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The continuum limit of three-dimensional SU(2) lattice Yang–Mills theory on a torus I: Foundations and classical fields

P. M. Aronow

Patrick Lopatto



WARNING!

We have extensively audited this manuscript using large language models, including adversarial reviews by multiple models. However, the proof is too complex for Lean formalization at the present time, and it has not yet undergone verification by a human expert.

Image: detail from Abraham Ortelius, *Theatrum orbis terrarum* (1570).
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AUTHORS' NOTE

With the exception of this note, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited.

The manuscript began as an attempt to pursue the program of Magnen, Rivasseau, and Sénéor [132] using large language models. Our hope was that understanding the problem on the torus might offer some insight into the general case. The construction that resulted from this starting point is three-dimensional and lattice-based and draws extensively on Bałaban's renormalization group methods.

As we were verifying these results, we learned that Alpöge and Sussman announced that they appear to have a YM4 existence argument, following Bałaban's program (`https://github.com/ethanwsussman/Yang--Mills-LLM-experiment`). We are thrilled and look forward to learning more about these exciting developments. We hope that this manuscript nonetheless contributes to understanding constructive approaches to three-dimensional Yang–Mills theory. The first version of these results was posted to the first author's GitHub on October 17, 2026 (`https://github.com/pmaronow/3d-ym-torus`). Prior to posting the first version of this work, we did not inspect the ongoing work of Alpöge and Sussman. Further, to the best of our knowledge, the LLMs we used did not access their work during the mathematical development and drafting of this manuscript. This version contains additional clarifications relative to the first version; it has also been partitioned into five volumes.

We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We consider the Wilson lattice gauge measure for SU(2) on a periodic three-dimensional torus of fixed size, with bare coupling $g^2 = g_3^2 a$ proportional to the lattice spacing a, as dictated by the superrenormalizability of three-dimensional Yang–Mills theory. Observables are Wilson loops of the lattice configuration evolved by the Wilson flow, the lattice gradient flow of the Wilson action, at positive flow time. We prove that, at every fixed value of the coupling, the joint laws of finitely many flowed Wilson loops converge as the lattice spacing tends to zero along the sequences of lattice sides nM^k, for fixed odd $M \geq 3$ and fixed positive n divisible by four. For each such refinement class, the limits are realized by a unique probability measure on the space of gauge orbits of Yang–Mills heat-flow trajectories at positive times, a version of the state space introduced by Cao and Chatterjee. At each fixed positive time, the energies of the selected trajectories $\pi_k(U)$, with U distributed under the lattice Wilson measures, are tight uniformly in the lattice spacing. The limiting measure is invariant under the lattice symmetries and the center symmetries. We also show that, at short distances, the rescaled deficits $(1 - W)/(g_3^2 \varepsilon)$ of the normalized fundamental Wilson loops W of loops $x + \varepsilon \ell$, flowed for time $\sigma \varepsilon^2$ with fixed $\sigma > 0$, converge to explicit quadratic functions of the heat-smoothed Gaussian free field. As the coupling tends to zero, the measure concentrates on flat connections, with Gaussian fluctuations of explicit covariance near regular commuting holonomies and fluctuations governed by the constant-mode quartic form near the central holonomies. The proof is a multiscale renormalization group analysis in the tradition of Bałaban, carried out on a small-field event whose complement we show to have small probability, uniformly in the lattice spacing, using reflection positivity and chessboard estimates. It is combined with stochastic estimates for the lattice heat flow and a deterministic analysis of the Wilson flow for rough initial data.

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PART I

Introduction

SECTION 1

Introduction

1.1 Yang–Mills theory and its lattice formulation

Yang and Mills introduced non-Abelian gauge theories in 1954 [156]. In the Euclidean formulation of the quantum theory, one would like to define a probability measure on connections A on a three- or four-dimensional manifold, formally given by

$$d\mu(A) = Z^{-1} \exp\Big(-\frac{1}{2g^2} \int |F_A|^2 \, dx\Big) \, \mathcal{D}A. \tag{1.1}$$

Here F_A is the curvature of A, g is the coupling constant, and $\mathcal{D}A$ stands for a Lebesgue measure on the infinite-dimensional space of connections, which does not exist. The measure should be invariant under gauge transformations, so it is really a measure on gauge orbits. The natural observables are gauge-invariant functions, such as traces of holonomies around closed loops, which are called Wilson loops.

Wilson's lattice regularization [153] replaces the connection by variables U_e in the gauge group, one for each link e of a lattice with mesh a. The action becomes the sum of $1 - \operatorname{tr} U_p/2$ over the plaquettes p of the lattice, where U_p is the product of the link variables around p. For every $a > 0$ this defines a probability measure that is exactly invariant under lattice gauge transformations. Osterwalder and Seiler established reflection positivity and other basic properties of these measures [134]; see also Seiler's monograph [141]. The central problem is to remove the regularization: one must show that suitable observables have limits as $a \to 0$, with the bare coupling tuned appropriately, and describe the limit. In four dimensions this is part of the problem described by Jaffe and Witten [114]; see [77] for a recent review by Douglas.

In this paper we consider the gauge group $\mathrm{SU}(2)$ in three dimensions, on a periodic torus $\mathbb{T}^3_L = (\mathbb{R}/L\mathbb{Z})^3$ of fixed side L. In three dimensions the coupling g_3^2 in (1.1) has the dimension of inverse length. The dimensionless strength of the interaction at length scale ε is $g_3^2\varepsilon$. In the sense of power counting, the theory becomes weakly coupled at short distances; it is superrenormalizable. The bare lattice coupling at mesh a is $g^2 = g_3^2 a$. With $N = L/a$ lattice sites along each side, this means that

$$g_N^2 = \frac{\gamma}{N}, \qquad \gamma = g_3^2 L, \tag{1.2}$$

where the dimensionless number γ is the coupling measured in units of the size of the torus.

Three-dimensional Yang–Mills theory is of independent physical interest. It describes the magnetic sector of four-dimensional gauge theory at high temperature and long distances; see Ginsparg [98] and Appelquist and Pisarski [5], and also Linde [126] and Jackiw and Templeton [113] on the infrared problems of its perturbative treatment. It is also expected to have a mass gap and to confine. Feynman gave qualitative arguments for both [88]. Karabali, Kim, and Nair developed a gauge-invariant Hamiltonian analysis of the gap and an approximate prediction of the string tension [115, 116], and Teper computed the spectrum and string tensions numerically, with continuum extrapolations [150]. Our results concern only the ultraviolet problem in a fixed finite volume.

Even in three dimensions, the ultraviolet problem is not only a question of bounds. In the Gaussian (Maxwell) approximation to (1.1), after gauge fixing and with the constant modes treated separately, a connection in three dimensions is a random distribution of Hölder–Besov regularity α for every $\alpha < -1/2$, and no better. Holonomies along general fixed loops then require an additional regularization [42, 43]. A convenient way to obtain gauge-invariant observables with limits is to smooth the field first. The gradient flow of the Yang–Mills action, $\partial_t A = -D_A^* F_A$, smooths connections at length scale $\sqrt{t}$ and commutes with gauge transformations. Its lattice version is the gradient flow of the Wilson action. Narayanan and Neuberger studied continuum limits of smeared Wilson loops numerically [133], and Lüscher developed this lattice flow systematically under the name Wilson flow [128, 129]. In [131], Lüscher and Weisz showed that in four dimensions the correlation functions of the flowed field are finite to all orders in renormalized perturbation theory. Wilson loops of the flowed field at positive flow time are bounded gauge-invariant functions of the lattice configuration, and we use them as our observables.

The geometric study of Yang–Mills connections and their flow has a long history, including work of Atiyah and Bott [6] and Donaldson [75, 76]. In three dimensions, global existence and regularity for smooth initial data are due to Råde [139]. Charalambous and Gross developed the flow for rough initial data and proposed it as a way to define gauge fields [47–49, 105]. In [43], Cao and Chatterjee made this proposal precise. They defined a state space for three-dimensional Yang–Mills theory, consisting of gauge orbits of trajectories of the Yang–Mills heat flow at positive times. They showed that it carries a natural topology in which Wilson loops of the flowed field are continuous, and they gave a tightness criterion for measures on it. Their arguments rest on Uhlenbeck's compactness theorem [151] and on Sengupta's result that holonomies separate gauge orbits [144]. In [42] they also proved local existence of the flow in the DeTurck gauge for a class of random distributional initial data that includes the Gaussian free field. Their decomposition into a rough linear part and a smoother nonlinear part follows the random-data method of Bourgain [32] and of Da Prato and Debussche [63]. The construction of a Yang–Mills measure on this state space from the lattice theory was left open. Our main result is such

a construction.

1.2 Previous rigorous results

We summarize the rigorous results most closely related to ours. The literature is large, and Chatterjee's survey [51] gives an overview from the point of view of probability.

Abelian theories, Higgs models, and two dimensions. In [104], Gross proved that three-dimensional pure $U(1)$ lattice gauge theory converges to the free electromagnetic field, for the Villain action and, in the electric sector, for the Wilson action. Brydges, Fröhlich, and Seiler constructed the Abelian Higgs model in two dimensions [36, 37, 40, 41]. King constructed the Abelian Higgs model in two and three dimensions by proving convergence of effective actions [117, 118]. Chatterjee constructed a Gaussian scaling limit of $\mathrm{SU}(2)$ lattice Yang–Mills theory coupled to a fixed-length Higgs field in the fundamental representation, in every dimension $d \geq 2$ [53]. The result uses unitary gauge, sends the Higgs length to infinity and the gauge coupling to zero rapidly, and yields independent massive Proca fields; it concerns a different scaling regime from the pure gauge theory considered here. In two dimensions, Yang–Mills theory is exactly solvable. Gross, King, and Sengupta [106] and Driver [78] gave probabilistic constructions in the plane, and Driver proved that the lattice Wilson loop expectations converge. Sengupta [142, 143, 145] and Lévy [124, 125] treated compact surfaces, and Fine gave a different geometric approach [89, 90]. In [154], Witten related the weak-coupling limit on surfaces to volumes of moduli spaces of flat connections. In [55], Chevyrev realized the two-dimensional measure on the torus as a random distributional connection.

The renormalization group for lattice gauge theories. Bałaban developed a renormalization group method for lattice gauge theories in a long series of papers. He proved estimates for lattice Green's functions and propagators [9–11, 13], introduced a gauge-covariant averaging operation for group-valued fields [12], and analyzed regular configurations and the variational problem that defines the background field [14, 16]. The culmination is ultraviolet stability on a fixed torus: bounds on renormalized effective densities and suitably normalized partition functions that are uniform in the lattice spacing. He proved it in three dimensions in [15], and the later papers on effective actions and large fields developed the four-dimensional program [17, 19–21]. His treatment of the complete effective densities also gave convergent renormalization expansions for superrenormalizable gauge models [18]. Bałaban, Imbrie, and Jaffe explained the exact renormalization group for gauge theories and treated Higgs models [22–25]. Federbush developed a different phase-cell approach to the stability problem [81–86]. Ultraviolet stability is a statement about bounds, and it does not by itself give convergence of observables or a limiting measure.

In [132], Magnen, Rivasseau, and Sénéor developed a constructive analysis of the Schwinger functions of four-dimensional pure $\mathrm{SU}(2)$ Yang–Mills theory with a fixed infrared cutoff, in the trivial topological sector. They work in a regularized axial gauge, with counterterms that restore gauge invariance, and they prove large-field estimates and nonperturbative Slavnov identities. Dimock gave a detailed exposition of Bałaban's method for the scalar ϕ_3^4 model [67–69], and applied related ideas to gauge fixing and to quantum electrodynamics in three dimensions [70–73]. More broadly, our methods descend from the block-spin renormalization group of Gawędzki and Kupiainen [94, 95, 97]. They also use the cluster expansion methods described by Glimm and Jaffe [99], Brydges [38, 39], and Rivasseau [140]. More specifically, we use the convergence criterion of Kotecký and Preiss [120] and the tree estimates going back to Penrose, as explained by Fernández and Procacci [87] and recorded in the graph estimate of Dimock [68].

Stochastic quantization. A different approach constructs the Yang–Mills measure as the invariant measure of a Langevin dynamics. This idea goes back to Parisi and Wu [135], with the gauge-fixing term introduced by Zwanziger [157]. Using regularity structures [107], Chandra, Chevyrev, Hairer, and Shen constructed a gauge-covariant Langevin dynamics and a state space of gauge orbits in two dimensions [45]. They also constructed local-in-time dynamics for the Yang–Mills–Higgs model in three dimensions [46]. Chevyrev and Shen proved that the Yang–Mills measure on the trivial bundle over the two-dimensional torus is invariant for this dynamics [57]. They also proved that the mass renormalization of the Yang–Mills field that yields gauge-covariant solutions of the stochastic three-dimensional Yang–Mills–Higgs equations is unique [58]. Bringmann and Cao gave a paracontrolled approach in two dimensions [35]. For the non-Abelian theory in three dimensions, these results give local dynamics, and they do not yet construct an invariant measure. For the Abelian theory, Chevyrev, Qu, and Shen proved convergence of gauge-fixed three-dimensional lattice Langevin dynamics to the one-form stochastic heat equation for a class of plaquette actions including Wilson, Manton, and Villain actions, under assumptions on the initial data [59]. We refer to Chevyrev's survey [56] for more on this approach.

Lattice estimates. Many recent results concern lattice gauge theories at a fixed lattice spacing, for instance at strong coupling or large rank [2, 26, 44, 52, 54, 146]. Closer to the present work are results on the free energy and on bounds that are uniform in the lattice spacing. In [50], Chatterjee identified the leading term of the $U(N)$ lattice Yang–Mills free energy, and in [34], Brennecke computed its Maxwell constant. Faria da Veiga and O'Carroll proved stability bounds and bounds for generating functions of scaled plaquette fields [64, 65]. These results control the normalization and local plaquette observables. They do not identify the continuum laws of macroscopic flowed Wilson loops, and asymptotics of the free energy that are extensive in the volume do not

by themselves control normalized observables.

1.3 Main results

We now describe our results informally; the precise statements are in Section 2. Let $\mu_{N,\gamma}$ denote the Wilson measure for $\mathrm{SU}(2)$ on the periodic lattice in $\mathbb{T}_L^3$ with N^3 sites and mesh $a_N = L/N$, with the coupling (1.2). For a configuration U and a physical time $t > 0$, we write $U(t)$ for the Wilson flow of U run for the lattice time $4t/a_N^2$; with this normalization, $U(t)$ approximates the continuum Yang–Mills heat flow at time t. Let ℓ be a piecewise C^1 closed loop in $\mathbb{T}_L^3$, and let ℓ_N be a lattice approximation of ℓ of bounded length, in the sense of Definition 2.1. The flowed Wilson loop is $\operatorname{tr} \mathrm{Hol}_{\ell_N} U(t)/2$, the normalized trace of the product of the flowed link variables along ℓ_N.

We fix an odd integer $M \geq 3$ and an integer n divisible by four, and consider the lattice sides $N_k = nM^k$ for $k \geq 0$. The restriction to such sequences comes from the renormalization group, which divides the lattice into blocks of side M.

- *Existence of the continuum limit* (Theorem 2.3). Fix $L > 0$ and $\gamma > 0$. Consider finitely many flowed Wilson loops, for arbitrary piecewise C^1 loops, characters of $\mathrm{SU}(2)$, and positive flow times. As $k \to \infty$, their joint laws under $\mu_{N_k,\gamma}$ converge, for every choice of admissible lattice approximations, and the limits do not depend on this choice. All these limits are realized by a single probability measure λ on the space $\mathcal{X}_L$ of gauge orbits of Yang–Mills heat-flow trajectories at positive times. That is, the limiting law of each finite family of flowed Wilson loops is the law of the corresponding continuum holonomies under λ. At each fixed positive flow time, the energies of $\pi_k(U)$, with U distributed according to $\mu_{N_k,\gamma}$, are tight uniformly in k. The measure λ depends only on L, γ, and the class of n under multiplication by powers of M, and it is invariant under translations, rotations, and reflections of the lattice, and under the center symmetries (Theorem 2.5(3)).
- *A bound on large fields that is uniform in the coupling* (Theorem 2.7). We carry out the renormalization group analysis on a small-field event, an explicit set of configurations that are close to a gauge transform of the identity at every scale of a multiscale decomposition. Let u be the side of the coarsest lattice in the decomposition. We prove that the complement of the event has probability at most $Cu^3 \exp(-cu/\gamma)$, plus a term of order $u^3 \exp(-c \log^2(u/\gamma))$. The bound is uniform in the number of renormalization group steps and in γ below any fixed ceiling.
- *Short distances* (Theorem 2.8). Fix a smooth loop ℓ in $\mathbb{R}^3$ and $\sigma > 0$, and let W be the flowed Wilson loop of the loop $x + \varepsilon\ell$ at flow time $\sigma\varepsilon^2$. As $\varepsilon \to 0$, the normalized deficit $(1 - W)/(g_3^2\varepsilon)$ converges in law to $|C|^2/8$, where C is an explicit Gaussian vector given by the free field. This holds at every fixed coupling and uniformly in the position of the loop. Jointly for finitely many fixed loop shapes, the limit is a quadratic

function of a Gaussian vector, and it is asymptotically independent of every fixed bounded measurable function on $\mathcal{X}_L$. This makes precise, for flowed Wilson loops, the statement that the theory is weakly coupled at short distances. In three dimensions this is a consequence of superrenormalizability, rather than of the logarithmic running of the coupling found in four dimensions by Gross and Wilczek [102] and Politzer [137].

- *Weak coupling* (Theorems 2.9 to 2.11). As $\gamma \to 0$, every subsequential limit of the measures λ_γ is supported on stationary flat connections. In particular, the holonomies (H_1, H_2, H_3) around the three coordinate cycles concentrate on commuting triples. On compact regular patches near the identity, the four normal components of the logarithms of the flowed holonomies have asymptotically Gaussian fluctuations of size $\sqrt{\gamma}$, with an explicit covariance, in a sense made precise in Theorem 2.10. At the central triples the flat connections are singular, and the logarithms of the holonomies live on the scale $\gamma^{1/4}$. Ratios of expectations of compactly supported invariant functions of the rescaled logarithms converge to the corresponding ratios of integrals against $e^{-Q(Y)}\,dY$, where $Q(Y) = \sum_{i<j} |y_i \times y_j|^2/2$ is the constant-mode quartic form. The error is $O(\gamma^{1/4})$ for Lipschitz test functions. The measure $e^{-Q(Y)}\,dY$ has infinite total mass, so this is not a statement about a limiting probability measure. In both regimes we obtain explicit statistics that distinguish λ_γ from specific Gaussian and Abelian reference measures; the leading coefficient in the regular regime is itself the coefficient of the Gaussian approximation around a regular flat connection.

Section 2 also contains further results, including continuity of λ in γ, with convergence that is uniform on compact sets of couplings, and exact scaling in L. We also prove convergence of the energy and loop processes on compact intervals of positive times, and convergence in the Sobolev topology of Cao and Chatterjee. Finally, for $0 < \gamma \leq 1$ and along the subsequences of Remark 2.4, the limit does not change when the Wilson flow is replaced by the gradient flow of $\sum_p [q(U_p) + c\, q(U_p)^2]$, uniformly for c in a fixed bounded interval of nonnegative numbers.

We emphasize that the main theorem holds at every fixed coupling $\gamma > 0$, not only at small coupling. This reflects superrenormalizability: at lattice side u the effective coupling is γ/u, so we can run the renormalization group from the finest lattice down to a coarse lattice whose side u is large compared to γ, where the field is still weakly coupled. By the large-field bound, the probability discarded by this procedure tends to zero as $u \to \infty$, uniformly in the lattice spacing. The results on weak coupling concern the opposite limit, $\gamma \to 0$ at fixed u, taken after the continuum limit.

1.4 Relation to previous work

Compared to Bałaban's three-dimensional theorem [15], which bounds renormalized effective densities and suitably normalized partition functions uniformly in the cutoff, our main theorem identifies the limits of normalized observables and constructs a limiting measure. Our proof follows Bałaban's general strategy. We use several of his constructions: the averaging operation, the background field obtained by minimizing the action subject to block constraints, and estimates for the associated propagators. However, we organize the analysis differently. Bałaban decomposes the configurations into small-field and large-field regions at every scale. Instead, we condition on a single small-field event. We bound the probability of its complement directly under the Wilson measure, using the reflection positivity of Osterwalder and Seiler [134] and the chessboard estimates of Fröhlich, Israel, Lieb, and Simon [92, 93], in the form given by Biskup [29]. Large fields enter only in this probability estimate, whose proof uses a localized version of the renormalization group. Beyond stability, we also need convergence of the conditioned measures at every fixed number of renormalization group steps. Further, we need estimates for the reconstruction of the fine field from the coarse fields, which control the flowed observables. The localization and fluctuation estimates adapt Combes–Thomas localization [61], the Brascamp–Lieb inequality [33], the Bakry–Émery criterion [8], the Helffer–Sjöstrand representation of covariances [110], and the Herbst argument as described by Ledoux [123]. The quadratic fluctuation bounds are of Hanson–Wright type [108], in the form of Adamczak [1], and the maximal estimates use chaining [74, 79].

Compared to the work of Magnen, Rivasseau, and Sénéor [132], our setting is three-dimensional and superrenormalizable. They work in four dimensions, with gauge-fixed Schwinger functions, a fixed infrared cutoff, the trivial topological sector, and counterterms that restore gauge invariance. We work with the Wilson lattice measure without counterterms, and our observables are gauge-invariant flowed Wilson loops. Cao and Chatterjee [43] constructed the state space and proved conditional tightness results; we construct a measure on a version of their state space from the lattice theory. We use their deterministic results on the flow and on the topology of the state space, but not their results for random initial data. The stochastic quantization approach gives local dynamics in three dimensions for the Yang–Mills–Higgs model, including the pure gauge case [46, 58]. We construct an equilibrium measure for the pure gauge theory and prove nothing about dynamics. Gross's three-dimensional Abelian result [104] and King's construction of the Abelian Higgs model [117, 118] are earlier continuum limits of three-dimensional lattice gauge theories; our result concerns a non-Abelian theory.

Our weak-coupling theorems are related to the analysis of Yang–Mills theory on a small torus. The center symmetries fit into the framework of electric and magnetic flux of 't Hooft [149]; we divide out strictly periodic gauge transformations, treat the center operations as symmetries, and include no twist. In the Hamiltonian formulation of the

four-dimensional theory on a small spatial torus, Lüscher [127], Lüscher and Münster [130], and Koller and van Baal [119] showed that the constant modes must be treated nonperturbatively. The leading classical interaction of the constant modes is quartic at the trivial flat connection, and integrating out the other modes adds further terms to the effective Hamiltonian; see van Baal's review [152]. Near regular flat connections our analysis is a version of Laplace's method around a nondegenerate critical manifold, in the sense of Bott [31]. González-Arroyo, Jurkiewicz, and Korthals Altes [101], and Coste and these authors [62], studied the zero-momentum modes of the Euclidean lattice theory in a finite box. In this literature the periodic flat connections are often called torons [100]. Fodor and collaborators combined an exact treatment of the constant modes with perturbation theory for gradient-flow observables on the four-dimensional torus [91]. The integral of e^{-Q} over the constant modes is the bosonic Yang–Mills integral of three matrices, studied by Krauth, Nicolai, and Staudacher [122], by Krauth and Staudacher [121], and by Austing and Wheater [7]. For $\mathrm{SU}(2)$ and three matrices it diverges, and for this reason our theorem at the central triples is stated for compactly supported test functions. Our weak-coupling theorems give rigorous versions of these pictures for the measure we construct, at positive flow time and in the order of limits described above.

1.5 Scope

We record what we do not prove. The limit is taken along the sequences $N_k = nM^k$. We do not compare the limits for different classes of n or different M; we expect, but do not prove, that λ does not depend on these choices. The torus is fixed: we do not take the infinite-volume limit, and we prove nothing about a mass gap or confinement. The observables are flowed Wilson loops at positive flow time. We do not construct a random distributional connection at time zero, and we do not prove reflection positivity of the limit or the Osterwalder–Schrader axioms. We do not give a rate of convergence in the lattice spacing. The large-field bound and the weak-coupling results are uniform in γ below fixed ceilings. The convergence in the main theorem, however, is not shown to be uniform as $\gamma \to 0$ or as $\gamma \to \infty$.

1.6 Organization

Section 2 states our results precisely, and Section 3 gives a detailed outline of the proof. Section 4 collects notation and conventions. The proof occupies the remaining sections, grouped into parts. Part II contains finite-dimensional tools: integration over $\mathrm{SU}(2)$, the block averaging map, reflection positivity, gauge fixing, and the linearized theory. Part III constructs the background fields. Part IV constructs the renormalization group transformation on the small-field event, and proves convergence at each fixed resolution.

Part V estimates the reconstruction of the fine field. Part VI proves the bound on large fields. Part VII studies the lattice and continuum gradient flows, and Part VIII proves the stochastic estimates for the flowed field. Part IX constructs the limiting measure and proves the main theorem. Parts X and XI prove the results on short distances and on weak coupling.

SECTION 2

Statement of results

This section states our results precisely. Section 2.1 defines the lattice model, the Wilson flow, and the lattice approximations of loops, and Section 2.2 defines the space on which the limiting measure lives. The main theorem is stated in Section 2.3, and further properties of the limit in Section 2.4. Section 2.5 states the bound on large fields that underlies the proof. The results on short distances and on weak coupling are in Sections 2.6 and 2.7.

2.1 The lattice model

Fix a side length $L > 0$ and an integer $N \geq 4$, and let $a_N = L/N$. Let $\Lambda_N = a_N\mathbb{Z}^3/L\mathbb{Z}^3$ be the periodic cubic lattice of mesh a_N in the torus $\mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3$. A positive link is a pair $(x, x + a_N e_i)$ with $x \in \Lambda_N$ and $i \in \{1,2,3\}$, and a positive plaquette is the positively oriented boundary of the square spanned by $a_N e_i$ and $a_N e_j$ at x, with $i < j$. Let $\mathcal{E}_N$ and $\mathcal{P}_N$ denote the sets of positive links and positive plaquettes; each has $3N^3$ elements. A configuration is an element $U = (U_e)_{e\in\mathcal{E}_N}$ of $\mathcal{U}_N = \mathrm{SU}(2)^{\mathcal{E}_N}$. For the reversed link we set $U_{\bar{e}} = U_e^{-1}$, and for a path $e_1 \cdots e_m$ in the lattice we write $U_{e_1} \cdots U_{e_m}$ for the transport along it. For a plaquette p, U_p is the transport around its boundary.

Let H be the Haar probability measure on $\mathrm{SU}(2)$, and let H_N be the product of copies of H over $\mathcal{E}_N$. Set

$$q(U) = 1 - \tfrac{1}{2}\operatorname{tr} U, \qquad U \in \mathrm{SU}(2).$$

For $\gamma > 0$ we define the Wilson measure

$$d\mu_{N,\gamma}(U) = \frac{1}{Z_{N,\gamma}} \exp\Big(-\frac{4N}{\gamma} \sum_{p\in\mathcal{P}_N} q(U_p)\Big)\, dH_N(U), \tag{2.1}$$

where $Z_{N,\gamma}$ is the normalizing constant. In the usual notation the inverse coupling is $\beta = 4/g_N^2$ with $g_N^2 = \gamma/N$, as in (1.2). A lattice gauge transformation $k \in \mathrm{SU}(2)^{\Lambda_N}$ acts by $(U^k)_{(x,y)} = k_x U_{(x,y)} k_y^{-1}$, and $\mu_{N,\gamma}$ is invariant under this action.

We write $\mathfrak{su}(2)$ for the Lie algebra of $\mathrm{SU}(2)$, with the inner product $\langle X, Y\rangle = -2\operatorname{tr}(XY)$ and norm $|X| = \langle X, X\rangle^{1/2}$. With this normalization $q(e^X) = |X|^2/8 + O(|X|^4)$. On $\mathcal{U}_N$ we use the product of the corresponding bi-invariant Riemannian metrics.

The Wilson flow. Let Φ^{W}_s be the flow on $\mathcal{U}_N$ of the vector field $-\nabla \sum_{p\in\mathcal{P}_N} q(U_p)$, the negative gradient of the Wilson action. It is defined for all $s \geq 0$ and commutes with lattice gauge transformations. For a physical time $t \geq 0$ we write

$$\Psi^{(N)}_t = \Phi^{\mathrm{W}}_{4t/a_N^2},$$

and call $\Psi^{(N)}_t U$ the flowed configuration at time t. We choose the factor $4t/a_N^2$ so that $\Psi^{(N)}_t U$ approximates the Yang–Mills heat flow at time t: if $U_e = \exp(a_N A(x)\cdot e_i)$ for a smooth connection A, then $\Psi^{(N)}_t U$ approximates the heat flow of A; see Section 24. This is the flow introduced by Lüscher [128], with time measured in physical units. We also use the lattice energy

$$\mathsf{E}_{a_N}(U) = \frac{8}{a_N} \sum_{p\in\mathcal{P}_N} q(U_p), \tag{2.2}$$

which is nonincreasing along the flow and approximates the continuum energy (2.3) below.

Lattice approximations of loops. A loop is a piecewise C^1 map $\ell : [0,1] \to \mathbb{T}^3_L$ with $\ell(0) = \ell(1)$. We approximate loops by closed paths in the lattice in the following sense.

Definition 2.1. Let ℓ be a loop, and let $(N_k)_{k\geq 0}$ be an increasing sequence of lattice sides. A sequence $(\ell_k)_{k\geq 0}$ of closed lattice paths, with ℓ_k in Λ_{N_k}, is an *admissible approximation* of ℓ if the physical lengths of the ℓ_k are bounded, and if for all sufficiently large k there exists a piecewise C^1 based homotopy between ℓ_k and ℓ (together with a path joining their base points, if these differ) whose swept area, counted with absolute value, tends to zero as $k \to \infty$. Here a piecewise C^1 homotopy is continuous on the parameter square and C^1 on the cells of a finite C^1 subdivision of that square.

Every loop has an admissible approximation (Lemma 29.10), and a sequence of lattice paths of bounded length that converges uniformly to ℓ, after reparametrization, is an admissible approximation of ℓ (Theorem 29.23). For a closed lattice path ℓ_k with base point x we write $\mathrm{Hol}_{\ell_k} U \in \mathrm{SU}(2)$ for the transport along ℓ_k; it changes by conjugation under gauge transformations, so $\chi(\mathrm{Hol}_{\ell_k} U)$ is gauge invariant for every character χ of a finite-dimensional representation of $\mathrm{SU}(2)$. The basic observables of this paper are the flowed Wilson loops

$$\chi\big(\mathrm{Hol}_{\ell_k} \Psi^{(N_k)}_t U\big), \qquad t > 0.$$

2.2 The trajectory space

Let $\mathcal{A}_L = C^\infty(\mathbb{T}^3_L; \mathfrak{su}(2)^3)$ be the space of smooth connections on the trivial bundle over $\mathbb{T}^3_L$, and let $\mathcal{G}_L = C^\infty(\mathbb{T}^3_L; \mathrm{SU}(2))$ act by $A^k = kAk^{-1} - (dk)k^{-1}$. For $A \in \mathcal{A}_L$ we write

$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and

$$E_L(A) = \sum_{i<j} \int_{\mathbb{T}^3_L} |F_{ij}(x)|^2 \, dx \tag{2.3}$$

for the Yang–Mills energy; with this normalization the exponent in (2.1) approaches $E_L/(2g_3^2)$ for smooth configurations. The Yang–Mills heat flow $\partial_t A_i = \sum_j D_j F_{ji}$, with $D_j = \partial_j + [A_j, \cdot]$, has a global smooth solution for every smooth initial datum on $\mathbb{T}^3_L$, and it induces a semigroup $(\Phi_t)_{t\geq 0}$ on the orbit space $\mathcal{A}_L/\mathcal{G}_L$ (Råde [139]; see Section 24). Following Charalambous and Gross [47] and Cao and Chatterjee [43], we consider trajectories of this flow at positive times.

Definition 2.2. The trajectory space is

$$\mathcal{X}_L = \big\{ X : (0,\infty) \to \mathcal{A}_L/\mathcal{G}_L : \ X(t) = \Phi_{t-s} X(s) \text{ for all } 0 < s < t \big\}.$$

For a loop ℓ, a character χ, and $t > 0$, let $W_{\ell,\chi,t}(X) = \chi(\mathrm{Hol}_\ell\, X(t))$, where Hol_ℓ is the holonomy of any representative of $X(t)$ along ℓ. The Wilson topology $\mathcal{T}_W$ on $\mathcal{X}_L$ is the weakest topology for which all the functions $W_{\ell,\chi,t}$ are continuous.

The space $\mathcal{X}_L$ is a version of the state space of Cao and Chatterjee [43]. In Theorem 29.3 we choose a countable family $(\ell_j, t_j)_{j\geq 1}$ of loops and positive times and set $w_j(X) = \operatorname{tr} \mathrm{Hol}_{\ell_j} X(t_j)/2$. We show that

$$d_c(X,Y) = \sum_{j\geq 1} 2^{-j} |w_j(X) - w_j(Y)|$$

is a metric on $\mathcal{X}_L$ whose topology is coarser than $\mathcal{T}_W$, that both topologies are separable and have the same Borel σ-algebra, and that a Borel probability measure on $\mathcal{X}_L$ is determined by the joint laws of finitely many of the w_j. For this reason we call the w_j determining coordinates. The energy $X \mapsto E_L(X(t))$ is a Borel function for every $t > 0$. For every sequence $(R_j)_{j\geq 1}$ of finite nonnegative numbers, the set $\{X : E_L(X(L^2 2^{-j})) \leq R_j \text{ for all } j \geq 1\}$ is compact for both topologies (Theorem 29.6).

2.3 The main theorem

Fix an odd integer $M \geq 3$. For integers $n, n' \geq 1$ we write $n \sim_M n'$ if $nM^a = n'M^b$ for some integers $a, b \geq 0$, and $[n]_M$ for the equivalence class of n.

Theorem 2.3. *Fix $L > 0$, $\gamma > 0$, an odd integer $M \geq 3$, and a positive integer n divisible by four. Set $N_k = nM^k$ and $\mu_k = \mu_{N_k,\gamma}$. Then the following statements hold.*

1. *Let $\ell_1, \dots, \ell_m$ be loops, $\chi_1, \dots, \chi_m$ characters of finite-dimensional representations of $\mathrm{SU}(2)$, and $t_1, \dots, t_m > 0$, and for each i let $(\ell_{i,k})_{k\geq 0}$ be an admissible approximation of ℓ_i. Then the law under μ_k of the random vector*

$$\Big(\chi_i\big(\mathrm{Hol}_{\ell_{i,k}} \Psi_{t_i}^{(N_k)} U\big)\Big)_{i=1}^{m} \tag{2.4}$$

   converges as $k \to \infty$, and the limit does not depend on the choice of admissible approximations. For every $R > 0$, the expectations of $\exp\big(\sum_i z_i \chi_i(\mathrm{Hol}_{\ell_{i,k}} \Psi_{t_i}^{(N_k)} U)\big)$ converge uniformly in $z \in \mathbb{C}^m$ with $\sum_i |z_i| \leq R$.
2. *There exists a unique Borel probability measure $\lambda = \lambda_{L,\gamma,[n]_M}$ on $\mathcal{X}_L$ such that, for every choice of data as in (1), the limit law of (2.4) is the law of $(W_{\ell_i,\chi_i,t_i})_{i=1}^m$ under λ.*
3. *There exist Borel maps $\pi_k : \mathcal{U}_{N_k} \to \mathcal{X}_L$, invariant under all lattice gauge transformations, such that the measures $(\pi_k)_\# \mu_k$ converge to λ against every bounded $\mathcal{T}_W$-continuous function and weakly with respect to d_c, and such that*

$$\lim_{k\to\infty} \mathbb{E}_{\mu_k} \big| w_j(\pi_k U) - \tfrac{1}{2} \operatorname{tr} \mathrm{Hol}_{\ell_{j,k}} \Psi_{t_j}^{(N_k)} U \big| = 0$$

   for every $j \geq 1$, where $(\ell_{j,k})_k$ is a fixed admissible approximation of ℓ_j.
4. *The energies are tight at positive times: for every $t > 0$,*

$$\lim_{R\to\infty} \sup_{k\geq 0} \mu_k\big(E_L((\pi_k U)(t)) > R\big) = 0.$$

5. *The measure λ depends only on L, γ, and the class $[n]_M$; in particular it does not depend on the auxiliary parameters used in its construction.*

The proof is completed in Theorem 29.28. The maps π_k do not preserve holonomies exactly: they are measurable selections of smooth trajectories whose Wilson loops approximate the flowed lattice Wilson loops, and we construct them in Theorem 29.19. Their role is to realize all limits on one probability space. Part (2) and the determining property of the coordinates w_j show that λ is characterized by the limits in part (1).

Remark 2.4. The proof first establishes the theorem along the subsequences $N_K = n_* B^K$ with $B = M^J$, for a fixed integer J chosen at the start, and for n_* larger than a threshold depending on γ; see Theorem 29.24. For every finite Γ, the parameters of the construction, including J and the threshold, can be chosen before $\gamma \in (0, \Gamma]$ is fixed. This uniformity is used in the results on weak coupling. The extension to the full sequence nM^k and to every n uses two consecutive values of J and the fact that the corresponding sequences of lattice sides share infinitely many terms.

2.4 Further properties of the limit

The following theorem collects further properties of the limit. For $0 < \tau < T$ we consider the lattice energy (2.2) and the flowed Wilson loops as continuous functions of the flow time.

Theorem 2.5. *Assume the hypotheses of Theorem 2.3, and let $\lambda = \lambda_{L,\gamma,[n]_M}$.*

1. *(Energy and loop processes.) Let $0 < \tau < T < \infty$, and let ℓ_i, χ_i, and $(\ell_{i,k})_k$, $1 \le i \le m$, be as in Theorem 2.3. Under μ_k, the processes*

$$t \mapsto \Big(\mathsf{E}_{a_{N_k}}(\Psi_t^{(N_k)} U),\ (\chi_i(\mathrm{Hol}_{\ell_{i,k}} \Psi_t^{(N_k)} U))_{i=1}^m \Big), \qquad t \in [\tau, T],$$

   converge in distribution in $C([\tau,T]; \mathbb{R} \times \mathbb{C}^m)$ to $t \mapsto (E_L(X(t)), (W_{\ell_i,\chi_i,t}(X))_i)$ with X distributed according to λ. For two admissible approximations of the same loops, the expected supremum over $[\tau, T]$ of the difference of the corresponding loop processes tends to zero.
2. *(Sobolev topology.) The image measures of $(\pi_k)_\# \mu_k$ and λ in the space of trajectories of Sobolev connections in H^1 modulo gauge transformations in $W^{2,2}$, with the topology of Cao and Chatterjee described in Definition 29.1, converge against every bounded continuous function.*
3. *(Symmetries.) The measure λ is invariant under the maps induced by the isometries $x \mapsto Qx + y$ of $\mathbb{T}_L^3$ with $Q \in O(3,\mathbb{Z})$ and $y \in \mathbb{T}_L^3$. It is also invariant under the three center symmetries: for $z \in \{\pm I\}^3$, the map $[A] \mapsto [A^h]$, where $h : \mathbb{R}^3 \to \mathrm{SU}(2)$ is smooth with $h(x + Le_i) = z_i h(x)$. In particular, if $\ell_1, \dots, \ell_m$ are loops with winding vectors $\nu(\ell_j) \in \mathbb{Z}^3$, $t_1, \dots, t_m > 0$, and $b_1, \dots, b_m \ge 0$ are integers with $\sum_j b_j \nu(\ell_j) \neq 0$ in $(\mathbb{Z}/2\mathbb{Z})^3$, then $\mathbb{E}_\lambda \prod_j \big(\mathrm{tr}\, \mathrm{Hol}_{\ell_j} X(t_j)/2\big)^{b_j} = 0$.*
4. *(Scaling.) Let D_L map a trajectory X on the unit torus to the trajectory $(D_L X)(t) = L^{-1} X(t/L^2)(\cdot/L)$ on $\mathbb{T}_L^3$. Then $\lambda_{L,\gamma,[n]_M} = (D_L)_\# \lambda_{1,\gamma,[n]_M}$.*
5. *(Continuity in the coupling.) On every compact interval $I \subset (0,\infty)$ of couplings, the convergence in Theorem 2.3(1) holds uniformly in $\gamma \in I$ in the bounded-Lipschitz distance, and the maps π_k can be chosen independently of $\gamma \in I$ so that the convergence in Theorem 2.3(3) and the tightness in Theorem 2.3(4) hold uniformly in $\gamma \in I$. The map $\gamma \mapsto \lambda_{L,\gamma,[n]_M}$ is weakly continuous on $(0,\infty)$ for both topologies on $\mathcal{X}_L$.*

Part (1) is Theorem 30.4, part (2) is Theorem 30.5, part (3) is Theorem 30.6, part (4) is Theorem 30.1, and part (5) is Theorem 30.10. By parts (4) and (5), at fixed physical coupling $g_3^2 = \gamma/L$ the measures on different tori, transported to the unit torus, depend continuously on L. Since $\gamma = g_3^2 L$, part (4) is not an infinite-volume statement.

Other smoothing flows. The limit does not depend on the details of the smoothing within the following family of flows. For $c \ge 0$, let $\Psi_t^{(N,c)}$ be the flow of the vector field

$-\nabla \sum_p [q(U_p) + c\, q(U_p)^2]$ run for the lattice time $4t/a_N^2$; for $c = 0$ this is the Wilson flow. The measure $\mu_{N,\gamma}$ is unchanged.

Theorem 2.6. *Fix $L > 0$ and $0 < \gamma \le 1$, and let $N_K = n_* B^K$ be as in Remark 2.4. Set $\lambda = \lambda_{L,\gamma,[n_*]_M}$. Let $C_0 \ge 0$, let ℓ_i, χ_i, t_i, and $(\ell_{i,K})_K$ be as in Theorem 2.3, and write $W^c_{K,i} = \chi_i(\mathrm{Hol}_{\ell_{i,K}} \Psi_{t_i}^{(N_K,c)} U)$. Then*

$$\lim_{K\to\infty} \mathbb{E}_{\mu_{N_K,\gamma}} \sup_{0\le c\le C_0} \max_i |W^c_{K,i} - W^0_{K,i}| = 0.$$

Further, for $0 < \tau < T$, the joint fields of lattice energies and Wilson loops along the modified flows converge in distribution in $C([0,C_0]\times[\tau,T];\mathbb{R}\times\mathbb{C}^m)$ to the field $(c,t) \mapsto (E_L(X(t)), (W_{\ell_i,\chi_i,t}(X))_i)$, $X \sim \lambda$, which does not depend on c.

This is Theorems 30.13 and 30.15.

2.5 The bound on large fields

The proof of Theorem 2.3 rests on a renormalization group analysis of $\mu_{N,\gamma}$ restricted to a small-field event. We describe the event briefly; the precise definition is Definition 4.4. Fix an odd integer $M \ge 3$ and an integer $J \ge 1$, and set $B = M^J$. For $N = uB^K$, let T_M be the block averaging map of Definition 4.3, which maps configurations on $\Lambda_{N'}$ to configurations on $\Lambda_{N'/M}$; it is a version of Bałaban's averaging operation [12]. Let $V^j = T_M^j U$ be the averaged configurations. A renormalization group step consists of J consecutive averaging steps. At the step from V^{kJ} to $V^{(k+1)J}$, the fine configuration is described by the coarse configuration and by normalized fluctuation variables w_i, one for each of the J intermediate scales. In free logarithmic coordinates z_i on the fiber of the averaging map, in a fixed gauge-fixed chart, they are defined by $z_i = z_i^* + g_i K_i^{-1/2} w_i$. Here z_i^* is the conditional constrained minimizer of the effective classical action, K_i is its constrained Hessian, and g_i is the coupling at that scale; see Section 3.2. The small-field event $S_{N,u,\gamma}$ is the set of configurations U such that U and each $V^{(k+1)J}$, $0 \le k < K$, are gauge equivalent to configurations all of whose link variables are close to the identity, such that each renormalization group step is defined at U, and such that every normalized fluctuation variable of the k-th step has size at most $A_{\mathrm{w}} \log(1/t_k)$, where $t_k^2 = \gamma/(M n_{(k+1)J})$ is the largest effective coupling of that step and $n_j = N/M^j$.

Theorem 2.7. *Fix $L > 0$, an odd integer $M \ge 3$, and $0 < \Gamma < \infty$. There exist an integer $J \ge 1$, a choice of the remaining parameters in the definition of $S_{N,u,\gamma}$, and an integer n_Γ such that the following holds with $B = M^J$. There exist constants $C, C_{\mathrm{w}}, c_*, a_{\mathrm{w}} > 0$ and $H_0, H_2 \ge 0$ such that, for every $\gamma \in (0,\Gamma]$, every $u \ge n_\Gamma$ divisible by four, and every $K \ge 0$,*

$$\mu_{uB^K,\gamma}\big(S^c_{uB^K,u,\gamma}\big) \le C u^3 \exp\Big(H_0 + H_2 \log\frac{\Gamma}{\gamma} - \frac{c_* u}{\gamma}\Big) + C_{\mathrm{w}} u^3 \exp\Big(-a_{\mathrm{w}} \log^2 \sqrt{\frac{Mu}{\gamma}}\Big). \quad (2.5)$$

The constants depend only on L, M, Γ, and the chosen parameters. In particular, for every fixed $u \geq n_\Gamma$ and every $q > 0$ there exists $C_{u,q}$ with $\sup_{K\geq 0} \mu_{uB^K,\gamma}(S^c_{uB^K,u,\gamma}) \leq C_{u,q}\gamma^q$ for all $\gamma \in (0,\Gamma]$, and the right side of (2.5) tends to zero as $u \to \infty$, uniformly in $\gamma \in (0,\Gamma]$. Further, for each such u there exist $c_u, C_u > 0$ and $\gamma_u > 0$ such that

$$e^{-C_u(1+\log(1/\gamma))^2} \leq \sup_{K\geq 0} \mu_{uB^K,\gamma}\big(S^c_{uB^K,u,\gamma}\big) \leq e^{-c_u \log^2(1/\gamma)} \qquad (0 < \gamma \leq \gamma_u).$$

This is proved in Theorems 23.5 and 23.10 and Corollary 23.13. By the last display, at fixed u the supremum over K of the discarded probability lies between $\exp(-C_u(1+\log(1/\gamma))^2)$ and $\exp(-c_u \log^2(1/\gamma))$; in particular it is much larger than $\exp(-c/\gamma)$. The lower bound comes from configurations that violate the cutoff on the normalized fluctuation variables, and it already holds for $K = 1$. The first term in (2.5) bounds the probability that some averaged configuration has a plaquette with large curvature. We prove it using reflection positivity and chessboard estimates under the unconditioned Wilson measure. The second term bounds the probability that some normalized fluctuation variable exceeds its cutoff.

2.6 Short distances

For a loop ℓ in $\mathbb{R}^3$, a point $x \in \mathbb{T}^3_L$, and $\varepsilon > 0$ small, let $x + \varepsilon\ell$ be the scaled and translated loop in $\mathbb{T}^3_L$, and let

$$W_{x+\varepsilon\ell,s}(X) = \tfrac{1}{2} \operatorname{tr} \operatorname{Hol}_{x+\varepsilon\ell} X(s)$$

be the normalized fundamental Wilson loop under the measure λ of Theorem 2.3.

Theorem 2.8. *Fix $L > 0$, $\gamma > 0$, an odd integer $M \geq 3$, and n divisible by four, let $\lambda = \lambda_{L,\gamma,[n]_M}$, and set $g_3^2 = \gamma/L$. Let $\ell_1, \dots, \ell_m$ be smooth loops in $\mathbb{R}^3$ and $\sigma > 0$. For $x \in \mathbb{T}^3_L$ and $\varepsilon > 0$ set*

$$Z_\varepsilon(x) = \Big(\frac{1 - W_{x+\varepsilon\ell_i,\sigma\varepsilon^2}}{g_3^2\varepsilon}\Big)_{i=1}^m.$$

Let $C^a = (C^a_i)_{i=1}^m$, $a = 1,2,3$, be independent centered Gaussian vectors with covariance

$$\Sigma_{ij} = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{e^{-2\sigma|k|^2}}{|k|^2} \overline{\widehat{J}_i(k)} \cdot \widehat{J}_j(k)\, dk, \qquad \widehat{J}_i(k) = \int_0^1 e^{-ik\cdot\ell_i(s)} \dot{\ell}_i(s)\, ds, \tag{2.6}$$

and set $Z_G = (|C_i|^2/8)_{i=1}^m$, where $C_i = (C^1_i, C^2_i, C^3_i)$. Then the following statements hold.

1. *As $\varepsilon \to 0$, the law of $Z_\varepsilon(x)$ under λ converges to the law of Z_G in the bounded-Lipschitz distance, uniformly in $x \in \mathbb{T}^3_L$.*
2. *If $m = 1$ and $v = \Sigma_{11} > 0$, then*

$$\liminf_{\varepsilon\to 0} \inf_{x\in\mathbb{T}^3_L} \frac{\operatorname{Var}_\lambda W_{x+\varepsilon\ell_1,\sigma\varepsilon^2}}{(g_3^2\varepsilon)^2} \geq \frac{3v^2}{32}.$$

3. *For every bounded Borel function* $G : \mathcal{X}_L \to \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{T}_L^3} \sup_f \left| \mathbb{E}_\lambda[f(Z_\varepsilon(x))G] - \mathbb{E}[f(Z_G)]\, \mathbb{E}_\lambda[G] \right| = 0,$$

where the supremum is over functions f *with* $\|f\|_\infty \le 1$ *and Lipschitz constant at most one.*

This is proved in Theorems 32.13 and 32.17 and Corollary 32.14. In the notation of Section 31, $C_i = G_i$ and $(Z_G)_i = \mathsf{Z}_i$. The vector C is the heat-smoothed circulation of a Gaussian free field (Maxwell field) with three independent colors, so the theorem says that at short distances the flowed Wilson loops of λ behave as those of the free field, up to the quadratic map $V \mapsto |V|^2/8$ coming from $1 - \operatorname{tr} e^V/2 = |V|^2/8 + O(|V|^4)$. As an intermediate step, we prove a Gaussian limit for the normalized heat-smoothed circulations of the reconstructed field. This holds in a fixed gauge, under the conditioned Wilson measure, at every sufficiently large fixed coarse side (Theorem 32.8). These circulations are not gauge-invariant functions on $\mathcal{X}_L$; the gauge-invariant conclusion is the theorem above. Part (2) is a lower bound only, since the proof does not give uniform integrability.

2.7 Weak coupling

We now let $\gamma \to 0$ with L, M, and n fixed, and write $\lambda_\gamma = \lambda_{L,\gamma,[n]_M}$. Fix $\theta > 0$ and the flow time $t = \theta L^2$. For $X \in \mathcal{X}_L$, let $H = (H_1, H_2, H_3) \in \mathrm{SU}(2)^3$ be the holonomies of $X(t)$ around the three coordinate cycles of $\mathbb{T}_L^3$ based at the origin; the triple H is defined up to simultaneous conjugation. Every continuous function of H that is invariant under simultaneous conjugation is a continuous function of the seven traces of H_i, $H_i H_j$ $(i < j)$, and $H_1 H_2 H_3$ (Lemma 5.6). Such functions are bounded and $\mathcal{T}_W$-continuous on $\mathcal{X}_L$. In the proofs we fix, before γ varies, a coarse side in the class $[n]_M$ that is larger than certain thresholds; by Theorem 2.3(5) this choice does not change λ_γ. Each of the following theorems holds for all sufficiently small γ, with a threshold that depends on the theorem, and in each the continuum limit is taken first, at fixed γ.

A connection is flat if $F_A = 0$. Flat connections on $\mathbb{T}_L^3$ modulo gauge are classified by their holonomies, which form a commuting triple; the moduli space is

$$\mathrm{Hom}(\mathbb{Z}^3, \mathrm{SU}(2))/\mathrm{SU}(2) \simeq (\mathbb{R}/4\pi\mathbb{Z})^3/\{\boldsymbol{\vartheta} \sim -\boldsymbol{\vartheta}\},$$

where $\boldsymbol{\vartheta}$ parametrizes the triple $(e^{\vartheta_1 u}, e^{\vartheta_2 u}, e^{\vartheta_3 u})$ for a unit vector $u \in \mathfrak{su}(2)$. The eight central triples $H \in \{\pm I\}^3$ are the singular points.

Theorem 2.9. *Fix* $L > 0$, *an odd integer* $M \ge 3$, *and* n *divisible by four. Then the following statements hold.*

1. *For every* $t > 0$ *and* $\delta > 0$, $\lambda_\gamma(E_L(X(t)) > \delta) \to 0$ *as* $\gamma \to 0$.

2. *There exists* $\gamma_0 \in (0,1]$ *such that the family* $(\lambda_\gamma)_{0<\gamma\le\gamma_0}$ *is tight: for every* $\eta > 0$ *there exist* $R_j < \infty$ *such that* $\lambda_\gamma(E_L(X(L^2 2^{-j})) \le R_j$ *for all* $j \ge 1) \ge 1-\eta$ *for all* $0 < \gamma \le \gamma_0$. *The sets in this display are compact for* $\mathcal{T}_W$ *and for* d_c, *so every sequence* $\gamma_k \to 0$ *has a subsequence along which* λ_{γ_k} *converges against bounded* $\mathcal{T}_W$*-continuous functions and weakly for* d_c.
3. *Every subsequential limit of* λ_γ *as* $\gamma \to 0$, *against bounded* $\mathcal{T}_W$*-continuous functions or weakly for* d_c, *is supported on the stationary flat trajectories* $\{X : X(t) = [A]$ *for all* $t > 0,\ F_A = 0\}$.

This is Theorem 34.13. The theorem does not identify the limiting distribution on the moduli space of flat connections. The next two theorems describe the fluctuations near the regular and the singular points of this moduli space.

Regular commuting triples. In a fixed neighborhood of the identity, write $X = (x_1, x_2, x_3) = (\operatorname{Log} H_1, \operatorname{Log} H_2, \operatorname{Log} H_3) \in \mathfrak{su}(2)^3 \simeq \mathbb{R}^9$. The commuting triples correspond to the cone $\mathcal{M} = \{(\vartheta_1 u, \vartheta_2 u, \vartheta_3 u) : \boldsymbol{\vartheta} \in \mathbb{R}^3,\ u \in S^2\}$. On a regular patch $0 < r_0 \le |\boldsymbol{\vartheta}| \le r_1 < \pi$ the normal space to $\mathcal{M}$ at $y = (\vartheta_i u)_i$ is

$$\mathcal{N}_y = \Big\{ b = (b_1, b_2, b_3) :\ b_i \perp u,\ \sum_i \vartheta_i b_i = 0 \Big\},$$

which has dimension four, and near the patch we write $X = y + b$ with $y \in \mathcal{M}$ and $b \in \mathcal{N}_y$. Let $\Sigma_\theta(y)$ be the normal covariance, the positive definite covariance on $\mathcal{N}_y$ defined in (4.12). It is the covariance, divided by γ, of the linear response of the flowed logarithms X at the flat connection y, and it does not depend on γ. Let ψ be a smooth function of X that is invariant under simultaneous conjugation and supported in a small tube around a compact regular patch. We assume that ψ has the form (tangential bump)$\times$(normal cutoff equal to one near $b = 0$), and that its tangential factor takes values in $[0,1]$, vanishes on a nonempty open subset, and is positive on another. Set

$$q_*(y+b) = \tfrac{1}{2}\, b^{\mathsf{T}} \Sigma_\theta(y)^{-1} b, \qquad B_{\mathrm{obs}} = \psi, \qquad A_{\mathrm{obs}} = \psi\, q_*,$$

where the formula for q_* is used on a fixed normal tube that contains the supports of all the functions ψ and $\widetilde{w}$ below, and q_* is extended outside this tube by a bounded nonnegative invariant function. The functions A_{obs} and B_{obs} are extended by zero outside the chart. We define the normalized regression residual variance

$$\mathscr{C}_\gamma = \frac{\operatorname{Var}(A_{\mathrm{obs}}) - \operatorname{Cov}(A_{\mathrm{obs}}, B_{\mathrm{obs}})^2 / \operatorname{Var}(B_{\mathrm{obs}})}{\mathbb{E}(B_{\mathrm{obs}}^2)},$$

with all moments under λ_γ. This is the mean square error of the best affine predictor of A_{obs} from B_{obs}, normalized by $\mathbb{E} B_{\mathrm{obs}}^2$.

Theorem 2.10. *Fix $L > 0$, an odd integer $M \geq 3$, n divisible by four, $\theta > 0$, and ψ as above with a sufficiently small tube. Then the following statements hold.*

1. *There exists $c > 0$ such that $\mathbb{E}_{\lambda_\gamma} B_{\mathrm{obs}}^2 \geq c$ and $\mathrm{Var}_{\lambda_\gamma} B_{\mathrm{obs}} \geq c$ for all sufficiently small $\gamma > 0$, and*
$$\mathscr{C}_\gamma = 2\gamma^2 + o(\gamma^2) \qquad (\gamma \to 0).$$
2. *Let $w \geq 0$ be a nonzero smooth invariant function on $\mathcal{M}$ supported in the interior of the regular patch, let χ_w be a smooth normal cutoff invariant under simultaneous conjugation, with $0 \leq \chi_w \leq 1$, supported in the tube and equal to one near $b = 0$, and set $\widetilde{w}(y + b) = w(y)\chi_w(b)$. Then $\mathbb{E}_{\lambda_\gamma} \widetilde{w} \geq c_w > 0$ for small γ, and for every bounded continuous $f : [0, \infty) \to \mathbb{R}$,*
$$\frac{\mathbb{E}_{\lambda_\gamma}[\widetilde{w}\, f(q_*/\gamma)]}{\mathbb{E}_{\lambda_\gamma} \widetilde{w}} \longrightarrow \int_0^\infty f(s)\, s\, e^{-s}\, ds \qquad (\gamma \to 0).$$
3. *More generally, on $\{\widetilde{w} \neq 0\}$ write $X = y_\gamma + b_\gamma$ and $z_\gamma = \gamma^{-1/2} \Sigma_\theta(y_\gamma)^{-1/2} b_\gamma$, and define products with $\widetilde{w}$ to be zero where $\widetilde{w} = 0$. For every bounded continuous function $F(y, z)$ on the normal bundle that is invariant under simultaneous conjugation,*
$$\frac{\mathbb{E}_{\lambda_\gamma}\left[\widetilde{w}\left(F(y_\gamma, z_\gamma) - \overline{F}(y_\gamma)\right)\right]}{\mathbb{E}_{\lambda_\gamma} \widetilde{w}} \longrightarrow 0, \qquad \overline{F}(y) = \int F(y, z)\, \mathcal{N}(0, I_{\mathcal{N}_y})(dz).$$

This is Theorems 35.2 to 35.4. Part (3) is a Gaussian approximation for the normal component of the flowed holonomies near a regular commuting triple, with covariance $\gamma \Sigma_\theta(y)$, in the sense of the displayed ratio of weighted expectations. The tangential component keeps its own law under the weighting, and we do not claim that this law converges. Part (2) is the special case $F = f(|z|^2/2)$, since $|z|^2/2$ has density se^{-s} when z is a standard Gaussian vector in four dimensions. The coefficient 2 in part (1) comes from the first two moments 2 and 6 of this density. It is the coefficient of the Gaussian approximation around a regular commuting flat background, and the theorem does not identify corrections beyond it. For comparison, consider the reference measures of Definition 4.13. For the Abelian Maxwell measure with Haar cycle holonomies, the holonomy triples commute and the statistic vanishes identically. For the Gaussian holonomies with independent Haar factors, $H_i = U_i^0 \operatorname{Hol}_i(a)$, where a is a smoothed Gaussian field and U_1^0, U_2^0, U_3^0 are independent Haar distributed elements of $\mathrm{SU}(2)$, independent of a, the triple is exactly Haar distributed, and the statistic is a positive constant independent of γ (Lemma 35.17).

Central triples. Near the central triples the cone of commuting triples is singular, and the constant modes are no longer Gaussian. Let
$$Q(Y) = \tfrac{1}{2} \sum_{i<j} |y_i \times y_j|^2, \qquad Y = (y_1, y_2, y_3) \in \mathbb{R}^9, \qquad r_\gamma = \gamma^{1/4},$$

where $\times$ is the cross product, which corresponds to the Lie bracket under $\mathfrak{su}(2) \simeq \mathbb{R}^3$. We call Q the constant-mode quartic form. The measure $e^{-Q(Y)}dY$ on $\mathbb{R}^9$ is locally finite but has infinite total mass. For a function φ on $\mathbb{R}^9$ we write $\mathcal{I}_{\gamma,\varphi} = \varphi(X/r_\gamma)$, extended by zero outside a fixed neighborhood $\mathcal{V}$ of the identity triple, invariant under simultaneous conjugation, in which X is defined. A radial cutoff is a function $\chi(|Y|)$ with $\chi \geq 0$ smooth, compactly supported, not identically zero, and constant near zero.

Theorem 2.11. *Fix $L > 0$, an odd integer $M \geq 3$, n divisible by four, and $\theta > 0$. Then the following statements hold.*

1. *For every continuous compactly supported function φ on $\mathbb{R}^9$ invariant under simultaneous conjugation and every radial cutoff χ,*

$$\lim_{\gamma\to 0} \frac{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\varphi}}{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\chi(|\cdot|)}} = \frac{\int_{\mathbb{R}^9} \varphi(Y)e^{-Q(Y)}\,dY}{\int_{\mathbb{R}^9} \chi(|Y|)e^{-Q(Y)}\,dY}. \tag{2.7}$$

   If $\varphi \geq 0$ is not identically zero, then $c_\varphi r_\gamma \leq \mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\varphi} \leq C_\varphi r_\gamma$ for small γ.

2. *For every $R < \infty$ and radial cutoff χ there exist $C, \gamma_0 > 0$ such that for all $0 < \gamma \leq \gamma_0$ and all invariant Lipschitz functions φ supported in the ball of radius R,*

$$\left| \frac{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\varphi}}{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\chi(|\cdot|)}} - \frac{\int_{\mathbb{R}^9} \varphi(Y)e^{-Q(Y)}\,dY}{\int_{\mathbb{R}^9} \chi(|Y|)e^{-Q(Y)}\,dY} \right| \leq C(\|\varphi\|_\infty + \operatorname{Lip}\varphi)\gamma^{1/4}.$$

3. *Let $W_\chi(Y) = [4Q(Y) - 9]\chi(|Y|) - |Y|\chi'(|Y|)$. Then*

$$\frac{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,W_\chi}}{\mathbb{E}_{\lambda_\gamma}\mathcal{I}_{\gamma,\chi(|\cdot|)}} = O(\gamma^{1/4}).$$

4. *The same statements hold, with the same constants, near each of the eight central triples $z \in \{\pm I\}^3$, with X replaced by $(\operatorname{Log}(z_i^{-1}H_i))_{i=1}^3$; using the translated neighborhoods $z\mathcal{V}$; at each fixed γ the corresponding expectations near different central triples are equal.*

This is proved in Theorems 36.7, 36.10, and 36.12. Part (3) is the dilation integration-by-parts identity, $\int \operatorname{div}(Y\chi(|Y|)e^{-Q(Y)})\,dY = 0$, which follows from the homogeneity $Y \cdot \nabla Q = 4Q$. For the reference measures of Definition 4.13, and for radial cutoffs with $\chi(0) > 0$, the same statistic has the following limits (Theorem 36.19). It tends to -6 for the Abelian Maxwell measure with Haar cycle holonomies, and to -9 for the Gaussian holonomies with Gaussian constant modes. For the mixture of normalized conditional normal Gaussian measures over commuting backgrounds, with a fixed prior density that is continuous and positive at the tip of the cone of commuting triples, its limit is strictly larger than 4. Hence the statistic distinguishes λ_γ from these comparison measures. In Section 36 we also prove joint versions of (2.7) at finitely many flow times, for test functions that are invariant under independent simultaneous conjugations at the different

times (Theorem 36.15). Finally, let $0 < \tau < T$, and weight λ_γ by $\mathcal{I}_{\gamma,\chi(|\cdot|)}$ evaluated at time τ, normalized to a probability measure. Under this measure, the rescaled holonomy paths on $[\tau, T]$, defined with a fixed smooth equivariant extension of the logarithm and taken modulo conjugation at each time, converge in the uniform topology to constant paths. The value of the limiting path has the law with density proportional to $\chi(|Y|)e^{-Q(Y)}$, and the error against bounded Lipschitz path functionals is $O(\gamma^{1/4})$ (Theorem 36.18).

Constants in Theorems 2.10 and 2.11 may depend on L, M, n, θ, and the test functions, but not on γ.

SECTION 3

Outline of the proof

This section explains the main steps of the proof and the ideas behind them. It is meant to be read before the technical parts, and it fixes the vocabulary used there. We describe the proof of Theorem 2.3 in Sections 3.1 to 3.7. The proofs of the results on short distances and on weak coupling are outlined in Sections 3.8 and 3.9. Throughout, L and γ are fixed, $g_3^2 = \gamma/L$, and $M \geq 3$ is a fixed odd integer.

3.1 Strategy

The Wilson measure $\mu_{N,\gamma}$ at fine mesh a_N lives on a space whose dimension grows like N^3. The observables we care about, flowed Wilson loops at a fixed positive time t, depend on the configuration at length scales of order $\sqrt{t}$ and larger. The renormalization group organizes the integration over the configuration scale by scale, from the finest lattice to a coarse lattice of fixed side. At each scale, we only need to control a weakly coupled and approximately Gaussian integral. In three dimensions the effective coupling at lattice side m is $g_m^2 = \gamma/m$, which is small when m is large. For this reason the renormalization group is a convergent procedure in the ultraviolet, as Bałaban showed [15]. However, this is true only as long as the field is close to a pure gauge at every scale. Configurations for which this fails, the large fields, have to be treated separately.

We organize the proof as a limit in two stages.

1. We fix a coarse side u and a large integer K, and let $N = uB^K$, where $B = M^J$ is the blocking factor of one renormalization group step. We condition $\mu_{N,\gamma}$ on the small-field event $S_{N,u,\gamma}$ of Section 2.5, which gives the conditioned Wilson measure $\nu_{N,u,\gamma} = \mu_{N,\gamma}(\,\cdot \mid S_{N,u,\gamma})$. On this event we can run the renormalization group exactly from side N down to side u, without any analysis of large fields. We prove that, for each fixed u, the laws of flowed Wilson loops under $\nu_{uB^K,u,\gamma}$ converge as $K \to \infty$.
2. By Theorem 2.7, we have $\mu_{uB^K,\gamma}(S^c_{uB^K,u,\gamma}) \leq F(u)$ for all K, where $F(u) \to 0$ as $u \to \infty$. For a bounded observable G we have the elementary inequality

$$\left|\mathbb{E}_{\mu_{N,\gamma}} G - \mathbb{E}_{\nu_{N,u,\gamma}} G\right| \leq 2\|G\|_\infty \, \mu_{N,\gamma}(S^c_{N,u,\gamma}). \tag{3.1}$$

If $u = nB^r$, the sequence of sides uB^K is the sequence nB^K shifted by r. Then the first stage shows that, under $\mu_{nB^K,\gamma}$, the expectations of a fixed bounded continuous function of finitely many flowed Wilson loops form a Cauchy sequence up to an

error $4\|G\|_\infty F(u)$. Letting $u \to \infty$, we obtain convergence under the unconditioned Wilson measure.

This order of limits, first $K \to \infty$ at fixed u and then $u \to \infty$, appears throughout the paper. Configurations outside the small-field event enter only through the bound on large fields. We prove that bound directly under $\mu_{N,\gamma}$, without conditioning. In particular, we never need to show that a conditioned measure is reflection positive.

The rest of the proof of Theorem 2.3 consists of three tasks. The first is the renormalization group analysis on the small-field event, and its consequences at a fixed number of steps (Sections 3.2 and 3.3). The second is the bound on large fields (Section 3.4). The third is the passage from the renormalization group variables to flowed Wilson loops, and then to a measure on the trajectory space (Sections 3.5 to 3.7).

