y_i - x_i + d_i)[u], \\ [L_i(-3)u] &= (x_i - 3y_i - d_i)[u]. \end{aligned} \tag{21.2}$$

For example, the first follows already from

$$[L_i(0), \mathcal{I}(u, z)] = z\mathcal{I}(L_i(-1)u, z) + \mathcal{I}(L_i(0)u, z).$$

The remaining formulas follow from the left/right Zhu products and the relation $[L_i(-3)u] + 2[L_i(-2)u] + [L_i(-1)u] = 0$. Successive applications of these relations give the PBW reduction of the Zhu bimodule of R_j : it is generated by its highest class under the commuting x_i, y_i . The argument uses this necessary leading-coefficient map, rather than a converse fusion-rule statement.

The reduction is performed separately at each Virasoro factor. Indeed:

$$A(\mathcal{B}) = \mathbb{C}[x_0, \dots, x_{N-1}], \quad x_i = [e_i],$$

and modes from distinct factors commute. Each factor of the root bimodule is generated by its highest class under its own left and right zero modes, so the tensor bimodule $A(R_j)$ is generated by the full root highest class. All partial variables remain independent. Lemma 21.1 identifies the target annihilation space $\Omega(Y)$ with T .

At an active factor use the parameter a , so that its root highest weight is $d = 2a - 1$ and its singular vector is

$$(L_{-1}^3 - 4aL_{-2}L_{-1} + 2a(2a - 1)L_{-3})u = 0.$$

Set $s = x - y - d$. Its Zhu image, from (21.2), is the polynomial

$$P_a(x, y) = s(s - 1)(s - 2) - 4a(y - s + 1)s + 2a(2a - 1)(s - 2y). \tag{21.3}$$

For $h_a(p) = (p^2 - (a - 1)^2)/(4a)$, direct factorization gives

$$P_a(x, h_a(p)) = \prod_{m=-1}^1 (x - h_a(p + 2ma)). \tag{21.4}$$

If $p = aR - S$ with the stated rational labels, the differences of these three roots are $a + p, a - p, 2p$, all nonzero; indeed each is nonintegral by irrationality and the odd-denominator condition.

Apply (21.4) to $t(u_j)$, where u_j is the highest vector of R_j . If λ_i is not one of the three roots, the polynomial in $\lambda_i + N_i$ is invertible and $t(u_j) = 0$. If λ_i is a root, it is simple, and

$$P_a(\lambda_i + N_i, h_i) = N_i U_i(N_i), \quad U_i(0) \neq 0.$$

The nilpotent polynomial $U_i(N_i)$ is invertible, so $N_i t(u_j) = 0$. At the other active factor apply the same calculation with reciprocal parameter, converting $(1, 3)$ to $(3, 1)$. At every inactive vacuum factor the Zhu relation is $x_i = y_i$; it likewise makes the top coefficient zero unless $\lambda_i = h_i$, and otherwise forces $N_i t(u_j) = 0$. Thus $t(u_j) \in T_0 := \bigcap_i \ker N_i$.

Cyclicity of the root Zhu bimodule now puts *every* $t(u)$ in T_0 , because each partial zero mode preserves T_0 . Pass to the quotient $\bar{Y} = Y/(U(\mathfrak{n}_-)T_0)$. All highest coefficients of the resulting intertwiner vanish. This implies that the intertwiner itself vanishes: F is generated by its highest

vector, and $\bar{Y}'$ is generated by its highest space by Lemma 21.1. Move their negative Virasoro modes across the matrix coefficient by the intertwining Jacobi identity. Every matrix coefficient is thereby reduced to a highest-target/highest-source coefficient with some descendant in the first input, all of which vanish. Equivalently, the generic Shapovalov matrices remain invertible after the nilpotent highest weight substitution, giving the same induction at each degree.

Thus the original image lies in $U(\mathfrak{n}_-)T_0$, the ordinary frame socle in (21.1). $\square$

Theorem 21.3 (Semisimple frame restriction in a nonzero rational sector). *Let X be a simple grading-restricted generalized W_N -module containing an ordinary simple frame module*

$$F = \bigotimes_i L(c_i, h_i(R_i, S_i)),$$

where every R_i, S_i is rational with odd denominator and no pair (R_i, S_i) consists of two integers. Then

$$X|_{\mathcal{B}}$$

is a direct sum of ordinary irreducible Verma tensors, with finite multiplicity at each joint highest weight. In particular the conclusion holds for every simple module in any of the nonzero periodic or antiperiodic odd coarse sectors.

Proof. The nilpotent part of total $L(0)$ is a natural W_N -endomorphism. On a simple grading-restricted generalized module it vanishes, so X is ordinary for total $L(0)$.

Let $X_{\text{rat}}^{\text{ord}}$ be the sum of the ordinary simple frame submodules whose Kac pairs satisfy the hypotheses. It contains F , and hence is nonzero. We show that every elementary generator preserves this sum.

Take a simple frame summand F_0 in this sum and project the action of R_j on F_0 to a joint frame block Y . By Lemma 12.1, the block Y has finite length. If the projected intertwiner is nonzero, the frame submodule spanned by its coefficients has a simple quotient Q , and the induced intertwiner to Q is nonzero. Both F_0 and Q' are generated by their highest spaces. The necessary leading-coefficient Zhu map is therefore injective, and the relations (21.2)–(21.4) apply to its highest coefficient. At each active factor the target momentum is p , $p + 2a$, or $p - 2a$; at each inactive factor the highest weight is unchanged. Thus the Kac pairs of Q are obtained from those of F_0 by even integral shifts and simultaneous sign changes. Each still has odd denominators and at least one nonintegral entry.

Consequently Q lies in a generic singleton block. Since Q is a constituent of Y , the target block Y is that same generic block. Lemma 21.2 now puts the entire projected image in the ordinary frame socle of Y , all of whose simple summands satisfy the required rationality condition. Each coefficient has only finitely many block components, so the action of R_j preserves $X_{\text{rat}}^{\text{ord}}$.

The frame and the elementary generators vertex-generate W_N . Thus $X_{\text{rat}}^{\text{ord}}$ is a nonzero W_N -submodule of X , and simplicity gives $X_{\text{rat}}^{\text{ord}} = X$. Finite-dimensional total conformal-weight spaces bound the multiplicity of each fixed joint highest weight. $\square$

Remark 21.4 (The leading-coefficient map). The Zhu-map input above is the necessary leading-coefficient map of Huang–Yang [18, Lemma 5.5 and Propositions 5.6–5.8], in their corrected formulation at level 0. Its injectivity applies when the second input and the contragredient of the target are generated by their highest spaces, as verified above. This use does not require an unrestricted Frenkel–Zhu fusion-rule isomorphism. The direct residue calculation also keeps the partial conformal operators separate. Frame semisimplicity is the conclusion of this section; multiplicity one and uniqueness require the transport argument that follows.

22 Multiplicity one and uniqueness in rational sectors

We now determine the multiplicities and the module structure on each regular rational grid. Besides frame semisimplicity and the BPZ calculation, we use the following property of the canonical diagonal algebra: multiplication is nonzero in every allowed diagonal frame fusion channel. In particular, with $R_j = M_{e_j}$ in its usual diagonal labeling,

$$R_j \cdot R_j \longrightarrow \mathbf{1}, \quad R_j, \quad M_{2e_j} \quad \text{are all nonzero}, \quad (22.1)$$

and, for $i \neq j$,

$$R_i \boxtimes_{\mathcal{B}} R_j = M_{e_i+e_j}, \quad R_i \cdot R_j \longrightarrow M_{e_i+e_j} \neq 0. \quad (22.2)$$

The fusion in (22.2) has just one simple summand. At a shared Virasoro site it is $(3, 1) \boxtimes (1, 3) = (3, 3)$; at every other site one factor is vacuum. The canonical diagonal tensor-category description also proves this directly. Finally, the frame and the R_j *vertex-generate* W_N , by the construction above. This vertex-generation statement suffices for the submodule arguments below.

22.1 Regular rational label grids

Choose signs $\sigma_i \in \{1, -1\}$, and define

$$(K_{\sigma} r)_j = \kappa_j r_j - \sigma_{j-1} r_{j-1} - \sigma_j r_{j+1}, \quad \kappa_0 = N + 2, \quad \kappa_j = 2 \ (j \neq 0).$$

After a diagonal sign change, this is K_+ when $\prod_i \sigma_i = 1$, and K_- when $\prod_i \sigma_i = -1$. Its determinant is respectively N^2 or $N^2 + 4$, so it is invertible. For $r \in \mathbb{Q}^N$ put

$$E(r) = \bigotimes_i L(c_i, h_i(r_i, \sigma_i r_{i+1})).$$

Call r regular here if its entries have odd denominators and no two cyclically adjacent entries are both integral. Every factor of $E(r)$ is then a generic irreducible Verma module in a singleton frame block. The term “regular” in this section refers only to this explicit arithmetic condition, not to regularity of a VOA.

Fix such an $r^{(0)}$ with $K_{\sigma} r^{(0)} \in \mathbb{Z}^N$, and set

$$\mathcal{O}(r^{(0)}) = r^{(0)} + 2\mathbb{Z}^N.$$

Every point of this grid has the same regularity property. The frame simples $E(r)$, for $r \in \mathcal{O}(r^{(0)})$, are pairwise distinct. Indeed equality of weights forces simultaneous sign equality of the two rational Kac labels at each irrational Virasoro parameter. A negative sign at a site, for two points in the same grid, would force both adjacent coordinates to be integral. This is excluded. The argument permits isolated integral or zero coordinates.

The nonzero odd periodic and antiperiodic discriminant classes are regular in this sense. Periodic residue vectors are the nonzero affine progressions modulo N . The antiperiodic representatives, verified below, have coordinates $s(N - 2i)/(N^2 + 4)$ modulo $\mathbb{Z}$, where $s \neq 0$ in $\mathbb{Z}/(N^2 + 4)\mathbb{Z}$. Two adjacent periodic entries can vanish only when the progression is zero. In the antiperiodic case, adjacent numerators have difference 2, except across the closing edge, where their sum is 2. Since $N^2 + 4$ is odd, two vanishing adjacent entries would again force $s = 0$.

Lemma 22.1 (The root action stays on one grid). *Suppose a simple W_N -module X contains the frame simple $E(r^{(0)})$ with the above hypotheses. Then its frame restriction is semisimple, and every frame summand belongs to $\{E(r) : r \in \mathcal{O}(r^{(0)})\}$.*

Proof. The generic frame-semisimplicity theorem applies to the nonintegral rational coarse tuple. Alternatively apply its top-polynomial argument successively from the given simple frame submodule: shifting a Kac label by $0, 2, -2$ preserves its nonintegral residue, so every possible target block remains generic. The frame socle is stable under each root, hence under their vertex-generated VOA.

At a point r set

$$a = \psi_j, \quad b = \psi_{j-1}^{-1} = \kappa_j - a, \quad p = ar_j - \sigma_j r_{j+1}, \quad q = b\sigma_{j-1}r_j - r_{j-1}.$$

Then $p + \sigma_{j-1}q = (K_\sigma r)_j \in \mathbb{Z}$. All nine elementary combinations of the two three-channel null polynomials are available as necessary frame fusion possibilities. An actual module mode changes the total weight by an integer. The mixed middle/corner channels would require $p \in \pm a + \mathbb{Z}$ or $q \in \pm b + \mathbb{Z}$; the wrongly paired corner channels would require $2p \in \mathbb{Z}$. All are excluded by regularity and irrationality, as in the BPZ irreducibility lemma. Thus the only possible targets are

$$E(r - 2e_j), \quad E(r), \quad E(r + 2e_j).$$

The grid is closed under all root actions. Its sum of frame summands in X is a nonzero W_N -submodule, so simplicity makes it all of X . $\square$

Write the resulting frame decomposition as

$$X|_{\mathcal{B}} = \bigoplus_{r \in \mathcal{O}(r^{(0)})} E(r) \otimes T_r, \quad \dim T_r < \infty,$$

initially allowing $T_r = 0$. Fix normalized highest vectors of all $E(r)$ and normalized frame intertwiners for every allowed channel. Each such intertwiner space has dimension at most one, by the highest-weight Ward recursion and the nondegenerate generic Shapovalov forms. Denote the three multiplicity maps of the root action by

$$A_j^\epsilon(r) : T_r \longrightarrow T_{r+2\epsilon e_j}, \quad \epsilon \in \{-1, 0, 1\}.$$

22.2 The two-root invariant tensor with multiplicity spaces

Lemma 22.2 (Return maps and middle maps are scalar). *For every r with $T_r \neq 0$, all adjacent spaces $T_{r \pm 2e_j}$ are nonzero and the maps $A_j^\pm(r)$ are isomorphisms. Their oppositely oriented compositions are nonzero scalar multiples of the identity. Moreover $A_j^0(r)$ is a scalar multiple of the identity. All these scalars depend only on the normalized W_N multiplication and the frame labels, and not on X or on the dimension of T_r .*

Proof. Use the operator-valued four-point coefficient

$$\Phi_j(z) = \text{pr}_{E(r)^{\text{top}} \otimes T_r} Y(v_j, 1)Y(v_j, z)|_{E(r)^{\text{top}} \otimes T_r}.$$

By choosing a vector and a functional in T_r one obtains ordinary rational matrix coefficients. The factorization and nine-derivative Wronskian proof apply to each matrix entry. Thus its coefficient tensor belongs to

$$(V_a \otimes V_b)^{\pi_1} \otimes \text{End}(T_r),$$

where V_a, V_b are the two irreducible rank-three BPZ local systems. The existence of the nonzero vacuum coefficient shows these systems are dual; Schur's lemma gives $\dim(V_a \otimes V_b)^{\pi_1} = 1$.

Normalize a generator of this one-dimensional invariant space by its vacuum-channel coefficient. The vacuum term of Φ_j is the fixed nonzero number ν_j times id_{T_r} . It follows that the entire invariant tensor equals this normalized scalar tensor multiplied by $\nu_j \text{id}_{T_r}$.

In the channel at $z = 0$ the invariant tensor has the three allowed entries, one for each $E(r + 2\epsilon e_j)$. It is invertible as a 3×3 tensor, so all three entries are nonzero. The two extreme entries are, with the fixed three-point normalizations,

$$A_j^{-\epsilon}(r + 2\epsilon e_j)A_j^{\epsilon}(r) = \lambda_{j,r,\epsilon} \text{id}_{T_r}, \quad \lambda_{j,r,\epsilon} \neq 0.$$

Applying the same assertion at the neighboring point gives the reverse composition as a nonzero scalar identity. Hence these maps are isomorphisms and every neighboring multiplicity is present.

For the middle map use the channel at $z = 1$, where the two roots fuse first. By (22.1) the three possible frame intermediates are $\mathbf{1}, R_j, M_{2e_j}$. The individual-factor local exponents are distinct modulo integers, making the three paired tensor channels independent: in one factor they are $2 - 4a, 1 - 2a, 2a$, whose differences are nonintegral because a is irrational. The other factor has parameter $b = \kappa_j - a$. The tensor in this basis is again diagonal and invertible. Its R_j entry is the nonzero multiplication constant in $R_j \cdot R_j \rightarrow R_j$ times $A_j^0(r)$. Since every tensor entry is a fixed scalar times id_{T_r} , the middle map is scalar too. The scalar is nonzero. $\square$

22.3 Mixed generators and scalar square transport

Lemma 22.3 (Every elementary square has scalar transport). *For $i \neq j$ there is a nonzero number $\gamma_{ij}(r)$, depending only on the normalized frame and algebra data, such that*

$$A_i^+(r + 2e_j)A_j^+(r) = \gamma_{ij}(r)A_j^+(r + 2e_i)A_i^+(r). \quad (22.3)$$

Proof. Project a two-root matrix coefficient from the highest space of $E(r)$ to that of $E(r + 2e_i + 2e_j)$. In the product ordering where R_j acts first, there is just one possible intermediate frame simple, namely $E(r + 2e_j)$. The other ordering similarly has only $E(r + 2e_i)$. The three-channel grid closure and the injectivity of the label map give these unique intermediate summands.

By (22.2), the product of the two root states belongs to the single simple frame module $P = M_{e_i+e_j}$, and its multiplication intertwiner is nonzero. The frame intertwiner space

$$\mathcal{I}_{\mathcal{B}} \left(\begin{array}{c} E(r + 2e_i + 2e_j) \\ P \quad E(r) \end{array} \right)$$

has dimension at most one: all modules are highest-weight modules, the target Verma factors are generic, and Ward recursion determines all three-point coefficients from the highest coefficient. Let $D_{ij}(r) : T_r \rightarrow T_{r+2e_i+2e_j}$ be the multiplicity map for this channel of the P action, if present.

Associativity expresses each ordered product through the same P action and the same one-dimensional frame intertwiner space. Thus each ordered composition in (22.3) is a scalar multiple of $D_{ij}(r)$. Those scalars are the ordinary frame change-of-channel constants together with the fixed multiplication in (22.2); they do not depend on the module. Both ordered compositions are invertible by the preceding lemma. Hence $D_{ij}(r)$ and both scalar constants are nonzero. Dividing gives (22.3), with $\gamma_{ij}(r) \neq 0$.

In particular the relevant frame intertwiner exists: otherwise associativity would make an invertible ordered product vanish. $\square$

Theorem 22.4 (One copy of every grid point). *If a simple W_N -module contains $E(r^{(0)})$ as above, then*

$$X|_{\mathcal{B}} = \bigoplus_{r \in r^{(0)} + 2\mathbb{Z}^N} E(r)$$

with multiplicity one at every point.

Proof. The scalar-return lemma makes every grid point present, all multiplicity spaces have a common finite dimension, and transport along every edge is invertible. Every closed lattice walk is a product of elementary squares and backtracking pairs. By (22.3) and the return identities, its transport is a scalar automorphism. These scalar holonomies need not equal one.

Choose a nonzero subspace $U \subseteq T_{r(0)}$, and transport it to $U_r \subseteq T_r$ along any grid path. The result is path-independent as a *subspace*, because scalar automorphisms preserve every subspace. The sum

$$\bigoplus_r E(r) \otimes U_r$$

is stable under the frame and under each root: the two extreme maps are the edge transports and their scalar inverses, while the middle map is scalar. Vertex generation makes it a W_N -submodule. Simplicity implies that U cannot be a proper nonzero subspace. Hence $\dim T_{r(0)} = 1$ and all multiplicities equal one. $\square$

Theorem 22.5 (Uniqueness on a fixed regular grid). *Any two simple W_N -modules containing the same regular frame simple $E(r^{(0)})$, with $K_\sigma r^{(0)} \in \mathbb{Z}^N$, are isomorphic.*

Proof. They have the same multiplicity-one frame decomposition by the preceding theorem. Fix the same normalized frame intertwiners for both. The scalar return identities, middle maps, and square constants are determined by the same frame local systems and the same fixed W_N multiplication, so they agree in both modules.

Choose a nonzero linear isomorphism between the multiplicity lines at one base point and extend it by the positive and negative edge maps. It is independent of the path: the ratio of the two transports around any square or backtracking pair is one, because the scalar holonomies agree. It intertwines the extreme root maps by construction and the middle maps because their scalars agree. It therefore intertwines every root and the frame, hence their vertex-generated VOA W_N . $\square$

22.4 Theta characters and the global sign

The simple module in a regular grid, whenever it exists, has character

$$\text{ch } X_{[r^{(0)}, \sigma]}(\tau) = \frac{1}{\eta(\tau)^N} \sum_{m \in \mathbb{Z}^N} q^{\frac{1}{4}(r^{(0)} + 2m)^T K_\sigma (r^{(0)} + 2m)}. \quad (22.4)$$

Indeed the oscillator-normalized exponent of $E(r)$ is

$$\sum_i \frac{(\psi_i r_i - \sigma_i r_{i+1})^2}{4\psi_i} = \frac{1}{4} r^T K_\sigma r.$$

Consequently, once the modular-support argument establishes the existence of a nonzero antiperiodic grid module, its entire character is given by (22.4).

The global sign gives the same set of frame simples: $E(r) = E(-r)$ and negation bijects the two grids. The uniqueness theorem consequently identifies their simple modules. For the periodic lattice restriction $r = \mathbf{1} - 2k$, this global sign is

$$k \mapsto \mathbf{1} - k.$$

On lattice vectors this is the reflection in $B/2$. It sends the lattice coset ξ to $-\xi$. Thus, for nonzero odd component,

$$M_\xi \cong M_{-\xi} \quad \text{and hence} \quad M'_\xi \cong M_\xi.$$

Different regular grids can agree as sets of frame simples only under the sign/gauge identifications described next. A coarse label alone does not identify different parity lifts modulo $2\mathbb{Z}^N$.

22.5 Alignment and canonical sign gauges

The next proposition derives rational alignment from local monodromy. It applies to generic blocks with the stated nonresonance conditions; integral and resonant blocks are treated separately later.

Proposition 22.6 (Alignment from the local monodromy spectra). *Suppose a generic singleton-block frame simple in a semisimple frame restriction has momenta p_i , so that the preceding highest-weight factorization applies, and at each elementary edge both rank-three BPZ local systems are irreducible with nonresonant local exponents. Suppose in addition that their middle eigenvalues cannot match a corner eigenvalue; it suffices that*

$$p_j \notin \pm\psi_j + \mathbb{Z}, \quad -p_{j-1}/\psi_{j-1} \notin \pm\psi_{j-1}^{-1} + \mathbb{Z}.$$

Then signs σ_i and integers n_i exist such that

$$p_j - \sigma_{j-1}p_{j-1}/\psi_{j-1} = n_j.$$

There is a rational vector r with

$$p_i = \psi_i r_i - \sigma_i r_{i+1}, \quad K_\sigma r = n.$$

In particular its denominators are odd, and its sign holonomy is periodic or antiperiodic. If this r has no adjacent integral pair, the preceding classification applies.

Proof. The same rational-derivative argument gives a nonzero invariant tensor between the two irreducible local systems at each edge. Their local eigenvalues at zero therefore match inversely. The middle eigenvalues match each other because $\psi_j + \psi_{j-1}^{-1} = \kappa_j \in \mathbb{Z}$. The stated exclusions rule out matching middle with corner. Matching the two remaining eigenvalues gives $p_j \pm q_j \in \mathbb{Z}$, with $q_j = -p_{j-1}/\psi_{j-1}$, proving the first assertion.

Define linear maps

$$(Dr)_i = \psi_i r_i - \sigma_i r_{i+1}, \quad (Lp)_j = p_j - \sigma_{j-1}p_{j-1}/\psi_{j-1}.$$

The cyclic parameter identity gives $LD = K_\sigma$. Since K_σ is invertible, both square matrices L, D are invertible. Setting $r = K_\sigma^{-1}n$ gives $Dr = p$. The matrix K_σ is integral with odd determinant, so $r \in \mathbb{Q}^N$ has odd denominators. $\square$

Corollary 22.7 (Exhaustion of the regular rational sectors). *Let X be a simple W_N -module containing a frame simple with rational Kac pairs of odd denominator, at least one entry of each pair being nonintegral. Then X is one of the regular periodic or antiperiodic grid modules just classified.*

Proof. The generic frame-semisimplicity theorem applies. For both the row and reciprocal-column parametrizations, the rational-label irreducibility and nonresonance criterion holds at every site. Proposition 22.6 therefore gives a canonical periodic or antiperiodic rational vector $r \in K_\pm^{-1}\mathbb{Z}^N$. Irrationality identifies its Kac pairs with the original pairs up to simultaneous sign, so no adjacent coordinates are both integral. Its odd-discriminant component is consequently nonzero. The periodic Smith form gives one of the nonzero affine-progressions modulo N ; the antiperiodic Smith form gives $s(N - 2i)/(N^2 + 4)$ modulo $\mathbb{Z}$ with $s \neq 0$. The grid classification and uniqueness theorems apply. $\square$

For the regular rational labels used in this section, $2p_i$ and $2p_i/\psi_i$ are nonintegral. Hence precisely one of the two possible edge-sign matching conditions can hold. Changing the independently chosen momentum signs $p_i \mapsto \varepsilon_i p_i$ consequently changes the unique edge signs by

$$\sigma_i \mapsto \varepsilon_i \varepsilon_{i+1} \sigma_i.$$

Their product is invariant, including at points where a coordinate r_i is zero or integral. Once σ is fixed to the standard periodic or antiperiodic representative, the only remaining gauge is global sign. This establishes the sign identification using local monodromy and integrality; it does not assume sign propagation through nonzero coordinates.