3.2 The renormalization group on the small-field event

Averaging. The renormalization group step is built from Bałaban's averaging operation for lattice gauge fields [12]. Fix a configuration on a lattice of side Mm. We divide the lattice into cubic blocks of M^3 sites, centered at the points of a sublattice of side m. For each link c of the coarse lattice, we consider the M^3 paths of length M parallel to c that start at the sites of the block at the initial point of c. They are joined to the endpoints of c by fixed connecting paths along the coordinate directions. We take the logarithms of their transports relative to the transport along c, average them, and multiply the exponential of the average by the transport along c; see (4.3). When some relative transport equals $-I$, the logarithm is not defined, and we use the reference transport instead. In this way the block averaging map T_M is a Borel map on the whole configuration space. It is gauge covariant, and it commutes with translations of the coarse lattice and with reflections of coordinates that preserve it. (Because the connecting paths change coordinates in a fixed order, we do not claim that it commutes with permutations of coordinates.) Since M is odd, the blocks are centered, and the blocks of successive averaging steps are nested. As a consequence, iterated averages of a configuration supported on one side of a reflection plane depend only on that side (Section 6). In Part II we prove the basic properties of the block averaging map, including an exact formula for the integral over a fiber $\{U : T_M U = b\}$ in local coordinates and an estimate that controls the curvature of $T_M U$ by that of U (Theorem 7.3). One renormalization group step consists of J consecutive averaging steps, $V^{(k+1)J} = T_M^J V^{kJ}$. The integer J is fixed at the start of the proof.

Background fields. Let c be a coarse configuration close to the identity. The integral over fine configurations U with $T_M^J U = c$ is dominated by a neighborhood of the minimizer of the Wilson action subject to this constraint. Following Bałaban [16], we call this minimizer the background field. We construct it in a gauge-fixed chart near the identity, by solving the constrained Euler–Lagrange equation with a contraction argument (Part III). It is a

specific branch of local constrained minimizers; we do not claim that it minimizes the action over the whole fiber of the averaging map. We prove that it depends analytically on c, with estimates that are uniform in the number of averaging steps between the fine and coarse lattices. The value of the constrained minimum defines the effective classical action $E_d(c)$ at depth d, the number of averaging steps. Minimizing in stages gives the nested minimization identity

$$E_{d+1}(c) = M \min_z E_d(b(c,z)).$$

Here $b(c,z)$ parametrizes, in the chart, the configurations on the intermediate lattice whose average is c, the minimum is over that chart, and the factor M is the change of units in three dimensions. The effective classical action E_0 is the Wilson action. Since $g_{d+1}^2 = Mg_d^2$, the ratio E_d/g_d^2 is invariant under this change of scale. The main technical point is that the analytic norms of E_d and of the background field do not grow with d. This rests on two facts. The first is the linear theory: the effective Maxwell quadratic form has an exponentially decaying kernel, and its restriction to the next constrained fluctuation space has a positive lower bound and an exponentially decaying inverse, uniformly in the depth and the lattice side (Theorem 8.3). Bałaban's propagator estimates [10, 11, 13] are used separately, in the constrained linear extension and the scalar gauge inverse of Theorem 9.3 and Lemma 9.8. The second is that the linearized averaging operator, acting on connections after a gauge correction, has row sums one and column sums M^{-3} ((10.12)). As a result the nonlinear errors are summable over scales.

Fluctuations and the effective interaction. We fix a renormalization group step from side $M^J m$ to side m. We parametrize the fine configuration by the coarse configuration c and by free logarithmic coordinates z_i at each of the J intermediate scales. We then center these coordinates at the background field and normalize them by the constrained Hessian:

$$z_i = z_i^*(c, z_{>i}) + g_i K_i(c, z_{>i})^{-1/2} w_i.$$

Here z_i^* is the conditional constrained minimizer, K_i is the constrained Hessian of the action, and g_i is the coupling at that scale. We call the w_i the normalized fluctuation variables. The change of variables is triangular, from the coarsest intermediate scale to the finest. We compute its Jacobian exactly, including the Haar density and the Jacobian of the constraint (Section 12). In the variables w the integrand is $\exp(-V(c,w))$. The interaction $U(c,w) = V(c,w) - |w|^2/2$ contains the higher-order terms of the action, the Jacobians, and the effective interaction inherited from finer scales. On the small-field event every w_i is bounded by $A_{\mathrm{w}} \log(1/t)$, where t is the largest coupling of the step. On this product of balls the interaction U is small, and we expand the integral over w in a convergent polymer expansion. We use the cluster expansion in the form of Kotecký and Preiss [120], with the tree estimates that go back to Penrose [136], as presented by Fernández

and Procacci [87] and Dimock [68]. After imposing the small-field tests through the k-th renormalization group step, we obtain an exact formula, relative to product Haar measure, for the unnormalized stopped density of the averaged configuration:

$$\mathbf{1}_{D_r}(c)\exp\Big(\Lambda_k - \frac{E_{kJ,\mathrm{sat}}(c)}{g_{kJ}^2} - \overline{R}_{kJ,\mathrm{sat}}(c)\Big). \tag{3.2}$$

Here $g_{kJ}^2 = \gamma/n_{kJ}$ is the coupling of the lattice of side n_{kJ}, and D_r, the gauge-invariant small-field domain, is the set of configurations that are gauge equivalent to configurations with all link variables close to the identity. The subscript sat denotes the gauge-invariant extension to D_r of a function defined near the identity. The function E_{kJ} is the effective classical action. The centered effective interaction $\overline{R}_{kJ}$ is generated by the integration of the fluctuations, and it is normalized so that $\overline{R}_{kJ}(I) = 0$. The normalization constant Λ_k collects all factors that do not depend on the field, including the vacuum energy (Sections 13 and 14). At $k = K$ the normalized stopped density is the law of the coarse field V^{JK} under the conditioned Wilson measure. At intermediate steps it does not include the remaining coarser restrictions of the small-field event.

Analytic norms and contraction. We control $\overline{R}_{kJ}$ by a norm that sums the absolute values of all Taylor coefficients at the identity (Definition 4.6). The coefficients of order k are weighted by $\rho^{k-2}/k!$ and by $\exp(a_{\mathrm{sp}}\mathrm{t})$, where t is the length of the shortest network that connects the lattice points on which the coefficient depends. The weighted coefficients measure spatial locality. Directed vector substitutions and products with a common output are controlled by output row bounds and volume-normalized input column bounds, under the required image conditions for compositions (Lemma 13.1). For general connected contractions, the coefficient calculus of Lemma 5.3(1)–(3) retains the attachment and external-normalization volume ratios in the majorants; connectedness alone does not cancel these ratios. Derivative estimates hold on strictly smaller analytic domains. Products of interactions whose coefficient networks are not connected, such as products of two extensive interactions, are not controlled in this way. The key estimate is Theorem 13.5, which says that one renormalization group step maps a centered effective interaction R to

$$R_+ = Q + \mathcal{P}R + \mathcal{D}R, \qquad \|R_+\| \le C_G + q\|R\| + C_T t^2 \|R\|(p^4 + \|R\|p^2).$$

In the notation of that theorem, $Q = Q^{\mathrm{w}}_{a_0,J}$, $\mathcal{P} = \mathcal{P}_\delta$, and $\mathcal{D} = \mathcal{D}_J$ for the step under consideration; in this paragraph only, Q does not denote the constant-mode quartic form. The term Q is the newly generated effective interaction, produced by the step itself. The term $\mathcal{P}R$, the pullback at zero fluctuation, is R evaluated on the background field. It contracts by a factor $q = O(M^{-J})$, because R is gauge invariant, invariant under translations, global color rotations, and inversion, and has no linear term at the identity

(Theorem 11.10). The term $\mathcal{D}R$ is the fluctuation correction to the pullback. It gains two powers of the coupling, because the linear term of the fluctuation integral vanishes on each connected support. We choose J so that $q < 1/4$, and then we choose the coupling small. The ball $\|R\| \leq E_*$ is then invariant, uniformly in the number of steps. Gauge invariance also enters in another way. At each step we choose a gauge transformation that depends only on the coarse configuration, and that brings the whole reconstructed fine configuration, fluctuations included, into a fixed analytic neighborhood of the identity (Theorem 13.2).

The conditional law of the fluctuations. Given the coarse configuration c, the normalized fluctuation variables w of one step have a conditional law, the conditional fluctuation kernel, with density proportional to $\mathbf{1}_{\Omega_p}(w)e^{-V_c(w)}$. Here Ω_p is a product of three-dimensional balls, and $D_w^2 V_c \geq I/2$ (Theorem 13.8). We emphasize that this law is not Gaussian, and that the variables are not independent. It is, however, uniformly log-concave on a convex set. Using the Bakry–Émery criterion [8] with reflecting boundary conditions and the Helffer–Sjöstrand representation of covariances [110], we obtain a logarithmic Sobolev inequality and exponentially decaying covariance estimates, with constants that do not depend on the dimension (Theorem 5.4). These estimates, together with bounds on the conditional means, are the probabilistic input for everything that follows. The conditional means are small, of order $t\|\log c\|_\infty$; at $c = I$ they vanish by the global $\mathrm{SU}(2)$ symmetry (Theorem 13.10).

3.3 Convergence at fixed resolution and reconstruction

Convergence of the effective interactions. The effective interaction after one step depends on the effective interaction at finer scales only through the contraction above. The influence of the finest scales therefore decays geometrically with the number of steps. In Section 15 we use this to prove convergence at fixed resolution. We fix the coarse side and a number of steps counted from the coarse side. Then the effective interactions, the conditional laws of the fluctuations, and the joint laws of the coarse configuration and the fluctuations of these steps converge as the total number of steps $K \to \infty$. This is a statement about finitely many variables. A subtle point is that gauge invariance allows different choices of gauge representatives at different K. We fix, once and for all, a family of gauge transformations that depend only on the coarse configuration and are chosen by a compactness argument. For this family the reconstructed configurations at each fixed resolution converge (Theorem 15.13).

Reconstruction of the fine field. From the coarse configuration and the fluctuations of all steps, we reconstruct the fine configuration step by step, from coarse to fine. We write A for the logarithm of the reconstructed fine configuration in the chosen gauge,

divided by the mesh, so that A approximates a connection. In Part V we estimate how A changes when we add the fluctuations of one more step. The change has three parts: a conditionally centered linear part, a nonlinear part, and the difference between the background field and the interpolation of the coarser field. The key deterministic fact concerns the last part. For each input direction and color, the linearized background field has the same signed volume-normalized column integral as the piecewise constant embedding of the coarse field. Their difference therefore has mean zero in each input column, and after heat smoothing at time s it gains a factor $\min(1, a/\sqrt{s})$, where a is the coarse mesh (Section 17). The key probabilistic fact is concentration for Lipschitz functions of the strongly log-concave law of the fluctuations (Section 16). Together, these two facts show that the heat evolution of A is controlled in norms of negative regularity, with summable contributions from the fine scales.

3.4 The bound on large fields

Theorem 2.7 has two parts. The second term in (2.5) accounts for configurations whose curvature is small at every scale, but for which some normalized fluctuation variable exceeds its cutoff $A_{\mathrm{w}} \log(1/t)$. We first concentrate the enclosing convex-product comparison law, transfer the bound to the incoming fiber law, and take a union bound over its b_{w} fluctuation blocks. The resulting conditional failure probability is at most $6b_{\mathrm{w}} \exp\{-m_{\mathrm{cv}} p_{\mathrm{w}}(t)^2/24\}/(1-d_t)$, with $p_{\mathrm{w}}(t) = A_{\mathrm{w}} \log(1/t)$ and d_t the enclosing-product exterior error of Theorem 14.7(2). The sum over steps converges, because the effective coupling t decreases geometrically toward the ultraviolet. The point that needs care is that we condition only on the preceding scales being good, and never on good behavior at later scales. The probability of the preceding event is multiplied, never divided (Sections 22 and 23).

The first term accounts for configurations with a plaquette of large curvature at some scale. Here the argument has three steps.

Step 1: exponential moments of blocked actions. Under the fixed-parameter hypotheses of Theorem 22.1, fix $0 < \Gamma < \infty$, take $u \in 4\mathbb{N}$ with $u \geq n_{\min}(\Gamma)$, and let $N = uB^K$ and $1 \leq j \leq JK$. Fix a tiling of the lattice of side n_j into cells. Let Q_j be the Wilson action, divided by the coupling, of the averaged configuration V^{j-1} on the plaquettes that determine the curvature of the faces inside the cells. We prove that

$$\log \mathbb{E}_{\mu_{N,\gamma}} e^{sQ_j} \leq C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b, \qquad 0 \leq s \leq s_0,$$

with constants that do not depend on the number of averaging steps or on $\gamma \in (0, \Gamma]$ (Theorem 22.1). The surface term comes from the difference between the numbers of independent plaquette variables of the periodic and the open lattice, in a comparison of partition functions, and it has to be kept. This is a stability bound in the spirit of Bałaban,

but for an exponential moment under the unconditioned measure. Its proof is the most involved part of the paper. We partition the configurations according to the regions where some averaged plaquette has large curvature, recorded scale by scale; we call this partition the large-field decomposition (Theorem 21.5). Outside these regions we run the renormalization group in a localized form. We construct background fields, fluctuation integrals, and effective interactions on arbitrary unions of blocks, with the link variables in the bad regions held fixed as parameters (Sections 18 to 20). The estimates are uniform in the shape, the number of components, and the age of the bad regions. In Section 21 we bound the density of the fine configurations on each piece of this partition by the localized effective density with the same normalization. The bound holds pointwise on gauge sections that contain every point of the relevant fibers (Theorems 21.10 and 21.13). The difference between the localized effective interactions and those on the full torus decays exponentially away from the bad regions. The missing coordinates, the boundaries, and the frozen coordinates contribute to the normalization constant, and we keep track of these contributions separately. Inside the bad regions we use only a crude bound: the effective classical action of the coarse configuration dominates both the Wilson action of the omitted links and a positive cost for each recorded bad plaquette (Section 22). This cost grows linearly in n_j/γ, while the entropy of the bad regions grows only like the square of a logarithm.

Step 2: chessboard estimates. The Wilson measure is reflection positive with respect to reflections in lattice planes through sites and through links, as Osterwalder and Seiler showed [134]. We use this separately for each of the eight shifted tilings of the lattice of side n_j into cells of $2 \times 2 \times 2$ sites. For a cell C, let I_0 be the event that some internal plaquette of C at level j has large curvature. Let R_0 be the event that every plaquette at a finer level whose full dependence set on the fine lattice lies in C has small curvature; plaquettes whose dependence sets cross the boundary of the cell are not tested. Both indicators are bounded, gauge-invariant Borel functions of the fine links in the cell. Because the blocks of the averaging map are centered and nested, the reflections between cells map these dependence sets and tests to their reflected placements in the other cells. Then the chessboard estimate of Fröhlich, Israel, Lieb, and Simon [92, 93], in the form given by Biskup [29], applies to this tiling (Theorem 6.6 and Lemma 22.11). Let R and I_{all} be the intersections of the reflected placements of R_0 and of I_0 over all cells of the tiling. The chessboard estimate gives

$$\mu_{N,\gamma}(R_0 \cap I_0) \le \mu_{N,\gamma}(R \cap I_{\mathrm{all}})^{8/n_j^3}.$$

The regularity events stay inside the root, so no division by their probability occurs. On R, the estimate on the curvature of averaged configurations shows that each cell with a plaquette of large curvature contributes at least a fixed multiple of n_j/γ to Q_j. By Step 1 and Markov's inequality, the probability of $R \cap I_{\mathrm{all}}$ is then at most $\exp(-cn_j^4/\gamma +$

$Cn_j^3 + Cn_j^2 \log(1/g_{n_j}))$. Taking the root, we find that $R_0 \cap I_0$ has probability at most $\exp(-c' n_j/\gamma)$, once n_j/γ is large.

Step 3: first failure. We decompose the event that some scale has a bad cell according to the first scale, counted from the finest lattice toward the coarse lattice, at which a bad cell occurs. On each piece of this decomposition all finer scales are good, which implies the regularity events R_0 of Step 2. We sum the bounds of Step 2 over cells and scales (Theorem 23.6). Since the bound at side n_j decays like $\exp(-cn_j/\gamma)$ and the sides grow geometrically toward the ultraviolet, the sum is dominated by the coarsest side u, which gives the first term in (2.5). On the complementary event the curvature is small at every scale. By Theorem 7.7, at every scale there exists a gauge representative with all link variables close to the identity. The remaining conditions of the small-field event reduce to the cutoffs on the fluctuations treated above.

The parameters are the integer J, the analytic radii, the cutoff amplitude A_{w}, the widths of the boundary layers in the localized analysis, and the minimum coarse side. We choose them in a fixed order: first the quantities that are uniform in J, then J, then the constants that depend on J, and finally the minimum side (Theorem 23.2). One of the tasks of Section 23 is to check that this order is free of circularity.

3.5 The gradient flow

To pass from the renormalization group variables to flowed Wilson loops, we need a theory of the lattice gradient flow for rough initial data that is uniform in the mesh. We write a lattice configuration near the identity as $U_e = \exp(aA_e)$. We apply a time-dependent gauge transformation that adds a damping term, in the spirit of the method of Zwanziger [157], DeTurck [66], and Donaldson [76]. Then the Wilson flow becomes a parabolic equation whose linear part is the lattice heat equation,

$$\partial_t A = \Delta_a A + \mathcal{N}_a(A, A) + (\text{higher order}),$$

where $\mathcal{N}_a$ is a quadratic nonlinearity involving one lattice derivative. For the initial data of interest, A has regularity just below $-1/2$, and the quadratic term is not defined by naive multiplication. We follow the approach of Da Prato and Debussche [63], which Cao and Chatterjee used for the Yang–Mills heat flow with random data [42], and write

$$A(t) = x(t) + y(t) + r(t), \qquad x(t) = e^{t\Delta_a} A(0), \qquad y(t) = \int_0^t e^{(t-s)\Delta_a} \mathcal{N}_a(x(s), x(s))\, ds.$$

We then solve for the remainder r, which is more regular. We call the pair (x, y), consisting of the linear heat evolution and the quadratic Picard term, the flow data. For a fixed $\alpha \in (1/2, 2/3)$ and horizon T, we measure it in the norms

$$\sup_{0 \le t \le T} \big\{ (t + a^2)^{\alpha/2} \|x(t)\|_\infty + (t + a^2)^{(\alpha+1)/2} \|D_a x(t)\|_\infty \big\}$$

and

$$\sup_{0\le t\le T}\left\{(t+a^2)^{\alpha-1/2}\|y(t)\|_\infty+(t+a^2)^{\alpha}\|D_a y(t)\|_\infty\right\},$$

where D_a includes the lattice differences and shifts that the equation requires (Definition 4.8). Part VII proves the deterministic theory. It consists of local existence of r on a short time interval with bounds uniform in the mesh (Theorem 24.5); smoothing after a positive time; continuation to any fixed time interval, by comparison with finitely many smooth solutions of the continuum flow (Theorems 24.10 and 25.7); and Lipschitz dependence of the flowed holonomies on the flow data (x, y) (Corollary 25.11). For smooth initial data the lattice flow converges to the continuum flow. Here we use the deterministic results of Cao and Chatterjee [42, 43], which rest on the global theory of Råde [139] and of Charalambous and Gross [47].

3.6 Stochastic estimates and convergence of flowed loops

In Part VIII we prove two facts about the flow data (x, y) of the reconstructed fine field under the conditioned Wilson measure. First, it is bounded in the norms of Section 3.5 with high probability, uniformly in the number of steps. Second, it is well approximated by the flow data of the field reconstructed from finitely many steps. At each step the new fluctuations contribute four terms: a conditionally centered linear term, a conditionally centered nonlinear term, an interpolation defect, and a predictable term (Lemma 26.2). We estimate all sixteen ordered products of these terms in the quadratic Picard term y (Sections 26 and 27). We keep their conditional means, which are nonzero, and we treat both orders of each product, since $\mathcal{N}_a$ is not symmetric. At mesh a the linear fluctuation contributes $g_3 a^{\alpha-1/2}\sqrt{\log(2+L/a)}$, which is summable over the geometric sequence of meshes since $\alpha > 1/2$.

Given these estimates, we compare the flowed Wilson loops of the fine configuration with those of the configuration reconstructed from the coarse field and the fluctuations of a fixed number q of steps. They are close in probability, uniformly in the total number of steps. The latter configuration is a smooth interpolation of a finite-dimensional random variable whose law converges as $K \to \infty$, by Section 3.3. Further, the flow of a fixed smooth field converges to the continuum flow. Letting first $K \to \infty$ and then $q \to \infty$, we find that the laws of flowed Wilson loops under the conditioned Wilson measure converge (Theorem 28.1). Together with (3.1) and the bound on large fields, this gives convergence under the Wilson measure (Theorem 28.2). We first prove these results when the coupling of the coarse lattice is below a small ceiling, and then extend them to every fixed γ by taking the coarse side large (Lemma 28.11 and Theorem 29.21). This proves part (1) of Theorem 2.3 along the subsequences $N_K = n_* B^K$.

3.7 Construction of the limiting measure

Convergence of finite families of flowed Wilson loops does not by itself produce a measure on $\mathcal{X}_L$. We obtain one as follows (Section 29). For each fixed number q of steps, the field reconstructed from q steps is smooth after interpolation, and its continuum flow defines a random element of $\mathcal{X}_L$. These random trajectories have energies that are tight uniformly in q at each of the times $s_j = L^2 2^{-j}$. Indeed, the lattice energy along the flow is controlled by the stochastic estimates, and it does not increase in time. The sets of trajectories whose energies at the times s_j are bounded by given numbers R_j, for all j, are compact, by results of Cao and Chatterjee that rest on Uhlenbeck's compactness theorem. Hence these laws converge along subsequences, and the joint laws of flowed Wilson loops identify the limit. The joint convergence of energies and Wilson loops follows from a comparison with the smooth fields reconstructed from finitely many steps, together with the continuity of the energy after a positive time in the Sobolev setting of Cao and Chatterjee (Theorem 29.18).

We also want to realize the limit as a limit of the lattice measures themselves. For this purpose we construct measurable gauge-invariant maps π_K from lattice configurations to $\mathcal{X}_L$. We enumerate the trajectories of the continuum flow started from trigonometric polynomials with rational coefficients. Given a lattice configuration, π_K selects the first trajectory in this list with the following two properties: its first m determining coordinates are within 2^{-m} of the flowed lattice Wilson loops, and its energies at m fixed positive times are at most the lattice energies plus 2^{-m}. If there is no such trajectory, π_K selects the zero trajectory. With $m = m(K)$ increasing slowly, these maps have the properties in part (3) of Theorem 2.3, and the energy constraint gives the tightness in part (4) (Theorems 29.19 and 29.20).

Finally, the passage from the subsequences $n_* B^K$ to the full sequence nM^k is arithmetic. Since the construction works for every sufficiently large J, we may take two consecutive values J and $J+1$. Every residue class of k modulo J contains infinitely many k for which nM^k also belongs to a sequence of the form $n' M^{(J+1)K}$, and hence the limits along the different residue classes coincide (Theorem 29.28). We prove the remaining parts of Theorem 2.5 in Section 30.

3.8 Short distances

For the short-distance theorem we fix γ and a coarse side. We study the heat-smoothed circulation of the reconstructed field around a small loop $x + \varepsilon\ell$, normalized by $\sqrt{g_3^2 \varepsilon}$. Adding the fluctuations step by step, we write this circulation as the sum of three terms: a contribution from the coarse field, a martingale whose increments are linear in the new normalized fluctuations, and a small nonlinear error (Section 31). A useful observation is that the linearized successive background fields differ from a single background field

by a periodic gradient, which the closed circulation does not see. We compare each martingale increment to a Gaussian, by an integration by parts adapted to the cutoff on the fluctuations, in the spirit of Stein's method [147]. The sum of the conditional covariances converges to the covariance (2.6) of the free field, as $\varepsilon \to 0$ after the lattice spacing (Theorem 31.6 and Section 32). The contribution of the coarse field vanishes in the limit because the loop shrinks. We then compare the flowed Wilson loop with the circulation, using the stability of the flow and the expansion $1 - \operatorname{tr} e^V/2 = |V|^2/8 + O(|V|^4)$ (Lemma 32.11). We carry out this analysis under the conditioned Wilson measure and in the fixed gauge of the reconstruction. To obtain the gauge-invariant statement under λ, we take the limits in the following order: the lattice spacing to zero; then $\varepsilon \to 0$ at a fixed coarse side and with a fixed cutoff on the flow data; then the removal of that cutoff; and finally the coarse side to infinity. The asymptotic independence from macroscopic observables has a simple reason: the Gaussian approximation holds uniformly conditionally on any fixed number of coarse steps, while a macroscopic flowed observable is close to a function of finitely many coarse steps.

3.9 Weak coupling

For the weak-coupling results we fix, before γ varies, the parameters of the construction and a coarse side u in the class $[n]_M$, larger than certain thresholds; by Theorem 2.3(5) this does not change λ_γ. As $\gamma \to 0$ at this fixed coarse side, every scale of the hierarchy becomes weakly coupled, and the analysis takes place on the coarse lattice. Section 33 develops the geometry near flat connections and Laplace's method on the coarse lattice. The density of the coarse configuration at the end of the renormalization group is proportional to $\mathbf{1}_{D_r} \exp(-uE_d/\gamma - R + R(I))$, and its normalizing constant is comparable to γ^{3u^3-1}, uniformly in the number of finer steps (Theorem 33.10). Near a regular flat connection, the integral over the directions transverse to the flat connections is Gaussian. In Section 34 we compare the reconstructed field with the field in which all fluctuations beyond a fixed number of steps are set to zero. Under the conditioned Wilson measure, their flowed holonomies differ by $O(\sqrt{\gamma})$ in L^p, uniformly in the number of steps and in small γ (Theorems 34.9 and 34.10). This gives the concentration on flat connections in Theorem 2.9.

For Theorem 2.10, recall that the normal covariance Σ_θ is the covariance of the linear response of the flowed logarithms at a regular flat connection. We compute it explicitly in Theorem 33.11. The theorem then follows from Gaussian integration at finitely many scales and the transfer estimates of Section 34 (Section 35). We take the limits in the following order: the continuum limit, then $\gamma \to 0$ at a fixed number of scales, and then the number of scales to infinity.

For Theorem 2.11, the starting point is an exact Taylor expansion of the effective classical action of a constant configuration, $E_d(e^{X/u}) = Q(X)/u + O(|X|^5)$, where Q is the constant-mode quartic form and the remainder is uniform in the depth d (Theorem 36.2). We integrate out the nonconstant modes with Gaussian estimates. This leaves a nine-dimensional integral against $e^{-Q(Y)}$ on the scale $Y = X/\gamma^{1/4}$, multiplied by a positive factor that is bounded above and below (Theorem 36.6). This factor is the same for all test functions, so it cancels in the ratios (2.7). The expectation of a compactly supported test function at this scale is of order $\gamma^{1/4}$, which comes from the volume $\gamma^{9/4}$ of the constant modes, compared with the order γ^2 of the normalization. We then transfer the result from the reconstructed holonomies to the flowed holonomies of the field. For this we use an estimate that is conditional on the coarse configuration (Theorem 34.11). A direct unconditional L^1 comparison loses the window factor when divided by this small expectation. Averaged higher moments still suffice for the limit and give every rate $\gamma^{1/4-\varepsilon}$, $\varepsilon > 0$; the conditional estimate supplies the stated sharp rate $\gamma^{1/4}$.

3.10 Order of choices and dependence of constants

Many statements in the paper are uniform in some parameters and not in others, and the order in which parameters are chosen matters. We describe the convention used throughout in Definition 4.14. In brief, we choose the integer J first, using only estimates that are uniform in J. Then we compute all local constants at that J. They are uniform in the lattice side, the number of averaging steps, the coupling below the relevant ceiling, and, in the localized analysis, the shape of the regions. Finally we choose the minimum coarse side. Statements at fixed γ may have constants that depend on γ; these include the convergence of flowed loops, the construction of λ, and the short-distance theorem. The bound on large fields and the weak-coupling estimates are uniform in γ below fixed ceilings.

SECTION 4

Notation and conventions

This section fixes the notation and conventions used in the rest of the paper. Section 2 introduced the lattice model, the Wilson flow, admissible approximations of loops, and the trajectory space, and Section 3 described the objects of the renormalization group analysis informally. Here we give the precise definitions that the later sections cite. A definition here fixes the meaning of a symbol; the existence of the object, its analytic properties, and the probability estimates for it are proved where it is constructed, and we say where in each case. A choice of letter never implies a uniformity: the parameters on which a constant depends are listed in each statement, or by reference to an item of Definition 4.14. We write $\mathbb{N} = \{1, 2, \dots\}$; depths and the index K may also be zero.

Sections 4.1 to 4.3 fix the lattice model, the block averaging map, and the measures of the renormalization group analysis: the Wilson measure, the small-field event, the conditioned Wilson measure, the stopped measure, and the conditional fluctuation kernel. Sections 4.4 to 4.9 fix the analytic coefficient norms, the conventions for polymer expansions and exponential moments, the flow conventions, the heat-smoothed circulation, the continuum trajectory space, and the weak-coupling observables, and Section 4.10 describes the dependence of constants.

4.1 The periodic lattice and the Wilson measure

The following definition restates the model of Section 2.1 in the notation of the later sections, where n denotes a lattice side.

Definition 4.1. Let $L > 0$ and let $n \geq 4$ be an integer. Set

$$\Lambda_n = (L/n)\mathbb{Z}^3/(L\mathbb{Z})^3, \qquad a_n = L/n.$$

A positive link is a pair $(x, x + a_n e_i)$ with $x \in \Lambda_n$ and $i \in \{1, 2, 3\}$, and $\mathcal{E}_n$ is the set of these $3n^3$ links. A positive plaquette is the positively oriented boundary $p_{ij}(x)$ of the square spanned by $a_n e_i$ and $a_n e_j$ at x, with $i < j$, and $\mathcal{P}_n$ is the set of these $3n^3$ plaquettes. A configuration is an element U of $\mathcal{U}_n = \mathrm{SU}(2)^{\mathcal{E}_n}$. For the reversed link $\bar{e}$ of a positive link e we set $U_{\bar{e}} = U_e^{-1}$, and the transport along a path $e_1 \cdots e_m$ is the ordered product $U_{e_1} \cdots U_{e_m}$. Writing $U_i(x)$ for the variable on the link $(x, x + a_n e_i)$, we have

$$U_{p_{ij}(x)} = U_i(x) U_j(x + a_n e_i) U_i(x + a_n e_j)^{-1} U_j(x)^{-1}.$$

Let H be the Haar measure on $\mathrm{SU}(2)$ with $H(\mathrm{SU}(2)) = 1$, and let $H_n = \bigotimes_{e\in\mathcal{E}_n} H$. Let $\gamma > 0$, and set

$$q(U) = 1 - \frac{1}{2}\operatorname{tr} U, \qquad S_W(U) = 4\sum_{p\in\mathcal{P}_n} q(U_p), \qquad h_n = g_n^2 = \frac{\gamma}{n}, \qquad \gamma = g_3^2 L.$$

The Wilson measure is the probability measure

$$d\mu_{n,\gamma}(U) = Z_{n,\gamma}^{-1} e^{-S_W(U)/h_n}\, dH_n(U)$$

on $\mathcal{U}_n$, where $Z_{n,\gamma}$ is the normalizing constant.

Since $S_W/h_n = (4n/\gamma)\sum_p q(U_p)$, this is the measure (2.1). The parameter γ is the dimensionless coupling, $g_3^2 = \gamma/L$ is the continuum coupling, and $h_n = g_n^2 = g_3^2 a_n$ is the lattice coupling at side n; in the usual notation the inverse coupling is $\beta_n = 4/h_n$. The letter n always denotes a lattice side, the number of sites in each coordinate direction, and never a depth. We write d and d_0 for classical depths (numbers of preceding averaging steps over which the classical action has been minimized), and ℓ for the index of an averaging step. The plaquette action is Wilson's [153]; the factor four and the identification $h_n = g_3^2 a_n$ are our normalization, explained in the following remark.

Remark 4.2. Let A be a smooth $\mathfrak{su}(2)^3$-valued function on $\mathbb{T}_L^3$, and set $U_e = \exp(a_n A_i(x))$ for $e = (x, x + a_n e_i)$. By the Baker–Campbell–Hausdorff formula, $\operatorname{Log} U_{p_{ij}(x)} = a_n^2 F_{ij}(x) + O(a_n^3)$ uniformly in x, where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. By (4.2) below, $q(e^X) = |X|^2/8 + O(|X|^4)$. Since $h_n = g_3^2 a_n$ and $|\mathcal{P}_n| = 3n^3$, we obtain

$$\frac{S_W(U)}{h_n} = \frac{1}{2g_3^2}\sum_{x\in\Lambda_n}\sum_{i<j} a_n^3 |F_{ij}(x)|^2 + O(a_n) \longrightarrow \frac{E_L(A)}{2g_3^2} \qquad (n\to\infty),$$

where $E_L(A) = \sum_{i<j}\int_{\mathbb{T}_L^3} |F_{ij}|^2\, dx$ is the Yang–Mills energy (2.3). This computation concerns smooth configurations only; it says nothing about typical configurations under $\mu_{n,\gamma}$.

Gauge transformations. A lattice gauge transformation is an element k of $\mathcal{G}_n = \mathrm{SU}(2)^{\Lambda_n}$. It acts on configurations by

$$(U^k)_{xy} = k_x U_{xy} k_y^{-1}, \qquad (U^k)^l = U^{lk}. \tag{4.1}$$

A based gauge transformation is a lattice gauge transformation equal to the identity at specified vertices, its roots. Under a gauge transformation, the transport around a closed path is conjugated by the value at its basepoint. Hence, among the functions of a tuple of holonomies along closed paths with a common basepoint, only those invariant under simultaneous conjugation are gauge invariant, while a based gauge transformation whose roots include the basepoint leaves these holonomies unchanged. All gauge transformations in this paper are periodic; we do not consider twisted boundary conditions.

The Lie algebra. Let $\tau_a = -i\sigma_a/2$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. For $X = x^a\tau_a$ and $Y = y^a\tau_a$ in $\mathfrak{su}(2)$ we use

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau_c, \qquad \langle X, Y\rangle = -2\,\mathrm{tr}(XY), \qquad |X| = |x|.$$

Then (τ_a) is an orthonormal basis, and the bracket corresponds to the cross product in $\mathbb{R}^3$. This is the inner product of Section 2.1, and on $\mathcal{U}_n$ we use the product of the corresponding bi-invariant Riemannian metrics. The group distance from the identity is

$$\rho_G(U) = \min\{|X| : e^X = U\} \in [0, 2\pi].$$

In analytic calculations, Log denotes the logarithm near the identity, continued holomorphically only inside a specified complex neighborhood of the identity. The principal real logarithm is defined on $\mathrm{SU}(2) \setminus \{-I\}$ and satisfies $|\mathrm{Log}\, U| < 2\pi$. There is no continuous logarithm on all of $\mathrm{SU}(2)$.

In these coordinates the Haar measure and the plaquette function are given by

$$\begin{aligned} dH(e^X) &= \frac{1}{16\pi^2}\,\mathrm{sinc}^2(|X|/2)\,d^3X \quad (|X| < 2\pi), \\ q(e^X) = 1 - \cos(|X|/2) &= \frac{|X|^2}{8} + O(|X|^4), \qquad \mathrm{sinc}\, z = \frac{\sin z}{z}, \quad \mathrm{sinc}\, 0 = 1. \end{aligned} \tag{4.2}$$

Integrating the first formula in polar coordinates, we find $H\{\rho_G < \varepsilon\} = (\varepsilon - \sin\varepsilon)/(2\pi)$ for $0 \le \varepsilon \le 2\pi$; for $\varepsilon \ge 2\pi$ this probability is one. These identities, and the estimates for products of finitely many group elements near the identity, are proved in Lemma 5.1. We write $c_{\mathrm{Haar}} = 1/(16\pi^2)$ for the constant in the coordinate density and $j_H(X) = \mathrm{sinc}^2(|X|/2)$ for its variable factor. The constant c_{Haar} is unrelated to the lower-bound constants in partition-function estimates with open boundary conditions.

4.2 Lattice refinements and the block averaging map

Fix an odd integer $M = 2R_M + 1 \ge 3$ and an integer $J \ge 1$, and set $B = M^J$. At total depth $K \ge 0$ the fine side is $N = n_* B^K$ with $n_* \in 4\mathbb{N}$, and for $0 \le \ell \le JK$ we set

$$n_\ell = N/M^\ell, \qquad a_{n_\ell} = L/n_\ell, \qquad g_\ell^2 = h_\ell = \gamma/n_\ell, \qquad V^\ell = T_M^\ell U,$$

where T_M is the block averaging map of Definition 4.3 below. Increasing ℓ moves toward the infrared. Since M is odd, all the sides n_ℓ are divisible by four. Schematically,

$$U \in \mathcal{U}_N \xrightarrow{T_M} V^1 \in \mathcal{U}_{n_1} \xrightarrow{T_M} \cdots \xrightarrow{T_M} V^{JK} \in \mathcal{U}_{n_*},$$

and the scales are as follows.

averaging step ℓ	side n_ℓ	mesh a_{n_ℓ}	coupling g_ℓ^2
0	N	L/N	γ/N
ℓ	$NM^{-\ell}$	LM^ℓ/N	$\gamma M^\ell/N$
JK	n_*	L/n_*	γ/n_*

As in Section 3.2, one application of T_M is an averaging step, and a renormalization group step (RG step) consists of J consecutive averaging steps, $V^{(k+1)J} = T_M^J V^{kJ}$, with blocking factor B.

We also use compositions of s averaging steps, $1 \le s \le J$, from an input side $M^s m$ to an output side $m \ge 4$, with an arbitrary preceding classical depth d_0. The fields on the s lattices of sides $Mm, \dots, M^s m$ are integrated. We call these lattices the integrated scales of the composition, and reserve the term intermediate scales for the J lattices of an RG step. The largest coupling of an integrated scale is attained at side Mm and is denoted by

$$t_{\max}^2 = \frac{\gamma}{Mm}.$$

If the integrated scales are numbered $i = 0, \dots, s-1$, from side $M^s m$ for $i = 0$ to side Mm for $i = s-1$, then the coupling g_i of scale i (not its square) satisfies $g_i/t_{\max} = M^{(i-s+1)/2}$. These ratios lie between $\lambda_* = M^{-(J-1)/2}$ and 1. The physical mesh is always denoted by a, never by h.

The equivalence relation $n \sim_M n'$ and the classes $[n]_M$ were defined in Section 2.3, and the index $[n_*]$ of the continuum measure $\lambda_{L,\gamma,[n_*]}$ refers to the class $[n_*]_M$. The measure is first constructed along the sides $n_* B^K$; by Theorem 29.28, it is the measure attached to the whole class. The small-field event and the bounds on its complement are always formulated with RG steps of J averaging steps.

Definition 4.3. We work in fine lattice units, so that the fine lattice is $\mathbb{Z}^3$ with its periodic identification and the coarse vertices are the points of $M\mathbb{Z}^3$. For a coarse vertex v, the centered block of v is

$$\mathsf{B}_M(v) = v + \big([-R_M, R_M]^3 \cap \mathbb{Z}^3\big).$$

For $x \in \mathsf{B}_M(v)$, let $\sigma_{v,x}$ be the lattice path from v to x that changes the coordinates in the order $1, 2, 3$, each monotonically. For the coarse link $c = (v, v + Me_i)$, let C_c be the straight path from v to $v + Me_i$, called the reference path, and for $x \in \mathsf{B}_M(v)$ let

$$F_{c,x} = \sigma_{v,x}[x, x + Me_i]\,\sigma^{-1}_{v+Me_i,x+Me_i}.$$

The paths $F_{c,x}$ and C_c both go from v to $v + Me_i$, so $U(F_{c,x})U(C_c)^{-1}$ is the holonomy of the closed path $F_{c,x}C_c^{-1}$ based at v; we call these matrices the based loop holonomies of c. If none of the based loop holonomies $U(F_{c,x})U(C_c)^{-1}$, $x \in \mathsf{B}_M(v)$, equals $-I$, define

$$(T_M U)_c = \exp\Big\{M^{-3} \sum_{x \in \mathsf{B}_M(v)} \operatorname{Log}\big(U(F_{c,x})U(C_c)^{-1}\big)\Big\} U(C_c), \tag{4.3}$$

where Log is the principal logarithm. If one of these based loop holonomies equals $-I$, set $(T_M U)_c = U(C_c)$. The variable on a reversed coarse link is the inverse of the variable on the positive link. For $J \geq 1$, T_M^J denotes the J-fold iterate of T_M.

The second clause, the Borel extension of the block averaging map, is part of the definition: T_M is defined on the whole configuration space. The iterate T_M^J is never replaced by a separately defined average with blocking factor B.

This construction follows Bałaban's averaging operation [12, (15), p. 19, and (42), p. 23]: one transports the fine link variables to a reference path, averages the logarithms of the based loop holonomies, and exponentiates. The centered blocks, the particular paths, and the Borel extension are specific to this paper. Under a gauge transformation k, every based loop holonomy is conjugated by k_v. The principal logarithm commutes with conjugation, and the exceptional value $-I$ is invariant under conjugation. Hence $T_M(U^k) = (T_M U)^{k'}$, where k' is the restriction of k to the coarse vertices. The following properties are proved in Proposition 6.2: T_M is Borel and gauge covariant; it is covariant under translations by coarse vectors and under coordinate reflections $x_j \mapsto a_j - x_j$, $a_j \in M\mathbb{Z}$, including at the exceptional values; the identity $T_M U_0(b) = b$ below holds; and the blocks of iterates of T_M are nested. The constraint coordinates described next are constructed in Lemma 6.3.

Constraint coordinates. Let b be a coarse configuration. The coarse-field lift $U_0(b)$ is the fine configuration equal to I on every link inside a block and to b_c on every link from $\mathsf{B}_M(v)$ to $\mathsf{B}_M(v + Me_i)$, for $c = (v, v + Me_i)$. Each path $F_{c,x}$, and the path C_c, crosses exactly one such interface link, so all based loop holonomies of $U_0(b)$ equal I and $T_M U_0(b) = b$. In each block, the links of the paths $\sigma_{v,x}$ form a spanning tree rooted at the coarse vertex (Proposition 6.2). A based gauge transformation with roots at the coarse vertices sets the variables on these block trees equal to I; this is the block axial gauge. For the coarse link $c = (v, v + Me_i)$, the eliminated edge is the interface link

$$e_c^* = (v + R_M e_i,\, v + (R_M + 1)e_i)$$

on the reference path. We write $U = e^X U_0(b)$ on the links outside the block trees. The block averaging constraint for c reads

$$\mathsf{C}_{b,c}(X) = \mathrm{Log}\big((T_M U)_c b_c^{-1}\big) = 0, \qquad D_{X_{e_c^*}} \mathsf{C}_{b,c}(0) = M^{-2} I_3.$$

To compute the derivative, note that $X_{e_c^*}$ enters $U(C_c)$ and, through $U(C_c)^{-1}$, the $M^3 - M$ based loop holonomies with x off the line through v in direction e_i; those with x on this line do not depend on $X_{e_c^*}$. At $X = 0$ the derivative is therefore $(1 - (M^3 - M)M^{-3})I_3 = M^{-2} I_3$. Near $X = 0$ the constraints are solved by local constraint coordinates $X = \Gamma_b(z)$, where the free logarithmic coordinates z are the logarithms on the links outside the block

trees other than the eliminated edges. On this branch, the Haar disintegration density with respect to $dH_m(b)\,dz$ is

$$\mathscr{J}_b(z) = c_{\text{Haar}}^f \frac{\prod_{e \text{ outside the block trees}} j_H(\Gamma_{b,e}(z))}{\prod_c \det D_{X_{e_c^*}} \mathsf{C}_{b,c}(\Gamma_b(z))}, \tag{4.4}$$

where f is the number of independent free group variables; for one averaging step from side Mm to side m, $f = 2((Mm)^3 - m^3)$. The numerator includes the Haar factors j_H of the eliminated edges, and there is no additional Haar density for the coarse field, no Gram determinant, and no residual gauge-volume factor. The formula is used only on the local branch, on which the determinants are positive, and with all domain indicators retained; the precise statement, with its domain, is Lemma 6.3. This change of variables is the spanning-tree-gauge form of the disintegration of Haar measure for this averaging constraint. Spanning-tree gauges are standard in lattice gauge theory, and Bałaban used them in the renormalization group context [14].

Under product Haar measure, the integral over the gauge transformations at the vertices of the block trees other than their roots has mass one. We keep this integral explicitly until the gauge invariance of the integrand has been established. In the constructions on block regions of Sections 18 to 21, some group variables are frozen coordinates. The based transport of a frozen coordinate b_e, which carries it to its basepoint, is $\widehat{b}_e = U(\sigma_e)b_e$. If the variables of the block region are held fixed, the change $b_e \mapsto \widehat{b}_e$ is a triangular Haar translation, and it preserves product Haar measure. Frozen coordinates remain indexed by the lattice on which they were frozen.

4.3 Effective actions, the small-field event, and conditioned measures

The effective classical action and the effective interaction. For $d \geq 0$, E_d denotes the effective classical action at preceding classical depth d, and $E_0 = S_W$. It is the value of the constrained minimization problem on the local branch constructed in Sections 9 and 10, and it satisfies

$$E_{d+1}(c) = M \min_z E_d(b(c,z))$$

on the common domain of these local constructions (Theorem 10.10). Here $b(c,z)$ parametrizes the configurations on the intermediate lattice whose average is c, and the factor M is the change of units in three dimensions. This notation does not refer to an unrestricted minimum over the compact fiber of T_M^d. Following Bałaban [16], we call the constrained minimizer the background field. We write R_d for the effective interaction that accompanies E_d in the effective density, and Λ for the normalization constant in the log density, which collects the factors independent of the field. The analytic norm of R_d controls no additive constant.

Normalized fluctuation variables. In an RG step, the free logarithmic coordinates z_i of the intermediate scale i are written as

$$z_i = z_i^*(c, z_{>i}) + g_i K_i(c, z_{>i})^{-1/2} w_i, \tag{4.5}$$

where c is the coarse field, $z_{>i}$ are the coordinates of the coarser intermediate scales, and z_i^* is the conditional constrained minimizer. Further, K_i is the constrained Hessian (a positive definite matrix) in the free coordinates of scale i at z_i^*, $K_i^{-1/2}$ is the inverse of its positive square root, and g_i is the coupling of scale i. The w_i are the normalized fluctuation variables; they differ from z_i/g_i. The map $w \mapsto z$ is triangular, with diagonal blocks $g_i K_i^{-1/2}$. Its Jacobian is therefore exactly

$$\prod_i g_i^{3f_i} \det K_i(c, z_{>i})^{-1/2},$$

where f_i is the number of free group variables of scale i. Its value at $w = 0$ and the logarithm of its ratio to that value are two parts of this single factor; neither is counted twice. The change of variables is carried out in Lemma 12.3.

Reconstruction maps. We write $\widehat{\mathscr{R}}$ for the fixed family of reconstruction maps of Theorem 15.13. These maps differ from the reconstructions in the depth-dependent gauge representatives by gauge transformations that depend only on the coarse field and equal the identity at the coarse roots. On each fixed lattice, the depths at which the family switches representatives are finite and deterministic, and they do not depend on the coupling, on configurations, or on observables. The family preserves the conditional fluctuation kernels and the fluctuation cutoffs defined below. Its construction, and the preservation of the analytic coefficient bounds on the larger domains, are in Section 15. The hat does not assert equality with the depth-dependent representatives.

The small-field event. The following definition makes the description in Section 2.5 precise.

Definition 4.4. Let $0 < r < \rho$, where ρ is the analytic radius of Section 4.4, and set

$$B_r = \{\exp A : \max_e |A_e| < r\}, \qquad D_r = \mathcal{G} B_r,$$

where $\mathcal{G}B_r$ is the set of all U^k with $U \in B_r$ and k a lattice gauge transformation. We call B_r the small-link domain and D_r the gauge-invariant small-field domain. Fix the parameters of Definition 14.4; in particular J, r, ρ, and $A_{\mathrm{w}} > 0$ are fixed. Let $u \in 4\mathbb{N}$, $K \geq 0$, $N = uB^K$, $V^\ell = T_M^\ell U$, and let $\gamma > 0$ be such that $t_k < 1$ for $0 \leq k < K$, where

$$t_k^2 = \frac{\gamma}{M n_{(k+1)J}}, \qquad p_{\mathrm{w}}(t) = A_{\mathrm{w}} \log(1/t).$$

For $0 \le k < K$, let $\mathcal{A}_{k,J}(U)$ be the event that the k-th RG step, from V^{kJ} to $V^{(k+1)J}$ at preceding classical depth kJ, is defined at U. This means that each pair formed by an intermediate coarse field and the free coordinates of the next finer scale lies in the local branch of the constraint coordinates and in the domains of the background field and of the Haar disintegration density, with positive Jacobian determinants, as specified in Definition 14.4. On $\mathcal{A}_{k,J}(U)$, let $w_{k,i,e}(U)$ be the normalized fluctuation variables of this step, obtained by inverting the step. The small-field event is

$$S_{N,u,\gamma} = \{U \in D_r\} \cap \bigcap_{k=0}^{K-1} \Big[\mathcal{A}_{k,J}(U) \cap \Big\{ \max_{i,e} |w_{k,i,e}(U)| \le p_{\mathrm{w}}(t_k) \Big\} \cap \{V^{(k+1)J} \in D_r\} \Big]. \tag{4.6}$$

For $K = 0$, the event is $\{U \in D_r\}$.

The domain conditions in $\mathcal{A}_{k,J}$ are the inverse-branch, classical-domain, and positive-Jacobian conditions of the construction, with their fixed radii. There is no additional D_r condition at an intermediate scale inside an RG step. The coupling t_k is the largest coupling of an integrated scale in the k-th RG step, and it is largest for $k = K - 1$, where $t_{K-1}^2 = \gamma/(Mu)$. The condition $t_k < 1$ for all k therefore means $\gamma < Mu$ when $K \ge 1$, and it ensures that the thresholds $p_{\mathrm{w}}(t_k)$ are positive. The analysis in Section 14 restricts γ further, by requiring $t_k \le t_*$ for a small upper bound t_*.

Lemma 4.5. *Under the assumptions of Definition 4.4, $\mu_{N,\gamma}(S_{N,u,\gamma}) > 0$.*

Proof. We show that $S_{N,u,\gamma}$ contains a nonempty open set. Near the identity configuration I all based loop holonomies in (4.3) are close to I, so the first clause of Definition 4.3 applies and the iterates T_M^ℓ are real analytic on a neighborhood of I, with $T_M^\ell(I) = I$. At $U = I$, every intermediate coarse field equals I and every free coordinate vanishes. The background field of the coarse field I is I, since I minimizes the Wilson action, so the conditional constrained minimizers vanish as well. Each domain condition in Definition 14.4 is a strict inequality on the logarithms of the coarse fields, on the free coordinates, on $\Gamma_c(z)$, and on the classical data, or the positivity of a determinant. At $U = I$ these quantities equal zero, and the determinants equal $\det(M^{-2} I_3) = M^{-6} > 0$. By Lemma 6.3 and the constructions of Sections 10 and 12, the inverse coordinates of the RG steps are continuous functions of U near I. Hence there exists a neighborhood O_1 of I contained in $\bigcap_{k<K} \mathcal{A}_{k,J}$. On O_1, the variables $w_{k,i,e}$ are continuous and vanish at I. Since $p_{\mathrm{w}}(t_k) > 0$ for every k, there exists a smaller neighborhood $O_2 \subset O_1$ of I on which $\max_{i,e} |w_{k,i,e}| \le p_{\mathrm{w}}(t_k)$ for all $k < K$. Finally, B_r is open, $I \in B_r \subset D_r$, and the maps $T_M^{(k+1)J}$ are continuous at I with value I. Hence there exists a neighborhood $O \subset O_2$ of I on which U and all $V^{(k+1)J}$ lie in B_r, and $O \subset S_{N,u,\gamma}$.

Since $0 \le q \le 2$, we have $S_W \le 24N^3$, and $Z_{N,\gamma} \le 1$. The density of $\mu_{N,\gamma}$ with respect to H_N is therefore at least e^{-24N^3/h_N}. The Haar measure of a compact group gives positive mass to every nonempty open set, so

$$\mu_{N,\gamma}(S_{N,u,\gamma}) \ge e^{-24N^3/h_N} H_N(O) > 0. \qquad \square$$

The three measures. Let $m \ge 4$ be a lattice side and $d \ge 0$ a preceding classical depth for which E_d and R_d have been constructed. Let $E_{d,\mathrm{sat}}$ be the gauge-invariant extension of E_d to D_ρ, $E_{d,\mathrm{sat}}(U) = E_d(U^k)$ for $U^k \in B_\rho$ (Theorem 11.6), and let $\overline{R}_d = R_d - R_d(I)$ be the centered effective interaction, extended to D_r in the same way. The stopped measure at side m and depth d is

$$d\widehat{\nu} = \widehat{Z}^{-1} \mathbf{1}_{D_r} e^{-E_{d,\mathrm{sat}}/g_m^2 - \overline{R}_d} \, dH_m, \qquad \overline{R}_d = R_d - R_d(I).$$

Under the hypotheses of Theorem 14.8, we have $E_{d,\mathrm{sat}} \ge \kappa_* E_0 \ge 0$ and $|\overline{R}_d| \le C$ on D_r, for constants κ_*, C depending on the fixed lattice. The density, defined to be zero outside D_r, is a nonnegative bounded Borel function and is strictly positive on D_r. Since D_r contains the nonempty open set B_r, we have $0 < \widehat{Z} < \infty$. The stopped measure is defined only in this situation.

Whenever $\mu_{N,\gamma}(S_{N,u,\gamma}) > 0$, in particular under the assumptions of Definition 4.4, we compare the three measures

$$\mu_{N,\gamma}, \qquad \nu_{N,u,\gamma} = \mu_{N,\gamma}(\,\cdot \mid S_{N,u,\gamma}), \qquad \widehat{\nu}, \tag{4.7}$$

namely the Wilson measure, the conditioned Wilson measure, and the stopped measure just defined. At an intermediate stopping side, the marginal of the conditioned Wilson measure generally differs from the stopped measure, because it also includes the restrictions of the remaining coarser RG steps. At the terminal side $m = u$ no such restriction remains, and the law of V^{KJ} under $\nu_{N,u,\gamma}$ is the stopped measure at depth KJ (Theorem 14.8(9)). An exponential-moment estimate over a range of scales that starts from the unconditioned Wilson measure uses $\mu_{N,\gamma}$ itself, without a D_r restriction.

The auxiliary integral. For coarse fields outside B_r, in the larger analytic domain, the effective interaction is defined recursively by an analytic, positive product integral, the auxiliary integral of Theorem 14.6. It is not the conditional integral of the preceding restricted measure, although the two agree for coarse fields in B_r. The reconstructions it uses for coarse fields in the larger domain need only take values in B_ρ. This distinction between the radii r and ρ is part of the construction in Section 14.

The conditional fluctuation kernel. Fix a coarse field c in the domain of an RG step. The conditional fluctuation kernel of the normalized fluctuation variables $w = (w_{i,e})$ of

the step is

$$k(c, dw) = \frac{\mathbf{1}_{\Omega_p}(w) e^{-V(c,w)}\, dw}{\int_{\Omega_p} e^{-V(c,w)}\, dw}, \qquad \Omega_p = \prod_{i,e} \{w_{i,e} \in \mathbb{R}^3 : |w_{i,e}| \le p_i\}. \tag{4.8}$$

Here the conditional fluctuation potential $V(c, w)$ contains $|w|^2/2$, all higher-order terms of the classical action in the fluctuations, the terms from the Haar disintegration density and from the Jacobian of (4.5) that depend on w, and the input effective interaction, produced by the integrations at finer scales, evaluated at the reconstructed field. Terms that depend on c but not on w cancel in the normalized kernel, and only there. For the kernels arising from the small-field event, $p_i = p_{\mathrm{w}}(t_k)$ for all intermediate scales of the k-th RG step. Products of truncated Gaussian measures serve as reference measures in expansions; they are not the conditional law. All conditional means, covariances, and derivatives of the conditional log-density refer to $k(c, dw)$ on its convex domain Ω_p. In particular, for a differentiable parameter λ on which V depends, with the coupling, the radii p_i, and the coordinates w held fixed, the normalized log-density derivative on Ω_p is $\partial_\lambda \log k = -\partial_\lambda V + \mathbb{E}_k \partial_\lambda V$ (Lemma 13.6). The kernel and its potential are constructed in Theorem 13.8.

The comparison lattice. For the index $s \ge 0$ of the comparison lattice we write $n_s = n_* B^s$ for its side, $H_s = L/n_s$ for its mesh, and $\nu_{K,s} = \nu_{N_K, n_s, \gamma}$ for $K \ge s$, where $N_K = n_* B^K$. The corresponding backward reconstruction sequence has physical meshes $a_j = H_s B^{-j}$, $0 \le j \le K - s$, and preceding classical depths $J(K - s - j)$. Once the coarse field of a step is fixed, its conditional fluctuation kernel is not changed by the restrictions at coarser scales, since these are functions of the coarse field alone.

4.4 Analytic coefficient norms

Effective interactions, background fields, and reconstruction maps are controlled by weighted sums of absolute values of Taylor coefficients. A bound of this kind controls derivatives on smaller domains (Lemma 5.3), is stable under products and substitutions, and measures locality through an exponential weight in the length of a network connecting the locations of the coefficient. Norms of this type are used throughout constructive renormalization group methods [17, 38, 96]; the coefficient norm with boundary decay (4.11) is specific to this paper. All indices below are scalar indices, which include orientation and color.

Definition 4.6. *Indices and networks.* Let I_σ be finite index sets, each with a location map $x : I_\sigma \to \mathbb{T}^3_L$, a mesh $a_\sigma > 0$, and the volume $v_\sigma = a_\sigma^3$, and write $v_i = v_\sigma$ for $i \in I_\sigma$. The assigned location $x(i)$ of an index is its position on the lattice on which it was introduced. Different indices may have the same location, and index sets introduced at different scales

remain different. Distances and network lengths in the coefficient weights are measured in units of the current output lattice spacing a_{ref}, that is, $d(x,y) = d_{\mathrm{phys}}(x,y)/a_{\mathrm{ref}}$, where d_{phys} is the periodic Euclidean distance on $\mathbb{T}^3_L$; all index sets are embedded in this common metric, and dist below denotes d. The boundary graph distance d_Ω and the boundary widths below are also expressed in output-mesh units. For a finite set S of locations let

$$\mathrm{t}(S) = \inf\{\mathrm{length}(\mathcal{N}) : \mathcal{N} \text{ a finite connected metric network containing } S\}.$$

Lengths are measured with d. Internal vertices are allowed, networks may wrap around the torus, and $\mathrm{t}(\{x\}) = 0$.

Tensor norms. Let $a_{\mathrm{sp}} \geq 0$. For a tensor $T(o; i_1, \dots, i_k)$ with an output index o and input indices $i_1, \dots, i_k$, set

$$\begin{aligned} R_{a_{\mathrm{sp}}}(T) &= \sup_{o} \sum_{i_1,\dots,i_k} e^{a_{\mathrm{sp}}\mathrm{t}(x(o),x(i_1),\dots,x(i_k))} |T(o; i_1, \dots, i_k)|, \\ C_{a_{\mathrm{sp}},r}(T) &= \sup_{i_r} \sum_{o,(i_s)_{s\neq r}} \frac{v_o}{v_{i_r}} e^{a_{\mathrm{sp}}\mathrm{t}(x(o),x(i_1),\dots,x(i_k))} |T(o; i_1, \dots, i_k)|, \qquad 1 \leq r \leq k, \\ |T|_{a_{\mathrm{sp}}} &= \max\{R_{a_{\mathrm{sp}}}(T), C_{a_{\mathrm{sp}},1}(T), \dots, C_{a_{\mathrm{sp}},k}(T)\}. \end{aligned} \tag{4.9}$$

The row norm $R_{a_{\mathrm{sp}}}$ fixes the output and sums over all inputs; the volume-normalized column sum $C_{a_{\mathrm{sp}},r}$ fixes the r-th input and sums over all other indices, including the output. For a kernel $K(x,y)$ from mesh a_{in} to mesh a_{out}, this reads

$$\|K\|_{\mathrm{row},a_{\mathrm{sp}}} = \sup_x \sum_y e^{a_{\mathrm{sp}}\,\mathrm{dist}(x,y)} |K(x,y)|, \qquad \|K\|_{\mathrm{col},a_{\mathrm{sp}}} = \sup_y \frac{a_{\mathrm{out}}^3}{a_{\mathrm{in}}^3} \sum_x e^{a_{\mathrm{sp}}\,\mathrm{dist}(x,y)} |K(x,y)|.$$

For positive input radii r_i we write $T[r](o;\mathbf{i}) = T(o;\mathbf{i}) \prod_s r_{i_s}$, so that every input, including a fixed one, contributes its radius. Normalizations of fields by couplings are written separately.

Majorant series. For an analytic map

$$F_o(u) = \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\dots,i_k} F_k(o; i_1, \dots, i_k) u_{i_1} \cdots u_{i_k}, \qquad F_k = D^k F(0),$$

an absolute majorant is a convergent series $\widehat{F}(s) = \sum_k f_k s^k$ with $f_k \geq 0$ and $|F_k|_{a_{\mathrm{sp}}}/k! \leq f_k$. There is exactly one Taylor factorial.

Scalar functions. For a symmetric tensor $A(i_1, \dots, i_k)$ without output, $k \geq 1$, set

$$|A|^{\mathrm{sc}}_{a_{\mathrm{sp}}} = \max_r \sup_{i_r} \sum_{(i_s)_{s\neq r}} e^{a_{\mathrm{sp}}\mathrm{t}(x(i_1),\dots,x(i_k))} |A(i_1, \dots, i_k)|, \tag{4.10}$$

and write $\|D^k R(0)\|_{a_{\mathrm{sp}},\mathrm{root}} = |D^k R(0)|^{\mathrm{sc}}_{a_{\mathrm{sp}}}$. For an analytic scalar function R of the link coordinates on the torus with $DR(0) = 0$, set

$$\mathcal{B}_{\rho,a_{\mathrm{sp}}}(R) = \sum_{k\geq 2} \frac{\rho^{k-2}}{k!} \|D^k R(0)\|_{a_{\mathrm{sp}},\mathrm{root}}.$$

Functions on a block region. Let Ω be a set of scalar labels of the variables that are still varied or integrated in a block region, let W be its boundary as in Section 18.1, and let $0 < \phi_i \leq \rho$ be a radius profile. Here $d_\Omega(i)$ is the graph distance, on the refined position graph of Section 18.2, from the position of i to the set of positions outside Ω, as in Definition 18.1; it is expressed in units of the current output mesh. It differs from the periodic Euclidean distance d used for the connecting networks. Sections 18–20 use the graph-metric variant of (4.11) fixed before Definition 18.1, in which network lengths are also measured in the ℓ^1 metric of that graph; a coefficient bound at a Euclidean decay rate is used there at that rate divided by $\sqrt{3}$. Let b_{rec} range over the admissible set of real frozen coordinates of Definition 18.1. For $f(v; b_{\mathrm{rec}}) = \sum_\nu f_\nu(b_{\mathrm{rec}}) v^\nu$, with $v = (v_i)_{i\in\Omega}$ and $\phi^\nu = \prod_i \phi_i^{\nu_i}$, set

$$\mathcal{X}^{\phi}_{\Omega}(f) = \sup_{b_{\mathrm{rec}}\ \mathrm{real}} \rho^{-2} \sum_{k\geq 1} \sup_{i\in\Omega} e^{\alpha_{\mathrm{bd}} d_\Omega(i)} \sum_{|\nu|=k} \frac{\nu_i}{k} |f_\nu(b_{\mathrm{rec}})| e^{a_{\mathrm{sp}}\mathrm{t}(\nu)} \phi^\nu, \tag{4.11}$$

where $\mathrm{t}(\nu)$ is the network length of the locations of the indices in the support of ν, and $a_{\mathrm{sp}}, \alpha_{\mathrm{bd}} > 0$. The order of the operations in (4.11), namely the supremum over frozen coordinates, the sum over degrees, the supremum over the fixed label, and the sum over coefficients, is part of the definition. Constants have norm zero, and degree one is included. If Ω is empty, the norm is zero. If Ω has no boundary, (4.11) is applied only to functions with vanishing nonconstant part, whose norm is then zero; effective interactions on the full torus are measured by $\mathcal{B}_{\rho,a_{\mathrm{sp}}}$. The classical version of these norms has its own radius ρ_{cl} and its own profile and rate parameters.

A bound on $|T|_{a_{\mathrm{sp}}}$ controls the row and every volume-normalized column; it is stronger than a bound on the output rows alone or on an operator norm of T contracted with given directions. In (4.10) the sums with one index fixed are counting sums. We write $|A|^{\mathrm{sc},*}_{a_{\mathrm{sp}}}$ for the convention that multiplies each summand with the index i_r fixed by v_*/v_{i_r}, for a reference volume v_*. Then

$$|A|^{\mathrm{sc}}_{a_{\mathrm{sp}}} \leq \max_i \frac{v_i}{v_*} |A|^{\mathrm{sc},*}_{a_{\mathrm{sp}}}, \qquad |A|^{\mathrm{sc},*}_{a_{\mathrm{sp}}} \leq \max_i \frac{v_*}{v_i} |A|^{\mathrm{sc}}_{a_{\mathrm{sp}}},$$

where the maxima are over the possible fixed indices. On one mesh we choose v_* equal to its volume, and the two conventions agree; on several meshes the conversion is kept explicitly. The assumption $DR(0) = 0$ in the definition of $\mathcal{B}_{\rho,a_{\mathrm{sp}}}$ concerns the effective

interactions. A constant is stored separately in Λ and is never bounded by $\mathcal{B}_{\rho,a_{\mathrm{sp}}}$. The precise definitions on block regions, including the frozen coordinates and the labels, are in Definition 18.1.

Remark 4.7. The weight ν_i/k in (4.11) is the sum (4.10) with the index i fixed, for a single monomial. Indeed, let $|\nu| = k$ and $\nu_i > 0$. The derivative $D_i D_{i_2} \cdots D_{i_k}(f_\nu v^\nu)(0)$ is nonzero for exactly $(k-1)!/[(\nu_i - 1)! \prod_{j \neq i} \nu_j!]$ ordered tuples $(i_2, \dots, i_k)$, and each nonzero value has modulus $|f_\nu| \prod_j \nu_j!$. Hence

$$\frac{1}{k!} \sum_{i_2,\dots,i_k} |D_i D_{i_2} \cdots D_{i_k}(f_\nu v^\nu)(0)| = \frac{\nu_i}{k} |f_\nu|.$$

Multiplying by ϕ^ν shows that (4.11) is the profile version of the scalar norm (4.10), with the index i fixed. Replacing ν_i/k by ν_i, or adding a factor $1/k!$ to the multiindex coefficients, would change the norm.

Rescaling an input transfers its locations, its network lengths, and its volume factors explicitly. Replacing a shortest connecting network by a tree on the locations of the external indices reduces the exponent once, from a_{sp} to $a_{\mathrm{sp}}/2$ (Lemma 5.3); there is no loss at every derivative order. Derivatives in analytic series are divided by $k!$. A bound on the values of a holomorphic function, or a pointwise bound on its Hessian, does not replace a bound on its absolute coefficients.

Radius profiles. For a boundary width w_{bd}, the spatially varying analytic radius is

$$\phi(i) = \begin{cases} \rho_\partial, & d_\Omega(i) < w_{\mathrm{bd}}, \\ \rho, & d_\Omega(i) \geq w_{\mathrm{bd}}, \end{cases} \qquad \psi_{\mathrm{prof}} = \sqrt{K_*}\,\phi, \qquad \widehat{\phi} = K_* \phi,$$

where $0 < \rho_\partial \leq \rho$ and $K_* \geq 2$ is a fixed enlargement factor of the analytic radii. The classical profile is defined separately. The symbol ψ without a subscript is reserved for an observable cutoff (Definition 4.11). We keep the following radii distinct.

r	radius of the small-link domain B_r in the small-field event
ρ, ρ_{cl}	analytic radii for the effective interaction and the effective classical action
R_{raw}	larger radius of the initial construction
$R_{\mathrm{buf}} = C_{\mathrm{map},J} r$	radius of the analytic extension around coarse fields
$R_{\mathrm{cell}} = \lceil A_8/r \rceil$	side length of the enlarged cells
R_W	radius imposed on the variables $g_i w_i$
R_{en}	radius of the enclosing product domain
β_{rec}	bound on the frozen coordinates

Here A_8 is the sufficiently large constant of Theorem 21.3. None of these radii is the inverse coupling $\beta_n = 4/h_n$. The width s_{er} of a removed boundary layer, the separation widths, and the numbers of elapsed RG steps are distinct geometric variables. Frozen coordinates from earlier steps are fixed real parameters; they are neither differentiated nor continued to complex values. Newly frozen variables were analytic variables before the substitution.

Normalization constants. Effective densities have the form $\exp(\Lambda - E/g^2 - R)$, multiplied by domain indicators. The centered effective interaction is $R - R(I)$, and in such a density the constant $R(I)$ is moved into Λ. If some variables are fixed at the value c by substitution, we set

$$R_{\mathrm{new}}(v) = R(v,c) - R(0,c), \qquad \Lambda_{\mathrm{new}} = \Lambda - R(0,c),$$

which is the identity $e^{\Lambda - R(v,c)} = e^{\Lambda - R(0,c)} e^{-[R(v,c) - R(0,c)]}$. The classical action is kept in full. A substitution does not remove a first derivative by translating a conditional minimizer, so R_{new} has in general terms of degree one, which (4.11) includes. The full determinant and the normalization factor of every eliminated Gaussian coordinate are included exactly once in the density. Their field-independent factors are collected in Λ; their remaining field dependence is kept in the effective interaction and in the conditional fluctuation potential. This is proved in Lemma 18.9 and Theorem 20.13. A comparison of densities on block regions is first a statement at representatives satisfying the reconstruction identities of Theorem 21.13; only the resulting gauge-invariant bound in terms of the Wilson action is afterwards extended to all representatives (Theorem 22.10).

4.5 Polymers and exponential moments of blocked actions

Polymer expansions. A polymer $\mathfrak{X}$ is a nonempty connected set of the specified block labels. Polymer activities $\mathfrak{z}(\mathfrak{X}; c)$ arise from the finite product expansion displayed at each use. The hard-core polymer partition function is

$$\Xi(c) = \sum_{\{\mathfrak{X}_1,\dots,\mathfrak{X}_m\} \text{ pairwise compatible}} \prod_{i=1}^{m} \mathfrak{z}(\mathfrak{X}_i; c),$$

where the empty family contributes one. Compatibility is determined by the incompatibility relation specified at each expansion (disjointness of supports, or the exclusion relation). The norm of the polymer activities combines absolute coefficient majorants on the given radii, the spatial weight $e^{a_{\mathrm{sp}}\mathfrak{t}}$, and a size weight $e^{\kappa_{\mathrm{pol}}|\mathfrak{X}|}$, and it takes the supremum over sites v of the sum over the polymers containing v, $\sup_v \sum_{\mathfrak{X} \ni v}$. The precise incompatibility relation and size weight are part of the corresponding theorem (Lemma 12.9 and Theorem 12.6). Boundary visits and fixed inputs of a coefficient are kept in the networks

through the expansion of the product into polymer activities (Lemma 12.9) and the cluster expansion of $\log \Xi$ (Theorem 12.6), and through their analogues on block regions in Section 20. The support of a polymer activity alone does not give a new decay rate. We use the cluster expansion in the form of Kotecký and Preiss [120], with the tree-graph bound as presented by Fernández and Procacci [87] and by Dimock [68, Appendix B, Theorem 27, p. 39, and (323), p. 42]; see also Brydges [38].

Reflected block tilings. These conventions are used in the bound on large fields in Sections 22 and 23. At averaging step j, a reflected block tiling $\mathcal{T}$ consists of $\mathfrak{m}_j = n_j^3/8$ disjoint coarse cells of $2 \times 2 \times 2$ sites. Each cell contains the sets of finer plaquettes on which the internal coarse plaquettes of the cell depend; the plaquettes crossing between cells are not tested in this tiling. The eight translates of $\mathcal{T}$ cover all plaquettes. We write

$$Q_j^{\mathcal{T}} = \frac{4}{h_{n_{j-1}}} \sum_{p \in \mathcal{P}_{j-1}^{\mathcal{T}}} q(V_p^{j-1})$$

for the blocked action of the finer plaquettes of the cells, and $R_{<j}^{\mathcal{T}}(d_{\mathrm{cut}})$ for the product over the cells of the indicators of the earlier small-field conditions. The threshold for a bad cell is $d_* = d_{\mathrm{cut}}/2$. The indicator stays inside the expectation that is raised to the power $1/\mathfrak{m}_j$ in the chessboard estimate. Reflection positivity of the unconditioned Wilson measure was proved by Osterwalder and Seiler [134]. By Theorem 6.6, it holds for the bounded Borel gauge-invariant functions of the fine links with both endpoints in the union of the blocks over the coarse vertices of one open half torus. For an observable supported in a reflection half, the pullback by the iterated block averaging map belongs to this class and intertwines the reflection, including at the exceptional values. This gives reflection positivity of the averaged measure. We never use reflection positivity for a fixed fiber or for a conditioned measure. The chessboard estimate is that of Fröhlich, Israel, Lieb, and Simon [92, 93], in the form of Biskup [29, Theorem 5.8, p. 44].

Exponential moments. The bound on the exponential moments of blocked actions has the form

$$\log \mathbb{E}_{\mu_{N,\gamma}}\left[e^{s Q_j^{\mathcal{T}}}; R_{<j}^{\mathcal{T}}\right] \le 3 C_v n_{j-1}^3 + C_s n_{j-1}^2 \log(1/g_{n_{j-1}}) + C_b', \qquad 0 \le s \le s_0,$$

with fixed constants C_v, C_s, C_b', s_0. It is proved, together with the induction over averaging steps under the Wilson measure, in Theorem 22.1 and Section 23. If a stopped side is $m = \chi_{\mathrm{stop}} n_{j-1}$, a coefficient 9 of m^2 becomes $9\chi_{\mathrm{stop}}^2$ in units of n_{j-1}^2. At the finest, unconditioned scale the bound contains the correction $36 \log \beta_N/N$, with $\beta_N = 4/h_N$, which is kept until the parameters are chosen uniformly in the coupling (Theorem 6.6).

4.6 Heat semigroup, Wilson flow, and flow data

For a physical mesh a, let $D^{\pm}_{a,i}$ be the componentwise forward and backward differences in direction i, in physical units, and let $\Delta_a = \sum_i D^-_{a,i} D^+_{a,i}$. The lattice heat semigroup is the componentwise scalar semigroup $P^a_t = e^{t\Delta_a}$, which retains the constant mode. The cardinal interpolation of directed-edge fields (Lemma 16.3) preserves constants and integrals of columns, and it is the identity when the input and output meshes coincide; its volume-normalized columns use its own input mesh.

The Wilson flow Φ^{W}_s is the flow of the vector field $-\nabla \sum_p q(U_p)$, where the gradient is taken with respect to the product metric on $\mathcal{U}_n$ given by $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$. Its version in physical time is

$$\Psi^{(N)}_t = \Phi^{\mathrm{W}}_{4t/a_N^2},$$

as in Section 2.1. This conversion of time contains no coupling factor; the factor $4/a_N^2$ is the normalization for which the linear part of the flow is the heat equation in physical units. The flow is Lüscher's [128]. In logarithmic coordinates $U = e^{aA}$, after we add the gauge-damping term of Lemma 24.2 (a lattice version of the gauge fixing of Zwanziger [157], DeTurck [66], and Donaldson [76]), the linear part of the equation is the scalar heat equation $\partial_t A = \Delta_a A$ in each component. We write $\mathcal{N}_a$ for the bilinear coefficient of this gauge-damped lattice equation, in the order in which its arguments appear and with all its lattice shifts (Lemma 24.2). Set

$$\mathcal{Q}_a(u,v)(t) = \int_0^t P^a_{t-s}\mathcal{N}_a(u(s),v(s))\,ds, \qquad \mathcal{B}_a(u,v) = \mathcal{Q}_a(u,v) + \mathcal{Q}_a(v,u).$$

The coefficient $\mathcal{N}_a$ and the bilinear Duhamel term $\mathcal{Q}_a$ need not be symmetric in their two arguments; their symmetrization $\mathcal{B}_a$ is symmetric. The letter Q alone is reserved for the constant-mode quartic form of Section 4.9.

Definition 4.8. Fix $1/2 < \alpha_{\mathrm{fl}} < 2/3$, and set $\beta_{\mathrm{fl}} = \alpha_{\mathrm{fl}} - 1/2$ and $\kappa_{\mathrm{fl}} = 1 - 3\alpha_{\mathrm{fl}}/2 > 0$. For a fixed time horizon $T > 0$ and a mesh a, let $v_a(t) = t + a^2$, and define, with $\alpha = \alpha_{\mathrm{fl}}$,

$$\mathsf{X}^a_\alpha(x) = \sup_{0\le t\le T}\big\{v_a(t)^{\alpha/2}\|x(t)\|_\infty + v_a(t)^{(\alpha+1)/2}\|D_a x(t)\|_\infty\big\},$$
$$\mathsf{Y}^a_\alpha(y) = \sup_{0\le t\le T}\big\{v_a(t)^{\beta_{\mathrm{fl}}}\|y(t)\|_\infty + v_a(t)^{\alpha}\|D_a y(t)\|_\infty\big\}.$$

Here $\|D_a x\|_\infty$ is the maximum over all componentwise differences $D^{\pm}_{a,i}x$ and their shifts occurring in the Wilson-flow equation. For $0 < \sigma_{\mathrm{sm}} < 1 - \alpha_{\mathrm{fl}}$ and $\sigma = \sigma_{\mathrm{sm}}$, set

$$\mathsf{H}^a_\sigma(f) = \sup_{0\le t\le T}\big\{v_a(t)^{\sigma/2}\|P^a_t f\|_\infty + v_a(t)^{(\sigma+1)/2}\|D_a P^a_t f\|_\infty\big\},$$
$$\mathsf{V}^a_\sigma(f) = \sup_{0\le t\le T} v_a(t)^{\sigma/2}\|P^a_t f\|_\infty.$$

For a physical initial potential A_0 on mesh a, the linear heat evolution is $x(t) = P_t^a A_0$, the quadratic Picard term is $y = \mathcal{Q}_a(x, x)$, and together they form the linear and quadratic flow data, measured in the norms X_α^a and Y_α^a.

The weights express heat scaling: $x(t)$ is of size $v_a(t)^{-\alpha/2}$ and $y(t)$ of size $v_a(t)^{1/2-\alpha}$, and each difference costs a further factor $v_a(t)^{-1/2}$. Flow data on different meshes are compared by the cross-mesh comparison (24.11) of Definition 24.3, whose four components are interpolated separately and weighted in time.

Further exponents. The constant C_{CP}, which controls finitely many derivatives of the conditional constrained minimizers and of their propagated contributions, is uniform in J. It is chosen before J, including every enlargement used later in an amplitude bound or a derivative bound, and J is then chosen with $C_{\mathrm{CP}}^2/B \le 1/4$. We write $a_{\mathrm{cap}} = C_{\mathrm{CP}}/B$ for the contraction factor in the amplitude bound (15.20) of the backward reconstruction (Lemma 15.10). The decay exponent at the coarse end of the reconstruction is a different number: $\kappa_{\mathrm{term}} = -\log a_{\mathrm{cap}}/\log B > 1/2$ when $a_{\mathrm{cap}}\sqrt{B} < 1$, and the contraction factor of the propagated contributions of coarser scales is $q_{\mathrm{anc}} = Ba_{\mathrm{cap}}^2 < 1$. A localization exponent is denoted by ξ, not by γ. A tail exponent τ_{tail} is chosen with

$$0 < \tau_{\mathrm{tail}} < \min\{\sigma_{\mathrm{sm}}, 1/2, -\log q_{\mathrm{anc}}/\log B\}.$$

Logarithms along the flow. Let $A = x + y + r$ be the solution on $[0, T]$ of the gauge-damped flow equation on mesh a whose linear and quadratic flow data satisfy $\mathsf{X}_\alpha^a(x) + \mathsf{Y}_\alpha^a(y) \le R$, with the remainder r bounded as in Theorem 24.5. Then

$$a \sup_{t \le T} |A(t)| \le C_R a^{1-\alpha_{\mathrm{fl}}} + C_R a^{2-2\alpha_{\mathrm{fl}}} + C_R a (T + a^2)^{1-3\alpha_{\mathrm{fl}}/2} \longrightarrow 0 \qquad (a \to 0),$$

at fixed T. The first two terms come from $a\|x(t)\|_\infty \le Ra\, v_a(t)^{-\alpha/2}$ and $a\|y(t)\|_\infty \le Ra\, v_a(t)^{1/2-\alpha}$, evaluated at $t = 0$. The last term increases with time; its maximum is $O_R(a)$, attained at $t = T$, and does not keep the mesh power of its value at $t = 0$. The smooth solutions of the continuum flow are constructed from deterministic results for the flow with smooth data and from heat-kernel calculus [42, Theorem 1.2 and Lemma 1.3, p. 3; Lemmas C.2–C.3, pp. 55–56; Corollary C.6, p. 58; Lemmas C.12–C.13, p. 61]. We do not use the theorem of Cao and Chatterjee on random initial data as a hypothesis on Wilson fields.

4.7 The heat-smoothed circulation

The short-distance results are proved for the following linear functional of the reconstructed field.

Definition 4.9. Fix $L > 0$, $\gamma > 0$, $g_3^2 = \gamma/L$, $\sigma > 0$, smooth loops $\ell_1, \dots, \ell_m$ in $\mathbb{R}^3$, and the index s of the comparison lattice, and let $K \geq s$ and $N_K = n_* B^K$. Let $\widehat{\theta}_{K-s}$ be the link logarithm, in the gauge fixed by the reconstruction maps $\widehat{\mathscr{R}}$, of the configuration on Λ_{N_K} reconstructed from the lattice of side n_s by $K - s$ RG steps, and let $A^{K,s} = a_{N_K}^{-1} \widehat{\theta}_{K-s}$ be the corresponding physical initial potential. For $a_{N_K} \leq \varepsilon$ and $x \in \mathbb{T}_L^3$, let $\ell_{i,K,\varepsilon}$ be the closed lattice path in Λ_{N_K} approximating $x + \varepsilon \ell_i$ that is specified in Definition 31.1. The heat-smoothed circulation is

$$I_i^{K,s}(\varepsilon, x) = \frac{a_{N_K}}{\sqrt{g_3^2 \varepsilon}} \sum_{e \in \ell_{i,K,\varepsilon}} \operatorname{sgn}(e) \big(P^{a_{N_K}}_{\sigma\varepsilon^2} A^{K,s}\big)_{\operatorname{dir}(e)} \big(x_{\text{positive}}(e)\big).$$

Here $\operatorname{dir}(e)$ is the direction of the link e, $x_{\text{positive}}(e)$ is the starting point of its positive orientation, and $\operatorname{sgn}(e) = 1$ if e is traversed in the positive direction and -1 otherwise; a backward edge thus contributes the negative of the value on its positive edge.

In this range, the paths $\ell_{i,K,\varepsilon}$ have physical length and diameter at most $C_\ell \varepsilon$, and they have zero winding for $\varepsilon \leq \varepsilon_{\text{loop}}$ of Definition 31.1. The functional annihilates periodic lattice gradients. Indeed, if $A_j(y) = D^+_{a,j} f(y)$ with $a = a_{N_K}$, then $P_t^a A = D_a^+ P_t^a f$. Each traversed edge contributes $\pm(P_t^a f(\text{end}) - P_t^a f(\text{start}))/a$, with the sign making it the increment along the path, and these increments sum to zero around a closed path. The zero winding also removes the pairing with constant one-forms. The scalar heat semigroup acts on the field at the finest mesh. It is not the nonlinear Wilson flow, and the comparison with flowed Wilson loops is a separate result (Lemma 32.11). The normalization $1/\sqrt{g_3^2 \varepsilon}$ is the scale of the circulation of the Gaussian free field after heat smoothing at time $\sigma\varepsilon^2$; the analysis is in Sections 31 and 32.

4.8 Gauge orbits and the trajectory space

Definition 4.10. A smooth connection is an element A of $\mathcal{A}_L = C^\infty(\mathbb{T}_L^3; \mathfrak{su}(2)^3)$. We write

$$F_{ij}(A) = \partial_i A_j - \partial_j A_i + [A_i, A_j], \qquad D_i = \partial_i + [A_i, \cdot\,].$$

A gauge transformation $k \in \mathcal{G}_L = C^\infty(\mathbb{T}_L^3; \mathrm{SU}(2))$ acts by $A^k = kAk^{-1} - (dk)k^{-1}$. The parallel transport along a piecewise C^1 path ℓ is the solution of $H'(s) = H(s) A(\ell(s)) \dot{\ell}(s)$, $H(0) = I$; it transforms by $H^k(s) = k(\ell(0)) H(s) k(\ell(s))^{-1}$. The Yang–Mills heat flow is $\partial_t A_i = \sum_j D_j F_{ji}$, and $(\Phi_t)_{t \geq 0}$ is the semigroup it induces on $\mathcal{A}_L/\mathcal{G}_L$. The trajectory space $\mathcal{X}_L$, the regularized Wilson loops $W_{\ell,\chi,t}$, and the Wilson topology $\mathcal{T}_W$ are as in Definition 2.2. Let $(\ell_j, t_j)_{j \geq 1}$ be the fixed enumeration used in Theorem 29.3, formed from its countable dense family of piecewise C^1 loops, which contains loops in every winding class, and the times in $L^2\mathbb{Q}_{>0}$. Set $w_j(X) = \operatorname{tr} \operatorname{Hol}_{\ell_j} X(t_j)/2$ and

$$d_c(X, Y) = \sum_{j \geq 1} 2^{-j} |w_j(X) - w_j(Y)|, \qquad X, Y \in \mathcal{X}_L.$$

Differentiating $k(\ell(0))H(s)k(\ell(s))^{-1}$ shows that it solves the transport equation for A^k, which confirms the sign of the gauge action and the order of the transport. Our gauge transformation k is the inverse of the gauge variable used by Cao and Chatterjee [43, (1.1), p. 2]. The trajectory space follows the proposal of Charalambous and Gross [47] and is a version of the state space of Cao and Chatterjee [43]. By definition d_c is a pseudometric. In Theorem 29.3 we prove that it separates points of $\mathcal{X}_L$, that both topologies are separable, and that they have the same Borel σ-algebra. We do not assert that $(\mathcal{X}_L, d_c)$ is complete.

The Yang–Mills energy is

$$E_L(A) = \sum_{i,j} \int_{\mathbb{T}^3_L} -\operatorname{tr}\big(F_{ij}(A)^2\big)\, dx = \sum_{i<j} \int_{\mathbb{T}^3_L} |F_{ij}(A)|^2\, dx,$$

since $-\operatorname{tr}(F_{ij}^2) = |F_{ij}|^2/2$. It carries no coupling factor; for smooth configurations the exponent of the Wilson measure tends to $E_L/(2g_3^2)$ (Remark 4.2). Under the rescaling to the unit torus, $\widetilde{A}(y) = LA(Ly)$, the flow time becomes t/L^2 and $E_1(\widetilde{A}) = LE_L(A)$.

The maps π_K of Theorem 29.19 are Borel maps on the lattice configuration space, invariant under all periodic lattice gauge transformations. They are chosen by a deterministic diagonal sequence of matching levels, fixed before any configuration is sampled. Their Wilson-loop coordinates approximate the flowed lattice Wilson loops in the sense of Theorem 2.3(3); the maps do not embed lattice holonomies exactly. We use the compactness of energy sublevel sets and the continuity of the flow semigroup under the hypotheses of the cited results, which concern the quotient of Sobolev connections in H^1 by gauge transformations in $W^{2,2}$ [43, Theorem 3.51, pp. 35–36; Lemma 3.68, p. 40].

4.9 Observables near commuting holonomies

Definition 4.11. Fix $\theta > 0$ and the flow time $t = \theta L^2$. Let $H_1, H_2, H_3 \in \mathrm{SU}(2)$ be the holonomies of $X(t)$ around the three coordinate cycles, based at a common point. Their seven trace coordinates are the normalized traces of H_i, of H_iH_j for $i < j$, and of $H_1H_2H_3$. In a fixed neighborhood of the identity write $X = (x_i)_{i=1}^3 = (\operatorname{Log} H_i)_{i=1}^3 \in \mathfrak{su}(2)^3 \simeq \mathbb{R}^9$. The cone of commuting triples is

$$\mathcal{M} = \{(y_i) = (\vartheta_i u)_{i=1}^3 : \boldsymbol{\vartheta} \in \mathbb{R}^3,\ u \in S^2\}, \qquad (\boldsymbol{\vartheta}, u) \sim (-\boldsymbol{\vartheta}, -u).$$

Fix $0 < r_0 < r_1 < \pi$, and let $\mathsf{P} = \{(\vartheta_i u)_i \in \mathcal{M} : r_0 \le |\boldsymbol{\vartheta}| \le r_1\}$, a compact subset of the regular part of $\mathcal{M}$ with nonempty relative interior. For $y = (\vartheta_i u)_i \in \mathsf{P}$, the normal space to $\mathcal{M}$ at y is

$$\mathcal{N}_y = \Big\{ b = (b_1, b_2, b_3) : b_i \perp u,\ \sum_i \vartheta_i b_i = 0 \Big\}, \qquad \dim \mathcal{N}_y = 4,$$

and on a fixed tubular neighborhood of P we use the unique coordinates $X = y + b$ with $y \in \mathcal{M}$ and $b \in \mathcal{N}_y$. Let $\Pi_{\boldsymbol{\vartheta}} = I - \boldsymbol{\vartheta}\boldsymbol{\vartheta}^{\mathsf{T}}/|\boldsymbol{\vartheta}|^2$. In unit-torus variables put, for $m \in \mathbb{Z}^3$,

$$k_m = 2\pi m + \boldsymbol{\vartheta}, \qquad \lambda_m = |k_m|^2, \qquad R_i(m, \boldsymbol{\vartheta}) = \begin{cases} 1, & m_i = 0, \\ \vartheta_i/(2\pi m_i + \vartheta_i), & m_i \neq 0, \end{cases}$$

$$d_i(\boldsymbol{\vartheta}, \theta) = \sum_{m \in \mathbb{Z}^3} \frac{e^{-2\theta\lambda_m}}{\lambda_m} R_i(m, \boldsymbol{\vartheta})^2, \qquad D = \operatorname{diag}(d_1, d_2, d_3),$$

and define the normal covariance

$$\Sigma_\theta(y) = \big(\Pi_{\boldsymbol{\vartheta}} D \Pi_{\boldsymbol{\vartheta}}\big)\big|_{\boldsymbol{\vartheta}^\perp} \otimes I_{u^\perp}. \tag{4.12}$$

Let $0 \le \psi_0 \le 1$ be a smooth function on P, invariant under simultaneous conjugation and compactly supported in the relative interior of P, that vanishes on a nonempty open set and is bounded below by a positive constant on another nonempty open set. Multiply it by a fixed smooth function χ of the normal variable, invariant under simultaneous conjugation, supported in the tube, with $0 \le \chi \le 1$ and $\chi = 1$ near $b = 0$ (for example a function of $|b|$), to obtain $\psi(y + b) = \psi_0(y)\chi(b)$. Fix a larger tube containing the support of ψ and the supports of all tangential insertions used later, and set

$$q_*(y + b) = \frac{1}{2}\, b^{\mathsf{T}} \Sigma_\theta(y)^{-1} b$$

throughout this larger tube, with a fixed bounded, continuous, nonnegative extension outside it that is invariant under simultaneous conjugation. All these functions are invariant under simultaneous conjugation, and they are extended by zero outside the identity chart, so that they define invariant functions of (H_1, H_2, H_3). The observables $B_{\mathrm{obs}} = \psi$ and $A_{\mathrm{obs}} = \psi q_*$ are fixed bounded functions, and the normalized regression residual variance $\mathscr{C}_\gamma$ is defined from them as in Section 2.7.

The tangent vectors to $\mathcal{M}$ at a regular point are $(\delta\vartheta_i u + \vartheta_i \delta u)_i$ with $\delta u \perp u$; their orthogonal complement is $\mathcal{N}_y$. Hence the regular part of the cone has dimension five and its normal space has dimension four. By the tubular neighborhood theorem, the coordinates $X = y + b$ near P are unique; the tip of the cone is excluded. The matrix $\Sigma_\theta(y)$ is the covariance, divided by γ, of the linearized response of the flowed logarithms at the flat connection y. It is derived and compared with the lattice in Theorem 33.11; its definition does not assert that the nonlinear law of b is Gaussian. The regression coefficient $\operatorname{Cov}(A_{\mathrm{obs}}, B_{\mathrm{obs}})/\operatorname{Var}(B_{\mathrm{obs}})$ in $\mathscr{C}_\gamma$ depends on the measure, and $\mathscr{C}_\gamma$ is defined for measures under which $\operatorname{Var}(B_{\mathrm{obs}}) > 0$ and $\mathbb{E}B_{\mathrm{obs}}^2 > 0$. We prescribe q_* on the larger tube, and not only on the support of ψ, so that q_* equals the prescribed quadratic form on the support of every insertion of Theorem 2.10(2).

Remark 4.12. The same formula (4.12) defines $\Sigma_\theta(y)$ throughout the punctured cone $0 < |\boldsymbol{\vartheta}| < \pi$. The series defining d_i converges uniformly on each compact regular patch such as P, and $\Sigma_\theta(y)$ is positive definite there. Indeed, $|\vartheta_i| \le |\boldsymbol{\vartheta}| < \pi$, so $|k_m| \ge 2\pi|m| - \pi \ge \pi|m|$ for $m \ne 0$, while $k_0 = \boldsymbol{\vartheta} \ne 0$. For $m_i \ne 0$ we have $|2\pi m_i + \vartheta_i| > \pi > |\vartheta_i|$, so $|R_i| \le 1$. Since $\theta > 0$, the sum of $e^{-2\theta\lambda_m}/\lambda_m$ over $m \ne 0$ converges uniformly. The summand with $m = 0$ gives $d_i \ge e^{-2\theta|\boldsymbol{\vartheta}|^2}|\boldsymbol{\vartheta}|^{-2} > 0$ for each i, hence $\Pi_{\boldsymbol{\vartheta}} D \Pi_{\boldsymbol{\vartheta}} \ge e^{-2\theta|\boldsymbol{\vartheta}|^2}|\boldsymbol{\vartheta}|^{-2}$ on $\boldsymbol{\vartheta}^\perp$. These unscaled covariance bounds are uniform only on compact regular patches away from the tip. As $|\boldsymbol{\vartheta}| \downarrow 0$, Theorem 33.11(4) instead gives $\Sigma_\theta(y) = |\boldsymbol{\vartheta}|^{-2} I_{\mathcal{N}_y} + O_\theta(1)$, uniformly in direction. The hypothesis $\theta > 0$ is needed: at $\theta = 0$, the summands with $m_i = 0$ have $R_i = 1$, and the sum of $1/|2\pi m + \boldsymbol{\vartheta}|^2$ over this plane diverges logarithmically.

Central triples. The central triples are the triples $(\pm I, \pm I, \pm I)$ of central elements; they are points of the moduli space of flat connections. Near them we set $r_\gamma = \gamma^{1/4}$ and define the constant-mode quartic form

$$Q(Y) = \frac{1}{2}\sum_{i<j}|y_i \times y_j|^2, \qquad Y = (y_1, y_2, y_3) \in \mathbb{R}^9.$$

The quartic reference measure $m_Q = e^{-Q(Y)}\,dY$ is locally finite and has infinite total mass; it is never normalized globally. The divergence of the corresponding reduced SU(2) integral is known from the matrix-integral literature [7, 122]; we give a proof in our coordinates in Sections 33 and 36. Test functions are compactly supported and invariant under the simultaneous adjoint rotation of the three vectors y_i. A radial cutoff is a function $\chi(|Y|)$ with χ smooth, nonnegative, not identically zero, compactly supported, and constant near zero; the comparison of the dilation statistic with the reference measures (Theorem 36.19) additionally requires $\chi(0) > 0$. Every observable expressed in logarithmic coordinates vanishes outside a fixed identity chart. We write $\rho_{\mathrm{rad}} = |X|/r_\gamma$ for the rescaled radial variable, to distinguish it from the analytic radius ρ.

Definition 4.13. The following comparison measures are distinct constructions. Each depends on the fixed choices stated in it, and the results about them hold for every such fixed choice.

1. *Abelian Maxwell measure with Haar cycle holonomies.* Embed a fixed $U(1)$ subgroup in SU(2). Take the transverse Maxwell Gaussian field of the nonconstant modes with values in this subgroup, with a fixed normalization of its covariance, and independent uniformly distributed constant phases in $(\mathbb{R}/4\pi\mathbb{Z})^3$. Smooth at the fixed observation time and take the Abelian holonomies around the three coordinate cycles. All these triples commute. A common random conjugation does not change invariant test functions.
2. *Gaussian holonomies with independent Haar factors.* Let a be a centered Gaussian field with three colors and covariance γC, for a fixed covariance C, and let S_t be a fixed

smoothing. Let U_1^0, U_2^0, U_3^0 be independent Haar distributed elements of $\mathrm{SU}(2)$, independent of a, and set $H_i = U_i^0 \operatorname{Hol}_i(\mathsf{S}_t a)$. The triple (H_1, H_2, H_3) has the product Haar law at every coupling.

3. *Gaussian holonomies with Gaussian constant modes.* Set $H_i = \operatorname{Hol}_i(\mathsf{S}_t a)$, without the Haar factors, where a includes Gaussian constant modes with covariance proportional to γ. Assume that the scalar color projections $(x, i, v) \mapsto \langle v, (\mathsf{S}_t a)_i(x)\rangle / \sqrt{\gamma}$, indexed by $\mathbb{T}_L^3 \times \{1,2,3\} \times S^2$, have a centered separable Gaussian realization with continuous paths and bounded variance, whose law does not depend on γ. Its supremum is then measurable and almost surely finite. The resulting Gaussian supremum estimate is Lemma 35.16, and it is used only for this measure.
4. *Conditional normal Gaussian measure.* On the patch P of Definition 4.11, choose a fixed smooth positive probability density for y that charges two nonempty open sets O_0 and O_+, invariant under simultaneous conjugation, with $\psi_0 = 0$ on O_0 and ψ_0 bounded below by a positive constant on O_+, as in Definition 35.1. Conditionally on y, let $b \sim N(0, \gamma\Sigma_\theta(y))$ on $\mathcal{N}_y$, and push the joint law forward by $X = y + b$. A fixed tube cutoff equal to one near the zero section, followed by normalization, changes bounded moments by exponentially small amounts. For these choices $\mathscr{C}_\gamma$ is defined for all sufficiently small γ and has leading term $2\gamma^2$ (Lemma 35.17).
5. *Mixture of normalized conditional normal Gaussian measures.* Let y have a fixed probability density $\alpha(y)\, d\mathrm{vol}_{\mathcal{M}}(y)$ on a small neighborhood of the tip of $\mathcal{M}$, with $\alpha \geq 0$ continuous and strictly positive at the tip. Given $y \neq 0$, let $b \sim N(0, \gamma\Sigma_\theta(y))$ be the normalized Gaussian measure on $\mathcal{N}_y$, and push the joint law forward by $X = y + b$. Assume that $|y|^2\Sigma_\theta(y) \to I_{\mathcal{N}_y}$ as $y \to 0$, uniformly in the direction of y, with the corresponding uniform upper and lower bounds on $|y|^2\Sigma_\theta(y)$; here $|y| = |\boldsymbol{\vartheta}|$. The associated weighted normal-bundle measure is

$$d\mu_{13} \;\propto\; |y|^4 e^{-|y|^2|b|^2/2}\, d\mathrm{vol}_{\mathcal{M}}(y)\, db.$$

Expectations of the form $\langle\cdot\rangle_{\chi,\mu_{13}}$ are normalized by $\int \chi(|y+b|)\, d\mu_{13}$.

The factor $|y|^4$ in μ_{13} comes from the normalization of the four-dimensional Gaussian measure: $\det(\gamma\Sigma_\theta(y))^{-1/2}$ is of order $\gamma^{-2}|y|^4$, since $|y|^2\Sigma_\theta(y) \to I$. Under the scaling $(y, b) = \gamma^{1/4}(Y, B)$, the tangential volume, the normal volume, and this normalization contribute the factor $\gamma^{5/4}$. After division by $\gamma^{5/4}$, the joint densities converge for $Y \neq 0$ to $(2\pi)^{-2}\alpha(0)|Y|^4 e^{-|Y|^2|B|^2/2}$ relative to $d\mathrm{vol}_{\mathcal{M}}(Y)\, dB$. Since $y \perp b$, we have $|y+b|^2 = |y|^2 + |b|^2$, and the uniform covariance bounds give dominated convergence on compact sets of outputs $Y + B$; this is the scaling limit of Theorem 36.19. The weighted volume element has dilation degree $5 + 4 + 4 = 13$, which explains the index of μ_{13}. The mixture differs from the quartic reference measure m_Q, and omitting the conditional normalization would change the comparison. The value of α at $y = 0$ is irrelevant, since $\{y = 0\}$ is a null set.

4.10 Dependence of constants

Each quantitative statement in this paper either lists the parameters on which its constants depend, or refers to one of the items of the following definition, with any additional fixed parameters listed in the statement. In quantifier form, all fixed data precede the choice of the constants, and every quantity said to vary follows that choice.

Definition 4.14. A constant has the dependence described in item (k) if the corresponding statement holds as follows.

(1) *Finite-dimensional estimates.* Algebraic identities hold in every finite dimension stated. Constants in analytic estimates for products of a bounded number of group elements depend only on the number of factors and on the distance to the boundary of the logarithmic chart. Estimates for measures on products of convex balls depend only on the stated bounds on the Hessian of the potential in the free coordinates and on the weighted interaction, and not on the dimension or on the radii of the individual balls.

(2) *Classical estimates on the torus.* Constants depend on M, on the fixed larger physical and analytic domains, on the spatial decay rates, and on fixed upper bounds for the norms involved. Bounds stated uniformly in depth are independent of the lattice side, of the preceding classical depth, and of the coarse field in the stated domain. The explicitly named constants controlling finitely many derivatives, and the Taylor coefficient majorant Z of the aligned zero-fluctuation minimizer in Lemma 11.9, together with its radius, its coefficient constant, and its spatial rate, are uniform in J: they are fixed before J is chosen. Joint coefficient bounds in the coarse field and the free coordinates, and coefficient bounds for the gauge transformations of the reconstruction maps, are bounds at fixed J; for the reconstruction maps, only the specified finitely many derivatives of the zero-fluctuation reconstruction are uniform in J.

(3) *Local estimates on block regions.* Each constant C_J, c_J, λ_J, local radius, or upper bound for an output norm depends only on M and J, on the fixed larger initial and analytic domains and their radius ratios, on the spatial and boundary decay rates and the stated margins, on a bound on the number of indices per lattice cell, and on fixed upper bounds for the norms on the torus and of the perturbations. It is independent of L, γ, m, N, K, d_0, of the number of elapsed RG steps, of the shape and topology of the block region, of the numbers of connected components of the block region and of the fixed complement, and of the values of the real frozen coordinates. Where a theorem says so, it is also independent of the later boundary widths, widths of removed boundary layers, separation widths, and enlarged-cell widths; such a constant is evaluated on the larger domain before these widths are chosen. This

applies only to the theorems on block regions proved in this paper; it does not extend an estimate on the full torus to arbitrary regions.

(4) *Estimates for the small-field failure probability.* Constants may depend on L, on the coupling ceiling γ_0 and the parameter tuple Θ of Theorem 23.2, on M and J, and on the fixed allowed number of averaging steps and enlargement of the stopping scale, but not on the coupling γ, on the lattice side, on the number of RG steps on the fine side or the preceding depth, or on the age, geometry, or widths of a large-field decomposition. The constant $C_{u,q}$ may in addition depend on fixed u and q. In a statement uniform in $0 < \gamma \le \Gamma$, these constants and the minimal admissible side may in addition depend on the fixed ceiling $\Gamma < \infty$; they are independent of $\gamma \in (0, \Gamma]$. This does not change the dependence of the constants in items (3) and (7).

(5) *Short-distance estimates.* Constants may depend on fixed L, γ, M, J, on the family of loops, on σ, and on the fixed index s of the comparison lattice. They are uniform in the remaining number of RG steps on the fine side, in the point x, and in the allowed lattice offsets, in the range $a \le \varepsilon \le \varepsilon_0(s)$. No uniformity as $s \to \infty$ or as $\gamma \downarrow 0$ is asserted.

(6) *Weak-coupling estimates.* Constants may depend on fixed n, L, θ, M, J, on the compact subsets of the regular part of $\mathcal{M}$, on the test functions, and on the larger margins, all fixed before γ. They are independent of the number of RG steps on the fine side, of the preceding depth, and of the coupling below the ceiling. Constants C_m attached to a fixed number m of RG steps may grow with m; constants in tail bounds at the coupling-dependent scale may not. These are estimates on the full torus.

(7) *Flow estimates.* Deterministic estimates for the flow on a set where the linear and quadratic flow data are bounded by R may depend on R, on positive waiting times, on the time horizon, and on the margins within the chosen coordinate charts. Constants in stochastic estimates and in convergence statements at fixed coupling, and the deterministic diagonal sequence of matching levels, may also depend on that coupling, on the coarse side, and on the observables. No rate that is uniform as a restart time tends to zero, or as the coupling varies, is implied.

We claim no useful bound as $J \to \infty$ for the joint constants at fixed J or for the constants on block regions; this concerns only these fixed-J constants, and not the constants in item (2) that are stated to be uniform in J. The order in which the parameters are chosen is established in Theorem 23.2: first the constants uniform in J, then J, then the local constants at fixed J, and then the geometric widths. In particular, J is not enlarged after the constants on block regions at fixed J have been computed.

Probabilistic conventions. For a random vector in a finite-dimensional space, the bounded-Lipschitz distance uses the test functions bounded by one with Lipschitz constant at most one with respect to the Euclidean norm. For probability measures we

use $\|\mu - \nu\|_{\mathrm{TV}} = \sup_A |\mu(A) - \nu(A)|$, so that $|\mathbb{E}_\mu f - \mathbb{E}_\nu f| \le 2\|f\|_\infty \|\mu - \nu\|_{\mathrm{TV}}$ for every bounded measurable f. The covariance $\mathrm{Cov}(F, G) = \mathbb{E}(FG) - \mathbb{E}F\,\mathbb{E}G$ is complex bilinear when complex sources are used. Whenever a real minimum or a positive density is asserted, the holomorphic continuations involved take real values at real arguments, that is, $f(\overline{z}) = \overline{f(z)}$ in the chosen coordinates.

PART II

Lattice gauge measures and block averaging

SECTION 5

Finite-dimensional integration and concentration inequalities

This section collects the finite-dimensional tools used throughout the renormalization group analysis. Each statement concerns a fixed finite lattice, a finite set of coordinates, or a fixed finite-dimensional measure; where we say so, the constants do not depend on the dimension. The results fall into three groups. The first group concerns gauge integration and Gaussian algebra. Lemma 5.1 records the normalization of Haar measure on $\mathrm{SU}(2)$, exact integration in a spanning-tree gauge, and analytic estimates for products of group elements near the identity, and Lemma 5.2 decomposes closed periodic one-cochains and eliminates positive definite quadratic blocks. The second group is the calculus of the analytic coefficient norms of Definition 4.6: compositions, localized inverses and square roots, logarithms of determinants, and implicit functions (Lemma 5.3). The third group is probabilistic. Theorem 5.4 proves logarithmic Sobolev, covariance, and quadratic concentration inequalities for uniformly log-concave measures on products of balls, Lemma 5.5 controls conditioning, parameter derivatives, and gauge-invariance identities, and Lemma 5.6 shows that seven traces determine a triple of $\mathrm{SU}(2)$ matrices up to simultaneous conjugation.

The only inputs are the definitions of Section 4: the Wilson measure and the normalization of the Lie algebra (Definition 4.1), the coefficient norms (Definition 4.6), and the convention on constants (Definition 4.14). Spanning-tree gauges and the estimates for products of group elements enter the construction of the block averaging map and the gauge fixing in Sections 6 and 7, and the local Wilson flow in Section 24. The Gaussian elimination is used for the linearized theory in Section 8. The coefficient calculus underlies all analytic estimates in Sections 6, 8 to 10, 12, and 18. The concentration theorem is the probabilistic input for the conditional fluctuation kernels in Sections 13, 16, 17, 20, 26 to 28, and 34. The conditioning lemma is used in Sections 15, 28, 29, and 32, and the seven traces in Sections 33, 35, and 36.

5.1 Haar measure, spanning-tree gauges, and products of group elements

The following facts are standard for the compact group $\mathrm{SU}(2)$ and for lattice gauge theory; we record them with the normalizations of Section 4.1, which fix the constants

used later. Gauge fixing along a spanning tree is the maximal-tree gauge, which we call the spanning-tree gauge. Bałaban used such gauges systematically in the renormalization group context [14].

Lemma 5.1. *Let $\mathcal{K}$ be a finite oriented cubical complex, with vertex set $\mathcal{V}$, set of positive links $\mathcal{E}$, and set of positive plaquettes $\mathcal{P}$. Then the following statements hold.*

1. *For every $\beta \geq 0$,*

$$Z_{\mathcal{K}}(\beta) = \int e^{-\beta \sum_{p \in \mathcal{P}} q(U_p)} \prod_{e \in \mathcal{E}} dH(U_e) \quad \text{satisfies} \quad e^{-2\beta|\mathcal{P}|} \leq Z_{\mathcal{K}}(\beta) \leq 1,$$

   and the normalized density defines a gauge-invariant Borel probability measure.

2. *The measure H is the Haar probability measure, and for $X = x^a \tau_a$ with $0 \leq r = |X| < 2\pi$,*

$$\frac{1}{2} \operatorname{tr} e^X = \cos(r/2), \qquad dH(e^X) = c_{\mathrm{Haar}} j_H(X)\, d^3x,$$

   where $c_{\mathrm{Haar}} = 1/(16\pi^2)$ and $j_H(X) = \operatorname{sinc}^2(r/2)$. For $0 \leq u \leq 2\pi$ and $U, V \in \mathrm{SU}(2)$,

$$H\{\rho_G < u\} = \frac{u - \sin u}{2\pi}, \qquad \frac{\rho_G(U)^2}{2\pi^2} \leq q(U) \leq \frac{\rho_G(U)^2}{8},$$

   and $\rho_G(UV) \leq \rho_G(U) + \rho_G(V)$. Further, $q(e^X) \geq (r^2/8)(1 - r^2/48)$ for $0 \leq r \leq 2\pi$, and on every ball $\{|X| \leq r_0\}$ with $r_0 < 2\pi$ we have $q(e^X) = r^2/8 + O(r^4)$ and $\log j_H(X) = -r^2/12 + O(r^4)$, with constants depending only on r_0. For $0 < u \leq 1$,

$$c_{\mathrm{ball}} u^3 \leq H\{\rho_G < u\} \leq \frac{u^3}{12\pi}, \qquad c_{\mathrm{ball}} = \frac{\operatorname{sinc}^2(1/2)}{12\pi} \geq \frac{1}{16\pi}.$$

3. *Left and right translations preserve Haar measure. For every $k \in \mathrm{SU}(2)$, $\mathrm{Ad}(k) \in \mathrm{SO}(3)$; it preserves the norm of $\mathfrak{su}(2)$, Lebesgue measure, and the balls centered at the origin. The map $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective. These statements concern $\mathrm{SU}(2)$, not conjugation by elements of its complexification $\mathrm{SL}(2, \mathbb{C})$.*

4. *Choose a spanning tree in each connected component of $\mathcal{K}$, with one root per component. Let h_x be the transport from the root to x along the tree, and set $V_{xy} = h_x U_{xy} h_y^{-1}$. The map*

$$U \longmapsto \big((h_x)_{x \text{ not a root}}, (V_e)_{e \text{ not in the forest}}\big)$$

   is a smooth bijection, and it carries product Haar measure to product Haar measure; $V_e = I$ on the forest. Hence, for an integrand invariant under the gauge transformations equal to I at the tree roots, integration over the tree variables contributes a factor one and leaves the same integrand expressed in the remaining variables V. If $\mathcal{K}$ is connected, there remain $|\mathcal{E}| - |\mathcal{V}| + 1$ independent group variables, together with the simultaneous conjugation at the root, and no stabilizer volume is divided out. The same statements hold for a partition of $\mathcal{K}$ into blocks, each equipped with a rooted spanning tree, with gauge transformations equal to I at every tree root.

5. *Fix a length ℓ, a distance from the singularity of the logarithm (the eigenvalue -1), and a smaller complex coordinate polydisc inside the larger domain. Every map given by a product of at most ℓ factors of the form $e^{\pm X_j}$, and every logarithm of such a product relative to a reference product on the specified chart, has absolute Taylor derivatives of order s bounded by $s!C^s$ for all $s \geq 1$, where C depends only on ℓ, these domains, and the norm of $\mathfrak{su}(2)$. Factors in $\mathrm{SU}(2)$ may be included if the relative products keep the same distance from the singularity. For small complex $u, v, a \in \mathfrak{su}(2)_{\mathbb{C}}$ and $0 \leq \epsilon \leq 1$,*

$$\epsilon^{-1} \operatorname{Log}(e^u e^{\epsilon a} e^{-u-\epsilon v}) = \operatorname{Ad}(e^u)a - \int_0^1 \operatorname{Ad}(e^{tu})v \, dt + \epsilon \mathscr{R}_\epsilon(u, v, a),$$

where the left side at $\epsilon = 0$ is defined by analytic continuation, and the remainder $\mathscr{R}_\epsilon$ is holomorphic, has bounds of the same type, vanishes to order at least two in (a, v), and satisfies $\mathscr{R}_\epsilon(u, 0, a) = 0$.

All constants have the dependence described in item (1) of Definition 4.14; for a fixed length ℓ they do not depend on the number of cells, on L, on the coupling, or on the depth.

Proof. Quaternions and Haar measure. Let $t_a = 2\tau_a$. Multiplying the Pauli matrices, we find

$$t_a t_b = -\delta_{ab} I + \epsilon_{abc} t_c, \qquad (aI + v \cdot t)(bI + w \cdot t) = (ab - v \cdot w)I + (aw + bv + v \times w) \cdot t. \tag{5.1}$$

Then $\mathrm{SU}(2)$ is the unit sphere $\{a^2 + |v|^2 = 1\}$ in $\mathbb{R}^4$, and inversion is $(a, v) \mapsto (a, -v)$. Left and right multiplications by a unit quaternion are linear maps of $\mathbb{R}^4$ preserving the norm, so the normalized surface measure on the sphere is left and right invariant. It is the unique invariant probability measure. Indeed, if H_1, H_2 are two such measures, integrating a continuous function $f(uv)$ against $H_1(du)H_2(dv)$ in the two orders gives $H_1(f) = H_2(f)$. Equality on continuous functions identifies the Borel measures, since continuous approximations of the indicators of closed and open sets give equality on these sets, and hence on the σ-algebra they generate.

Since $(x \cdot \tau)^2 = -r^2 I/4$, we have $e^{x \cdot \tau} = \cos(r/2)I + \sin(r/2)(x/r) \cdot t$, with the value at $r = 0$ obtained by continuity; in particular $\operatorname{tr} e^X/2 = \cos(r/2)$. In the polar coordinates $a = \cos\vartheta$, $v = \sin\vartheta\,\omega$, $0 < \vartheta < \pi$, the area element of the sphere is $\sin^2\vartheta \, d\vartheta \, d\omega$, where ω ranges over the unit two-sphere of area 4π; the total area is $2\pi^2$. Substituting $r = 2\vartheta$ and $d^3x = r^2 dr \, d\omega$, we obtain

$$\frac{\sin^2(r/2)}{4\pi^2 r^2} \, d^3x = \frac{1}{16\pi^2} \operatorname{sinc}^2(r/2) \, d^3x.$$

The omitted point $-I$ has measure zero. The radial density is $\pi^{-1}\sin^2(r/2)\,dr$, and integrating it from 0 to u, we obtain the formula for $H\{\rho_G < u\}$, including $u = 0$ and $u = 2\pi$.

The distance. The quantity r is twice the round-sphere distance from I. By the exponential formula, every logarithm of $U = aI + v \cdot t$ has norm at least $2 \arccos a$, and

the logarithm displayed above attains this value. Great-circle arcs are shortest: for a curve $z(s)$ on the sphere, the derivative of its polar angle has absolute value at most $|z'(s)|$, as one sees by differentiating its scalar product with the initial point. Left multiplication preserves the round metric. Concatenating a shortest arc from I to U with the left translate by U of a shortest arc from I to V, we obtain $\rho_G(UV) \le \rho_G(U) + \rho_G(V)$.

The plaquette function. For $r \in [0, 2\pi]$ we have $q(e^X) = 2\sin^2(r/4)$. On $[0, \pi/2]$, the sine lies above its chord by concavity, $\sin x \ge 2x/\pi$, and $\sin x \le x$. These two inequalities imply the quadratic lower and upper bounds. Integrating $\sin x \ge x - x^3/6$, we obtain $1 - \cos x \ge x^2/2 - x^4/24$, and the inequality for the sine follows by integrating $x - \sin x \ge 0$ twice. Setting $x = r/2$ yields the additional lower bound. By Taylor's formula with integral remainder, $|q(e^X) - r^2/8| \le r^4/384$. The function $g(r) = \log \operatorname{sinc}^2(r/2) + r^2/12$ is even and satisfies $g(0) = g''(0) = 0$, as the expansion $\operatorname{sinc}(r/2) = 1 - r^2/24 + O(r^4)$ shows; its odd derivatives at 0 vanish as well. Its fourth derivative is bounded on every interval $[-r_0, r_0]$ with $r_0 < 2\pi$, and by the same remainder formula $|g(r)| \le C(r_0) r^4$. For $0 < u \le 1$, the function $\operatorname{sinc} s$ decreases on $[0, 1/2]$, since $(s\cos s - \sin s)' = -s \sin s \le 0$. Writing $\sin^2(r/2) = (r^2/4)\operatorname{sinc}^2(r/2)$ in the radial density, we obtain the two bounds on $H\{\rho_G < u\}$. Finally $\operatorname{sinc}(1/2) \ge 1 - 1/24 = 23/24$, whose square exceeds $3/4$; this implies $c_{\mathrm{ball}} \ge 1/(16\pi)$.

Partition functions. The function q is continuous on $\mathrm{SU}(2)$ with values in $[0, 2]$. Every plaquette transport is a product of link variables and their inverses, and its trace is invariant under gauge transformations, since the intermediate vertex factors cancel. Under a gauge transformation each link variable undergoes a left and a right translation, so product Haar measure is invariant. The integrand lies between $e^{-2\beta|\mathcal{P}|}$ and 1. Integrating, we obtain the bounds on $Z_{\mathcal{K}}(\beta)$, and dividing by $Z_{\mathcal{K}}(\beta)$ gives a gauge-invariant probability measure. Neither connectedness nor periodicity of $\mathcal{K}$ is needed.

The adjoint representation. Conjugation preserves the inner product $-2\operatorname{tr}(XY)$ and the bracket, so $\mathrm{Ad}(k)$ is orthogonal on $\mathfrak{su}(2)$. Its determinant is $+1$, since $\mathrm{SU}(2)$ is connected and the determinant equals 1 at I. In particular, $\mathrm{Ad}(k)$ preserves the norm, Lebesgue measure, and the balls centered at the origin. For surjectivity, note that $R(\vartheta) = \mathrm{Ad}(e^{\vartheta n \cdot \tau})$ solves $R'v = n \times Rv$; it is the rotation by ϑ about the unit vector n. Since every element of $\mathrm{SO}(3)$ has a fixed unit vector (its nonreal eigenvalues come in conjugate pairs, and the remaining real eigenvalue is 1) and acts on the orthogonal complement as a plane rotation, it is one of these rotations.

Spanning-tree gauges. We orient each forest link from the parent to the child, replacing U_e by U_e^{-1} when the given orientation is opposite; inversion preserves Haar measure. Set $h_o = I$ at every root and recursively $h_y = h_x U_{xy}$. Then $V_{xy} = h_x U_{xy} h_y^{-1} = I$ on the forest. Conversely, arbitrary h_x at the vertices that are not roots and arbitrary V_e off the forest determine U uniquely by $U_{xy} = h_x^{-1} V_{xy} h_y$, with $V_{xy} = I$ on forest links. All these maps are finite products and inverses, hence smooth. We list the vertices in increasing distance

from their roots along the forest. Conditionally on the earlier forest links, $h_y = h_x U_{xy}$ is Haar distributed and independent of all earlier h's. By induction, the joint law of the forest variables is product Haar measure. Conditionally on them, every link off the forest is translated on the left and on the right by fixed matrices. Its Haar factor is therefore unchanged, and these factors remain independent. This proves the product identity, first for products of functions and then by approximation for all integrable functions.

If the integrand is invariant under the gauge transformations equal to I at the roots, substituting the gauge transformation h makes it a function of V alone, and the product Haar integral over h contributes a factor one. A connected complex has $|\mathcal{V}| - 1$ tree links, which gives the stated number of remaining group variables. For a gauge transformation k we have $h_x(U^k) = k_o h_x(U) k_x^{-1}$ and $V_e(U^k) = k_o V_e(U) k_o^{-1}$, which leaves the conjugation at the root. We do not take a quotient by it, and no stabilizer factor appears. Applying the recursion in every block, with the links between blocks among the variables off the forest, proves the version for blocks with rooted spanning trees.

Products of group elements. Matrix exponentials are given by their absolutely convergent series, and inversion on $\mathrm{SL}(2,\mathbb{C})$ is polynomial in the matrix entries. In a complex neighborhood of a compact set of unitary matrices whose eigenvalues stay at a fixed distance from -1, the formula

$$\operatorname{Log} W = \int_0^1 (W - I)\big((1-s)I + sW\big)^{-1}\, ds \tag{5.2}$$

defines a holomorphic logarithm. Indeed, on the compact real set the matrices $(1-s)I + sW$ have smallest singular value bounded below uniformly in $s \in [0,1]$, and half of this bound persists on a fixed smaller complex neighborhood, by the identity for differences of inverses. For diagonal unitary matrices the formula reduces to the scalar principal logarithm, by direct integration; conjugation extends it to all unitary matrices of the set, and the identity $e^{\operatorname{Log} W} = W$ continues to the neighborhood by analytic continuation. On a connected neighborhood of I its trace vanishes, since $\det W = 1$ and the trace is continuous with value zero at I.

A product of at most ℓ factors involves at most 3ℓ scalar coordinates. On the larger polydisc its matrix norm and the norms of the relevant logarithms are bounded by a constant B depending on ℓ and the domains: each exponential is bounded by the exponential of the matrix norm of its exponent, each logarithm by (5.2), and factors in $\mathrm{SU}(2)$ have matrix norm one. If the smaller polydisc has coordinate margin $\delta > 0$, the Cauchy integral in each coordinate bounds a derivative of multi-index α by $B\alpha!\delta^{-|\alpha|}$. To see this, we insert the absolutely convergent power series into the circle integrals; the integral of z^k/z^{j+1} over a circle around zero vanishes for $k \neq j$ and equals $2\pi i$ for $k = j$. Summing all ordered derivatives of total order s introduces at most a factor $(3\ell)^s$, which gives the bound $s!C^s$ for the absolute derivative tensor. Where the branches are

defined, $\operatorname{Log}(kWk^{-1}) = k(\operatorname{Log} W)k^{-1}$ and $\operatorname{Log}(W^{-1}) = -\operatorname{Log} W$, by the same formula and analytic continuation.

The expansion with a removable singularity. Let $B(\epsilon) = \operatorname{Log}(e^{u}e^{\epsilon a}e^{-u-\epsilon v})$. For small fixed complex u, v, a it is holomorphic in $|\epsilon| < 1 + \delta_0$, with uniform bounds, and $B(0) = 0$. Differentiating the exponential series, we obtain

$$\frac{d}{d\epsilon} e^{-u-\epsilon v}\Big|_{\epsilon=0} = -\int_0^1 e^{-(1-t)u} v e^{-tu}\, dt.$$

Since the derivative of Log at I is the identity, $B'(0) = \operatorname{Ad}(e^{u})a - \int_0^1 \operatorname{Ad}(e^{tu})v\, dt$. The product depends on a, v only through $(\epsilon a, \epsilon v)$, and the coefficient of ϵ^k in B is therefore homogeneous of degree k in (a, v). Hence

$$\mathscr{R}_\epsilon = \int_0^1 (1-t) B''(t\epsilon)\, dt$$

is holomorphic, vanishes to order at least two in (a, v), and satisfies the bounds just proved. Using Taylor's formula along the segment $t(a, v)$ and the bound on second derivatives, we find on a fixed smaller complex ball

$$\|\mathscr{R}_\epsilon(u, v, a)\| \le C(|a| + |v|)^2, \qquad 0 \le \epsilon \le 1, \tag{5.3}$$

with C depending only on the product, the larger complex domain, and its margin. At $v = 0$ the product is $\exp(\epsilon \operatorname{Ad}(e^{u})a)$, its logarithm is $\epsilon \operatorname{Ad}(e^{u})a$, and the remainder vanishes. Every argument concerns a fixed product or an exact identity, and the stated dependence of the constants follows. □

5.2 Closed one-cochains and Gaussian elimination

Parts (1) and (2) of the next lemma are the discrete Poincaré lemma and the fact that the first real cohomology of the three-torus is $\mathbb{R}^3$; part (3) is Gaussian elimination by Schur complements [27]. The harmonic part c is kept as data throughout.

Lemma 5.2. *The following statements hold.*

1. *On every finite contractible rectangular box in $\mathbb{Z}^3$, a real closed one-cochain is the coboundary of a function on the vertices.*
2. *On the periodic cubic lattice of side $n \ge 4$, every real closed one-cochain α has a unique decomposition $\alpha = \nabla f + c$, where f is periodic with mean zero and c is constant in each link direction. Moreover $c_i = n^{-3} \sum_x \alpha_i(x)$. The decomposition holds in each color separately and after rescaling to physical units. Thus requiring the spatial mean of α to vanish removes exactly c, and not a gauge degree of freedom.*

3. *Let a real symmetric nonnegative quadratic form on a finite-dimensional space of fixed coordinates c and integration coordinates $z \in \mathbb{R}^f$ have matrix*

$$\mathsf{A} = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & K \end{pmatrix}, \qquad K > 0.$$

For every c, including all specified harmonic coordinates,

$$\frac{1}{2}(c,z)^{\mathsf{T}}\mathsf{A}(c,z) = \frac{1}{2}c^{\mathsf{T}}(A - BK^{-1}B^{\mathsf{T}})c + \frac{1}{2}(z + K^{-1}B^{\mathsf{T}}c)^{\mathsf{T}}K(z + K^{-1}B^{\mathsf{T}}c),$$

and

$$\int_{\mathbb{R}^f} e^{-\frac{1}{2}(c,z)^{\mathsf{T}}\mathsf{A}(c,z)}\,dz = (2\pi)^{f/2}(\det K)^{-1/2}e^{-\frac{1}{2}c^{\mathsf{T}}(A-BK^{-1}B^{\mathsf{T}})c}.$$

Successive minimization, or successive Gaussian integration, over the blocks of any ordered decomposition of z gives the same Schur complement and the same product of determinants as simultaneous elimination. Conditionally on c, the Gaussian measure in z has mean $-K^{-1}B^{\mathsf{T}}c$ and covariance K^{-1}. A Gaussian measure defined on a specified orthogonal complement of a null space, by a quadratic form positive there, has covariance equal to the inverse on that complement and zero on the null space.

These are exact finite-dimensional statements; in particular, no lower bound on eigenvalues uniform in the side n is asserted.

Proof. Boxes. We regard a one-cochain α as an antisymmetric function on oriented links; closedness means that its signed sum around every unit square vanishes. On $\mathbb{Z}^3$, a finite path can be sorted by coordinate direction by adjacent interchanges of steps in different directions. Each interchange crosses exactly one unit square and preserves the integral of α, and adjacent opposite steps in one direction cancel. Then every closed path has integral zero, since after sorting its total displacement in each direction is zero and all steps cancel. Inside a rectangular box we integrate from a fixed corner, first in direction 1, then in direction 2, then in direction 3. Interchanging an additional final step in direction i with the later coordinate segments uses only the squares of the intervening rectangle, which lies in the box, and closedness makes the two path integrals equal. The resulting vertex function has coboundary α on every link. Boxes with a side of length zero are covered by omitting the absent direction.

The periodic lattice. We lift a periodic closed cochain to $\mathbb{Z}^3$. The argument just given produces a function f_0 on $\mathbb{Z}^3$ with $f_0(x + e_i) - f_0(x) = \alpha_i(x)$. Since α is periodic, $f_0(x + ne_i) - f_0(x) = P_i$ does not depend on x, because its difference across every link is zero; equivalently, P_i is the integral of α along any straight cycle winding once in direction i. Set $c_i = P_i/n$. The function $f(x) = f_0(x) - \sum_i c_i x_i$ is periodic and satisfies $\alpha = \nabla f + c$; subtracting its average makes its mean zero. If $\nabla f + c = 0$, then summing over a winding cycle gives $nc_i = 0$. Then all differences of f vanish, and the mean-zero condition implies

$f = 0$, which proves uniqueness. Because the sum of $\nabla_i f$ over all periodic vertices telescopes to zero, we have $c_i = n^{-3} \sum_x \alpha_i(x)$, and the mean condition removes exactly the three constants. For cochains with values in $\mathfrak{su}(2)$, the argument applies to each real component. For physical potentials we use $\nabla_a = a^{-1}\nabla$ and integrate $a\alpha$ along paths; the conclusion has the same form, with $c_i = P_i/(na)$.

Gaussian elimination. If we expand the right side of the completion of the square, its cross term is $c^{\mathsf{T}}Bz$, and the added constant $c^{\mathsf{T}}BK^{-1}B^{\mathsf{T}}c/2$ cancels the subtracted one. After the translation $z \mapsto z - K^{-1}B^{\mathsf{T}}c$, an orthogonal diagonalization of the positive matrix K reduces the integral to products of $\int_{\mathbb{R}} e^{-kt^2/2}dt = \sqrt{2\pi/k}$. To diagonalize K, we maximize its Rayleigh quotient on the unit sphere to obtain an eigenvector, restrict to the orthogonal complement of this eigenvector, and induct on the dimension. The value of the one-dimensional integral follows by squaring it and using polar coordinates in the plane. We obtain the Gaussian formula. For $f = 0$, the centered Gaussian integral and the empty determinant $\det K$ are one; the full integral is $\exp(-c^{\mathsf{T}}Ac/2)$. The same completion of the square with a linear term in z, differentiated at zero, gives the conditional mean and covariance.

For successive elimination, write $z = (z_1, z_2)$ and $K = \left(\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right)$. Both K_{11} and $K_{22} - K_{21}K_{11}^{-1}K_{12}$ are positive definite, since their quadratic forms are the restriction of the form of K and the minimum of that form over z_1. The change of variables $z_1 \mapsto z_1 + K_{11}^{-1}K_{12}z_2$ has determinant one, and by block multiplication we have

$$\det K = \det K_{11}\ \det\bigl(K_{22} - K_{21}K_{11}^{-1}K_{12}\bigr).$$

The unique joint minimizer in z solves $Kz = -B^{\mathsf{T}}c$; solving the first block and substituting gives the successive minimizers. The minimum value is therefore the same Schur form. By the determinant identity and the Gaussian formula, successive integration gives the same normalization. Induction over the blocks proves the statement for every ordered decomposition; for nonnegative integrands the repeated integrals are justified directly. No harmonic coordinate is discarded, since the vector c was arbitrary throughout.

On the given orthogonal complement of a null space, we diagonalize the restricted positive form. In the resulting coordinates the Gaussian measure is a product of independent one-dimensional Gaussian measures with variances equal to the inverse eigenvalues; embedding this measure by setting the null coordinates equal to zero, we obtain the stated covariance. This construction lives on the complement; it is not a Lebesgue integral along null directions. □

5.3 Analytic coefficient bounds and implicit functions

The following lemma is the calculus of the coefficient norms of Definition 4.6. Its first parts are combinatorial. A Taylor coefficient of a composite map is a finite or absolutely

convergent sum of products of Taylor coefficients of the constituents. We bound the absolute value of the sum by the sum of absolute values. The spatial weight survives because the connecting networks of the factors can be joined at the contracted indices, and the volume factors in the column sums telescope. The simplest instance is a product of two kernels, T from mesh a_{mid} to mesh a_{out} and S from mesh a_{in} to mesh a_{mid}. In counting coordinates the kernel of TS is $\sum_j T_{ij}S_{jk}$, without any volume factor, and its column sum at a fixed input k carries the factor

$$\frac{a_{\mathrm{out}}^3}{a_{\mathrm{in}}^3} = \frac{a_{\mathrm{out}}^3}{a_{\mathrm{mid}}^3} \cdot \frac{a_{\mathrm{mid}}^3}{a_{\mathrm{in}}^3},$$

so that column norms are submultiplicative. For a composition with one fixed input, we use column sums along the chain of factors that joins this input to the output, and row sums for all other factors. The later parts concern inverses and square roots of matrices with a spectral gap. The exponential decay of an inverse follows from the conjugation argument of Combes and Thomas [61], and square roots are represented by the standard resolvent integral [27]; the proof keeps track of the spatial weights and of all fixed indices. Bounds by coefficientwise majorant series are a classical tool of constructive renormalization group methods; the volume-normalized columns, the arbitrary fixed inputs, and the traces with a fixed external index are specific to our use.

Throughout this section, all indices are scalar indices and all sums are counting sums. A contraction identifies two indices of the same kind (same lattice and scale of introduction); an identification of indices of different kinds must first be represented by an explicitly bounded transfer kernel. We fix a rate $a_{\mathrm{sp}} \geq 0$ and write $|T| = |T|_{a_{\mathrm{sp}}}$ for the norm (4.9), $R(T)$ for its row norm, and $C_r(T)$ for its volume-normalized column sums. For a scalar tensor we use the sums (4.10) with one index fixed; on several meshes, a scalar tensor may instead be normalized by a reference volume v_*, which multiplies each summand with the index i_r fixed by v_*/v_{i_r}. Distances and network lengths are those of Definition 4.6, measured in units of the current output lattice spacing. For a matrix $B = (B_{xy})$ of blocks indexed by a finite metric space (X, d), we write

$$\|B\|_c = \max\Big\{\sup_x \sum_y e^{cd(x,y)}\|B_{xy}\|,\ \sup_y \sum_x e^{cd(x,y)}\|B_{xy}\|\Big\}, \qquad c \geq 0,$$

where $\|B_{xy}\|$ is the operator norm of the block.