23 Refined modular traces after regularity

Regularity gives finitely many simple W_N -modules and ordinary trace functions on each of them. We refine these traces by recording the commuting partial conformal weights. To avoid a separate convergence problem in several variables, we work with formal Taylor germs along the diagonal. The Ward identities determine all their coefficients from convergent one-variable traces.

Let V be a regular, C_2 -cofinite CFT-type VOA with a commuting conformal frame $\omega = \sum_{i=0}^{N-1} e_i$, and write $L_i(n)$ for its modes and c_i for its central charges. For a simple V -module M , put

$$F_M(v; \tau) = \text{Tr}_M \left(o(v) \prod_i e^{2\pi i \tau_i (L_i(0) - c_i/24)} \right).$$

Initially this expression is interpreted coefficientwise in the total q -grading and as a formal Taylor germ at $\tau = (\tau, \dots, \tau)$. The Ward recursion below proves that each Taylor coefficient is holomorphic for τ in the upper half-plane. All the partial zero modes commute and preserve each finite-dimensional total-weight space. For a vector that is not ordinary homogeneous, $o(v)$ is extended linearly from its homogeneous components. The analytic theorem in [26, Theorem 1] assumes convergence on the full product of upper half-planes. We use its Ward recursion at the formal diagonal germ, and prove that version below; no such global convergence hypothesis is silently applied to the unknown modules.

For ordinary homogeneous v , use the square-bracket coordinate

$$Y[v, z] = Y(v, e^{2\pi i z} - 1) e^{2\pi i z \text{wt}(v)}, \quad Y[v, z] = \sum_{m \in \mathbb{Z}} v[m] z^{-m-1},$$

extended linearly, and set

$$\tilde{e}_i = (2\pi i)^2 (e_i - c_i \mathbf{1}/24), \quad L_i[m] = \tilde{e}_i[m+1].$$

The Eisenstein normalization is

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

For a bracket-frame homogeneous vacuum descendant v of partial weights w_i , the trace Ward identities are

$$\begin{aligned} F_M(L_i[-1]v) &= 0, \\ F_M(L_i[-2]v) &= 2\pi i \partial_i F_M(v) + w_i G_2(\tau_i) F_M(v) + \sum_{k \geq 2} G_{2k}(\tau_i) F_M(L_i[2k-2]v). \end{aligned} \quad (23.1)$$

Only finitely many terms occur for each v . The residue proof uses mode commutators and cyclicity of trace on finite total-weight spaces; it is valid as an identity of formal germs. For a Taylor coefficient of a fixed order, solve this identity for $\partial_i F_M(v)$ and iterate. Only finitely many frame descendants are introduced at that order, and the diagonal values are convergent ordinary one-point functions. Derivatives of the Eisenstein series are holomorphic as well. This proves the claimed convergence of individual Taylor coefficients, without convergence of the full multivariable series. The identities are [26, Lemmas 4–5 and equation (2.5)].

Lemma 23.1 (Formal covariance with the ordinary modular matrix). *Let v be a bracket-frame homogeneous vector in the vacuum subalgebra generated by the commuting e_i , with partial weights w_i . Let $S_{M,X}^{\text{an}}$ be the coefficients in the full insertion-dependent one-point S -transformation of V . Then, as formal germs along the diagonal,*

$$F_M(v; -1/\tau_0, \dots, -1/\tau_{N-1}) = \prod_i \tau_i^{w_i} \sum_{X \in \text{Irr}(V)} S_{M,X}^{\text{an}} F_X(v; \tau).$$

Proof. We prove the analogous identity for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Put $j_i = c\tau_i + d$ and subtract the proposed right-hand side from $\prod_i j_i^{-w_i} F_M(v; \gamma\tau)$, calling the result H_v . Full one-point modularity makes H_v vanish on the diagonal for every frame vacuum descendant v . In this normalization,

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d).$$

The anomaly cancels the derivative of $j_i^{-w_i}$ in the covariant operator $2\pi i \partial_i + w_i G_2$. Higher Eisenstein series have their usual modular weights. Thus (23.1) implies

$$2\pi i \partial_i H_v = H_{L_i[-2]v} - w_i G_2(\tau_i) H_v - \sum_{k \geq 2} G_{2k}(\tau_i) H_{L_i[2k-2]v}.$$

Induct simultaneously for every frame descendant v on the total order of a multi-index of partial derivatives. Order zero is the ordinary modular identity on the diagonal. Apply a derivative of order m to the displayed recursion. Every term on its right is an Eisenstein coefficient times a derivative of some H_u of order at most m , and hence vanishes on the diagonal by the induction hypothesis. Thus every derivative of order $m+1$ vanishes there. All transverse Taylor coefficients therefore vanish, proving the formal identity. This is the formal Ward-recursion argument underlying [26]; it requires convergence of the individual one-variable coefficients only. $\square$

Lemma 23.2 (Ordinary trace support in a frame block). *For a grading-restricted W_N -module, multiplication of its refined ordinary character by $\eta_\Pi = \prod_i \eta(\tau_i)$ leaves only the joint endpoints of the irrational Virasoro linkage blocks in its frame restriction. In a singleton block with irreducible Verma tensor of joint highest weight λ , its contribution is*

$$m_\lambda \prod_i q_i^{\lambda_i - (c_i - 1)/24}, \quad m_\lambda \in \mathbb{Z}_{\geq 0}.$$

For a finite linear combination of refined characters of simple W_N -modules, a nonzero coefficient at such a singleton-block endpoint implies that at least one simple module with a nonzero coefficient in that linear combination contains the corresponding frame simple. This assertion is not made for a subtracted endpoint of a two-element linkage block.

Proof. The frame-block lemma gives a finite-length restriction in each fixed joint block, with finitely many simple types. Ordinary trace is additive on a composition series, including when partial zero modes are not semisimple: in a simultaneous upper triangular basis the exponential has the prescribed diagonal eigenvalues, and the strictly upper triangular terms have zero trace. Its block trace is therefore the sum of the ordinary frame-simple characters with their composition multiplicities.

At an irrational Virasoro parameter, an ordinary simple is either an irreducible Verma module or the quotient of a Verma module by its single irreducible singular submodule. Its character is accordingly one Verma character or the difference of two. Multiplication by eta cancels the oscillator factor, leaving just the one or two linkage endpoints. Tensoring gives the stated support description. A singleton block with positive multiplicity is a nonzero finite-length module and has a simple socle of that unique type. Only the same linkage block can contribute that endpoint. Hence, even in a linear combination with complex coefficients, a nonzero singleton endpoint requires a nonzero block in one of its summands, proving the stated submodule conclusion.

The possibly infinite sum of blocks causes no ambiguity. At each fixed total q -degree only finitely many joint weight spaces occur. To see that the formal germ distinguishes them, fix a total exponent and let $\lambda^{(1)}, \dots, \lambda^{(r)}$ be the distinct joint exponent tuples occurring there. They have the same coordinate sum. Choose a transverse direction u with $\sum_i u_i = 0$ such that $u \cdot \lambda^{(1)}, \dots, u \cdot \lambda^{(r)}$ are distinct; this avoids finitely many proper hyperplanes. The directional derivatives of orders $0, \dots, r-1$ of

$$\sum_{a=1}^r c_a \exp(2\pi i z u \cdot \lambda^{(a)})$$

at $z = 0$ form a Vandermonde system in the c_a , so its vanishing forces every $c_a = 0$. The same argument applies after multiplication by eta, which contributes only finitely many terms at each bounded degree. Coefficients can therefore be compared degree by degree in the formal germs of Lemma 23.1. $\square$

The theta transforms used below have real, positive-definite quadratic exponents. Comparing all transverse derivatives separates joint weight tuples even when their total conformal weights coincide. This distinction is necessary at composite $N^2 + 4$: a nonzero discriminant label can have square zero modulo $N^2 + 4$, so its conformal residue alone need not separate it from the zero label.

23.1 The trace normalization and categorical dimensions

We next relate the analytic modular matrix to categorical dimensions, fixing compatible conventions for the Hopf link. The argument applies to a simple, self-contragredient, regular, C_2 -cofinite, nonnegatively graded VOA V ; it does not require $V_0 = \mathbb{C}\mathbf{1}$. Its finite module category is modular by [22, Corollary 5.19].

Let S^{an} be the matrix acting on the full insertion-dependent ordinary traces. Its square is the contragredient permutation C . Here this is an identity of full trace functions, not an inference from scalar characters. Indeed the defining contragredient vertex operator uses the coordinate inversion $x \mapsto x^{-1}$. At the insertion $x = 1$, the coordinate $f(z) = e^{2\pi i z} - 1$ satisfies

$$f(-z) = -\frac{f(z)}{1 + f(z)}.$$

Thus inversion in the exponential coordinate becomes $z \mapsto -z$. For a bracket-homogeneous insertion of weight w , the contragredient trace is consequently $(-1)^w$ times the original trace. This is exactly the slash action of $-I \in \text{SL}_2(\mathbb{Z})$. Linear independence with all insertions identifies its matrix with C , proving $(S^{\text{an}})^2 = C$.

Write $d_X = \dim X$ and $\Delta = \sum_X d_X^2$. Choose the orientation of the categorical Hopf-link matrix $\tilde{S}$ compatible with the analytic generator. For clarity, McRae’s geometrically modified convention uses $(-1/\tau)^{L(0)}$, whereas the ordinary bracket convention uses τ^{-w} . These represent the two inverse lifts of $\tau \mapsto -1/\tau$; their matrices differ by C . Reversing the Hopf-link orientation likewise dualizes one index. Both choices of $\tilde{S}$ are symmetric and satisfy

$$\tilde{S}_{\mathbf{1}, X} = d_X, \quad \tilde{S}^2 = \Delta C.$$

These are the usual ribbon-category identities [27, Section 4], with no positivity condition on the d_X .

In the matching convention, apply the open-Hopf-link formula [22, Theorem 5.2, equation (5.5)] directly to V . All simple V -modules are rigid, and regularity makes the modular transform

of the vacuum trace a sum of ordinary simple-module traces. On a simple Y , the open Hopf link is the scalar $\tilde{S}_{X,Y}/d_Y$. Linear independence of the full insertion-dependent traces therefore gives

$$S_{X,Y}^{\text{an}} = S_{1,Y}^{\text{an}} \frac{\tilde{S}_{X,Y}}{d_Y}.$$

This argument applies separately to $V = W_N$ and to the deformed lattice VOA $V = A$: both are simple, self-contragredient, regular, C_2 -cofinite and nonnegatively graded, as proved above. The theorem requires no one-dimensional degree-zero space. In particular this application to A does not use the orbifold-specific extension setting of [22, Remark 5.21]. Set

$$D = \text{diag} \left(S_{1,Y}^{\text{an}}/d_Y \right)_Y.$$

Spherical dimensions of simple objects are nonzero, so D is well-defined. The preceding identity reads $S^{\text{an}} = \tilde{S}D$. Using the two square identities gives

$$D\tilde{S}D = \Delta^{-1}\tilde{S}.$$

Its $(1,1)$ entry gives $D_1^2 = \Delta^{-1}$; its $(1,Y)$ entries then give $D_Y = D_1$ for every Y . Consequently

$$S_{1,X}^{\text{an}} = d_X S_{1,1}^{\text{an}}, \quad S_{1,1}^{\text{an}} \neq 0. \quad (23.2)$$

In particular the analytic matrix is symmetric, in either compatible orientation. If J centralizes X , its closed Hopf link is $d_J d_X$ in both orientations, and hence

$$\frac{S_{J,X}^{\text{an}}}{S_{1,X}^{\text{an}}} = d_J. \quad (23.3)$$

These two identities are independent of the compatible orientation chosen. They provide the lattice-index calculation and the categorical dimension ratios extracted from the refined parity rows, without requiring a unique module of minimal conformal weight.

24 The multivariable theta identity and its Fourier transform

The difference of two theta quotients is the multivariable analogue of Flohr’s one-variable character formula [35, equations (4.2)–(4.5) and Table 4.2]. The periodic and antiperiodic metric sectors are the two sectors in the Evans–Gannon grafting [2, Section 3.3], proved for untwisted HI centers in [5, Theorem 5.1]. Here the separate frame variables are retained to resolve labels that the single total conformal weight does not separate. Let $\sigma = (\sigma_i)$ have all signs positive, or just its closing sign negative. Write these two choices as $+$ and $-$, and define

$$(B_\sigma r)_i = \sqrt{\psi_i} r_i - \frac{\sigma_i r_{i+1}}{\sqrt{\psi_i}}.$$

Direct multiplication, using the cyclic parameter relations, gives

$$B_\sigma^T B_\sigma = K_\sigma, \quad \det B_+ = t^{-1} - t = N, \quad \det B_- = t^{-1} + t = \sqrt{\mu}.$$

For $v \in \frac{1}{2}\mathbb{Z}^N$, put

$$\Theta_{\sigma,v}(\tau) = \sum_{x \in \mathbb{Z}^N + v} \exp \left(2\pi i \sum_i \tau_i (B_\sigma x)_i^2 \right), \quad \eta_\Pi(\tau) = \prod_i \eta(\tau_i).$$

Lemma 24.1 (All-parity refined identity). *For $v_\epsilon = (1 + \epsilon)/2$ modulo $\mathbb{Z}^N$,*

$$\text{ch } J_\epsilon(\tau) = \frac{\Theta_{+,v_\epsilon}(\tau) - \Theta_{-,v_\epsilon}(\tau)}{2\eta_\Pi(\tau)}.$$

Proof. Expand the product of generic Virasoro character differences in the known frame decomposition of J_ϵ . Its positive labels have parity $1 + \epsilon_i$. Each term is indexed by edge signs. Independent changes of signs of the labels gauge all edge signs except their product; they preserve *each* partial exponent, so separate τ_i are permitted. Overall label reversal gives the factor two.

Completing positive labels to full signed theta sums introduces zero coordinates for even labels. They cancel. If j is the least zero coordinate, reverse the labels at $0, \dots, j-1$. This toggles the closing sign and the sign on an edge incident to that zero, preserving each partial square. If $j=0$ the closing cross term already vanishes. This is an involutive, parity-preserving cancellation of the extra terms. $\square$

Define

$$\mathcal{F}_a(\tau) = 2^{-N} \sum_{\epsilon \in (\mathbb{Z}/2)^N} (-1)^{\epsilon \cdot a} \text{ch } J_\epsilon(-1/\tau_0, \dots, -1/\tau_{N-1}).$$

Poisson summation with the matrix $B_\sigma^T \text{diag}(\tau_i) B_\sigma$, followed by finite Fourier orthogonality, gives

$$\eta_{\Pi} \mathcal{F}_a = (-1)^{|a|} 2^{-N/2-1} \left[\frac{1}{N} \sum_{\substack{z \in \mathbb{Z}^N \\ z \equiv a \pmod{2}}} e^{2\pi i \sum_i \tau_i (B_+^{-T} z)_i^2 / 4} - \frac{1}{\sqrt{\mu}} \sum_{\substack{z \in \mathbb{Z}^N \\ z \equiv a \pmod{2}}} e^{2\pi i \sum_i \tau_i (B_-^{-T} z)_i^2 / 4} \right]. \quad (24.1)$$

The determinant factors cancel the eta automorphy factors. The shift phase is $(-1)^{z \cdot (1+\epsilon)}$, so the Fourier sum keeps exactly $z \equiv a \pmod{2}$. All coefficients within either bracketed branch have the same sign. The theta expressions converge on $\mathbb{H}^N$; for comparison to ordinary traces, only their formal diagonal germs are needed.

25 Exceptional modules for all nonzero discriminant labels

Set

$$v_i = \frac{N-2i}{\mu}, \quad v_N = -v_0 = -\frac{N}{\mu}.$$

Direct multiplication gives

$$K_- v = e_0.$$

The interior equations are second differences, the final equation is $2(N-2N+2) - (N-2N+4) + N = 0$, and the zeroth equation is $(N+2)N - (N-2) + (2-N) = \mu$. The class of v has order μ , since $v_{(N-1)/2} = 1/\mu$. It generates the odd discriminant group.

For $s \not\equiv 0 \pmod{\mu}$ define the joint highest weights

$$\lambda_i(s) = h_i(sv_i, sv_{i+1}), \quad 0 \leq i < N, \quad (25.1)$$

where the definition of v_N includes the closing minus sign. Let

$$\mathcal{Q}(s) = \left([(sv_i \bmod \mathbb{Z}, sv_{i+1} \bmod \mathbb{Z})]_{\pm} \right)_i.$$

The sign quotient at each site means simultaneous sign reversal of both entries.

Lemma 25.1 (Genericity and separation). *For $s \not\equiv 0 \pmod{\mu}$, every factor in $\otimes_i M(c_i, \lambda_i(s))$ is a nondegenerate irreducible Verma module. Moreover*

$$\mathcal{Q}(s) = \mathcal{Q}(s') \iff s' \equiv \pm s \pmod{\mu}.$$

Every such tuple is disjoint from all periodic lattice coarse tuples.

Proof. No adjacent entries sv_i, sv_{i+1} can both be integral: their difference is $2s/\mu$, also at the closing edge with the signed extension. Since μ is odd, both integral would imply $\mu \mid s$. For rational R, S , an equality $h_i(R, S) = h_i(m, n)$ with positive integral m, n forces $(R, S) = \pm(m, n)$, by taking square roots and using irrationality of ψ_i . Thus every factor is nondegenerate.

At the middle edge $j = (N - 1)/2$, the pair is exactly

$$(sv_j, sv_{j+1}) = (s/\mu, -s/\mu).$$

Equality of this coarse pair already forces $s' \equiv \pm s$. The converse holds at every site. Periodic odd lattice coordinates have denominators dividing N . Their coarse Kac pairs therefore cannot contain $s/\mu \notin \mathbb{Z}$, because $\gcd(N, \mu) = 1$. Parity shifts change Kac labels by integers and preserve these coarse tuples. $\square$

Theorem 25.2 (All exceptional labels). *For every integer s with $\mu \nmid s$ there is a simple grading-restricted W_N -module X_s whose frame restriction contains*

$$\bigotimes_{i=0}^{N-1} M(c_i, \lambda_i(s)).$$

At least $(\mu - 1)/2$ distinct parity orbits are thereby forced, indexed by nonzero s modulo global sign, and disjoint from the periodic orbits.

Proof. In (24.1) take $z = se_0$ and $a = z \bmod 2$. Since $B_\sigma^{-T} = B_\sigma K_\sigma^{-1}$, its antiperiodic joint exponent is

$$\left((B_- sv)_i^2 / 4 \right)_i = (\lambda_i(s) - (c_i - 1)/24)_i.$$

It cannot cancel inside the antiperiodic sum, where all coefficients have one sign.

Suppose a periodic term had the same joint exponent. Its rational labels are the entries of $K_+^{-1} z'$ with denominator dividing N^2 . Equality of each squared Kac expression, by irrationality of ψ_i , forces simultaneous-sign equality of the corresponding rational pairs. The middle antiperiodic pair then forces $s/\mu \in \frac{1}{N^2}\mathbb{Z}$, contrary to $\mu \nmid s$ and $\gcd(N^2, \mu) = 1$. Thus the tuple survives the entire transform. This argument also works when $s^2 \equiv 0 \pmod{\mu}$.

Apply full ordinary one-point modularity and Lemma 23.1 to the parity-module traces. The result is a finite linear combination of ordinary simple-module traces with identical formal frame jets. Lemma 23.2 and finite exponential independence at each total q exponent force the surviving tuple to be a linkage endpoint of one of these modules. By Lemma 25.1 its joint block is a singleton. Its finite-length block component has a nonzero simple socle, necessarily the displayed irreducible Verma tensor. Thus that simple W_N -module contains the stated frame module.

The necessary level-three BPZ constraints for the elementary generators change rational Kac labels by $0, \pm 2$; the parity-current constraints change them by integers. The ordered coarse tuple modulo simultaneous sign is preserved by these operations and by frame linkage. It therefore labels an entire simple W -module and its parity orbit. Lemma 25.1 supplies all the claimed distinctions. $\square$

The same proof gives every exceptional parity lift directly. For $u \in \mathbb{Z}^N$, take $z = se_0 + K_- u$ in (24.1). Its antiperiodic label vector is $r = sv + u$. The no-adjacent-integral argument and the middle-pair noncancellation proof are unchanged. Thus a simple module contains the corresponding generic frame tensor for every u . Reducing u modulo $2\mathbb{Z}^N$ supplies all parity grids. The multiplicity-one and uniqueness theorems give their full frame decompositions and theta characters. Fusion with J_ϵ shifts the rational grid by ϵ modulo $2\mathbb{Z}^N$, whereas the elementary roots shift it by even integers. For a nonzero odd-order discriminant label, a parity shift cannot identify the grid with its global negative: that would make twice the odd-order label zero. Thus the parity action is free, and the 2^N lifts of a fixed nonzero label form one parity orbit. The local monodromy

sign argument identifies two such grids precisely by global sign after fixing the antiperiodic gauge. There are consequently $2^N(\mu - 1)/2$ exceptional simples, or $(\mu - 1)/2 = (N^2 + 3)/2$ parity orbits, in the detected family. Exhaustion of all W_N -simples will follow from the nonnegative dimension budget below.

For example, the middle pair separates labels 1 and 16 at $N = 9$ even though their squares agree modulo 85. At $N = 11$ the label 25 is nonzero modulo 125 and has square zero, but its middle pair $(1/5, -1/5)$ remains nonintegral. The argument covers both phenomena uniformly and does not count square residues alone.

Part IV

Condensation and the Haagerup–Izumi center

26 The lattice extension and its categorical index

By Section 17, the category $\mathcal{C} = \text{Rep}(W_N)$ is modular. We now regard the conformal lattice extension A as an algebra object in $\mathcal{C}$. The first task is to show that its categorical dimension is nonzero; separability will then follow from the extension theorems.

26.1 The deformation preserves the braided category

Let $B = \sum_i \beta_i$ and $\varepsilon = e^B$. The two conformal vectors have the form

$$\omega_A = \omega_L + Dv, \quad v = \frac{1}{2}B(-1)\mathbf{1} + \lambda D\varepsilon.$$

Here $v_0\omega_L = 0$: the Heisenberg zero mode annihilates ω_L , and $(D\varepsilon)_0 = 0$. Both conformal gradings are nonnegative by the construction. McRae’s Proposition 2.17 [22] therefore identifies the entire braided tensor structures for the two conformal vectors, including associators. The twist can change, as explained in his Remark 2.18.

The possible failure of CFT type for A causes no modularity problem. McRae’s Corollary 5.19 applies to a simple, self-contragredient, C_2 -cofinite, nonnegatively graded VOA with semisimple Zhu algebra. Regularity supplies the last property. Thus $\text{Rep}(A)$ is modular and its braided category is the pointed lattice category.

Its simple objects are indexed by

$$D_L = L^*/L, \quad |D_L| = p_N N^2, \quad D_L[2] = (\tfrac{1}{2}\mathbb{Z}^N)/\mathbb{Z}^N, \quad |D_L[2]| = p_N.$$

Every entry of its categorical S -matrix is nonzero, because all simple objects are invertible. Duality acts by $\xi \mapsto -\xi$: the conformal shift changes the contragredient charge by $B \in L$, which does not change the discriminant class.

26.2 Disjointness of the integral branches

The lattice charge labels at coset ξ are

$$r_i = 1 - 2(\xi_i + n_i), \quad n_i \in \mathbb{Z}.$$

For $\xi = u/2 \in D_L[2]$, all labels are integral, with parity $r = \mathbf{1} - u \pmod{2}$. Irrational Virasoro linkage changes their signs but preserves their parities. The root label shifts are $0, \pm 2$, so they preserve the same invariant.

The exact block projections and triangular charge argument consequently separate the simple supports of the restrictions of all distinct two-torsion lattice modules. They also separate these restrictions from the nonzero odd branches. On the latter branches, the periodic-grid classification gives

$$M_\xi|_W \text{ simple}, \quad M_\xi|_W \simeq M_\eta|_W \iff \eta = \xi \text{ or } -\xi.$$

The two-primary coordinate is retained in this equivalence.

In particular, W occurs only in the restriction of the zero lattice module A . Its multiplicity there is one. For example, a homomorphism $W \rightarrow A$ sends $\mathbf{1}$ to a vector annihilated by translation; in this underlying positive lattice vertex algebra, $\ker D = \mathbb{C}\mathbf{1}$.

26.3 A nonzero index before invoking separability

Let $b_{\xi,Z}$ be the multiplicity of the W -simple Z in the restriction of M_ξ . The inclusion is conformal, so each restriction is grading restricted. The C_2 -cofinite module category has finite length, and regularity makes these restrictions semisimple; thus b is a finite matrix. The preceding paragraph gives

$$b_{\xi,\mathbf{1}} = \delta_{\xi,0}.$$

For every insertion $w \in W$, restriction of ordinary one-point traces gives

$$F_\xi^A(w; \tau) = \sum_Z b_{\xi,Z} F_Z^W(w; \tau).$$

The inclusion is conformal, so its coordinate changes agree. Modular transformation and linear independence of the full insertion-dependent W -trace functions imply

$$S_L^{\text{an}} b = b S_W^{\text{an}}. \quad (26.1)$$

Scalar characters alone would not justify this inference.

Apply the normalization argument of Section 23.1 separately to W and A . Symmetry and (23.2) give, for either VOA,

$$(S^{\text{an}})_{M,V} = \dim_{\text{Rep}(V)}(M) (S^{\text{an}})_{V,V}.$$

Both vacuum entries are nonzero by the same argument. Taking the zero row and vacuum column of (26.1) yields

$$\dim_{\mathbb{C}} A = \frac{(S_L^{\text{an}})_{0,0}}{(S_W^{\text{an}})_{0,0}} \neq 0. \quad (26.2)$$

This is a categorical index identity. It uses no high-temperature limit and no minimal-weight conjecture.

26.4 Connected étale algebra and local modules

The conformal extension gives a commutative algebra object A in $\mathcal{C}$, connected by the vacuum multiplicity above and with trivial twist by integer grading. It is simple as its own algebra module because the vertex algebra A is simple. Equation (26.2) and Kirillov–Ostrik’s Lemma 1.20 [27] make it a rigid algebra. Their Theorem 3.3 supplies an algebra bimodule splitting of multiplication, so A is separable.

The induced category $\mathcal{C}_A$ is spherical, and induction preserves spherical dimensions by [27, Theorem 1.18 and Remark 1.19]. The local module category is the VOA category $\text{Rep}(A)$, with its braiding and tensor structure, by [28, Theorem 3.65]; the twists agree on the underlying modules. As a further check, the modern extension theorem [29, Theorem 4.10] independently gives fusion-category semisimplicity in this nonnegatively graded setting.

27 Branching determines the rank and parity locality

27.1 Determining the branching Gram matrix

For $\xi \in D_L[2]$, set

$$d_\xi = \dim \text{End}_{\mathcal{C}}(M_\xi|_W).$$

Disjointness and the opposite-coset identification give

$$bb^t = I + C_L + D_0, \quad (27.1)$$

where C_L is inversion on D_L , and D_0 is diagonal, supported on $D_L[2]$, with entries $d_\xi - 2$. This formula includes all unknown integral W -simples in the columns of b .

Use the analytic matrices in (26.1); both are symmetric by Section 23.1. Transposing that identity gives $b^t S_L^{\text{an}} = S_W^{\text{an}} b^t$, and hence

$$S_L^{\text{an}} b b^t = b S_W^{\text{an}} b^t = b b^t S_L^{\text{an}}.$$

Write $S_L = S_L^{\text{an}}$ in the following calculation. Since $I + C_L$ commutes with S_L , so does D_0 . Thus

$$(D_{0,\eta} - D_{0,\xi})(S_L)_{\xi,\eta} = 0 \quad (\xi, \eta \in D_L).$$

Every pointed S -entry is nonzero. Hence D_0 is scalar. It vanishes on the nonzero odd labels, so it vanishes everywhere. Therefore

$$d_\xi = 2 \quad (\xi \in D_L[2]).$$

At $\xi = 0$, connectedness and the simple top-zero constituent T constructed from A/W now force

$$A = \mathbf{1} \oplus T. \quad (27.2)$$

No further constituent of the lattice restriction remains.

27.2 The full module-category rank

We use right modules with the tensor product convention of [30, Section 3.3]: a right module acquires its left action using the inverse braiding. This fixes the orientation in the center equivalences below.

Lemma 27.1. *Let $\mathcal{C}$ be a nondegenerate braided fusion category over $\mathbb{C}$, let A be connected étale, and put $\mathcal{B} = \mathcal{C}_A$ and $\mathcal{E} = \mathcal{C}_A^0$. If $U : \mathcal{E} \rightarrow \mathcal{C}$ is restriction and $b_{e,X} = \dim \text{Hom}_{\mathcal{C}}(X, U(e))$, then*

$$\text{rank } \mathcal{B} = \sum_{e \in \text{Irr } \mathcal{E}} \dim \text{End}_{\mathcal{C}} U(e) = \text{Tr}(bb^t).$$

No unitarity or positivity of categorical dimensions is required.

Proof. Write $I_{\mathcal{B}}$ for the right adjoint of $\text{For} : Z(\mathcal{B}) \rightarrow \mathcal{B}$. The center equivalence [30, Corollary 3.30 and Remark 3.31(i)] identifies $Z(\mathcal{B})$ with $\mathcal{C} \boxtimes \mathcal{E}^{\text{rev}}$. Under its forgetful functor, $X \boxtimes e$ goes to $F_A(X) \otimes_A e$. Adjunction and rigidity therefore give

$$\text{Hom}_{\mathcal{B}}(F_A(X) \otimes_A e, A) \cong \text{Hom}_{\mathcal{B}}(F_A(X), e^*) \cong \text{Hom}_{\mathcal{C}}(X, U(e^*)).$$

Consequently

$$I_{\mathcal{B}}(A) \cong \bigoplus_{e \in \text{Irr } \mathcal{E}} U(e^*) \boxtimes e, \quad \dim \text{End}_{Z(\mathcal{B})} I_{\mathcal{B}}(A) = \sum_e \dim \text{End}_{\mathcal{C}} U(e).$$

On the other hand, the ordinary center-adjunction formula [31, Proposition 9.2.2] gives

$$\text{For } I_{\mathcal{B}}(A) \cong \bigoplus_{M \in \text{Irr } \mathcal{B}} M \otimes_A M^*.$$

Adjunction now identifies the preceding endomorphism dimension with

$$\dim \text{Hom}_{\mathcal{B}}(\text{For } I_{\mathcal{B}}(A), A) = |\text{Irr } \mathcal{B}|.$$

The last equality in the statement follows by decomposing each $U(e)$. $\square$

Applying Lemma 27.1 to the algebra A gives:

$$\text{rk}(\mathcal{C}_A) = \text{Tr}(bb^t) = p_N(N^2 + 1). \quad (27.3)$$

The local category has $p_N N^2$ invertible simples, so exactly p_N simples of $\mathcal{C}_A$ are nonlocal. Here “nonlocal” means that the right A -action fails the trivial-monodromy condition; this count is taken in the full fusion category of right A -modules.

27.3 An orbit argument forces the parity factor to be local

If a nonlocal simple Q in $\mathcal{C}_A$ were invertible, tensoring it by all $p_N N^2$ local invertibles would give $p_N N^2$ distinct nonlocal simples. Distinctness follows by cancellation; locality of any one of them would imply locality of Q . This contradicts the count p_N , since $N \geq 3$.

Every induced parity current $F(J_\epsilon)$ is invertible because induction is a tensor functor. Hence it is local. Locality of the free module $A \otimes J_\epsilon$ implies that J_ϵ centralizes A : inserting the unit into the inner A -factor of the locality equation gives precisely the trivial monodromy equation. Thus

$$A \in \mathcal{D} = \mathcal{P}'. \quad (27.4)$$

Induction is fully faithful on $\mathcal{P}$. Indeed,

$$\text{Hom}_{\mathcal{C}_A}(F(J_\epsilon), F(J_\eta)) = \text{Hom}_{\mathcal{C}}(J_\epsilon, A \otimes J_\eta) = \delta_{\epsilon, \eta},$$

since $A = \mathbf{1} \oplus T$ lies in $\mathcal{P}'$ and $\mathcal{P} \cap \mathcal{P}' = \text{Vec}$. Its local image is the full two-primary pointed subgroup of $\text{Rep}(A)$.

Algebra-module formation now factors:

$$\mathcal{C}_A \simeq \mathcal{P} \boxtimes \mathcal{D}_A, \quad \mathcal{C}_A^{\text{loc}} \simeq \mathcal{P} \boxtimes \mathcal{D}_A^{\text{loc}}.$$

Equation (27.3) gives

$$\text{rk}(\mathcal{D}_A) = N^2 + 1. \quad (27.5)$$

27.4 The odd local ribbon category

The integral map

$$m \longmapsto ((m_i - m_{i+1})_i, m_0)$$

identifies the lattice with

$$L \simeq \sqrt{2}A_{N-1} \oplus \langle 2N \rangle.$$

The difference vector has sum zero and, together with m_0 , reconstructs m , so this is an integral isometry, not merely a rational diagonalization.

The two odd discriminant factors have quadratic values $-b^2/N$ and a^2/N modulo $\mathbb{Z}$. Thus the odd metric group is

$$E = (\mathbb{Z}/N)^2, \quad q_E(a, b) = \exp\left(\frac{2\pi i(a^2 - b^2)}{N}\right).$$

Because N is odd, $x = a + b$, $y = a - b$ identifies this with $\exp(2\pi ixy/N)$. The conformal shift pairs integrally with odd representatives:

$$(B/2, \xi) = \mathbf{1}^T K_+ \xi = N\xi_0 \in \mathbb{Z}.$$

This follows directly from the odd discriminant representatives $\xi_i = a + ib$, $Na = Nb = 0$ modulo $\mathbb{Z}$. The $D^2\varepsilon$ term does not alter $L(0)$. Consequently $\mathcal{D}_A^{\text{loc}}$ is the standard hyperbolic ribbon category, including its cyclic Lagrangian subgroup and its complementary subgroup with trivial associator.

28 Modular coefficients, multiplicities, and exhaustion

28.1 The remaining near-group multiplicity

By (27.5), the category $\mathcal{M} = \mathcal{D}_A$ has the N^2 local invertibles E and one additional simple R . The object R is fixed on both sides by E , is self-dual, and obeys

$$R^2 = \sum_{e \in E} e + mR$$

for an integer $m \geq 0$. Rigidity makes each invertible multiplicity exactly one.

Set $r = \text{FPdim } R$ and $a = \text{FPdim } A$. The standard dimension formulas for algebra-module categories give

$$r^2 = N^2 + mr, \quad a = 1 + r^2/N^2.$$

For the local unit, restriction gives $U(\mathbf{1}) = A = \mathbf{1} \oplus T$; for a nonzero local invertible e , restriction gives the simple $U(e) = Y_{[e]}$. These constituents are disjoint by the branching calculation. Consequently Frobenius reciprocity gives

$$[F(T) : e] = [T : U(e)] = \delta_{e, \mathbf{1}}.$$

There is only one nonlocal simple R , so

$$F(T) = \mathbf{1} + kR, \quad k \in \mathbb{Z}_{\geq 0}.$$

Since induction preserves FP dimension,

$$a - 1 = 1 + kr, \quad a = 2 + kr, \quad r^2 = N^2 + N^2kr, \quad m = N^2k.$$

For the periodic simple indexed by $[e] = \{e, -e\}$, reciprocity similarly gives exactly the two local summands e and $-e$, each once. They are distinct because N is odd and $e \neq 0$. The module-dimension formula gives $\text{FPdim}_{\mathcal{D}} U(e) = a \text{FPdim}_{\mathcal{M}} e = a$. Thus $\text{FPdim } Y_{[e]} = a = 2 + kr$, and the remaining multiplicity is k :

$$F(Y_{[e]}) = e \oplus (-e) \oplus kR. \tag{28.1}$$

Every other simple Z of $\mathcal{D}$ has

$$F(Z) = n_Z R, \quad n_Z \in \mathbb{Z}_{>0}. \tag{28.2}$$

There is no local summand, and a tensor functor between fusion categories cannot annihilate a nonzero object.

28.2 A sum-of-squares dimension identity

The global dimension identity is

$$N^2 a^2 = 1 + (a - 1)^2 + \frac{N^2 - 1}{2} a^2 + r^2 \sum_Z n_Z^2.$$

Substituting $a = 2 + kr$ and $r^2 = N^2(1 + kr)$ gives

$$\boxed{\sum_Z n_Z^2 = 2 + \frac{N^2 - 1}{2} k^2.} \quad (28.3)$$

The sum runs over all neutral simples other than $\mathbf{1}$, T , and the periodic family. In particular, it includes both the exceptional modules constructed in Section 25 and any additional simples. Thus the identity is available before exhaustion has been proved.

28.3 Extracting the modular coefficient ratio

For a regular module Z , let $a_Z = z \bmod 2$ be its parity Fourier label in the refined dual-lattice formula. The refined Poisson identity has coefficient

$$2^{-N/2-1} N^{-1} (-1)^{(1+\epsilon) \cdot a_Z} \quad \text{or} \quad -2^{-N/2-1} \mu^{-1/2} (-1)^{(1+\epsilon) \cdot a_Z}.$$

More explicitly, for a regular label choose z_0 in the appropriate discriminant class and let $\Gamma = z_0 + 2K_{\pm} \mathbb{Z}^N$ and $-\Gamma = -z_0 + 2K_{\pm} \mathbb{Z}^N$. These grids are distinct: equality would make the nonzero odd part of the class two-torsion. Their parity labels agree because $-z_0 \equiv z_0 \pmod{2}$, and $z \mapsto -z$ preserves every squared partial exponent. Consequently their contributions combine as

$$\frac{c_{\pm}}{2} \left(\sum_{z \in \Gamma} q^{\mathbf{h}(z)} + \sum_{z \in -\Gamma} q^{\mathbf{h}(z)} \right) = c_{\pm} \sum_{z \in \Gamma} q^{\mathbf{h}(z)},$$

where $q^{\mathbf{h}(z)}$ denotes the multivariable exponential in (24.1). Here $c_+ = 2^{-N/2} N^{-1} (-1)^{(1+\epsilon) \cdot a_Z}$ and $c_- = -2^{-N/2} \mu^{-1/2} (-1)^{(1+\epsilon) \cdot a_Z}$. This accounts for the factor two before a module coefficient is read off. The exact nonintegral frame support uniquely isolates its regular simple and excludes every unknown integral module. Hence

$$S_{J_{\epsilon}, Y}^W = 2^{-N/2} N^{-1} (-1)^{(1+\epsilon) \cdot a_Y}, \quad (28.4)$$

$$S_{J_{\epsilon}, X}^W = -2^{-N/2} \mu^{-1/2} (-1)^{(1+\epsilon) \cdot a_X}. \quad (28.5)$$

Rationality ensures that the full one-point modular action here is the ordinary modular action; the formal frame-germ covariance uses the same coefficients. This argument does not depend on linear independence of scalar characters.

For a representative $Z \in \mathcal{D}$, trivial monodromy with $\mathcal{P}$ implies

$$\frac{S_{J_{\epsilon}, Z}^W}{S_{\mathbf{1}, Z}^W} = \dim J_{\epsilon}.$$

Equations (28.4)–(28.5) therefore force the same a_Z for every neutral representative in both regular families. Removing the common parity vacuum entry gives the exact ratio

$$\boxed{\frac{S_{\mathbf{1}, X_s}^{\mathcal{D}}}{S_{\mathbf{1}, Y_{[e]}}^{\mathcal{D}}} = -\frac{N}{\sqrt{N^2 + 4}}.} \quad (28.6)$$

28.4 Determining the integer multiplicities

Let $x = \dim_{\mathcal{M}} R$. The local invertibles have spherical dimension one: in a spherical category their dimensions are ± 1 , and their group has odd order. Dimension-preserving induction gives

$$x^2 = N^2 + N^2 kx, \quad \dim Y_{[e]} = 2 + kx, \quad \dim Z = n_Z x.$$

The number x may be negative. The quadratic relation implies

$$\left(\frac{x}{2 + kx} \right)^2 = \frac{N^2}{N^2 k^2 + 4}.$$

For each of the $M = (N^2 + 3)/2$ known exceptional simples, equation (28.6) yields

$$n_s^2(N^2 + 4) = N^2 k^2 + 4. \quad (28.7)$$

All n_s are positive integers, hence have the same value n . Subtract their contribution from (28.3):

$$\sum_{\text{additional } Z} n_Z^2 = 2 + \frac{N^2 - 1}{2} k^2 - \frac{N^2 + 3}{2} n^2 = \frac{2(1 - k^2)}{N^2 + 4}. \quad (28.8)$$

Every term on the left is the square of a positive integer. Nonnegativity therefore gives $k \leq 1$. If $k = 0$, (28.7) gives $n^2 = 4/(N^2 + 4) < 1$, impossible. Hence $k = n = 1$, and the sum over additional simples vanishes. No additional simple can occur, since it would contribute a positive integer square. This proves both the asserted multiplicities and exhaustion.

The negative sign of (28.6) selects

$$x = -Nt, \quad \dim T = t^2, \quad \dim Y = 1 + t^2, \quad \dim X = -Nt.$$

The positive FP root is $r = N\delta$, giving

$$\text{FPdim } T = \delta^2, \quad \text{FPdim } Y = 1 + \delta^2, \quad \text{FPdim } X = N\delta.$$

Neutral simple	Number	Induction to $\mathcal{D}_A$	Spherical dimension
1	1	1	1
T	1	$\mathbf{1} \oplus R$	t^2
$Y_{[e]}$	$(N^2 - 1)/2$	$e \oplus (-e) \oplus R$	$1 + t^2$
X_s	$(N^2 + 3)/2$	R	$-Nt$

29 Reconstruction of the untwisted ribbon center

29.1 Cyclic de-equivariantization gives the HI rules

The near-group setting originates in Izumi's work [1] and the constructions and doubles of Evans–Gannon [40, 3]. The equivariantization construction in [5, Section 6] relates cyclic HI categories to near-group categories. For the converse used here, the near-group fusion rules must be accompanied by a specified central Tannakian algebra. The next theorem records these hypotheses and reconstructs the fusion rules after de-equivariantization.

Theorem 29.1. *Let $N \geq 3$ be odd. Let $\mathcal{B}$ be a complex fusion category whose simple objects are the group $E = (\mathbb{Z}/N)^2$ of invertibles and one object R , with*

$$R^2 = \bigoplus_{e \in E} e \oplus N^2 R.$$

Suppose a central Tannakian functor $\text{Rep}(\Gamma) \rightarrow Z(\mathcal{B})$, with $\Gamma \cong \mathbb{Z}/N$, has an underlying fully faithful functor identifying its simple objects with a subgroup $H \subset E$ of order N . Let D_H be the image of its regular algebra and put $\mathcal{H} = \mathcal{B}_{D_H}$, using its specified central structure. Then $\mathcal{H}$ has precisely the simple objects g and $g\rho$, $g \in G = E/H \cong \mathbb{Z}/N$, and

$$gh = g + h, \quad \rho g = (-g)\rho, \quad \rho^2 = \mathbf{1} \oplus \bigoplus_{g \in G} g\rho.$$

In particular, $\mathcal{H}$ is a fusion category with the cyclic Haagerup–Izumi fusion rules.

Proof. De-equivariantization by the stated central Tannakian algebra is a fusion category, with a tensor action of Γ whose equivariantization is $\mathcal{B}$; the underlying full faithfulness is precisely what ensures a simple unit [24, Theorem 4.18, Proposition 4.19, Remark 4.21]. Let $F : \mathcal{B} \rightarrow \mathcal{H}$ be free induction. The subgroup H is cyclic of order N , because its simple objects are the characters of the cyclic group Γ . A generator of $H \subset (\mathbb{Z}/N)^2$ is a primitive vector modulo N and extends to a basis; hence $E/H \cong \mathbb{Z}/N$. For invertibles e, f ,

$$\text{Hom}_{\mathcal{H}}(F(e), F(f)) \cong \bigoplus_{h \in H} \text{Hom}_{\mathcal{B}}(e, fh).$$

Thus the $F(e)$ give N distinct simple invertibles, indexed by G . Moreover,

$$\text{End}_{\mathcal{H}} F(R) \cong \bigoplus_{h \in H} \text{Hom}_{\mathcal{B}}(R, Rh).$$

Every summand is one-dimensional. Choosing an isomorphism $R \rightarrow Rh$ in each degree, composition multiplies such basis vectors by nonzero scalars. The endomorphism algebra is therefore a twisted group algebra of H . A cyclic group has trivial $H^2(H, \mathbb{C}^\times)$: after choosing a lift of its generator, rescale that lift so that its N th power is 1. Hence this algebra is $\mathbb{C}[H] \cong \mathbb{C}^N$. It follows that $F(R) = Q_1 \oplus \cdots \oplus Q_N$, with all Q_i simple, distinct and of multiplicity one. Adjunction excludes the $F(e)$ among the Q_i . Dominance of F shows that this list exhausts the simples of $\mathcal{H}$.

Put $\delta = (N + \sqrt{N^2 + 4})/2$. Then $\text{FPdim } R = N\delta$. Tensor functors preserve Frobenius–Perron dimensions, and de-equivariantization divides the global dimension by N . Writing $d_i = \text{FPdim } Q_i$ gives

$$\sum_i d_i = N\delta, \quad \sum_i d_i^2 = \frac{N^2 + N^2\delta^2}{N} - N = N\delta^2.$$

Equality in Cauchy–Schwarz forces every $d_i = \delta > 1$. For any Q_i , write the total invertible and noninvertible multiplicities of $Q_i Q_i^*$ as a, b . Since $\delta^2 = 1 + N\delta$ is irrational, $a + b\delta = 1 + N\delta$ gives $a = 1$, $b = N$. Frobenius reciprocity identifies a with the order of the left G -stabilizer. Thus the left action is free, and hence transitive; the same argument with $Q_i^* Q_i$ handles the right action. Choose $\rho = Q_1$ and write $\rho g = \varphi(g)\rho$, where $\varphi \in \text{Aut}(G)$.

The deck action fixes the isomorphism class of each $F(e)$: twisting the regular-algebra action is undone by rescaling its H -graded free factor. It is transitive on the Q_i . Indeed each deck orbit of noninvertible simples supplies a noninvertible equivariant simple, whereas $\mathcal{B}$ has just one. Since there are N deck elements and N such simples, the action is free and transitive. As it fixes every invertible, it commutes with left translation by G . Thus the deck action on the noninvertible labels consists of all translations $g\rho \mapsto (g + a)\rho$.