Lemma 5.3. *The following statements hold.*

1. Substitution. *For tensors $T(o; j_1, \dots, j_k)$ and $S_r(j_r; \mathbf{l}_r)$, $1 \leq r \leq k$, the tensor*

$$U(o; \mathbf{l}_1, \dots, \mathbf{l}_k) = \sum_{j_1,\dots,j_k} T(o; j_1, \dots, j_k) \prod_{r=1}^{k} S_r(j_r; \mathbf{l}_r)$$

has every output-row norm and every volume-normalized input-column norm bounded by $|T|\prod_r |S_r|$. *In particular compatible matrix products are submultiplicative in the row norm and in the volume-normalized column norm; for counting-coordinate kernels from mesh* a_{in} *to mesh* a_{out} *the column factor is exactly* $a_{\mathrm{out}}^3/a_{\mathrm{in}}^3$, *and a contraction introduces no further volume factor. The same bound holds for a scalar outer tensor normalized by a reference volume. For a scalar outer tensor with counting sums on mesh* a_{old}, *the sum with a new input* l *fixed acquires the factor* v_l/v_{old}, *maximized over the possible kinds of the replaced inputs. Pointwise products with a common output index satisfy the product bound, and a shared index may be represented by the equality tensor* $\mathbf{1}_{o=i_1=\cdots=i_r}$, *whose norm is one when all its ports are of the same kind.*

2. Contraction diagrams. *Consider a finite connected diagram of tensors, in which shared indices are represented by equality tensors and every contracted index identifies two ports, and fix one external index. Choose a spanning tree of the graph of factors, rooted at the factor containing the fixed index, and attach every other factor* V *to its parent through a port* $j(V)$. *For a vector factor* V *with output* $o(V)$, *let* $A_V = |V|$ *if* $j(V) = o(V)$ *and* $A_V = (v_{j(V)}/v_{o(V)})|V|$ *if* $j(V)$ *is an input; for the factor containing the fixed index use the same rule at that index; for a scalar factor use its sum with the corresponding port fixed. Then the sum over all indices except the fixed one of the absolute value of the diagram, weighted by* $e^{a_{\mathrm{sp}}\mathfrak{t}}$ *of the locations of the external indices, is at most* $\prod_V A_V$. *If the fixed index is an external input* l, *the volume-normalized column of the diagram is obtained by multiplying by* v_o/v_l, *where* o *is the output of the diagram; for the substitution trees of part (1) the volume ratios cancel exactly. A diagram without a fixed external index is not covered; for example* $\mathrm{tr}\, I$ *equals the number of indices. An externally differentiated trace is covered if the differentiated tensor has the bound with the input fixed and* all *of its matrix ports and other Taylor inputs summed; a row bound alone does not suffice.*

3. Majorant series. *Products, substitutions, and summable connected diagrams of analytic maps with absolute majorants (in the sense of Definition 4.6) are bounded coefficientwise by the corresponding products, compositions, and diagram sums of the nonnegative scalar majorants, with the bounds of parts (1) and (2). The domain of the conclusion is the common product of polydiscs on which these scalar series converge absolutely. For* $F \circ G$ *it suffices that the scalar majorant of* G *maps the domain strictly inside the coefficient domain of* F; *for formal substitution take* $G(0) = 0$, *and otherwise require this absolute image condition, so that the substitutions into the constant terms also converge. The coefficients with one fixed input are bounded by the derivative of the same scalar majorant; no second Taylor factorial occurs.*

4. Smaller domains. *If* $A(s) = \sum_n A_n s^n$ *with* $A_n \geq 0$ *converges at* $R > 0$, *then for* $0 < r < R$ *and every fixed* $j \geq 0$,

$$A^{(j)}(r) \leq R^{-j} \sup_{n \geq j} (n)_j (r/R)^{n-j} A(R),$$

where $(n)_j = n(n-1)\cdots(n-j+1)$. *If the majorant* $\sum_k b_k s^k$ *satisfies* $\sum_k b_k R^k \le B$, *the coefficients of degree* s *of its translate to a background of size* $r_1 < R$ *are at most* $B(R-r_1)^{-s}$. *Thus evaluation, translation to a background, and a fixed finite number of derivatives are controlled on a smaller coefficient domain, at the same spatial rate. These statements concern absolute coefficient series; they are not Cauchy estimates for directional values.*

5. Resolution into occurrences. *Every expression built by these operations has an exact coefficient resolution, indexed by its constituent coefficient occurrences, its contracted indices, and its partitions of derivatives. Assign to each constituent occurrence a connected network through the locations of its ports and Taylor indices, and to each contraction the common location; the network of a resolved occurrence is the union of these networks. Its external monomial has its exact multi-index, and its fluctuation index set is the set of blocks of fluctuation variables with a positive exponent; the locations of background variables need not lie in that set. The weighted total variation of the resolution satisfies the same scalar diagram bounds, provided the assigned networks satisfy the constituent bounds. Let* W *be a specified boundary. If a designated constituent occurrence is assigned, by hypothesis, a connected network meeting* W, *then every resolved occurrence containing this designated factor has a connected network meeting* W, *and its weighted total variation satisfies the corresponding diagram bound, provided the constituent bounds include the weights of these assigned networks. Marking an input for differentiation does not by itself provide such a visit.*

6. Networks and trees. *Let* $\mathrm{t}_{\mathrm{MST}}(S)$ *be the length of a minimal spanning tree on a finite set* S *of locations. Then* $\mathrm{t}(S) \le \mathrm{t}_{\mathrm{MST}}(S) \le 2\mathrm{t}(S)$. *A bound at rate* a_{sp} *with shortest networks therefore gives a bound at rate* $a_{\mathrm{sp}}/2$ *with spanning trees on the locations of the external indices; this reduction is made once, independently of the order of the tensors.*

7. Inverses. *Let* (X,d) *be a finite metric space, with blocks of a fixed finite dimension, such that* $\#\{y : d(x,y) \le R\} \le D(1+R)^v$ *for all* $x \in X$ *and* $R \ge 0$. *Let* $K = K^* \ge \lambda I > 0$ *satisfy, for some* $a > 0$,

$$\max\Big\{\sup_x \sum_{y\neq x} e^{ad(x,y)}\|K_{xy}\|,\ \sup_y \sum_{x\neq y} e^{ad(x,y)}\|K_{xy}\|\Big\} \le H.$$

Let $b = \min\{a/2, a\lambda/(4H)\}$ *if* $H > 0$ *and* $b = a/2$ *if* $H = 0$; *then the same maximum with weights* $e^{bd(x,y)} - 1$ *is at most* $\lambda/2$. *For every* $0 < b' < b$,

$$\|(K^{-1})_{xy}\| \le 2\lambda^{-1}e^{-bd(x,y)}, \qquad \|K^{-1}\|_{b'} \le 2\lambda^{-1}\mathcal{S}(b-b';D,v),$$

where $\mathcal{S}(c;D,v) = D\sum_{k\ge 0}(k+2)^v e^{-ck}$. *No upper bound on the diagonal of* K *is needed, and the bounds do not depend on* $|X|$.

8. Perturbations and square roots. *In the setting of part (7), put* $C_{\mathrm{inv}} = \|K^{-1}\|_{b'}$. *For every complex matrix* E *with* $\|E\|_{b'} \le (2C_{\mathrm{inv}})^{-1}$, *we have* $\|(K+E)^{-1}\|_{b'} \le 2C_{\mathrm{inv}}$, *and the*

inverse is a holomorphic function of E. If also $K \le \Lambda I$, there exists $\eta_0 > 0$, depending only on $a, H, \lambda, \Lambda, D, v, b'$, such that for $\|E\|_a < \eta_0$ the branches $(K+E)^{1/2}$ and $(K+E)^{-1/2}$ continuing the positive roots exist and are bounded in $\|\cdot\|_{b'/2}$ by constants depending only on these data. If $E = E(\theta)$ has an absolute majorant with all fixed indices on a fixed analytic ball, the coefficients of these roots in θ have absolute bounds with every fixed index on a smaller coefficient ball. For volume-normalized coefficient columns the constants and the smaller radius may also depend on $\Gamma_X = \max_{x,y\in X} v_x/v_y$ and on the stated majorant. The coefficient ball is contained in the root domain $\|E(\theta)\|_a < \eta_0$, and its output spatial rate is no larger than the supplied coefficient rate. The counting-norm bounds for the inverse and roots, and the preceding choice of η_0, are unchanged.

9. Logarithms of determinants. *In the setting of part (8), assume the coefficient majorant for $E(\theta)$ is available at a positive spatial rate. Then $D \log\det K = \operatorname{tr}(K^{-1}DK)$, and the derivatives of positive order of $\log\det(K + E(\theta))$ in θ have absolute bounds with any fixed external index, with one reduction of the rate independent of the order. The undifferentiated $\log\det K$ is not bounded independently of $|X|$.*

10. Implicit functions. *Let $\Phi(\theta, u)$ be an analytic map of input variables $\theta = (\theta_j)$ and unknowns $u = (u_o)$, with values indexed like u, defined on a product of polydiscs, with $\Phi(0,0) = 0$. Assume that for all $k, l \ge 0$ the output-row norm and every input-column norm, at any fixed input θ_j or u_o, of the coefficient tensor $D_\theta^k D_u^l \Phi(0,0)/(k!\,l!)$ are at most the coefficient of $t^k Y^l$ in*

$$\Phi^+(t, Y) = At + \kappa Y + \frac{A(t+Y)^2}{1 - C(t+Y)}, \qquad A, C > 0, \quad 0 \le \kappa < 1.$$

In particular the linear map $D_u\Phi(0,0)$ has row and column norms at most κ. Let $\delta = 1 - \kappa$, $r = \min\{1, (4C)^{-1}, \delta/(32A)\}$, and $t_0 = \min\{r, \delta r/(4A)\}$, and assume that the domain of Φ contains a neighborhood of the closed product $\{\max_j|\theta_j| \le t_0,\ \max_o|u_o| \le r\}$. Then the scalar equation $Y = \Phi^+(t, Y)$ has a unique solution $Y(t)$ with $|Y(t)| \le r$ for $|t| < t_0$; it is analytic, $Y(0) = 0$, and its Taylor coefficients are nonnegative. For $\max_j|\theta_j| < t_0$ the system $u = \Phi(\theta, u)$ has a unique solution $u(\theta)$ with $\max_o|u_o| \le r$, and for every k the output-row norm and every input-column norm of $D^k u(0)/k!$ are at most the coefficient of t^k in $Y(t)$. For a fixed input θ_j, the derivative $\partial_{\theta_j} u$ has coefficient tensors bounded by the coefficients of

$$Y'(t) = \frac{A + A\varphi'(t + Y(t))}{1 - \kappa - A\varphi'(t + Y(t))}, \qquad \varphi(s) = \frac{s^2}{1 - Cs},$$

whose denominator satisfies $|1 - \kappa - A\varphi'(t + Y(t))| \ge 5\delta/8$ for $|t| < t_0$. If only the output-row hypotheses are assumed, the existence, uniqueness, holomorphy, and output-row conclusions remain valid; no input-column conclusion is then asserted. In the respective asserted norms, all derivatives of positive order satisfy bounds $s!C_1^s$ at the same spatial rate,

also after translation to a background in a fixed smaller ball, where C_1 depends only on A, C, κ and the radius ratios, and not on the number of coordinates. If a parameter enters Φ with a variation of majorant $H(t, Y)$, the variation of the solution has majorant $Z(t)$ with $Z(t) \le H(t, Y(t))/(1 - \partial_Y \Phi^+(t, Y(t)))$ for $0 \le t < t_0$; this bounds the value of the majorant series at t, and it is not a coefficientwise inequality.

The constants in parts (1)–(6) are as displayed. In parts (7)–(10) they have the dependence described in item (1) of Definition 4.14, through the displayed data. The volume-normalized coefficient conclusions of part (8), and the conclusions of part (9) using them, include the stated dependence on Γ_X and on the smaller coefficient ball. When a scalar trace is normalized by a reference volume v_*, the corresponding conversion $\max_{x \in X} v_*/v_x$ is retained as well. On one matrix mesh $\Gamma_X = 1$. When the counting bound for the indices depends on J (with $D = D_J$), the constants depend on this fixed J, and no uniformity as $J \to \infty$ follows. Bounds obtained from part (10) for a class of expressions require one common scalar majorant for the whole class; expressions of arbitrary algebraic complexity are not covered without such a majorant.

Proof of Lemma 5.3. We first prove parts (1)–(6), which are bookkeeping for absolute sums, then the inverse estimates (7)–(9), and finally the implicit function statement (10).

Step 1: substitution. For matrices, the row norm of $\sum_j T_{ij} S_{jk}$ satisfies

$$\sup_i \sum_k e^{a_{\mathrm{sp}} d(i,k)} \Big\| \sum_j T_{ij} S_{jk} \Big\| \le R(T) R(S),$$

since $d(i,k) \le d(i,j) + d(j,k)$. For columns, the exact identity

$$\frac{v_i}{v_k} = \frac{v_i}{v_j} \frac{v_j}{v_k}, \qquad \sup_k \sum_i \frac{v_i}{v_k} e^{a_{\mathrm{sp}} d(i,k)} \Big\| \sum_j T_{ij} S_{jk} \Big\| \le C(T) C(S) \tag{5.4}$$

holds. If indices of several kinds occur, the sums include them with their volumes. This proves submultiplicativity and explains why the column factor of a counting-coordinate kernel must be exactly $a_{\mathrm{out}}^3 / a_{\mathrm{in}}^3$.

For the substitution tensor U, the union of the networks of T and of the S_r, joined at the contracted indices, connects the locations of all external indices, which implies

$$\mathsf{t}(\text{locations of the external indices}) \le \mathsf{t}(o, \mathbf{j}) + \sum_r \mathsf{t}(j_r, \mathbf{l}_r).$$

We take absolute values before any summation. For an output row, we sum each inner branch by its row norm and then the outer tensor by its row norm. If an external input l belonging to branch r is fixed, we sum every other branch by its row norm, use the column of branch r at l, and the j_r-column of the outer tensor. The volume factors telescope, since $v_o/v_l = (v_o/v_{j_r})(v_{j_r}/v_l)$ by (5.4). For a scalar outer tensor normalized by a reference volume the same computation applies, and for counting sums the only

change is the factor v_l/v_{old}. At a common output index the networks meet at that output, and a bound with an input fixed uses the column of the factor carrying the input and rows of the other factors. This proves part (1).

Step 2: diagrams. We replace every edge of the diagram outside the chosen spanning tree by two independently summed copies of its index, deleting the equality constraint between them; for nonnegative sums this can only increase the result. When a split edge is present we may enlarge the external network weight to include the locations of both copies, since the connected tree of factors still spans the locations of the given external indices. We orient the tree from the factor containing the fixed index. Starting at its leaves, we sum each factor over all its ports except the port of attachment to its parent, using the bounds A_V; we continue toward this factor and finally sum it over all ports except the fixed index. This proves the bound without any count of the internal indices. In a substitution tree, the factors other than rows lie on the path from the fixed input to the output, and the product of their volume ratios is v_l/v_o, which the external column normalization cancels. A diagram without a fixed external index has no factor at which to start, and it is excluded. This proves part (2).

Step 3: majorant series and smaller domains. We first consider Taylor polynomials and finite diagrams. For $G(0) = 0$, the ordered derivative formula reads

$$D^N(F \circ G)(0)[u_1, \dots, u_N] = \sum_{n=0}^{N} \frac{1}{n!} \sum_{(B_1,\dots,B_n)} F_n\big[G_{|B_1|}[u_{B_1}], \dots, G_{|B_n|}[u_{B_n}]\big],$$

where $(B_1, \dots, B_n)$ runs over the ordered partitions of $\{1, \dots, N\}$ into nonempty blocks, and the term $N = n = 0$ is understood separately. For block sizes $m_1 + \cdots + m_n = N$ the number of ordered partitions is $N!/\prod_r m_r!$. Dividing by $N!$ therefore gives exactly the coefficient of s^N in $\widehat{F}(\widehat{G}(s))$, and parts (1) and (2) bound each summand. Marking one input chooses its block and its position in the block, exactly as in $(\widehat{F} \circ \widehat{G})' = (\widehat{F}' \circ \widehat{G})\widehat{G}'$. This shows that the coefficient of order N with one marked input is bounded by Nh_N, where h_N is the Taylor-divided coefficient of the composite majorant. Inputs of different kinds may be marked by separate scalar variables. For infinite orders and diagrams, the nonnegative majorant series converge on the stated domain, and Tonelli's theorem and absolute convergence justify every grouping, contraction, substitution, and derivative; equality of the analytic functions follows from equality of their coefficients. The case $G(0) \neq 0$ is handled in the same way under the absolute image condition. This proves part (3).

For part (4), we write $r^{n-j} = R^{n-j}(r/R)^{n-j}$ termwise; then $A^{(j)}(r) = \sum_{n \geq j} (n)_j A_n R^{n-j} (r/R)^{n-j} \leq R^{-j} \sup_{n \geq j} (n)_j (r/R)^{n-j} \sum_n A_n R^n$. The coefficient of degree s of the translate of the majorant to a background of size $r_1 < R$ is the binomial sum $\sum_{k \geq s} \binom{k}{s} b_k r_1^{k-s}$. By the binomial theorem, $\binom{k}{s} r_1^{k-s} (R - r_1)^s \leq R^k$ for each $k \geq s$; multi-

plying by b_k, summing, and using $\sum_k b_k R^k \le B$, we obtain $\sum_{k\ge s} \binom{k}{s} b_k r_1^{k-s} \le B(R-r_1)^{-s}$. Evaluating extra inputs at a background can only remove locations of external indices from a network, and the same spatial rate therefore remains valid.

For part (5), we keep a separate signed occurrence for every summand of the derivative formula, every tuple of contracted indices, and every refinement of a constituent network. Its coefficient is the product of the constituent coefficients and the displayed combinatorial factor, and its monomial is the product of their monomials. Grouping occurrences with equal multi-indices or equal networks is optional and can only decrease the total variation. The network inequality of Step 1 implies the weighted bound, and the convergence argument of this step justifies the infinite sums. If a designated constituent occurrence has an assigned network meeting W, the network of a resolved occurrence containing it is a union that contains this assigned network, and it retains the visit to W.

Step 4: networks and trees. The inequality $\mathsf{t}(S) \le \mathsf{t}_{\mathrm{MST}}(S)$ holds since a spanning tree on S is a connecting network. Conversely, we delete the cycles of a competing network without increasing its length, traverse the resulting tree with every edge twice, shortcut the tour between successive points of S by the triangle inequality, and delete its last edge. The result is a path through S of length at most twice the length of the network. Taking infima, we find $\mathsf{t}_{\mathrm{MST}}(S) \le 2\mathsf{t}(S)$, which implies $e^{(a_{\mathrm{sp}}/2)\mathsf{t}_{\mathrm{MST}}} \le e^{a_{\mathrm{sp}}\mathsf{t}}$. Joining the constituent trees once proves the assertion that the rate is halved only once. This proves part (6).

Step 5: inverses. For a block matrix B with absolute row sums at most r and column sums at most c, the Cauchy–Schwarz inequality gives

$$\sum_x \Big| \sum_y B_{xy} u_y \Big|^2 \le r \sum_{x,y} \|B_{xy}\| |u_y|^2 \le rc \sum_y |u_y|^2,$$

which implies $\|B\|_{2\to 2} \le \sqrt{rc}$. For $0 < b \le a/2$ and $s \ge 0$ we have $e^{bs} - 1 \le bse^{bs} \le (2b/a)e^{as}$, and with the choice of b in part (7) the weighted off-diagonal sums are at most $(2b/a)H \le \lambda/2$; any smaller positive choice also works. Fix $y_0 \in X$ and let $W_x = e^{bd(x,y_0)} I$ on the block of x. The matrix

$$B = WKW^{-1} - K, \qquad B_{xz} = \big(e^{b(d(x,y_0)-d(z,y_0))} - 1\big) K_{xz},$$

has zero diagonal, and by the triangle inequality each scalar factor is at most $e^{bd(x,z)} - 1$ in absolute value. By the bound just proved, $\|B\|_{2\to 2} \le \lambda/2$. Since $K \ge \lambda I$, we have $\|K^{-1}\|_{2\to 2} \le 1/\lambda$, and the geometric series for $(I + K^{-1}B)^{-1}$ yields $\|(WKW^{-1})^{-1}\|_{2\to 2} \le 2/\lambda$. The (x, y_0) block of this inverse is $e^{bd(x,y_0)}(K^{-1})_{xy_0}$, which proves the pointwise bound. The largest value of the weight never enters, and the diagonal of K cancels from B.

For $c > 0$, decomposing into the shells $k \le d(x,y) < k+1$ and using the counting bound, we obtain

$$\sup_x \sum_y e^{-cd(x,y)} \le D \sum_{k=0}^{\infty} (k+2)^v e^{-ck} = \mathcal{S}(c; D, v) < \infty. \tag{5.5}$$

With $c = b - b'$ this bounds both weighted row and column sums of K^{-1} by $2\lambda^{-1}\mathcal{S}(b - b'; D, v)$. The fixed block dimension does not change the argument, since all entries were bounded in block operator norm. This proves part (7).

Step 6: perturbations and square roots. By submultiplicativity of $\|\cdot\|_{b'}$ (Step 1, without volume factors), when $\|E\|_{b'} \le (2C_{\mathrm{inv}})^{-1}$ the series

$$(K+E)^{-1} = \sum_{r=0}^{\infty} (-K^{-1}E)^r K^{-1} \tag{5.6}$$

converges absolutely in this norm, with sum bounded by $2C_{\mathrm{inv}}$. Uniform convergence on every smaller ball implies holomorphy by termwise differentiation, and multiplication of partial sums shows that the sum is the inverse.

For the square roots, let $c_2 = b'/2$. We apply the pointwise bound of Step 5 to $K + tI \ge (\lambda + t)I$, whose off-diagonal part is that of K and whose commutator bound $\lambda/2 \le (\lambda + t)/2$ persists. This gives, for all $t \ge 0$,

$$\|(K+tI)^{-1}\|_{c_2} \le \frac{C_0}{\lambda + t}, \qquad C_0 = 2\mathcal{S}(b - c_2; D, v). \tag{5.7}$$

Fix $\eta_0 < \min\{\lambda/(2C_0), \lambda/2\}$. Since $c_2 < a$, we have $\|E\|_{c_2} \le \|E\|_a < \eta_0$, and expanding the shifted resolvents as in (5.6), we obtain $\|(K+E+tI)^{-1}\|_{c_2} \le 2C_0/(\lambda+t)$. Hence the absolutely convergent integral

$$S(E) = \frac{1}{\pi} \int_0^{\infty} t^{-1/2} (K+E+tI)^{-1}\, dt \tag{5.8}$$

has $\|S(E)\|_{c_2} \le 2C_0\lambda^{-1/2}$ and is holomorphic in E. The scalar identity $\pi^{-1}\int_0^{\infty} t^{-1/2}(k+t)^{-1}dt = k^{-1/2}$ (substitute $t = ks^2$ and integrate $2/(1+s^2)$) and diagonalization show that $S(E)$ is the positive inverse square root when $K+E$ is real symmetric and positive. For an arbitrary complex, possibly nonsymmetric E, let $M = K + E$. By the bound of Step 5, $\|E\|_{2\to 2} < \eta_0 < \lambda/2$, which implies $\mathrm{Re}\langle u, Mu\rangle \ge (\lambda/2)|u|^2$. For real $d > \|M\|_{2\to 2}^2/\lambda$,

$$\|(I - M/d)u\|^2 \le \big(1 - \lambda/d + \|M\|_{2\to 2}^2/d^2\big)|u|^2 < |u|^2.$$

With $T = I - M/d$, expanding $(M+tI)^{-1}$ in powers of T and integrating termwise in (5.8), which is justified by $\|T\|_{2\to 2} < 1$, identifies $S(E)$ with $d^{-1/2}\sum_{r\ge 0} c_r T^r$, where $c_r = 4^{-r}\binom{2r}{r}$ are the coefficients of $(1-z)^{-1/2}$; indeed, differentiating $(1-z)^{-1/2}$ at zero

gives $c_0 = 1$ and $c_{r+1} = (r + 1/2)c_r/(r + 1)$. The square of this scalar series is $(1 - z)^{-1}$, which gives $S(E)^2 M = I$; all factors commute with M, being limits of its resolvents. This proves the root identity without a symmetry assumption on E. Multiplying by $K + E$ gives the square root; since $K \le \Lambda I$ implies $\|K\|_{c_2} \le \Lambda + H$, this multiplication is bounded uniformly. All choices depend only on $a, H, \lambda, \Lambda, D, v, b'$.

Suppose now that $E = E(\theta)$ has an absolute majorant with all fixed indices on a fixed analytic ball inside the stated root domain. Write $E_0 = E(0)$ and $\Delta E(\theta) = E(\theta) - E_0$. The preceding perturbation bound gives $R_t = (K + E_0 + tI)^{-1}$ with counting row and column bounds $2C_0/(\lambda + t)$ at rate c_2. Its volume-normalized matrix columns are bounded by $2\Gamma_X C_0/(\lambda + t)$. Work at a rate $c \le c_2$ no larger than the supplied coefficient rate of ΔE; if that rate is smaller, decrease c accordingly. Let $A(r)$ be the common full absolute coefficient majorant of ΔE there. Since $\Delta E(0) = 0$, $A(r) \to 0$ as $r \downarrow 0$. Choose a smaller coefficient ball, still in the root domain, with $2\Gamma_X C_0 A(r) < \lambda/2$, and expand

$$(K + E(\theta) + tI)^{-1} = \sum_{j \ge 0} (-R_t \Delta E(\theta))^j R_t.$$

The substitution and contraction rules apply with these row and column bounds, retaining their volume factors. For j insertions, integration against $t^{-1/2}/\pi$ is bounded by

$$\frac{1}{\pi} \int_0^\infty t^{-1/2} \Big(\frac{2\Gamma_X C_0}{\lambda + t} \Big)^{j+1} dt \le (2\Gamma_X C_0)^{j+1} \lambda^{-j-1/2}.$$

Indeed, the same integral without the constant equals $\lambda^{-j-1/2} \binom{2j}{j} / 4^j$. The resulting scalar coefficient series is geometrically summable on this ball, uniformly in the number of matrix labels. It bounds every fixed-index coefficient of the inverse square root. Its constant matrix coefficient has the preceding counting bound and the corresponding volume-normalized bound with factor Γ_X. Multiplication by $K + E(\theta)$ gives the positive square root; the coefficient norm of the constant matrix K costs at most the same mesh-ratio factor. This proves part (8), with no identification of counting and volume-normalized columns.

Step 7: logarithms of determinants. Expanding the determinant in columns gives $D \det K = (\det K) \operatorname{tr}(K^{-1} DK)$, since the linear term of $\det(I + tB)$ is $t \operatorname{tr} B$. On the smaller coefficient ball just chosen, put $R_0 = (K + E_0)^{-1}$. The centered identity is

$$\log \det(K + E(\theta)) = \log \det(K + E_0) + \sum_{j \ge 1} \frac{(-1)^{j+1}}{j} \operatorname{tr}\big((R_0 \Delta E(\theta))^j \big).$$

To see it, differentiate in a scalar multiple of ΔE, use the convergent geometric inverse series, and integrate from zero. The logarithm is the branch continuing the value at $\theta = 0$. The same coefficient gate as in Step 6 ensures absolute convergence before the trace is

closed. Only positive-degree coefficients are bounded with a fixed external input; the constant logarithmic determinant is retained separately.

For these coefficients, keep the two matrix ports x, y open. Steps 1 and 2 and the centered series give an output-row majorant B_s for the degree-s tensor $T_s(x, y; u_1, \dots, u_s)$ at some positive rate c, no larger than the supplied coefficient rate. Fix an external input $u_j = u$ and close the trace by setting $y = x$. The unclosed network length is at least both the connecting length of the external inputs and $d(x, u)$. Splitting the rate in half bounds the sum over the other external inputs, with their external weight at rate $c/2$, by $B_s e^{-cd(x,u)/2}$ for fixed x. If $u \notin X$, choose a nearest $x_0 \in X$, so $d(x, x_0) \le 2d(x, u)$; otherwise set $x_0 = u$. Equation (5.5) sums over x with a factor at most $\mathcal{S}(c/4; D, v)$. This proves the counting fixed-external-index bound with one rate reduction independent of s. For a scalar reference-volume normalization, apply the distinguished-input column and diagram rules, keeping their attachment and external-normalization ratios, including the stated conversion from matrix volumes to the reference volume. No count of all internal labels is inserted. Without an external input the bound fails: $K = 2I$ gives $\log\det K = |X| \log 2$ times the block dimension. This proves part (9).

Step 8: the scalar equation. We write $\varphi(s) = s^2/(1 - Cs)$, so that $\Phi^+(t, Y) = At + \kappa Y + A\varphi(t + Y)$. For $|t| \le t_0$ and $|Y| \le r$ we have $|t + Y| \le 2r$, $C|t + Y| \le 1/2$, and

$$|At| + A|\varphi(t + Y)| \le \frac{\delta r}{4} + 8Ar^2 \le \frac{\delta r}{2}.$$

Moreover $|\varphi'(s)| \le 2|s|/(1 - C|s|) + C|s|^2/(1 - C|s|)^2 \le 12r$ for $|s| \le 2r$. The map $Y \mapsto \Phi^+(t, Y)$ therefore maps the closed disc of radius r into itself, since $\kappa r + \delta r/2 < r$, and its derivative is at most $\kappa + 12Ar \le 1 - 5\delta/8 < 1$. The iteration started at $Y = 0$ converges uniformly, since successive differences decrease by this ratio, and the same estimate implies uniqueness in the disc. Since each iterate is holomorphic, the limit is holomorphic, and $Y(0) = 0$. Every iterate has nonnegative Taylor coefficients, and they increase along the iteration; the limit therefore has nonnegative coefficients. Equivalently, the coefficient of degree s of Y is $(1 - \kappa)^{-1}$ times a nonnegative expression in the earlier coefficients, with the source term at degree one.

Step 9: the system. For $\max_j |\theta_j| \le t$ and $\max_o |u_o| \le Y$, the row bounds of the hypothesis imply $\max_o |\Phi_o(\theta, u)| \le \Phi^+(t, Y)$, since the weights $e^{a_{\mathrm{sp}}\mathsf{t}}$ are at least one; similarly the Lipschitz constant of $u \mapsto \Phi(\theta, u)$ in the maximum norm is at most $\partial_Y \Phi^+(t, Y)$. By Step 8, the Picard iteration $u^{(0)} = 0$, $u^{(n+1)} = \Phi(\theta, u^{(n)})$ stays in the ball of radius r and converges, and the small solution is unique. By part (3), each iterate $u^{(n)}$ has coefficient tensors bounded, in every output-row norm and every input-column norm, by the coefficients of the scalar iterate $Y^{(n)}$, where $Y^{(n+1)} = \Phi^+(t, Y^{(n)})$; the factorials cause no loss, since after division by $s!$ the counting of ordered partitions in Step 3 reproduces ordinary composition of power series. Passing to the limit coefficientwise, we obtain the

bounds for $u(\theta)$. For the derivative in a fixed input θ_j, we mark this input in the same expansion. The branch containing it uses column sums and all other branches use row sums. By Step 3, the resulting scalar recursion is the derivative of the scalar equation, which is solved by the displayed formula for $Y'(t)$. By the reverse triangle inequality and $A|\varphi'| \le 12Ar$, its denominator has modulus at least $1 - \kappa - 12Ar \ge 5\delta/8$, and no further loss of the spatial rate occurs. All comparisons are made coefficient by coefficient, before summing over degrees.

Translation to a background of size r_1 in a smaller ball is handled by part (4): the coefficients of order s are bounded by $B(R - r_1)^{-s}$, and multiplying back by $s!$ and absorbing the prefactor for $s \ge 1$ gives $s!C_1^s$, with C_1 depending only on the majorants and radius ratios. For a parameter entering Φ with variation majorant $H(t, Y)$, differentiating the fixed-point equation, we obtain the coefficientwise inequality $Z \preceq H + (\partial_Y \Phi^+)Z$ between majorant series, and repeated substitution bounds Z by the Neumann series in $\partial_Y \Phi^+(t, Y(t))$. Evaluating at $0 \le t < t_0$ gives the value bound $Z(t) \le H(t, Y(t))/(1 - \partial_Y \Phi^+(t, Y(t)))$; it does not give $Z_n \le H_n/(1 - q)$ for the individual coefficients. Since the only counting estimate in the proof was (5.5), all constants depend on the fixed J when the counting constant is D_J. This proves part (10). □

5.4 Concentration on products of balls

The conditional fluctuation kernels of (4.8) are measures of the form $Z^{-1}e^{-V}dw$ on a product of Euclidean balls, with a potential whose Hessian is bounded below. The next theorem gives the concentration inequalities for such measures used in later sections, with constants depending only on the lower Hessian bound. The variance, entropy, and gradient arguments are those of Brascamp and Lieb [33], Gross [103], and Bakry and Émery [8]. The representation of covariances by a one-form operator is the method of Helffer and Sjöstrand [110], whose relation to the Brascamp–Lieb inequality is discussed by Helffer [109]. The passage from entropy to exponential moments is the Herbst argument [123, Section 2.3, Proposition 2.3, p. 149, and Corollary 2.4, p. 151]. We must check that these arguments survive the boundary of the product of balls, including the intersections of faces. They do because the balls are convex: in the Bochner identity for the Neumann Laplacian, the second fundamental form of each sphere contributes the nonnegative term $p_x^{-1}|(\nabla_x h)^{\tan}|^2$, and the faces of different factors meet transversally. We prove this realization explicitly rather than importing a statement on the whole space.

Theorem 5.4. *Let X be a finite set, and for $x \in X$ let $q_x \ge 1$ be integers and $p_x > 0$. Let*

$$\Omega = \prod_{x \in X} \{w_x \in \mathbb{R}^{q_x} : |w_x| \le p_x\}, \qquad d\mu = Z^{-1}e^{-V(w)}\,dw \quad \text{on } \Omega,$$

where V is real and of class C^2 on a neighborhood of Ω, and $D^2V \geq mI$ for some $m > 0$. Let $q = \sum_x q_x$ and $\xi = w - \mathbb{E}_\mu w$. Then the following statements hold.

1. *For every real $f \in H^1(\mu)$ and every real function F that is Lipschitz for the Euclidean norm,*

$$\operatorname{Ent}_\mu(f^2) \leq \frac{2}{m}\mathbb{E}_\mu|\nabla f|^2, \qquad \operatorname{Var}_\mu(f) \leq \frac{1}{m}\mathbb{E}_\mu|\nabla f|^2,$$

$$\log \mathbb{E}_\mu e^{t(F-\mathbb{E}_\mu F)} \leq \frac{t^2 \operatorname{Lip}(F)^2}{2m} \qquad (t \in \mathbb{R}),$$

   where $\operatorname{Ent}_\mu(h) = \mathbb{E}_\mu[h \log h] - \mathbb{E}_\mu h \log \mathbb{E}_\mu h$. These constants do not depend on $|X|$, on the q_x and p_x, or on upper bounds for the diagonal blocks of the Hessian.

2. *Let d be a metric on X and $b > 0$, and write $V_{xy} = D_{w_x} D_{w_y} V$. If*

$$\max\Big\{\sup_x \sum_{y \neq x} (e^{bd(x,y)} - 1) \sup_\Omega \|V_{xy}\|,\ \sup_y \sum_{x \neq y} (e^{bd(x,y)} - 1) \sup_\Omega \|V_{xy}\|\Big\} \leq \frac{m}{2},$$

   then for all real or complex $F, G \in H^1(\mu)$,

$$|\operatorname{Cov}_\mu(F, G)| \leq \frac{2}{m} \sum_{x,y} e^{-bd(x,y)} \|\nabla_x F\|_{L^2(\mu)} \|\nabla_y G\|_{L^2(\mu)},$$

   where the covariance is bilinear for complex functions. In particular, if $V_{xy} = 0$ for $d(x, y) > r_0$ and the mixed blocks have row and column sums at most H_0, the hypothesis holds whenever $H_0(e^{br_0} - 1) \leq m/2$; exponentially decaying mixed Hessians are covered by the displayed condition.

3. *For every real symmetric $q \times q$ matrix A, the quadratic form $Q_A = \xi^{\mathsf{T}} A \xi$ satisfies*

$$\log \mathbb{E}_\mu e^{t(Q_A - \mathbb{E}_\mu Q_A)} \leq \frac{8t^2}{m^2}\|A\|_{\mathrm{HS}}^2, \qquad |t| \leq \frac{m}{4\|A\|_{\mathrm{op}}},$$

   and for every $\zeta \in \mathbb{R}^q$,

$$\log \mathbb{E}_\mu e^{\zeta \cdot \xi + t(Q_A - \mathbb{E}_\mu Q_A)} \leq \frac{|\zeta|^2}{m} + \frac{16t^2}{m^2}\|A\|_{\mathrm{HS}}^2, \qquad |t| \leq \frac{m}{8\|A\|_{\mathrm{op}}},$$

   where the range of t is all of $\mathbb{R}$ if $A = 0$.

4. *Suppose $q_x = 3$ for all x. Let $\mathcal{F}$ be a family of distinct pairs (e, f) of blocks such that every block belongs to at most D_0 pairs of $\mathcal{F}$, counting both orientations if both occur, and let $j_{ef} \in \mathbb{R}^3$. Then $\sum_{(e,f)\in\mathcal{F}} j_{ef} \cdot (\xi_e \times \xi_f) = \xi^{\mathsf{T}} A_j \xi$ for a real symmetric matrix A_j with*

$$\|A_j\|_{\mathrm{op}} \leq \frac{D_0}{2}\|j\|_\infty, \qquad \|A_j\|_{\mathrm{HS}}^2 \leq D_0 \sum_{(e,f)\in\mathcal{F}} |j_{ef}|^2.$$

   If μ is invariant under the simultaneous rotation $w \mapsto (Rw_x)_x$ for every $R \in \mathrm{SO}(3)$, then this quadratic form has mean zero.

All statements concern the measure μ on the fixed convex set Ω, with its full potential V, and they are centered at the mean under μ. They give no convexity statement for marginals of μ or for measures on nonconvex sets.

Steps 1–6 construct the Neumann realization $\mathcal{L}$ of $-\Delta + \nabla V \cdot \nabla$ and a realization $\mathcal{A}_1$ of the corresponding operator on vector fields, whose form contains the Hessian of V and a nonnegative curvature term from the spheres. They also prove the commutation $\nabla \mathcal{L}^{-1} = \mathcal{A}_1^{-1} \nabla$, and establish the semigroup bound $|\nabla P_t h| \le e^{-mt} P_t |\nabla h|$. Steps 7 and 8 derive the entropy bound and, from $\mathrm{Cov}(F, G) = \langle \nabla F, \mathcal{A}_1^{-1} \nabla G \rangle$ and exponential conjugation, the covariance bound. Step 9 derives the quadratic moments by comparison with an auxiliary Gaussian vector and absorption of the relative entropy, without independent coordinates or a Hessian bound for the tilted measure, and Step 10 treats cross products. The construction of the operators in Steps 1 and 2 uses constants that may depend on the dimension, the radii, and the bounds of e^{-V} on the fixed compact set Ω; these constants do not enter the inequalities.

Proof of Theorem 5.4. All Hilbert spaces in this proof use the measure μ unless indicated otherwise. We write $B_x = \{|w_x| < p_x\}$, and n_x for the outer normal of the face $\partial B_x \times \prod_{z \neq x} B_z$.

Step 1: the scalar operator. The Sobolev space $H^1(\Omega)$ consists of the L^2 functions with first distributional derivatives in L^2; the weighted and unweighted norms are equivalent, since e^{-V} is bounded above and below on Ω. Smooth functions on $\overline{\Omega}$ are dense in $H^1(\Omega)$, and the embedding $H^1(\Omega) \to L^2(\Omega)$ is compact. For both facts we extend functions to a fixed ambient rectangle. On a ball of radius p, we reflect the radial coordinate $r \in (p, 3p/2)$ to $2p - r \in (p/2, p)$, multiply the reflected function by a smooth cutoff equal to one near $r = p$ and zero near $r = 3p/2$, and set it zero beyond. The Jacobians on the annulus are bounded above and below and the values match at $r = p$; differentiation in the radial and tangential directions shows that this extension is bounded on H^1. For an interval we reflect at each endpoint. Applying these extensions successively in the finitely many factors gives a bounded extension operator into H^1 of a rectangle, and convolution with smooth approximate identities proves density. For compactness, we replace an extended function u by its averages over a grid of cubes of side ε. Integrating $u(t) - u(s) = \int_s^t u'$ in each coordinate and using Cauchy–Schwarz, we bound the squared L^2 error by $C\varepsilon^2 \|\nabla u\|_2^2$. The averages lie in a fixed finite-dimensional space. From a bounded sequence in H^1 we may therefore extract, by a diagonal argument over $\varepsilon = 2^{-k}$, a subsequence converging in L^2.

The form $q_0(h, k) = \mathbb{E}_\mu \nabla h \cdot \nabla k$ on $H^1(\mu)$ is closed. For $f \in L^2(\mu)$, minimizing $q_0(h, h) + \|h\|_2^2 - 2\langle f, h \rangle$ gives a unique weak solution of $(I + \mathcal{L})h = f$. Indeed, a minimizing sequence is bounded, and the parallelogram identity shows that it is Cauchy in the form norm. Its limit lies in H^1 since distributional derivatives pass to L^2 limits, the

first variation gives the weak equation, and strict positivity of the form gives uniqueness. The solution operator $T = (I + \mathcal{L})^{-1}$ is positive, self-adjoint, and compact by the compact embedding. It has a complete orthonormal eigenbasis. Indeed, we maximize $\langle Tu, u\rangle$ on the unit sphere. A maximizing sequence has a weakly convergent subsequence, and the compactness of T lets us pass to the limit in $\langle Tu, u\rangle$; positivity forces the limit to have norm one, and the first variation shows that it is an eigenvector. We repeat this in the orthogonal complement of the eigenvectors already found. The eigenvalues tend to zero, since otherwise the images under T of these orthonormal vectors would have no convergent subsequence. On the orthogonal complement of all the eigenvectors the supremum of $\langle Tu, u\rangle$ is zero. By polarization T vanishes there, and since T is injective this complement is trivial. Hence $\mathcal{L}$ has an orthonormal basis of eigenfunctions with eigenvalues $0 \le \lambda_\nu \to \infty$. Where its derivatives are defined, $\mathcal{L} = -\Delta + \nabla V \cdot \nabla$, with the weak Neumann condition. The kernel of $\mathcal{L}$ consists of the constants. Indeed, a function with zero energy has zero distributional gradient. It is therefore constant on each ball contained in Ω, and these constants agree since Ω is connected. There also exists a positive gap on functions of mean zero, since otherwise normalized functions of mean zero with gradients tending to zero would converge, by compactness, to a constant of mean zero and norm one. This preliminary gap is used only for the existence of solutions of Poisson equations and for convergence to equilibrium.

Step 2: a core for the unweighted Laplacian. Let A_0 be the unweighted Neumann Laplacian on Ω, with respect to $L^2(dw)$. On a single ball, the construction of Step 1 gives a Neumann eigenbasis. Its eigenfunctions are smooth up to the boundary, and weak Neumann solutions are in H^2. In the interior this is the standard difference-quotient argument. Testing the weak equation with $-D_{-h,j}(\chi^2 D_{h,j} u)$, where χ is a cutoff and $D_{h,j}$ a difference quotient, gives

$$c\|\chi D_{h,j}\nabla u\|_2^2 \le \frac{c}{2}\|\chi D_{h,j}\nabla u\|_2^2 + C\big(\|f\|_2^2 + \|u\|_{H^1}^2\big)$$

for uniformly elliptic Lipschitz coefficients a. Here the leading term on the left is $\int \chi^2 a(\cdot + he_j) D_{h,j}\nabla u \cdot D_{h,j}\nabla u$, and all remaining terms contain either $D_{h,j}a$, which is bounded by the Lipschitz constant of a, or a derivative of χ. The right side $\int f[-D_{-h,j}(\chi^2 D_{h,j}u)]$ is bounded by $C\|f\|_2(\|\chi D_{h,j}\nabla u\|_2 + \|\nabla u\|_2)$, by the line-integral bound for a difference quotient, and the inequality $rs \le \epsilon r^2 + s^2/(4\epsilon)$ absorbs these terms. Uniformly bounded difference quotients yield weak second derivatives, since a subsequence converges weakly in L^2, and testing against smooth compactly supported functions identifies the limit as the distributional derivative. Near the boundary we use the inward normal distance $s \ge 0$ and coordinates y on the sphere; the metric has no mixed normal-tangential term and normal coefficient one, and its volume density J is smooth and positive on a thin collar. The weak Neumann equation reads $-\partial_s(J\partial_s u) - \partial_{y_\alpha}(J g^{\alpha\beta}\partial_{y_\beta} u) = Jf$. Reflecting

$u, J, g^{\alpha\beta}, f$ evenly in s gives Lipschitz, uniformly elliptic coefficients and a weak equation across $s = 0$, since adding its tests on the two half-collars gives exactly the Neumann weak equation. The interior estimate then gives H^2 up to the boundary, and integration by parts identifies the zero normal derivative. For an eigenfunction, $f = \lambda u$, we repeat the tangential difference-quotient argument in the half-collar; at each order the commutators differentiate only the smooth metric coefficients and lower derivatives already controlled, so all tangential derivatives gain one more H^1 derivative. The equation then solves for $\partial_s^2 u$, and differentiating it recovers successively the derivatives with two, three, and more normal indices; by induction $u \in H^k$ for every k. To deduce smoothness, we extend u from a half-chart by a finite linear combination $\sum_j c_j u(y, -js)$ of reflections, whose coefficients solve the invertible Vandermonde system $\sum_j c_j (-j)^l = 1$, $0 \le l < k$; this gives an H^k extension. We multiply it by an interior cutoff in an ambient rectangle with q coordinates. Repeated integration in the q coordinate directions and the Cauchy–Schwarz inequality bound the supremum of a derivative of order r of a smooth function by $C\|u\|_{H^{r+q}}$ on the rectangle. For example, the function itself is the iterated integral of $\partial_1 \cdots \partial_q u$ from each coordinate to the far side of the support. Applying this bound to smooth approximations of the extension and to their differences, we see that their derivatives through order r converge uniformly. It follows that the H^{r+q} function has continuous derivatives of order r. Since k is arbitrary, u is smooth. For an interval the same conclusions follow from $-u'' = f$ and the Neumann conditions at the endpoints.

Finite products of these eigenfunctions are smooth, satisfy the Neumann condition on every face, and form an orthonormal basis of $L^2(dw)$ on Ω, as one sees by expanding successively in each factor using Fubini's theorem. Their eigenvalues for A_0 are sums of eigenvalues of the factors. They form a core for A_0. Indeed, the weak equation shows that every $h \in D(A_0)$ has coefficients with $\sum_\nu (1 + \lambda_\nu^2)|h_\nu|^2 < \infty$; conversely, if this sum is finite, the partial sums solve the weak equation and converge in the graph norm, which shows that $h \in D(A_0)$. The same argument with $1 + \lambda_\nu$ describes the form domain; in particular, a function orthogonal to all product eigenfunctions in the form norm has $(1 + \lambda_\nu)h_\nu = 0$ for all ν and hence vanishes, which proves density in the form norm. Truncating the expansion therefore gives convergence in the graph norm and in the form norm.

Step 3: the boundary identity. For a function h in this core, two integrations by parts, one factor at a time, give

$$\|D^2 h\|^2_{L^2(dw)} + \sum_x \int_{\partial B_x \times \prod_{z \neq x} B_z} p_x^{-1} |(\nabla_x h)^{\tan}|^2 \, dS = \|A_0 h\|^2_{L^2(dw)}. \tag{5.9}$$

Indeed, the difference of the two bulk terms is the boundary integral of $(\Delta h)\partial_n h - D^2 h(n, \nabla h)$. The first term vanishes. Differentiating $\partial_{n_x} h = 0$ tangentially along the face of x, we find $D^2 h(n_x, \nabla h) = -p_x^{-1} |(\nabla_x h)^{\tan}|^2$, because $D_v n_x = p_x^{-1} v_x^{\tan}$. For an interval

the curvature term is zero. By Fubini's theorem on the individual balls, the identity holds on the product. Since intersections of distinct faces have zero surface measure, there is no integral over a corner. Applied to differences of core functions, the identity shows that convergence in the graph norm of A_0 implies convergence in H^2 and in the boundary term.

Step 4: the operator on vector fields. Let $\mathcal{H}_1$ be the space of H^1 vector fields $u = (u_x)_{x\in X}$ with $u_x \cdot n_x = 0$ on the face of x, in the sense of traces. Traces are continuous on H^1: in the collar, $|u(y,0)|^2 \le C\int_0^\varepsilon (|u(y,s)|^2 + |u(y,s)||\partial_s u(y,s)|)\,ds$, and Cauchy–Schwarz applies. Smooth vector fields satisfying the face conditions are dense in $\mathcal{H}_1$. To see this, we reflect the normal component of u_x oddly and its tangential components evenly in the local normal frame of the face of x. The odd component has zero trace, and no jump occurs. Near an intersection of faces we do this separately in each active ball factor; the reflections commute, since the normal frames of different factors depend on different variables. Convolving with a kernel that is even in every reflected normal coordinate and restricting to the product half-chart preserves the zero normal components. A partition of unity, constant in the normal coordinate near each face, and ordinary convolution in the interior complete the approximation, including near all corners.

On $\mathcal{H}_1$ we define the symmetric form

$$\begin{aligned} q_1(u,v) = \mathbb{E}_\mu \sum_i \partial_i u \cdot \partial_i v + \mathbb{E}_\mu u^{\mathsf{T}}(D^2V)v \\ + \sum_x \int_{\partial B_x \times \prod_{z\neq x} B_z} p_x^{-1} u_x^{\tan}\cdot v_x^{\tan}\, Z^{-1} e^{-V}\, dS. \end{aligned} \tag{5.10}$$

By the trace estimate, the boundedness of D^2V, and $D^2V \ge mI$, its norm is equivalent to the H^1 norm, and the form is therefore closed. The associated self-adjoint operator $\mathcal{A}_1$, obtained as in Step 1, satisfies

$$q_1(u,u) \ge m\|u\|_2^2, \qquad \|\mathcal{A}_1^{-1}\|_{2\to 2} \le m^{-1}. \tag{5.11}$$

Only this lower bound enters the final constants. Formally $\mathcal{A}_1 = \mathcal{L}I + D^2V$; on the face of x the tangential part of the block u_x satisfies the Robin condition $\partial_{n_x} u_x^{\tan} + p_x^{-1}u_x^{\tan} = 0$, while the other blocks satisfy the Neumann condition. At intersections of faces, the closed form (5.10) is the definition.

Step 5: gradient commutation. Let $\mathcal{L}h = g$ weakly. Testing with e^V times an unweighted H^1 test function, we find that $A_0 h = g - \nabla V\cdot\nabla h$ weakly in $L^2(dw)$. Since the right side is in L^2, we have $h \in D(A_0)$, and there exist core functions h_j converging to h in the graph norm of A_0. By (5.9), $h_j \to h$ in H^2 and in the boundary term, and $\mathcal{L}h_j \to g$ in $L^2(\mu)$. For a smooth vector field v satisfying the face conditions, differentiation and integration by parts give

$$q_1(\nabla h_j, v) = \langle \mathcal{L}h_j, d_V^* v\rangle, \qquad d_V^* v = -\operatorname{div} v + \nabla V\cdot v. \tag{5.12}$$

Indeed, $\nabla \mathcal{L} h_j = \mathcal{L} \nabla h_j + D^2 V \nabla h_j$, and on the face of x the normal derivative of the gradient, paired with v, equals $-p_x^{-1}(\nabla_x h_j)^{\tan} \cdot v_x^{\tan}$, while the other blocks have zero normal derivative there; these are exactly the boundary terms in (5.10). The scalar integration by parts produces no further term, since the normal traces of v vanish. By density, (5.12) extends to $v \in \mathcal{H}_1$. Passing to the limit $j \to \infty$, and taking $g \in H^1$, we obtain

$$\nabla h = \mathcal{A}_1^{-1} \nabla g. \tag{5.13}$$

This uses only the C^2 regularity of V: ∇V is bounded, and the differentiation in (5.12) uses only continuous second derivatives of V.

Step 6: semigroups and the gradient bound. Using the eigenbasis of Step 1, we define $P_t = e^{-t\mathcal{L}}$ by multiplying the coefficients by $e^{-t\lambda_\nu}$. It preserves constants, positivity, and the interval $[0,1]$. For the last two properties, we solve $(I + s\mathcal{L})h = f$ and test with h^- and $(h-1)^+$. If $0 \le f \le 1$, the first test gives $h^- = 0$ and the second $(h-1)^+ = 0$; since $P_t = \lim_k (I + (t/k)\mathcal{L})^{-k}$ in L^2, these properties pass to P_t. Symmetry and $P_t 1 = 1$ imply the invariance of μ, and the preliminary gap implies $P_t h \to \mathbb{E}_\mu h$ in L^2.

The variational construction applied to q_1 on vector-valued L^2, with form domain $\mathcal{H}_1$ (dense, since it contains the compactly supported smooth vector fields, and compactly embedded by the scalar argument), gives the semigroup $e^{-t\mathcal{A}_1}$. For $h \in H^1$ and $t > 0$, the spectral expansion puts $P_t h$ and $\mathcal{L} P_t h$ in the required domains, and (5.12) shows that $u_t = \nabla P_t h$ solves $\partial_t u_t = -\mathcal{A}_1 u_t$ weakly, with initial value ∇h. Uniqueness follows by differentiating the squared norm of the difference of two solutions and using (5.11). We conclude that $\nabla P_t h = e^{-t\mathcal{A}_1} \nabla h$.

We claim that

$$|e^{-t\mathcal{A}_1} u| \le e^{-mt} P_t |u|. \tag{5.14}$$

Let $u_t = e^{-t\mathcal{A}_1} u$ and $h_\varepsilon = (|u_t|^2 + \varepsilon^2)^{1/2}$. In the weak equation for u_t we use the test field $\psi u_t / h_\varepsilon$, with $\psi \in H^1$ bounded and nonnegative; it belongs to $\mathcal{H}_1$, since multiplication by a scalar preserves the face conditions. The kinetic term equals

$$\nabla h_\varepsilon \cdot \nabla \psi + \psi \Big(\frac{|\nabla u_t|^2}{h_\varepsilon} - \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon} \Big),$$

whose second summand is nonnegative by Cauchy–Schwarz. The Hessian term is at least $m\psi |u_t|^2 / h_\varepsilon \ge m\psi (h_\varepsilon - \varepsilon)$, and every boundary curvature term is nonnegative. Hence $(\partial_t + \mathcal{L} + m)(h_\varepsilon - \varepsilon) \le 0$ in the weak scalar sense. We compare with $e^{-mt} P_t |u|$ and test the difference with its positive part; the squared L^2 norm of this positive part does not increase from zero. This comparison is justified first on intervals $[\delta, T]$, where the spectral solutions have time derivatives in L^2 and lie in the form domains, and then by continuity in L^2 as $\delta \downarrow 0$; truncation of the tests and approximation in the form norm justify the chain

rule, whose derivatives are bounded for fixed ε. Letting $\varepsilon \downarrow 0$ proves (5.14). Combined with the previous paragraph, this gives

$$|\nabla P_t h| \le e^{-mt} P_t |\nabla h|.$$

Step 7: entropy, variance, and exponential moments. Let h be smooth, positive, and bounded above and below. Applying the positive operator P_t to $(f - ch)^2/h \ge 0$ for every rational c and minimizing over c, we obtain the pointwise inequality $(P_t f)^2 \le P_t(f^2/h)\, P_t h$. With $f = |\nabla h|$ and the gradient bound of Step 6, we obtain

$$\frac{|\nabla P_t h|^2}{P_t h} \le e^{-2mt} P_t\Big(\frac{|\nabla h|^2}{h}\Big). \tag{5.15}$$

By the weak equation and the chain rule for $\log P_t h$, we have

$$-\frac{d}{dt}\mathbb{E}_\mu[(P_t h)\log P_t h] = \mathbb{E}_\mu \frac{|\nabla P_t h|^2}{P_t h}.$$

Since the bounds on h persist under P_t, convergence of $P_t h$ to its mean in L^2 implies convergence of the entropy integral to $(\mathbb{E}h)\log \mathbb{E}h$. Integrating in time and using (5.15) and the invariance of μ, we find $\mathrm{Ent}_\mu(h) \le (2m)^{-1}\mathbb{E}_\mu|\nabla h|^2/h$. For bounded smooth f, substituting $h = f^2 + \varepsilon$ and letting $\varepsilon \downarrow 0$, we obtain $\mathrm{Ent}_\mu(f^2) \le (2/m)\mathbb{E}_\mu|\nabla f|^2$. For general $f \in H^1$, we truncate, approximate by smooth functions, and use that the entropy is lower semicontinuous under convergence in L^2. Indeed, $s \log s$ is bounded below, and Fatou's lemma applies along an almost everywhere convergent subsequence, while the mean term converges in L^1. This proves the logarithmic Sobolev inequality.

Expanding at $f = 1 + \varepsilon g$ with g bounded, we find $\mathrm{Ent}((1+\varepsilon g)^2) = 2\varepsilon^2 \mathrm{Var}(g) + o(\varepsilon^2)$. Then the logarithmic Sobolev inequality implies the Poincaré inequality with constant $1/m$; truncation and density extend it to H^1. For a real Lipschitz F let $\varphi(t) = \log \mathbb{E}_\mu e^{tF}$. Applying the logarithmic Sobolev inequality to $e^{tF/2}$, we find $t\varphi'(t) - \varphi(t) \le t^2 \mathrm{Lip}(F)^2/(2m)$. Integrating $(\varphi(t)/t)'$ from zero proves the exponential moment bound for $t > 0$, and applying it to $-F$ gives $t < 0$. To justify the chain rule, we extend F to a neighborhood by $\inf_{y\in\Omega}(F(y) + \mathrm{Lip}(F)|x - y|)$ and smooth; this preserves the Lipschitz constant and converges uniformly on Ω; no differentiability of F at the boundary is needed. This proves part (1).

Step 8: covariances. For real $G \in H^1$, we solve $\mathcal{L}h = G - \mathbb{E}G$ with $\mathbb{E}h = 0$, using the preliminary gap. By weak integration by parts and (5.13), we have

$$\mathrm{Cov}_\mu(F, G) = \langle \nabla F, \mathcal{A}_1^{-1} \nabla G \rangle_{L^2(\mu)}, \tag{5.16}$$

first for smooth F and then for $F \in H^1$ by density. Complexifying gives the bilinear identity for complex F, G, with neither function conjugated; its absolute value is bounded by the Cauchy–Schwarz inequality in the complex Hilbert space.

Fix $y_0 \in X$ and set $W_x = e^{bd(x,y_0)} I_{q_x}$ on the components of the block x. This weight does not depend on w; it preserves the face conditions and commutes with the kinetic and boundary forms and with every diagonal block of the Hessian. In the closed-form realization, we obtain

$$W\mathcal{A}_1 W^{-1} = \mathcal{A}_1 + B, \qquad B_{xz} = \big(e^{b(d(x,y_0)-d(z,y_0))} - 1\big) V_{xz}(w).$$

Each block multiplier has norm at most $(e^{bd(x,z)} - 1) \sup_\Omega \|V_{xz}\|$. By the row-column bound of Step 5 in the proof of Lemma 5.3 and the hypothesis of part (2), we have $\|B\|_{2\to 2} \le m/2$ on the direct sum of the block spaces $L^2(\mu)$. By (5.11) and a Neumann series, $\|(W\mathcal{A}_1 W^{-1})^{-1}\| \le 2/m$, and the (x, y_0) block satisfies

$$\|(\mathcal{A}_1^{-1})_{xy_0}\|_{L^2\to L^2} \le \frac{2}{m} e^{-bd(x,y_0)}.$$

Inserting this bound into (5.16) and applying Cauchy–Schwarz to each block pairing, we obtain the covariance bound of part (2), for real and complex functions. For an interaction of range r_0, every off-diagonal weight is at most $e^{br_0} - 1$, which gives the stated criterion. Neither the individual p_x, q_x nor an upper bound on the diagonal Hessian blocks entered.

Step 9: quadratic moments. The classical Hanson–Wright inequality concerns independent coordinates [108], and extensions based on concentration allow dependent vectors [1, Theorem 2.5, p. 4]; here we derive the required bounds directly from the logarithmic Sobolev inequality for μ. Applying part (1) to the linear function $v \cdot w$, we obtain

$$\mathbb{E}_\mu e^{v\cdot\xi} \le e^{|v|^2/(2m)}. \tag{5.17}$$

If $A = 0$, all quadratic statements follow from this. Let $A \neq 0$, and let Z_0 be a standard Gaussian vector in $\mathbb{R}^q$, independent of w. Since $e^{\alpha|y|^2} = \mathbb{E} e^{\sqrt{2\alpha} Z_0 \cdot y}$ (completion of the square, as in Lemma 5.2) and all integrands are positive, we have for $\alpha \ge 0$

$$\mathbb{E}_\mu e^{\alpha|A\xi|^2} = \mathbb{E}_{Z_0}\mathbb{E}_\mu e^{\sqrt{2\alpha}(AZ_0)\cdot\xi} \le \mathbb{E}_{Z_0} e^{(\alpha/m)|AZ_0|^2} = \det\big(I - (2\alpha/m)A^2\big)^{-1/2},$$

provided $2\alpha\|A\|_{\mathrm{op}}^2 < m$. At $\alpha = m/(4\|A\|_{\mathrm{op}}^2)$ all eigenvalues of $(2\alpha/m)A^2$ are at most $1/2$, and since $-\log(1-s) \le 2s$ on $[0, 1/2]$,

$$\log \mathbb{E}_\mu e^{\alpha|A\xi|^2} \le \frac{2\alpha}{m}\|A\|_{\mathrm{HS}}^2. \tag{5.18}$$

Now let $Q = \xi^{\mathsf{T}} A\xi$, $\psi(t) = \log \mathbb{E}_\mu e^{tQ}$, and $d\mu_t = e^{tQ-\psi(t)} d\mu$. All these quantities are finite and smooth in t, since the domain is compact. The relative entropy of μ_t with respect to μ is $D_t = t\psi'(t) - \psi(t)$; with $r = d\mu_t/d\mu$, integrating $r\log r \ge r - 1$ against μ, we find $D_t \ge 0$. Applying the logarithmic Sobolev inequality to $e^{tQ/2}$ and using $\nabla Q = 2A\xi$, we find $D_t \le (2t^2/m)\mathbb{E}_{\mu_t}|A\xi|^2$. For any real random variable Y, the entropy

inequality $\alpha \mathbb{E}_{\mu_t} Y \le D_t + \log \mathbb{E}_\mu e^{\alpha Y}$ holds (Jensen's inequality for the logarithm, applied to $e^{\alpha Y} d\mu / d\mu_t$ under μ_t). With $Y = |A\xi|^2$ and (5.18), we obtain

$$D_t \le \frac{8t^2 \|A\|_{\mathrm{op}}^2}{m^2} D_t + \frac{4t^2}{m^2} \|A\|_{\mathrm{HS}}^2.$$

For $|t| \le m/(4\|A\|_{\mathrm{op}})$ the first coefficient is at most $1/2$, which implies $D_t \le 8t^2 \|A\|_{\mathrm{HS}}^2 / m^2$. For $t > 0$, integrating $(\psi(t)/t)' = D_t/t^2$ from zero, where $\psi(t)/t \to \mathbb{E}_\mu Q$, proves the first bound of part (3); replacing A by $-A$ gives $t < 0$. For the joint bound, by the Cauchy–Schwarz inequality and (5.17), we have

$$\log \mathbb{E}_\mu e^{\zeta \cdot \xi + t(Q - \mathbb{E}Q)} \le \frac{1}{2} \log \mathbb{E}_\mu e^{2\zeta \cdot \xi} + \frac{1}{2} \log \mathbb{E}_\mu e^{2t(Q - \mathbb{E}Q)} \le \frac{|\zeta|^2}{m} + \frac{16t^2}{m^2} \|A\|_{\mathrm{HS}}^2,$$

for $|t| \le m/(8\|A\|_{\mathrm{op}})$. This proves part (3).

Step 10: cross products. We write $J(j)v = j \times v$. A pair (e, f) contributes the symmetric matrix with blocks $A_{ef} = -J(j_{ef})/2$ and $A_{fe} = A_{ef}^{\mathsf{T}}$, because $\xi_e^{\mathsf{T}} [-J(j_{ef})] \xi_f = j_{ef} \cdot (\xi_e \times \xi_f)$. Since $\|J(j)\|_{\mathrm{op}} = |j|$ and $\|J(j)\|_{\mathrm{HS}}^2 = 2|j|^2$, a single pair has block row sums $|j_{ef}|/2$ and squared Hilbert–Schmidt norm $|j_{ef}|^2$. Since at most D_0 pairs meet a block, the row-column bound implies $\|A_j\|_{\mathrm{op}} \le D_0 \|j\|_\infty / 2$; using $|\sum_{k=1}^r b_k|^2 \le r \sum_k |b_k|^2$ with $r \le D_0$ in each repeated matrix entry, we obtain $\|A_j\|_{\mathrm{HS}}^2 \le D_0 \sum |j_{ef}|^2$. Pairs with $e = f$ contribute zero. If μ is invariant under simultaneous rotations, then $\mathbb{E}_\mu w$ is invariant and ξ transforms as a vector. The law of ξ therefore has the same invariance. The vector $\mathbb{E}(\xi_e \times \xi_f)$ is then fixed by every rotation, hence zero (a rotation by π about an axis perpendicular to a nonzero vector reverses it). This proves part (4). □

5.5 Conditioning, exponential moments, and gauge identities

The next lemma gives conditioning bounds and identities for parameter derivatives and gauge transformations; the bound on large fields enters the proof through the comparison of a measure with its conditioning (see (3.1)). For a link e and $a \in \{1, 2, 3\}$ we write L_e^a and R_e^a for the derivatives of a function of U along $s \mapsto e^{s\tau_a} U_e$ and $s \mapsto U_e e^{s\tau_a}$ at $s = 0$, all other link variables being fixed.

Lemma 5.5. *Let μ be a probability measure on a measurable space, and let S be a measurable event with $p_{\mathrm{fail}} = \mu(S^c) < 1$. Put $\nu = \mu(\cdot \mid S)$. Then the following statements hold.*

1. *With $\|\mu - \nu\|_{\mathrm{TV}} = \sup_A |\mu(A) - \nu(A)|$, we have $\|\mu - \nu\|_{\mathrm{TV}} = p_{\mathrm{fail}}$, and*

$$|\mathbb{E}_\mu F - \mathbb{E}_\nu F| \le 2 p_{\mathrm{fail}} \|F\|_\infty$$

for every bounded complex measurable F; if $0 \le F \le A$, the bound is $p_{\mathrm{fail}} A$. The image measures of μ and ν under a common measurable map satisfy the same bound on the total

variation distance. If $d\mu = Z^{-1} f\, d\omega$ *with* $f \geq 0$*, then the restricted partition function* $Z_S = \int_S f\, d\omega$ *satisfies* $Z_S/Z = 1 - p_{\mathrm{fail}}$*, and* $|\log(Z_S/Z)| \leq 2p_{\mathrm{fail}}$ *if* $p_{\mathrm{fail}} \leq 1/2$*.*

2. *For nested events* $S_0 \supset S_1 \supset \cdots \supset S_k$ *of positive probability and* $0 \leq i \leq k$*,*

$$\mu(S_k) = \mu(S_i) \prod_{j=i}^{k-1} \mu(S_{j+1} \mid S_j).$$

With $r_j = \mu(S_j \setminus S_{j+1} \mid S_j)$*, it follows that* $1 - \mu(S_k) \leq 1 - \mu(S_i) + \sum_{j=i}^{k-1} r_j$*; the successive conditional failure probabilities are added without dividing by* $\mu(S_i)$*.*

3. *For a real function* J *with* $|J| \leq R$*, the tilted probability measure* $\mu_J = e^J \mu / \mathbb{E}_\mu e^J$ *satisfies* $\mu_J(A) \leq \exp(2R)\mu(A)$ *for every event* A*.*

4. *Let* $O_1, \ldots, O_r$ *be real measurable functions with* $|O_i| \leq 1$*, and set* $M_\mu(z) = \mathbb{E}_\mu e^{\sum_i z_i O_i}$ *and* $M_\nu(z) = \mathbb{E}_\nu e^{\sum_i z_i O_i}$ *for* $z \in \mathbb{C}^r$*. If* $\sum_i |z_i| \leq R$*, then*

$$|M_\mu(z) - M_\nu(z)| \leq 2\exp(R) p_{\mathrm{fail}}.$$

If moreover $R < \log 2$*, the branches of* $\log M_\mu$ *and* $\log M_\nu$ *vanishing at* $z = 0$ *satisfy*

$$|\log M_\mu(z) - \log M_\nu(z)| \leq \frac{2\exp(R)}{2 - \exp(R)}\, p_{\mathrm{fail}},$$

and their derivatives satisfy the corresponding Cauchy bounds on every smaller polydisc, with constants depending only on the two polydiscs and the order of the derivative.

5. *Let* ω *be a measure on a fixed domain, let* $V(\theta, w)$ *be real with* $0 < Z(\theta) = \int e^{-V(\theta,w)}\, d\omega(w) < \infty$*, and let* $F(\theta, w)$ *be a function such that both* e^{-V} *and* Fe^{-V} *can be differentiated in* θ *under the integral. Assume also that*

$$\int e^{-V} \big(|\partial_\theta V| + |\partial_\theta F| + |F \partial_\theta V| \big)\, d\omega < \infty$$

at each parameter under consideration. A sufficient condition for these assumptions is pointwise differentiability in θ *together with an* ω*-integrable envelope, uniform for* θ *near the given parameter, for*

$$e^{-V} \big(1 + |\partial_\theta V| + |F| + |\partial_\theta F| + |F \partial_\theta V| \big);$$

joint C^1 *regularity of* V *and* F *on a fixed compact domain and a neighborhood of the parameter suffices. Then, with* $p_\theta = Z(\theta)^{-1} e^{-V(\theta,\cdot)}$*,*

$$\partial_\theta \log Z = -\mathbb{E}_\theta \partial_\theta V, \qquad \partial_\theta \log p_\theta = -\partial_\theta V + \mathbb{E}_\theta \partial_\theta V,$$

$$\partial_\theta \mathbb{E}_\theta F = \mathbb{E}_\theta \partial_\theta F - \mathrm{Cov}_\theta(F, \partial_\theta V).$$

These are identities for the full normalized density, and the covariance is bilinear for complex F*.*

6. *Let μ be the Wilson measure on a periodic lattice, or more generally a probability measure of the form in Lemma 5.1(1), and let $\mathscr{G}_v^a = \sum_{e \text{ starts at } v} L_e^a - \sum_{e \text{ ends at } v} R_e^a$ be the gauge generator at the vertex v. Then for all C^1 functions F and J, possibly complex,*

$$\mathbb{E}_\mu\left[e^J(\mathscr{G}_v^a F + F\mathscr{G}_v^a J)\right] = 0.$$

In a spanning-tree gauge, only the corresponding identity for the simultaneous conjugation at the root remains. If S is invariant under the gauge transformations, the same identity holds under $\mu(\cdot \mid S)$, without any regularity of the boundary of S. The same conclusion holds in any finite joint coordinate representation whose full normalized measure is invariant. In particular, a triangular change of variables $w_i' = O_i(\theta, w_{i+1}, \dots, w_r)w_i$ by orthogonal matrices O_i, independent of w_i, has Jacobian of absolute value one, preserves every product of balls centered at the origin, and its infinitesimal generator has zero flux through every face; the parameter-dependent normalizing factors remain in the joint derivative of the log-density.

7. *If F is integrable but possibly unbounded and $p_{\text{fail}} > 0$, there is only the exact identity*

$$\mathbb{E}_\mu F - \mathbb{E}_\nu F = p_{\text{fail}}\big(\mathbb{E}_{\mu(\cdot|S^c)}F - \mathbb{E}_\nu F\big);$$

an error tending to zero requires additional control of these integrals.

All constants are explicit as displayed.

Proof. Parts (1)–(3). If $p_{\text{fail}} = 0$, then $\mu = \nu$ and there is nothing to prove. For $p_{\text{fail}} > 0$ let $\nu^c = \mu(\cdot \mid S^c)$. Splitting every integral over S and S^c, we obtain

$$\mu = (1 - p_{\text{fail}})\nu + p_{\text{fail}}\nu^c. \tag{5.19}$$

For an event A, we have $\mu(A) - \nu(A) = p_{\text{fail}}(\nu^c(A) - \nu(A))$, where both $\nu(A)$ and $\nu^c(A)$ lie in $[0,1]$. This bounds the total variation distance by p_{fail}, and $A = S$ attains it. For complex F, the two conditional expectations have absolute value at most $\|F\|_\infty$, which gives $2p_{\text{fail}}\|F\|_\infty$; if $0 \le F \le A$, their difference is at most A. A common measurable map replaces each test event by its preimage and therefore does not increase the total variation distance. In the density notation, $Z_S = \int_S f\,d\omega = Z\mu(S)$, and for $0 \le p_{\text{fail}} \le 1/2$ we have $-\log(1 - p_{\text{fail}}) = \int_0^{p_{\text{fail}}}(1-t)^{-1}dt \le 2p_{\text{fail}}$.

For nested events, consecutive conditional probabilities cancel:

$$\prod_{j=i}^{k-1}\mu(S_{j+1} \mid S_j) = \prod_{j=i}^{k-1}\frac{\mu(S_{j+1})}{\mu(S_j)} = \frac{\mu(S_k)}{\mu(S_i)}.$$

Induction on the number of factors gives $1 - \prod_j(1 - r_j) \le \sum_j r_j$, the induction step being $1 - (1-a)(1-b) = a + b - ab \le a + b$. Since $\mu(S_k) = \mu(S_i)\prod_j(1 - r_j) \ge \mu(S_i) - (1 - \prod_j(1 - r_j))$, we obtain $1 - \mu(S_k) \le 1 - \mu(S_i) + \sum_j r_j$. For $|J| \le R$, the density of μ_J with respect to μ is at most $\exp(R)/\exp(-R) = \exp(2R)$.

Part (4). Let $T(z) = \sum_i z_i O_i$; on the ball $\sum_i |z_i| \le R$ we have $|T(z)| \le R$ and $|e^{T(z)}| \le \exp(R)$. Applying part (1) to $e^{T(z)}$, we obtain $|M_\mu - M_\nu| \le 2\exp(R)p_{\text{fail}}$. By the exponential series, we have $|M_\mu(z) - 1| \le \exp(R) - 1$ and $|M_\nu(z) - 1| \le \exp(R) - 1$. If $R < \log 2$, these values lie in the open disc of center 1 and radius $\exp(R) - 1 < 1$, where the series of $\log(1 + w)$ defines the branch vanishing at the origin. Integrating its derivative $1/u$ along the segment between the two values, on which $|u| \ge 2 - \exp(R)$, we find

$$|\log M_\mu(z) - \log M_\nu(z)| \le (2 - \exp(R))^{-1} |M_\mu(z) - M_\nu(z)|.$$

The functions are holomorphic, since bounded observables allow termwise integration of the exponential series. If a smaller polydisc has coordinate distances $\delta_i > 0$ to a larger one inside the ball, the Cauchy formula gives

$$|\partial^\alpha (\log M_\mu - \log M_\nu)| \le \alpha! \prod_i \delta_i^{-\alpha_i} \frac{2\exp(R)}{2 - \exp(R)}\, p_{\text{fail}}$$

at every point of the smaller polydisc.

Part (5). Write $g_\theta = e^{-V(\theta,\cdot)}$, $h_\theta = F(\theta,\cdot)e^{-V(\theta,\cdot)}$, and $N(\theta) = \int h_\theta \, d\omega$. Their pointwise derivatives are $\partial_\theta g_\theta = -(\partial_\theta V)e^{-V}$ and $\partial_\theta h_\theta = (\partial_\theta F - F\partial_\theta V)e^{-V}$. The assumed interchanges give

$$\partial_\theta Z = -\int (\partial_\theta V) e^{-V}\, d\omega, \qquad \partial_\theta N = \int (\partial_\theta F - F\partial_\theta V) e^{-V}\, d\omega. \tag{5.20}$$

The added integrability assumption permits splitting the second integral into its two finite terms. Since $0 < Z < \infty$, dividing by Z gives the first identity, differentiating $\log p_\theta = -V - \log Z$ gives the second, and the quotient rule for $\mathbb{E}_\theta F = N/Z$ gives the covariance identity, with every term finite. For a vector parameter, this argument applies to each coordinate derivative.

To verify the sufficient envelope condition, fix θ_0 and a neighborhood on which the integrable envelope $B(w)$ is valid. It bounds $|g_\theta|$, $|h_\theta|$, and both derivatives; the mean value theorem, applied to real and imaginary parts, bounds the difference quotients by $2B$. Dominated convergence proves the interchanges in (5.20). The same envelope gives the additional integrability of the separated terms. For the sufficient condition on a compact domain Ω, choose a compact neighborhood of the parameter inside the domain of C^1 regularity. Joint continuity bounds $|V|$, $|F|$, $|\partial_\theta V|$, and $|\partial_\theta F|$ by some $C \ge 1$ there. Moreover $e^{-V(\theta_0,w)} \ge e^{-C}$ and $Z(\theta_0) < \infty$ imply $\omega(\Omega) \le e^C Z(\theta_0) < \infty$. Then the constant $e^C(1 + 3C + C^2)$ is an integrable envelope, uniformly on this neighborhood. No boundary term occurs, since the domain is fixed. In particular, a term of V that depends only on θ cancels in the derivative of $\log p_\theta$ only together with its contribution to $\log Z$; it cannot be deleted from a joint density.

Part (6). Let $k(s)$ equal $e^{s\tau_a}$ at v and I at all other vertices. By Lemma 5.1, μ is invariant under $U \mapsto U^{k(s)}$, which gives

$$\int (Fe^J)(U^{k(s)})\, d\mu(U) = \int Fe^J\, d\mu.$$

Differentiation at $s = 0$ is valid for C^1 functions on the compact manifold of configurations. On a link starting at v the derivative is L_e^a, and on a link ending at v it is $-R_e^a$; their sum is $\mathscr{G}_v^a$, and the product rule gives the identity, for real or complex J. In a spanning-tree gauge, the conjugation at the root described in Lemma 5.1 acts on every remaining link by $U_e \mapsto kU_ek^{-1}$; its derivative is $\sum_{e\notin T}(L_e^a - R_e^a)$, and the same change of variables gives the remaining identity. No generator at a vertex other than the root is used on the gauge-fixed configurations. If S is invariant, we insert $\mathbf{1}_S$ into the integral before the change of variables and divide by $\mu(S) > 0$. Neither the regularity of ∂S nor a derivative of $\mathbf{1}_S$ is needed, since the indicator is unchanged for every s.

For the triangular change of variables, given the transformed variables of higher index, we solve for w_i using O_i^{-1}, proceeding downward. This shows that the map is invertible, and it preserves the product of balls. Its Jacobian matrix is block triangular with orthogonal diagonal blocks O_i, and its determinant therefore has absolute value one. For a differentiable one-parameter family of such maps, the generator is $Y_i = B_i(\theta, w_{>i})w_i$ with $B_i^{\mathsf{T}} = -B_i$. Hence

$$\operatorname{div}_{w_i} Y_i = \operatorname{tr} B_i = 0, \qquad w_i \cdot Y_i = 0 \quad \text{on } |w_i| = p_i,$$

and the flux through every face vanishes; intersections of faces carry no additional flux, since the map preserves each radius $|w_i|$. Under the assumed invariance of the full joint probability measure, we obtain the gauge-invariance identity by differentiating the change-of-variables identity. If this measure is written with conditional factors $Z(\theta)^{-1}e^{-V(\theta,w)}$, their parameter derivatives are the derivatives of the normalized log-densities computed in part (5); a derivative of an unnormalized conditional density is never substituted into a joint identity.

Part (7). The identity (5.19) applies to an integrable F when $p_{\text{fail}} > 0$, and both conditional integrals are finite since both events have positive probability. Without further control there is no small error: on a two-point space, give S^c probability p_{fail} and take $F = p_{\text{fail}}^{-2}\mathbf{1}_{S^c}$; then $\mathbb{E}_\nu F = 0$ and $\mathbb{E}_\mu F = p_{\text{fail}}^{-1}$. □

5.6 Invariant functions of three holonomies

The weak-coupling results concern continuous functions of the three holonomies H_1, H_2, H_3 around the coordinate cycles of the torus, with a common basepoint, that are invariant under simultaneous conjugation. The next lemma shows that such functions

are exactly the continuous functions of seven traces. If we write $H_i = a_i I + v_i \cdot t$, the traces of H_i and H_iH_j determine the scalar parts a_i and the Gram matrix of the vectors v_i. The trace of $H_1H_2H_3$ determines in addition the orientation $\det(v_1, v_2, v_3)$, and a triple of vectors is determined up to rotation by its Gram matrix and its orientation. These seven coordinates are the case of three generators in the classical theory of trace coordinates of Horowitz [111]; the argument below proves the separation and the continuous factorization for all triples in $\mathrm{SU}(2)$, including those of deficient rank.

Lemma 5.6. *Define* $\mathfrak{p} : \mathrm{SU}(2)^3 \to [-1,1]^7$ *by the normalized traces of* $H_1, H_2, H_3, H_1H_2, H_1H_3, H_2H_3, H_1H_2H_3$, *in this order. For all* $H, H' \in \mathrm{SU}(2)^3$,

$$\mathfrak{p}(H) = \mathfrak{p}(H') \quad \Longleftrightarrow \quad \text{there exists } k \in \mathrm{SU}(2) \text{ with } H_i' = kH_ik^{-1} \text{ for } i = 1,2,3.$$

This includes all triples of deficient rank and the central triples. Explicitly, if $H_i = a_i I + 2v_i^a \tau_a$ *with* $a_i^2 + |v_i|^2 = 1$, *the normalized trace of* H_iH_j *is* $a_ia_j - v_i \cdot v_j$, *and that of* $H_1H_2H_3$ *is*

$$a_1a_2a_3 - a_3 v_1 \cdot v_2 - a_1 v_2 \cdot v_3 - a_2 v_1 \cdot v_3 - \det(v_1, v_2, v_3).$$

The map induced by $\mathfrak{p}$ *from the quotient of* $\mathrm{SU}(2)^3$ *by simultaneous conjugation onto* $\mathfrak{p}(\mathrm{SU}(2)^3)$ *is a homeomorphism. Every continuous function* $F : \mathrm{SU}(2)^3 \to \mathbb{C}$ *invariant under simultaneous conjugation has the form* $F = f \circ \mathfrak{p}$, *where* f *is continuous on* $\mathfrak{p}(\mathrm{SU}(2)^3)$ *and extends to a continuous function on* $[-1,1]^7$ *with* $\|f\|_\infty \le \|F\|_\infty$. *In particular, if the trace vectors* $\mathfrak{p}(H^{(k)})$ *of random triples* $H^{(k)}$ *converge jointly in distribution, then for every such* F *the expectations* $\mathbb{E}F(H^{(k)})$ *converge, to the expectation of* f *under the limiting law, which is supported on* $\mathfrak{p}(\mathrm{SU}(2)^3)$.

The lemma concerns one triple of matrices and its seven traces; it does not by itself relate lattice holonomies to continuum observables.

Proof. By (5.1), $H_i = a_i I + v_i \cdot t$ with $t_a = 2\tau_a$; the scalar part a_i is the normalized trace, and $a_i^2 + |v_i|^2 = 1$. The scalar part of H_iH_j is $a_ia_j - v_i \cdot v_j$. Multiplying once more, we find that the scalar part of $H_1H_2H_3$ is

$$(a_1a_2 - v_1 \cdot v_2)a_3 - (a_1 v_2 + a_2 v_1 + v_1 \times v_2) \cdot v_3,$$

which expands to the stated formula, with the minus sign in front of $\det(v_1, v_2, v_3) = (v_1 \times v_2) \cdot v_3$ for our choice of τ_a.

Suppose $\mathfrak{p}(H) = \mathfrak{p}(H')$. Then $a_i = a_i'$, the Gram matrices $(v_i \cdot v_j)$ and $(v_i' \cdot v_j')$ agree (including the diagonal entries $1 - a_i^2$), and $\det(v_1, v_2, v_3) = \det(v_1', v_2', v_3')$. We define a map between the spans by $T(\sum_i c_i v_i) = \sum_i c_i v_i'$. Equality of the Gram matrices shows that T is well defined, since if $\sum_i c_i v_i = 0$ then $|\sum_i c_i v_i'|^2$ is the same quadratic form in c and vanishes, and that T is an isometry onto the span of the v_i'. Completing orthonormal bases of the two spans to bases of $\mathbb{R}^3$, we extend T to an orthogonal map R. If the

common rank is three, equality of the nonzero determinants implies $\det R = 1$. If the rank is at most two and $\det R = -1$, we compose R with the reflection in a direction perpendicular to the span of the v_i'; this fixes every v_i' and changes the determinant to $+1$. Rank zero covers the central triples. In every case some $R \in \mathrm{SO}(3)$ maps each v_i to v_i'. By the surjectivity of Ad (Lemma 5.1) there exists $k \in \mathrm{SU}(2)$ with $\mathrm{Ad}(k) = R$. Since conjugation does not change the scalar parts, we have $H_i' = kH_ik^{-1}$. Conversely, simultaneous conjugation preserves the trace of every product, by cyclicity of the trace.

Let Q_0 be the quotient of $\mathrm{SU}(2)^3$ by simultaneous conjugation, with the quotient topology, and let $A = \mathfrak{p}(\mathrm{SU}(2)^3)$. Then Q_0 is compact, as a continuous image of a compact space, and so is A. Since the polynomial map $\mathfrak{p}$ is continuous and constant on equivalence classes, it induces a continuous map $Q_0 \to A$, which is bijective by what we just proved. A closed subset of Q_0 is compact, and its image is compact and hence closed in the metric space A. Thus the inverse map is continuous, and the induced map is a homeomorphism.

A continuous invariant function F descends to a continuous function on Q_0, which defines a continuous function f_A on A with $\|f_A\|_\infty = \|F\|_\infty$. We construct a continuous extension to the cube. For a real continuous function g on the compact set A with $|g| \le B$, set $g_T(x) = \inf_{y \in A}\{g(y) + T|x - y|\}$ for $x \in [-1,1]^7$. By the triangle inequality g_T is T-Lipschitz, and $g_T \le g$ on A. Given $\varepsilon > 0$, choose $\delta > 0$ from the uniform continuity of g, and T with $T\delta \ge 2B$. For $x \in A$, the terms with $|x - y| \le \delta$ are at least $g(x) - \varepsilon$, and the remaining terms are at least $-B + T\delta \ge B \ge g(x)$. Thus $g - \varepsilon \le g_T \le g$ on A, and clipping g_T to $[-B, B]$ preserves its continuity and this approximation, and it bounds the result by B on the cube. Let $M = \|f_A\|_\infty$. If $M = 0$, extend by zero. Otherwise let $r_0 = f_A$. If $\|r_k\|_\infty \le B_k = 2^{-k}M$, apply this construction to the real and imaginary parts of r_k, with errors at most $B_k/(2\sqrt{2})$; this gives a continuous complex function h_k on the cube with $\|h_k\|_\infty \le \sqrt{2}B_k$ and $\|r_k - h_k|_A\|_\infty \le B_k/2$, and we set $r_{k+1} = r_k - h_k|_A$. Since $\sum_k \sqrt{2}B_k < \infty$, the series $h = \sum_k h_k$ converges uniformly on the cube. Then h is continuous, and $h = f_A$ on A since the residuals tend to zero uniformly. Composing h with the radial retraction of $\mathbb{C}$ onto the closed disc of radius M,

$$z \longmapsto \begin{cases} z, & |z| \le M, \\ Mz/|z|, & |z| > M, \end{cases}$$

which is continuous and fixes the values of f_A, gives a continuous extension f with $\|f\|_\infty \le M$. Finally, $\mathbb{E}F(H^{(k)}) = \mathbb{E}f(\mathfrak{p}(H^{(k)}))$. Since f is bounded and continuous on the cube, convergence in distribution of the traces implies convergence of the expectations. The limiting law, a probability measure on the cube, is supported on the closed set A, since the nonnegative continuous function $\mathrm{dist}(\cdot, A)$ has expectation zero under every approximating law and hence under the limit. The limit of the expectations is therefore the integral of the intrinsically defined function f_A. □

SECTION 6

Block averaging and chessboard estimates

In this section, we prove the finite-dimensional properties of the block averaging map T_M of Definition 4.3 that are used throughout the renormalization group analysis, and the reflection positivity estimates for the Wilson measure on which the bound on large fields rests. Section 6.1 shows that T_M is Borel, gauge covariant, and covariant under translations by coarse vectors and coordinate reflections $x_j \mapsto a_j - x_j$, $a_j \in M\mathbb{Z}$, also where its logarithm is not defined, and that its iterates have nested local dependence sets (Proposition 6.2). Section 6.2 constructs local constraint coordinates near the coarse-field lift $U_0(b)$ for every coarse configuration b, restores the block gauge variables, and computes Haar measure on the resulting coordinate domain exactly (Lemma 6.3); this formula is the starting point of every fluctuation integral in Parts IV and VI. Section 6.3 proves that the curl bounds the ℓ^2 norm on the linearized constrained fluctuation space, uniformly in the volume (Lemma 6.5); this is an input of Section 8. Section 6.4 contains reflection positivity and chessboard estimates for the Wilson measure and its images under T_M^ℓ, and exponential moment bounds for plaquette energies on the finest lattice (Theorem 6.6); these are used in Sections 7, 22, and 23. The inputs are the conventions of Definitions 4.1 and 4.3, the Haar measure, tree gauge, and finite-word estimates of Lemma 5.1, and the coefficient calculus of Lemma 5.3. No later result is used.

Throughout this section, we measure distances and dependence sets in fine lattice units; physical rescaling multiplies every spatial coordinate by the fine mesh. Unless a more specific dependence is stated, constants depend only on M, on the stated coordinate radii, and, for derivative estimates, on the prescribed finite derivative order. They are independent of L, the lattice side, the coupling, and the number of preceding averaging steps; this is the dependence described in item (1) of Definition 4.14.

6.1 The block averaging map

The block averaging map is an instance of Bałaban's averaging operation for lattice gauge fields [12, eq. (15)]: the fine link variables of a block are transported to a common reference path, their logarithms are averaged, and the average is exponentiated. It is defined in Definition 4.3; we fix the additional notation used in this section and the next.

Definition 6.1. Let $M = 2R_M + 1 \geq 3$ be odd, let the fine periodic lattice have side Mm with $m \geq 4$, work in fine lattice units, and let T_M be the block averaging map of Definition 4.3, with the centered blocks $\mathsf{B}_M(v)$, $v \in M\mathbb{Z}^3$, the root paths $\sigma_{v,x}$, the reference paths C_c, and the paths

$$F_{c,x} = \sigma_{v,x}\,[x, x + Me_i]\,\sigma^{-1}_{v+Me_i,x+Me_i}, \qquad c = (v, v + Me_i),\ x \in \mathsf{B}_M(v),$$

defined there; we call the $F_{c,x}$ translated paths with endpoint connectors. The logarithm in the first clause of (4.3) is the principal logarithm on $\mathrm{SU}(2) \setminus \{-I\}$, with $|\mathrm{Log}\, V| < 2\pi$, and we call the second clause, $(T_M U)_c = U(C_c)$ when some based loop holonomy $U(F_{c,x})U(C_c)^{-1}$ equals $-I$, the Borel extension of T_M. For odd $s \geq 1$ and a vertex v set

$$\mathsf{B}_s(v) = v + \{-(s-1)/2, \dots, (s-1)/2\}^3,$$

so that $\mathsf{B}_1(v) = \{v\}$. The links of the paths $\sigma_{v,x}$, $x \in \mathsf{B}_M(v)$, form a spanning tree of the block (Proposition 6.2), which we call the block tree, with root v. For $\ell \geq 1$, T_M^ℓ denotes the ℓ-fold composition of T_M; one renormalization group step is the composition T_M^J, not an average over blocks of side M^J. For a configuration b on the coarse lattice, the coarse-field lift $U_0(b)$ is the fine configuration that equals I on every link with both endpoints in one block, and equals b_c on every link from $\mathsf{B}_M(v)$ to $\mathsf{B}_M(v + Me_i)$, where $c = (v, v + Me_i)$.

Figure 6.1 shows these paths for two neighboring blocks.

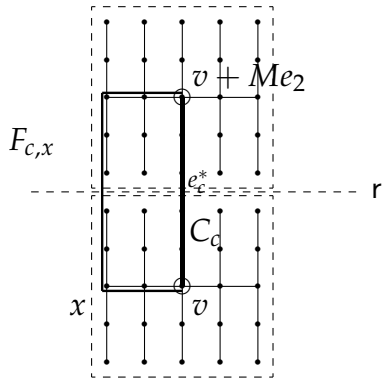


Figure 6.1. Two neighboring centered blocks for $M = 5$, in the plane spanned by e_1 and e_2, with $c = (v, v + Me_2)$. Dashed squares: the blocks $\mathsf{B}_M(v)$ and $\mathsf{B}_M(v + Me_2)$. Thin lines: the block trees. Thick line: the reference path C_c; it contains the eliminated edge e_c^* of Section 6.2. Medium line: the translated path with endpoint connectors $F_{c,x}$ for $x = v - 2e_1$. Dotted line: the plane of a reflection r bisecting the links between the two blocks; each block lies on one side of it.

Proposition 6.2. *Let M and m be as in Definition 6.1. Then the following statements hold.*

1. *The centered blocks partition the fine vertices. In each block the links of the paths $\sigma_{v,x}$ form a spanning tree with $M^3 - 1$ links, and the paths from the root v in this tree are the $\sigma_{v,x}$.*
2. *T_M is a Borel map on the whole configuration space $\mathcal{U}_{Mm}$.*
3. *For every periodic gauge transformation k, $T_M(U^k) = (T_M U)^{k|_{M\mathbb{Z}^3}}$.*
4. *T_M commutes with the translations by vectors of $M\mathbb{Z}^3$ and with every coordinate reflection $x_j \mapsto a_j - x_j$ with $a_j \in M\mathbb{Z}$, in particular with the spatial inversion $x \mapsto -x$. This includes the configurations at which the Borel extension is used. These reflections map block trees onto block trees; no invariance under permutations of the coordinates is needed or claimed.*
5. *For every coarse configuration b, without any restriction on its holonomies, $T_M \mathcal{U}_0(b) = b$.*
6. *Let $\ell \geq 0$, $m_\ell = M^\ell$, and let the fine side be $N = m_\ell n$ with $n \geq 4$. The value of $T_M^\ell U$ on the coarse link $(v, v + m_\ell e_i)$ depends only on the fine links with both endpoints in*

$$\mathsf{B}_{m_\ell}(v) \cup \mathsf{B}_{m_\ell}(v + m_\ell e_i).$$

For $\ell \geq 1$ the blocks are nested:

$$\bigcup_{\delta \in \{-R_M, \dots, R_M\}^3} \mathsf{B}_{M^{\ell-1}}\big(v + M^{\ell-1}\delta\big) = \mathsf{B}_{M^\ell}(v). \tag{6.1}$$

All statements are deterministic. The sets in (6) do not depend on L, the coupling, the number of preceding averaging steps, or the fields outside them.

Proof. Blocks and trees. In one coordinate, the intervals $Mk + \{-R_M, \dots, R_M\}$, $k \in \mathbb{Z}$, consist of M consecutive integers and have centers at distance M. They therefore partition $\mathbb{Z}$, and also $\mathbb{Z}/Mm\mathbb{Z}$. Taking products, we see that the blocks partition the vertices, including across the periodic identifications. For a vertex $x \neq v$ of $\mathsf{B}_M(v)$, we define the parent of x by decreasing the absolute value of the last nonzero component of $x - v$ by one. Since the ℓ^1 distance to v decreases strictly along parents and every vertex reaches v, the links from vertices to their parents form a tree with $M^3 - 1$ links. Read from v, the path to x in this tree changes first the first coordinate, then the second, then the third; it is $\sigma_{v,x}$. This proves (1).

Measurability and gauge covariance. Every transport along a translated path with endpoint connectors or a reference path is a finite product of link matrices and their inverses. The set of configurations at which none of the finitely many based loop holonomies equals $-I$ is open. Since the principal logarithm is continuous on $\mathrm{SU}(2) \setminus \{-I\}$, the first clause of (4.3) defines a continuous map on this open set. On its closed complement, $(T_M U)_c = U(C_c)$ is continuous. This proves (2); no value of $\mathrm{Log}(-I)$ is used. Let k be a periodic gauge transformation. The paths $F_{c,x}$ and C_c both start at v and end at $v + Me_i$, so each based loop holonomy $U(F_{c,x})U(C_c)^{-1}$ is conjugated by k_v. On $\mathrm{SU}(2) \setminus \{-I\}$ the principal logarithm satisfies $\mathrm{Log}(kVk^{-1}) = \mathrm{Ad}(k)\,\mathrm{Log}\,V$, since both sides exponentiate to kVk^{-1} and lie in the ball of radius 2π on which the exponential

map is injective. The condition that some based loop holonomy equals $-I$ is invariant under conjugation. Multiplying by the reference transport $k_v U(C_c) k_{v+Me_i}^{-1}$ gives (3), in both cases of the definition.

Translations and reflections. A translation by a vector of $M\mathbb{Z}^3$ maps blocks, root paths, translated paths with endpoint connectors, and reference paths to the corresponding objects. For reflections we use the following identity. Let F_x, $x \in \mathsf{B}_M(v)$, and C be elements of $\mathrm{SU}(2)$ with $F_x C^{-1} \neq -I$ for all x, and let $A = M^{-3} \sum_x \operatorname{Log}(F_x C^{-1})$. Then

$$\begin{aligned} \operatorname{Log}(F_x^{-1} C) &= -\operatorname{Ad}(C^{-1}) \operatorname{Log}(F_x C^{-1}), \\ \exp\Big(M^{-3} \sum_x \operatorname{Log}(F_x^{-1} C)\Big) C^{-1} &= C^{-1} e^{-A} = (e^A C)^{-1}. \end{aligned} \tag{6.2}$$

The first line follows from $F_x^{-1} C = C^{-1} (F_x C^{-1})^{-1} C$, the conjugation property of Log just used, and $\operatorname{Log}(V^{-1}) = -\operatorname{Log} V$; the second follows from the first. If one of the elements $F_x C^{-1}$ equals $-I$, then its conjugated inverse $F_x^{-1} C$ also equals $-I$, and the two values given by the Borel extension are C and C^{-1}, which are again inverse to each other.

We write a coordinate reflection preserving $M\mathbb{Z}^3$ as $x_j \mapsto a_j - x_j$ with $a_j \in M\mathbb{Z}$. It maps blocks to blocks and centers to centers, and it changes the signs of the steps of a path without changing the order in which the coordinates are changed. In particular, it maps each block tree and each root path $\sigma_{v,x}$ to the corresponding block tree and root path. If $j \neq i$, it maps $F_{c,x}$ and C_c to the corresponding paths of the reflected positive link. If $j = i$, let s denote the reflection, let $c' = (s(v + Me_i), s(v))$ be the positive link associated with the image of c, and let $y = s(x + Me_i)$, which lies in the initial block of c'. Substituting the reflected endpoint paths, we find

$$s F_{c,x} = F_{c',y}^{-1}, \qquad s C_c = C_{c'}^{-1},$$

and $x \mapsto y$ is a bijection between the two initial blocks. Then (6.2), including its exceptional case, implies reflection covariance. Products of the three coordinate reflections give the spatial inversion. This proves (4).

Right inverse. Each straight segment $[x, x + Me_i]$ passes from the initial block of c into the final block exactly once; its transverse coordinates stay in the common transverse range of the two blocks, and both root paths stay in their blocks. The same holds for the reference path. For $U_0(b)$, every link inside a block has transport I and the single interface link crossed has transport b_c. This gives $U_0(b)(F_{c,x}) = U_0(b)(C_c) = b_c$ for every x. Every based loop holonomy then equals I, and $(T_M U_0(b))_c = b_c$. This proves (5).

Nesting and dependence sets. Let $s = M^{\ell-1}$. In one coordinate, the intervals $s\delta + \{-(s-1)/2, \dots, (s-1)/2\}$, $-R_M \le \delta \le R_M$, are consecutive disjoint integer intervals, with extreme endpoints $\pm(sR_M + (s-1)/2) = \pm(Ms - 1)/2$. Taking products in three coordinates yields (6.1), including the case $s = 1$. For $\ell = 1$, the dependence statement in (6) follows from the description of the three pieces of each translated path with endpoint

connectors in the previous paragraph, and from the fact that the reference path lies in the two blocks. Suppose that it holds for $\ell - 1$. The last averaging step reads only intermediate links with both endpoints in the two centered blocks of intermediate vertices around the final endpoints. By the induction hypothesis, each of these intermediate links depends only on fine links in the blocks of side $M^{\ell-1}$ around its endpoints. By (6.1) the union of these blocks lies in the two claimed blocks of side M^ℓ. The decision whether the Borel extension is used reads the same based loop holonomies and does not enlarge the dependence set. By induction, this proves (6) for the composition T_M^ℓ. Since $n \geq 4$, the union of two blocks of side M^ℓ has extent $2M^\ell < M^\ell n$ in each coordinate, and all these sets lift to boxes in the covering lattice $\mathbb{Z}^3$, also across the periodic identifications. □

6.2 Local constraint coordinates and Haar measure

We now parametrize the configurations near a level set $\{T_M U = b\}$. In each block we fix the gauge by the block tree, with gauge transformations equal to I at the block centers; we call this the block axial gauge. For each positive coarse link $c = (v, v + Me_i)$ we then eliminate one link variable, on the central interface link

$$e_c^* = \big(v + R_M e_i,\, v + (R_M + 1)e_i\big),$$

by solving the equation $(T_M U)_c = b_c$. This edge, which we call the eliminated edge, lies on the reference path C_c (Figure 6.1), and C_c consists of e_c^* and of links of the two block trees.

The equation can be solved with radii independent of b. To see this, write $U = e^X U_0(b)$ on the links not in the block trees, and suppose that only $X_{e_c^*} = \omega$ is nonzero. Exactly M of the M^3 paths $F_{c,x}$ cross e_c^*, namely those with $x - v \in \mathbb{Z}e_i$, while the reference path always crosses it. Then $M^3 - M$ of the based loop holonomies equal $e^{-\omega}$, the others equal I, and

$$(T_M U)_c = e^{-(1-M^{-2})\omega} e^{\omega} b_c = e^{M^{-2}\omega} b_c.$$

The derivative of the constraint in the eliminated variable is therefore $M^{-2} I_3$ at every real coarse field b, including coarse fields with large holonomies. We construct the local inverse in a neighborhood of $U_0(b)$ rather than of the identity configuration.

The variables are counted as follows; here f denotes the number of remaining SU(2) variables.

fine links	$3(Mm)^3$
links of the block trees	$(Mm)^3 - m^3$
links not in the block trees	$2(Mm)^3 + m^3$
eliminated edges e_c^*, one for each positive coarse link	$3m^3$
remaining SU(2) variables	$f = 2((Mm)^3 - m^3)$
real dimension of the free logarithmic coordinates z	$3f$

We obtain the Haar formula in Lemma 6.3 by two successive changes of variables. First, with the free link matrices fixed, we replace the eliminated link matrices by the coarse output $T_M U$. Then we translate the free link matrices on the right by $U_0(b)^{-1}$ and pass to their logarithms. In the first change, the constant c_{Haar} of the output chart cancels that of the eliminated edges, but their Haar density j_H remains. For this reason the numerator of the Haar disintegration density $\mathscr{J}_b$ below contains j_H for every link not in the block trees. Recall from Lemma 5.1 that $dH(e^X) = c_{\mathrm{Haar}}\, j_H(X)\, d^3X$ for $|X| < 2\pi$, with $c_{\mathrm{Haar}} = (16\pi^2)^{-1}$ and $j_H(X) = \mathrm{sinc}^2(|X|/2)$.

Lemma 6.3. *Let M, m, and the block trees be as in Definition 6.1, and fix the block axial gauge. For a coarse configuration b write $U_e = e^{X_e} U_{0,e}(b)$ on the links e not in the block trees, and $U_e = I$ on the links of the block trees. For each positive coarse link c define*

$$\mathsf{C}_{b,c}(X) = \mathrm{Log}\big((T_M(e^X U_0(b)))_c\, b_c^{-1}\big).$$

Call the variables X_e with e not in the block trees and not an eliminated edge the free logarithmic coordinates, and denote their vector by z.

1. *Among the eliminated variables $X_{e_{c'}^*}$, the function $\mathsf{C}_{b,c}$ depends only on $X_{e_c^*}$. It depends on at most $6M^3$ links, all with both endpoints in the two blocks of c, and each link enters at most six of the functions $\mathsf{C}_{b,c}$. For every real b,*

$$\mathsf{C}_{b,c}(0) = 0, \qquad D_{X_{e_c^*}} \mathsf{C}_{b,c}(0) = M^{-2} I_3.$$

2. *There exist $r_M, \widetilde{r}_M > 0$, depending only on M, with the following property. For every real b and every vector z of free logarithmic coordinates with $\max_e |z_e| < r_M$, there exists a unique X whose free entries equal z, whose eliminated entries have norm less than $\widetilde{r}_M$, and which satisfies $\mathsf{C}_{b,c}(X) = 0$ for all c. Writing $X = \Gamma_b(z)$, we have*

$$\Gamma_b(0) = 0, \qquad (\Gamma_b(z))_{\mathrm{free}} = z, \qquad T_M\big(e^{\Gamma_b(z)} U_0(b)\big) = b.$$

   The map Γ_b is real analytic and has a holomorphic extension, jointly in the free logarithmic coordinates and in local complex coordinates of the target around every real b. The eliminated entry of Γ_b at e_c^* depends only on the free logarithmic coordinates in the two blocks of c. For each order $s \geq 1$, the derivatives of order s of Γ_b on a fixed smaller complex polydisc have absolute row and column sums bounded in terms of M, s, and the ratio of the radii of the two polydiscs. These bounds do not depend on m, on b, or on the number of preceding averaging steps. All determinants $\det D_{X_{e_c^*}} \mathsf{C}_{b,c}(\Gamma_b(z))$ are positive.

3. *Let $f = 2((Mm)^3 - m^3)$ be the number of free $\mathrm{SU}(2)$ variables, and set*

$$\mathscr{J}_b(z) = c_{\mathrm{Haar}}^f \, \frac{\prod_{e \notin \text{block trees}} j_H\big((\Gamma_b(z))_e\big)}{\prod_c \det D_{X_{e_c^*}} \mathsf{C}_{b,c}(\Gamma_b(z))}.$$

For a Borel set $\mathcal{Z} \subset \{\max_e |z_e| < r_M\}$, let $\mathcal{D}_{\mathcal{Z}}$ be the set of configurations $(e^{\Gamma_b(z)} U_0(b))^k$ with b a coarse configuration, $z \in \mathcal{Z}$, and k a gauge transformation equal to I at every block center. Then for every nonnegative Borel function F on $\mathcal{U}_{Mm}$,

$$\int_{\mathcal{D}_{\mathcal{Z}}} F(U)\, dH_{Mm}(U) = \int dH_m(b) \int_{\mathcal{Z}} \mathcal{J}_b(z)\, dz \int F\big((e^{\Gamma_b(z)} U_0(b))^k\big)\, dH_{\mathrm{for}}(k),$$

where dz is Lebesgue measure on $\mathbb{R}^{3f}$ and H_{for} is the product Haar probability measure on the values of k at the vertices other than block centers. For gauge-invariant F the inner integral equals $F(e^{\Gamma_b(z)} U_0(b))$. The numerator of $\mathcal{J}_b$ contains the Haar density of every eliminated edge, and no further factor (no density in a coarse logarithm, no Gram determinant, and no volume of a gauge orbit) occurs.

4. *Let $g > 0$. The substitution $z = g\eta$ turns $\mathcal{J}_b(z)\, dz$ into $g^{3f}\, \mathcal{J}_b(g\eta)\, d\eta$. Put $V(b, z) = e^{\Gamma_b(z)} U_0(b)$, and let A be a Borel event. If $F(U) = \exp[-4g^{-2} \sum_p q(U_p)] \mathbf{1}_A(U)$, then after integrating the forest gauges the joint density relative to $dH_m(b)\, d\eta$ is*

$$g^{3f}\, \mathcal{J}_b(g\eta) e^{-4g^{-2} \sum_p q(V(b, g\eta)_p)} \mathbf{1}_{\{g\eta \in \mathcal{Z}\}} \int \mathbf{1}_A\big(V(b, g\eta)^k\big)\, dH_{\mathrm{for}}(k).$$

For an event invariant under the forest gauge transformations, the last integral equals $\mathbf{1}_A(V(b, g\eta))$. For an arbitrary event the gauge average is retained. After integrating over η and normalizing, provided the restricted mass is positive, one obtains the image under T_M of the restricted measure; in general it differs from the image of the unrestricted Wilson measure, since the mass of the restriction depends on the target.

5. *At fixed real b, a coarse gauge transformation h, extended to be constant on each block, acts on the free logarithms by $X_e \mapsto \mathrm{Ad}(h_v) X_e$, where v is the center of the block containing the initial vertex of e. The spatial inversion acts, together with the induced permutation of links, by*

$$X'_{e'} = -\,\mathrm{Ad}\big(U_{0,e}(b)^{-1}\big) X_e.$$

These orthogonal maps commute with Γ and preserve the Wilson action and $\mathcal{J}$ exactly. They map products of three-dimensional Euclidean balls to the products with permuted radii; products of balls with a common radius are invariant. Although the maps depend on b, the joint change of variables (b, z) has a block-triangular derivative whose diagonal block in z has absolute determinant one.

All constants depend only on M, the stated coordinate radii, and, where applicable, the prescribed derivative order. The only restriction involving g is that $g\eta$ lies in the coordinate domain of (2).

The uniqueness in (2) holds in the stated neighborhood of $U_0(b)$, not on the whole level set $\{T_M U = b\}$, and the formula in (3) concerns only the set $\mathcal{D}_{\mathcal{Z}}$. We construct Γ_b by a contraction argument in one coarse link at a time; this is why the radii do not depend on the number of blocks.

Proof. We proceed in six steps.

Step 1: locality of the constraints. We apply the block version of the tree identity of Lemma 5.1, with every block center fixed; the block trees have $(Mm)^3 - m^3$ links. In this gauge every reference path has exactly one link not in a block tree, namely its eliminated edge. Since a straight segment crosses only the interface of c and a root path never crosses an interface, no constraint contains the eliminated variable of another constraint. The remaining variables of $\mathsf{C}_{b,c}$ are links in the two blocks of c and on their common interface, at most $6M^3$ positive links. A free link inside a block can affect only the constraints of the six coarse links incident to the center of that block, and a free interface link affects only its own constraint. In particular, every variable enters at most six constraints.

Step 2: the linear coefficient. If all free variables vanish and only $X_{e_c^*} = \omega$ is varied, then exactly M of the M^3 straight segments cross e_c^*: their two transverse offsets vanish and their longitudinal offset has M possible values. The translated paths with endpoint connectors that contain them, and the reference path, have transport $e^{\omega} b_c$; all other such paths have transport b_c. For ω small every based loop holonomy different from I equals $e^{-\omega}$. Hence

$$(T_M(e^X U_0(b)))_c = e^{-(1-M^{-2})\omega} e^{\omega} b_c = e^{M^{-2}\omega} b_c, \qquad \mathsf{C}_{b,c}(\omega, 0) = M^{-2}\omega. \tag{6.3}$$

This gives the value and the derivative in (1) at every real b, including coarse fields whose holonomies do not commute.

Step 3: analytic neighborhoods. In local target coordinates we write $b'_c = e^{\theta_c} b_c$ with b_c real. The word $F_{c,x} C_c^{-1}$ has at most $5M - 3$ fine factors, and at $X = 0$ it equals I, also for the complex target b', since the identity $U_0(b)(F_{c,x}) = U_0(b)(C_c)$ is algebraic. There exists $\epsilon_M > 0$, depending only on M, such that for $\max_e |X_e| < \epsilon_M$ and $\max_c |\theta_c| < \epsilon_M$ all these words and the outer transport $T_M(e^X U_0(b'))_c b_c'^{-1}$ have matrix distance less than $1/4$ from I. Indeed, we insert the factors e^{X_e} one at a time into the fixed word: real compact factors have operator norm one, the complex target factors and their inverses have norm at most $e^{|\theta_c|}$, and $\|e^Y - I\| \le e^{|Y|} - 1$. The resulting bound tends to zero uniformly in the real background. The outer transport tends to I by the same bound applied to the finite sum of inner logarithms. On this domain every logarithm is given by the convergent series $\sum_{k \ge 1} (-1)^{k+1} (V - I)^k / k$. By the finite-word estimates of Lemma 5.1, we have first and second derivative bounds there, uniform in the real background.

Step 4: construction of the inverse. We solve one constraint at a time. In one constraint let $\omega = X_{e_c^*}$ and let z denote its free logarithmic coordinates, with $\|z\|_\infty$ the maximum of their norms. After decreasing ϵ_M, we fix $A_M^{\mathrm{an}}, K_M^{\mathrm{an}} \ge 1$ such that, for all local targets with $\max|\theta| < \epsilon_M/2$ and $|\omega|, \|z\|_\infty < \epsilon_M/2$,

$$\begin{aligned} |\mathsf{C}_{b',c}(0, z)| &\le A_M^{\mathrm{an}} \|z\|_\infty, \\ \|D_\omega \mathsf{C}_{b',c}(\omega, z) - M^{-2} I\| &\le K_M^{\mathrm{an}} \big(|\omega| + \|z\|_\infty\big). \end{aligned} \tag{6.4}$$

The first line follows from the first derivative bound and $\mathsf{C}_{b',c}(0,0) = 0$. The identity (6.3) continues to the complex target coordinates, so $D_\omega \mathsf{C}_{b',c}(0,0) = M^{-2} I$, and the second line follows from the second derivative bound. All these norms concern at most $6M^3$ links, independently of m. Fix

$$\delta_M \le \min\{\epsilon_M/4,\ (8M^2 K_M^{\mathrm{an}})^{-1},\ 1/4\}, \qquad \widetilde{r}_M = \delta_M, \qquad r_M = \frac{\delta_M}{4M^2 A_M^{\mathrm{an}}}.$$

For $\|z\|_\infty < r_M$, the second line of (6.4) implies that the map $\Psi(\omega) = \omega - M^2 \mathsf{C}_{b',c}(\omega, z)$ has Lipschitz constant at most $1/4$ on $\{|\omega| \le \delta_M\}$, and by the first line we have $|\Psi(0)| \le \delta_M/4$. Then Ψ maps this closed ball into the ball of radius $\delta_M/2$. Starting from $\omega_0 = 0$, the iterates satisfy $|\omega_{j+1} - \omega_j| \le 4^{-j}\delta_M/4$ and converge uniformly to a solution of $\mathsf{C}_{b',c}(\omega, z) = 0$. Two solutions in the ball of radius δ_M differ by at most a quarter of their distance, so they coincide. We define Γ_b by solving these independent equations, one for each c, and keeping the free entries z.

Each iterate is holomorphic. On every strictly smaller polydisc in the free logarithmic coordinates and the target variables the uniform limit is holomorphic, since the one-variable Cauchy formula in each coordinate passes to the uniform limit. Iterated Cauchy formulas give the derivative bounds. If the margins between the two polydiscs are $d_z, d_\theta > 0$, a coefficient of order s in one constraint is bounded by $\delta_M s!\,(C_M/\min(d_z, d_\theta))^s$, after enlarging C_M to account for the at most $3(6M^3+1)$ scalar variables of the constraint. The free entries of Γ_b are coordinate functions. Each eliminated entry uses only the variables of its own constraint, and each variable belongs to at most six constraints, by Step 1. This proves the row and column bounds of every fixed order in (2); it is also the finite-range case of Lemma 5.3. Real coefficients and uniqueness imply real analyticity in real variables. The same construction applies in target coordinates centered at every real b, with the radii just chosen. On the solution, $\|M^2 D_\omega \mathsf{C} - I\| \le 1/4$, so the determinant does not vanish; by continuity from $X = 0$ it is positive for real data, and it lies between $M^{-6}(3/4)^3$ and $M^{-6}(5/4)^3$. This proves (2).

Step 5: the Haar formula. There are $d_{\mathrm{nf}} = 3(Mm)^3 - ((Mm)^3 - m^3) = 2(Mm)^3 + m^3$ links not in the block trees, of which $3m^3$ are eliminated edges, so $f = d_{\mathrm{nf}} - 3m^3 = 2((Mm)^3 - m^3)$ and dz is Lebesgue measure in dimension $3f$.

We first keep the untranslated free link matrices V_e fixed and replace the matrices on the eliminated edges by the output $T_M V$. At a configuration with output b we use the output chart $y_c = \mathrm{Log}((T_M V)_c b_c^{-1})$. Since $y_c = 0$ at that point, the coarse Haar measure has density $c_{\mathrm{Haar}}\, d^3 y_c$ there. In the input chart $V_{e_c^*} = e^{X_{e_c^*}} b_c$ the Haar measure of the eliminated edge has density $c_{\mathrm{Haar}}\, j_H(X_{e_c^*})\, d^3 X_{e_c^*}$. With the free matrices fixed, the derivative $D_{X_{e^*}} y$ is block diagonal with blocks $D_{X_{e_c^*}} \mathsf{C}_{b,c}$, by Step 1, and its determinant is

positive. This gives the density factor

$$\prod_c \frac{j_H(X_{e_c^*})}{\det D_{X_{e_c^*}} \mathsf{C}_{b,c}(X)} \tag{6.5}$$

relative to the coarse Haar probability measure and the Haar measure of the untranslated free matrices. The constants c_{Haar} of the eliminated edges cancel against those of the output chart, not against the factors j_H.

At fixed b we next write $V_e U_{0,e}(b)^{-1} = e^{z_e}$ for the free links. Right translation preserves Haar measure. Made jointly with b, this change of variables is triangular, so the derivatives of z in b do not affect its determinant. Passing to logarithms gives the factor $c_{\mathrm{Haar}}^f \prod_{e \text{ free}} j_H(z_e)\, dz$. Combining this with (6.5), we obtain $\mathscr{J}_b$, with the factor j_H of every link not in the block trees in its numerator. Since no coarse logarithm has been integrated against Lebesgue measure, there is no additional coarse factor j_H. Since we used independent coordinates throughout, no surface measure or Gram determinant occurs.

The parametrization is injective: the target is recovered by applying T_M, and then the free logarithms are recovered from the free links. The image of a Borel set $\mathcal{Z}$ is Borel. Indeed, with $b = T_M V$, a configuration in the block axial gauge belongs to the image if and only if its logarithms relative to $U_0(b)$ on the links not in the block trees exist in the neighborhoods above, its free logarithms belong to $\mathcal{Z}$, and its eliminated logarithms have norm less than $\widetilde{r}_M$. The constraint then holds, and uniqueness identifies these logarithms with Γ_b; each condition is Borel. The pointwise Jacobian computation therefore gives the integral formula, first on coordinate neighborhoods and then on a disjoint measurable refinement of countably many of them. Countably many suffice, since a finite product of matrix groups has a countable base of coordinate neighborhoods. By monotone approximation, the formula holds for all nonnegative Borel F. Finally we restore the block-tree links by the tree identity of Lemma 5.1; they contribute the integral over k against H_{for}. For a gauge-invariant integrand this integral equals the value at $k = I$. Since the tree gauge and its inverse are finite products of transports, restoring it preserves the Borel description. This proves (3).

The substitution $z = g\eta$ multiplies Lebesgue measure in dimension $3f$ by g^{3f}. The Wilson weight is gauge invariant and can be pulled out of the forest integral in part (3), while the event indicator remains inside that integral. This gives the stated density, including the indicator of $\{g\eta \in \mathcal{Z}\}$. Since the forest gauges fix the coarse vertices, $T_M(V(b, g\eta)^k) = b$. Integrating out η and normalizing when the restricted mass is positive therefore gives the image of the restricted measure. This proves (4).

Step 6: symmetries. We extend a coarse gauge transformation h to be constant on each block. The coarse-field lift transforms into the lift of b^h, the block-tree links remain equal to I, and the relative logarithm on a link whose initial vertex lies in the block of v becomes

$\mathrm{Ad}(h_v)X_e$. This is an orthogonal map of $\mathbb{R}^3$ with determinant one. It acts in the same way on an eliminated input and on the output of its constraint, so $D_{X_{e_c^*}}\mathsf{C}_{b,c}$ is conjugated and its determinant is unchanged; uniqueness implies the equivariance of Γ.

Under the spatial inversion, let e' be the positive link representing the reversed image of e. The lift is covariant under inversion, and

$$\left(e^{X_e}U_{0,e}\right)^{-1} = e^{-\mathrm{Ad}(U_{0,e}^{-1})X_e}\,U_{0,e}^{-1},$$

which gives the formula in (5). At fixed real b this is an orthogonal signed permutation of the free and the eliminated variables. The reflected constraint for the reversed coarse link is

$$\mathrm{Log}\big((T_MV)_c^{-1}b_c\big) = -\,\mathrm{Ad}(b_c^{-1})\,\mathsf{C}_{b,c}(X),$$

and its eliminated input is transformed by the same orthogonal map, because $U_{0,e_c^*} = b_c$. Its derivative is therefore conjugated, and its determinant is unchanged. Norms and the Haar factors j_H are unchanged, and so is the Wilson action, since a reflection changes a plaquette transport only by conjugation and possibly inversion, both of which preserve q. Uniqueness implies the intertwining of Γ. A product of three-dimensional balls is mapped to the product with relabeled radii, and a product with a common radius is invariant. Although these orthogonal matrices depend on the target, the joint transformation of (b, z) has a triangular derivative, and its diagonal block in z has absolute determinant one. This proves (5). □

6.3 Coercivity on the constrained fluctuation space

At the identity configuration the constraint $T_MU = I$ linearizes to $DT_M(I)a = 0$, where a is a real one-cochain with values in $\mathfrak{su}(2) \simeq \mathbb{R}^3$. On this space, after the block axial gauge fixing, the curl controls the ℓ^2 norm with a constant depending only on M, for a local reason. The value of a gauge-fixed cochain on a link e not in a block tree is the circulation of a around the short loop formed by e and two root paths. On interface links, the loop has to be closed through the reference path C_c and therefore also involves the eliminated edge e_c^*. The constraint $DT_M(I)a = 0$ expresses $a_{e_c^*}$ as an average of circulations around the short loops $F_{c,x}C_c^{-1}$. Each circulation around a short loop is bounded by the curl in a small box, by an explicit filling (Lemma 6.4). The tree gauge and the constraint remove the directions in which the curl vanishes; the coarse field itself is not controlled.

Lemma 6.4. *Let P be a box in $\mathbb{Z}^3$ with side lengths d_1, d_2, d_3, measured in links. Every closed lattice path ξ of length h in P is the boundary of an integer combination $\sum_p n_p p$ of oriented elementary plaquettes of P with*

$$\sum_p |n_p| \le h(d_1 + d_2 + d_3).$$

In particular, every real $\mathfrak{su}(2)$*-valued one-cochain* a *satisfies*

$$|a(\xi)| \le h(d_1+d_2+d_3)\Big(\sum_{p\subset P}|(\mathrm{d}a)_p|^2\Big)^{1/2},$$

where $a(\xi)$ *is the signed sum of* a *along* ξ *and* $(\mathrm{d}a)_p$ *is the signed sum of* a *around* p*.*

Proof. Let o be the lower corner of P and let τ_x be the path from o to x that changes the coordinates in the order $1,2,3$. For a positive link $e=(x,y)$ in direction i, we move the final step of $\tau_x e$ backwards through the steps of τ_x in the directions $j>i$. Each interchange crosses one elementary ij plaquette inside P; the result is τ_y, and the number of plaquettes crossed is $\sum_{j>i}(x_j-o_j)\le d_1+d_2+d_3$. This gives an integer filling of the closed chain $\tau_x+e-\tau_y$ with that mass bound. For a negatively oriented step we use the negative of the filling of the reversed positive link. When we sum over all steps of a closed path, the terms τ_x cancel and the oriented edge chain of ξ remains. Adding the mass bounds, we obtain the first assertion, also for paths that retrace links or repeat plaquettes. Pairing the chain identity with a gives $a(\xi)=\sum_p n_p(\mathrm{d}a)_p$, and the second assertion follows from the triangle inequality and $\max_p|(\mathrm{d}a)_p|\le(\sum_{p\subset P}|(\mathrm{d}a)_p|^2)^{1/2}$. The constant does not depend on the components of a. □

Lemma 6.5. *Fix an odd* $M\ge 3$*. There exist* $0<c_M\le C_M<\infty$*, depending only on* M*, with the following properties. Let* $m\ge 4$*, and let* a *be a real* $\mathfrak{su}(2)$*-valued one-cochain on the periodic fine lattice of side* Mm *that vanishes on the block trees and satisfies* $DT_M(I)a=0$*. Then*

$$\sum_e |a_e|^2 \le C_M\sum_p|(\mathrm{d}a)_p|^2,$$

where $\mathrm{d}a$ *is the signed elementary curl in fine lattice units. Further, let* Γ_I *be the map of Lemma 6.3(2) at the coarse configuration all of whose links equal* I*, and let* $E_0=D_z\Gamma_I(0)$ *be the induced linear embedding of the free logarithmic coordinates. Then*

$$c_M I \le E_0^{\mathsf{T}}\mathrm{d}^{\mathsf{T}}\mathrm{d}E_0 \le C_M I.$$

The rows and columns of E_0 *have finite range and absolute sums bounded in terms of* M*. These estimates include all periodic identifications and all three components of* $\mathfrak{su}(2)$*, and the constants do not depend on* m *or on the number of preceding averaging steps.*

Proof. The linearized constraint. We differentiate (4.3) at I. The contribution of the reference path to the mean of the logarithms of the based loop holonomies cancels the derivative of the final reference factor, and we find

$$(DT_M(I)a)_c = M^{-3}\sum_{x\in \mathsf{B}_M(v)} a(F_{c,x}). \tag{6.6}$$

Since a vanishes on the block trees, $a(C_c) = a_{e_c^*}$, and the constraint gives

$$a_{e_c^*} = -M^{-3} \sum_x \big(a(F_{c,x}) - a(C_c) \big). \tag{6.7}$$

Local bounds. The closed path $F_{c,x} C_c^{-1}$ lies in the box P_c formed by the two blocks of c, and its length is at most $5M-3$: each root path has at most $3R_M$ links, the straight segment has M links, and C_c has M links. The side lengths of P_c, in links, sum to $(2M-1) + 2(M-1) = 4M-3$. By Lemma 6.4 and (6.7), with the common constant $A_M^{\mathrm{fill}} = 20M^2 \geq (5M-3)(4M-3)$, we have

$$|a_{e_c^*}| \leq A_M^{\mathrm{fill}} \Big(\sum_{p \subset P_c} |(\mathrm{d}a)_p|^2 \Big)^{1/2}. \tag{6.8}$$

A link $e = (x, y)$ not in the block tree with both endpoints in the block of v has the closed path $\sigma_{v,x} e \sigma_{v,y}^{-1}$, of length at most $3M-2$. Its circulation is a_e, since a vanishes on the tree. The side lengths of the block sum to $3M-3$, and $(3M-2)(3M-3) \leq A_M^{\mathrm{fill}}$, so the same filling bound controls $|a_e|$ by the curl in this single block. For an interface link $e = (x, y)$ from the block of v to the block of $v + Me_i$, the closed path $\sigma_{v,x} e \sigma_{v+Me_i,y}^{-1} C_c^{-1}$ has length at most $4M-2$ and circulation $a_e - a_{e_c^*}$, and $(4M-2)(4M-3) \leq A_M^{\mathrm{fill}}$. Combining these bounds with (6.8), we find, for every fine link e,

$$|a_e|^2 \leq 4 \big(A_M^{\mathrm{fill}} \big)^2 \sum_{p \subset P_e} |(\mathrm{d}a)_p|^2, \tag{6.9}$$

where P_e is the block containing e or, for an interface link, the box of the two blocks. Links of the block trees satisfy the same inequality, since a vanishes on them.

Summation. A plaquette has its vertices in at most four blocks. A box of one or two blocks containing the plaquette contains one of these blocks, and for each block there are at most seven such boxes: the block itself, or its union with one of its six neighbors. Each such box has at most $2M^3$ vertices and hence at most $6M^3$ positive links to which it is assigned. When (6.9) is summed over e, a fixed plaquette is counted at most $D_M^{\mathrm{ov}} = 4 \cdot 7 \cdot 6M^3 = 168M^3$ times, and

$$\sum_e |a_e|^2 \leq C_M^{\mathrm{gap}} \sum_p |(\mathrm{d}a)_p|^2, \qquad C_M^{\mathrm{gap}} = 4 \big(A_M^{\mathrm{fill}} \big)^2 D_M^{\mathrm{ov}}. \tag{6.10}$$

Since a box of two blocks has longest side $2M-1 < Mm/2$ for $m \geq 4$, it lifts to a box in the covering lattice even when it meets the periodic identifications. The fillings above are therefore valid with the same bounds. All arguments are identities for chains, applied to each of the three components of a, with the Euclidean norm on $\mathfrak{su}(2)$.

The embedding. In the block axial gauge we write (6.6) as

$$M^{-2} a_{e_c^*} + \sum_{e \text{ free}} q_{ce} a_e = 0.$$

The coefficient of $a_{e_c^*}$ is M^{-2}, since e_c^* occurs in exactly M of the paths $F_{c,x}$. The sum of the absolute values of the other coefficients is at most $M^{-3}\sum_x \mathrm{length}(F_{c,x}) \le 4M$. Differentiating the identity $\mathsf{C}_{I,c}(\Gamma_I(z)) = 0$ of Lemma 6.3 at $z = 0$, we find that the free rows of E_0 are the identity and that the eliminated rows are

$$(E_0 z)_{e_c^*} = -M^2 \sum_e q_{ce} z_e,$$

with absolute row sums at most $4M^3$; in particular each entry of an eliminated row is at most $4M^3$ in absolute value. A free link inside a block occurs only in the six constraints incident to its block, and a free interface link only in its own constraint. Hence the absolute row and column sums of E_0 are at most $1 + 4M^3$ and $1 + 24M^3$, respectively, and its rows and columns have finite range. For any matrix A, by the weighted Cauchy–Schwarz inequality, we have

$$\sum_x \Big|\sum_y A_{xy} z_y\Big|^2 \le \Big(\sup_x \sum_y |A_{xy}|\Big)\Big(\sup_y \sum_x |A_{xy}|\Big) \sum_y |z_y|^2,$$

and this also applies to scalar coefficients tensored with the identity on $\mathbb{R}^3$. In particular, $\|E_0\|_2^2 \le (1 + 4M^3)(1 + 24M^3)$. Since the curl matrix has four signed entries in each row and each link lies in four plaquettes, we have $\|\mathrm{d}\|_2 \le 4$. This proves the upper bound with the constant $16(1 + 4M^3)(1 + 24M^3)$.

For the lower bound, note that the free entries of $E_0 z$ are the entries of z, and that $E_0 z$ vanishes on the block trees and satisfies the linearized constraints. Applying (6.10) to $a = E_0 z$, we obtain

$$\|\mathrm{d} E_0 z\|_2^2 \ge \big(C_M^{\mathrm{gap}}\big)^{-1} \|E_0 z\|_2^2 \ge \big(C_M^{\mathrm{gap}}\big)^{-1} \|z\|_2^2.$$

We take $c_M = (C_M^{\mathrm{gap}})^{-1}$ and let C_M be the larger of C_M^{gap} and $16(1 + 4M^3)(1 + 24M^3)$. Both depend only on M. □

6.4 Reflection positivity and chessboard estimates

Osterwalder and Seiler proved in [134] that the Wilson measure is reflection positive. The chessboard estimate is the consequence of reflection positivity developed by Fröhlich, Israel, Lieb, and Simon in [92, 93]; we follow the formulation and proof of Biskup in [29, Theorem 5.8 and (5.34)–(5.35)]. We need these results for the images of the Wilson measure under T_M^ℓ, for gauge-invariant observables, for reflections through planes bisecting links, and for disjoint cells of two sites in each direction. The treatment in [29] concerns spin systems with blocks that share boundary sites. We therefore verify reflection positivity directly for the observables we need, show that it survives the block averaging map because the blocks are centered and nested, and give the chessboard argument in full. At

the end of the section, we prove exponential moment bounds for plaquette energies under the Wilson measure by combining the chessboard estimate with elementary bounds on partition functions.

Reflections and observables. Let $n \in 4\mathbb{N}$ and consider the periodic lattice of side n, in its own lattice units. A bond reflection is a map r of the vertices of the form $x_j \mapsto 2b + 1 - x_j$ for one coordinate j and some $b \in \mathbb{Z}$, with the other coordinates unchanged. It fixes the two planes $x_j = b + 1/2$ and $x_j = b + 1/2 + n/2$, which bisect links, and these planes divide the vertices into two open halves of width $n/2$ that are exchanged by r. It acts on configurations by $(\mathsf{r}U)_{(x,y)} = U_{(\mathsf{r}x,\mathsf{r}y)}$, with the convention $U_{(y,x)} = U_{(x,y)}^{-1}$, and on functions by $(F \circ \mathsf{r})(U) = F(\mathsf{r}U)$. All observables below are bounded complex-valued Borel functions of the link variables. An observable supported in a reflection half of r is such a function that is invariant under all periodic gauge transformations and depends only on links with both endpoints in one of the two open halves of r. In particular, it does not depend on the links crossing the two planes.

We tile the vertices by the cells $Q_t = 2t + \{0,1\}^3$, $t \in (\mathbb{Z}/(n/2)\mathbb{Z})^3$. There are $\mathfrak{m} = n^3/8$ cells, and the cell indices form a torus of side $n/2$, which is even. An observable supported in Q_t is a bounded Borel function, invariant under all periodic gauge transformations, of the links with both endpoints in Q_t. For a cell index t, let g_t be the map of the vertices with coordinate maps

$$g_{t,j}(x_j) = x_j + 2t_j \quad \text{if } t_j \text{ is even}, \qquad g_{t,j}(x_j) = 2t_j + 1 - x_j \quad \text{if } t_j \text{ is odd},$$

modulo n. It maps Q_0 onto Q_t, and in each coordinate j it is the composition of the t_j reflections $x_j \mapsto 4k + 3 - x_j$, $0 \le k < t_j$, through the planes $x_j = 2k + 3/2$ between consecutive cells; since $n/2$ is even, g_t depends only on t modulo $n/2$. For an observable f supported in Q_0, the reflected translate $\vartheta_t f$ is the observable supported in Q_t defined by $\vartheta_t f(U) = f(U \circ g_t)$, where $(U \circ g_t)_{(x,y)} = U_{(g_t x, g_t y)}$ for the links (x,y) of Q_0. If r is the reflection through the planes between two consecutive cells and t' is the image of the cell index t under r, a direct computation gives $\mathsf{r} \circ g_t = g_{t'}$, and hence $(\vartheta_t f) \circ \mathsf{r} = \vartheta_{t'} f$. For a shift $\eta \in \{0,1\}^3$ we use in the same way the shifted tiling by the cells $\eta + 2t + \{0,1\}^3$, with $g_{t,j}(x_j) = x_j + 2t_j$ if t_j is even and $g_{t,j}(x_j) = 2\eta_j + 2t_j + 1 - x_j$ if t_j is odd.

For $\beta \ge 0$ and an even side $N \ge 4$, let

$$Z_N(\beta) = \int e^{-\beta \sum_{p \in \mathcal{P}_N} q(U_p)} \, dH_N(U), \qquad d\mu_N^\beta = Z_N(\beta)^{-1} e^{-\beta \sum_{p \in \mathcal{P}_N} q(U_p)} \, dH_N.$$

The Wilson measure of Definition 4.1 is $\mu_{N,\gamma} = \mu_N^\beta$ with $\beta = 4N/\gamma$.

Theorem 6.6. *Let $M \ge 3$ be odd, $\ell \ge 0$, $n \in 4\mathbb{N}$, and $N = M^\ell n$. For $\beta \ge 0$ let $\mu^{(\ell)} = (T_M^\ell)_* \mu_N^\beta$ be the image of μ_N^β on the coarse lattice of side n; for $\ell = 0$ this is μ_n^β.*

1. Reflection positivity. *For every bond reflection* r *of the coarse lattice and every observable* F *supported in a reflection half of* r,

$$\mathbb{E}_{\mu^{(\ell)}}\left[F\,\overline{F\circ\mathsf{r}}\right]\geq 0.$$

The same holds under μ_N^β *for every bounded Borel function on* $\mathcal{U}_N$ *that is invariant under all periodic gauge transformations and depends only on fine links with both endpoints in the union of the blocks* $\mathsf{B}_{M^\ell}(v)$ *over the coarse vertices* v *of one open half; this class contains* $F\circ T_M^\ell$ *for every observable* F *supported in a reflection half of* r. *The configurations at which the Borel extension of* T_M *is used are included.*

2. Chessboard estimates. *For nonnegative observables* $f_1,\dots,f_k$ *supported in* Q_0 *and distinct cell indices* $t_1,\dots,t_k$,

$$\mathbb{E}_{\mu^{(\ell)}}\prod_{j=1}^{k}\vartheta_{t_j}f_j\leq\prod_{j=1}^{k}\Big[\mathbb{E}_{\mu^{(\ell)}}\prod_{t}\vartheta_t f_j\Big]^{1/\mathfrak{m}}.$$

The same holds for each of the eight shifted tilings, and every coarse plaquette lies in a cell of at least one of them. In particular, writing $V=T_M^\ell U$ *and* $D_t(u)=\{\max_{p\subset Q_t}\rho_G(V_p)>u\}$ *for* $u\geq 0$, *where the maximum is over the plaquettes with all vertices in* Q_t, *we have for every set* T *of cell indices*

$$\mu^{(\ell)}\Big(\bigcap_{t\in T}D_t(u)\Big)\leq\mu^{(\ell)}\Big(\bigcap_{t}D_t(u)\Big)^{|T|/\mathfrak{m}},$$

and

$$\mu_N^\beta\Big\{\max_p\rho_G\big((T_M^\ell U)_p\big)>u\Big\}\leq n^3\Big[\frac{1}{Z_N(\beta)}\int e^{-\beta\sum_p q(U_p)}\mathbf{1}_{\cap_t D_t(u)}(T_M^\ell U)\,dH_N(U)\Big]^{8/n^3}.$$

More generally, the chessboard inequality holds under μ_N^β *for the cells* $C_t^\eta=\bigcup_{\zeta\in\{0,1\}^3}\mathsf{B}_{M^\ell}(M^\ell(\eta+2t+\zeta))$ *of the fine lattice, for every shift* η, *and for nonnegative observables supported in* C_0^η, *with the reflected translates defined in Section 6.4. The inequality for* $\mu^{(\ell)}$ *is the special case of the functions* $f\circ T_M^\ell$.