Write $\rho^2 = g_0 \oplus \bigoplus_g m_g g\rho$. The same dimension calculation shows there is exactly one invertible term and $\sum_g m_g = N$. Applying the deck translation a , and comparing unique invertible terms with $(a\rho)^2 = (a + \varphi(a))\rho^2$, gives $a + \varphi(a) = 0$ for every a . Hence $\varphi = -1$. Now $(a\rho)^2 = \rho^2$, so the same comparison of noninvertible terms gives $m_{g+a} = m_g$; consequently all $m_g = 1$. Finally associativity of ρ^3 gives $g_0\rho = (-g_0)\rho$ after cancelling the common sums. The free G -action yields $2g_0 = 0$. Oddness of N gives $g_0 = 0$, as required. $\square$

29.2 The complementary subgroup fixes the pointed associator

Corollary 29.2. *In Theorem 29.1, suppose the specified central Tannakian category is one Lagrangian factor of a braided embedding*

$$\mathcal{E}^{\text{rev}} \longrightarrow Z(\mathcal{B}), \quad \mathcal{E} \simeq \mathcal{C}(G \times \widehat{G}, q), \quad q(g, \chi) = \chi(g),$$

whose underlying functor identifies all N^2 invertibles of $\mathcal{B}$. Then the pointed subcategory of $\mathcal{H}$ is tensor equivalent to Vec_G with trivial associator.

Proof. The hyperbolic category has a skeletal realization with associator 1 and braiding $c_{(g, \chi), (h, \psi)} = \psi(g)$; replacing this braiding by its inverse-transpose realizes the reverse category. Both factors $H = \{0\} \times \widehat{G}$ and $K = G \times \{1\}$ are therefore tensor subcategories with trivial associator. Use the regular algebra on H . The underlying central embedding and free induction restrict to a tensor functor $\text{Vec}_K \rightarrow \mathcal{H}$. The adjunction calculation in the theorem shows that its N simple images are distinct and exhaust the invertibles of $\mathcal{H}$; it is fully faithful and hence an equivalence onto $\mathcal{H}_{\text{pt}}$. This proves the associator assertion. It does not require K to remain a central subcategory after induction. $\square$

29.3 Preservation of the ribbon structure

Theorem 29.3. *Let $\mathcal{D}$ be a modular tensor category, and let A be a connected étale algebra with $\theta_A = 1$ and $\dim_{\mathcal{D}} A \neq 0$. Suppose $\mathcal{B} = \mathcal{D}_A$ has the near-group rules in Theorem 29.1 and its local category is the standard hyperbolic ribbon category of order N^2 , with all its objects invertible in $\mathcal{B}$. Then there is a connected étale algebra A' over A such that $\mathcal{H} = \mathcal{D}_{A'}$ has the untwisted HI rules and*

$$\mathcal{D} \simeq Z(\mathcal{H})$$

as ribbon categories. If $\dim_{\mathcal{B}} R = -Nt$, then $\dim_{\mathcal{H}} \rho = -t$.

Proof. The central embedding $(\mathcal{D}_A^0)^{\text{rev}} \rightarrow Z(\mathcal{D}_A)$ is part of the center equivalence [30, Corollary 3.30]. It is fully faithful after forgetting to $\mathcal{D}_A$ on the local subcategory. Choose either cyclic Lagrangian factor H and its regular algebra. By [30, Section 3.6 and Proposition 3.16], this algebra in $\mathcal{D}_A^0$ gives a connected étale algebra A' over A in $\mathcal{D}$, and algebra-module formation in stages identifies $\mathcal{D}_{A'}$ with the corresponding central de-equivariantization. Our inverse-braiding right-module convention agrees with the reverse local factor in the displayed center embedding. Theorem 29.1 and Corollary 29.2 apply to it.

The local modules after the second condensation have group $H^\perp/H = 0$, so $\mathcal{D}_{A'}^0 \simeq \text{Vec}$. Equivalently $\text{FPdim}(A')^2 = \text{FPdim } \mathcal{D}$. The central lift of free induction is therefore a braided equivalence $\mathcal{D} \rightarrow Z(\mathcal{D}_{A'})$ [30, Corollary 4.1].

For completeness, this equivalence preserves the required ribbon structures, not only the braiding. Local restriction preserves twists, so $\theta_{A'} = 1$. Each local invertible has dimension 1, and the module-dimension formula gives $\dim_{\mathcal{D}} A' = N \dim_{\mathcal{D}} A \neq 0$. A connected étale algebra has nondegenerate multiplication pairing against its normalized projection to the unit: the radical is a right ideal, while its regular module is simple since its endomorphism algebra is $\text{Hom}(\mathbf{1}, A') = \mathbb{C}$ in a semisimple module category; the pairing is nonzero on the unit. Thus A' is a rigid algebra in the sense of [27, Definition 1.11]. Theorems 1.17–1.18 and Remark 1.19 of that paper give a spherical structure on $\mathcal{D}_{A'}$ with

$$\dim_{\mathcal{D}_{A'}} F_{A'}(X) = \dim_{\mathcal{D}} X.$$

The induced spherical structure on its center has the dimension of the underlying object. Hence the braided center equivalence preserves the dimension of every simple. Two pivotal structures

on a fusion category differ by a tensor automorphism of the identity, scalar on each simple; that scalar multiplies its nonzero pivotal dimension. Equality of all simple dimensions therefore makes the two pivotal structures equal. In a braided category the pivotal structure and braiding determine the twist, proving the ribbon assertion.

The same dimension formulas in two stages give $\dim F_H(R) = \dim R$. Its N noninvertible summands form one orbit under the odd-order invertibles, all of dimension 1. Their common dimension is consequently $\dim R/N = -t$. $\square$

Scope of the identification. The theorem constructs the particular fusion category $\mathcal{H} = \mathcal{D}_{A'}$ and identifies its full ribbon center. Here “untwisted” means that its pointed associator is trivial. It does not identify $\mathcal{H}$ with every category having these fusion rules, or with a separately prescribed HI solution. Such uniqueness cannot be inferred from modular data; distinct order-fifteen HI categories with identical center modular data are exhibited in [5, Corollary 5.9].

Corollary 29.4. *For the algebra $A = \mathbf{1} \oplus T$ constructed in $\mathcal{D} = \mathcal{P}'$ above, the category $\mathcal{D}$ is ribbon equivalent to the modular tensor category $Z(\mathcal{H}_N)$, where $\mathcal{H}_N$ has the untwisted cyclic HI fusion rules and $\dim \rho = -t$.*

Proof. The branching and multiplicity budget prove that $\mathcal{D}_A$ has exactly the invertibles $E = (\mathbb{Z}/N)^2$ and R , with $R^2 = \sum_E e + N^2 R$ and $\dim R = -Nt$. Its local ribbon category is the standard hyperbolic category, and $\dim A = 1 + t^2 \neq 0$. Theorem 29.3 applies. In particular, $\mathcal{H}_N$ is a spherical fusion category over $\mathbb{C}$. Its Drinfeld center is therefore a modular tensor category with the induced ribbon structure by the general center theorem [24, Section 2.9]. $\square$

Galois consequence. Choose a number-field form of the fusion category and its spherical structure, and extend the automorphism $\sqrt{\mu} \mapsto -\sqrt{\mu}$ to its algebraic closure. Applied to the associativity and duality maps, it preserves the fusion rules and the trivial pointed associator, and sends $-t$ to δ . All simple spherical dimensions of the conjugate are therefore its Frobenius–Perron dimensions. Hence the conjugate is pseudo-unitary. This is the same consequence already included in [5, Theorem A(1) and Section 1.3]; pseudo-unitarity here means equality of global and Frobenius–Perron dimensions. It does not assert a positive-definite unitary structure or a Galois-conjugate VOA construction.

30 Modular data of the center

The center equivalence in Corollary 29.4 determines a ribbon category. We now identify its complete modular data in the notation of Gannon–Schopieray–Yadav [5, Section 5 and Appendix F.1]. The comparison is uniform in odd N , including when $N^2 + 4$ is composite or has repeated prime factors. In their notation the answer is

$$\mathcal{MD}_-(\mathbb{Z}/N; \mathbb{Z}/(N^2 + 4), b \mapsto b^2/(N^2 + 4)).$$

The subscript $-$ specifies the spherical dimension $\dim \rho = -t$. The cyclic group and its quadratic form are additional information; the spherical sign alone does not determine them in general.

30.1 Quadratic forms and the exceptional labels

Write $e(x) = \exp(2\pi i x)$ and $\mu = N^2 + 4$. In this section quadratic forms are written additively in $\mathbb{Q}/\mathbb{Z}$, with their e -values giving the twists. The local odd lattice category has metric group

$$E = (\mathbb{Z}/N)^2, \quad q_E(x, y) = \frac{xy}{N} \pmod{\mathbb{Z}}, \quad B_E((x, y), (x', y')) = \frac{xy' + x'y}{N} \pmod{\mathbb{Z}}.$$

Here the coordinates are obtained from the difference-of-squares coordinates in Part IV by $x = a + b$, $y = a - b$. The letters a, b below will instead label the exceptional cyclic group.

Recall $v = K_-^{-1}e_0$, with $v_i = (N - 2i)/\mu$ and $v_N = -v_0$. For each nonzero $a \in \mathbb{Z}/\mu$, choose the unique odd residue s modulo 2μ satisfying $s \equiv 2a \pmod{\mu}$. Denote by $X_{[a]}^{\text{neu}}$ the exceptional simple with frame grid

$$r \in sv + 2\mathbb{Z}^N.$$

The subscript $[a]$ means the orbit $\{a, -a\}$. Its dual-theta label has parity e_0 , which is exactly the neutral label forced by (28.4)–(28.5) and $\dim J_\epsilon = (-1)^{e_0}$. Changing s by 2μ preserves the grid, while replacing s by $-s$ gives the isomorphic opposite-grid module. This notation fixes the neutral representative as well as its odd discriminant label.

The total conformal weight at $r = sv$ is

$$h(s) = \frac{N}{4} \left(\frac{s^2}{\mu} - 1 \right).$$

Writing $s = 2a + \mu k$, with k odd, gives

$$h(s) - \frac{Na^2}{\mu} = Nak + \frac{N}{4}(\mu k^2 - 1) \in \mathbb{Z},$$

because $\mu \equiv 1 \pmod{4}$. Thus the exceptional twists are governed by

$$H_N^{\text{met}} = \mathbb{Z}/\mu, \quad q_N(a) = \frac{Na^2}{\mu} \pmod{\mathbb{Z}}, \quad B_N(a, a') = \frac{2Na a'}{\mu} \pmod{\mathbb{Z}}. \quad (30.1)$$

Nondegeneracy follows from $\gcd(2N, \mu) = 1$.

Lemma 30.1 (Uniform metric isometry). *For every odd $N \geq 3$, put*

$$u_N = \frac{N^2 - N + 2}{2}.$$

Then multiplication by u_N identifies $(\mathbb{Z}/\mu, q_N)$ with $(\mathbb{Z}/\mu, q_0)$, where $q_0(b) = b^2/\mu \pmod{\mathbb{Z}}$. In particular,

$$\sum_{a \bmod \mu} e(q_N(a)) = \sqrt{\mu}. \quad (30.2)$$

Proof. There is the integer identity

$$u_N^2 - N = \mu \left(\frac{N-1}{2} \right)^2.$$

Hence $u_N^2 \equiv N \pmod{\mu}$. Since N is a unit modulo μ , so is u_N , and $q_0(u_N a) = q_N(a)$ for every a . The ordinary quadratic Gauss sum for the positive odd integer $\mu \equiv 1 \pmod{4}$ is $\sum_{b \bmod \mu} e(b^2/\mu) = \sqrt{\mu}$; the formula holds without a square-free hypothesis. Applying the isometry proves (30.2). This is the sign required in [5, Theorem 5.1], where $r_- = -\sqrt{\mu}$. $\square$

30.2 The complete modular-data theorem

We use the normalized categorical Hopf-link matrix with

$$S_{1,1} = \frac{1}{\dim \mathcal{H}_N} = \frac{1}{N(1+t^2)} > 0. \quad (30.3)$$

Write $T^{\text{rib}} = \text{diag}(\theta_Z)$ for the ribbon twist matrix. This is the T used in the center formulas of [5]. The object called T earlier in the manuscript remains the simple summand of $A = \mathbf{1} \oplus T$; the superscript distinguishes the matrix.

Theorem 30.2 (Negative-branch GSY modular data). *The ribbon categories $\mathcal{D} = \mathcal{P}'_N$, $Z(\mathcal{H}_N)$, and $\text{Rep}(E_N)$ have modular data*

$$(S, T^{\text{rib}}) = \mathcal{MD}_-(\mathbb{Z}/N; \mathbb{Z}/\mu, q_0), \quad q_0(b) = \frac{b^2}{\mu} \pmod{\mathbb{Z}}, \quad (30.4)$$

under the following bijection of simple labels:

Simple of $\mathcal{D}$	GSY label	Ribbon twist
$\mathbf{1}$	0_+	1
T	0_-	1
$Y_{[(x,y)]}$	$\mathfrak{a}_{[x,y]}$	$e(xy/N)$
$X_{[a]}^{\text{neu}}$	$\mathfrak{d}_{[u_N a]}$	$e(Na^2/\mu)$

The periodic labels range over $((\mathbb{Z}/N)^2 \setminus \{0\})/\{\pm 1\}$ and the exceptional labels over $(\mathbb{Z}/\mu \setminus \{0\})/\{\pm 1\}$. To display every entry, set

$$\alpha = \frac{1}{2} \left(\frac{1}{N} + \frac{1}{\sqrt{\mu}} \right), \quad \beta = \frac{1}{2} \left(\frac{1}{N} - \frac{1}{\sqrt{\mu}} \right), \quad r_G = \frac{N^2 - 1}{2}, \quad r_H = \frac{\mu - 1}{2}.$$

In the ordered GSY basis $(0_+, 0_-), (\mathfrak{a}), (\mathfrak{d})$,

$$S = \begin{pmatrix} \alpha & \beta & N^{-1}\mathbf{1}_{1 \times r_G} & -\mu^{-1/2}\mathbf{1}_{1 \times r_H} \\ \beta & \alpha & N^{-1}\mathbf{1}_{1 \times r_G} & \mu^{-1/2}\mathbf{1}_{1 \times r_H} \\ N^{-1}\mathbf{1}_{r_G \times 1} & N^{-1}\mathbf{1}_{r_G \times 1} & C_E & 0 \\ -\mu^{-1/2}\mathbf{1}_{r_H \times 1} & \mu^{-1/2}\mathbf{1}_{r_H \times 1} & 0 & C_H \end{pmatrix}, \quad (30.5)$$

where

$$(C_E)_{[x,y],[x',y']} = \frac{2}{N} \cos \left(\frac{2\pi(xy' + x'y)}{N} \right), \quad (30.6)$$

$$(C_H)_{[b],[b']} = \frac{2}{\sqrt{\mu}} \cos \left(\frac{4\pi bb'}{\mu} \right). \quad (30.7)$$

The diagonal entries of T^{rib} in this basis are $1, 1, e(xy/N), e(b^2/\mu)$. Thus the comparison fixes the tensor unit, the full S -matrix, and the twists, rather than only their spectra or the fusion rules.

Proof. It suffices to work in $\mathcal{D}$: the two ribbon equivalences are Corollary 29.4 and Theorem 31.1. We first keep the natural exceptional coordinate a of (30.1).

Normalization and distinguished rows. The dimensions computed in Section 28 are $1, t^2, 1 + t^2, -Nt$ for $\mathbf{1}, T, Y, X$, respectively. Since $1 + t^2 = t\sqrt{\mu}$, the vacuum row in normalization (30.3) is

$$S_{\mathbf{1}, \mathbf{1}} = \alpha, \quad S_{\mathbf{1}, T} = \beta, \quad S_{\mathbf{1}, Y} = \frac{1}{N}, \quad S_{\mathbf{1}, X} = -\frac{1}{\sqrt{\mu}}.$$

The remaining distinguished entries also follow directly from the branching matrix. Write $\mathcal{E} = \mathcal{D}_A^{\text{loc}}$ and let b^0 be restriction from its odd local simples to $\mathcal{D}$. Its row at the local unit is $\mathbf{1} + T$, and its row at a nonzero $e \in E$ is $Y_{[e]}$. In the categorical normalizations $S_{0,0}^{\mathcal{E}} = 1/N$ and $S_{\mathbf{1}, \mathbf{1}}^{\mathcal{D}} = 1/[N \dim A]$, one has

$$S^{\mathcal{E}} b^0 = b^0 S^{\mathcal{D}}.$$

Indeed, factor the analytic branching identity (26.1) by the common parity matrix and use (26.2). If $S_W^{\text{an}} = c_W \tilde{S}_W$ and $S_L^{\text{an}} = c_L \tilde{S}_L$, that index identity is $c_L = c_W \dim A$; the unnormalized neutral

and local Hopf-link matrices are $N \dim A S^{\mathcal{D}}$ and $NS^{\mathcal{E}}$, respectively. The scalar factors therefore cancel. The local unit row now gives

$$S_{\mathbf{1},T} + S_{T,T} = \frac{1}{N}, \quad S_{\mathbf{1},Y} + S_{T,Y} = \frac{2}{N}, \quad S_{\mathbf{1},X} + S_{T,X} = 0.$$

Together with the vacuum row these imply

$$S_{T,T} = \alpha, \quad S_{T,Y} = \frac{1}{N}, \quad S_{T,X} = \frac{1}{\sqrt{\mu}}.$$

These are precisely the tractable entries of [5, Section 5.1]. The two distinguished labels are fixed by their dimensions and by $A = \mathbf{1} \oplus T$; the periodic and exceptional families have dimensions $1 + t^2$ and $-Nt$. The twists of $\mathbf{1}$ and T are 1 because A is the conformal extension algebra. The local lattice twists give $e(xy/N)$ for Y , and the calculation preceding Lemma 30.1 gives $e(Na^2/\mu)$ for $X_{[a]}^{\text{neu}}$.

Comparing the regular Fourier blocks. For a rational vector w , write

$$f_{\sigma,w}(\tau) = \frac{1}{\eta_{\Pi}(\tau)} \sum_{m \in \mathbb{Z}^N + w} e \left(\sum_i \tau_i (B_{\sigma} m)_i^2 \right).$$

Multiplicity one and the grid classification identify the refined exceptional character with $f_{-,sv/2}$. Poisson summation gives the finite transform

$$f_{\sigma,w}(-1/\tau) = \frac{1}{\sqrt{\det(2K_{\sigma})}} \sum_{z \in \mathbb{Z}^N / (2K_{\sigma}\mathbb{Z}^N)} e(-w^{\dagger}z) f_{\sigma,K_{\sigma}^{-1}z/2}(\tau). \quad (30.8)$$

Here $2K_{\sigma}w$ is integral for the grids being transformed. The formula follows by Poisson summation for the quadratic matrix $B_{\sigma}^{\dagger} \text{diag}(\tau_i) B_{\sigma}$; its inverse sends z to the displayed dual grid. The determinant factor is $2^{-N/2}/N$ in the periodic case and $2^{-N/2}/\sqrt{\mu}$ in the antiperiodic case. It holds on the convergent theta functions, hence on their formal diagonal germs.

To pass to the normalization of $\mathcal{D}$, there is one common scalar κ such that $S_{Z,Z'}^{W,\text{an}} = \kappa S_{Z,Z'}^{\mathcal{D}}$ for neutral simples Z, Z' . This follows from the analytic/Hopf-link comparison in Section 23.1 and the ribbon factorization by $\mathcal{P}_N$. For neutral Y , its Fourier label is e_0 , so (28.4) and $S_{\mathbf{1},Y}^{\mathcal{D}} = 1/N$ give

$$\kappa = -2^{-N/2}. \quad (30.9)$$

In particular the analytic sign is determined, not replaced by a positivity assumption.

For two neutral exceptional labels choose odd s, s' with $s \equiv 2a$ and $s' \equiv 2a'$ modulo μ . The two opposite target grids in (30.8) contribute

$$\frac{2^{1-N/2}}{\sqrt{\mu}} \cos \left(\frac{\pi N s s'}{\mu} \right) = -\frac{2^{1-N/2}}{\sqrt{\mu}} \cos \left(\frac{4\pi N a a'}{\mu} \right).$$

For the equality, write $s = 2a + \mu k$, $s' = 2a' + \mu k'$ with k, k' odd: the additional phase in $N s s' / (2\mu)$ is $N \mu k k' / 2$ modulo integers, a half-integer. Division by (30.9) therefore gives

$$S_{X_{[a]}^{\text{neu}}, X_{[a']}^{\text{neu}}}^{\mathcal{D}} = \frac{2}{\sqrt{\mu}} \cos \left(\frac{4\pi N a a'}{\mu} \right). \quad (30.10)$$

For the periodic block, let ξ_e be the odd lattice class indexed by $e \in E$. The neutral character uses the theta coset $\xi_e - \mathbf{1}/2$. Pairing an odd-order class with the two-primary class $\mathbf{1}/2$ is integral, and

$$2(\mathbf{1}/2)^{\dagger} K_{+}(\mathbf{1}/2) = N/2.$$

Thus the Fourier phase between two neutral periodic cosets is $-e(-B_E(e, e'))$. Pairing the opposite grids and dividing by κ gives (30.6). A periodic theta transform has no antiperiodic regular support, and conversely, so $S_{Y,X} = 0$. These coefficient comparisons use the disjoint joint frame supports and uniqueness of regular grid modules from Sections 22 and 25, together with Lemma 23.1. They do not assume linear independence of scalar characters, nor do they distinguish labels merely by their total conformal weight. They therefore also apply at composite and nonsquare-free μ .

Finally put $b = u_N a$. Lemma 30.1 turns (30.10) into (30.7) and turns the exceptional twist into $e(b^2/\mu)$. In Appendix F.1 of [5], substituting $\sigma = -1$, $y = N/\sqrt{\mu}$, and $B_H(b, b') = 2bb'/\mu$ gives exactly (30.5)–(30.7). Equation (30.2) supplies their required Gauss-sum normalization. Every block and every twist agrees under the displayed bijection. $\square$

30.3 Ribbon twists and the VOA character normalization

The theorem compares ribbon modular data. For an ordinary VOA trace, the translation matrix includes the central-charge scalar:

$$T^{\text{an}} = e(-c/24)T^{\text{rib}}.$$

Consequently, for the parity-cancelled realization E_N ,

$$T^{E_N, \text{an}} = e(2N/3)T^{\text{rib}}.$$

One must remove this scalar when comparing character transformations with the center matrices in [5]. It has cube 1, consistently with $c(E_N) = -16N \equiv 0 \pmod{8}$. For W_N , the parity factor must also be retained:

$$T^{W_N, \text{an}} = e(5N/24)(T^{\mathcal{P}_N, \text{rib}} \otimes T^{\text{rib}}).$$

The full character representation of W_N is therefore not the center representation by itself.

The vacuum row of (30.5) recovers precisely $1, t^2, 1 + t^2, -Nt$. In particular, the negative branch is fixed with the vacuum at 0_+ ; no interchange of 0_+ and 0_- is being used. All the labels are self-dual, so the categorical normalization obeys $S^2 = I$ and $(ST^{\text{rib}})^3 = I$.

30.4 Matching the tabulated categories and the scope of the result

The compact GSY modular label for Theorem 30.2 is μ_1^- . The explicit isometry in Lemma 30.1 removes the need for separate square-root searches at small orders. For completeness, the following are all rows of [5, Appendix F, Table 2] with this modular label at each listed N . A range of superscripts includes every row in that range. The table is a comparison of exact modular labels, not a calculation from the rounded numerical HI matrices printed there.