3. Plaquette energies under the Wilson measure. *Let* $\ell=0$ *and* $C_0=48+24\log(1/c_{\mathrm{ball}})$, *where* $c_{\mathrm{ball}}=\operatorname{sinc}^2(1/2)/(12\pi)$ *is the constant of Lemma 5.1. For* $\beta\geq 2$ *and* $0<s\leq 1/2$ *set*

$$A_s(n,\beta)=C_0+24\log\frac{1}{1-s}+\frac{36\log\beta}{n}.$$

Then for every deterministic set S *of plaquettes,*

$$\mathbb{E}_{\mu_n^\beta}\exp\Big(\frac{s}{8}\sum_{p\in S}\beta q(U_p)\Big)\leq e^{A_s(n,\beta)|S|/8}.$$

For $K \geq 0$ and a cell Q let $D_Q(K) = \{\max_{p \subset Q} \beta q(U_p) \geq K\}$. For distinct cells $Q_{t_1}, \dots, Q_{t_k}$ of one tiling,

$$\mu_n^\beta\Big(\bigcap_{j=1}^{k} D_{Q_{t_j}}(K)\Big) \leq \big[6e^{A_s(n,\beta)-sK}\big]^k$$

and

$$\mu_n^\beta\Big\{\max_p \beta q(U_p) \geq K\Big\} \leq \min\big\{1, 6n^3 e^{A_s(n,\beta)-sK}\big\}.$$

If $\beta \leq b_+ n$ for some $b_+ > 0$, then $36 \log \beta / n \leq 36 b_+ / \mathrm{e}$.

4. Conditioning. *If D is an event with $\mu_n^\beta(D) \geq 1 - \varepsilon_D > 0$, then $\mathbb{E}_{\mu_n^\beta}[X \mid D] \leq (1 - \varepsilon_D)^{-1} \mathbb{E}_{\mu_n^\beta} X$ for every nonnegative random variable X.*

The constant C_0 is absolute; no constant in (1)–(4) depends on L, β, n, N, or ℓ.

Remark 6.7. Reflection positivity is asserted here for the Wilson measure and its images. For $\ell > 0$, the integral in part (2) is estimated in Sections 22 and 23. The term $36 \log \beta / n$ in A_s comes from the comparison of the periodic lattice with the open cube in Lemma 6.11. After tree gauge fixing, the periodic lattice has $r_P = 2n^3 + 1$ independent link variables and the open cube with n vertices per side has $r_O = 2n^3 - 3n^2 + 1$. Each variable contributes a factor of order $\beta^{-3/2}$, and the chessboard exponent is $1/\mathfrak{m} = 8/n^3$, so that

$$\frac{8}{n^3} \cdot \frac{3}{2} (r_P - r_O) \log \beta = \frac{36 \log \beta}{n}.$$

This term cannot be absorbed into C_0 when n and β vary independently, so we keep it.

We prove part (1) in Section 6.4, part (2) in Section 6.4, and parts (3) and (4) in Section 6.4.

Reflection positivity

Lemma 6.8. *Let $N \geq 4$ be even, $\beta \geq 0$, and let r be a bond reflection of the periodic lattice of side N. Then μ_N^β is invariant under r, and $\mathbb{E}_{\mu_N^\beta}[F \overline{F \circ \mathsf{r}}] \geq 0$ for every observable F supported in a reflection half of r.*

Proof. Since the two planes of r divide the vertices into halves of width $N/2 \geq 2$, a vertex is an endpoint of at most one link crossing the planes, and the crossing links form a matching. We write the integral of the gauge-invariant function $G(U) = F(U)\overline{F(\mathsf{r}U)} e^{-\beta \sum_p q(U_p)}$ against H_N as an iterated integral, first over the links not crossing the planes and then over the crossing links. For fixed values of the crossing links, let k be the gauge transformation with $k_y = U_e$ at the endpoint y in the plus half of each crossing link e, oriented from the minus half to the plus half, and $k = I$ at all other vertices. It is well defined

because the crossing links form a matching. Then $(U^k)_e = I$ on every crossing link, while every other link undergoes a left or right translation by a matrix depending only on the crossing links. These translations preserve the product Haar measure of the noncrossing links, and $G(U^k) = G(U)$. Hence the integral equals the integral of G with all crossing links set to I, against the Haar measure of the noncrossing links. The integrals over the crossing links give one. No cycle around the torus is removed by this gauge fixing.

A plaquette either has all its vertices in one half or crosses one of the planes. After the crossing links are set to I, the transport around a crossing plaquette is, up to conjugation and inversion, UV^{-1}, where U is a link of the plus half parallel to the plane and adjacent to it, and V is the reflected link of the minus half. We identify $\mathrm{SU}(2)$ with the unit quaternions, $U \leftrightarrow u \in S^3 \subset \mathbb{R}^4$ and $V \leftrightarrow v$. Then $\mathrm{tr}(UV^{-1})/2 = u \cdot v$, so

$$e^{-\beta q(UV^{-1})} = e^{-\beta} \sum_{r=0}^{\infty} \frac{\beta^r}{r!} \sum_{a_1,\dots,a_r=1}^{4} u_{a_1} \cdots u_{a_r}\, v_{a_1} \cdots v_{a_r}. \tag{6.11}$$

For each r the sum of the absolute values of the terms is at most $(\sum_a |u_a v_a|)^r \le 1$. The series and its finite products over the crossing plaquettes therefore converge absolutely and uniformly, and every coefficient is nonnegative for $\beta \ge 0$. The plaquettes inside the minus half and the Haar measure of its links are the reflected copies of those of the plus half. Integrating $F\overline{F \circ \mathsf{r}}$ against the expanded weight, we obtain

$$Z_N(\beta)\, \mathbb{E}_{\mu_N^\beta}\left[F\,\overline{F \circ \mathsf{r}}\right] = e^{-\beta|\mathcal{P}_{\mathrm{cross}}|} \sum_{\alpha} c_\alpha \left| \int F(U_+) e^{-\beta A_+(U_+)} \phi_\alpha(U_+)\, dH_+(U_+) \right|^2 \ge 0. \tag{6.12}$$

Here $\mathcal{P}_{\mathrm{cross}}$ is the set of crossing plaquettes, α runs over the degrees and component indices in (6.11) for all crossing plaquettes, c_α is the product of the corresponding coefficients $\beta^r/r!$, U_+ denotes the links with both endpoints in the plus half, H_+ their product Haar measure, A_+ the sum of $q(U_p)$ over the plaquettes in the plus half, and ϕ_α the product of the corresponding real quaternion components of the plus links adjacent to the planes. Absolute convergence justifies the interchange of sum and integral for bounded Borel F. The partition function $Z_N(\beta)$ is positive by Lemma 5.1, and the term $r = 0$ covers $\beta = 0$. Finally, μ_N^β is invariant under r, since Haar measure is preserved by permutations of links and by inversion, and q is preserved by conjugation and inversion. □

Proof of Theorem 6.6(1). For $\ell = 0$ this is Lemma 6.8. Let $\ell \ge 1$ and $a = M^\ell$, so that $N = an$. In fine units, a bond reflection of the coarse lattice acts in the reflected coordinate by $x \mapsto (2b+1)a - x$ for some $b \in \mathbb{Z}$. Since a is odd, both planes are bond planes of the fine lattice. For $0 \le k < \ell$, the intercept divided by the mesh M^k of the lattice after k averaging steps is $(2b+1)M^{\ell-k}$, an odd multiple of M. By Proposition 6.2(4), each averaging step therefore commutes with this reflection, including at the configurations where the Borel extension is used, and so does T_M^ℓ. By Proposition 6.2(6), a coarse link with both

endpoints in one open half depends only on fine links in the union of the blocks of side a of that half. Since the block nearest a plane ends half a fine lattice spacing before it, these fine links do not cross the fine planes. Since T_M^ℓ is Borel, the composition with T_M^ℓ of an observable supported in a reflection half is bounded and Borel. By gauge covariance (Proposition 6.2(3)), it is invariant under all periodic gauge transformations of the fine lattice. Applying Lemma 6.8 to $F \circ T_M^\ell$ and to the other functions in the stated class proves (1). By the same covariance, $\mu^{(\ell)}$ is invariant under the translations of the coarse lattice and under its bond reflections, since μ_N^β is invariant under the corresponding fine maps. □

The chessboard estimate

For an odd integer $a \geq 1$, $n \in 4\mathbb{N}$, and a shift $\eta \in \{0,1\}^3$, consider the periodic lattice of side an and the cells

$$C_t^\eta = \bigcup_{\zeta \in \{0,1\}^3} \mathsf{B}_a\big(a(\eta + 2t + \zeta)\big), \qquad t \in (\mathbb{Z}/(n/2)\mathbb{Z})^3.$$

They are cubes with $2a$ vertices in each direction, and they partition the vertices, since the blocks $\mathsf{B}_a(av)$, $v \in \mathbb{Z}^3$, do; for $a = 1$ they are the cells $\eta + 2t + \{0,1\}^3$ above. In the coordinate j, the plane between the cells with indices k and $k+1$ is $x_j = (2\eta_j + 4k + 3)a/2$. It bisects links, since a is odd, and the reflection through it, $x_j \mapsto (2\eta_j + 4k + 3)a - x_j$, maps each cell onto a cell. The second fixed plane of this reflection lies at distance $an/2 = 2a \cdot (n/4)$ from the first, which is a multiple of the side $2a$ of the cells. It is also a plane between consecutive cells, and every cell lies in one open half of the reflection. Observables supported in C_t^η are defined as for $a = 1$; they depend only on the links with both endpoints in C_t^η. The maps g_t now have the coordinate maps $g_{t,j}(x_j) = x_j + 2at_j$ if t_j is even and $g_{t,j}(x_j) = (2\eta_j + 2t_j + 1)a - x_j$ if t_j is odd, modulo an; they map C_0^η onto C_t^η and are compositions of t_j reflections through planes between consecutive cells in each coordinate j. Reflected translates are defined by $\vartheta_t f(U) = f(U \circ g_t)$ for observables f supported in C_0^η. If r is the reflection through the plane between the cells with indices k and $k+1$ in the coordinate j, it maps the index t to the index t' with $t'_j = 2k + 1 - t_j$ and the other components unchanged. Further, $\mathsf{r} \circ g_t = g_{t'}$, since for even t_j, $(2\eta_j + 4k + 3)a - (x_j + 2at_j) = (2\eta_j + 2t'_j + 1)a - x_j$, and for odd t_j, $(2\eta_j + 4k + 3)a - ((2\eta_j + 2t_j + 1)a - x_j) = x_j + 2at'_j$. This implies $(\vartheta_t f) \circ \mathsf{r} = \vartheta_{t'} f$.

Lemma 6.9. *Let $a \geq 1$ be odd, $n \in 4\mathbb{N}$, and $\eta \in \{0,1\}^3$. Let ν be a probability measure on $\mathcal{U}_{an}$ such that, for every bond reflection r through the planes between consecutive cells C_t^η, ν is invariant under r and $\mathbb{E}_\nu[F\,\overline{F \circ \mathsf{r}}] \geq 0$ for every observable F supported in a reflection half of r.*

Then for nonnegative observables $f_1, \dots, f_k$ supported in C_0^η and distinct cell indices $t_1, \dots, t_k$,

$$\mathbb{E}_\nu \prod_{j=1}^{k} \vartheta_{t_j} f_j \le \prod_{j=1}^{k} \Big[\mathbb{E}_\nu \prod_t \vartheta_t f_j \Big]^{1/\mathfrak{m}}, \qquad \mathfrak{m} = n^3/8.$$

The hypotheses concern only the reflections between the cells of the given tiling. In the applications below, all bond reflections are available.

Proof. The Schwarz inequality. Let r be a reflection through planes between consecutive cells, and call one of its open halves the plus half. For observables G, H supported in the plus half, let $B_\mathsf{r}(G,H) = \mathbb{E}_\nu[G\,\overline{H \circ \mathsf{r}}]$. Since ν is invariant under the involution r, we have $B_\mathsf{r}(H,G) = \mathbb{E}_\nu[(H \circ \mathsf{r})\,\overline{G}] = \overline{B_\mathsf{r}(G,H)}$, so B_r is Hermitian, and by assumption it is positive semidefinite. Expanding $B_\mathsf{r}(G + zH, G + zH) \ge 0$ for complex z, we find $|B_\mathsf{r}(G,H)|^2 \le B_\mathsf{r}(G,G) B_\mathsf{r}(H,H)$. If a diagonal entry vanishes, varying the real and imaginary parts of z shows that the mixed entry vanishes as well, so no nondegeneracy is needed. For an assignment $\boldsymbol{f} = (f_t)_t$ of nonnegative observables supported in C_0^η to the cell indices, we set

$$\mathcal{F}(\boldsymbol{f}) = \mathbb{E}_\nu \prod_t \vartheta_t f_t, \qquad \mathcal{F}(\boldsymbol{f})^2 \le \mathcal{F}(R_+ \boldsymbol{f})\, \mathcal{F}(R_- \boldsymbol{f}), \tag{6.13}$$

where $R_+ \boldsymbol{f}$ assigns f_t both to each index t of the plus half and to its image t' under r, and $R_- \boldsymbol{f}$ is defined in the same way from the minus half. To prove the inequality, let $G = \prod_t \vartheta_t f_t$ and $H = \prod_t \vartheta_t f_{t'}$, both products over the indices t of the plus half. Since every cell lies in one open half of r, the functions G and H are observables supported in the plus half: they are bounded, Borel, gauge invariant, and depend only on links with both endpoints in that half. Since $(\vartheta_t f_{t'}) \circ \mathsf{r} = \vartheta_{t'} f_{t'}$, the product $H \circ \mathsf{r}$ is the product over the minus half, so that $\mathcal{F}(\boldsymbol{f}) = B_\mathsf{r}(G,H)$, $\mathcal{F}(R_+ \boldsymbol{f}) = B_\mathsf{r}(G,G)$, and $\mathcal{F}(R_- \boldsymbol{f}) = B_\mathsf{r}(H,H)$. These identities show that (6.13) is the Schwarz inequality.

One direction. Consider first a cyclic array of q positions, q even, in which the object at each position may be an entire product over a transverse slab of cells, and reflections through the planes between positions act as above. Fix a finite collection of strictly positive bounded functions, and for its assignments $(f_0, \dots, f_{q-1})$ set

$$\mathcal{G}(f_0, \dots, f_{q-1}) = \frac{\mathcal{F}(f_0, \dots, f_{q-1})}{\prod_{j=0}^{q-1} \mathcal{F}(f_j, \dots, f_j)^{1/q}},$$

where $\mathcal{F}(f_j, \dots, f_j)$ is the value at the assignment that places f_j at every position. The denominator is positive. Under each of the two copies in (6.13) every function of one half appears twice. The product of the two new denominators therefore equals the square of the old one, and $\mathcal{G}$ satisfies (6.13). Let $\mathcal{G}_*$ be the maximum of $\mathcal{G}$ over the finite set of assignments; $\mathcal{G}_* \ge 1$, since a constant assignment gives the value one. Among the

maximizers we choose one with a longest cyclic run of one function, of length k. If $k < q$, we cut just before this run and at the opposite position; these two planes are the two fixed planes of one reflection through planes between cells. We copy the half that begins with the run to the other half. If $k < q/2$, the new array has a run of length at least $2k$; if $k \geq q/2$, that half is constant and the new array is constant. The two copied arrays have values at most $\mathcal{G}_*$, and (6.13) at the maximizer forces both values to equal $\mathcal{G}_* > 0$. This contradicts the maximality of the run unless $k = q$. Hence a constant assignment is a maximizer, and $\mathcal{G}_* = 1$. This proves the one-dimensional inequality $\mathcal{F}(f_0, \dots, f_{q-1}) \leq \prod_j \mathcal{F}(f_j, \dots, f_j)^{1/q}$ for every even q, without assuming that q is a power of two.

Three directions. Let $q = n/2$. We group the cell functions into the q products over the transverse slabs with fixed first cell index and apply the one-dimensional inequality. Each resulting expectation repeats one slab, by reflected translates, in the first direction, and carries the exponent $1/q$. We apply the inequality in the second direction to each of these expectations, grouping over the first and third indices. Now the first two directions are homogeneous, and each exponent is $1/q^2$. Applying it in the third direction, grouping over the first two, leaves the products over all reflected translates of single cell functions, with exponent $1/q^3 = 1/\mathfrak{m}$. Every slab product used is bounded, nonnegative, gauge invariant, and supported in a reflection half of the reflections in question. To allow functions with zeros, we replace each f by $f + \varepsilon$ and let $\varepsilon \downarrow 0$; the finite products converge boundedly. For partial assignments, we place the constant function one in the unused cells. This proves the inequality for the tiling with shift η. □

Proof of Theorem 6.6(2). We apply Lemma 6.9 in two ways.

Coarse cells. For every shift η, the reflections through the planes between consecutive cells $\eta + 2t + \{0,1\}^3$ are bond reflections of the coarse lattice. By part (1) and the invariance proved with it, $\nu = \mu^{(\ell)}$ is invariant under every bond reflection of the coarse lattice and reflection positive for observables supported in its reflection halves. Then the lemma with $a = 1$ implies the functional inequality for $\mu^{(\ell)}$ and every shifted tiling.

Fine cells. Let $a = M^\ell$. For every shift η, the planes between consecutive cells C_t^η are $x_j = (2\eta_j + 4k + 3)M^\ell/2$, and the reflections through them are bond reflections of the fine lattice, since $(2\eta_j + 4k + 3)M^\ell$ is odd. An observable supported in a reflection half of such a reflection is a bounded Borel gauge-invariant function of the fine links with both endpoints in one open half. Since the blocks partition the vertices and the planes run between blocks, these are the fine links with both endpoints in the union of the blocks $\mathsf{B}_{M^\ell}(v)$ over the coarse vertices v of that half. By Lemma 6.8, $\nu = \mu_N^\beta$ is invariant under every bond reflection of the fine lattice and reflection positive for these observables. By the lemma with $a = M^\ell$, the inequality holds for nonnegative observables supported in C_0^η, for every shift η.

The coarse inequality as a special case. Let f be an observable supported in the coarse cell $\eta + \{0,1\}^3$. By Proposition 6.2(2), (3), and (6), $f \circ T_M^\ell$ is bounded, Borel, invariant under all periodic gauge transformations of the fine lattice, and depends only on fine links with both endpoints in the union of the blocks $\mathsf{B}_{M^\ell}(M^\ell v)$ over the vertices v of the coarse cell, that is, in C_0^η. Written in fine units, the coarse map g_t is the fine map g_t with $a = M^\ell$: in each coordinate it is a translation by $2M^\ell t_j$ or a reflection $x_j \mapsto (2\eta_j + 2t_j + 1)M^\ell - x_j$. After $k < \ell$ averaging steps these are, in the units of that lattice, a translation by a multiple of M or a reflection with intercept $(2\eta_j + 2t_j + 1)M^{\ell-k} \in M\mathbb{Z}$. By Proposition 6.2(4), each averaging step commutes with them, and so does T_M^ℓ. We conclude that $(\vartheta_t f) \circ T_M^\ell = \vartheta_t(f \circ T_M^\ell)$ and

$$\mathbb{E}_{\mu^{(\ell)}} \prod_j \vartheta_{t_j} f_j = \mathbb{E}_{\mu_N^\beta} \prod_j \vartheta_{t_j}(f_j \circ T_M^\ell),$$

so the inequality for $\mu^{(\ell)}$ is the special case of the functions $f \circ T_M^\ell$ of the fine inequality.

Bad cells. The reflections permute the six plaquettes of a cell and conjugate or invert their transports, which preserves ρ_G; hence the reflected translate of $\mathbf{1}_{D_0(u)}$ is $\mathbf{1}_{D_t(u)}$. As a function of V, this indicator is a nonnegative observable supported in Q_0, since ρ_G is continuous and invariant under conjugation. Its composition with T_M^ℓ is Borel and gauge invariant by Proposition 6.2(2) and (3). Taking $f_j = \mathbf{1}_{D_0(u)}$ for the cells of T gives the bound for $\bigcap_{t \in T} D_t(u)$. A coarse ij plaquette is contained in a cell of the tiling whose shift matches the parities of its starting vertex in the directions i and j, with either shift in the third direction. The eight tilings therefore cover all plaquettes, with $8\mathfrak{m} = n^3$ cells in total. The event that all cells of a shifted tiling are bad is a translate of the corresponding event for $\eta = 0$, and $\mu^{(\ell)}$ is invariant under translations of the coarse lattice. The bound for a single cell and a union bound over the n^3 cells give the maximum inequality, since

$$\mu^{(\ell)}\Big(\bigcap_t D_t(u)\Big) = \frac{1}{Z_N(\beta)} \int e^{-\beta \sum_p q(U_p)} \mathbf{1}_{\cap_t D_t(u)}(T_M^\ell U)\, dH_N(U).$$

□

Partition functions and plaquette energy moments

We now derive the moment bounds of part (3). We combine the chessboard estimate with bounds on partition functions obtained by tree gauge fixing, by integrating independent plaquette variables, and by dropping nonnegative plaquette terms. Elementary bounds of this kind are in the tradition of the stability estimates of Faria da Veiga and O'Carroll in [64, 65]; we derive the precise periodic correction needed here.

Lemma 6.10. *For every $\beta > 0$,*

$$z_1(\beta) = \int_{\mathrm{SU}(2)} e^{-\beta q(U)}\, dH(U) = \frac{2}{\pi}\int_0^\pi e^{-\beta(1-\cos t)} \sin^2 t\, dt$$
$$= \frac{2e^{-\beta} I_1(\beta)}{\beta} \le c_G \beta^{-3/2}, \qquad c_G = \sqrt{2/\pi},$$

where $I_1(\beta) = \sum_{k\ge 0} (\beta/2)^{2k+1}/(k!(k+1)!)$, and $z_1(0) = 1$ by continuous extension. If S is a random variable with density $s^{1/2}e^{-s}/\Gamma(3/2)$ on $s > 0$, then

$$\frac{z_1(\beta)}{c_G \beta^{-3/2}} = \mathbb{E}\sqrt{(1 - S/(2\beta))_+}.$$

For an integer $n \ge 1$ let O_n be the open cube with vertices $\{0,\dots,n\}^3$, all links and plaquettes between them, and no prescribed boundary links, and let $Z_{O_n}(\beta)$ be its partition function with respect to the product Haar probability measure. Let $r_n = 2n^3 + 3n^2$ be the number of links of O_n not in a spanning tree, and let $P_n = 3n^2(n+1)$ be the number of its plaquettes. Then for every $\beta \ge 1$,

$$-\tfrac{3}{2} r_n \log\beta + r_n \log c_{\mathrm{ball}} - 2P_n \le \log Z_{O_n}(\beta) \le r_n \log z_1(\beta) \le -\tfrac{3}{2} r_n \log\beta + r_n \log c_G,$$

where $c_{\mathrm{ball}} = \mathrm{sinc}^2(1/2)/(12\pi)$. The constants are absolute and the inequalities hold uniformly in n and β; the bound $Z_{O_n}(\beta) \le z_1(\beta)^{r_n}$ holds for every $\beta > 0$. For an open cube with n vertices in each direction the number of links not in a spanning tree is $2n^3 - 3n^2 + 1$. Finally, the partition function of the periodic lattice of side $n+1$ at any $\beta > 0$ is at most $Z_{O_n}(\beta)$.

The last statement compares unnormalized integrals. It bounds a periodic partition function from above, but it does not replace that partition function in the denominator of a normalized expectation.

Proof. One plaquette. By the radial form of Haar measure in Lemma 5.1, with $t = |X|/2$, we have $z_1(\beta) = (2/\pi)\int_0^\pi e^{-\beta(1-\cos t)} \sin^2 t\, dt$. To compute the series, let $J_k = \int_0^\pi \sin^2 t \cos^{2k} t\, dt$, so $J_0 = \pi/2$. For $k \ge 1$, integrating the derivative of $\sin^3 t \cos^{2k-1} t$, whose boundary values vanish, gives $(2k+2)J_k = (2k-1)J_{k-1}$, and therefore $J_k = \pi(2k)!/(2\cdot 4^k k!(k+1)!)$. Odd powers of $\cos t$ integrate to zero against $\sin^2 t$, by the substitution $t \mapsto \pi - t$. Since the exponential series converges uniformly on $[0,\pi]$, integrating termwise, we find

$$z_1(\beta) = e^{-\beta} \sum_{k\ge 0} \frac{(\beta/2)^{2k}}{k!(k+1)!} = \frac{2e^{-\beta} I_1(\beta)}{\beta}.$$

The integral is at most one and tends to one as $\beta \downarrow 0$, which gives the continuous extension. For $\beta > 0$, we substitute $s = \beta(1-\cos t)$. Since $\sin t = (2s/\beta)^{1/2}(1 - s/(2\beta))^{1/2}$ and $ds = \beta \sin t\, dt$, we obtain

$$z_1(\beta) = \frac{2\sqrt{2}}{\pi}\, \beta^{-3/2} \int_0^{2\beta} s^{1/2} e^{-s} \sqrt{1 - \frac{s}{2\beta}}\, ds. \tag{6.14}$$

Dividing by $c_G\beta^{-3/2}$ and using $\Gamma(3/2) = \sqrt{\pi}/2$ gives the identity with the Gamma variable S. The integrand $\sqrt{(1 - S/(2\beta))_+}$ lies between zero and one, which proves $z_1(\beta) \le c_G\beta^{-3/2}$ for every $\beta > 0$.

Counting. The open cube has $(n+1)^3$ vertices, $3n(n+1)^2$ links, and $P_n = 3n^2(n+1)$ plaquettes, and $3n(n+1)^2 - (n+1)^3 + 1 = r_n$. The following links form a spanning tree: all links in direction 1, the links in direction 2 in the plane $x_1 = 0$, and the links in direction 3 on the line $x_1 = x_2 = 0$. Indeed every vertex reaches $(0,0,0)$ by decreasing first x_1, then x_2, then x_3, and there are $n(n+1)^2 + n(n+1) + n = (n+1)^3 - 1$ such links. By Lemma 5.1, integrating out the gauge in this tree gives the factor one. In this gauge we write $Y_{x,y,z}$ and $Z_{x,y,z}$ for the links in directions 2 and 3 at the vertex (x,y,z); then $Y_{0,y,z} = I$ and $Z_{0,0,z} = I$.

Upper bound. Consider the set of plaquettes consisting of all 12 plaquettes, all 13 plaquettes, and the 23 plaquettes in the plane $x_1 = 0$. With positive orientations, their transports are

$$A_{x,y,z} = Y_{x+1,y,z}Y_{x,y,z}^{-1}, \qquad B_{x,y,z} = Z_{x+1,y,z}Z_{x,y,z}^{-1}, \qquad D_{y,z} = Z_{0,y+1,z}Z_{0,y,z}^{-1}.$$

For A and B the coordinate x runs from 0 to $n-1$ and the other coordinates over their full ranges, so there are $n^2(n+1)$ plaquettes of each of these two kinds, and there are n^2 plaquettes D; in total r_n. From the D variables we reconstruct the links $Z_{0,y,z}$ successively in y, starting from $Z_{0,0,z} = I$. From the A variables we reconstruct every Y successively in x, starting from $Y_{0,y,z} = I$. From the B variables, with $Z_{0,y,z}$ already known, we reconstruct the remaining Z successively in x. In each step a new link equals a new plaquette variable times a link already fixed. By invariance of Haar measure and iterated integration, these plaquette variables are independent and Haar distributed. The plaquettes not in this set contribute factors at most one, since $q \ge 0$. Dropping them, we find $Z_{O_n}(\beta) \le z_1(\beta)^{r_n}$ for every $\beta > 0$.

Lower bound. Let $\beta \ge 1$. We restrict each of the r_n links not in the tree to $\{\rho_G(U_e) < \beta^{-1/2}\}$, where $\beta^{-1/2} \le 1$. By Lemma 5.1 each such ball has Haar measure at least $c_{\text{ball}}\beta^{-3/2}$. A plaquette contains at most four links not in the tree, so by the triangle inequality for ρ_G we have $\rho_G(U_p) \le 4\beta^{-1/2}$, and $q(U_p) \le \rho_G(U_p)^2/8 \le 2/\beta$; the first inequality is an inequality between group distances and holds even if its right side exceeds the radius of the logarithm chart. Hence the Boltzmann factor is at least e^{-2P_n} on this set, whose product Haar measure is at least $(c_{\text{ball}}\beta^{-3/2})^{r_n}$. Taking logarithms gives the lower bound, and the bound of the previous paragraph together with $z_1(\beta) \le c_G\beta^{-3/2}$ gives the remaining inequality. Since the boundary spheres of the balls have Haar measure zero, open and closed balls give the same estimate.

Replacing n by $n-1$, we find that the number of links not in a spanning tree of an open cube with n vertices in each direction is $2(n-1)^3 + 3(n-1)^2 = 2n^3 - 3n^2 + 1$. Finally, deleting from the periodic lattice of side $n+1$ the plaquettes that contain a link

joining the vertex layers $x_j = n$ and $x_j = 0$, for some j, leaves the open cube O_n and the links joining those layers, which no remaining plaquette uses. The integrals of the latter against Haar probability measure equal one, and deleting plaquettes with nonnegative cost can only increase the integral. This proves the last statement. □

Lemma 6.11. *Let $n \in 4\mathbb{N}$, put $V = n^3$ and $\mathfrak{m} = V/8$, and let $Z_P(n,\beta) = Z_n(\beta)$ be the partition function of the periodic lattice of side n. Put $r_P = 2V + 1$ and $r_O = 2V - 3n^2 + 1$. Then*

$$\log Z_P(n,\beta) \geq -\tfrac{3}{2} r_P \log\beta + r_P \log c_{\mathrm{ball}} - 6V, \qquad \beta \geq 1, \tag{6.15}$$

$$Z_P(n,\beta') \leq Z_{O_{n-1}}(\beta') \leq \bigl(c_G(\beta')^{-3/2}\bigr)^{r_O}, \qquad \beta' > 0. \tag{6.16}$$

As a consequence, for $\beta \geq 2$ and $0 < s \leq 1/2$,

$$\begin{aligned} \frac{1}{\mathfrak{m}} \log \frac{Z_P(n,(1-s)\beta)}{Z_P(n,\beta)} &\leq \frac{12(r_P - r_O)}{V} \log\beta + \frac{12 r_O}{V} \log\frac{1}{1-s} \\ &\quad + \frac{8}{V}\bigl(r_O \log c_G - r_P \log c_{\mathrm{ball}} + 6V\bigr) \\ &\leq \frac{36\log\beta}{n} + 24 \log\frac{1}{1-s} + C_0 = A_s(n,\beta), \end{aligned} \tag{6.17}$$

with

$$C_0 = 48 + 24\log(1/c_{\mathrm{ball}}). \tag{6.18}$$

Proof. A spanning tree of the periodic lattice has $V - 1$ links, so $3V - (V-1) = r_P$ links are not in it; by Lemma 5.1 the gauge in the tree integrates to one. We restrict all links not in the tree, including those carrying the holonomies around the torus, to balls $\{\rho_G(U_e) < \beta^{-1/2}\}$ with $\beta^{-1/2} \leq 1$. As in the proof of Lemma 6.10, each ball has Haar measure at least $c_{\mathrm{ball}}\beta^{-3/2}$ and each plaquette costs at most $q(U_p) \leq 2/\beta$. There are $3V$ plaquettes, which gives (6.15). By the last statement of Lemma 6.10, applied with side $n = (n-1)+1$, the periodic partition function is at most that of the open cube with n vertices in each direction, which has r_O links not in a spanning tree. The upper bound of Lemma 6.10 then implies (6.16). This upper bound needs only $\beta' > 0$; below we use it with $\beta' = (1-s)\beta \geq 1$, where every inequality of the lemma applies.

Subtracting (6.15) from the logarithm of (6.16) with $\beta' = (1-s)\beta$ and multiplying by $1/\mathfrak{m} = 8/V$, we obtain the first inequality in (6.17). For the second we use $r_P - r_O = 3n^2$, $r_O/V \leq 2$, $r_P/V \leq 3$, $\log\beta \geq 0$, and $\log c_G < 0$: the first term equals $36\log\beta/n$, the second is at most $24\log(1/(1-s))$, and the third is at most $(8/V)(3V\log(1/c_{\mathrm{ball}}) + 6V) = C_0$. We keep the surface term $36\log\beta/n$ in this form, before any relation between n and β is imposed. □

Proof of Theorem 6.6(3) and (4). We write $Y_p = \beta q(U_p) \geq 0$ and $\mathfrak{m} = n^3/8$. For a cell Q_0 let $f = \exp(s\sum_{p\subset Q_0} Y_p)$, the sum running over the six plaquettes of the cell. This is a

nonnegative bounded observable supported in Q_0. The product of its reflected translates over all cells uses each plaquette inside a cell exactly once, so its exponent is at most $s\sum_p Y_p$, the sum over all plaquettes. By Lemma 6.11, we have

$$\mathbb{E}_{\mu_n^\beta} \prod_t \vartheta_t f \le \frac{Z_P(n,(1-s)\beta)}{Z_P(n,\beta)} \le e^{A_s(n,\beta)\mathfrak{m}}.$$

By part (2), the product of f over h distinct cells of one tiling has expectation at most $e^{A_s(n,\beta)h}$. Now let S be a deterministic set of plaquettes. We assign each $p \in S$, by a fixed rule, to a cell containing it in one of the eight shifted tilings. Let S_a be the plaquettes assigned to the tiling a, and let h_a be the number of cells of that tiling to which some plaquette is assigned. Enlarging each assigned sum to all six plaquettes of the cell, we find $\mathbb{E}\exp(s\sum_{p\in S_a} Y_p) \le e^{A_s(n,\beta)h_a}$. By Hölder's inequality with eight exponents equal to eight, we have

$$\mathbb{E}e^{(s/8)\sum_{p\in S} Y_p} \le \prod_{a=1}^{8} \Big[\mathbb{E}e^{s\sum_{p\in S_a} Y_p}\Big]^{1/8} \le e^{A_s(n,\beta)\sum_a h_a/8} \le e^{A_s(n,\beta)|S|/8}.$$

In the last inequality we used $\sum_a h_a \le |S|$ and $A_s(n,\beta) \ge 0$.

For the bad cells, note the pointwise bound $\mathbf{1}_{D_Q(K)} \le \sum_{p\subset Q} e^{s(Y_p-K)}$. Multiplying over the $\mathfrak{m}$ cells of a tiling and expanding, we obtain $6^{\mathfrak{m}}$ terms, one for each choice of a plaquette in every cell. The chosen plaquettes are distinct, and the sum of their energies is at most the sum over all plaquettes. Therefore

$$\mu_n^\beta\Big(\bigcap_t D_{Q_t}(K)\Big) \le 6^{\mathfrak{m}} e^{-sK\mathfrak{m}} \frac{Z_P(n,(1-s)\beta)}{Z_P(n,\beta)} \le \big[6e^{A_s(n,\beta)-sK}\big]^{\mathfrak{m}}.$$

Applying part (2) to the indicator of $D_{Q_0}(K)$ in k distinct cells, we obtain the joint bound. A union bound over the n^3 cells of the eight tilings, together with the trivial bound by one, gives the bound for the maximum. All constants in these bounds are absolute.

If $\beta \le b_+ n$, then

$$\frac{\log\beta}{n} = \frac{\beta}{n}\cdot\frac{\log\beta}{\beta} \le \frac{b_+}{\mathrm{e}},$$

since $(\log x)/x$ attains its maximum at $x = \mathrm{e}$; this simplification uses the additional relation between β and n. Finally, for a nonnegative X and an event D with $\mu_n^\beta(D) \ge 1-\varepsilon_D > 0$,

$$\mathbb{E}[X \mid D] = \frac{\mathbb{E}[X\mathbf{1}_D]}{\mu_n^\beta(D)} \le (1-\varepsilon_D)^{-1}\mathbb{E}X.$$

This bound does not depend on the number of factors in X, and it does not use reflection positivity of the conditioned measure. This completes the proof of Theorem 6.6. □

SECTION 7

Periodic gauge fixing and energy estimates

In this section, we prove deterministic estimates for the block averaging map and for gauge fixing, and one probability bound. Together they show that configurations with small curvature at every scale lie in the domains where the renormalization group analysis applies. The two main estimates are the following. We write $\delta(U) = \max_p \rho_G(U_p)$ for the maximal curvature and $\mathscr{E}(U) = \sum_p q(U_p)$ for the Wilson energy of a configuration U. If $\delta(U)$ is small, then

$$\mathscr{E}(T_M U) \le M\big(1 + C_M \delta(U)\big)\mathscr{E}(U), \qquad \delta(T_M U) \le M^2 \delta(U) + c_M \delta(U)^2$$

(Theorem 7.3). Since the squared coupling grows by the factor M in each averaging step, the first inequality bounds the output Wilson action divided by the squared coupling by $1 + C_M\delta(U)$ times its input value. The product of these upper-bound factors is bounded under the squared-mesh curvature hypothesis of part (5) of that theorem. Second, on a periodic cubic lattice of side $n \ge 4$, every configuration with $\delta(U) \le \delta \le \delta_{\mathrm{SG}}$ has a periodic gauge transform U^k with

$$\max_e \rho_G(U^k_e) \le C_{\mathrm{SG}}\big(\sqrt{\delta} + n^{-1}\big)$$

(Theorem 7.7). In particular, $\delta(U) \le \delta_r$ and $n \ge 4C_{\mathrm{SG}}/r$ imply that U lies in the gauge-invariant small-field domain D_r, with δ_r defined in that theorem, and the plaquette moments of Theorem 6.6 bound the probability of the complement of D_r under the Wilson measure.

Section 7.1 gives sufficient conditions, small links or small plaquettes, under which the logarithms of a configuration relative to the coarse-field lift are the local constraint coordinates of Lemma 6.3 (Lemma 7.2), and Section 7.2 proves the energy estimate; these results are used in the classical analysis of Sections 8, 9, and 11 and in Sections 14, 22, and 23. Section 7.3 describes the dependence sets of iterated averages and the eight shifted tilings by which the chessboard estimates are applied to blocked plaquettes (Lemma 7.4). Section 7.4 constructs periodic gauges by gluing local axial gauges (Lemmas 7.5 and 7.6), and Section 7.5 proves the small-curvature gauge estimate and the probability bound; these results are used in the large-field analysis of Sections 18 and 21 to 23. The inputs are the block averaging map, its local constraint coordinates, and the chessboard estimates of Section 6, together with the metric, word, and conditioning estimates of Lemmas 5.1 and 5.5. No result on conditional constrained minimizers or on normalized fluctuation variables is used.

Conventions. The distance between group elements is $\operatorname{dist}(g,h) = \rho_G(gh^{-1})$. We work in lattice units until a physical mesh is displayed. A rectangular patch contains all its vertices, all links with both endpoints among those vertices, and all plaquettes with their four vertices among them; in particular its boundary links are included. A plaquette is counted once in every sum of energies, and a maximum over an empty set of plaquettes is zero. When an estimate concerns only the norm of a logarithm of a group element V, it refers to a shortest logarithm, whose norm is $\rho_G(V)$; every logarithm that is later used as an analytic coordinate lies in a fixed ball strictly inside the domain of the principal logarithm. Constants depend only on M and on such fixed radii, with additional dependencies stated explicitly; this is the dependence described in item (1) of Definition 4.14.

7.1 Local inverse coordinates for block averaging

We begin with the elementary discrete form of the non-Abelian Stokes theorem in a box: in an axial gauge, every link is a product of transported plaquette variables along strips of width one.

Lemma 7.1. *Let $Q = \prod_{i=1}^{3}\{0,\dots,b_i\}$ with integers $b_i \geq 0$, assign arbitrary values in $\mathrm{SU}(2)$ to its links, and put $b = b_1 + b_2 + b_3$. Let $q_Q(x)$ be the transport from the lower corner of Q to x along the path that changes the coordinates in the order $1,2,3$. After the gauge transformation q_Q, every link of Q is a product of at most b conjugates of plaquette variables of Q or their inverses. In particular, writing $d_p = \rho_G(U_p)$,*

$$\max_{e\subset Q} \rho_G\big(U_e^{q_Q}\big) \leq b \max_{p\subset Q} d_p, \qquad \Big(\sum_{e\subset Q} \rho_G\big(U_e^{q_Q}\big)^2\Big)^{1/2} \leq b\sqrt{3|Q|}\Big(\sum_{p\subset Q} d_p^2\Big)^{1/2}.$$

If the first bound is smaller than a fixed radius below 2π, these are bounds for the principal logarithms of the links. Every closed path of s links in Q has a transport that is a product of at most sb conjugated plaquette variables. These statements also hold in a periodic lattice for a box that lifts injectively to the covering lattice.

Proof. For two successive orthogonal positive steps we have the identity

$$U_i(x)U_k(x+e_i) = U_{p_{ik}(x)}U_k(x)U_i(x+e_k), \qquad i<k.$$

Changing an orientation replaces the plaquette factor by a conjugate of its inverse, and in a longer path the transport of the preceding subpath conjugates the plaquette factor. Each interchange of two adjacent orthogonal steps therefore introduces exactly one transported plaquette and leaves the other factors unchanged. Repeating it gives the corresponding identity for a strip of width one, including its boundary links.

The path from the corner to x first runs in direction 1, then 2, then 3. For a link in direction 3, the path to its endpoint is the path to its initial point followed by the link,

and its transformed value is I. For a link in direction 2, we move it backwards through the direction-3 part of the path, of length x_3. This uses the x_3 plaquettes of a 23 strip of width one, and the remaining path is the path to the endpoint, which cancels. For a link in direction 1, we move it first through the direction-3 part, using x_3 plaquettes of a 13 strip, and then through the direction-2 part in the lower plane, using x_2 plaquettes of a 12 strip. The direction-1 parts then cancel. These moves stay in the closed box and use at most b plaquettes. Reversing a link inverts its word. Using the bi-invariance of ρ_G, the triangle inequality $\rho_G(UV) \le \rho_G(U) + \rho_G(V)$ of Lemma 5.1, and invariance under inversion, we obtain the bound on the maximum.

Since each d_p is at most $(\sum_{p \subset Q} d_p^2)^{1/2}$, each transformed link is bounded by b times this quantity, and there are at most $3|Q|$ links. This gives the second bound. In a closed path the gauge factors cancel except for a conjugation at the starting vertex, and substituting the words just obtained for its links gives at most sb factors. In an injective periodic lift every interchange uses a lifted plaquette, and the same proof applies. If Q has no plaquettes, all words are empty. □

For a coarse link $c = (v, v + Me_i)$, its two blocks form a rectangular patch with $2M$ vertices in direction i and M vertices in the other directions. For a coarse ik plaquette, the four blocks at its corners form a rectangular patch with $2M$, $2M$, and M vertices in the directions i, k, and the third direction. Their diameters in the lattice path metric are at most $4M$ and $5M$. Since on a coarse lattice of side $m \ge 4$ every coordinate extent of these patches is smaller than the period Mm, each patch lifts injectively to a box of the covering lattice, also when it meets the periodic identifications. Every translated path with endpoint connectors, every reference path, and every strip used below lies in such a lift. Figure 7.1 shows these patches.

We now recall the setting of Lemma 6.3. Let h be the block axial gauge transformation, $h_x = U(\sigma_{v,x})$ for $x \in \mathsf{B}_M(v)$, so that $h_v = I$ at every block center and $U^h = I$ on the block trees. By gauge covariance, $b = T_M U^h = T_M U$. On the links not in the block trees we write

$$U_e^h = e^{X_e} U_{0,e}(b), \qquad z = X_{\text{free}}.$$

Lemma 6.3 provides radii r_M and $\widetilde{r}_M$ for the free and the eliminated coordinates, independent of the coarse configuration b, within which the constraint has the unique solution $X = \Gamma_b(z)$. The next lemma identifies the logarithms X of the given configuration with $\Gamma_b(z)$. The existence of a solution of the constraint near $U_0(b)$ would not suffice, since later sections integrate functions of the given configuration in these coordinates. We distinguish three normalizations: the relative logarithms X and their free part z, which are controlled here; the coupling-rescaled coordinates z/g; and the normalized fluctuation variables w of Lemma 12.3, defined by $z_i = z_i^* + g_i K_i^{-1/2} w_i$ after the conditional constrained minimizer has been subtracted, which do not appear in this section.

Lemma 7.2. *Fix an odd $M \geq 3$ and a fine side Mm with $m \geq 4$, and let T_M, the block trees, and Γ_b be as in Definition 6.1 and Lemma 6.3. There exist $u_M, \delta_M > 0$ and $C_M < \infty$, depending only on M and on fixed radii strictly inside the coordinate domain of Lemma 6.3 and the domain of the principal logarithm, with the following properties.*

1. *If $\max_e \rho_G(U_e) \leq u$ and $(5M-3)u < 2\pi$, then no based loop holonomy $U(F_{c,x})U(C_c)^{-1}$ in one application of T_M equals $-I$, and*
$$\max_c \rho_G\big((T_M U)_c\big) \leq (6M-3)u \leq 10Mu.$$
In the block axial gauge, the logarithms relative to $U_0(b)$, $b = T_M U$, of the links not in the block trees satisfy $\max_e |X_e| \leq C_M u$, where shortest logarithms are meant. If moreover $u \leq u_M$, then all these logarithms, including those on the eliminated edges, are principal logarithms in the coordinate domain of Lemma 6.3, and $X = \Gamma_b(X_{\text{free}})$.
2. *Let $J \geq 1$ be fixed, and let the fine side be $M^J m_{\text{out}}$ with an integer $m_{\text{out}} \geq 4$. Suppose that $\max_e \rho_G(U_e) \leq u$ and $(10M)^J u \leq u_M$. Then the conclusions of (1) with $u \leq u_M$ hold for each of the J averaging steps, with u replaced by $(10M)^s u$ at the step from $T_M^s U$ to $T_M^{s+1} U$, and*
$$\max_e \rho_G\big((T_M^s U)_e\big) \leq (10M)^s u, \qquad 0 \leq s \leq J.$$
3. *Suppose only that $\max_p \rho_G(U_p) \leq \delta \leq \delta_M$; no bound on the links is assumed. Then all based loop holonomies in one application of T_M lie in a fixed ball strictly inside the domain of the principal logarithm, and in the block axial gauge*
$$\max_e |X_e| \leq C_M \delta, \qquad X = \Gamma_{T_M U}(X_{\text{free}}), \qquad \max_P \rho_G\big((T_M U)_P\big) \leq C_M \delta,$$
where P runs over the coarse plaquettes. For every $g > 0$ the coupling-rescaled coordinate $\eta = z/g$ satisfies $\|\eta\|_\infty \leq C_M \delta / g$.
4. *Statements (1) and (3) are local: they hold for a given coarse link or plaquette if their hypothesis is imposed only on the patch formed by the two blocks of the link, or by the four blocks at the corners of the plaquette, with all their links and plaquettes. In (2), locality through J steps uses the corresponding nested ancestor patch of two or four blocks of side M^J in the original lattice. Links outside these patches and the holonomies around the noncontractible cycles of the torus do not affect the estimates.*

All constants are independent of m, of the physical scale, of the coupling, and of the number of preceding averaging steps.

Proof. Small links. Recall that $R_M = (M-1)/2$. Since each root path has at most $3R_M$ links, the path $F_{c,x}$ has at most $M + 6R_M = 4M - 3$ links and $F_{c,x} C_c^{-1}$ at most $5M - 3$. By the triangle inequality, a product of s links of distance at most u from I has distance at most su. If $(5M-3)u < 2\pi$, every based loop holonomy therefore differs from $-I$. With
$$A_c = M^{-3} \sum_{x \in \mathsf{B}_M(v)} \operatorname{Log}\big(U(F_{c,x})U(C_c)^{-1}\big), \qquad b_c = e^{A_c} U(C_c),$$

we have $|A_c| \le (5M-3)u$ and hence $\rho_G(b_c) \le (6M-3)u \le 10Mu$.

In the block axial gauge every transformed link not in a block tree is a product of at most $1 + 6R_M = 3M-2$ links, and its distance from I is at most $(3M-2)u$. Since its value under the coarse-field lift is I or one of the b_c, every relative element $U_e^h U_{0,e}(b)^{-1}$ has distance at most $(9M-5)u$ from I. Taking shortest logarithms gives the first bound in (1) with any $C_M \ge 9M$.

Let $r_M, \widetilde{r}_M$ be the radii of Lemma 6.3 for the free and the eliminated coordinates. Fix smaller positive radii inside them and inside a fixed ball strictly within the domain of the principal logarithm, and choose $u_M > 0$ with

$$(5M-3)u_M < 2\pi, \qquad C_M u_M < \tfrac{1}{2}\min\{r_M, \widetilde{r}_M, 1\},$$

decreasing u_M further if the fixed ball requires it. For $u \le u_M$ all relative logarithms are unique principal logarithms. By gauge covariance, and since $h = I$ at the block centers, $T_M U^h = b$. Hence the vector X solves the constraint at the target b, and both its free and its eliminated entries lie in the uniqueness neighborhood of Lemma 6.3. Uniqueness implies $X = \Gamma_b(X_{\mathrm{free}})$; this identifies every eliminated entry, not only the free ones. This proves (1).

Several averaging steps. Let the initial side be $M^J m_{\mathrm{out}}$, and define

$$n_s = M^{J-s} m_{\mathrm{out}} \quad (0 \le s \le J), \qquad n_s = M n_{s+1}, \quad n_{s+1} \ge m_{\mathrm{out}} \ge 4 \quad (0 \le s < J). \tag{7.1}$$

At each step, the input side is a multiple of M and the output side is at least four, so the hypotheses of Definition 6.1 and Lemma 6.3 hold and every patch used above lifts injectively. Let $u_s = (10M)^s u$. The assumed bound $\max_e \rho_G(U_e) \le u$ gives the link bound for $s = 0$, and the hypothesis $(10M)^J u \le u_M$ implies $u_s \le u_M$, and hence $(5M-3)u_s < 2\pi$, for $0 \le s \le J$. We apply (1) to $T_M^s U$ on the lattice of side n_s, for $s = 0, 1, \dots, J-1$ in turn. This identifies the relative coordinates of each step with the solution Γ at its target and gives the link bound $10Mu_s = u_{s+1}$ for the next step. The constants are the same at every step, since they do not depend on n_s. This proves (2).

Small plaquettes. We now assume only a bound on the plaquettes, and let $K_M^{(0)} = 100M^2$. Every based loop holonomy is the transport around a closed path of at most $5M-3$ links in a patch of two blocks, whose side lengths sum to at most $4M$. By Lemma 7.1, it is a product of at most $20M^2 \le K_M^{(0)}$ conjugated plaquette variables, so

$$\rho_G\big(U(F_{c,x})U(C_c)^{-1}\big) \le K_M^{(0)}\delta, \qquad |A_c| \le K_M^{(0)}\delta. \tag{7.2}$$

Here and below, δ need only bound the plaquettes of the indicated patch. It is chosen so small that the first bound lies in a fixed ball strictly inside the domain of the principal logarithm.

Consider a link not in the block tree with both endpoints in one block. Its gauge-transformed value is the transport around the closed path formed by the link and two

root paths, of length at most $6R_M + 1$ in one block. The same argument bounds its distance from I by $K_M^{(0)}\delta$. For an interface link $e = (x, y)$ between the blocks of c, we write

$$W_e = U(\sigma_{v,x})\, U_e\, U(\sigma_{v+Me_i,y})^{-1} = U_e^h.$$

The closed path for $W_e U(C_c)^{-1}$ lies in the patch of the two blocks and has length at most $6R_M + 1 + M$. As before, we have $\rho_G(W_e U(C_c)^{-1}) \le K_M^{(0)}\delta$. Since $b_c = e^{A_c}U(C_c)$,

$$\rho_G\big(W_e b_c^{-1}\big) \le \rho_G\big(W_e U(C_c)^{-1}\big) + |A_c| \le 2K_M^{(0)}\delta. \tag{7.3}$$

This includes the eliminated edge. The coarse plaquette formed by the four reference paths has distance at most $M^2\delta$ from I, since its M by M square can be reduced by successive interchanges of steps that use each of its M^2 plaquettes once. Replacing its four reference transports by the values b_c changes this distance by at most $4K_M^{(0)}\delta$. Here we used that the bi-invariant metric satisfies $\operatorname{dist}(\prod_i a_i, \prod_i b_i) \le \sum_i \operatorname{dist}(a_i, b_i)$, as one sees by replacing one factor at a time, and that inverting both elements preserves each term. We obtain

$$\rho_G\big((T_M U)_P\big) \le \big(M^2 + 4K_M^{(0)}\big)\delta. \tag{7.4}$$

We increase C_M so that it dominates the three constants in (7.2), (7.3), and (7.4), and choose $\delta_M > 0$ with $C_M\delta_M < \min\{r_M, \widetilde{r}_M, 1\}/2$, also respecting the fixed logarithm radius. Then the relative vector lies in the uniqueness neighborhood of Lemma 6.3, and, as in the proof of (1), we obtain $X = \Gamma_{T_M U}(X_{\mathrm{free}})$. Its free part satisfies $\|z\|_\infty \le C_M\delta$, and dividing by $g > 0$ gives the bound for η. This proves (3).

Locality. All paths and all fillings used above lie in the patches of two or four blocks. A gauge transformation on such a patch can be extended arbitrarily to the other vertices of the torus, and by covariance of T_M no link outside the patch enters these computations; the block axial gauge uses only the paths in its own block. This proves the one-step locality in (1) and (3). Iterating the dependence sets by (6.1) gives the ancestor patches stated for (2). The local statements therefore do not constrain any holonomy around a noncontractible cycle. For a local statement on the inverse coordinates, the equality with Γ_b refers to the coordinates in question: we extend the local free data by zero to a free vector in the coordinate domain. Each eliminated coordinate is determined by its own constraint, which involves only its two blocks. By the locality and uniqueness statements of Lemma 6.3, the corresponding entry of Γ_b does not depend on the extension. The local eliminated value solves the same constraint in the same uniqueness neighborhood; by uniqueness, the two agree. This proves (4). All radii and constants were chosen from fixed finite patches and from the radii of Lemma 6.3; they depend only on M and the fixed radii. □

7.2 Energy estimates and physical curvature

The leading factor M in the energy estimate has a simple origin. At the linear level, the curvature of a coarse plaquette is an average over M^3 translates of the sum of the M^2 fine curvatures in an M by M square ((7.5) below). A coarse curvature is therefore about M^2 times a typical fine curvature, while the number of plaquettes decreases by the factor M^3. The sum of the squared curvatures, and with it the Wilson energy, is multiplied by $M^4/M^3 = M$. Since the squared coupling satisfies $g_{j+1}^2 = Mg_j^2$, this factor cancels in the action $4\mathscr{E}/g^2$. Part (3) bounds the nonlinear error by the maximal curvature times the local ℓ^2 norm of the curvature, which keeps the estimate uniform in the volume. The bound on the maximal curvature in part (1) and its iteration in parts (5) and (6) adapt the estimates of Bałaban for his averaging operation in [12, Section B, Propositions 1 and 2], which also use the four blocks at the corners of a coarse plaquette. We give the proofs for the paths and the metric used here. The estimate for the Wilson energy with leading factor M requires in addition the exact column sums of the linearized kernel and the local form of the remainder.

Theorem 7.3. *For each odd $M \geq 3$ there exist $\delta_*(M) > 0$ and $C_M, c_M < \infty$ with the following properties, for every $m \geq 4$ and every configuration U on the fine lattice of side Mm. Write $\delta(U) = \max_p \rho_G(U_p)$ and $\mathscr{E}(U) = \sum_p q(U_p)$.*

1. *If $\delta(U) \leq \delta_*$, then*

$$\mathscr{E}(T_M U) \leq M\big(1 + C_M\delta(U)\big)\mathscr{E}(U), \qquad \delta(T_M U) \leq M^2\delta(U) + c_M\delta(U)^2.$$

2. *The derivative of T_M at the identity acts on curvatures by the nonnegative kernel*

$$(\mathsf{L}_M f)_{ik}(v) = M^{-3} \sum_{r \in \{-R_M,\dots,R_M\}^3} \sum_{s,t=0}^{M-1} f_{ik}(v + r + se_i + te_k),$$

   that is, $\mathrm{d}\, DT_M(I) = \mathsf{L}_M \mathrm{d}$ on one-cochains. Its row sums equal M^2 and its column sums equal M^{-1}, and $\|\mathsf{L}_M f\|_2^2 \leq M\|f\|_2^2$ for every array f indexed by plaquettes with values in a Euclidean space.

3. *For a set $\mathcal{A}$ of positive coarse plaquettes, let $\mathrm{Pa}(\mathcal{A})$ be the union, counted once, of the fine plaquettes in the patches of four blocks of the plaquettes in $\mathcal{A}$. If $\delta = \max_{p \in \mathrm{Pa}(\mathcal{A})} \rho_G(U_p) \leq \delta_*$, then*

$$\sum_{P \in \mathcal{A}} q\big((T_M U)_P\big) \leq M(1 + C_M\delta) \sum_{p \in \mathrm{Pa}(\mathcal{A})} q(U_p).$$

   For each $P \in \mathcal{A}$ there is a rooted axial gauge on its patch $Q(P)$ of four blocks in which

$$\mathrm{Log}\big((T_M U)_P\big)^{(P)} = \sum_p \mathsf{L}_M(P,p)\,\mathrm{Log}(U_p)^{(P)} + e_P,$$

with

$$|e_P| \le C_M \delta \Big(\sum_{p \subset Q(P)} \rho_G(U_p)^2 \Big)^{1/2},$$

where the superscript (P) refers to this gauge; the gauges for different P need not agree. Each fine plaquette lies in at most 48 of the patches $Q(P)$, and the restriction of L_M to output plaquettes in $\mathcal{A}$ has row sums at most M^2 and column sums at most M^{-1}.

4. *Let $U^{j+1} = T_M U^j$, $0 \le j < d$, be configurations on lattices all of whose coarse sides are at least four, and let $g_j > 0$ satisfy $g_{j+1}^2 = M g_j^2$. If $\delta_j = \delta(U^j) \le \delta_*$ for $j < d$, then*

$$\frac{4\mathscr{E}(U^d)}{g_d^2} \le \prod_{j<d} (1 + C_M \delta_j) \, \frac{4\mathscr{E}(U^0)}{g_0^2} \le e^{C_M \sum_{j<d} \delta_j} \, \frac{4\mathscr{E}(U^0)}{g_0^2}.$$

5. *Measure lengths in units of the mesh of the coarsest lattice, so that U^j lives on a lattice of mesh $\epsilon_j = M^{j-d}$. Let*

$$0 < \varepsilon \le \varepsilon_M, \qquad \varepsilon_M \le \min\big\{ \delta_*/2,\ M^2(M^2-1)/(4c_M) \big\}.$$

If $\delta(U^0) \le \varepsilon \epsilon_0^2$, then each of $U^0, \dots, U^{d-1}$ satisfies the hypothesis of Lemma 7.2(3), and

$$\delta(U^j) \le 2\varepsilon \epsilon_j^2 \quad (0 \le j \le d), \qquad \sum_{j<d} \delta(U^j) \le \frac{2\varepsilon}{M^2 - 1},$$

and

$$\frac{4\mathscr{E}(U^d)}{g_d^2} \le e^{2C_M \varepsilon/(M^2-1)} \, \frac{4\mathscr{E}(U^0)}{g_0^2}.$$

6. *Let U^0 be obtained by sampling a fixed smooth periodic connection, with $\delta(U^0) \le C_A a_0^2$, where a_j denotes the physical mesh of U^j and $a_* = a_d$, so that $a_j = a_* \epsilon_j$. If $\varepsilon = C_A a_*^2 \le \varepsilon_M$, the conclusions of (5) hold, and the free logarithmic coordinates z_j of U^j in Lemma 7.2(3) satisfy*

$$\|z_j / g_j\|_\infty \le C_{A,M} \, a_j^{3/2} / g_3, \qquad g_j = g_3 \sqrt{a_j}, \qquad 0 \le j < d,$$

with $C_{A,M} = 2 C_M C_A$.

All constants are independent of d, of the volume, of the physical side, and of the coupling, except for the constant C_A of the smooth connection. The iterated bound in (4) is uniform under a bound on the sum of the curvatures, not merely under a bound on each curvature separately.

In the proof, we first compute the derivative of T_M at the identity and the row and column sums of its kernel on curvatures. Then, in an axial gauge on the patch of each coarse plaquette, we compare $\operatorname{Log}((T_M U)_P)$ with its linearization by Taylor's formula. The remainder is quadratic in the logarithms of the links, and the axial gauge bounds these by the curvatures in the patch. Summing over P with the bounded overlap of the patches gives the energy estimate, and a single row gives the curvature recursion. The iteration in (4)–(6) is then elementary.

Proof. The linearized kernel. Let a be a real $\mathfrak{su}(2)$-valued one-cochain and let $U_e(t) = e^{ta_e}$. For a coarse vertex v define

$$r(v) = M^{-3} \sum_{x \in \mathsf{B}_M(v)} \sum_{e \in \sigma_{v,x}} a_e, \qquad k_i(v) = M^{-3} \sum_{r \in \{-R_M, \dots, R_M\}^3} \sum_{s=0}^{M-1} a_i(v + r + se_i),$$

where the sum along a path takes orientations into account. We differentiate the formula for T_M. The derivative of the reference transport cancels its negative occurrence in the logarithm of every based loop holonomy, and we find

$$(DT_M(I)a)_i(v) = r(v) + k_i(v) - r(v + Me_i).$$

Around a coarse plaquette the four terms involving r cancel. For each r, the four segment sums in k form the boundary of the translated M by M square, and summing the signed curls of its plaquettes cancels every interior link. With d the signed curl in the respective lattice units, this gives

$$\big(\mathrm{d}\, DT_M(I)a\big)_{ik}(v) = M^{-3} \sum_{r \in \{-R_M, \dots, R_M\}^3} \sum_{s,t=0}^{M-1} (\mathrm{d}a)_{ik}(v + r + se_i + te_k). \tag{7.5}$$

This proves the kernel identity in (2); repeated occurrences of a fine plaquette are added in its entry. Since each row has $M^3 M^2$ terms of weight M^{-3}, the row sums equal M^2. Fix a fine plaquette at x with orientation ik. For each $(s,t) \in \{0, \dots, M-1\}^2$ the vertex $x - se_i - te_k$ has exactly one representation $v + r$ with v a coarse vertex and $r \in \{-R_M, \dots, R_M\}^3$. This gives exactly M^2 contributions in the column, and its sum is M^{-1}; this count includes contributions across the periodic identifications. For an array f with values in a Euclidean space, by the weighted Cauchy–Schwarz inequality, we have

$$\sum_P \Big| \sum_p \mathsf{L}_M(P,p) f_p \Big|^2 \le M^2 \sum_{P,p} \mathsf{L}_M(P,p) |f_p|^2 = M \sum_p |f_p|^2. \tag{7.6}$$

Restricting to output plaquettes in $\mathcal{A}$ keeps the row sums equal to M^2 and decreases the column sums, so the restricted kernel satisfies the stated bounds and the same inequality.

The local Taylor estimate. Fix an output plaquette P. We use the rooted axial gauge of Lemma 7.1 on its patch $Q(P)$ of four blocks and set $A_e = \operatorname{Log}(U_e^{\vec{q}_{Q(P)}})$. The patch has at most $12M^3$ links and at most $12M^3$ plaquettes, and the sum of its side lengths is at most $5M$. By Lemma 7.1, with constants depending only on M,

$$\|A\|_\infty \le C_M \delta, \qquad \|A\|_2 \le C_M \Big(\sum_{p \subset Q(P)} \rho_G(U_p)^2 \Big)^{1/2}. \tag{7.7}$$

The local gauge can be extended to the torus, and by Proposition 6.2(6) its values outside the patch do not affect $(T_M U)_P$. In this gauge, we consider the functions

$$\Phi_P(A) = \operatorname{Log}\big((T_M e^A)_P\big), \qquad f_p(A) = \operatorname{Log}\big((e^A)_p\big),$$

and

$$G_P(A) = \Phi_P(A) - \sum_p \mathsf{L}_M(P,p) f_p(A).$$

Let $\delta_* \le \delta_M$ be so small that all link, path, and plaquette transports in the patch stay in a fixed ball strictly inside the domain of the principal logarithm whenever $\|A\|_\infty \le C_M \delta_*$. Since the distance of a product from I is at most its length times $\|A\|_\infty$, this choice is made on a fixed finite patch and does not depend on the side of the torus. On a slightly larger fixed ball, the finite-word bounds of Lemma 5.1 bound the Hessian of G_P in the Euclidean operator norm by a constant H_M. Indeed, each averaged link is a composition of finitely many word maps, logarithms, an arithmetic mean, and an exponential, whose first two derivatives are bounded on these balls, and the chain rule applies. The value and the first derivative of G_P vanish at zero, the latter by (7.5). By Taylor's formula along the segment tA, we have

$$|G_P(A)| \le \int_0^1 (1-t) H_M \|A\|_2^2 \, dt \le \tfrac{1}{2} H_M \sqrt{12M^3}\, \|A\|_\infty \|A\|_2,$$

where we used $\|A\|_2 \le \sqrt{12M^3}\|A\|_\infty$. Inserting (7.7), we obtain the estimate for $e_P = G_P(A)$ in (3). Here $\mathrm{Log}(U_p)^{(P)} = f_p(A)$ is the logarithm of the plaquette transport in the gauge of this patch, at the usual base point of p. Its norm is $\rho_G(U_p)$, although its direction in $\mathfrak{su}(2)$ may depend on P. By covariance of T_M, the coarse plaquette is conjugated by the axial gauge at its starting vertex, and we also have $|\Phi_P(A)| = \rho_G((T_M U)_P)$.

Overlap. A fine vertex belongs to exactly one block, and a block center is a corner of twelve positive coarse plaquettes: three orientations and four choices of the corner. If a fine plaquette lies in $Q(P)$, its vertices lie in the four corner blocks of P, so one of the at most four block centers of its vertices is a corner of P. Hence there are at most $4 \cdot 12 = 48$ such P; this count may be redundant, and it holds on every periodic lattice considered. Combining this count with the error estimate in (3), we find

$$\sum_{P \in \mathcal{A}} |e_P|^2 \le C_M \delta^2 \sum_{p \in \mathrm{Pa}(\mathcal{A})} \rho_G(U_p)^2, \tag{7.8}$$

where the sum on the right is over the union, without multiplicity.

Energy and curvature bounds. Let $F^2 = \sum_{p \in \mathrm{Pa}(\mathcal{A})} \rho_G(U_p)^2$. Although the gauges differ from one P to another, the Cauchy–Schwarz inequality in each row uses only the norms $|f_p| = \rho_G(U_p)$, so (7.6) bounds the squared norm of the linear term by MF^2. By (7.8), the error term has norm at most $C_M \delta F$. By the triangle inequality in the finite-dimensional Euclidean space of outputs, we have

$$\sum_{P \in \mathcal{A}} \rho_G\big((T_M U)_P\big)^2 \le \big(\sqrt{M} + C_M \delta\big)^2 F^2 \le M(1 + C_M \delta) F^2, \tag{7.9}$$

after enlarging C_M, for $0 \le \delta \le \delta_* \le 1$. On a single row, the error estimate and the row sum give

$$\rho_G\big((T_M U)_P\big) \le M^2\delta + C_M\delta\sqrt{12M^3}\,\delta \le M^2\delta + c_M\delta^2, \tag{7.10}$$

with $c_M \ge 1$. The linear coefficient M^2 is exact.

To pass to the Wilson energy, recall from Lemma 5.1 that, for a fine plaquette with $\rho_G(U_p) \le \delta$ and for every coarse plaquette,

$$\frac{\rho_G(U_p)^2}{8}\Big(1 - \frac{\delta^2}{48}\Big) \le q(U_p), \qquad q\big((T_M U)_P\big) \le \frac{\rho_G((T_M U)_P)^2}{8}.$$

The energy of the output plaquettes in $\mathcal{A}$ is therefore at most $M(1 + C_M\delta)(1 - \delta^2/48)^{-1}$ times the energy of $\mathrm{Pa}(\mathcal{A})$. Decreasing $\delta_* \le 1$ and enlarging C_M, we absorb the last factor into $1 + C_M\delta$. This proves the restricted inequality in (3); with all output plaquettes it proves the energy inequality in (1), and (7.10) proves the curvature inequality in (1). All estimates used only the plaquettes in the patches.

Iteration. Dividing the one-step energy inequality by $g_{j+1}^2 = Mg_j^2$, multiplying by 4, and iterating, we obtain the product in (4); since $1 + x \le e^x$ for $x \ge 0$, the exponential bound follows. For $d = 0$ the product is empty and (4) is an equality.

For (5), let $y_j = \delta(U^j)/\epsilon_j^2$. As long as the preceding inputs satisfy $\delta(U^i) \le \delta_*$, (7.10) and $\epsilon_{j+1} = M\epsilon_j$ imply

$$y_{j+1} \le y_j + \frac{c_M}{M^2}\epsilon_j^2 y_j^2, \qquad \sum_{j=0}^{d-1} \epsilon_j^2 = \frac{1 - M^{-2d}}{M^2 - 1} \le \frac{1}{M^2 - 1}. \tag{7.11}$$

The initial value is at most ε. Suppose that $y_i \le 2\varepsilon$ for all $i \le j$. Summing the recurrence up to j bounds y_{j+1} by

$$\varepsilon + \frac{4c_M\varepsilon^2}{M^2(M^2 - 1)} \le 2\varepsilon,$$

by the choice of ε_M. Each preceding curvature is then at most $2\varepsilon\epsilon_i^2 \le 2\varepsilon \le \delta_* \le \delta_M$, which verifies the hypothesis needed for the next application of the recurrence and of Lemma 7.2(3). This induction reaches $j = d$. Summing $\delta(U^j) \le 2\varepsilon\epsilon_j^2$ over $j < d$, we find the bound on the sum of the curvatures, and inserting it into (4) gives the last bound in (5).

For (6), note that $a_0 = a_* M^{-d} = a_*\epsilon_0$, so the hypothesis $\delta(U^0) \le C_A a_0^2$ is the hypothesis of (5) with $\varepsilon = C_A a_*^2$. Part (5) then implies $\delta(U^j) \le 2C_A a_j^2$. Applying Lemma 7.2(3) to each U^j, $j < d$, we find

$$\|z_j/g_j\|_\infty \le \frac{C_M\delta(U^j)}{g_j} \le \frac{2C_M C_A}{g_3} a_j^{3/2}, \qquad 0 \le j < d,$$

where $g_j = g_3\sqrt{a_j}$, in agreement with $g_n^2 = \gamma/n = g_3^2 a_n$ from Definition 4.1. If $C_A = 0$, the curvature recursion gives $\delta(U^j) = 0$ at every step and Lemma 7.2(3) gives $z_j = 0$, and the formulas hold directly. Every constant in the proof came from fixed patches of blocks, from the radii of the logarithm and the inverse coordinates, or from the constant C_A. □

7.3 Dependence sets and reflected cell partitions

To apply the chessboard estimates of Theorem 6.6 to plaquettes of the blocked configurations $V^j = T_M^j U$, we need to know on which fine links such a plaquette depends. We also need a partition of the lattice into cells, invariant under reflections, such that each cell contains these dependence sets. The relevant sets are shown in Figure 7.1. Because the blocks are centered and nested, the iterated dependence set of a coarse plaquette stays inside the four blocks at its corners, at every scale. Since a single tiling by cells of two coarse sites in each direction does not contain every plaquette in a cell, we use eight tilings, shifted by the parities of the starting vertex.

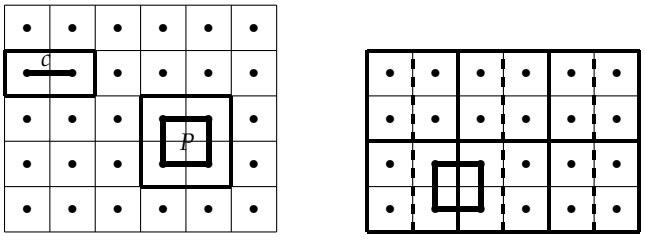


Figure 7.1. Left: the blocks of side m_j (thin squares) around the vertices of the level-j lattice, in a plane spanned by two coordinate directions. The patch of the coarse link c consists of its two blocks, and the patch of the coarse plaquette P consists of the four blocks at its corners (thick outlines). Right: cell boundaries of two of the eight shifted tilings, with shifts $(0,0)$ (solid) and $(1,0)$ (dashed) in the two directions shown; each cell is the union of the blocks of $2 \times 2 \times 2$ level-j vertices. The thick plaquette crosses a cell boundary of the first tiling, but lies in a cell of the second.

Lemma 7.4. *Fix an odd $M \geq 3$, integers $D \geq 0$ and $0 \leq j \leq D$, and the fine side $N = M^D n$ with $n \in 4\mathbb{N}$. Work in fine lattice units, put $m_j = M^j$ and $n_j = N/m_j$, and write $V^\ell = T_M^\ell U$.*

1. *For a plaquette P of the lattice at level j, define recursively $\mathcal{P}_j(P) = \{P\}$ and $\mathcal{P}_i(P) = \mathrm{Pa}(\mathcal{P}_{i+1}(P))$ for $i < j$, with* Pa *defined in Theorem 7.3(3); we call $\mathcal{P}_0(P)$ the energy-estimate predecessor set of P. Every fine plaquette in $\mathcal{P}_0(P)$ has its vertices in the union of the four blocks B_{m_j} at the corners of P. This union also contains all fine links on which V_P^j depends,*

including through the Borel extension of T_M, and all paths and plaquettes used in the patches of the intermediate steps. Further,

$$|\mathcal{P}_0(P)| \le C_M m_j^3, \qquad \operatorname{diam} \mathcal{P}_0(P) \le C_M m_j, \qquad \sup_p \#\{P : p \in \mathcal{P}_0(P)\} \le 48.$$

The same statements hold between any two intermediate levels, with the ratio of their meshes in place of m_j.

2. *For each shift $\eta \in \{0,1\}^3$, tile the vertices of the level-j lattice by the sets $\eta + 2t + \{0,1\}^3$, and define the cells of the fine lattice*

$$C_{j,t}^{\eta} = \bigcup_{\zeta \in \{0,1\}^3} \mathsf{B}_{m_j}\big(m_j(\eta + 2t + \zeta)\big).$$

These are disjoint cubes with $2m_j$ fine vertices in each direction, and there are $\mathfrak{m}_j = n_j^3/8$ of them in each tiling. If a level-j plaquette has all its vertices in the level-j vertices of a cell, then the plaquette and its energy-estimate predecessor set depend only on fine links with both endpoints in the cell. The reflections between adjacent cells are $x_i \mapsto (2\eta_i + 4t_i + 3)m_j - x_i$; their planes and the opposite planes bisect fine links, and these reflections map the blocks, the patches, and the sets $\mathcal{P}_\ell(P)$ at every level $\ell \le j$ onto the corresponding sets. Every level-j plaquette lies in a cell of at least one of the eight tilings.

3. *Let $j \ge 1$. For a cell C of one tiling $\mathcal{T}$, let $\mathcal{P}_{j-1}(C)$ be the union of the sets $\mathrm{Pa}(\{P\})$ of the six level-j plaquettes P in C. Then*

$$|\mathcal{P}_{j-1}(C)| \le 24M^3.$$

The sets obtained from $\mathcal{P}_{j-1}(C)$ by successive applications of Pa*, for distinct cells C of $\mathcal{T}$, are disjoint, and for individual level-j plaquettes the overlap bound 48 of (1) applies. For $0 \le \ell < j$ let*

$$\mathcal{F}_\ell(C) = \Big\{ p \in \mathcal{P}_{n_\ell} : \bigcup_{v \text{ a vertex of } p} \mathsf{B}_{m_\ell}(m_\ell v) \subset C \Big\},$$

where the vertices v are written in level-ℓ lattice units, and for $d_{\mathrm{cut}} > 0$ let

$$R_{<j}(C; d_{\mathrm{cut}}) = \bigcap_{\ell<j} \bigcap_{p \in \mathcal{F}_\ell(C)} \big\{ q(V_p^\ell) < d_{\mathrm{cut}} \big\}, \qquad R_{<j}^{\mathcal{T}}(d_{\mathrm{cut}}) = \bigcap_{C \in \mathcal{T}} R_{<j}(C; d_{\mathrm{cut}}).$$

These events impose the conditions on every $\mathcal{P}_\ell(P)$, $\ell < j$, for the six level-j plaquettes P in C. Further, put

$$\mathcal{P}_{j-1}^{\mathcal{T}} = \bigcup_{C \in \mathcal{T}} \mathcal{P}_{j-1}(C), \qquad Q_j^{\mathcal{T}} = \frac{4}{h_{n_{j-1}}} \sum_{p \in \mathcal{P}_{j-1}^{\mathcal{T}}} q(V_p^{j-1}),$$

where $h_n = g_n^2 = \gamma/n$.

4. *Put* $Q_{j,C} = (4/h_{n_{j-1}}) \sum_{p \in \mathcal{P}_{j-1}(C)} q(V_p^{j-1})$. *The following functions are nonnegative observables supported in the cell* C*, in the sense of Theorem 6.6(2), that is, bounded Borel functions of the fine links with both endpoints in* C *that are invariant under all periodic gauge transformations: the indicator of* $R_{<j}(C; d_{\mathrm{cut}})$*, the plaquette-energy sum* $Q_{j,C}$*, the functions* $\exp(\lambda Q_{j,C})$ *for* $\lambda \geq 0$*, and their products. Plaquettes crossing the boundaries of the cells of a tiling are not tested by the events of that tiling.*

All bounds are deterministic and do not depend on the coupling or on D*; the constant* C_M *depends only on* M*, not on any region of the lattice or on widths chosen later.*

Proof. Nesting. For odd $s \geq 1$ and $-R_M \leq \alpha \leq R_M$, the intervals of s consecutive integers centered at $v + s\alpha$ are disjoint and adjacent, with extreme endpoints $v - (Ms-1)/2$ and $v + (Ms-1)/2$. Taking products gives

$$\bigcup_{\alpha \in \{-R_M, \dots, R_M\}^3} \mathsf{B}_s(v + s\alpha) = \mathsf{B}_{Ms}(v). \tag{7.12}$$

This is the nesting identity of Proposition 6.2(6); it holds on the periodic lattice, since the blocks partition its vertices.

Containment. At a single step, a plaquette of $\mathrm{Pa}(\{P\})$ has its four vertices in the four corner blocks of P. Replacing each of these vertices by its own finer block keeps the union inside the same four larger blocks, by (7.12). By induction backwards from level j, we obtain the containment for $\mathcal{P}_0(P)$ and, in the same way, at every intermediate level. The same induction shows that the dependence sets are contained: at each step the dependence set of a link given by Proposition 6.2(6), its reference path, and the patch of four blocks used in Theorem 7.3 are included. The decision whether the Borel extension is used reads only the same based loop holonomies, and the value it assigns reads only the reference path. No continuity of T_M at such configurations is used.

Counting and overlap. The four corner blocks of side m_j contain $4m_j^3$ vertices and fit into a box with $2m_j$, $2m_j$, and m_j vertices in the three directions. Since at most three positive plaquettes start at each vertex, we have $|\mathcal{P}_0(P)| \leq 12m_j^3$, and the diameter in the lattice path metric is at most $5m_j$. Periodic identifications can only decrease these bounds. For the overlap, fix a fine plaquette p; each of its four vertices lies in one block of side m_j. If $p \in \mathcal{P}_0(P)$, all its vertices lie in corner blocks of P, so one of the at most four centers of these blocks is a corner of P. A center is a corner of twelve positive plaquettes, so there are at most $4 \cdot 12 = 48$ such P, also when some centers coincide. Starting the argument at an intermediate level replaces m_j by the ratio of the meshes and changes nothing else.

Cells and reflections. For the shift η, the coordinate interval of $C_{j,t}^{\eta}$ in direction i starts at $l_i = m_j(\eta_i + 2t_i) - (m_j - 1)/2$ and ends at $l_i + 2m_j - 1$; it contains $2m_j$ vertices, and the next interval starts at $l_i + 2m_j$. Since n_j is even, these intervals partition the coordinate circle, and their products give $(n_j/2)^3 = n_j^3/8$ disjoint cells. A level-j plaquette with all its vertices in a cell has its four corner blocks in the cell. By the containment just proved,

its energy-estimate predecessor set and all its dependencies are in the cell. Every link involved has both endpoints in the cell, including links in the boundary planes of the cell. The plane between the interval above and the next one is $x_i = l_i + 2m_j - 1/2$, and the reflection in it is $x_i \mapsto c_i - x_i$ with

$$c_i = 2l_i + 4m_j - 1 = (2\eta_i + 4t_i + 3)m_j.$$

Since m_j is odd, this number is odd, and the plane bisects fine links. The opposite plane has intercept larger by N, which is even, and it also bisects fine links. Since, for every level $\ell \le j$, the intercept divided by m_ℓ is still an odd integer, the reflection preserves the partition into blocks of side m_ℓ. By Proposition 6.2(4) it preserves the order of the coordinates in the root paths, and it maps a closed rectangular patch, with all its links and plaquettes, onto the reflected patch. Repeating this at every level proves the statement for all the sets $\mathcal{P}_\ell(P)$.

Coverage. Given a level-j *ik* plaquette with starting vertex v, we choose $\eta_i = v_i \bmod 2$ and $\eta_k = v_k \bmod 2$. Its two vertices in each of these directions then lie in one interval of the tiling. In the third direction either shift works, since the interval containing v contains the plaquette. Hence every plaquette lies in a cell of at least one of the eight tilings.

The sets of one cell. Let $j \ge 1$. Every plaquette of $\mathcal{P}_{j-1}(C)$ has its vertices among the $8M^3$ level-$(j-1)$ vertices whose blocks of side m_{j-1} form C, and there are at most $3 \cdot 8M^3 = 24M^3$ positive plaquettes with vertices there; the bound is for the union of the six patches, whose overlaps are not counted. Every backward expansion of this set stays in C, by (7.12). Since distinct cells have disjoint sets of vertices, the sets obtained from them by successive applications of Pa, and the links on which these depend, are disjoint. For individual target plaquettes the bound 48 of (1) still applies. When a plaquette occurs in one of the sets $\mathcal{P}_\ell(P)$, $\ell < j$, of a plaquette P in C, the union of the blocks of side m_ℓ of its four vertices lies in C, by the same nesting argument; this is the defining condition of $\mathcal{F}_\ell(C)$. The events $R_{<j}(C; d_{\mathrm{cut}})$ therefore impose the conditions on all these sets, and $Q_j^{\mathcal{T}}$ is the sum over the union of the sets $\mathcal{P}_{j-1}(C)$, each plaquette counted once, with the normalization $4/h_{n_{j-1}}$ of the Wilson action.

Observables. Each function $q(V_p^\ell)$ is Borel, since T_M is Borel (Proposition 6.2(2)), and gauge invariant, by gauge covariance of T_M and conjugation invariance of q. For $p \in \mathcal{F}_\ell(C)$ it depends only on links with both endpoints in C. The indicator of $R_{<j}(C; d_{\mathrm{cut}})$ is a finite product of Borel indicators of such events; it is Borel with values in $\{0, 1\}$. For $j \ge 1$, the sum $Q_{j,C}$ is a Borel gauge-invariant function of the links in C, and $0 \le q \le 2$ gives

$$0 \le Q_{j,C} \le \frac{8|\mathcal{P}_{j-1}(C)|}{h_{n_{j-1}}} \le \frac{8 \cdot 24M^3}{h_{n_{j-1}}}. \tag{7.13}$$

Hence, for every fixed $\lambda \ge 0$, $\exp(\lambda Q_{j,C})$ is bounded by the exponential of λ times this bound, and the indicator, the exponential, and their product are nonnegative bounded

Borel observables supported in C. The reflections between cells map each set of tested plaquettes onto the reflected set; they may reverse a plaquette or move its base point, and $q(HWH^{-1}) = q(W^{-1}) = q(W)$. Finally, the parity condition of Theorem 6.6 holds, since $n_j = M^{D-j}n \in 4\mathbb{N}$. The constants do not depend on the coupling, on D, or on any region or width. □

7.4 Lipschitz extensions and periodic gauge fixing

Regular lattice configurations and gauge fixing conditions are part of the renormalization group program of Bałaban for lattice gauge theories [14]. Here we need a particular quantitative statement: a configuration with small curvature on a periodic lattice has a periodic gauge transform with uniformly small links, with no restriction on its holonomies around the noncontractible cycles. The following example shows the form such a bound must take. Let $\theta \in \mathfrak{su}(2)$ with $0 < |\theta| < 2\pi$, and let U be the configuration on the lattice of side n with $U_e = e^{\theta/n}$ on the links in direction 1 and $U_e = I$ on all other links. Its curvature vanishes, and the transport around a cycle in direction 1 is e^{θ}. For every gauge transform U^k this transport is conjugate to e^{θ}. By the triangle inequality, some link satisfies $\rho_G(U_e^k) \geq |\theta|/n$. A bound of the form $C\sqrt{\delta}$ in terms of the maximal curvature δ alone is therefore impossible, and a term of order $1/n$ is needed. Spreading the holonomy evenly over the links of the cycle, as in the example, attains this order. At positive curvature δ the relevant length scale is $\delta^{-1/2}$. An axial gauge on a box of side R produces links of size about δR, while gluing such boxes requires gauge transformations varying by about $1/R$ per link; both are of order $\sqrt{\delta}$ when R is of order $\delta^{-1/2}$.

The construction glues local axial gauges on a partition of the torus into cells. In the gluing we extend group-valued maps successively over the edges, faces, and interiors of the cells. This is possible because $\mathrm{SU}(2) \simeq S^3$ has vanishing first and second homotopy groups, and Lemma 7.5 below gives Lipschitz bounds for the extensions. We prove the bounds on the extension, interpolation, and gluing directly, without an external extension theorem.

Lemma 7.5. *Use the bi-invariant metric of* $\mathrm{SU}(2) \simeq S^3$*, which is twice the round metric. There is an absolute constant C with the following properties.*

1. *Every map from the unit circle S^1 to* $\mathrm{SU}(2)$ *with Lipschitz constant B_0 extends to the unit disk with Lipschitz constant at most $C(1 + B_0)^2$, and every such map from the unit sphere S^2 extends to the unit ball of $\mathbb{R}^3$ with Lipschitz constant at most $C(1 + B_0)^5$; the extensions agree with the given maps on the boundary. For $A_0 \geq 1$ and a rectangle or a rectangular box with all side lengths between R/A_0 and A_0R, the corresponding boundary-preserving extensions of a map with Lipschitz constant B_0 on the boundary have Lipschitz constants at most*

$$C(A_0)R^{-1}(1 + B_0R)^2 \qquad \text{and} \qquad C(A_0)R^{-1}(1 + B_0R)^5.$$

2. *Fix the triangulation of every lattice cube into the six simplices determined by the orderings of the coordinates; it is consistent on common faces and invariant under translations, so it descends to periodic lattices, and it induces triangulations of the faces of lattice boxes. There is an absolute $c_0 > 0$ such that, if unit quaternions are prescribed at the vertices of a simplex with pairwise distances at most c_0, their normalized affine interpolation $\sum_i \lambda_i u_i / |\sum_i \lambda_i u_i|$ is defined, with denominator at least $1/2$. If vertex data on a lattice of mesh a satisfy $\operatorname{dist}(u_x, u_y) \le B_0 a$ for all nearest neighbors x, y, and $B_0 a \le c(A_0)$, then the interpolation has Lipschitz constant at most $C(A_0)B_0$, and the interpolations on two simplices agree on their common face.*
3. *For every $L > 0$, $\mathscr{K} \ge 0$, integer $n \ge 4$, and $a = L/n$, every periodic configuration on the lattice of side n and mesh a with $\max_p \rho_G(U_p) \le \mathscr{K} a^2$ admits a periodic gauge transformation k with*

$$\max_e \rho_G(U_e^k) \le \frac{Ca}{L}\bigl(1 + \mathscr{K} L^2\bigr)^{10}.$$

All holonomies around the cycles of the torus are allowed. If $a \le cL(1 + \mathscr{K} L^2)^{-2}$ for an absolute $c > 0$, the construction uses a partition of the torus into closed rectangular cells Q with side lengths between $L/8$ and $L/2$, each with a rooted axial gauge q_Q such that $\rho_G(U_e^{q_Q}) \le C\mathscr{K} La$ on every link of its closure, and the transition functions $t_{QR} = q_Q q_R^{-1}$ on common links (x, y) satisfy

$$t_{QR}(y) = \bigl(U_{xy}^{q_Q}\bigr)^{-1} t_{QR}(x)\, U_{xy}^{q_R}, \qquad \operatorname{dist}\bigl(t_{QR}(x), t_{QR}(y)\bigr) \le C\mathscr{K} La.$$

On coarser meshes the bound on $\max_e \rho_G(U_e^k)$ follows from the diameter of $\mathrm{SU}(2)$, after an absolute enlargement of C. When this bound is smaller than 2π, it bounds the principal logarithms of the links of U^k.

The constants in (1) and (2) depend only on A_0 and the triangulation, and the constant in (3) is absolute: it does not depend on n, L, $\mathscr{K}$, or the holonomies. No regularity of k as a function of U is claimed, and no estimate for differences of the link logarithms beyond the bound on each link is asserted.

We prove parts (1) and (2) first. Part (3) follows from the gluing estimate of Lemma 7.6 below.

Proof of Lemma 7.5(1) and (2). We identify $\mathrm{SU}(2)$ with the unit quaternions in $\mathbb{R}^4$. Its metric is twice the round metric of the unit sphere: the quaternion $\cos(t/2) + \omega \sin(t/2)$ is at distance $|t|$ from I for $0 \le |t| \le 2\pi$. Comparing arc and chord on a circle, we obtain, for $u, v \in S^3$,

$$2|u - v| \le \operatorname{dist}(u, v) \le \pi |u - v|. \tag{7.14}$$

All constants in the arguments on spheres below are absolute.

Covering. A unit sphere S^m, $m = 1, 2$, can be covered by at most $C_m r^{-m}$ balls of radius $r \le 1$. Indeed, consider a maximal set of points with mutual distances greater than r.

The balls of radius $r/2$ around them are disjoint, and each has volume at least $c_m r^m$, as one sees by integrating the polar density $(\sin t)^{m-1}$ and using $2t/\pi \le \sin t \le t$ for $0 \le t \le \pi/2$. A separated set is finite, since it has at most one point in each cube of a sufficiently fine subdivision of a cube containing the sphere, and maximality gives the cover. The same polar integration shows that a ball of radius 2ε in S^3 has volume at most $C\varepsilon^3$ for $\varepsilon \le c$.

Extension from a sphere. Let $f : S^m \to S^3$ have Lipschitz constant B_0. A cover of S^m by balls of radius $\varepsilon/(1+B_0)$ gives a cover of the image by at most $C((1+B_0)/\varepsilon)^m$ balls of radius ε, so the ε-neighborhood of the image has volume at most $C(1+B_0)^m \varepsilon^{3-m}$. Let

$$\varepsilon = c_m (1+B_0)^{-m/(3-m)}$$

with $c_m > 0$ so small that this volume is less than half the volume of S^3. Then some point p has distance at least ε from the image, and after a rotation we may take it to be the north pole. The stereographic projection from p is $\pi_p(u) = u_\perp/(1-u\cdot p)$, where $u_\perp$ is the component of u orthogonal to p. For points at round distance at least $\varepsilon/2$ from p it satisfies

$$|\pi_p(u)| \le C\varepsilon^{-1}, \qquad |\pi_p(u) - \pi_p(v)| = \frac{|u-v|}{\sqrt{(1-u\cdot p)(1-v\cdot p)}} \le C\varepsilon^{-2}|u-v|. \quad (7.15)$$

The equality follows by squaring both sides and using $|u| = |v| = 1$, and the inequalities follow from $1-\cos t = 2\sin^2(t/2) \ge ct^2$ for $0 \le t \le \pi$; the factor two between the group metric and the round metric only changes constants. We conclude that $\psi = \pi_p \circ f$ has supremum at most $C\varepsilon^{-1}$ and Lipschitz constant at most $C\varepsilon^{-2}B_0$. On the source sphere the arc distance is at most $\pi/2$ times the chord distance, so this is also a Lipschitz bound with respect to the Euclidean metric, after an absolute change of the constant.

We extend ψ to the unit ball by $\Psi(r\omega) = r\psi(\omega)$, with $\Psi(0) = 0$. For $r \ge s$,

$$|r\psi(\omega) - s\psi(\eta)| \le (r-s)\|\psi\|_\infty + s\,\mathrm{Lip}(\psi)|\omega - \eta|,$$

and $r - s \le |r\omega - s\eta|$, $s|\omega-\eta| \le 2|r\omega - s\eta|$, so the Lipschitz constant of Ψ is at most $\|\psi\|_\infty + 2\,\mathrm{Lip}(\psi)$. The inverse stereographic projection is

$$\pi_p^{-1}(z) = \frac{(2z, |z|^2 - 1)}{1+|z|^2}.$$

Its differential has Euclidean norm at most 2. Following the image of a line segment and using its round length, and doubling for the group metric, we find that π_p^{-1} has Lipschitz constant at most 4. Hence $\pi_p^{-1} \circ \Psi$ extends the given boundary map and has Lipschitz constant at most

$$C\big(\varepsilon^{-1} + B_0\varepsilon^{-2}\big) \le \begin{cases} C(1+B_0)^2, & m = 1, \\ C(1+B_0)^5, & m = 2. \end{cases}$$

This proves the extension bounds for spheres, including the case $B_0 = 0$.

Rectangles and boxes. In dimension $k = 2, 3$, the radial map $r\omega \mapsto r\omega/\|\omega\|_\infty$ takes the Euclidean unit ball onto $[-1,1]^k$ and the unit sphere onto its boundary. On the sphere, $1/\sqrt{k} \le \|\omega\|_\infty \le 1$ and $|\|\omega\|_\infty - \|\eta\|_\infty| \le |\omega - \eta|$, so the map is Lipschitz on the sphere, and so is its inverse, which has radial factor $\|\omega\|_\infty$ on each ray. The radial estimate used for Ψ shows that both maps are Lipschitz on the balls as well. A diagonal affine map onto a box with side lengths between R/A_0 and $A_0 R$ changes Lipschitz constants by factors at most $C(A_0)R$ and $C(A_0)/R$. Composing the extensions for spheres with these maps gives the bounds for rectangles and boxes. The same comparison bounds the intrinsic distance on the boundary of a box by $C(A_0)$ times the ambient distance: on a sphere, a shortest arc is at most $\pi/2$ times its chord, and the boundary maps just discussed transfer this inequality. This proves (1).

Interpolation. Let $c_0 > 0$ be so small that pairwise distances at most c_0 between the vertex values of a simplex imply $u_0 \cdot u_i \ge 3/4$; this follows from $u_0 \cdot u_i = 1 - |u_0 - u_i|^2/2$ and (7.14). For barycentric coordinates λ_i,

$$\Big|\sum_i \lambda_i u_i\Big| \ge u_0 \cdot \sum_i \lambda_i u_i \ge \frac{3}{4} \ge \frac{1}{2},$$

so the normalization is defined. The derivative of $z \mapsto z/|z|$ has operator norm at most 2 on $\{|z| \ge 1/2\}$. On a simplex of the triangulation, the affine interpolation has derivative at most $C(A_0)a^{-1}\max_{i,l}|u_i - u_l|$, as one sees by writing it in barycentric coordinates, whose gradients are bounded by $C(A_0)/a$ on the finitely many simplex shapes. Two vertices of a simplex are joined by at most three lattice links, so the maximum is at most CB_0a, and the derivative of the normalized interpolation, hence its Lipschitz constant for the group metric, is at most $C(A_0)B_0$. The condition $B_0 a \le c(A_0)$ ensures the smallness condition on the simplices. Since the restriction of the barycentric formula to a common face uses only the values at the common vertices, the interpolations agree there. This proves (2). □

Lemma 7.6. *Let $A_0 \ge 1$. There exist $c(A_0) > 0$ and $C(A_0) < \infty$ with the following property. Consider a periodic cubic lattice of mesh a and a face-to-face decomposition of the torus into closed coordinate boxes: each lifted cell has the form $\prod_{i=1}^3 [am_i, an_i]$, with integers $m_i < n_i$. Assume that every side length lies between R/A_0 and $A_0 R$ and that each cell lifts injectively to the covering lattice. Here face-to-face means that the intersection of two distinct cells is empty or a whole face, edge, or vertex of both cells. Suppose that each cell Q has a gauge transformation q_Q, defined on its vertices, such that*

$$\rho_G\big(U_{xy}^{q_Q}\big) \le B_0 a \qquad \text{for every link } (x,y) \text{ of the closure of } Q. \tag{7.16}$$

No bound on the gauge transformations q_Q themselves is assumed. Then there is a periodic gauge transformation k with

$$a \le c(A_0) R (1 + B_0 R)^{-2} \quad \Longrightarrow \quad \max_e \rho_G(U_e^k) \le \frac{C(A_0)\, a}{R} (1 + B_0 R)^{10}. \tag{7.17}$$

The constants do not depend on the number of cells.

Proof. On a link (x, y) common to two cells Q and R', let $t_{QR'} = q_Q q_{R'}^{-1}$. Direct multiplication gives

$$t_{QR'}(y) = \big(U_{xy}^{q_Q}\big)^{-1} t_{QR'}(x)\, U_{xy}^{q_{R'}}, \qquad \operatorname{dist}\big(t_{QR'}(x), t_{QR'}(y)\big) \le 2B_0 a. \tag{7.18}$$

This applies also to cells that meet only along an edge. We assign a single gauge transformation k at all vertices. Its coordinates in the frame of Q are $s_Q = k q_Q^{-1}$, so that $s_{R'} = s_Q t_{QR'}$. By bi-invariance, multiplying two group-valued maps adds their Lipschitz bounds, so passing from one incident frame to another increases the bound on neighbor distances by at most $2B_0 a$.

Vertices and edges of the cells. We set $k = I$ at every corner of a cell, once on the periodic lattice. For each edge of the cell partition, we choose one incident cell Q and join the two prescribed values of s_Q at its endpoints by a minimizing geodesic of $\mathrm{SU}(2)$, parametrized proportionally to arc length over the edge. Such a geodesic is an arc of a great circle in S^3; if its endpoints are antipodal, any great circle through them may be used. Since the diameter of $\mathrm{SU}(2)$ is 2π and the edge has length at least R/A_0, the speed is at most $C(A_0)/R$. At the lattice vertices of the edge, we set $k = s_Q q_Q$. These assignments reproduce the given values at the corners. In every incident frame the resulting vertex data satisfy $\operatorname{dist}(s(x), s(y)) \le B_1 a$ for neighbors on the edge, with

$$B_1 \le C(A_0)/R + 2B_0.$$

Faces of the cells. For each face of the cell partition, we choose one incident cell, express the values already assigned on the boundary of the face in its frame, and interpolate them along the lattice links of the boundary by Lemma 7.5(2). Provided $aB_1 \le c(A_0)$, this is a continuous loop with Lipschitz constant at most $C(A_0)B_1$. The comparison of intrinsic and ambient distances in the proof of Lemma 7.5(1) gives this bound also between points on different sides of the face. The extension over the disk, rescaled to the face, gives an extension with Lipschitz constant

$$B_2 \le \frac{C(A_0)}{R} (1 + B_1 R)^2.$$

We use it to assign $k = s_Q q_Q$ at the remaining lattice vertices of the face. Every vertex value assigned before is kept. Each face is treated once, including faces that meet the

periodic identifications. The vertices of an edge shared by several faces were assigned before and have a single value throughout.

Interiors of the cells. Consider a cell Q. On a face that was treated in the frame of a cell Q_F, the boundary values in the frame of Q are $s_Q = s_{Q_F} t_{Q_F Q}$, so by (7.18) neighboring values satisfy $\operatorname{dist}(s_Q(x), s_Q(y)) \le B_\partial a$ with

$$B_\partial \le B_2 + 2B_0 \le \frac{C(A_0)}{R}(1 + B_0 R)^2.$$

We interpolate the vertex values on the boundary of Q by the triangulation of Lemma 7.5(2). If $aB_\partial \le c(A_0)$, this is a continuous map on the boundary sphere with Lipschitz constant at most $C(A_0)B_\partial$. It is single valued at every edge and vertex, since the gauge transformation was assigned there only once. The extension over the ball gives an extension in the frame of Q with Lipschitz constant

$$B_3 \le \frac{C(A_0)}{R}(1 + B_\partial R)^5 \le \frac{C(A_0)}{R}(1 + B_0 R)^{10}.$$

At each interior vertex of Q, we assign $k = s_Q q_Q$ from this extension. Interior vertices belong to only one cell, and the boundary values are those assigned before. The result is a gauge transformation on the whole periodic lattice.

Links. Each lattice link lies in a closed cell, and the extension constructed for that cell represents both endpoint values, also when one or both endpoints lie on the boundary of the cell. Therefore

$$\rho_G\big(k_x U_{xy} k_y^{-1}\big) = \rho_G\big(s_Q(x) U_{xy}^{q_Q} s_Q(y)^{-1}\big) \le (B_0 + B_3)a.$$

Both smallness conditions $aB_1 \le c(A_0)$ and $aB_\partial \le c(A_0)$ follow from $a \le c(A_0)R(1 + B_0R)^{-2}$ after decreasing $c(A_0)$, and $(B_0 + B_3)a$ is at most the right side of (7.17). The constants do not depend on the number of cells: each change of frame compares two incident frames by a single transition (7.18), and no chain of comparisons through other cells is taken. Continuous extensions in different frames were required to agree only at the lattice vertices assigned earlier, and every later extension kept these values. This suffices for the bound on every link. □

Proof of Lemma 7.5(3). Suppose first that $n \ge 8$. We divide each coordinate circle at the vertices with indices 0, $\lfloor n/4 \rfloor$, $\lfloor n/2 \rfloor$, $\lfloor 3n/4 \rfloor$, and n; each interval has physical length between $L/8$ and $L/2$. The resulting 4^3 closed cells form a product, hence face-to-face, decomposition of the torus; they have bounded aspect ratios and lift injectively to the covering lattice, also when they meet the periodic identifications. On each cell, we use the rooted axial gauge of Lemma 7.1. Each link not in its axial tree is a product of at most CL/a conjugated plaquette variables along strips of width one, each at distance at most $\mathscr{K} a^2$ from I. Then (7.16) holds with $B_0 = C\mathscr{K} L$ and $R = L$. These gauges are rooted and

give the stated bound on every link of the closure of a cell; their transitions satisfy (7.18), which gives the stated transition bounds.

With the fixed aspect bound, the condition in (7.17) follows from $a \le cL(1+\mathscr{K}L^2)^{-2}$ after decreasing the absolute constant c; taking $c \le 1/8$ also guarantees $n \ge 8$. The conclusion of (7.17) is the asserted bound. If the condition on the mesh fails, we take $k = I$. Then $\rho_G(U_e) \le 2\pi$, while

$$\frac{a}{L}\big(1+\mathscr{K}L^2\big)^{10} > c\big(1+\mathscr{K}L^2\big)^{8} \ge c,$$

so an absolute enlargement $C \ge 2\pi/c$ gives the same bound on all such meshes, including $4 \le n < 8$. If the bound is smaller than 2π, the links of U^k differ from $-I$, and their principal logarithms have the same norms.

All assignments were made on the periodic lattice, not by identifying the values of independently chosen gauges afterwards, so arbitrary holonomies remain in the local gauges and in the assigned values throughout. □

7.5 Small-curvature gauge fixing and probability bounds

We apply the gluing lemma at cell size $R \approx \delta^{-1/2}$, and then estimate the probability of D_r^c using Theorem 6.6. Recall from Definition 4.4 the small-link domain and the gauge-invariant small-field domain; for $0 < r < 2\pi$ they are

$$B_r = \big\{V : \max_e \rho_G(V_e) < r\big\}, \qquad D_r = \big\{U : U^k \in B_r \text{ for some } k\big\},$$

where k runs over the periodic gauge transformations. The deterministic part of the following theorem, parts (1) and (2), is used at every scale in Sections 22 and 23; the probability bound in parts (3) and (4) concerns only the restriction of the Wilson measure on the finest lattice.

Theorem 7.7. *There exist absolute constants $C_{\mathrm{SG}} \ge 1$ and $\delta_{\mathrm{SG}} > 0$ with the following properties.*

1. *For every periodic cubic lattice of side $n \ge 4$, every $0 \le \delta \le \delta_{\mathrm{SG}}$, and every configuration U with $\max_p \rho_G(U_p) \le \delta$, there exists a periodic gauge transformation k with*

$$\max_e \rho_G(U_e^k) \le C_{\mathrm{SG}}\big(\sqrt{\delta} + n^{-1}\big).$$

   This includes $\delta = 0$ and arbitrary holonomies around the cycles of the torus.

2. *Fix $0 < r < 2\pi$ and set*

$$\delta_r = \min\big\{\delta_{\mathrm{SG}},\, (r/(4C_{\mathrm{SG}}))^2\big\}, \qquad d_r = \frac{\delta_r^2}{2\pi^2} = \min\Big\{\frac{\delta_{\mathrm{SG}}^2}{2\pi^2},\, \frac{r^4}{512\pi^2 C_{\mathrm{SG}}^4}\Big\}.$$

If $n \geq 4C_{SG}/r$, then

$$\{\max_p \rho_G(U_p) \leq \delta_r\} \subset D_r, \qquad \{\max_p q(U_p) \leq d_r\} \subset D_r.$$

Some lower bound on the side is necessary: for $n < 2\pi/r$, the flat configurations of the example at the beginning of Section 7.4 with $rn \leq |\theta| < 2\pi$ do not lie in D_r. The set D_r is open and gauge invariant, so these inclusions do not require a measurable choice of the gauge transformation.

3. *Let $n \in 4\mathbb{N}$ with $n \geq 4C_{SG}/r$, and let $\gamma > 0$ be such that $\beta_n = 4n/\gamma \geq 2$. Then the Wilson measure satisfies $\mu_{n,\gamma}(D_r^c) \leq \varepsilon_{n,r,\gamma}$, where*

$$\varepsilon_{n,r,\gamma} = \min\Big\{1,\ 3n^3 \exp\Big[\frac{1}{8}\Big(C_0 + 24\log 2 + \frac{36\log\beta_n}{n}\Big) - \frac{\beta_n \delta_r^2}{32\pi^2}\Big]\Big\}$$

and C_0 is the absolute constant of Theorem 6.6(3). The surface term $36\log\beta_n/n$ is kept for every (n,γ); no relation between n and γ is assumed.

4. *Under the assumptions of (3), $\mu_{n,\gamma}(D_r) > 0$, the conditioned measure $\mu_{n,\gamma}(\,\cdot \mid D_r)$ has total variation distance exactly $\mu_{n,\gamma}(D_r^c)$ from $\mu_{n,\gamma}$, and the partition function Z_{D_r} restricted to D_r satisfies*

$$\frac{Z_{D_r}}{Z} = \mu_{n,\gamma}(D_r), \qquad \Big|\log\frac{Z_{D_r}}{Z}\Big| \leq 2\varepsilon_{n,r,\gamma} \quad \text{if } \varepsilon_{n,r,\gamma} \leq 1/2.$$

The constants C_{SG} and δ_{SG} do not depend on L, γ, n, the number of averaging steps, or any widths of regions. The probability bound concerns the restriction to D_r of the Wilson measure on the full periodic lattice; it is not a conditional bound at a later scale.

Proof. Part (1). We use lattice spacing one. For $\delta = 0$, by Lemma 7.5(3) with $a = 1$, $L = n$, and $\mathscr{K} = 0$, we have $\max_e \rho_G(U_e^k) \leq C/n$ for every $n \geq 4$, including arbitrary flat holonomies. For $0 < \delta \leq 1$, let $R = \lceil \delta^{-1/2} \rceil$. Then

$$1 \leq \delta R^2 \leq 4. \tag{7.19}$$

If $n < 8R$, the same lemma with $\mathscr{K} = \delta$ gives

$$\max_e \rho_G(U_e^k) \leq \frac{C}{n}\big(1 + \delta n^2\big)^{10} \leq \frac{C'}{n},$$

with C' absolute, since $\delta n^2 < 64\delta R^2 \leq 256$.

Suppose now that $n \geq 8R$, and let $b = \lfloor n/R \rfloor \geq 8$. We divide each coordinate circle at the vertices $\lfloor in/b \rfloor$, $0 \leq i \leq b$. Since $R \leq n/b < R(1 + 1/b)$, the intervals have lengths between $R/2$ and $2R$ when $R \geq 2$. The closed cells form a product, hence face-to-face, decomposition; they have bounded aspect ratios, and they lift injectively since $2R \leq n/4$. By Lemma 7.1, the rooted axial gauges on these cells satisfy

$$\max_{e \subset Q} \rho_G\big(U_e^{q_Q}\big) \leq C\delta R.$$

Indeed, a link in direction 1 not in the axial tree is moved through one strip of width one in direction 3 and one in direction 2, of total length at most $4R$, and the counts for the other directions are smaller. Then Lemma 7.6 applies with mesh one, cell size R, and $B_0 = C\delta R$, and (7.19) implies $B_0 R \le 4C$. Fix an absolute $R_0 \ge 2$ so large that

$$1 \le c(A_0) R_0 (1 + 4C)^{-2}$$

for these fixed aspect bounds, and let $0 < \delta_{\mathrm{SG}} \le \min\{1, R_0^{-2}\}$. Then $R \ge R_0$, and the condition on the mesh in (7.17) holds. Its conclusion is

$$\max_e \rho_G(U_e^k) \le \frac{C}{R}(1 + B_0 R)^{10} \le \frac{C''}{R} \le C''\sqrt{\delta}.$$

As in the gluing lemma, these constants do not depend on the number b^3 of cells. Taking $C_{\mathrm{SG}} \ge \max\{1, C, C', C''\}$, we obtain (1) in all cases, with absolute constants and no condition on the holonomies.

Part (2). For δ_r and n as stated, the gauge transformation of part (1) satisfies

$$\max_e \rho_G(U_e^k) \le C_{\mathrm{SG}}\sqrt{\delta_r} + C_{\mathrm{SG}}/n \le r/4 + r/4 = r/2 < r < 2\pi.$$

These links have unique principal logarithms and lie in B_r, which proves the first inclusion, with a margin even at the closed threshold. By Lemma 5.1, $\rho_G(U_p)^2 \le 2\pi^2 q(U_p)$, so the event $\{\max_p q(U_p) \le d_r\}$ with $d_r = \delta_r^2/(2\pi^2)$ is contained in the first event. Squaring the two values in the minimum defining δ_r gives the formula for d_r. For fixed k the action of k is a homeomorphism of the configuration space, so D_r, a union of translates of the open set B_r, is open, and it is gauge invariant because gauge transformations compose. In particular, the existence of the constructed gauge transformation gives membership in a Borel set without a measurable choice. The example at the beginning of Section 7.4 shows that the term n^{-1} and the corresponding condition on the side are needed, also at zero curvature.

Part (3). Let $n \in 4\mathbb{N}$ and $\beta_n = 4n/\gamma \ge 2$, with $n \ge 4C_{\mathrm{SG}}/r$ as in part (2). In the notation of Definition 4.1 the exponent of the Wilson measure is $-\beta_n \sum_p q(U_p)$, so $\mu_{n,\gamma} = \mu_n^{\beta_n}$ in the notation of Theorem 6.6. We apply Theorem 6.6(3) with $\ell = 0$, the face set $S = \{p\}$, and $s = 1/2$; its hypotheses $n \in 4\mathbb{N}$ and $\beta_n \ge 2$ hold. The function $\exp(\beta_n q(U_p)/16)$ is gauge invariant, Borel, and between 1 and $e^{\beta_n/8}$, and p lies in a cell of one of the eight shifted tilings, by Lemma 7.4 with $j = 0$. We obtain

$$\mathbb{E}_{\mu_{n,\gamma}} \exp\big(\beta_n q(U_p)/16\big) \le \exp\Big[\frac{1}{8}\Big(C_0 + 24\log 2 + \frac{36\log\beta_n}{n}\Big)\Big].$$

Since $\{\rho_G(U_p) > \delta_r\} \subset \{q(U_p) > \delta_r^2/(2\pi^2)\}$, by Markov's inequality, we have

$$\mu_{n,\gamma}\{\rho_G(U_p) > \delta_r\} \le \exp\Big[\frac{1}{8}\Big(C_0 + 24\log 2 + \frac{36\log\beta_n}{n}\Big) - \frac{\beta_n \delta_r^2}{32\pi^2}\Big].$$

There are $3n^3$ positive plaquettes, and by part (2) the complement of D_r is contained in the union of these events. The union bound and the trivial bound by one give $\varepsilon_{n,r,\gamma}$. The calculation keeps the surface term for every n and γ and uses no bound on β_n/n.

Part (4). By Lemma 5.1, the open set B_r contains a product of balls of positive Haar measure. Since the density of the Wilson measure is strictly positive, we have $\mu_{n,\gamma}(D_r) > 0$. The identity for the total variation distance (in the convention $\|\mu - \nu\|_{\mathrm{TV}} = \sup_A |\mu(A) - \nu(A)|$), the identity $Z_{D_r}/Z = \mu_{n,\gamma}(D_r)$, and the bound $|\log(Z_{D_r}/Z)| \leq 2\mu_{n,\gamma}(D_r^c)$ when $\mu_{n,\gamma}(D_r^c) \leq 1/2$ are the conditioning identities of Lemma 5.5, applied with $S = D_r$. Since $\mu_{n,\gamma}(D_r^c) \leq \varepsilon_{n,r,\gamma}$, the last bound gives the stated one. The deterministic constants came only from the axial strips, the extension bounds, and the aspect bounds, and the probability constant is the absolute constant C_0. □

SECTION 8

Linearized averaging and effective quadratic forms

This section proves deterministic estimates for the block averaging map T_M of Definition 4.3. We compute the derivative of T_M^d at the identity, the transverse Gaussian covariance that it induces from an auxiliary Maxwell field, and the effective Maxwell quadratic form (Theorem 8.3). We also bound small translation-invariant kernels that annihilate gradients and commute with spatial inversion by the lattice Maxwell form (Lemma 8.6). We then construct the gauge-corrected iteration near a background with small curvature and prove locality and absolute derivative bounds at each fixed order, including volume-normalized column sums, uniformly in the number of averaging steps (Lemmas 8.10 and 8.11). The Gaussian fields in this section are auxiliary, and no statement concerns the Wilson measure. The estimates are used to construct the background fields in Part III.

We distinguish two statements with different proofs. The first is linear and exact: the derivative of T_M^d at the identity is the average K_{M^d} along straight segments of length M^d plus a lattice gradient,

$$D(T_M^d)(I) = K_{M^d} + \nabla \mathsf{R}_d$$

(Lemma 8.1). The second is nonlinear. Near a background configuration U_0 whose plaquettes are close to the identity, we multiply the output of each averaging step by gauge transformations at the two endpoints of each coarse link. In logarithmic coordinates divided by the lattice spacing, the resulting gauge-corrected step is a transported segment average plus errors that are summable over scales. The gauge-corrected iteration coincides with the iterated block averaging map after an explicit composed gauge transformation (Lemma 8.10). The gradient in the first statement and the gauge transformations in the second play analogous roles, but neither statement implies the other.

The inputs are the Hodge decomposition and Schur complement identities of Lemma 5.2, the weighted product and inverse estimates of Lemma 5.3, the coercivity of the curl on the constrained fluctuation space in Lemma 6.5, and the curvature iteration of Theorem 7.3. The lattice, the norms, and the conventions for constants are those of Definitions 4.1, 4.6, and 4.14. The results are used in Section 9 to construct coordinates on the constraint fibers (Theorems 9.3, 9.6, and 9.9), in Theorem 10.10 to identify the Hessian of the effective classical action, in Lemma 11.11 to factor perturbations of that Hessian, and in Section 31 for the normalization of the heat-smoothed circulations (Theorem 31.6).

The linear theory is part of Bałaban's analysis of Gaussian block integration for lattice gauge fields. The averages with and without endpoint paths and their action on gradients appear in [10, (1.8)–(1.13), p. 19]. Section B, (1.16)–(1.18), pp. 19–20 of that paper composes the segment average without endpoint paths; its scalar Fourier multiplier is given in (1.29)–(1.31), p. 23, and the vector Fourier formula is (1.61), p. 28. We use centered blocks, a fixed order of coordinate paths, and a Borel extension at the singularity of the logarithm, and we prove the estimates below for these choices. Two points require care. First, the meromorphic formula for the inverse of the projected covariance has apparent singularities on the complex characteristic set $\Lambda_m(z) = 0$, which meets every neighborhood of the origin. We cancel these singularities before deducing exponential decay of the kernel. Second, the nonnegative scalar kernel of the transported segment average has row sums exactly one and counting column sums exactly M^{-3}. These masses must be kept exactly, since a fixed factor larger than one at each scale would destroy the uniformity in the number of averaging steps.

8.1 Linearized block averaging

Conventions. Unless physical units are displayed, a Fourier calculation in this section uses lattice coordinates with spacing one. For a finite matrix whose entries are color blocks, the row norm is the supremum over outputs of the sum over inputs of the operator norms of the blocks. The column norm is the supremum over inputs of the sum over outputs. Both use counting measure, and the subscript row + col denotes their sum, which is at least their maximum and at most twice it. A rate in the subscript, as in $\|\cdot\|_{a,\mathrm{row+col}}$, means that the entry with indices x, y is multiplied by $e^{a\,\mathrm{dist}(x,y)}$ before summing. Distances between links are periodic ℓ^1 distances between their initial vertices. Moving the location of a link from its initial vertex to its midpoint changes these distances by at most one, and it changes any fixed exponentially weighted bound by at most a fixed factor. For the nonlinear maps in Sections 8.4 and 8.5, distances are measured in the physical units stated there.

Paths in the block averaging map. We recall the paths of Definition 4.3. We write $M = 2R_M + 1$. For a vertex $v \in M\mathbb{Z}^3$ of the coarse lattice, its block is $\mathsf{B}_M(v) = v + ([-R_M, R_M]^3 \cap \mathbb{Z}^3)$, and for $x \in \mathsf{B}_M(v)$ the coordinate path $\sigma_{v,x}$ from v to x changes the coordinates in the order $1, 2, 3$. For a coarse link $c = (v, v + Me_i)$, the reference path C_c is the straight path from v to $v + Me_i$, and for $x \in \mathsf{B}_M(v)$ the translated path with endpoint connectors is

$$F_{c,x} = \sigma_{v,x}\,[x, x + Me_i]\,\sigma^{-1}_{v+Me_i,x+Me_i};$$

see Figure 8.1. The block averaging map is

$$(T_M U)_c = \exp\Big\{ M^{-3} \sum_{x \in \mathsf{B}_M(v)} \operatorname{Log}\big(U(F_{c,x}) U(C_c)^{-1}\big) \Big\} U(C_c),$$

where $U(\eta)$ is the transport along a path η, and $(T_M U)_c = U(C_c)$ if one of the based loop holonomies $U(F_{c,x})U(C_c)^{-1}$ equals $-I$.

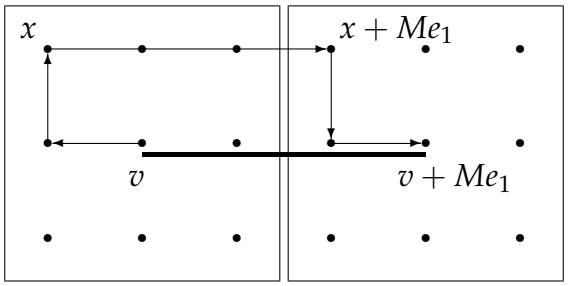


Figure 8.1. A two-dimensional slice of two neighboring blocks for $M = 3$. The thick line is the reference path C_c of the coarse link $c = (v, v + Me_1)$. The arrows show the translated path with endpoint connectors $F_{c,x}$: the coordinate path $\sigma_{v,x}$, the segment $[x, x + Me_1]$, and the reversed coordinate path $\sigma^{-1}_{v+Me_1,x+Me_1}$. At the identity the derivatives of the two connectors produce the gradient term in (8.1); in the gauge-corrected average of Section 8.4 they are canceled by gauge transformations at v and $v + Me_1$.

Lemma 8.1. *Fix an odd integer $M \geq 3$ and integers $d \geq 0$ and $n \geq 4$, and set $m = M^d$. On the fine torus of side mn, with coarse coordinates Y, let $\mathsf{I}_m = \{-(m-1)/2, \dots, (m-1)/2\}^3$ and define, for a one-cochain x and a scalar function f,*

$$(K_m x)_i(Y) = m^{-3} \sum_{r \in \mathsf{I}_m} \sum_{s=0}^{m-1} x_i(mY + r + se_i),$$
$$(\mathsf{A}_m f)(Y) = m^{-3} \sum_{r \in \mathsf{I}_m} f(mY + r).$$

For $d = 0$ both maps are the identity. Then the following statements hold.

1. *With the forward gradient $\nabla f(x, x + e_i) = f(x + e_i) - f(x)$,*

$$K_m \nabla_{\text{fine}} f = \nabla_{\text{coarse}} \mathsf{A}_m f, \qquad K_{m_2} K_{m_1} = K_{m_1 m_2}, \qquad \mathsf{A}_{m_2} \mathsf{A}_{m_1} = \mathsf{A}_{m_1 m_2},$$

   where in the last two identities m_1 and m_2 are odd, the first map averages by the factor m_1, and the second acts on the resulting lattice.

2. *There exists a linear map R_d from fine one-cochains to periodic scalar functions on the coarse torus such that*

$$D(T_M^d)(I)x = K_m x + \nabla_{\text{coarse}} \mathsf{R}_d x.$$

In particular $\langle J, D(T_M^d)(I)x\rangle = \langle J, K_m x\rangle$ for every divergence-free coarse current J, that is, for every one-chain with zero signed incidence at each vertex.

3. *In units in which the coarse spacing is one, so that the fine spacing is $\epsilon = M^{-d}$, define the average of physical potentials by*
$$Q_d = \frac{1}{m} K_m.$$
Then Q_d maps every constant vector field to itself, $Q_d \nabla_{\text{phys,fine}} = \nabla_{\text{phys,coarse}} Q_d'$ with the scalar average $Q_d' = \mathsf{A}_m$, and if $d = d_1 + d_2$, then Q_d is the composition of the physical averages with factors M^{d_1} and M^{d_2} on the corresponding lattices.
4. *Translating the averaging box from $\{0, \dots, m-1\}^3$ to the centered box I_m multiplies the Fourier multiplier of K_m by the common phase $\exp[-i(m-1)\sum_j k_j/2]$, which is the same for all output directions and cancels in every product of the multiplier with its adjoint.*

These are identities for the derivative of T_M^d at the identity connection $U = I$; they do not identify a nonlinear average with factor m with T_M^d. They involve no coupling constant and no volume factor.

Proof. For a path σ, let $p_\sigma(x)$ be the signed sum of the one-cochain x along σ, so that a reversed link contributes the negative of the value on the positively oriented link. At the identity, the derivative of a path transport is the path sum. In the formula for T_M, the derivative of the final factor $U(C_c)$ cancels the reference-path sums inside the mean of the logarithms. The derivative on $c = (v, v + Me_i)$ is

$$M^{-3} \sum_{r \in \mathsf{I}_M} \Big\{ p_{\sigma_{v,v+r}}(x) + \sum_{s=0}^{M-1} x_i(v + r + se_i) - p_{\sigma_{v+Me_i, v+Me_i+r}}(x) \Big\}.$$

Let $p_v(x) = M^{-3} \sum_r p_{\sigma_{v,v+r}}(x)$. In the forward gradient convention of the statement we obtain

$$DT_M(I)x = K_M x - \nabla p(x). \tag{8.1}$$

The scalar p is periodic and linear in x. Its sign matters only for the definition of R_d and is fixed by (8.1).

For a scalar f, the inner sum over s in $K_m \nabla f$ telescopes to $f(mY + r + me_i) - f(mY + r)$, and averaging over r proves the gradient identity. For two successive odd factors, we expand both sums. The offsets in the box are $m_1 r_2 + r_1$, and the offsets along the segment are $m_1 s_2 + s_1$. The latter run once through $0, \dots, m_1 m_2 - 1$. In one coordinate, the former run once through the centered interval of length $m_1 m_2$: the intervals of length m_1 centered at the points $m_1 r_2$ are adjacent, and their extreme endpoints are $\pm(m_1 m_2 - 1)/2$. Taking products over the three coordinates and multiplying the weights gives both composition identities, including across the periodic seams.

Let $F_j = D(T_M^j)(I)$, and let $\mathsf{R}_0 = 0$. Suppose that $F_j = K_{M^j} + \nabla \mathsf{R}_j$. Since $T_M^j(I) = I$, using the chain rule, (8.1) at the next lattice, and the gradient identity, we obtain

$$F_{j+1} = K_{M^{j+1}} + \nabla\{\mathsf{A}_M \mathsf{R}_j - pF_j\}.$$

This defines the periodic linear map R_{j+1}. If J has zero signed incidence at every vertex, then $\nabla^{\mathsf{T}} J = 0$ and $\langle J, \nabla \mathsf{R}_d x\rangle = \langle \nabla^{\mathsf{T}} J, \mathsf{R}_d x\rangle = 0$.

A constant link-angle cochain is multiplied by m under K_m. If a physical potential on the fine lattice of spacing a/m is A, its link angles are $(a/m)A$, and dividing the coarse angles by a, we obtain $Q_d A = K_m A/m$. For a physical scalar gradient, the fine difference is multiplied by m/a and the coarse difference by $1/a$; the gradient identity becomes $Q_d \nabla_{\text{phys,fine}} = \nabla_{\text{phys,coarse}} \mathsf{A}_m$. The composition identity, with these divisions by the lattice spacings, gives the composition rule for Q_d.

Finally, moving the averaging box from the lower-corner position to the centered position translates each of its three coordinates by $-(m-1)/2$. Its Fourier multiplier is multiplied by $\exp[-i(m-1)\sum_j k_j/2]$, independently of the output direction, and multiplication by the adjoint cancels this phase at each frequency. All identities were obtained by differentiating at the identity or by finite sums, and no coupling or volume enters their coefficients. □

8.2 The effective Maxwell quadratic form

Fix $L > 0$, $\gamma > 0$, an odd integer $M \geq 3$, and integers $d \geq 0$ and $n \geq 4$. Set $m = M^d$, let $a = L/n$ be the coarse spacing, so that the fine spacing is a/m, and write $g_n^2 = g_3^2 a = \gamma/n$ and $g_{mn}^2 = \gamma/(mn) = g_n^2/m$ for the couplings at the coarse and fine lattices. For one color, let x be the centered Gaussian field of link angles on the fine torus of side mn with covariance $g_{mn}^2(\mathrm{d}_{\text{fine}}^{\mathsf{T}}\mathrm{d}_{\text{fine}})^{\dagger}$, where d_{fine} is the elementary curl from links to plaquettes and † denotes the Moore–Penrose inverse. By Lemma 5.2, the kernel of d_{fine} consists of gradients and constant link fields, and this covariance vanishes on gradients and on constant fields. We push x forward by $D(T_M^d)(I)$ and project orthogonally onto the coarse transverse space, the orthogonal complement of the gradients and constant fields on the coarse torus. By Lemma 8.1, the projection removes the gradient term $\nabla \mathsf{R}_d x$, and the projected field is the projection of $K_m x$. By Lemma 5.2, the inverse covariance of a linear image of a Gaussian field is obtained by minimizing the quadratic form of the field under the corresponding linear constraint. This is why the inverse covariance computed below reappears as the Hessian at the identity of the effective classical action in Theorem 10.10.

Definition 8.2. Let $m = M^d$ with M and d as fixed above. For $z \in \mathbb{C}^3$ set

$$d_i(z) = 1 - e^{iz_i}, \qquad d_i^-(z) = 1 - e^{-iz_i}, \qquad \Lambda_m(z) = 4m^2 \sum_j \sin^2\big(z_j/(2m)\big),$$

and regard $d(z) = (d_i(z))_i$ as a column vector and $d^-(z) = (d_i^-(z))_i$ as a row vector; for real z we have $d^-(z) = d(z)^*$. We suppress the argument z in matrix formulas. The integer d in $m = M^d$ and in statements of uniformity remains the averaging depth. For real $\theta \notin 2\pi\mathbb{Z}^3$ set

$$\lambda(\theta) = \sum_i |d_i(\theta)|^2, \qquad P(\theta) = I - \frac{d(\theta)d(\theta)^*}{\lambda(\theta)},$$

and for real k

$$b_{m,i}(k) = m^{-1} \sum_{s=0}^{m-1} e^{isk_i}, \qquad B_m(k) = \prod_j b_{m,j}(k).$$

For real $\theta \notin 2\pi\mathbb{Z}^3$ define

$$D_{m,i}(\theta) = \sum_r \frac{|B_m(k)|^2 |b_{m,i}(k)|^2}{\Lambda_m(\theta + 2\pi r)}, \qquad k = \frac{\theta + 2\pi r}{m},$$

where r runs over a complete set of representatives of $\mathbb{Z}^3/m\mathbb{Z}^3$; the points k are the m^3 fine frequencies with $mk \equiv \theta$ modulo 2π, called the aliases of θ, and each summand depends only on the class of r. Write $D = \operatorname{diag}(D_{m,i})$. Let

$$\beta_m(w) = \Big[\frac{\sin(w/2)}{m \sin(w/(2m))}\Big]^2,$$

extended by continuity at its removable singularities, let $\mathsf{I}_m^{(1)} = \{-(m-1)/2, \dots, (m-1)/2\}$, and define for $z \in \mathbb{C}^3$, wherever the denominators do not vanish,

$$D_{m,i}(z) = \sum_{r \in (\mathsf{I}_m^{(1)})^3} \frac{\prod_j \beta_m(z_j + 2\pi r_j)\, \beta_m(z_i + 2\pi r_i)}{\Lambda_m(z + 2\pi r)}, \tag{8.2}$$

and let $R_i(z)$ be the same sum restricted to $r \neq 0$. Finally set

$$\Lambda_0 = \Lambda_m(z), \qquad q_m(z) = \prod_j \beta_m(z_j), \qquad c_i(z) = q_m(z)\beta_m(z_i),$$

$$A_i(z) = c_i(z) + \Lambda_0 R_i(z), \qquad \omega_m(z) = q_m(z)^{-1} - \sum_i \frac{d_i^-(z) d_i(z) R_i(z)}{c_i(z) A_i(z)},$$

and $c_{\mathrm{av}} = (2/\pi)^{10}$, $C_{\mathrm{av}} = 1 + 3(1 + \pi^2/3)^3$.

For real $\theta \notin 2\pi\mathbb{Z}^3$ the two definitions of $D_{m,i}(\theta)$ agree; this is checked in Step 4 of the proof below. In the following theorem the coarse curl is denoted d, and $\mathrm{d}^{\mathsf{T}}\mathrm{d}$ is the lattice Maxwell form on the coarse torus.

Theorem 8.3. *Let $L > 0$, $\gamma > 0$, let $M \geq 3$ be odd, let $d \geq 0$ and $n \geq 4$ be integers, and let m, a, g_n, and the Gaussian field x be as above. Then the following statements hold.*

1. Covariance. *In unitary discrete Fourier coordinates on the coarse torus, at every nonzero frequency $\theta \in (2\pi/n)\mathbb{Z}^3/(2\pi\mathbb{Z})^3$, the covariance of the projected coarse field is*

$$g_n^2 P(\theta)\,\mathrm{diag}_i\big(D_{m,i}(\theta)\big)P(\theta),$$

and its zero-frequency component vanishes. For all values of the parameters, all i, and all $\theta \neq 0$,

$$\frac{c_{\mathrm{av}}}{\lambda(\theta)} \le D_{m,i}(\theta) \le \frac{C_{\mathrm{av}}}{\lambda(\theta)}.$$

2. Inverse covariance. *Let $\mathsf{H}_m(\theta)$ be the inverse covariance of the projected coarse field after dividing the field by g_n, taken on the nonzero transverse space and extended by zero on gradients and constant fields. For $\theta \neq 0$,*

$$\mathsf{H}_m = D^{-1} - D^{-1}d\Big(\sum_i d_i^- d_i / D_{m,i}\Big)^{-1} d^- D^{-1}.$$

There exist absolute constants $\delta, a_0, C_0 > 0$ such that, for every m, this symbol has a 2π-periodic holomorphic continuation to the strip $\{z \in \mathbb{C}^3 : |\mathrm{Im}\, z_i| < \delta\}$, including a neighborhood of $z = 0$, bounded uniformly in m. On a neighborhood of the origin that does not depend on m, we have $A_i(z) = 1 + O(|z|^2)$ and $\omega_m(z) = 1 + O(|z|^2)$ uniformly in m, and

$$\mathsf{H}_m(z) = \Lambda_0\,\mathrm{diag}(A_i^{-1}) - \frac{\mathrm{diag}(A_i^{-1})\,d\,d^-\,\mathrm{diag}(A_i^{-1})}{\omega_m(z)}.$$

The continuation satisfies $\mathsf{H}_m(z)d(z) = 0$ and $d^-(z)\mathsf{H}_m(z) = 0$, and $\mathsf{H}_m(0) = 0$. We call the corresponding periodic kernel $\mathsf{H}_{m,n}$ on the coarse torus the effective Maxwell quadratic form; it satisfies

$$\sup_x \sum_y e^{a_0\,\mathrm{dist}_n(x,y)} \|\mathsf{H}_{m,n}(x,y)\| + \sup_y \sum_x e^{a_0\,\mathrm{dist}_n(x,y)} \|\mathsf{H}_{m,n}(x,y)\| \le C_0.$$

As quadratic forms on coarse link fields,

$$C_{\mathrm{av}}^{-1}\mathrm{d}^{\mathsf{T}}\mathrm{d} \le \mathsf{H}_{m,n} \le c_{\mathrm{av}}^{-1}\mathrm{d}^{\mathsf{T}}\mathrm{d},$$

both sides vanishing on gradients and constant fields. With $\mathsf{H}_{\mathrm{Max}} = \lambda I - dd^-$ there exist absolute constants $\theta_0, C > 0$ such that $\|\mathsf{H}_m(\theta) - \mathsf{H}_{\mathrm{Max}}(\theta)\| \le C|\theta|^4$ for $|\theta| \le \theta_0$ and all m.

3. Next constrained fluctuation space. *Suppose that $n = Mn_+$ with $n_+ \ge 4$, and let E_0 be the embedding of the next constrained fluctuation space of Lemma 6.5, on the torus of side n. Then*

$$\mathsf{K}_{m,n} = E_0^{\mathsf{T}}\mathsf{H}_{m,n}E_0$$

satisfies

$$c_M I \le \mathsf{K}_{m,n} \le C_M I, \qquad \|\mathsf{K}_{m,n}\|_{a_M,\mathrm{row+col}} + \|\mathsf{K}_{m,n}^{-1}\|_{a_M,\mathrm{row+col}} \le C_M$$

for constants $a_M, c_M, C_M > 0$ that depend only on M.

4. Normalization. *Let A be a smooth periodic connection on $\mathbb{T}^3_L$, and let $U^{(a)}$ be its link transports on the lattice of spacing a. Then*

$$\frac{S_W(U^{(a)})}{g_3^2 a} \longrightarrow \frac{1}{2g_3^2}\int_{\mathbb{T}^3_L}\sum_{i<j}|F_{ij}(A)|^2\,dx \qquad (a \downarrow 0).$$

Let $A = g_3\mathcal{A}$ be the centered Gaussian transverse random distribution with zero constant mode, constructed by requiring that each finite symmetric Fourier truncation have density proportional to the exponential of minus the quadratic part of this action at $A = 0$, restricted to its real transverse coordinates. With the Fourier convention $\widehat{\mathcal{A}}(k) = \int_{\mathbb{T}^3_L} e^{-ik\cdot x}\mathcal{A}(x)\,dx$, the three colors of $\mathcal{A}$ are independent and

$$\mathbb{E}\widehat{\mathcal{A}}^\alpha_i(k)\widehat{\mathcal{A}}^\beta_j(k') = L^3\mathbf{1}_{\{k'=-k\}}\delta_{\alpha\beta}|k|^{-2}\Big(\delta_{ij} - \frac{k_ik_j}{|k|^2}\Big), \qquad k,k' \in (2\pi/L)\mathbb{Z}^3 \setminus \{0\}.$$

The constants $c_{\mathrm{av}}, C_{\mathrm{av}}, \delta, a_0, C_0, \theta_0$, and C are absolute, and a_M, c_M, C_M depend only on M; none of them depends on L, γ, n, or d.