N	μ	Modular label	Matching GSY rows
3	13	13_1^-	$\alpha(3)$
5	29	29_1^-	$\alpha(5)$
7	53	53_1^-	$\alpha(7)$
9	85	85_1^-	$\alpha^{(1)}(9)$
11	125	125_1^-	$\alpha_A^{(1)}(11)$
13	173	173_1^-	$\alpha(13)$
15	229	229_1^-	$\alpha^{(1)}(15), \alpha^{(2)}(15), \alpha^{(3)}(15)$
17	293	293_1^-	$\alpha(17)$
19	365	365_1^-	$\alpha^{(1)}(19)$
21	445	445_1^-	$\alpha^{(1)}(21), \alpha^{(3)}(21)$
23	533	533_1^-	$\alpha^{(1)}(23)$
25	629	629_1^-	$\alpha^{(1)}(25)$
27	733	733_1^-	$\alpha^{(1)}(27), \alpha^{(2)}(27), \alpha^{(3)}(27)$
29	845	845_1^-	$\alpha_A^{(1)}(29)$
31	965	965_1^-	$\alpha^{(1)}(31)$
33	1093	1093_1^-	$\alpha^{(1)}(33), \dots, \alpha^{(5)}(33)$
35	1229	1229_1^-	$\alpha^{(1)}(35), \alpha^{(2)}(35), \alpha^{(3)}(35)$
37	1373	1373_1^-	$\alpha^{(1)}(37), \alpha^{(2)}(37), \alpha^{(3)}(37)$
39	1525	1525_1^-	$\alpha_A^{(2)}(39), \alpha_A^{(4)}(39)$
41	1685	1685_1^-	$\alpha^{(1)}(41)$
43	1853	1853_1^-	$\alpha^{(1)}(43)$

Thus the realization has the modular data of the listed negative spherical GSY centers. In particular it selects 85_1^- , rather than 85_3^- , at $N = 9$, and the cyclic 125_1^- , rather than the noncyclic $25_1^- \times 5_1^-$, at $N = 11$. For arbitrary odd N , the uniform assertion is the exact datum (30.4) in the GSY classification of center modular data. Their general theorem permits other metric groups and forms. It does not by itself assign μ_1^- to every category in their uniform double-sine construction, and no such additional assignment is needed or asserted here.

Nor does this theorem identify the associator of $\mathcal{H}_N$ with that of a separately prescribed GSY category. Already at $N = 15$, the three matching rows have the same modular data but are pairwise not categorically Morita equivalent [5, Corollary 5.9]. An equivalence with a named category would require comparison of associativity data, or an equivalence of centers retaining the canonical Lagrangian algebra. At $N = 3$ the present theorem gives the negative-branch Haagerup modular data; the classification results in [44, 43] are not being used to promote that equality to a named-category equivalence. The ribbon equivalence with the center of the category constructed here remains Corollary 29.4.

31 An explicit VOA realization of the parity centralizer

We use “vertex operator algebra” in the lower-bounded, integer-graded sense: negative conformal degrees are allowed, but there are only finitely many of them and every homogeneous space is finite-dimensional. The extension below is simple, self-contragredient, regular (hence rational), and C_2 -cofinite. Its regularity means that every weak module is a direct sum of irreducible ordinary modules; its C_2 -cofiniteness is proved separately from this assertion.

31.1 The parity coordinates

Let $W = W_N$ and retain the parity subcategory $\mathcal{P} \subset \text{Rep}(W_N)$ from Theorem 11.1. Its parity modules are

$$J_\epsilon = \bigoplus_{\substack{r_i \geq 1 \\ r_i \equiv 1 + \epsilon_i \pmod{2}}} \bigotimes_{i=0}^{N-1} L_i(r_i, r_{i+1}), \quad \epsilon \in P = (\mathbb{Z}/2)^N.$$

The parity-current construction gives $J_\epsilon \boxtimes_W J_\eta \cong J_{\epsilon+\eta}$. With the cycle matrix K of diagonal $(N+2, 2, \dots, 2)$ and off-diagonal cycle entries -1 , its self-braiding quadratic form, twist, and dimension are

$$Q(\epsilon) = i^{\epsilon^\dagger K \epsilon}, \quad \theta_\epsilon = i^{\epsilon^\dagger K \epsilon + 2N\epsilon_0}, \quad d_\epsilon = (-1)^{\epsilon_0}. \quad (31.1)$$

For $N = 3$ each off-diagonal entry is -1 , so that $x^\dagger K x = \sum_i (x_i - x_{i+1})^2 + Nx_0^2$ still holds.

Here is an explicit change of parity coordinates, needed to specify the cancellation algebra. Set

$$(S\epsilon)_i = \epsilon_i + \epsilon_{i+1} + \epsilon_0 \quad \text{in } \mathbb{F}_2.$$

If $S\epsilon = 0$, all successive differences equal ϵ_0 ; summing them and using oddness of N gives $\epsilon_0 = 0$, and then every coordinate is zero. Thus S is invertible. Let $T(\epsilon)$ be the number of cyclic transitions of the binary word ϵ ; it is even. Then

$$\epsilon^\dagger K \epsilon = T(\epsilon) + N\epsilon_0, \quad |S\epsilon| = \begin{cases} T(\epsilon), & \epsilon_0 = 0, \\ N - T(\epsilon), & \epsilon_0 = 1. \end{cases}$$

Substitution in (31.1) gives

$$Q(\epsilon) = i^{|S\epsilon|}, \quad \theta_\epsilon = (-i)^{|S\epsilon|}, \quad d_\epsilon = (-1)^{|S\epsilon|}. \quad (31.2)$$

The classification of pointed braided categories by their quadratic forms, with the additional specified ribbon twist, therefore identifies the parity ribbon category as $\mathcal{P} \simeq (\mathcal{S}^-)^{\boxtimes N}$. The nontrivial object of $\mathcal{S}^-$ has self-braiding i , twist $-i$, and dimension -1 .

31.2 A minuscule conformal shift of the E_7 lattice

Choose the standard even unimodular E_8 lattice in $\mathbb{R}^8$, put $\alpha = e_7 + e_8$, and let

$$K_7 = E_8 \cap \alpha^\perp \cong E_7, \quad u = e_6 + \frac{1}{2}e_7 - \frac{1}{2}e_8.$$

Writing $\beta = e_6 + e_7 \in E_8$, we have $u = \beta - \alpha/2$. Hence

$$u \in K_7^* \setminus K_7, \quad 2u \in K_7, \quad u^2 = \frac{3}{2}, \quad K_7^*/K_7 \cong \mathbb{Z}/2.$$

Define

$$U = (V_{K_7}, \omega_{K_7} + u(-2)\mathbf{1}).$$

The fields, vacuum and translation are those of the lattice vertex algebra. Its Virasoro modes and momentum weights are

$$\begin{aligned} L_u(n) &= L_{K_7}(n) - (n+1)u(n), & c(U) &= 7 - 12u^2 = -11, \\ \text{wt}_u(e^p) &= \frac{p^2}{2} - (u, p) = \frac{(p-u)^2}{2} - \frac{3}{4}. \end{aligned}$$

The shifted VOA is integer graded since $u \in K_7^*$. It is nonnegatively graded: for $p \in K_7$, the vector $\beta - p \in E_8$ has inner product 1 with α , and therefore has squared norm at least 2. Its

orthogonal projection $u - p$ has squared norm at least $2 - 1/2 = 3/2$. The positive definite square-completion formula also gives finite weight spaces.

The two simple weak modules are the shifted versions of V_{K_7} and V_{K_7+u} . Their vertex operators are unchanged. The ordinary lattice VOA is regular, and each lattice module remains ordinary for $L_u(0)$ by the same square-completion formula. Thus U is regular. It is C_2 -cofinite because the state-field map and hence its C_2 -space are unchanged. Self-contragredience follows from $2u \in K_7$ [32, Theorems 3.1, 4.1, 4.4].

Write $M = V_{K_7+u}$ with the shifted conformal structure. For $p = u + a$, $a \in K_7$, its weights are $a^2/2 - 3/4$ plus oscillator degrees. Thus it has a one-dimensional lowest space of weight $-3/4$ and twist i . The change of conformal vector preserves the braided category: apply [22, Proposition 2.17] to $v = u(-1)\mathbf{1}$, for which $v_0\omega_{K_7} = 0$; both conformal gradings are nonnegative. The lattice fusion and self-braiding are consequently

$$M \boxtimes_U M \cong U, \quad c_{M,M} = e^{\pi i u^2} = -i.$$

The order-two pointed ribbon category has $d(M) = \theta_M/c_{M,M} = -1$. Its associator on the triple of nontrivial labels is -1 in the usual semion gauge. These data give the ribbon equivalence

$$\text{Rep}(U) \simeq (\mathcal{S}^-)^{\text{rev}}. \quad (31.3)$$

In particular no theorem about the representation category of a twisted HI VOA is needed to construct this auxiliary factor.

31.3 The diagonal extension and all its weak modules

Theorem 31.1. *Let W_N be the VOA constructed above, with its established parity ribbon factor, and put $\mathcal{D} = \mathcal{P}' \subset \text{Rep}(W_N)$. Set $M^0 = U$, $M^1 = M$. Then*

$$E_N = \bigoplus_{\epsilon \in P} J_\epsilon \otimes \bigotimes_{i=0}^{N-1} M^{(S\epsilon)_i}$$

has a simple, self-contragredient, regular, C_2 -cofinite VOA extension structure with

$$c(E_N) = -16N, \quad \text{Rep}(E_N) \simeq \mathcal{D}$$

as ribbon categories. In particular, E_N is rational and has exactly $N^2 + 3$ irreducible ordinary modules. Its least conformal degree is $-(N-1)/2$, and its homogeneous spaces are finite-dimensional. Thus it is a lower-bounded integer-graded VOA, and is not of CFT type.

Proof. Let $V = W_N \otimes U^{\otimes N}$. Every tensor factor is nonnegatively graded and C_2 -cofinite. Their finite tensor product has these properties too. The Deligne product theorem [12, Theorem 5.1] gives

$$\text{Rep}(V) \simeq \mathcal{P} \boxtimes \mathcal{D} \boxtimes \mathcal{P}^{\text{rev}}.$$

The displayed object E_N is the diagonal regular algebra in the first and third factors, tensored with the unit of $\mathcal{D}$. Explicitly, all its self-braidings and twists cancel by (31.2)–(31.3). The diagonal pointed subcategory is Tannakian, and its regular algebra is connected, commutative and separable. Its multiplication is nonzero between every pair of its simple-current components. It has dimension $|P| = 2^N$, since each paired component has dimension 1.

The weights in this finite direct sum are integral and lower bounded, with finite-dimensional homogeneous spaces, as verified below. The extension correspondence [13, Theorem 7.5] therefore supplies the VOA structure. Its unit map embeds V since V is simple. Separability and connectedness make its regular algebra module simple, which is exactly simplicity of the extension VOA.

C₂-cofiniteness. Equip the same underlying vertex algebra as V with the unshifted lattice conformal vectors and write

$$V^{\text{un}} = W_N \otimes V_{K_7}^{\otimes N}.$$

This is a CFT-type, C_2 -cofinite VOA. As a weak V^{un} -module, E_N is the finite direct sum of the irreducible modules

$$J_\epsilon \otimes \bigotimes_{i=0}^{N-1} V_{K_7 + (S\epsilon)_i u}, \quad \epsilon \in P.$$

Irreducibility follows from the tensor-product decomposition, and the conformal shift has not changed the module vertex operators. Each summand is C_2 -cofinite as a V^{un} -module by [23, Proposition 5.2]. Consequently the subspace

$$C_2^{V^{\text{un}}}(E_N) = \text{span}_{\mathbb{C}}\{v_{-2}e : v \in V^{\text{un}}, e \in E_N\}$$

has finite codimension in E_N . The underlying vertex algebra V^{un} embeds in E_N , so

$$C_2^{V^{\text{un}}}(E_N) \subseteq C_2(E_N), \quad \dim E_N / C_2(E_N) \leq \dim E_N / C_2^{V^{\text{un}}}(E_N) < \infty.$$

Here V^{un} is used only to apply the CFT-type module theorem: its conformal vector need not agree with that of E_N . The vertex operators, and hence the displayed C_2 -spaces, are unchanged. This proves C_2 -cofiniteness with the stated conformal structure on E_N .

The module category and regularity. The local-module category of the diagonal algebra in $\mathcal{P} \boxtimes \mathcal{P}^{\text{rev}}$ is Vec ; this follows either from its Lagrangian dimension or directly from the diagonal subgroup's orthogonal complement. Hence the local algebra modules in $\text{Rep}(V)$ form the ribbon category $\mathcal{D}$. The VOA/local-algebra correspondence is a braided tensor equivalence [28, Theorem 3.65]; twists agree because both are $e^{2\pi i L(0)}$ on the same underlying module.

This identifies the full ordinary category, and also proves regularity rather than only semisimplicity within a chosen finite category. Indeed the displayed Deligne product is semisimple. By [22, Proposition 2.11], every cyclic weak V -module is grading-restricted generalized, since V is nonnegatively graded and C_2 -cofinite. It is therefore a finite direct sum of ordinary simples. Every weak V -module is the sum of its cyclic submodules, and hence a sum of simple submodules. Such a module is semisimple; this proves regularity of V in the weak-module sense. Every weak E_N -module restricts to a direct sum of ordinary simple V -modules, hence belongs to $\text{Ind}(\text{Rep}(V))$. The extension correspondence on direct limits [13, Theorems 7.5 and 7.7] identifies it with a local module for the same finite separable algebra there. Here the passage to arbitrary weak modules can be checked on finite pieces. Let Q be such a local algebra module and let $X \subset Q$ be any finite V -subobject. The action map

$$E_N \boxtimes_V X \longrightarrow Q$$

has finite image, since its source lies in the finite category $\text{Rep}(V)$. Associativity makes that image an E_N -submodule. Separability makes every finite algebra module semisimple. These finite images cover Q , because Q is a union of finite V -subobjects and the algebra action is unital. Consequently Q is a sum of simple algebra submodules, hence a direct sum of them. Each simple summand is finite as a V -object and is ordinary for the common conformal vector: its total $L(0)$ is diagonalizable, lower bounded, and has finite-dimensional eigenspaces. Locality passes to subobjects and direct summands. Thus every weak E_N -module is a direct sum of irreducible ordinary E_N -modules. In particular every admissible module is completely reducible, so E_N is rational. The ribbon equivalence and the previously established rank of $\mathcal{D}$ give its $N^2 + 3$ simple modules.

Self-contragredience. The conformal subalgebra $V = W_N \otimes U^{\otimes N}$ is self-contragredient, since all its tensor factors are. The finite V -module decomposition of E_N contains V as a direct summand.

Taking graded contragredients therefore shows that the ordinary E_N -module $E_N^\vee$, restricted to V , contains $V^\vee \cong V$. Induction–restriction adjunction [28, Lemma 2.61 and Theorem 3.65] gives

$$\mathrm{Hom}_{E_N}(E_N, E_N^\vee) \cong \mathrm{Hom}_V(V, E_N^\vee) \neq 0,$$

because induction sends V to E_N . The adjoint E_N -module is irreducible by simplicity, and its graded contragredient is also irreducible. A nonzero morphism between them is therefore an isomorphism. Thus $E_N \cong E_N^\vee$ for its stated conformal vector; this argument does not change that conformal vector or assume CFT type.

Conformal degrees and central charge. Write $r = \mathbf{1} + v$ in a frame summand of J_ϵ . Its lowest total weight is

$$\frac{1}{4} \sum_i (v_i - v_{i+1})^2 + \frac{N}{4} v_0(v_0 + 2).$$

The constraints $v_i \geq 0$ and $v_i \equiv \epsilon_i \pmod{2}$ imply a squared difference of at least 1 at every parity transition and $v_0 \geq \epsilon_0$; $v = \epsilon$ attains all these bounds. Consequently

$$h_{\min}(J_\epsilon) = \frac{T(\epsilon) + 3N\epsilon_0}{4}.$$

Adding the auxiliary minimum $-3|S\epsilon|/4$ gives the integral minimum of the corresponding extension component:

$$h_{\min}(E_{N,\epsilon}) = \begin{cases} -T(\epsilon)/2, & \epsilon_0 = 0, \\ T(\epsilon), & \epsilon_0 = 1. \end{cases}$$

All higher weights differ from these by integers. The maximum possible even transition number on an odd cycle is $N - 1$, achieved by a word with $\epsilon_0 = 0$. This proves the stated least degree. Central charges add to $-5N + N(-11) = -16N$. $\square$

Completion of the proof of Theorem 1.2. The cyclic construction, invariant form, and C_2 argument establish the stated properties of W_N ; Section 17 proves its regularity. The parity factorization is Section 18. The lattice branching and the vanishing remainder in the dimension identity give $|\mathrm{Irr} \mathcal{D}| = N^2 + 3$. The parity factorization then gives $|\mathrm{Irr}(W_N)| = 2^N(N^2 + 3)$. Corollary 29.4 constructs $\mathcal{H}_N$ and gives the ribbon-center identification with $\dim \rho = -t$. Theorem 31.1 supplies the explicit VOA E_N , its simplicity, self-contragredience, C_2 -cofiniteness, regularity (and hence rationality), its central charge, its least degree, and the required ribbon equivalence. Theorem 30.2, with the metric isometry of Lemma 30.1, proves the modular-data identification in part (4). These are all the assertions of the theorem. $\square$

Technical appendices

A Ordinary first images of proper cyclic intervals

Keep the hypothetical weight-one vacuum extension and put $H_* = 2N + 2$. For a proper connected binary root support J containing zero, let B_J be its primary and $r_J = (B_J)_{H_* - 2v}$.

Lemma A.1. *Every r_J is a sum of ordinary simultaneous partial eigenvectors. Its support consists only of coordinate spectra at the active frame sites of B_J .*

Proof. At a vacuum frame site, $L_i(-1)B_J = 0$ and $h_i^J = 0$. Equations (12.2)–(12.3) therefore give $H_i r_J = 0$ exactly. At its two endpoints, Lemma 12.5 makes the partial operator semisimple on r_J . Only internal $(3, 3)$ factors can initially have partial nilpotents. For $J = \{0\}$ there are no such internal factors, proving the base case.

First let $J = [0, k]$, with $0 \leq k \leq N - 2$, and write $B_k = B_J$. For $0 \leq a < k$, adjoin the ordinary interval $A_{a,k} = P_{[a+1,k]}$ to B_a . Unique diagonal fusion and its level-one Ward coefficient give

$$(A_{a,k})_1 B_a = \kappa_{a,k} B_k,$$

$$(A_{a,k})_0 B_a = \kappa_{a,k} \left(\sum_{i=a+1}^k L_i(-1) - (t+a)L_a(-1) \right) B_k, \quad \kappa_{a,k} \neq 0.$$

At a newly added site the right input is the vacuum, so its intertwiner is $e^{zL_i(-1)}$ and has first coefficient one. At the shared site, the ordered $(1, 3), (3, 1)$ coefficient is $-1/(\psi_a - 1) = -(t+a)$. This proves the displayed formulas, including their signs. The pairing of these coefficients with the output partial weights is

$$\sum_{i=a+1}^k h_i^{(k)} - (t+a)h_a^{(k)} = 1,$$

by telescoping the internal weights $2/((t+i)(t+i+1))$ and using $h_k^{(k)} = 1 + 2/(t+k)$.

For any weight-two A and weight- H_* state B , (12.4) and the zero-mode commutator give $o(B)o(A)w = -o(A_0B)w/H_*$. Applying this here gives the exact transport relation

$$\kappa_{a,k} \left(\sum_{i=a+1}^k H_i - (t+a)H_a - 1 \right) r_k = -o(B_a)o(A_{a,k})w. \quad (\text{A.1})$$

Indeed, $o(L_i(-1)B_k)w = H_*(H_i - h_i^{(k)})r_k$. Every ordinary interval preserves U , so if $o(A_{a,k})w = \sum_i \alpha_i w_i$, its right-hand side is

$$-\sum_i \alpha_i (h_i^{(a)} - H_i) r_a.$$

By induction this is ordinary and supported at $e_{N-1}, e_0, \dots, e_a$.

On a component e_j with $0 \leq j < k$, choose $a = j$. The transport operator has nonzero scalar value $-(t+j+1)$. On e_{N-1} every cut has scalar value -1 . A polynomial in the commuting partial operators with nonzero scalar part is invertible on the generalized component, and its inverse takes an ordinary vector to an ordinary vector. All these components of r_k are ordinary. On the remaining component e_k , every right-hand side is zero. Write $n_i = H_i - \delta_{ik}$. Endpoint semisimplicity gives $n_k r_k = 0$. The cut $a = k-1$ gives $n_{k-1} r_k = 0$, then $a = k-2$ gives $n_{k-2} r_k = 0$, and so on. Vacuum sites and the fixed endpoint were already treated. This proves all prefixes.

For left growth, adjoin an ordinary interval $[a, b]$ to a heavy interval starting at $b+1$ and wrapping through zero. The analogous level-one coefficients are one at $a-1, \dots, b-1$ and $t+b+1$ at the shared site b . The latter equals $\psi_b/(\psi_b - 1)$ for the ordered $(3, 1), (1, 3)$ input. Their partial-weight pairing is

$$\left(1 - \frac{2}{t+a}\right) + \sum_{i=a}^{b-1} \frac{2}{(t+i)(t+i+1)} + (t+b+1) \frac{2}{(t+b)(t+b+1)} = 1.$$

The transport operator is therefore

$$\tilde{D}_{a,b} = \sum_{i=a-1}^{b-1} H_i + (t+b+1)H_b - 1.$$

For the suffix support $\{a, \dots, N-1, 0\}$, a component e_j with $a \leq j \leq N-1$ uses the cut $b = j$, whose scalar is $t+j$. The component e_0 has scalar -1 . Both are nonzero, and induction on the

shorter suffix proves ordinariness there. On the new component e_{a-1} , the right-hand sides vanish. Endpoint semisimplicity and the cuts $b = a, a+1, \dots, N-1$ successively kill all its internal nilpotents. This proves every suffix together with root zero.

Finally take $J = \{\ell, \dots, N-1\} \cup \{0, \dots, k\}$ with $1 \leq k < \ell-1$. For each component e_j , $0 \leq j \leq k$, cut off the entire ordinary suffix $[\ell, N-1]$. The left-growth transport operator has scalar -1 , and the remaining prefix image is ordinary. For each component e_j , $\ell-1 \leq j \leq N-1$, cut off the entire ordinary prefix $[1, k]$. The right-growth operator again has scalar -1 , and its remaining suffix image is ordinary. These components exhaust the active sites, proving the lemma without assuming a cyclic symmetry of the distinguished root. $\square$

B Explicit cut polynomials

This appendix defines the polynomials used in the Jacobian and endpoint arguments by finite sums. Thus all their stated values and derivatives are identities of rational functions determined inside the article.

B.1 Local descendant evaluation

For a local parameter p , put $h(p) = 2(p-1)^2/p$. For a PBW word $\mathbf{n} = (n_1, \dots, n_s)$ representing $L(-n_1) \cdots L(-n_s)C$, define

$$Z_{\mathbf{n}}(x, y; p) = \prod_{k=1}^s (-1)^{n_k} \left(-x + n_k y + h(p) + \sum_{\ell > k} n_{\ell} \right).$$

This is the repeated partial Zhu identity (12.5), with the partial weight updated after each inner descendant. Let $V_d^+(p)$ and $V_d^-(p)$ be the local level- d coefficients for ordered inputs $(3, 1)$, $(1, 3)$ and $(1, 3)$, $(3, 1)$, respectively. Through level three they are the coefficients displayed in Section 13, with $V_0^+ = V_0^- = 1$. Write $Q_d^{\pm}(x, y; p)$ for the polynomial obtained by substituting $Z_{\mathbf{n}}$ into $V_d^{\pm}$.

For three disjoint groups of sites, let X, Y, Z denote the sums of their left partial conformal operators, let U, V, T denote independent right variables, and let h_X, h_Y, h_Z be the corresponding partial weights of C . For a polynomial P , write $P_{abc} = P(U=a, V=b, T=c)$ and put

$$\begin{aligned} \mathcal{F}(P) &= (H-1)P_{000} + (h_X - X)(P_{100} - P_{000}) \\ &\quad + (h_Y - Y)(P_{010} - P_{000}) + (h_Z - Z)(P_{001} - P_{000}), \quad H = 2N. \end{aligned}$$

The right partial operators on $U = \text{span}\{w_i\}$ are orthogonal coordinate idempotents. Since $o(C)w = (H-1)r_C$ and $o(C)w_i = (h_i - H_i)r_C$, the action of the descendant encoded by P on w is exactly $\mathcal{F}(P)r_C$. There is no division by r_C in this definition.

B.2 The prefix cuts

Fix $0 \leq a \leq N-2$ and put $r = t + a$,

$$\begin{aligned} X &= H_a, & Y &= H_{N-1}, & Z &= \sum_{i < a} H_i, \\ h_X &= \frac{2}{r(r+1)}, & h_Y &= \frac{2t^2}{1-t}, & h_Z &= \frac{2}{t} - \frac{2}{r}. \end{aligned}$$

The ordered product $B_a \times A_a$, with $B_a = P_{[0,a]}$ and $A_a = P_{[a+1, N-1]}$, has highest output C with coefficient $\kappa_a \neq 0$. Its degree- d coefficient, divided by κ_a , is represented by

$$P_d = \sum_{b+c+e=d} Q_b^+ \left(X, U; \frac{r+1}{r} \right) Q_c^- \left(Y, V; \frac{1}{1-t} \right) \frac{(Z-T-h_Z)_e}{e!}, \quad 0 \leq d \leq 3,$$

where $(z)_e = z(z-1)\cdots(z-e+1)$. The last factor is the simultaneous translated unit intertwiner at the prefix sites. By (12.4),

$$K_a = \sum_{\ell=0}^3 \frac{\binom{H+1}{\ell}}{H+2-\ell} \mathcal{F}(P_{3-\ell}), \quad \kappa_a K_a(H) r_C = o(B_a) o(A_a) w. \quad (\text{B.1})$$

For $a = 0$ the variable Z is absent. Direct differentiation of this finite expression gives all four gradients in Section 13. All denominators are nonzero at $0 < t < 1/3$ and $r = t + a$: their possible zeros in a local parameter are rational values excluded by its irrationality.

B.3 Every ordinary interval and its heavy complement

Let $A = P_{[a,b]}$, $1 \leq a \leq b \leq N-1$, let $B = A^c$ be its heavy complement, and put

$$r = t + a - 1, \quad s = t + b, \quad X = H_{a-1}, \quad Y = H_b, \quad Z = \sum_{a \leq i < b} H_i.$$

Here

$$h_X = \frac{2}{r(r+1)}, \quad h_Y = \frac{2}{s(s+1)}, \quad h_Z = \frac{2}{r+1} - \frac{2}{s}.$$

Use the simultaneous disjoint-support normalization proved in Appendix E. For the product ordered as $A \times B$, define

$$P_3^{a,b} = \sum_{d+e+f=3} Q_d^- \left(X, U; \frac{r+1}{r} \right) Q_e^+ \left(Y, V; \frac{s+1}{s} \right) \frac{(Z - T - h_Z)_f}{f!}.$$

Since A has weight two, the defect identity and zero-mode commutator give

$$o(B)o(A)w = -\frac{1}{H+2} o(A_0 B)w.$$

Consequently the full polynomial in the complementary-interval equation is

$$K_{a,b} = -\frac{\mathcal{F}(P_3^{a,b})}{H+2}. \quad (\text{B.2})$$

Evaluating and differentiating this formula at $(X, Y, Z) = (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ gives respectively the off-support, left, right, and internal rows in Appendix E. The formula retains both left and right Zhu variables at every intermediate step.

B.4 Reconstructing the first derivatives

The four evaluation points in the preceding formula use the coordinate order (X, Y, Z) . Their gradients are, respectively,

$$g_{\text{off}}, \quad g_{\text{left}}, \quad g_{\text{right}}, \quad g_{\text{inside}}.$$

For the prefix formula the same points give, after reordering components to (Z, X, Y) ,

$$G_{\text{suffix}}, \quad G_{\text{cut}}, \quad G_{\text{last}}, \quad G_{\text{prefix}}.$$

This is the gradient convention of Section 13.

For example, write $P = P_3^{a,b}$ and evaluate the following expression at $(X, Y, Z) = (0, 0, 0)$:

$$\begin{aligned} \partial_X K_{a,b} = & -\frac{1}{H+2} [(H-1)\partial_X P_{000} - (P_{100} - P_{000}) \\ & + h_X \partial_X (P_{100} - P_{000}) + h_Y \partial_X (P_{010} - P_{000}) + h_Z \partial_X (P_{001} - P_{000})]. \end{aligned}$$

Substituting the displayed Z_n and $V_d^\pm$ gives $r(r^2 + 4r + 1)/(3(r-1)(r+2))$, the first entry of g_{off} . The other entries use the same differentiation, with the selected coordinate equal to one at an endpoint or interior component. The prefix calculation uses (B.1) and $H = 2/t - 2t$.

C Maximum plateaux and the filtered Zhu span

The frame decomposition has highest vectors $v_m \in M_m$, with

$$h(m) = \sum_i (m_i - m_{i+1})^2 + Nm_0(m_0 + 1), \quad m \in \mathbb{Z}_{\geq 0}^N.$$

Its diagonal multiplication has a nonzero top channel $M_m \times M_n \rightarrow M_{m+n}$, and all other output labels ℓ obey $\ell_i \leq m_i + n_i$ coordinatewise.

Lemma C.1. *The Zhu algebra of W is generated by the frame classes, all connected proper binary interval primaries, and the full cyclic primary C . For a proper root subset I , the Zhu algebra of $W_I = \bigoplus_{\text{supp}(m) \subset I} M_m$ is generated by the frame and the binary interval primaries supported in I . If A_0 is the Zhu image of $W_{\{1, \dots, N-1\}}$ and $F_1 W = \bigoplus_{m_0 \leq 1} M_m$, then*

$$[F_1 W] \subseteq A_0 + \sum_{P \in \{C, \text{proper heavy interval primaries}\}} A_0[P]A_0.$$

Proof. We first prove a strong spanning assertion by induction on $|m| = \sum_i m_i$. If $\max_i m_i \geq 2$, choose one connected component of the maximum plateau, and let u be its binary indicator. If the maximum plateau is the whole cycle, take u to be the all-ones vector. Direct expansion gives

$$\begin{aligned} \Delta &= h(m) - h(u) - h(m - u) \\ &= 2 \sum_i (u_i - u_{i+1})((m_i - u_i) - (m_{i+1} - u_{i+1})) + 2Nu_0(m_0 - u_0) \geq 0. \end{aligned}$$

Each boundary term is nonnegative because a neighbor outside the maximum plateau has height at most one less than its maximum. The final term is nonnegative as well. The product

$$(v_u)_{-\Delta-1} v_{m-u}$$

has a nonzero v_m coefficient, and every other output label has strictly smaller total label. These other contributions are partial Virasoro descendants of their highest vectors. Induction expresses v_m in normal products of frame states and partial descendants of binary highest vectors. Every negative mode is a normal product after a total derivative of its left factor.

The same spanning statement holds for partial descendants. Each $L_i(-1) = (\omega_i)_0$ is a derivation of every vertex product, so it differentiates the normal-product factors. For $n \geq 2$, $L_i(-n)a = (\omega_i)_{1-n}a$ adds a frame factor by a negative product. This proves closure under all partial Virasoro descendants. It also proves the assertion within W_I , since the top-channel construction and all lower output labels keep their support in I .

A disconnected binary support is the disjoint union of its separated interval components. Their unique diagonal output is their union, its weight is the sum of their weights, and its highest coefficient is nonzero. All singular products between the separated components vanish. Their normal product consequently gives the union primary, and their Zhu classes multiply to its class up to a nonzero scalar. The full cyclic support is a separate connected case, with primary C . Repeated use of (12.5) reduces every partial descendant to its primary class and left/right frame classes. This proves the two Zhu-generation assertions.

For the filtered assertion, suppose $m_0 \leq 1$. Whenever the maximum is at least two, the chosen maximum plateau avoids root zero, so $u_0 = 0$. The first factor in the displayed negative product is therefore in the open-chain algebra. Moreover every lower output label has $\ell_0 = m_0$ exactly: at root coordinate zero the first factor is the tensor unit in the aligned fusion rule. The same induction on $|m|$ thus expresses every state of $F_1 W$ in normal products with at most one binary factor whose support contains zero. A terminal binary support has at most one connected

component containing zero, and that component is a proper heavy interval or the whole cycle. Partial derivatives and frame factors preserve this count by the preceding derivation argument.

Now use a second, independent induction on total conformal weight to pass to Zhu classes. The identity

$$[a_{-1}b] = [a] * [b] - \sum_{j=1}^{\text{wt}(a)} \binom{\text{wt}(a)}{j} [a_{j-1}b]$$

has strictly lower-weight singular terms. In the spanning expression just obtained, every such singular term remains in F_1W , because one factor has root-zero label zero. Its class is therefore covered by the conformal-weight induction. Higher negative modes reduce by total derivatives. Partial descendants reduce by (12.5), adding only left and right frame classes. The right-hand side of the claimed inclusion is an A_0 -bimodule, so these reductions preserve it without commuting any two noncommuting factors. This proves the filtered assertion. $\square$

D Uniform invertibility of the all-cut Jacobian

We prove that the gradients in Section 13 are independent on the hyperplane $\sum_i n_i = 0$. Fix the joint eigenvalue tuple e_j , let $n = (n_i)$ lie in the kernel of the corresponding Jacobian, and write

$$S_a = \sum_{i < a} n_i, \quad z = n_{N-1}, \quad r = t + a,$$

and denote the three appropriate gradient entries by A_a, B_a, D_a . The equations are

$$A_a S_a + B_a n_a + D_a z = 0, \quad S_{a+1} = (1 - A_a/B_a) S_a - (D_a/B_a) z.$$

The coefficient B_a is nonzero in each row. This is immediate for the suffix and cut rows; for the prefix and last rows it follows from the signs established below. If $z = 0$, induction from $S_0 = 0$ gives $n = 0$. Otherwise set $u_a = S_a/z$; the required terminal value is $u_{N-1} = -1$.

D.1 The last component

Here

$$A = \frac{2t-3}{6t}, \quad D = \frac{t-1}{3(t+1)}, \quad B(r) = \frac{(r+1)((2t-3)r^2 + (8t-9)r - 4t + 6)}{6t(r-1)(r+2)}.$$

Since $0 < t < 1/3$, the first recurrence multiplier is

$$q_0 = 1 - A/B(t) = \frac{t^2(2t+7)}{(t+1)(2t^2+7t-6)} \in (-1, 0).$$

For $a \geq 1$, $r > 1$ and

$$B(r) - A = \frac{r((2t-3)r^2 + (8t-9)r + 2t)}{6t(r-1)(r+2)} < 0.$$

Thus $B(r) < A < 0$, and $q_a = 1 - A/B(r) \in (0, 1)$. The recurrence has fixed point $-D/A$. Its terminal condition would require

$$\prod_{a=0}^{N-2} q_a = 1 - A/D = -\frac{3-t}{2t(1-t)} < -1.$$

The left side lies in $(-1, 0)$, a contradiction. For the first multiplier, the inequality $q_0 > -1$ follows from $4t^3 + 16t^2 + t - 6 < 0$ on $0 < t < 1/3$.

D.2 The other components for ranks at least five

Now $0 < t < 1/5$. For $a < j$, the suffix recurrence has multiplier $(t + a)/(t + a + 1)$ and fixed point $(1 - t)/t$, hence

$$u_j = \frac{j(1 - t)}{t(t + j)}.$$

At the cut row, with $r = t + j$, substitution gives

$$u_{j+1} = \frac{(t - 1)(6tr^2 + (6t^3 - t^2 - 5t + 2)r + 2t^3 - 3t^2 - 9t + 2)}{2t(r + 1)(t + 1)(2t - 1)}.$$

Both polynomial coefficients $6t^3 - t^2 - 5t + 2$ and $2t^3 - 3t^2 - 9t + 2$ are positive for $0 < t < 1/5$: they are decreasing on this interval and positive at $1/5$. Therefore $u_{j+1} > 0$.

Every subsequent prefix row has $r > 1$, multiplier

$$\frac{r(r^2 + 4r + 1)}{(r + 1)(r^2 + 4r - 2)} \in (0, 1),$$

and common fixed point

$$v_* = -\frac{(t - 1)(2t^2 + 4t - 1)}{t(t + 1)(2t - 1)} > 0.$$

Each step is a convex combination of its predecessor and v_* . Thus $u_{N-1} > 0$, contradicting the terminal value -1 .

D.3 The two remaining rank-three cases

Put $t = (\sqrt{13} - 3)/2$. At $j = 0$, exact reduction modulo $t^2 + 3t - 1$ gives

$$u_1 + 1 = \frac{53 - 175t}{2(16t - 5)} < 0, \quad v_* + 1 = \frac{7t - 2}{16t - 5} < 0.$$

The next step is a convex combination of two values below -1 , so cannot reach -1 . The signs follow from $2/7 < t < 53/175 < 5/16$. At $j = 1$ the cut is the last row, and

$$u_2 = \frac{37 - 120t}{2(7t - 2)} > 0,$$

using $2/7 < t < 37/120$. These inequalities can be checked exactly by evaluating $f(x) = x^2 + 3x - 1$ at the rational endpoints:

$$f(2/7) = -3/49, \quad f(53/175) = 9/30625, \quad f(37/120) = 289/14400.$$

All cases have zero homogeneous kernel, proving uniform invertibility.

E Uniform elimination of the first-image obstruction

We prove Theorem 13.2 using the semisimplicity of the proper-interval first images and the bound $\mathfrak{m}^2 r_{C,j} = 0$. First we compare complementary intervals to eliminate the two possible endpoint components of the closing-root image. We then propagate the vanishing to every proper heavy interval. All equations below are vector equations with scalar coefficients.

E.1 Common normalization and complementary equations

For disjoint binary root supports J, K , aligned fusion has a unique highest output on $J \cup K$, with nonzero coefficient $\kappa_{J,K}$. Choose every local Virasoro intertwiner with highest coefficient one. In a triple of pairwise disjoint binary supports, each Virasoro site sees at most two nonvacuum inputs, since its two labels depend on just the two adjoining roots. The local associativity comparison therefore includes a tensor unit. With its canonical left unit and translated right unit $e^{zL(-1)}$, its scalar is one. The translation contributes descendants but no extra constant. Thus the κ 's obey the ordinary cocycle identity on disjoint unions.

The only nontrivial crossing has local exponent $h_{33} - h_{31} - h_{13} = -2$, so its scalar is symmetric. Iterated products of the singleton highest vectors are consequently independent of order and parenthesization. They normalize all $\kappa_{J,K}$ to one while retaining prescribed singleton scales. Choose ordinary-root scales so that their left endpoint coefficient on w is one. The remaining free scale of root zero is chosen so that C has leading charge term e^{-B} ; every full-support disjoint product is linear in that root scale, so both normalizations can be imposed simultaneously.

For an ordinary interval $A = P_{[a,b]}$, $1 \leq a \leq b \leq N-1$, put

$$r = t + a - 1, \quad s = t + b, \quad D(x) = x(x-1)(x+1)(x+2), \quad d(r, s) = \frac{s(s+1)}{D(r)}.$$

The open-chain anticommutator induction gives

$$o(A)w = d(r, s) \left(r(r-1)w_{a-1} + 2 \sum_{i=a}^{b-1} w_i + (s+1)(s+2)w_b \right). \quad (\text{E.1})$$

For a singleton this is Lemma 12.7. Adjoining the next ordinary root multiplies old nonshared coefficients by $(s+1)/(s-1)$; the shared and new endpoint coefficients become $2d(r, s)$ and $d(r, s)(s+1)(s+2)$, proving the formula.

Let $B = A^c$ be the heavy complementary interval. The exact global equation is

$$K_{a,b}(H)r_C = o(B)o(A)w.$$

The polynomial vanishes at every e_j . With $X = H_{a-1}$, $Y = H_b$, $Z = \sum_{a \leq i < b} H_i$, its gradients for an off-support, left-endpoint, or right-endpoint component are respectively

$$\begin{aligned} g_{\text{off}} &= \left(\frac{r(r^2 + 4r + 1)}{3(r-1)(r+2)}, -\frac{(s+1)(s^2 - 2s - 2)}{3(s-1)(s+2)}, -\frac{1}{3} \right), \\ g_{\text{left}} &= \left(-\frac{r(3r+1)}{6(r-1)}, -\frac{(s+1)(3rs^2 - 3rs - 6r + 2s^2 - 4s - 4)}{6(s-1)(s+2)}, -\frac{3r+2}{6} \right), \\ g_{\text{right}} &= \left(-\frac{r(3r^2s + r^2 + 9rs + r - 2)}{6(r-1)(r+2)}, \frac{(s+1)(3s+2)}{6(s+2)}, \frac{3s+1}{6} \right). \end{aligned}$$

For an internal component it is $g_{\text{inside}} = (-r, s+1, 1)/6$. These follow from the level-three coefficients already displayed; the finite polynomial prescription is given in Appendix B.

The proper first image is ordinary. Its right-side coefficient on a component where it is supported is

$$d(r, s)\gamma_j(r, s)r_{B,j}, \quad \gamma_j = \begin{cases} 2r + s^2 + s - 4, & j = a-1, \\ r^2 + r - 2s - 6, & j = b, \\ r^2 + r + s^2 + s - 4, & j \notin [a-1, b]. \end{cases}$$

This is the dot product of (E.1)'s coefficient vector with $h_i^B - \delta_{ij}$. Since $\mathbf{m}^2 r_{C,j} = 0$, these equations use only the gradients, with $n_i = (H_i - \delta_{ij})r_{C,j}$ and $\sum_i n_i = 0$.

E.2 The endpoint components for ranks at least five

Write $\sigma = t + N - 1 = t^{-1} - 1$. On component e_0 , set $u = r_{R_0,0}$. Repeated proper transport gives

$$r_{B,0} = \begin{cases} D(\sigma)u/D(s), & a = 1, \\ \frac{2r(r+1)D(\sigma)}{t(t+1)^2(t+2)D(s)}u, & a \geq 2. \end{cases}$$

These formulas follow by removing ordinary tips on each side of zero, using the nonzero scalar parts of the previously established transport operators. Here is the scalar recursion explicitly. Write $r_{B,j} = T_j(a, b)u$, initialized by $T_j(1, N-1) = 1$. Removing one tip gives, whenever the indicated intervals are nonempty,

$$\begin{aligned} \frac{T_0(a+1, b)}{T_0(a, b)} &= \begin{cases} 2/[t(t+1)], & a = 1, \\ (r+2)/r, & a \geq 2, \end{cases} \\ \frac{T_0(a, b-1)}{T_0(a, b)} &= \frac{s+2}{s-2}. \end{aligned}$$

These are scalar equalities, not divisions by the possibly zero vector u . They follow by evaluating the proper-transport operator in (A.1) at e_0 , using the ordinary-interval coefficient vector (E.1); the left-tip operator has shared-site coefficient $s+1$ and coefficient one at each new site. For example, at an endpoint component $j \in \{0, N-1\}$ the right-growth scalar is

$$-\frac{1 + 2/r - \delta_{j,a-1}}{-1 - r\delta_{j,a-1}}.$$

The old heavy interval has partial weight $1 + 2/r$ at $a-1$ and zero at a ; the ordinary root has coefficient one at $a-1$. For left growth the scalar is

$$-\frac{(s+1)(s+2)}{(s-1)(s-2)} \frac{1 - 2/(s+1) - \delta_{j,b}}{(s+1)\delta_{j,b} - 1}.$$

Here the old heavy weight at b is $1 - 2/(s+1)$, and the first fraction is the right coefficient of the adjoining ordinary root. These two scalar evaluations also give the recurrences for the last component below. Products telescope by $D(s)/D(s-1) = (s+2)/(s-2)$ and $r(r+1)$, giving the stated $T_0(a, b)$. Set

$$z = \frac{2D(\sigma)}{t(t+1)^2(t+2)}u, \quad A_*(x) = (x-1)(x+2), \quad F(x) = x+3 + \frac{6}{A_*(x)}, \quad G(x) = x-2 - \frac{6}{A_*(x)}.$$

For every pair of off-component frame endpoints $1 \leq i < k \leq N-1$, the global equation becomes

$$F(t+i)n_i - G(t+k)n_k - \sum_{i < \ell < k} n_\ell = 3z \left(\frac{1}{A_*(t+i)} + \frac{1}{A_*(t+k)} \right). \quad (\text{E.2})$$

Add the equations for (i, k) and (k, ℓ) , then subtract the one for (i, ℓ) . Since

$$F(x) - G(x) + 1 = \frac{6x(x+1)}{A_*(x)},$$

we obtain $n_k = z/[(t+k)(t+k+1)]$ for $2 \leq k \leq N-2$. For $N \geq 5$, the set $\{2, \dots, N-2\}$ is nonempty. Using a known internal site in (E.2) extends the formula to $k = 1, N-1$, since $F(t+1) \neq 0$ and $G(\sigma) > 0$. The sum relation then gives $n_0 = z(t-1/(t+1))$. Substitution in any remaining interval with $a = 1$ leaves

$$-\frac{(3t+1)(t^2+1)}{6(t-1)}z = 0.$$

The scalar substitution behind this residual is

$$g_{\text{left}}(t, s) \cdot \left(t - \frac{1}{t+1}, \frac{1}{s(s+1)}, \frac{1}{t+1} - \frac{1}{s} \right) - \frac{(t+1)(2t+s^2+s-4)}{2(t-1)(s-1)(s+2)} = -\frac{(3t+1)(t^2+1)}{6(t-1)}.$$

The coefficient is nonzero, so $u = 0$ and every $n_i = 0$ on this component.

On component e_{N-1} , put $u = r_{R_0, N-1}$. The transport formulas are

$$r_{B, N-1} = \begin{cases} \frac{r(r+1)}{t(t+1)}u, & b = N-1, \\ \frac{2r(r+1)(\sigma+1)(\sigma+2)}{t(t+1)D(s)}u, & b \leq N-2. \end{cases}$$

For completeness, the corresponding scalar recurrences are

$$\frac{T_{N-1}(a+1, b)}{T_{N-1}(a, b)} = \frac{r+2}{r}, \quad \frac{T_{N-1}(a, b-1)}{T_{N-1}(a, b)} = \begin{cases} \frac{2(\sigma+2)}{(\sigma-2)(\sigma-1)\sigma}, & b = N-1, \\ \frac{s+2}{s-2}, & b \leq N-2. \end{cases}$$

They use the same operator at e_{N-1} and the same initial value. Set $z = 2(\sigma+1)(\sigma+2)u/[t(t+1)]$. The off-component equations are again (E.2), now for $0 \leq i < k \leq N-2$. Triangle differences give the same formula for n_k at $1 \leq k \leq N-3$. It extends to the two other off-component sites using $F(t) < 0$ and $G(t+N-2) > 0$. The sum fixes n_{N-1} . A remaining interval ending at $N-1$ gives

$$-\frac{(3\sigma+2)(\sigma^2+2\sigma+2)}{6(\sigma+2)}z = 0.$$

In this case the corresponding scalar identity, with $s = \sigma$, is

$$g_{\text{right}}(r, s) \cdot \left(\frac{1}{r(r+1)}, \frac{1}{s} - s - 1, \frac{1}{r+1} - \frac{1}{s} \right) - \frac{s(r^2+r-2s-6)}{2(r-1)(r+2)(s+2)} = -\frac{(3s+2)(s^2+2s+2)}{6(s+2)}.$$

Again $u = 0$ and every $n_i = 0$. The two endpoints are the only allowed components of r_{R_0} , so $r_{R_0} = 0$.

E.3 Exact rank-three endpoint calculation

For $N = 3$, use the four unknown vectors (n_0, n_1, n_2, u) , with $u = r_{R_0, j}$. The row order is the sum relation followed by the interval equations $[1, 1], [1, 2], [2, 2]$. Substituting the same gradients and transport formulas, and reducing modulo $t^2 + 3t - 1$, gives the matrices below. Explicitly, for $(a, b) = (1, 1), (1, 2), (2, 2)$, choose $g = (g_X, g_Y, g_Z)$ to be the left, right, or off row according as $j = a-1, j = b$, or neither. With $\mathbf{e}_i$ the coordinate row vectors, the corresponding matrix row is

$$\left(g_X \mathbf{e}_{a-1} + g_Y \mathbf{e}_b + g_Z \sum_{a \leq i < b} \mathbf{e}_i, -d(r, s) \gamma_j(r, s) T_j(a, b) \right).$$

The first row is $(1, 1, 1, 0)$. This gives every entry without a separate rank-three ansatz:

$$M_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -(5t-4)/18 & (19t+11)/6 & 0 & -(22t+73)/3 \\ -(5t-4)/18 & -(3t+2)/6 & 1/6 & (42t+139)/3 \\ 0 & (t+10)/3 & (t+2)/9 & -4(53t+175)/9 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -(t+1)/9 & -(t-7)/3 & 0 & -4(17t+56)/9 \\ 1/6 & (3t+7)/6 & (5t+19)/18 & -(102t+337)/3 \\ 0 & -(19t+46)/6 & (5t+19)/18 & (122t+403)/3 \end{pmatrix}.$$

Their exact determinants are

$$\det M_0 = -\frac{376t + 1253}{27}, \quad \det M_2 = \frac{4016t + 13265}{27}.$$

Both are nonzero for $t > 0$. Thus $r_{R_0} = 0$ also for $N = 3$. The displayed matrices follow by substituting the three intervals into (B.2) and the transport relations, then reducing modulo $t^2 + 3t - 1$.

E.4 Propagation and the final Jacobian step

Suppose a proper heavy interval B satisfies $r_B = 0$, so $o(B)U = 0$. Adjoin an ordinary interval A to form another proper heavy interval B' . Its unique order-two OPE has

$$A_1 B = \kappa B', \quad A_0 B = \kappa \sum_i c_i L_i(-1) B', \quad \sum_i c_i h_i^{B'} = 1.$$

For right growth, c_i is $-(t+a)$ at the shared site, one at each new site, and zero elsewhere. For left growth, its shared-site value is $t+b+1$, again with new-site values one. Put $Q = \sum_i c_i H_i$. On every top, the partial Zhu formula and the usual mode commutator give

$$[o(A), o(B)] = o(A_0 B) + o(A_1 B) = \kappa [Q, o(B')].$$

The scalar terms cancel because their partial-weight pairing is one. The left side kills U , as $o(A)U \subset U$.

The first image $r_{B'}$ is ordinary. Project the identity, applied to w_i , to a target component e_j :

$$(c_j - c_i)(h_i^{B'} - \delta_{ij})r_{B',j} = 0.$$

On the active support, shared and new sites have different coefficients. For any target j , choose an active i with $c_i \neq c_j$; then $i \neq j$. Every active partial weight is nonzero: internally it is $2/[(t+i)(t+i+1)]$, and at an endpoint it is an irrational $(3, 1)$ or $(1, 3)$ weight. Hence $r_{B',j} = 0$.

Starting at R_0 and adjoining roots on either side reaches every proper heavy cyclic interval, proving $r_B = 0$ for all of them. Every right side in (13.2) is now zero. Appendix D's invertible Jacobian therefore gives $\mathfrak{m}r_{C,j} = 0$. This proves Theorem 13.2 for every odd $N \geq 3$.

F The weight-zero top has vacuum open-chain action

We use the frame decomposition of W , its nonzero diagonal multiplication, the elementary cubic null relations, and Appendix C. Let M be a finite-dimensional representation of $A(W)$ whose commuting partial conformal operators H_i have joint generalized eigenvalue zero and satisfy

$$\sum_{i=0}^{N-1} H_i = 0.$$

These hypotheses hold on the relevant neutral aligned simple tops of lowest weight zero. They also hold on a vacuum extension by a distinct such simple: the total conformal nilpotent factors through a homomorphism from the vacuum to the distinct simple and hence vanishes. We do not assert the result for arbitrary logarithmic tops with a nonzero total conformal nilpotent.

F.1 Eliminating the partial nilpotents

For the ordinary elementary root R_j , $1 \leq j < N$, put

$$a = 1 + \frac{2}{t+j}, \quad b = 2 - a, \quad t^2 + Nt - 1 = 0, \quad t > 0.$$

Its two active partial weights are a, b at sites $j, j-1$. The local cubic relation is

$$(x-y)((x-y)^2 - 2\psi(x+y) + 2\psi - 1) = 0.$$

The second factor is invertible on left and right multiplication by a nilpotent H_i , since $2\psi - 1 \neq 0$. Thus $[H_i, R_j] = 0$ for every i . Let $\mathcal{R} \subseteq \text{End}(M)$ be the commutative algebra generated by the H_i , and let $\mathfrak{m} = (H_0, \dots, H_{N-1})$ be its nilpotent maximal ideal. The operators R_j commute with $\mathcal{R}$ by the preceding null relation. The partial Zhu formula

$$[L_i(-n)v] = (-1)^n(-H_i * [v] + n[v] * H_i + h_i[v]), \quad n \geq 1, \quad (\text{F.1})$$

gives $[D_a R_j] = -a R_j$. In particular $[R_j, D_a R_j] = 0$.

Only the vacuum and root self-channel occur in $R_j * R_j$ and in

$$[R_j, D_a R_j] = [(R_j)_0 D_a R_j] + [(R_j)_1 D_a R_j].$$

All states used have weight at most four, whereas the next aligned channel has lowest weight eight. Write $\nu_j, \sigma_j \neq 0$ for the vacuum and self-channel coefficients. Set

$$\begin{aligned} u(h) &= \frac{h(h+1)}{3(h-4)(3h-1)}, & d(a) &= (a-4)(a+2)(2a-3)(2a-1), \\ \kappa(a) &= -\frac{a(a-2)(5a^2-10a+2)}{4d(a)}, & \eta(a) &= -\frac{7a^2(a-2)^2(a-1)}{8d(a)}. \end{aligned}$$

The commutator has the form $P(H) + Q(H)R_j = 0$, where $P(0) = 0$ and $Q(0) = 2\sigma_j\eta(a) \neq 0$. Since $Q(H)$ is invertible in $\mathcal{R}$, this expresses R_j as $-Q(H)^{-1}P(H) \in \mathfrak{m}$. Modulo $\mathfrak{m}^2$, the product and half the commutator give

$$\begin{aligned} 0 &\equiv \nu_j(u(a)H_j + u(b)H_{j-1}) + \sigma_j\kappa(a)R_j, \\ 0 &\equiv \nu_j(-b u(a)H_j + a u(b)H_{j-1}) + \sigma_j\eta(a)R_j. \end{aligned}$$

Eliminating R_j uses the exact identity

$$((\eta + b\kappa)u(a), (\eta - a\kappa)u(b)) = \frac{a^2(a-2)^2}{24d(a)}(a+1, a-3).$$

All denominators, as well as σ_j , $\eta(a)$ and the common factor $a^2(a-2)^2/(24d(a))$, are nonzero: each possible zero would force a to be rational. No division by $\kappa(a)$ is needed. We obtain

$$(t+j+1)H_j - (t+j-1)H_{j-1} \in \mathfrak{m}^2. \quad (\text{F.2})$$

Iteration gives

$$H_j \equiv \frac{t(t+1)}{(t+j)(t+j+1)}H_0 \pmod{\mathfrak{m}^2}.$$

All coefficients are positive. Their sum, together with $\sum_i H_i = 0$, gives $H_0 \in \mathfrak{m}^2$, then $\mathfrak{m} = \mathfrak{m}^2$. Nilpotence forces $\mathfrak{m} = 0$. Therefore

$$H_i = 0 \quad (0 \leq i < N), \quad R_j = 0 \quad (1 \leq j < N). \quad (\text{F.3})$$

Derivation of the coefficients. For a local self-intertwiner with highest weight h in all three positions, the first evaluated coefficients at left and right Zhu weights zero are

$$f_0 = 1, \quad f_1 = -\frac{h}{2}, \quad f_2 = \frac{h(h+1)(h^2+3h-2)}{4(h+2)(2h-1)}.$$

The central charge is $c = (-3h^2 + 7h - 2)/(h + 1)$. The level-two Ward solution is

$$\frac{h(h+1)}{(h+2)(2h-1)}L(-2)v + \frac{(h-2)(h+1)}{4(h+2)(2h-1)}L(-1)^2v.$$

For a right-input derivative, $f_n^D = (h-n)f_n - (h+n-1)f_{n-1}$, with $f_0^D = h$. Thus $[z^2](1+z)^2F_aF_b = \kappa$ and $[z^2](1+z)F_a^DF_b = 2\eta$. For the vacuum channel the linear coefficients are $g_2 = 2h/c$, $g_3 = -2h/c$, and $g_4 = 3\alpha + 2\beta$, where

$$\alpha = \frac{3h(c-2h+4)}{c(5c+22)}, \quad \beta = \frac{2h(5h+1)}{c(5c+22)}.$$

Thus $g_4 - g_2 = u(h)$. Differentiation gives the vacuum linear pair $2(-bu(a), au(b))$. These rational identities use levels at most four and follow by substituting the displayed Ward coefficients and collecting the indicated powers of z .

F.2 All ordinary intervals vanish

Let $P = P_{[j,k]}$, $1 \leq j \leq k < N$, be an ordinary interval primary of weight two. Its endpoint weights are

$$a = 1 + \frac{2}{t+k}, \quad b = 1 - \frac{2}{t+j}.$$

By (F.3) and (F.1),

$$\frac{1}{a}[P, D_aP] - \frac{1}{b}[P, D_bP] = 0. \quad (\text{F.4})$$

We induct on the interval length. Every output has labels in $\{0, 1, 2\}$, supported in $[j, k]$, and lowest weight at most four. Strict-support outputs and their descendants vanish by induction and the maximum-plateau strong spanning of the smaller-support open-chain subalgebras. A full-support output is either the binary interval itself, or has one interior height-two plateau and unchanged endpoint labels one. Indeed, the weight is the sum of squared adjacent label differences: endpoint height two costs at least six, and each separated interior height-two plateau adds two over binary weight two.

The extra weight-four channels occur only as highest vectors in P_0D_iP . Their coefficients in the two commutators are respectively a and b times the same canonical channel coefficient, so they cancel in (F.4). For the self-channel, if F_2 is the degree-two coefficient in the product of all local self series, then the coefficient in $[P, D_aP]$, divided by a , is

$$F_2 - 2g(a) - \frac{1}{2}, \quad g(h) = \frac{f_2(h)}{h} - \frac{h}{8} = \frac{5h^2 + 4h - 4}{8(h+2)(2h-1)}.$$

All interior data cancel between endpoints. Thus $2\sigma_P(g(b) - g(a))P = 0$, where $\sigma_P \neq 0$ is the self-channel coefficient. The exact factorization is

$$g(a) - g(b) = \frac{(a-b)(7ab - 2a - 2b + 4)}{8(a+2)(2a-1)(b+2)(2b-1)}.$$

The denominators and $a - b$ are nonzero. The remaining numerator, after multiplying by $(t+j)(t+k)$ and using $t^2 + Nt - 1 = 0$, is

$$7(j+k-N)t + 7jk - 10(k-j) - 21.$$

If the expression vanished, irrationality of t would force both its coefficient of t and its constant term to vanish. For $j = 1$ the constant term is $-3k - 11 < 0$. For $j \geq 2$, using $k \geq j$, the constant term is $k(7j - 10) + 10j - 21 > 0$ (already $4k - 1 > 0$ when $j = 2$). It never vanishes. Hence every ordinary interval acts by zero. The full ordinary open-chain Zhu image therefore acts by its vacuum character on the entire top M .

G Completion of the weight-zero Zhu calculation

Retain the hypotheses and conclusion of Appendix F. We now show that the proper heavy intervals also act trivially. Only the full-cycle class C remains; its quadratic relation will determine the quotient governing lowest weight zero.

G.1 The known ground character and maximal heavy intervals

Lemma 14.1 constructs a simple ground character T_0 on which every proper interval vanishes and C acts by $2\lambda \neq 0$. The argument uses only the quotient A/W , charge support and the invariant pairing; semisimplicity of the lattice restriction is not required.

Fix $1 \leq j < N$ and put $q = t + j$, $Q = R_j$, $B_j = v_{\mathbf{c}-e_j}$, $K_j = v_{\mathbf{c}+e_j}$, where $\mathbf{c} = (1, \dots, 1)$ labels C . The exact aligned channels of $Q \times C$ are C, B_j, K_j , with weights $2N, 2N + 2, 2N + 2$. Since $Q = 0$ and $H_i = 0$, the identities $Q * C = 0$ and $[Q, D_j C] = 0$ give

$$\begin{aligned} 0 &= \mu_- B_j + \mu_+ K_j + \alpha C, \\ 0 &= \mu_- d_- B_j + \mu_+ d_+ K_j + \beta C. \end{aligned}$$

Here $\mu_{\pm} \neq 0$ are canonical highest-channel coefficients, and all partial descendants of C have reduced to scalar multiples of C . Right differentiation gives $d_{\pm} = h_j(Q) + h_j(C) - h_j(B_j \text{ or } K_j)$. The two output weights are

$$h_j(B_j) = \frac{q-1}{q+1}, \quad h_j(K_j) = \frac{(q+2)(q+3)}{q(q+1)}, \quad d_- - d_+ = \frac{6}{q} \neq 0.$$

Eliminating K_j expresses B_j as a fixed scalar multiple of C , universally on these tops. Evaluate this identity on T_0 : $B_j = 0$ and $C = 2\lambda \neq 0$, so the scalar is zero. Thus every maximal proper heavy interval vanishes. This argument does not assume that K_j vanishes on T_0 .

G.2 Descending through all proper heavy intervals

Let B be any nonmaximal proper heavy interval. Extend its right endpoint by an ordinary root j to a proper heavy interval B' . Suppose by descending induction that $B' = 0$. The three outputs of $R_j \times B'$ replace its endpoint label by $0, 1, 2$. The first two are B, B' , of weight $H = 2N + 2$; the third has weight $H + 4$. It cannot occur in $R_j * B'$, whose states have weight at most $H + 2$. After partial Zhu reduction,

$$0 = R_j * B' = \kappa_- \mu_- B + \kappa_0 \mu_0 B' = \kappa_- \mu_- B, \quad \kappa_- = -\frac{q+2}{(q+3)(3q+2)} \neq 0, \quad q = t + j > 0.$$

Therefore $B = 0$.

The scalar requires only level two. At site j the two inputs have weight $1 + 2/q$ and the output is the vacuum, whose evaluated series is 1. At site $j - 1$ the first, second, and output weights are

$$p = \frac{q-2}{q}, \quad x = \frac{2}{q(q-1)}, \quad w = \frac{q+1}{q-1}.$$

The evaluated local series is

$$1 - z + \frac{3q^2 + 10q + 4}{(q + 3)(3q + 2)}z^2 + O(z^3).$$

Indeed, writing the second descendant as $\alpha L(-2)v + \beta L(-1)^2v$, the Ward equations are

$$\begin{aligned} 3\alpha + 2(2w + 1)\beta &= \frac{(w + p - x + 1)(w + p - x)}{2w}, \\ (4w + c/2)\alpha + 6w\beta &= w + 2p - x, \quad c = 13 - 6\left(\frac{q}{q-1} + \frac{q-1}{q}\right). \end{aligned}$$

Its Zhu evaluation is $w\alpha + w(w+1)\beta$. Multiplication of the series by $(1+z)^2$ and extraction of its degree-two coefficient gives κ_- . All other first inputs are the tensor unit. This proves that every proper heavy interval vanishes.

G.3 The universal cyclic quadratic and its consequences

The full Zhu algebra is generated by the partial frame classes, all proper interval highest classes, and C . The preceding results leave only C . The exact lowest-weight formula

$$h(m) = \sum_i (m_i - m_{i+1})^2 + Nm_0(m_0 + 1)$$

shows that a frame summand with $m_0 \geq 2$ starts at weight at least $6N$. Every state in $C * C$ has weight at most $4N < 6N$, so $C * C$ lies in the root-zero filtration piece F_1W . The filtered maximum-plateau Zhu spanning lemma gives

$$[F_1W] \subseteq A_0 + \sum_{P \text{ proper heavy or } P=C} A_0[P]A_0,$$

where A_0 is the ordinary open-chain image. Its vacuum action and the heavy annihilation show that $C^2 = \gamma \mathbf{1} + \delta C$ with fixed scalars. The vacuum character gives $\gamma = 0$, and T_0 gives $\delta = 2\lambda$. Consequently

$$C(C - 2\lambda) = 0, \quad A(W_N)/\langle H_0, \dots, H_{N-1} \rangle \cong \frac{\mathbb{C}[X]}{X(X - 2\lambda)} \cong \mathbb{C} \oplus \mathbb{C}.$$

The vacuum and T_0 supply both distinct characters. Thus the surjection from $\mathbb{C}[X]/(X(X - 2\lambda))$ to the quotient is an isomorphism.

Every relevant finite top with generalized partial spectrum zero and exact total conformal operator zero first reduces to this quotient by Appendix F. It is therefore a direct sum of the known vacuum and T characters. The Zhu correspondence excludes any additional neutral aligned simple of lowest weight zero. In a vacuum extension by a distinct such simple, the top extension splits over this semisimple quotient; the previously proved injection of vacuum extensions into Zhu-top extensions then splits the full module extension. Vacuum self-extensions split by Proposition 12.4.

Thus every vacuum extension by a simple target of lowest weight zero splits.

H The heavy-root vacuum coefficient excludes a weight-one defect

This calculation uses the closing-root first-image annihilation from Theorem 13.2. Suppose a nonsplit vacuum extension has a simple target of lowest weight one. Lemma 12.7 gives nonzero vectors w_j with

$$U = \text{span}\{w_0, \dots, w_{N-1}\}, \quad H_i w_j = \delta_{ij} w_j.$$

Theorem 13.2 gives $o(R_0)U = 0$ for the closing root in every odd rank, including $N = 3$. Its total and active local weights are

$$H = 2N + 2, \quad a = 1 + \frac{2}{t}, \quad b = 1 - 2t, \quad a + b = H, \quad t^2 + Nt - 1 = 0.$$

Every partial Virasoro descendant of R_0 also annihilates U : iterate

$$[L_i(-n)v] = (-1)^n(-H_i * [v] + n[v] * H_i + h_i[v])$$

and use $H_i U \subset U$. The aligned channels of $R_0 \times R_0$ are the vacuum, R_0 , and $2e_0$. The last has lowest weight $6N + 8$, above the maximum weight $2H = 4N + 4$ in $R_0 * R_0$. Thus its self-channel annihilates U , and it remains to evaluate the vacuum channel on w_0 .

H.1 The local evaluated vacuum block

Let h be an irrational local $(3, 1)$ or $(1, 3)$ weight, with

$$c = \frac{-3h^2 + 7h - 2}{h + 1}.$$

Its highest vector has the null relation

$$(L(-1)^3 - 2(h + 1)L(-2)L(-1) + h(h + 1)L(-3))v_h = 0. \quad (\text{H.1})$$

Normalize the vacuum intertwiner as

$$I(v_h, z)v_h = z^{-2h} \sum_{n \geq 0} v_n z^n, \quad v_0 = \mathbf{1}.$$

Evaluate these vacuum descendants in the highest-weight-one Virasoro module, with normalized highest vector and dual ξ, ξ' , and put

$$V_h(z) = \sum_{n \geq 0} \langle \xi', o(v_n)\xi \rangle z^n, \quad F(x) = (x - 1)^{-2h} V_h(x - 1).$$

This is the formal four-point block with weights $1, h, h, 1$ at $\infty, x, 1, 0$, in the vacuum channel between the two h insertions. It is defined by the formal vacuum intertwiner and the ordinary Virasoro module action, and has normalization $V_h(0) = 1$.

Set

$$\begin{aligned} A(x) &= \frac{x - 1}{x} - \frac{x}{x - 1}, \\ B(x) &= \frac{1 + 2h}{x} - \frac{1 + 2h}{x - 1} + \frac{h}{(x - 1)^2} + \frac{1}{x^2}, \\ C(x) &= -\frac{x - 1}{x^2} + \frac{x}{(x - 1)^2}, \\ D(x) &= -\frac{2h}{x^2} + \frac{2h}{(x - 1)^2} - \frac{2h}{(x - 1)^3} - \frac{2}{x^3}. \end{aligned}$$

The null relation gives

$$F''' - 2(h + 1)AF'' + (-2(h + 1)B + h(h + 1)C)F' + h(h + 1)DF = 0. \quad (\text{H.2})$$

For a direct Ward verification, allow the other insertions to vary at y, s . At $y = 1, s = 0$, translation and scaling give

$$\partial_y F = -xF' - 2hF, \quad \partial_s F = (x - 1)F' + 2hF.$$

The $L(-2)$ insertion is $AF' + B_0F$, with

$$B_0 = \frac{1}{x^2} + \frac{h}{(x-1)^2} + \frac{2h}{x} - \frac{2h}{x-1}.$$

Differentiating the global identities for the first descendant gives $AF'' + BF'$ for $L(-2)L(-1)$. Since $L(-3) = [L(-1), L(-2)]$, its insertion is $CF' + DF$, with $C = A' + B_0 - B$ and $D = B'_0$. This proves (H.2), including the outer-weight-one terms.

H.2 Uniform solution of the vacuum channel

Put $u = (x-1)^2/x$, $\theta = u d/du$, and substitute $F = (x-1)^{-2h}G(u)$. After removing $(x-1)^{-2h}$, the left side of (H.2) equals exactly

$$\frac{4(x+1)}{x^2(x-1)^3} \left[\theta \left(\theta - \frac{h}{2} \right) \left(\theta - \frac{1+3h}{2} \right) + \frac{u}{4} (\theta - h - 1)(\theta - h)(\theta + 2) \right] G(u). \quad (\text{H.3})$$

One can verify the identity directly by expanding the bracket as an operator in d/du . Its coefficients on G''' , G'' , G' , G , respectively, are

$$u^3(1+u/4), \quad \frac{u^2}{2}(5-4h+(2-h)u), \quad \frac{u}{4}(3h^2-7h+2+h(h-5)u), \quad \frac{u}{2}h(h+1).$$

The chain rule with $u' = (x^2-1)/x^2$ gives exactly the four coefficients of the left-hand side after its common factor has been removed. Construct the ordinary power-series solution $G(0) = 1$. Its recurrence has no zero denominator because h is irrational, and gives

$$G(u) = \sum_{n \geq 0} d_n u^n, \quad d_n = \left(-\frac{1}{4} \right)^n \frac{(-h-1)_n (-h)_n (2)_n}{(1-h/2)_n ((1-3h)/2)_n n!}, \quad (\text{H.4})$$

where $(x)_n = x(x+1) \cdots (x+n-1)$. The exponents of (H.2) at $x = 1$ are $-2h, -h, h+1$, with nonintegral pairwise differences because h is irrational. The normalized Frobenius solution with exponent $-2h$ is therefore unique. Thus the series just constructed is the vacuum block, and its dependence on u follows from the equation and normalization. Consequently

$$V_h(z) = {}_3F_2 \left(\begin{matrix} -h-1, -h, 2 \\ 1-h/2, (1-3h)/2 \end{matrix}; -\frac{z^2}{4(1+z)} \right).$$

For comparison with the low-level Ward equations, $d_1 = 2h/c$ and

$$d_2 = \frac{h^2(h+1)}{(h-4)(h-2)(3h-1)}$$

agree with the level-two and level-four vacuum Ward equations.

H.3 The coefficient cannot vanish

On w_0 the two active partial Zhu weights are 1, 0. At weight zero, every positive vacuum descendant evaluates to zero; the second local series is exactly 1. Therefore, after dividing out the nonzero root vacuum coefficient ν_0 , the vacuum contribution to $R_0 * R_0$ is

$$P_H(h) = [z^{2H}](1+z)^H V_h(z), \quad h = a = 1 + 2/t.$$

Using (H.4),

$$(1+z)^H V_h(z) = \sum_{n \geq 0} d_n z^{2n} (1+z)^{H-n}.$$

For $n < H$ the term is a polynomial of degree $H + n < 2H$; for $n > H$ its lowest degree is $2n > 2H$. Hence only $n = H$ contributes:

$$P_H(h) = \left(-\frac{1}{4}\right)^H \frac{(-h-1)_H(-h)_H(2)_H}{(1-h/2)_H((1-3h)/2)_H H!} \neq 0. \quad (\text{H.5})$$

Every possible zero or pole of these finite Pochhammer products would make h rational, whereas $1 + 2/t$ is irrational. The scalar ν_0 is nonzero by the nondegenerate invariant form on the multiplicity-one self-contragredient summand M_{e_0} .

Finally apply $R_0 * R_0$ to $w_0 \neq 0$. Its left side is zero because $R_0 U = 0$. Its self-channel descendants annihilate U , and the higher channel is excluded by weight. Its vacuum channel equals $\nu_0 P_H(a) w_0 \neq 0$, a contradiction. Thus no nonsplit vacuum extension with a lowest-weight-one simple target exists. Together with the weight-zero analysis and Proposition 12.4, this proves that every extension of the vacuum by a simple module splits.

I The uniform symbolic frame certificate

The screening-frame construction has one computer-assisted algebraic step. The charge bound in Section 5 reduces the Virasoro relations to proper intervals, charge B , and finitely many tails $B + I$. The proper-interval and tail equations are proved there. The program reproduced below checks the remaining charge- B identities with N , t , and the ordered site positions left indeterminate.

A coefficient involves at most two frame indices and three oscillator indices. The program enumerates their equality and order patterns; its interval-count formula includes the end sites. After the denominators prescribed by the frame formulas are cleared, every residual numerator expands to zero. The frame proof separately establishes that those denominators are nonzero at the parameters used in the theorem. Together with the mathematical reduction to these patterns, this makes the calculation uniform in N .

The program checks the following numbers of coefficient identities:

Distinct site indices	1	2	3	4	5
Coefficient identities	7	40	69	46	10

All 172 residual numerators vanish identically. The calculation uses exact symbolic arithmetic, with no numerical substitution for the rank or site positions. It checks these identities, not the surrounding representation-theoretic arguments.

To reproduce the calculation, save the listing as `certify_uniform_epsilon.py` and run it with Python 3 and SymPy. It prints a JSON report and writes the same report beside the script. The program requires no external mathematical data and makes no network requests. The listing is self-contained and requires no supplementary files.

I.1 Complete program

```

1  """Uniform symbolic verification of the charge-B Virasoro-frame equations.
2
3  N, t, and all distinct site positions are indeterminates. This enumerates
4  order/equality types of at most five site indices, NOT numerical ranks.
5  Interval sums use the exact counting formula in the frame construction.
6  """
7  from collections import Counter
8  from functools import lru_cache
9  from itertools import combinations_with_replacement, product
10 from pathlib import Path

```

```

11 import json
12 import time
13 import sympy as s
14 # Polynomial operations for oscillator monomials.
15 def poly_add(*args):
16     out = {}
17     for p in args:
18         for k,v in p.items():
19             out[k] = out.get(k,0)+v
20     return {k:v for k,v in out.items() if v != 0}
21
22 def scale(p,c):
23     return {k:c*v for k,v in p.items() if c*v != 0}
24
25 def mul(p,q):
26     out={}
27     for a,x in p.items():
28         for b,y in q.items():
29             k=tuple(u+v for u,v in zip(a,b))
30             out[k]=out.get(k,0)+x*y
31     return {k:v for k,v in out.items() if v != 0}
32
33
34 N,t=s.symbols('N t')
35
36
37 def ba(*args):
38     ans={}
39     for p in args:
40         for key,c in p.items():ans[key]=ans.get(key,0)+c
41     return {key:c for key,c in ans.items() if c != 0}
42
43 def bm(p,q):
44     ans={}
45     for (M,H),a in p.items():
46         for (M2,H2),b in q.items():
47             key=(M|M2,H|H2);ans[key]=ans.get(key,0)+a*b
48     return {key:c for key,c in ans.items() if c != 0}
49
50 def bs(p,c):return {key:v*c for key,v in p.items() if v*c != 0}
51
52 BC={(frozenset(),frozenset()):s.S.One}
53
54
55 def setup(m):
56     positions=s.symbols(f'j0:{m}')
57     r=[t+j for j in positions];p=[1/(x+1) for x in r]
58     psi=[(x+1)/x for x in r];R=(t+N)/t
59     def bv(kind,i,offset=0):
60         point=(i,offset)
61         return {(frozenset([point]) if kind=='M' else frozenset(),
62                 frozenset([point]) if kind=='H' else frozenset()):s.S.One}
63     mom=[];u=[];v=[]
64     for i in range(m):
65         mi,mnext=bv('M',i),bv('M',i,1)
66         hi,hnext=bv('H',i),bv('H',i,1)
67         mom.append(ba(mi,bs(mnext,-r[i]/(r[i]+1))))
68         u.append(ba(bs(mi,1/r[i]),bs(mnext,-1/(r[i]+1))))
69         gi=ba(BC,bs(mi,-1),bs(hi,R-1))
70         gn=ba(BC,bs(mnext,-1),bs(hnext,R-1))
71         v.append(ba(bs(gi,1/r[i]),bs(gn,-1/(r[i]+1))))
72     comp=[ba(bs(BC,p[i]),bs(mom[i],-1)) for i in range(m)]
73     def val(point):return positions[point[0]]+point[1]
74     def interval_sum(pol):
75         terms=[]
76         for (M,H),c in pol.items():
77             upper=min(H) if H else (m,0)
78             uperval=val(upper) if H else N
79             if M:
80                 lo,hi=min(M),max(M)
81                 count=0 if upper<=hi else val(lo)*(uperval-val(hi))
82             else:count=uperval*(uperval-1)/2
83             if count!=0:terms.append(c*count)

```

```

84     return s.Add(*terms)
85 @lru_cache(None)
86 def source(i,j,indices):
87     left=bm(u[i],v[j]);right=bm(v[i],u[j])
88     for k in indices:
89         left=bm(left,mom[k]);right=bm(right,comp[k])
90     return interval_sum(ba(left,right))
91
92 variables=3*m
93 one={0,)*variables:s.S.One}
94 def atom(i,k):
95     key=[0]*variables;key[3*i+k-1]=1;return {tuple(key):s.S.One}
96 atoms={i,k):atom(i,k) for i in range(m) for k in (1,2,3)}
97 def D(pol,i):
98     ans=mul(scale(atoms[i,1],p[i]),pol)
99     for key,c in pol.items():
100         for k in (1,2):
101             power=key[3*i+k-1]
102             if power:
103                 kk=list(key);kk[3*i+k-1]-=1;kk[3*i+k]+=1
104                 ans=poly_add(ans,{tuple(kk):k*power*c})
105     return ans
106 def Dt(pol):return poly_add(*(D(pol,i) for i in range(m)))
107 def action(pol,i,mode):
108     if mode==-1:return D(pol,i)
109     if mode==0:return {key:c*sum(k*key[3*i+k-1] for k in (1,2,3)) for key,c in pol.items()}
110     ans={}
111     for key,c in pol.items():
112         if mode==1 and key[3*i+1]:
113             kk=list(key);kk[3*i+1]-=1;kk[3*i]+=1
114             ans=poly_add(ans,{tuple(kk):2*c*key[3*i+1]})
115         if mode==2:
116             if key[3*i]==2:
117                 kk=list(key);kk[3*i]-=2
118                 ans=poly_add(ans,{tuple(kk):2*psi[i]*c})
119             if key[3*i+1]:
120                 kk=list(key);kk[3*i+1]-=1
121                 ans=poly_add(ans,{tuple(kk):-2*psi[i]*p[i]*c})
122     return ans
123 corrections=[]
124 for i in range(m):
125     pol=scale(atoms[i,2],(1+R)*p[i]/2)
126     for j in range(m):
127         diag=(R*p[i]-r[i]*p[i]**2)/2 if j==i else -(R if j<i else 1)*r[j]*p[j]/(2*r[i]*(r[i]+1))
128         pol=poly_add(pol,scale(mul(atoms[j,1],atoms[j,1]),diag))
129         if j!=i:
130             cross=(R if j<i else 1)*p[i]*p[j]
131             pol=poly_add(pol,scale(mul(atoms[i,1],atoms[j,1]),cross))
132     corrections.append(pol)
133 def tag(pol,k):return {(k,)+key:c for key,c in pol.items()}
134 @lru_cache(None)
135 def linear(i,j):
136     ans={}
137     for k in range(4):
138         ans=poly_add(ans,tag(scale(action(corrections[j],i,k-1),s.S.One/s.factorial(k)),k))
139     base=action(corrections[i],j,k-1)
140     for zpower in range(k,-1,-1):
141         if zpower!=k:base=Dt(base)
142         coefficient=(-1)**k*s.binomial(k,zpower)/s.factorial(k)
143         ans=poly_add(ans,tag(scale(base,coefficient),zpower))
144     if i==j:
145         ans=poly_add(ans,tag(scale(Dt(corrections[i]),-1),0),tag(scale(corrections[i],-2),1))
146     return ans
147 return source,linear,variables
148
149
150 def cases(m):
151     labels=range(m)
152     # (lambda power, oscillator list, exponential-series coefficient)
153     shapes=[(3,(),s.Rational(1,6))]
154     for k in labels:
155         shapes.extend([(2,((k,1),),s.Rational(1,2)),
156                        (1,((k,2),),s.Rational(1,2)),

```

```

157         (0,((k,3),),s.Rational(1,3))])
158     for k,l in combinations_with_replacement(labels,2):
159         shapes.append((1,((k,1),(1,1)),s.Rational(1,2) if k==1 else s.S.One))
160     for k,l in product(labels,repeat=2):
161         shapes.append((0,((k,2),(1,1)),s.Rational(1,2)))
162     for inds in combinations_with_replacement(labels,3):
163         denominator=s.prod(s.factorial(c) for c in Counter(inds).values())
164         shapes.append((0,tuple((k,1) for k in inds),1/denominator))
165     for i in labels:
166         for j in range(i,m):
167             for zpower,osc,factor in shapes:
168                 if {i,j}|{k for k,_ in osc} != set(labels):continue
169                 yield i,j,zpower,osc,factor
170
171
172 def verify(max_sites=5):
173     results=[];start=time.time()
174     for m in range(1,max_sites+1):
175         source,linear,variables=setup(m)
176         checked=0
177         for i,j,zpower,osc,factor in cases(m):
178             key=[0]*variables
179             for k,degree in osc:key[3*k+degree-1]+=1
180             lhs=linear(i,j).get((zpower,)+tuple(key),0)
181             rhs=factor*source(i,j,tuple(sorted(k for k,_ in osc)))
182             residual=s.expand(s.together(lhs+rhs).as_numer_denom()[0])
183             if residual!=0:
184                 raise AssertionError({'sites':m,'i':i,'j':j,'lambda_power':zpower,
185                                     'oscillators':osc,'residual':str(s.factor(residual))})
186             checked+=1
187             results.append({'distinct_site_indices':m,'coefficient_order_types':checked})
188             print(json.dumps({'stage':'PASS','distinct_sites':m,'identities':checked,'seconds':round(time.time
189 (-start,2))},flush=True)
189 return {'status':'UNIFORM_SYMBOLIC_PASS','parameters':'Independent N,t,j0<...<jk; no numerical rank
190 substitution',
191         'interval_moment_formula':'min(M)*max(0,min(H or N)-max(M)); for empty M, U*(U-1)/2',
192         'all_order_types':results,'total_identities':sum(x['coefficient_order_types'] for x in results)
193 }
194
195 if __name__=='__main__':
196     result=verify()
197     Path(__file__).with_suffix('.json').write_text(json.dumps(result,indent=2)+'\n')
198     print(json.dumps(result,indent=2))

```

References

- [1] M. Izumi, *The structure of sectors associated with Longo–Rehren inclusions. II: Examples*, Rev. Math. Phys. 13 (2001), no. 5, 603–674. <https://doi.org/10.1142/S0129055X01000818>.
- [2] D. E. Evans and T. Gannon, *The exoticness and realisability of twisted Haagerup–Izumi modular data*, Commun. Math. Phys. 307 (2011), 463–512. <https://arxiv.org/abs/1006.1326>.
- [3] D. E. Evans and T. Gannon, *Non-unitary fusion categories and their doubles via endomorphisms*, Adv. Math. 310 (2017), 1–43. <https://arxiv.org/abs/1506.03546>.
- [4] T.-C. Huang, *Cyclic Haagerup–Izumi fusion categories at every odd order*, 2026. <https://arxiv.org/abs/2609.15986>.
- [5] T. Gannon, A. Schopieray and H. Yadav, *On Haagerup–Izumi fusion categories*, 2026. <https://arxiv.org/abs/2609.25185v1>. Theorem 5.1 and Appendix F.1 specify the center modular data; Appendix F, Table 2 gives the negative-branch comparison labels. Section 6 gives the forward HI-to-near-group construction and states the additional central data needed for an inverse. Corollary 5.9 gives the order-fifteen nonuniqueness example.
- [6] S. Koshida, *Module categories of the generic Virasoro VOA and quantum groups*, SIGMA 21 (2025), 039; <https://sigma-journal.com/2025/039/sigma25-039.pdf>.

- [7] T. Creutzig, C. Jiang, F. Orosz Hunziker, D. Ridout and J. Yang, *Tensor categories arising from the Virasoro algebra*, Adv. Math. 380 (2021), 107601. <https://arxiv.org/abs/2002.03180>. Numbering follows arXiv version 2.
- [8] T. Creutzig, S. Kanade and R. McRae, *Gluing vertex algebras*, Adv. Math. 396 (2022), 108174; [arXiv:1906.00119](https://arxiv.org/abs/1906.00119). Numbering follows arXiv version 2.
- [9] C. Dong, H. Li and G. Mason, *Regularity of rational vertex operator algebras*, Adv. Math. 132 (1997), 148–166; [arXiv:q-alg/9508018](https://arxiv.org/abs/q-alg/9508018), Theorem 3.16.
- [10] H. Li, *Symmetric invariant bilinear forms on vertex operator algebras*, J. Pure Appl. Algebra 96 (1994), 279–297.
- [11] A. Tsuchiya and Y. Kanie, *Fock space representations of the Virasoro algebra—intertwining operators*, Publ. Res. Inst. Math. Sci. 22 (1986), 259–327; [doi:10.2977/prims/1195178069](https://doi.org/10.2977/prims/1195178069). Proposition 2.7 and its proof in Section 2.5; [open publisher PDF](#).
- [12] R. McRae, *Deligne tensor products of categories of modules for vertex operator algebras*, [arXiv:2304.14023](https://arxiv.org/abs/2304.14023).
- [13] T. Creutzig, R. McRae and J. Yang, *Direct limit completions of vertex tensor categories*, Commun. Contemp. Math. 24 (2022), no. 2, 2150033. <https://arxiv.org/abs/2006.09711>. We use Theorems 7.5 and 7.7 of arXiv version 2.
- [14] S. Morrison, E. Peters and N. Snyder, *Categories generated by a trivalent vertex*, Selecta Math. (N.S.) 23 (2017), no. 2, 817–868; [arXiv:1501.06869](https://arxiv.org/abs/1501.06869).
- [15] Y.-Z. Huang, *Cofiniteness conditions, projective covers and the logarithmic tensor product theory*, J. Pure Appl. Algebra 213 (2009), 458–475; [arXiv:0712.4109](https://arxiv.org/abs/0712.4109).
- [16] Y. Moriwaki, *Quantum coordinate ring in WZW model and affine vertex algebra extensions*, Selecta Math. 28 (2022), 68; [arXiv:2111.11357](https://arxiv.org/abs/2111.11357).
- [17] M. Miyamoto, *C_1 -cofiniteness and fusion products for vertex operator algebras*, in *Conformal Field Theories and Tensor Categories*, Math. Lect. Peking Univ., Springer, Heidelberg, 2014, 271–279; [arXiv:1305.3008](https://arxiv.org/abs/1305.3008).
- [18] Y.-Z. Huang and J. Yang, *Logarithmic intertwining operators and associative algebras*, J. Pure Appl. Algebra 216 (2012), no. 6, 1467–1492. We use the corrected arXiv version 3, Lemma 5.5 and Propositions 5.6–5.8. <https://arxiv.org/abs/1104.4679>. See also the authors’ erratum: <https://sites.math.rutgers.edu/~yzhuang/rci/math/papers/HY-erratum.pdf>.
- [19] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. 9 (1996), no. 1, 237–302. <https://doi.org/10.1090/S0894-0347-96-00182-8>. Theorem 2.2.2 gives the correspondence between simple modules and simple modules for the Zhu algebra.
- [20] R. Allen, S. Lentner, C. Schweigert, and S. Wood, *Duality structures for representation categories of vertex operator algebras and the Feigin–Fuchs boson*, Selecta Math. (N.S.) 31 (2025), article 36. Numbering follows arXiv version 2. <https://arxiv.org/abs/2107.05718>.
- [21] P. Etingof and D. Penneys, *Rigidity of non-negligible objects of moderate growth in braided categories*, Forum Math. Pi 14 (2026), e7. Theorem numbering follows arXiv:2412.17681v3. <https://arxiv.org/abs/2412.17681>.
- [22] R. McRae, *On rationality for C_2 -cofinite vertex operator algebras*, Camb. J. Math. 14 (2026), no. 1, 1–115. <https://arxiv.org/abs/2108.01898>. Numbering follows arXiv version 3.
- [23] T. Abe, G. Buhl, and C. Dong, *Rationality, regularity, and C_2 -cofiniteness*, Trans. Amer. Math. Soc. 356 (2004), no. 8, 3391–3402. <https://arxiv.org/abs/math/0204021>. Proposition 5.2 gives C_2 -cofiniteness of irreducible weak modules; its CFT-type hypothesis is applied to V^{un} , not to E_N .
- [24] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik, *On braided fusion categories I*, Selecta Math. 16 (2010), 1–119. <https://arxiv.org/abs/0906.0620>. We use Theorem 4.18, Proposition 4.19 and Remark 4.21.

- [25] K. Mimachi, *The irreducibility of the monodromy representation associated with the Dotsenko–Fateev equation*, Commun. Math. Phys. 405 (2024), article 71, [doi:10.1007/s00220-024-04942-7](https://doi.org/10.1007/s00220-024-04942-7), especially equation (1.1) and Theorem 11.
- [26] M. Krauel and M. Miyamoto, *A modular invariance property of multivariable trace functions for regular vertex operator algebras*, J. Algebra 444 (2015), 124–142. <https://arxiv.org/abs/1501.00124>. Lemmas 4–5 and equations (2.2), (2.5) fix the normalization of the formal Ward recursion used here.
- [27] A. Kirillov, Jr., and V. Ostrik, *On a q -analog of the McKay correspondence and the ADE classification of $\widehat{\mathfrak{sl}}_2$ conformal field theories*, Adv. Math. 171 (2002), 183–227. <https://arxiv.org/abs/math/0101219>. Numbering follows arXiv version 3: Definition 1.11, Theorems 1.17–1.18, Remark 1.19 and Lemma 1.20.
- [28] T. Creutzig, S. Kanade and R. McRae, *Tensor categories for vertex operator superalgebra extensions*, Mem. Amer. Math. Soc. 295 (2024), no. 1472. <https://arxiv.org/abs/1705.05017>. We use Lemma 2.61 and Theorem 3.65 of arXiv version 2.
- [29] T. Creutzig, R. McRae, K. Shimizu, and H. Yadav, *Commutative algebras in Grothendieck–Verdier categories, rigidity, and vertex operator algebras*, Commun. Contemp. Math. 28 (2026), no. 10, 2550091. Numbering follows arXiv version 2 (18 October 2025). <https://arxiv.org/abs/2409.14618>.
- [30] A. Davydov, M. Müger, D. Nikshych and V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. Reine Angew. Math. 677 (2013), 135–177. <https://arxiv.org/abs/1009.2117>. We use Section 3.3, Section 3.6, Proposition 3.16, Corollary 3.30, Remark 3.31 and Corollary 4.1.
- [31] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, Mathematical Surveys and Monographs 205, American Mathematical Society, 2015. <https://math.mit.edu/~etingof/egnobookfinal.pdf>. We use Proposition 9.2.2.
- [32] C. Dong and G. Mason, *Shifted vertex operator algebras*, Math. Proc. Cambridge Philos. Soc. 141 (2006), 67–80. <https://arxiv.org/abs/math/0411526>. We use Theorems 3.1, 4.1 and 4.4.
- [33] D. E. Evans, *A vertex operator algebra for the S -fold realisation of a non-unitary Haagerup–Izumi modular fusion category*, 2026. <https://arxiv.org/abs/2610.26341>. Section 6.2 proposes the cyclic Virasoro sum used here.
- [34] T. Creutzig and D. Gaiotto, *Vertex algebras for S -duality*, Commun. Math. Phys. 379 (2020), 785–845. <https://arxiv.org/abs/1708.00875>.
- [35] M. Flohr, *W -algebras, new rational models and completeness of the $c = 1$ classification*, Commun. Math. Phys. 157 (1993), 179–212. <https://arxiv.org/abs/hep-th/9207019>.
- [36] M. Flohr, *More curiosities at effective $c = 1$* , Mod. Phys. Lett. A 9 (1994), 1071–1082. <https://arxiv.org/abs/hep-th/9312097>.
- [37] D. Gang and D. Kim, *Generalized non-unitary Haagerup–Izumi modular data from 3D S -fold SCFTs*, JHEP 03 (2023), 185. <https://arxiv.org/abs/2211.13561>.
- [38] D. Gang, D. Kim and S. Lee, *A non-unitary bulk-boundary correspondence: Non-unitary Haagerup RCFTs from S -fold SCFTs*, SciPost Phys. 17 (2024), 064. <https://arxiv.org/abs/2310.14877>.
- [39] K. Jeong and H. Sohn, *Non-unitary Haagerup-like TQFTs and RCFTs from generalized S -fold SCFTs*, 2026. <https://arxiv.org/abs/2608.11946>.
- [40] D. E. Evans and T. Gannon, *Near-group fusion categories and their doubles*, Adv. Math. 255 (2014), 586–640. <https://arxiv.org/abs/1208.1500>.
- [41] A. Riesen, *The Haagerup VOA at central charge eight*, announcement, 2026. <https://arxiv.org/abs/2610.13933>.

- [42] D. E. Evans, *Haagerup–Izumi modular data from hyperbolic words: S-fold TQFTs, central charge, torsion and the Bethe vacua*, 2026. <https://arxiv.org/abs/2610.18512>.
- [43] P. Grossman and M. Izumi, *Drinfeld centers of fusion categories arising from generalized Haagerup subfactors*, Quantum Topol. 13 (2022), 593–668. <https://arxiv.org/abs/1501.07679>.
- [44] P. Grossman and N. Snyder, *Quantum subgroups of the Haagerup fusion categories*, Commun. Math. Phys. 311 (2012), 617–643. <https://arxiv.org/abs/1102.2631>.