Remark 8.4. The constant modes are set to zero only in the auxiliary Gaussian field. In the classical minimization of Section 10, coarse fields with nonzero constant modes are allowed, and the Schur complement identities of Lemma 5.2 apply to them. Accordingly, an inverse of $\mathsf{H}_{m,n}$ is asserted in part (3) only on the next constrained fluctuation space, where the averaging constraint removes the constant modes; no bound on an inverse of $\mathsf{H}_{m,n}$ on all transverse coarse fields, uniform in n, is asserted.

Remark 8.5. At each fixed nonzero θ, the alias sums have the limit

$$\lim_{d\to\infty} D_{M^d,i}(\theta) = D_{\infty,i}(\theta) = \sum_{r\in\mathbb{Z}^3} \frac{\prod_j \operatorname{sinc}^2\big((\theta_j + 2\pi r_j)/2\big)\ \operatorname{sinc}^2\big((\theta_i + 2\pi r_i)/2\big)}{|\theta + 2\pi r|^2}.$$

Indeed, choose $\theta_j \in [-\pi, \pi)$ and aliases with $\kappa_j = \theta_j + 2\pi r_j \in [-\pi m, \pi m)$. For a fixed integer vector r, as $m = M^d \to \infty$, the four squared averaging factors $|b_{m,j}(k)|^2 = \sin^2(\kappa_j/2)/(m^2\sin^2(\kappa_j/(2m)))$ converge to the four squared sinc factors, and $\Lambda_m(\kappa) \to |\kappa|^2$; each fixed alias is present for large m. By Step 2 of the proof below, after extension by zero outside the finite set of aliases, every nonzero-alias summand is bounded by $\prod_j w(r_j)/4$, where $w(0) = 1$ and $w(r) = r^{-2}$ for $r \neq 0$, and this bound is summable over $\mathbb{Z}^3$. Fix a finite cube of aliases outside which the sum of these bounds is less than a given $\eta > 0$, pass to the limit on that cube, and let $\eta \downarrow 0$. The zero alias is nonsingular at the fixed $\theta \neq 0$. This proves the limit.

Proof of Theorem 8.3. We begin with an outline. Part (1) is a Fourier computation followed by elementary bounds on the alias sums. For part (2), the formula for H_m is the inverse of

the projected covariance on its range, and the comparison with the Maxwell form follows by inverting the bounds of part (1). The main point is the holomorphic continuation of H_m to a strip whose width does not depend on m; exponential decay of the kernel then follows by shifting contours. Near the origin, $D_{m,i} = A_i/\Lambda_0$, and the second term in the formula for H_m contains the inverse of $\sum_i d_i^- d_i / D_{m,i} = \Lambda_0 \sum_i d_i^- d_i / A_i$. The zero set of the holomorphic function $\Lambda_0 = \Lambda_m(z)$ near the origin is a complex hypersurface through 0, and it contains nonreal points arbitrarily close to the origin. Although $\mathsf{H}_m(\theta) \to 0$ as real $\theta \to 0$, this does not show that H_m is holomorphic near 0. We rewrite H_m exactly so that its only denominators are A_i and ω_m, which equal $1 + O(|z|^2)$ uniformly in m. Part (3) combines the Maxwell comparison with Lemma 6.5 and the weighted inverse estimate of Lemma 5.3, and part (4) is a direct calculation.

Step 1: Fourier conventions and the covariance. On a torus of side N we use the unitary transform

$$\widehat{x}_i(k) = N^{-3/2} \sum_{z \in \mathbb{Z}^3_N} e^{-ik\cdot z} x_i(z), \qquad x_i(z) = N^{-3/2} \sum_k e^{ik\cdot z} \widehat{x}_i(k).$$

The gradient has symbol $-d(k)$, and the elementary curl has symbol

$$(\mathrm{d}x)^{\wedge}_{ij}(k) = -d_i(k)\widehat{x}_j(k) + d_j(k)\widehat{x}_i(k), \qquad i < j.$$

Expanding the three squares, we find for $k \neq 0$ that

$$\mathrm{d}(k)^* \mathrm{d}(k) = \lambda(k) I - d(k) d(k)^* = \lambda(k) P(k). \tag{8.3}$$

This identifies the null modes used in the Gaussian convention: the multiples of $d(k)$ at $k \neq 0$, which are the gradients, and all of $k = 0$, which are the constant fields, in agreement with Lemma 5.2.

For the lower-corner box, before restriction to the coarse lattice, the operator K_m is diagonal in Fourier space with multiplier $T_i(k) = m B_m(k) b_{m,i}(k)$. The centered box adds the common phase computed in Lemma 8.1. We substitute the inverse fine transform into $K_m x(Y)$ and take the coarse transform. Only the frequencies with $mk \equiv \theta$ modulo 2π survive the sum over coarse vertices, and the restriction to the coarse lattice contributes a factor $m^{-3/2}$ for each of these m^3 aliases. Summing the finite geometric series, we obtain

$$m b_{m,i}(k) d_i(k) = d_i(\theta).$$

Then $T(k) d(k) = B_m(k) d(\theta)$, with the same common phase in the centered convention, and $P(\theta) T(k) d(k) = 0$. The longitudinal part of the fine covariance vanishes after the projections on both sides, alias by alias.

The fine coupling is $g^2_{mn} = g^2_n / m$. The fine frequency $k = (\theta + 2\pi r)/m$ has $\lambda(k) = \Lambda_m(\theta + 2\pi r)/m^2$, and the inverse of the fine Laplacian at k is $m^2/\Lambda_m(\theta + 2\pi r)$.

Combining the coupling, the factor from the restriction to the coarse lattice, the square of the segment multiplier m, and the inverse Laplacian, we obtain

$$\frac{g_n^2}{m} \cdot m^{-3} \cdot m^2 \cdot m^2 = g_n^2.$$

Since $P(\theta)$ removes the gradient term $\nabla \mathsf{R}_d x$ of Lemma 8.1, this is the covariance of the projection of $D(T_M^d)(I)x$, which proves the covariance formula. At $\theta = 0$, every alias with $r \not\equiv 0$ modulo m has a vanishing factor in B_m, since $\sum_{s=0}^{m-1} e^{2\pi i r_j s/m} = 0$ when $r_j \not\equiv 0$ modulo m. The remaining alias is the fine constant mode, which has zero covariance. This shows that the zero-frequency component vanishes.

Step 2: Real bounds for the alias sums. We take $\theta_j \in [-\pi, \pi)$ and aliases with $\kappa_j = \theta_j + 2\pi r_j \in [-\pi m, \pi m)$. For the alias $r = 0$ we have

$$|b_{m,j}(\theta/m)| = \frac{|\sin(\theta_j/2)|}{m|\sin(\theta_j/(2m))|} \geq \frac{2}{\pi}, \qquad \lambda(\theta) \leq \Lambda_m(\theta) \leq |\theta|^2 \leq \frac{\pi^2}{4}\lambda(\theta),$$

where the value at a zero coordinate is understood by continuity. The first inequality for Λ_m follows from the concavity of the sine, $\sin(t/m) \geq \sin(t)/m$ for $0 \leq t \leq \pi/2$, and the last from $|\sin t| \geq 2|t|/\pi$ for $|t| \leq \pi/2$. These inequalities, together with the factor $(2/\pi)^8$ from the four squared averaging factors, prove the lower bound. The contribution of $r = 0$ to the upper bound is at most $1/\lambda(\theta)$.

For every other alias, the same chord inequality implies that

$$|b_{m,j}(k)| \leq \min\{1, \pi/|\kappa_j|\}, \qquad \Lambda_m(\kappa) \geq (4/\pi^2)|\kappa|^2.$$

If $r_j \neq 0$, then $|\kappa_j| \geq \pi|r_j|$. With $w(0) = 1$ and $w(r) = r^{-2}$ for $r \neq 0$, the numerator without its extra factor $|b_{m,i}(k)|^2$ is at most $\prod_j w(r_j)$, the extra factor is at most one, and the denominator is at least four. Since $\sum_{r \geq 1} r^{-2} = \pi^2/6$, we have $\sum_{r \in \mathbb{Z}} w(r) = 1 + \pi^2/3$, and the nonzero aliases contribute at most $(1 + \pi^2/3)^3/4$. Since $\lambda \leq 12$, this is at most $3(1 + \pi^2/3)^3/\lambda$, and the upper bound with the constant C_{av} follows.

Step 3: The transverse inverse and the Maxwell comparison. Fix real $\theta \neq 0$, write $D^{-1} = \operatorname{diag}(D_{m,i}^{-1})$, and set $s = d^- D^{-1} d = \sum_i d_i^- d_i / D_{m,i}$, which is positive. The matrix

$$H = D^{-1} - D^{-1} d\, s^{-1} d^- D^{-1}$$

is Hermitian, satisfies $Hd = 0$ and $d^- H = 0$, and $HD = I - D^{-1} d s^{-1} d^*$, so that $HDP = P$ since $d^* P = 0$. Since $HP = PH = H$, the matrix H is the inverse of PDP on $\operatorname{ran} P$. This proves the inverse formula. Note that we invert the projected covariance PDP, not the diagonal matrix D. Inverting the inequalities

$$\frac{c_{\mathrm{av}}}{\lambda} I_{\operatorname{ran} P} \leq (PDP)|_{\operatorname{ran} P} \leq \frac{C_{\mathrm{av}}}{\lambda} I_{\operatorname{ran} P}$$

from part (1) and using (8.3), we obtain the comparison with the Maxwell form at each nonzero frequency. Both sides vanish on gradients and at zero frequency, which proves the comparison of quadratic forms.

Step 4: Uniform bounds for the continued alias sums. The identity

$$\beta_m(w) = \Big(m^{-1}\sum_{s=0}^{m-1} e^{isw/m}\Big)\Big(m^{-1}\sum_{t=0}^{m-1} e^{-itw/m}\Big)$$

shows that the singularities of β_m are removable and that β_m is entire and even; no complex conjugation enters. For real θ each squared modulus in the real alias sum is the product of the multipliers at opposite frequencies, $|b_{m,j}(k)|^2 = \beta_m(\theta_j + 2\pi r_j)$, and (8.2), with the residue set $(\mathsf{I}_m^{(1)})^3$, agrees with the real formula. Increasing a coordinate of z by 2π permutes the aliases; the alias that leaves the residue set changes its coordinate by $2\pi m$, and both β_m and Λ_m are invariant under this change. This shows that (8.2) is 2π-periodic in each coordinate.

Let $z = x + iy$ with $|x_j| \le \pi$ and $|y_j| \le \delta_0$, for a sufficiently small absolute $\delta_0 > 0$. For our residue set, $|x_j + 2\pi r_j|/(2m) \le \pi/2$. Using $|\sin(u+iv)|^2 = \sin^2 u + \sinh^2 v$, we find for $r_j \neq 0$ that

$$|m\sin\big((z_j + 2\pi r_j)/(2m)\big)| \ge m|\sin\big((x_j + 2\pi r_j)/(2m)\big)| \ge |x_j + 2\pi r_j|/\pi \ge |r_j|.$$

The numerator $\sin((z_j + 2\pi r_j)/2)$ is bounded on this tube, and for $r_j = 0$ the finite-sum representation shows that β_m is uniformly bounded there. We obtain

$$|\beta_m(z_j + 2\pi r_j)| \le C w(r_j). \tag{8.4}$$

The extra factor $\beta_m(z_i + 2\pi r_i)$ is bounded by the same finite-sum formula. Since $\operatorname{Re}\sin^2(u+iv) = \sin^2 u - \cos(2u)\sinh^2 v$ and $\sinh|v| \le 2|v|$ for $|v| \le 1$, we have

$$\operatorname{Re}\Lambda_m(z + 2\pi r) \ge (4/\pi^2)|x + 2\pi r|^2 - 4|y|^2.$$

For $r \neq 0$ this is at least $2|r|^2$ after choosing δ_0 small. In particular no denominator of a nonzero alias vanishes on the tube, and its summand is bounded by $C\prod_j w(r_j)/|r|^2$. Summing and using (8.4), we obtain a uniform bound for every R_i on the whole tube over the real cube, with holomorphy in its interior. Bounds for derivatives on smaller neighborhoods follow from Cauchy's formula on fixed circles in each coordinate. For $m = 1$ these remainders vanish. For the zero alias the lower bound reads

$$\operatorname{Re}\Lambda_m(z) \ge (4/\pi^2)|x|^2 - 4|y|^2.$$

On any real region $|x| \ge \rho > 0$, the functions $D_{m,i}$ and their first derivatives are bounded and holomorphic, uniformly in m, in a tube whose width depends only on ρ.

Step 5: Cancellation on the complex characteristic set. On a small polydisc about zero that does not depend on m, the finite-sum formula and evenness imply that $\beta_m(z_j) = 1 + O(|z_j|^2)$ uniformly in m. We obtain

$$q_m = 1 + O(|z|^2), \qquad c_i = 1 + O(|z|^2), \qquad \Lambda_0 = O(|z|^2), \qquad A_i = 1 + O(|z|^2);$$

for the last assertion we used the bounds for R_i from Step 4. Since $d_i^-(z)d_i(z) = 4\sin^2(z_i/2)$ and $\beta_m(z_i) = \sin^2(z_i/2)/(m^2\sin^2(z_i/(2m)))$, we have the exact identity

$$\sum_i \frac{d_i^- d_i}{c_i} = q_m^{-1} \sum_i \frac{4\sin^2(z_i/2)}{\beta_m(z_i)} = \frac{\Lambda_0}{q_m},$$

valid also at the removable values. Using $A_i^{-1} = c_i^{-1} - \Lambda_0 R_i/(c_i A_i)$, we obtain

$$\sum_i \frac{d_i^- d_i}{A_i} = \Lambda_0 \omega_m, \qquad \omega_m = q_m^{-1} - \sum_i \frac{d_i^- d_i R_i}{c_i A_i} = 1 + O(|z|^2). \tag{8.5}$$

We fix an absolute radius $\rho > 0$ such that q_m, c_i, A_i, and ω_m have modulus at least $1/2$ on the polydisc $|z_j| < 2\rho$; this is possible uniformly in m by the preceding quadratic estimates. The zero alias of (8.2) is c_i/Λ_0. Where the alias sum is regular, we have

$$D_{m,i} = \frac{A_i}{\Lambda_0}, \qquad \sum_i \frac{d_i^- d_i}{D_{m,i}} = \Lambda_0^2 \omega_m.$$

Substituting these identities into the inverse formula, we obtain exactly the expression for $\mathsf{H}_m(z)$ in part (2). Its denominators A_i and ω_m do not vanish on the polydisc, including the points where $\Lambda_0 = 0$. This proves holomorphy there and the bound $\|\mathsf{H}_m(z)\| \le C|z|^2$, since Λ_0 and dd^- are $O(|z|^2)$. Multiplying by d on the right and by d^- on the left and using (8.5), we obtain $\mathsf{H}_m d = 0$ and $d^- \mathsf{H}_m = 0$; the value at zero is zero.

Step 6: The strip, the periodic kernel, and the comparison at low frequency. On the real complement of the smaller neighborhood $|z_j| < \rho$, part (1) implies that $D_{m,i} \ge c_{\mathrm{av}}/12$ and

$$\sum_i d_i^- d_i / D_{m,i} \ge \lambda^2 / C_{\mathrm{av}} \ge c_\rho > 0.$$

By the bounds of Step 4, the first derivatives there have a common bound C_ρ. We shrink the tube so that $D_{m,i}$ changes by less than $c_{\mathrm{av}}/24$ from its real value; its reciprocal then has bounded value and bounded first derivatives. After one more shrinking, we may assume that $\sum_i d_i^- d_i / D_{m,i}$ changes by less than $c_\rho/2$. The inverse formula of part (2) is then holomorphic and uniformly bounded on this tube. Across the faces of the real cube we use the periodicity of (8.2); the same real lower bounds hold there.

We fix an absolute width $\delta > 0$ below these widths and so small that points whose real parts lie in $|x_j| < \rho$ remain in the polydisc $|z_j| < 2\rho$. The two expressions for H_m

agree on their common regular set; multiplying by their nonvanishing denominators shows that they agree throughout each overlap, including removable points. They define one bounded periodic holomorphic function on the strip of width δ.

Let its Fourier coefficients be

$$h_m(\zeta) = (2\pi)^{-3} \int_{[-\pi,\pi]^3} e^{i\theta\cdot\zeta} \mathsf{H}_m(\theta)\, d\theta, \qquad \zeta \in \mathbb{Z}^3.$$

For $0 < \delta_1 < \delta$, shift the three integration contours by $\pm i\delta_1$, with the signs chosen according to the signs of the components of ζ so as to decrease the exponential factor. The vertical sides cancel by periodicity, and from the uniform bound on the strip we obtain

$$\|h_m(\zeta)\| \le C e^{-\delta_1 |\zeta|_1}.$$

The Fourier series of H_m converges absolutely. Sampling it at the frequencies of the torus of side n and using the orthogonality of the finite characters, we obtain

$$\mathsf{H}_{m,n}(x,y) = \sum_{\zeta \equiv x-y \pmod{n}} h_m(\zeta).$$

For each representative, $\mathrm{dist}_n(x,y) \le |\zeta|_1$. For any $0 < a_0 < \delta_1$, each of the two weighted sums in part (2) is at most

$$C \sum_{\zeta\in\mathbb{Z}^3} e^{-(\delta_1 - a_0)|\zeta|_1} = C\Big(\frac{1 + e^{-(\delta_1-a_0)}}{1 - e^{-(\delta_1-a_0)}}\Big)^3,$$

which gives the kernel bound with a constant C_0 independent of n and d.

The comparison at low frequency follows from the same identities. On the real axis near zero, $\mathrm{diag}(A_i^{-1}) = I + O(|\theta|^2)$ and $\omega_m^{-1} = 1 + O(|\theta|^2)$. Since the Taylor remainder of the sine is uniform in $m \ge 1$, we have $\Lambda_m(\theta) - \lambda(\theta) = O(|\theta|^4)$. Since $dd^- = O(|\theta|^2)$, the two terms in the formula of part (2) differ from λI and dd^-, respectively, by $O(|\theta|^4)$. This proves $\|\mathsf{H}_m(\theta) - \mathsf{H}_{\mathrm{Max}}(\theta)\| \le C|\theta|^4$.

Step 7: Restriction to the next constrained fluctuation space. Let $a = E_0 z$. The embedding of Lemma 6.5 vanishes on the links of the centered block trees, which are the links of the coordinate paths $\sigma_{v,x}$, and every path sum $p_{\sigma_{v,x}}(a)$ in (8.1) vanishes. The linearized constraint $DT_M(I)a = 0$ says $K_M a = 0$. A fixed input link occurs exactly M times in the straight segments of the coarse sum, each time with weight M^{-3}. Summing over the coarse links, we find

$$\sum_Y (K_M a)_i(Y) = M^{-2} \sum_x a_i(x).$$

This shows that a has zero spatial mean in each component and has no constant mode. By the comparison with the Maxwell form in part (2) and the lower bound of Lemma 6.5, we have

$$\langle z, E_0^{\mathsf{T}} \mathsf{H}_{m,n} E_0 z\rangle \ge C_{\mathrm{av}}^{-1} \|\mathrm{d} E_0 z\|_2^2 \ge c_M \|z\|_2^2,$$

after renaming the constant. The upper comparison and the upper bound of the same lemma imply that $\mathsf{K}_{m,n} \le C_M I$. The rows and columns of E_0 have finite range and bounded sums depending only on M, by the same lemma. They transfer the weighted bounds for $\mathsf{H}_{m,n}$ to $\mathsf{K}_{m,n}$: each endpoint of a contributing stencil moves by at most a distance depending on M. The index set of the free coordinates has counting growth $\#\{z : \operatorname{dist}(z, z_0) \le R\} \le D_M(1+R)^3$. We now apply the inverse estimate of Lemma 5.3. Its hypotheses are a self-adjoint matrix with a positive lower bound, bounded off-diagonal row and column sums with an exponential weight, and polynomial counting growth of the index set. We have verified them with lower bound c_M, rate a_0, a bound on the weighted sums depending only on M, and growth exponent three. The lemma gives exponentially weighted bounds for $\mathsf{K}_{m,n}^{-1}$ at a smaller rate. Taking a_M to be the smaller of the two rates and enlarging C_M proves part (3).

Step 8: Normalization. Let $U_i^{(a)}(x)$ be the transport of the smooth connection A along the link $(x, x + ae_i)$. Iterating the integral equation of the transport twice, we find uniformly in x that

$$U_i^{(a)}(x) = I + aA_i(x) + \tfrac{1}{2}a^2\{\partial_i A_i(x) + A_i(x)^2\} + O_A(a^3).$$

The remaining ordered integrals are bounded by $C_A a^3$, uniformly in x. We insert the four expansions, with the shifted arguments on the two shifted sides, into the plaquette product. The terms of degree one cancel, and the terms of degree two are $\partial_i A_j - \partial_j A_i + [A_i, A_j]$; hence $\operatorname{Log} U_{p_{ij}(x)} = a^2 F_{ij}(x) + O_A(a^3)$. For real $X = x^a \tau_a$, the Pauli identity implies that $X^2 = -|x|^2 I/4$. Then $\operatorname{tr}(e^X)/2 = \cos(|x|/2)$ and $4q(e^X) = |X|^2/2 + O(|X|^4)$. We obtain

$$\frac{S_W(U^{(a)})}{g_3^2 a} = \frac{1}{2g_3^2} a^3 \sum_x \sum_{i<j} |F_{ij}(x)|^2 + O_{A,L,g_3}(a),$$

and the Riemann sum converges because the integrand is smooth on the fixed torus. This proves the first assertion of part (4).

The quadratic part of the continuum action at $A = 0$ is $(2g_3^2)^{-1} \int \sum_{i<j} |\partial_i A_j - \partial_j A_i|^2 \, dx$. With the Fourier convention of the statement, we have $A(x) = L^{-3} \sum_k e^{ik\cdot x} \widehat{A}(k)$. Integrating the products of exponentials, we find that on any finite symmetric set of nonzero frequencies the quadratic part is

$$\frac{1}{2g_3^2 L^3} \sum_{k \ne 0} \widehat{A}(k)^* \big(|k|^2 I - kk^{\mathsf{T}}\big) \widehat{A}(k).$$

We restrict to $\widehat{A}(k) \perp k$ and impose the reality condition $\widehat{A}(-k) = \overline{\widehat{A}(k)}$. For one representative of each pair $\{k, -k\}$, the corresponding term is $|k|^2 |\widehat{A}(k)|^2 / (g_3^2 L^3)$. For a transverse real or imaginary coordinate u, the one-variable density is therefore proportional to $\exp[-|k|^2 u^2 / (g_3^2 L^3)]$, and differentiating the Gaussian integral gives the variance

$g_3^2 L^3/(2|k|^2)$; different coordinates, pairs, and colors are independent. We realize the field by independent centered real Gaussian variables with these variances, one for each transverse real and imaginary coordinate of one representative of each pair and each color. The remaining coordinates are defined by the reality condition. Then $A = g_3\mathcal{A}$ has exactly the covariance displayed in part (4); in particular $\mathbb{E}\widehat{\mathcal{A}}(k)\widehat{\mathcal{A}}(k)^{\mathsf{T}} = 0$ for $k' = k \neq -k$, since the real and imaginary parts have equal variances.

These coordinates define a random distribution almost surely. Indeed,

$$\mathbb{E}\sum_{k\neq 0}(1+|k|^2)^{-2}|\widehat{\mathcal{A}}(k)|^2 < \infty :$$

the summands are nonnegative with expectations at most $C_L(1+|k|)^{-6}$, and the number of frequencies in the shell $R \le |k| < R+1$ is at most $C_L(1+R)^2$, as one sees by covering their disjoint cubes of fixed volume by a shell of fixed extra thickness. Markov's inequality applied to the increasing partial sums shows that their supremum is finite almost surely, and on that event the Cauchy–Schwarz inequality pairs the Fourier sequence continuously with every smooth test function. Alternatively, each such pairing converges in mean square, since repeated integration by parts bounds the Fourier coefficients of a smooth test function f by $C_{f,s}(1+|k|)^{-s}$ for every s. Limits of the characteristic functions of the finite truncations show that these pairings are jointly Gaussian with the stated covariance, and the covariance identity for arbitrary pairs of frequencies already holds at every finite truncation. The three colors are independent because their quadratic forms add.

The constants in Steps 1–6 are absolute; only the stencils, the lower bound, and the counting growth in Step 7 introduce dependence on M. The constants asserted in parts (1)–(3) are independent of L, γ, n, and d. The auxiliary remainders and summability bounds used in Step 8 may depend on the fixed connection A, the torus size L, and g_3. This completes the proof. □

8.3 Gradient annihilation and spatial inversion

The effective Maxwell quadratic form of Theorem 8.3 vanishes identically on the constant modes because of the convention for the auxiliary Gaussian field. The Hessians that occur later are perturbations of such forms by small translation-invariant kernels that annihilate gradients (by gauge invariance) and commute with spatial inversion, but need not vanish on the constant modes. The following lemma shows that such a kernel is bounded by the lattice Maxwell form, with an exponentially small error from its zero-frequency block. It is used in Lemma 11.11.

Spatial inversion and midpoint coordinates. Spatial inversion maps the vertex x of the periodic lattice of side n to $-x$. It maps the positive link $(x, x+e_i)$ to the reversed link of

$(-x-e_i,-x)$, so it acts on link fields by

$$(\mathsf{I}v)_i(x) = -v_i(-x-e_i).$$

Let H be a translation-invariant kernel on link fields, written $H_{ij}(x,y) = h_{ij}(y-x)$, where $h_{ij}(t)$ is the block from a j-link with initial vertex y to an i-link with initial vertex x. The midpoint displacement of this entry is

$$t = y - x + (e_j - e_i)/2 \mod n\mathbb{Z}^3.$$

We lift it to $\mathbb{R}^3$ by choosing in each coordinate a representative of smallest absolute value. If a coordinate has two such representatives, which happens when it equals $n/2$ modulo n, we divide the coefficient equally between them. With several such coordinates, we take the product of these equal divisions. For example, in one dimension with $n = 4$ a coefficient at displacement 2 is split into two halves placed at 2 and -2. Denote the lifted coefficients by $\widetilde{h}_{ij}(t)$, $t \in \mathbb{R}^3$, and define the midpoint interpolation

$$\widetilde{H}_{ij}(k) = \sum_t \widetilde{h}_{ij}(t) e^{ik\cdot t}, \qquad k \in \mathbb{R}^3.$$

At a frequency $k \in (2\pi/n)\mathbb{Z}^3$ of the torus, lifts differing by elements of $n\mathbb{Z}^3$ have the same phase. Then $\widetilde{H}(k)$ is the Fourier block of H in midpoint coordinates, which differs from the Fourier block in initial-vertex coordinates by diagonal unitary phases on the links.

Lemma 8.6. *Fix a rate $a > 0$ and a finite color dimension. For every periodic side $n \geq 4$, let H be a real, self-adjoint, translation-invariant kernel on link fields of the torus of side n such that*

$$H\nabla = 0 = \nabla^{\mathsf{T}} H, \qquad H\mathsf{I} = \mathsf{I}H, \qquad \|H\|_{a,\mathrm{row+col}} \leq \varepsilon.$$

Then there exist $C(a), c(a) > 0$, depending only on a and the color dimension, such that with $w_n = (1+n)^3 e^{-c(a)n}$ the following hold.

1. *The zero-frequency block satisfies $\|\widehat{H}(0)\| \leq C(a)\varepsilon w_n$.*
2. *For every real or complex link field v, with the Hermitian pairing for complex v,*

$$|\langle v, Hv\rangle| \leq C(a)\varepsilon \|\mathrm{d}v\|_2^2 + C(a)\varepsilon w_n \|v\|_2^2.$$

3. *The midpoint interpolation is even, $\widetilde{H}(-k) = \widetilde{H}(k)$ for $k \in \mathbb{R}^3$, and at every frequency k of the torus, represented with $k_j \in [-\pi, \pi]$,*

$$\|\widetilde{H}(k)\| \leq C(a)\varepsilon\Big(\sum_i 4\sin^2(k_i/2) + w_n\Big).$$

The proof uses inversion in all three coordinates; a reflection in a single coordinate would not make the first derivative of $\widetilde{H}$ vanish at zero. The zero-frequency block is estimated, not assumed to vanish.

Proof. Let r be the color dimension; constants below may depend on r. The weighted hypothesis bounds each row sum and each column sum of H by ε.

We first estimate the zero-frequency block, without using inversion. Let $n \geq 32$, represent vertex coordinates in $[-n/2, n/2)^3$, and fix a Lipschitz function χ on $\mathbb{R}$ with $0 \leq \chi \leq 1$, equal to one on $[-1/8, 1/8]$, and vanishing outside $(-1/4, 1/4)$. The periodic function

$$f_j(x) = x_j \prod_{\ell=1}^{3} \chi(x_\ell/n)$$

has discrete first differences bounded by a fixed constant. The difference of x_j contributes at most one, and a difference of a cutoff contributes at most $\mathrm{Lip}(\chi)/n$ times a coordinate of size at most $n/2$. Its gradient coincides with the constant j-directed link field $\mathbf{1}_j$ on links whose initial vertices have ℓ^∞ norm at most $n/8 - 1$. For a color vector b with $|b| = 1$, applying the identity $H\nabla = 0$ to $f_j b$ in the row of an i-link with initial vertex zero, we obtain

$$(H(\mathbf{1}_j b))_i(0) = \big(H(\mathbf{1}_j b - \nabla(f_j b))\big)_i(0).$$

Every nonzero input on the right is at periodic distance at least $n/16$ from this row and is bounded by a fixed constant. By the weighted row bound, the right side is at most $C\varepsilon e^{-an/16}$. The left side is the ij block of $\widehat{H}(0)b$. Summing over the finitely many blocks, we obtain $\|\widehat{H}(0)\| \leq C\varepsilon e^{-an/16}$. For $4 \leq n < 32$, the unweighted row bound implies that $\|\widehat{H}(0)\| \leq \varepsilon$, and enlarging the constant covers these finitely many sides. With $c(a) = a/32$ both ranges give part (1).

We next use inversion. Both oriented links of an entry acquire a minus sign, and their initial vertices become $-x - e_i$ and $-y - e_j$. In terms of the entries, $H\mathsf{I} = \mathsf{I}H$ reads

$$h_{ij}(y - x) = h_{ij}(-y + x + e_i - e_j),$$

and the midpoint displacement of the right side is $-t$. Since ties were split symmetrically, we have $\widetilde{h}_{ij}(t) = \widetilde{h}_{ij}(-t)$. Then $\widetilde{H}$ is even and $D\widetilde{H}(0) = 0$. The lifts of each coefficient sum to that coefficient, and the weighted absolute sums of the lifted coefficients are at most $C_a\varepsilon$, since the shortest midpoint distance exceeds the vertex distance by at most one. All representatives have $|t_\ell| \leq n/2$. Since $\sup_{u \geq 0} u^2 e^{-au} < \infty$, the second derivatives satisfy

$$\sup_{k \in \mathbb{R}^3} \|D^2\widetilde{H}(k)\| \leq C \sum_{i,j,t} |t|_1^2 \|\widetilde{h}_{ij}(t)\| \leq C(a)\varepsilon.$$

At $k = 0$ all phases are one, and $\widetilde{H}(0) = \widehat{H}(0)$. By Taylor's formula along the segment from 0 to k and part (1), we have $\|\widetilde{H}(k)\| \leq C(a)\varepsilon(w_n + |k|^2)$. For $k_j \in [-\pi, \pi]$ we have $|k|^2 \leq (\pi^2/4) \sum_j 4\sin^2(k_j/2)$, which proves part (3).

To pass to quadratic forms, set $q_i(k) = 2\sin(k_i/2)$ and $\lambda_q = \sum_i q_i^2$. After we move the phases from the initial vertices to the midpoints of the links and plaquettes, the gradient becomes multiplication by iq, and the curl from links to plaquettes becomes

$$(D(k)u)_{ij} = i(q_i u_j - q_j u_i), \qquad D(k)^* D(k) = \lambda_q I - qq^{\mathsf{T}}.$$

At a frequency $k \neq 0$, the identities $H\nabla = 0 = \nabla^{\mathsf{T}} H$ say that $\widetilde{H}(k)$ vanishes on q and maps into $q^{\perp}$. If u_T is the component of u orthogonal to q, then Hermitian symmetry and part (3) imply that

$$|\langle u, \widetilde{H}(k)u\rangle| \leq C(a)\varepsilon\lambda_q |u_T|^2 + C(a)\varepsilon w_n |u|^2 = C(a)\varepsilon |D(k)u|^2 + C(a)\varepsilon w_n |u|^2.$$

At $k = 0$ we use part (1). The unitarity of the finite Fourier transform and the triangle inequality over the frequencies imply part (2) for real and complex link fields. □

Remark 8.7. On the transverse fields with nonzero frequency, the quadratic form of H factors through the curl with a bounded factor: there is an operator T on plaquette fields, diagonal in Fourier space, with $\|T\|_{\mathrm{op}} \leq C(a)\varepsilon$ uniformly in n, such that $\langle v, Hv\rangle = \langle \mathrm{d}v, T\mathrm{d}v\rangle$ for every v orthogonal to the gradients and the constant fields. Indeed, at each frequency $k \neq 0$ set

$$T(k) = \lambda_q^{-2} D(k)\widetilde{H}(k)D(k)^*,$$

and set $T(0) = 0$. Since $D^*D = \lambda_q P_q$, where P_q is the orthogonal projection onto $q^{\perp}$, and $\widetilde{H}(k) = P_q \widetilde{H}(k) P_q$, we have $D(k)^* T(k) D(k) = \widetilde{H}(k)$, and $T(k)$ vanishes on the plaquette directions orthogonal to the range of $D(k)$. Since $\|D(k)\| = \lambda_q^{1/2}$, part (3) implies that $\|T(k)\| \leq C(a)\varepsilon(1 + w_n/\lambda_q)$. Every nonzero frequency has $\lambda_q \geq 4\sin^2(\pi/n) \geq 16/n^2$ by the chord inequality, and $\sup_{n\geq 4} n^2(1+n)^3 e^{-c(a)n} < \infty$, so $\|T\|_{\mathrm{op}} \leq C(a)\varepsilon$ uniformly in n. The zero-frequency block of H is not part of this factorization; it is estimated separately in part (1). This construction controls only the operator norm of T, and no exponential decay of the kernel of T follows from it. A factorization with exponentially local factors and an exponentially small remainder is proved separately in Lemma 11.11.

8.4 The gauge-corrected block average

In this section and the next we study the iterated block averaging map near a background configuration with small curvature. The background is fixed, and all derivatives are taken with respect to a fluctuation. Gauge-corrected averages, in which the output of each averaging step is multiplied by gauge transformations at the endpoints of the coarse links, occur in Bałaban's averaging construction [12, (61)–(63), p. 28], and their composition is treated in [12, (78)–(92), pp. 30–31; (97)–(100), p. 32]. We use the centered blocks of Definition 4.3, and we derive below the derivative of the corrected step, its endpoint terms, and the order of the composed gauge transformations.

We explain why the estimates are uniform in the number d of averaging steps. We use units in which the coarse spacing after d steps is one, so that the spacing at level j is $\epsilon_j = M^{j-d}$. If $\delta(U_0) \le \varepsilon\epsilon_0^2$, then by Theorem 7.3 the plaquette angles δ_j of $T_M^j U_0$ satisfy $\delta_j \le 2\varepsilon\epsilon_j^2$; they scale like the square of the spacing, as for a smooth field. One gauge-corrected step, written for the potential $A_j = \epsilon_j^{-1} \operatorname{Log} V_j$, has the form

$$A_{j+1} = Q_{U_j} A_j + O_M\big(\epsilon_{j+1}|A_j|^2\big),$$

where the row sums of Q_{U_j} are at most $1 + C_M\delta_j$. The relative errors at level j are of order $\delta_j + \epsilon_{j+1}|A_j|$, and

$$\sum_{j<d} \delta_j \le \frac{2\varepsilon}{M^2-1}, \qquad \sum_{j<d} \epsilon_{j+1} \le \frac{M}{M-1}$$

are bounded independently of d. This is why the constants below do not depend on d, and why the scalar weights of the transported segment average must sum to exactly one in each row, rather than to $1 + c$ for a fixed $c > 0$. All smallness assumptions concern plaquettes and holonomies of contractible closed paths; the background may have arbitrary holonomies around the cycles of the torus.

Definition 8.8. Let U be an $\mathrm{SU}(2)$-valued configuration on a periodic lattice of side Mn' with $n' \ge 4$, and let $V = (V_e)_e$ be a link field with values in $\mathrm{SL}(2,\mathbb{C})$. For a coarse vertex v set

$$\mathsf{h}_{v;U}(V) = \exp\Big[-M^{-3} \sum_{x \in \mathsf{B}_M(v)} \operatorname{Log}\big((VU)(\sigma_{v,x})U(\sigma_{v,x})^{-1}\big)\Big],$$

where $VU = (V_e U_e)_e$, and write $\mathsf{h}_U(V)$ for the gauge transformation $(\mathsf{h}_{v;U}(V))_v$ on the coarse vertices. For a coarse link $c = (v, s)$, $s = v + Me_i$, define the gauge-corrected block average

$$(\mathscr{M}_U(V))_c = \mathsf{h}_{v;U}(V)\,(T_M(VU))_c\,\mathsf{h}_{s;U}(V)^{-1}\,(T_M U)_c^{-1}.$$

For a fine spacing $\epsilon > 0$ and a link field A of potentials, set

$$\Psi_{U,\epsilon}(A) = (M\epsilon)^{-1} \operatorname{Log} \mathscr{M}_U(e^{\epsilon A}).$$

All logarithms are identity-chart logarithms; Lemma 8.9 shows that they are defined in the domains used below.

In other words, $\mathscr{M}_U(V)\,T_M U$ is the gauge transform of $T_M(VU)$ by $\mathsf{h}_U(V)$, and $\mathscr{M}_U(V)$ is its transport relative to the background output $T_M U$. The endpoint gauge transformation at v is chosen so that its derivative cancels the contribution of the connectors $\sigma_{v,x}$ in Figure 8.1. For a gauge transformation k on the vertices of a lattice, we write $k|_{\mathrm{roots}}$ for its restriction to the vertices of the next coarser lattice, which are the centers of the blocks.

Lemma 8.9. *Fix an odd $M \geq 3$, a radius $0 < \rho_{\log} < 2\pi$ inside the real identity chart of $\mathrm{SU}(2)$, and a radius $0 < r_{\log} < 1$ for the matrix logarithm series. There exist $b_M, \delta_M > 0$ and $q_M, C_M < \infty$, depending only on these data, such that the following holds for every application of T_M from a periodic lattice of side Mn' to the lattice of side n', $n' \geq 4$.*

1. *For each coarse link $c = (v, v + Me_i)$, the paths C_c, $F_{c,x}$, $\sigma_{v,x}$, and $\sigma_{v+Me_i,x+Me_i}$, $x \in \mathsf{B}_M(v)$, have at most q_M links and lie in a neighborhood of c of radius depending only on M, and each fine link belongs to at most a number depending only on M of these neighborhoods.*
2. *If U is $\mathrm{SU}(2)$-valued and $\delta(U) = \max_p \rho_G(U_p) \leq \delta$ with $0 \leq \delta < \delta_M$, then every based loop holonomy $U(F_{c,x})U(C_c)^{-1}$ has an identity-chart logarithm of norm at most $C_M\delta$.*
3. *For such U and complex link variables with $\max_e |X_e| < b_M$, the path products with links $e^{X_e}U_e$, their relative logarithms, and the logarithm $\operatorname{Log} \mathscr{M}_U(e^X)$ of the gauge-corrected average are holomorphic in X. On a fixed smaller ball, each partial derivative of fixed order in the finitely many relevant X_e is bounded by a constant depending only on M, the order, and the radii $\rho_{\log}$, $r_{\log}$. The logarithm $\operatorname{Log} \mathscr{M}_U(e^X)$ vanishes at $X = 0$.*

These bounds are independent of the holonomies of U around the cycles of the torus and of n', and they include reversed links.

Proof. Each coordinate path has length at most $3(M-1)/2$, and the middle segment has length M. The union of the two adjacent centered blocks has extent $2M-1$ in the direction of the link and $M-1$ in the other two. Since the fine side is $Mn' \geq 4M$, this union lifts to a box in $\mathbb{Z}^3$ without winding. All occurrences of a fixed fine link lie in a bounded number of such neighborhoods: their coarse vertices are within a distance depending on M, and there are three orientations and boundedly many paths.

We next prove a filling estimate. In a lifted box of maximal extent D, we choose its lower corner and the coordinate paths, ordered $1, 2, 3$, from it to every vertex. For a positive link in direction i, we append the link to the ordered path of its initial vertex and move it to the left past the portions of the path in directions greater than i. Each interchange replaces two adjacent perpendicular links by the other two links of their elementary square. There are at most $3D$ interchanges, all within the box. Inserting a square changes the transport by a conjugate of the plaquette transport or its inverse. Hence, in the frame given by the ordered paths, the transport of this link is a product of at most $3D$ conjugated plaquette transports. For a reversed link we take the inverse. Along a closed path of length q the frames cancel at successive vertices, and the transport around the path is, up to one conjugation, a product of at most $3Dq$ conjugated plaquette transports. We apply this to $F_{c,x}C_c^{-1}$ in the box above, with $D \leq 2M-1$ and $q \leq q_M$, which gives at most $s_M = 3(2M-1)q_M$ plaquette factors. A product of s elements of $\mathrm{SU}(2)$, each within group distance δ of the identity, is within group distance $s\delta$ of the identity, as one sees by concatenating the shortest one-parameter subgroups and using the bi-invariance of the metric. Let $0 < \delta_M \leq \rho_{\log}/(2s_M)$. For $0 \leq \delta < \delta_M$, every based

loop holonomy is within group distance $s_M\delta < \rho_{\log}$ of the identity, so its identity-chart logarithm exists and has norm at most $s_M\delta$. This proves part (2), and further decreases of δ_M below preserve it.

For the complex bounds, write a positive link as $e^{X_e}U_e$ and a reversed link as $U_e^{-1}e^{-X_e}$. For a path product W, moving the fixed SU(2)-valued factors to the right writes $W(X)W(0)^{-1}$ as a product of at most q_M exponentials of $\pm X_e$, each conjugated by an SU(2)-valued partial transport. On the complexified color space these conjugations are unitary, so in a fixed matrix norm

$$\|W(X)W(0)^{-1} - I\| \le \exp\bigl(Cq_M \max_e |X_e|\bigr) - 1,$$

as one sees by expanding each exponential absolutely and multiplying the bounds. For a based loop holonomy, the value at $X = 0$ is within $C_M\delta$ of the identity by part (2). Choosing b_M and δ_M small puts all these matrices in the ball $\|Z - I\| < r_{\log}$, on which $\operatorname{Log} Z = \sum_{\ell\ge1}(-1)^{\ell+1}(Z-I)^\ell/\ell$ converges absolutely. Its derivatives on smaller balls are bounded by the differentiated series, and the exponential series has the same property everywhere. The endpoint gauge transformations equal the identity at $X = 0$, and there the output of T_M equals the background output, so $\mathscr{M}_U(e^0) = I$. After decreasing b_M, its logarithm lies in the same chart. At most q_M distinct link variables enter, even when a link occurs repeatedly. Bounds for every partial derivative of fixed order follow by differentiating these finite products and series, or by Cauchy's formula on fixed circles in these finitely many complex variables. Neither method introduces a dependence on the side or the depth. □

Lemma 8.10. *Fix an odd $M \ge 3$ and the radii $\rho_{\log}, r_{\log}$ of Lemma 8.9. There exist $\varepsilon_*, a_*, \delta_{\mathrm{av}}, C_M > 0$, depending only on these data, with $2\varepsilon_* \le \delta_{\mathrm{av}} < \delta_M$, such that the following holds for all integers $d \ge 0$ and $n \ge 4$. Use units in which the spacing of the coarse lattice of side n is one, so that the lattice at level j has side $M^{d-j}n$ and spacing $\epsilon_j = M^{j-d}$, $0 \le j \le d$.*

1. One step. *Let U be an* SU(2)*-valued configuration on a periodic lattice of side Mn', $n' \ge 4$, with $\delta(U) = \delta \le \delta_{\mathrm{av}}$, and let $0 < \epsilon \le 1/M$. On the complex ball $\|A\|_\infty < 2a_*$,*

$$\Psi_{U,\epsilon}(A) = Q_U A + \mathscr{E}_{U,\epsilon}(A), \qquad \|Q_U\|_{\mathrm{row}} \le 1 + C_M\delta,$$
$$\|\mathscr{E}_{U,\epsilon}(A)\|_\infty \le C_M M\epsilon\|A\|_\infty^2, \qquad \|D\mathscr{E}_{U,\epsilon}(A)\|_{\mathrm{row}} \le C_M M\epsilon\|A\|_\infty,$$

where $Q_U = D\Psi_{U,\epsilon}(0)$ does not depend on ϵ. Its principal part is the transported segment average

$$(Q_U^{\mathrm{main}}A)_c = M^{-4}\sum_{x\in \mathsf{B}_M(v)}\ \sum_{e\in[x,x+Me_i]} \mathrm{Ad}\bigl(U(\sigma_{v,x}[x,e^-])\bigr)A_e, \qquad c = (v, v+Me_i),$$

where e^- is the initial vertex of e and $\sigma_{v,x}[x,e^-]$ is the path $\sigma_{v,x}$ followed by the segment from x to e^-. The scalar averaging weights in each row of Q_U^{main} sum to exactly one, the

norms of its color blocks in each row sum to at most one, and $Q_U - Q_U^{\mathrm{main}}$ has row norm at most $C_M\delta$.

2. Iteration. *Let U_0 be an $\mathrm{SU}(2)$-valued configuration on the lattice of side $M^d n$ with $\delta(U_0) \le \varepsilon\epsilon_0^2$ for some $0 \le \varepsilon \le \varepsilon_*$, and set $U_j = T_M^j U_0$. Then $\delta_j = \delta(U_j) \le 2\varepsilon\epsilon_j^2$ for $0 \le j \le d$. For $\|A\|_\infty < a_*$ define*

$$V_0 = e^{\epsilon_0 A}, \qquad V_{j+1} = \mathscr{M}_{U_j}(V_j), \qquad A_j = \epsilon_j^{-1} \operatorname{Log} V_j.$$

The recursion is defined and holomorphic on this ball, and for $0 \le j \le d$

$$\|A_j\|_\infty \le 2\|A\|_\infty, \qquad \|D_A A_j\|_{\mathrm{row}} \le \exp\Big[C_M \sum_{\ell<j} (\delta_\ell + \epsilon_{\ell+1}\|A_\ell\|_\infty)\Big] \le C_M.$$

The coordinate $(A_j)_c$ depends only on the variables A_e with $\operatorname{dist}(c,e) \le C_M\epsilon_j$, where distances are measured in the units above, in which the coarse spacing is one. For every fixed $a_{\mathrm{sp}} \ge 0$ we have $\|D_A A_j\|_{a_{\mathrm{sp}},\mathrm{row}} \le C_{M,a_{\mathrm{sp}}}$ on the same ball. For every fixed $s \ge 1$, on the ball $\|A\|_\infty < a_*/2$, every coarse link c and all link fields $B_1, \dots, B_s$,

$$\big|\big(D_A^s A_j(A)[B_1, \dots, B_s]\big)_c\big| \le C_{M,s,a_{\mathrm{sp}}} \prod_{\ell=1}^{s} \sup_e e^{-a_{\mathrm{sp}} \operatorname{dist}(c,e)} |(B_\ell)_e|.$$

3. Composed gauge transformation. *Define gauge transformations on the vertices of the lattice at level j by*

$$\mathsf{G}_0 = I, \qquad \mathsf{G}_{j+1} = \mathsf{h}_{U_j}(V_j)\,(\mathsf{G}_j|_{\mathrm{roots}}).$$

Then

$$V_j U_j = \big(T_M^j(e^{\epsilon_0 A} U_0)\big)^{\mathsf{G}_j}, \qquad 0 \le j \le d.$$

For real A the gauge transformations G_j are $\mathrm{SU}(2)$-valued. On a fixed smaller complex ball they and their inverses have uniformly bounded holomorphic logarithms; the factors satisfy $\|\operatorname{Log} \mathsf{h}_{U_j}(V_j)\| \le C_M\epsilon_j\|A_j\|_\infty$, whose sum over j is bounded.

The constants in parts (1)–(3) do not depend on n, n', d, j, the holonomies of the background around the cycles of the torus, L, or γ; the constants $C_{M,a_{\mathrm{sp}}}$ and $C_{M,s,a_{\mathrm{sp}}}$ depend in addition only on a_{sp} and s.

Proof. We begin with the curvature along the background trajectory. Let $0 < \varepsilon_* \le \varepsilon_M$, with ε_M as in Theorem 7.3. At level j the background has side $M^{d-j}n$, and every step $j < d$ goes from side $M \cdot M^{d-j-1}n$ to side $M^{d-j-1}n \ge n \ge 4$, with the factor M and spacings $\epsilon_j = M^{j-d}$. For $0 < \varepsilon \le \varepsilon_*$, the hypothesis $\delta(U_0) \le \varepsilon\epsilon_0^2$ is the hypothesis of the curvature iteration in Theorem 7.3, which shows that T_M is given by its logarithmic formula at every step and yields the curvature bounds below. If $\varepsilon = 0$, then $\delta(U_0) = 0$, and for every $\eta \in (0, \varepsilon_M]$ the same background satisfies $\delta(U_0) \le \eta\epsilon_0^2$. We apply the same theorem with η in place of ε. The iterates $U_j = T_M^j U_0$ do not depend on η, and each

application bounds the same fields by $\delta_j \le 2\eta\epsilon_j^2$; letting $\eta \downarrow 0$, we obtain $\delta_j = 0$ for every j, and any one application with positive η shows that the logarithmic formula applies. When $d = 0$ the bound is the hypothesis and the sums below are empty. In all cases, for $0 \le \varepsilon \le \varepsilon_*$,

$$\delta_j \le 2\varepsilon\epsilon_j^2, \qquad \sum_{j<d} \delta_j \le \frac{2\varepsilon}{M^2-1}. \tag{8.6}$$

We fix $0 < \delta_{\mathrm{av}} < \delta_M$, with δ_M from Lemma 8.9, and impose $2\varepsilon_* \le \delta_{\mathrm{av}}$. Every application below with $0 \le \delta \le \delta_{\mathrm{av}}$ then satisfies the curvature hypothesis $\delta < \delta_M$ of that lemma. We decrease ε_*, δ_{av}, and the fluctuation radius finitely many times below.

Step 1: The one-step differential and its endpoint terms. Fix an $\mathrm{SU}(2)$-valued U with $\delta(U) = \delta \le \delta_{\mathrm{av}}$, and write X for a link field of infinitesimal fluctuations. For a path η, the product rule and $(e^{tX_e}U_e)^{-1} = U_e^{-1}e^{-tX_e}$ give the left-trivialized derivative of the transport,

$$p_\eta(X) = \sum_{\substack{\text{positive}\\ \text{occurrences } e}} \mathrm{Ad}\big(U(\eta_{\text{before } e})\big)X_e - \sum_{\substack{\text{negative}\\ \text{occurrences } \bar e}} \mathrm{Ad}\big(U(\eta_{\text{before } \bar e})U_e^{-1}\big)X_e,$$

where $\eta_{\text{before } e}$ is the part of η preceding the occurrence. All transports in this formula are $\mathrm{SU}(2)$-valued, hence orthogonal on colors.

To differentiate T_M without discarding a term, let $F_x = U(F_{c,x})$, $C = U(C_c)$, $L_x = F_xC^{-1}$, $Z = M^{-3}\sum_x \operatorname{Log} L_x$, and $O = e^ZC = (T_MU)_c$. Let $f_x = \dot F_x F_x^{-1}$ and $c_0 = \dot C C^{-1}$, where the dot denotes the derivative in the direction X, and introduce the left-trivialized differentials

$$\mathcal{E}_Z(Y) = (D\exp_Z Y)e^{-Z}, \qquad \mathcal{L}_{L_x}(Y) = D\operatorname{Log}_{L_x}(YL_x).$$

By the product rule, we have

$$\dot O O^{-1} = \mathcal{E}_Z\Big[M^{-3}\sum_x \mathcal{L}_{L_x}\big(f_x - \mathrm{Ad}(L_x)c_0\big)\Big] + \mathrm{Ad}(e^Z)c_0. \tag{8.7}$$

By Lemma 8.9, $|\operatorname{Log} L_x| \le C_M\delta$ and $|Z| \le C_M\delta$. Using the convergent exponential and logarithm series, we obtain $\mathcal{E}_Z = I + O_M(\delta)$, $\mathcal{L}_{L_x} = I + O_M(\delta)$, $\mathrm{Ad}(L_x) = I + O_M(\delta)$, and $\mathrm{Ad}(e^Z) = I + O_M(\delta)$ as operators. Their absolute local coefficient sums have the same bounds, because at most q_M local labels occur. Replacing these four operators by I in (8.7) cancels the two terms containing c_0 and leaves $M^{-3}\sum_x f_x$. The replacement error, as a linear map of X, has absolute row norm at most $C_M\delta$.

We split the derivative along the path at its two endpoint connectors:

$$f_x = p_{\sigma_{v,x}}(X) + p^{\mathrm{seg}}_{v,x}(X) - \mathrm{Ad}(F_x)p_{\sigma_{s,x+Me_i}}(X), \qquad s = v + Me_i.$$

The middle term is the sum of the variations of the M links of the segment $[x, x + Me_i]$, transported first along $\sigma_{v,x}$ and then along the segment up to the link. Differentiating the endpoint gauge transformation, we find

$$\dot{\mathsf{h}}_{v;U}(I) = -M^{-3} \sum_x p_{\sigma_{v,x}}(X).$$

The left-trivialized derivative of the gauge-corrected output is the derivative of T_M minus this mean over the initial connectors, plus $\mathrm{Ad}(O)$ times the corresponding mean over the terminal connectors. The terms of the initial connectors cancel exactly. The remaining terminal terms are

$$M^{-3} \sum_x \{\mathrm{Ad}(O) - \mathrm{Ad}(F_x)\} p_{\sigma_{s,x+Me_i}}(X);$$

here the average over $x + Me_i$ is exactly the average over $\mathsf{B}_M(s)$ in the definition of $\mathsf{h}_{s;U}$, with the same paths. These terms have absolute row norm at most $C_M\delta$, since $OF_x^{-1} = e^Z L_x^{-1}$ is within $C_M\delta$ of the identity and $\mathrm{SU}(2)$-valued transports have norm one.

The remaining segment derivative is

$$M^{-3} \sum_{x \in \mathsf{B}_M(v)} \sum_{e \in [x, x+Me_i]} \mathrm{Ad}\big(U(\sigma_{v,x}[x, e^-])\big) X_e.$$

For $X = \epsilon A$, division by the coarse length $M\epsilon$ gives Q_U^{main}. There are M^4 occurrences of weight M^{-4}, each an orthogonal map on colors. The scalar weights in each row therefore sum to one, and the norms of the color blocks in each row sum to at most one. The endpoint and coefficient errors above, divided by M, are still bounded by $C_M\delta$. We conclude that $Q_U = D\Psi_{U,\epsilon}(0)$ does not depend on ϵ and satisfies the row estimates of part (1).

Step 2: The one-step remainder. Let $F_U(X) = \mathrm{Log}\, \mathscr{M}_U(e^X)$, which vanishes at $X = 0$. By Lemma 8.9, its second derivative has absolute row norm at most C_M on a fixed complex ball of X, uniformly in U. Here the row sums over both inputs, and the bounded size of the stencils justifies this absolute bound. Since $\Psi_{U,\epsilon}(A) = (M\epsilon)^{-1} F_U(\epsilon A)$, on a fixed complex ball of A with $\epsilon \le 1/M$,

$$\|D_A^2 \Psi_{U,\epsilon}\|_{\mathrm{row}} \le C_M \frac{\epsilon^2}{M\epsilon} = C_M \epsilon / M \le C_M M\epsilon.$$

The quadratic bound for $\mathscr{E}_{U,\epsilon}$ follows from Taylor's formula along the segment tA, $0 \le t \le 1$. Integrating the second derivative along the same segment gives the bound for its first derivative. This proves part (1) on $\|A\|_\infty < 2a_*$, after choosing $2a_*$ inside that ball.

Step 3: Uniform iteration. Let C dominate the constants of part (1). Since $A_{j+1} = \Psi_{U_j,\epsilon_j}(A_j)$, $M\epsilon_j = \epsilon_{j+1}$, and $\epsilon_j \le 1/M$ for $j < d$, the norms $b_j = \|A_j\|_\infty$ satisfy, whenever the iteration is defined,

$$b_{j+1} \le b_j\big(1 + C\delta_j + C\epsilon_{j+1}b_j\big), \qquad \sum_{j<d} \epsilon_{j+1} \le \frac{M}{M-1}.$$

We choose the radii so small that

$$C\Big(\frac{2\varepsilon_*}{M^2-1} + \frac{2a_*M}{M-1}\Big) < \log(3/2). \tag{8.8}$$

For an input with $\|A\|_\infty = t < a_*$, as long as $b_\ell \le 2t$ for $\ell < j$, multiplying the inequalities and using $\log(1+u) \le u$, we obtain

$$b_j \le t\exp\Big[C\sum_{\ell<j}\delta_\ell + 2Ct\sum_{\ell<j}\epsilon_{\ell+1}\Big] < \tfrac{3}{2}t.$$

In particular b_j never reaches $2t$, so the bound holds for all $j \le d$, and each step stays in the domain $b_j < 2a_*$ of part (1). Since a finite composition of holomorphic maps is holomorphic, the recursion is holomorphic on the input ball at every finite depth. For $t = 0$ it vanishes identically, and for $d = 0$ it is the identity.

The derivative of one step at A_j, including the nonlinear remainder, has row norm at most $1 + C_M(\delta_j + \epsilon_{j+1}b_j)$. By the submultiplicativity of absolute row sums, we obtain the bound for $D_A A_j$ in part (2). By (8.6) and the bootstrap, its exponent is at most

$$C_M\Big(\frac{2\varepsilon_*}{M^2-1} + \frac{2a_*M}{M-1}\Big),$$

which does not depend on d, n, or j.

Step 4: Locality and directional derivatives. An output of one step depends only on inputs within a neighborhood of radius $r_M^{\mathrm{loc}}\epsilon_j$, where $r_M^{\mathrm{loc}} < \infty$ is a fixed stencil radius. If r_j is the dependence radius at level j, measured in final-coarse units, in which $\epsilon_d = 1$, then $r_{j+1} \le r_j + r_M^{\mathrm{loc}}\epsilon_j$ and $r_0 = 0$. Summing the geometric series, we find $r_j \le C_M\epsilon_j$. The same recursion bounds the background links on which the coefficients depend, since each preceding application of T_M has a neighborhood of the same scale. The assertion holds on the torus, since each neighborhood can be lifted to a box before returning to the torus. Inserting the weight $e^{a_{\mathrm{sp}}\,\mathrm{dist}(c,e)}$ multiplies the bound for the first derivative by at most $e^{a_{\mathrm{sp}}C_M\epsilon_j} \le e^{a_{\mathrm{sp}}C_M}$.

For the higher derivatives, take the smaller ball $\|A\|_\infty < a_*/2$. For link fields $B_1, \dots, B_s$ with $\|B_\ell\|_\infty \le 1$, the map $(t_1, \dots, t_s) \mapsto A_j(A + \sum_\ell t_\ell B_\ell)$ is bounded by $2a_*$ for $|t_\ell| < a_*/(4s)$. Cauchy's formula on these circles bounds its mixed coefficient,

and hence $\|D_A^s A_j(A)[B_1,\dots,B_s]\|_\infty$, by $2a_*(4s/a_*)^s$. This bounds a multilinear operator norm; absolute sums of coefficients over the inputs are treated in Lemma 8.11. For the weighted bound, restrict each B_ℓ to the dependence neighborhood of the fixed output c, which does not change the derivative by locality. On that neighborhood $|(B_\ell)_e| \le e^{a_{\mathrm{sp}} C_M} \sup_{e'} e^{-a_{\mathrm{sp}}\,\mathrm{dist}(c,e')} |(B_\ell)_{e'}|$, so the multilinear bound acquires at most the factor $e^{s a_{\mathrm{sp}} C_M}$.

Step 5: The composed gauge transformation. For a gauge transformation k, a transport along a path from v to s transforms as $k_v U(\eta) k_s^{-1}$. The based loop holonomies $U(F_{c,x})U(C_c)^{-1}$ are conjugated by k_v. Within the identity chart the logarithm series and the exponential are equivariant under conjugation, and substituting into the formula for T_M, we obtain

$$T_M(W^k) = (T_M W)^{k|_{\mathrm{roots}}}.$$

For SU(2)-valued gauge transformations this also holds at the configurations where the Borel extension is used, since the exceptional matrix $-I$ is central. For complex gauge transformations we use only the domain where all logarithms are given by their series. Let $Z_j = T_M^j(e^{\epsilon_0 A} U_0)$. If $V_j U_j = Z_j^{\mathsf{G}_j}$, then by the definition of $\mathscr{M}$ and the gauge identity we have

$$V_{j+1}U_{j+1} = \big(T_M(V_jU_j)\big)^{\mathsf{h}_{U_j}(V_j)} = (T_M Z_j)^{\mathsf{h}_{U_j}(V_j)(\mathsf{G}_j|_{\mathrm{roots}})},$$

where we used $(W^k)^l = W^{lk}$. Starting at $j = 0$, induction proves the identity of part (3) with the stated order of the factors. For real A, every relative transport in an endpoint gauge transformation is SU(2)-valued with small logarithm in $\mathfrak{su}(2)$. Then every factor and every G_j is SU(2)-valued.

For complex A, Lemma 8.9 and the definition of h imply that

$$\|\mathrm{Log}\,\mathsf{h}_{U_j}(V_j)\| \le C_M \epsilon_j b_j, \qquad \sum_{j<d} \epsilon_j b_j \le \frac{2\|A\|_\infty}{M-1}.$$

It follows that all partial products and their inverses have norm at most $\exp(C_M\|A\|_\infty)$ and differ from the identity by at most $\exp(C_M\|A\|_\infty) - 1$. We decrease a_* once so that the last quantity is less than $1/2$. The logarithms then exist by the same convergent series and are holomorphic and uniformly bounded.

It remains to check that the gauge identities do not leave the logarithm chart. At level j, every based loop holonomy of V_jU_j differs from the identity by at most $C_M(\delta_j + \epsilon_j b_j)$, by Lemma 8.9. The corresponding based loop holonomy of Z_j is its conjugate by G_j^{-1}, and its distance to the identity in matrix norm is larger by at most the factor $\exp(C_M\|A\|_\infty)$. The bounds $\delta_j \le 2\varepsilon_* \epsilon_j^2$ and $b_j \le 2a_*$ allow one further decrease of ε_* and a_* after which both based loop holonomies lie strictly inside the chart. The conjugation identity for the logarithm series then holds term by term, and the induction above applies at every level on this ball. This shows that the complex gauge relation is an identity of holomorphic

maps, not only of Taylor coefficients. All choices involved only M and the radii $\rho_{\log}$, $r_{\log}$. □

8.5 Volume-normalized column sums and derivative bounds

We now prove bounds on absolute sums of derivative coefficients of the iterates $A_j = \Phi_j(A)$ of Lemma 8.10, with either the output or any one input held fixed. Two normalizations of column sums occur. For a matrix P from link fields at spacing ϵ_0 to link fields at spacing ϵ_j, the counting column norm is $\sup_e \sum_c \|P(c,e)\|$. The volume-normalized column norm of Definition 4.6, with $a_{\mathrm{out}} = \epsilon_j$ and $a_{\mathrm{in}} = \epsilon_0$, is $(\epsilon_j/\epsilon_0)^3$ times the counting column norm. A fine link contributes to few coarse links, each time with a small weight. The counting column sums of $D\Phi_j$ are therefore of order $(\epsilon_0/\epsilon_j)^3 = M^{-3j}$, and the volume-normalized column sums are bounded. For derivatives of higher order with one input fixed, exactly one such volume factor appears. For example, with one fixed input e_1,

$$D^2(\Psi \circ \Phi)[e_1, e_2] = D^2\Psi\big[D\Phi\, e_1, D\Phi\, e_2\big] + D\Psi\, D^2\Phi[e_1, e_2];$$

in the first term, the branch that contains e_1 is summed with a column sum of $D\Phi$ and the other branch with a row sum, and only one factor $(\epsilon_0/\epsilon_j)^3$ is produced. The proof below organizes this observation by an induction over the order, using the set-partition form of the chain rule.

Lemma 8.11. *Assume the hypotheses of part (2) of Lemma 8.10, and let A range over a fixed smaller complex ball on which the bounds of Lemmas 8.9 and 8.10 hold with a fixed margin. Write $\Phi_j(A) = A_j$ and, for $j < d$, $e_j = \delta_j + \epsilon_{j+1}\|A_j\|_\infty$. The derivative of one step decomposes as*

$$L_j = D\Psi_{U_j,\epsilon_j}(A_j) = K_j + E_j, \qquad K_j = Q^{\mathrm{main}}_{U_j},$$

and the nonnegative scalar kernel of occurrences

$$k_M(c,e) = M^{-4}\#\big\{(x,s) : x \in \mathsf{B}_M(v),\ 0 \le s < M,\ e = (x + se_i, x + (s+1)e_i)\big\}$$

for $c = (v, v + Me_i)$, has row sums exactly 1 and column sums exactly M^{-3}; the sums of the norms of the color blocks of K_j are no larger. With counting column norms,

$$\begin{aligned} \|L_j\|_{\mathrm{row}} &\le 1 + C_M e_j, & \|L_j\|_{\mathrm{col}} &\le M^{-3}(1 + C_M e_j),\\ \|E_j\|_{\mathrm{row}} &\le C_M e_j, & \|E_j\|_{\mathrm{col}} &\le M^{-3} C_M e_j. \end{aligned}$$

Set $S_j = \sum_{\ell<j} e_\ell$, $P_j = L_{j-1}\cdots L_0 = D\Phi_j$, and $P^0_j = K_{j-1}\cdots K_0$, empty products being the identity. Then

$$\begin{aligned} \|P_j\|_{\mathrm{row}} &\le e^{C_M S_j}, & \|P_j\|_{\mathrm{col}} &\le (\epsilon_0/\epsilon_j)^3 e^{C_M S_j},\\ \|P_j - P^0_j\|_{\mathrm{row}} &\le C_M S_j e^{C_M S_j}, & \|P_j - P^0_j\|_{\mathrm{col}} &\le C_M (\epsilon_0/\epsilon_j)^3 S_j e^{C_M S_j}, \end{aligned}$$

and these bounds follow from the identity

$$P_j - P_j^0 = \sum_{\ell<j} K_{j-1}\cdots K_{\ell+1}E_\ell L_{\ell-1}\cdots L_0.$$

For every fixed $a_{\mathrm{sp}} \geq 0$, the coefficients of the last iterate satisfy

$$\|D_{A_e}\Phi_d(c)\| \leq C_{M,a_{\mathrm{sp}}}\epsilon_0^3 e^{-a_{\mathrm{sp}}\,\mathrm{dist}(c,e)},$$

and $D\Phi_j$ is bounded from ℓ^2 with weight ϵ_0^3 to ℓ^2 with weight ϵ_j^3, uniformly. For every fixed $s \geq 1$ and $0 \leq j \leq d$, with $\mathtt{t}(c, e_1, \ldots, e_s)$ the length of a shortest connecting network of the locations of $c, e_1, \ldots, e_s$, measured in final-coarse units, in which $\epsilon_d = 1$,

$$\sup_c \sum_{e_1,\ldots,e_s} e^{a_{\mathrm{sp}}\mathtt{t}(c,e_1,\ldots,e_s)}\|D_{A_{e_1}}\cdots D_{A_{e_s}}\Phi_j(c)\| \leq C_{M,s,a_{\mathrm{sp}}},$$
$$\sup_{e_1} \sum_{c,e_2,\ldots,e_s} e^{a_{\mathrm{sp}}\mathtt{t}(c,e_1,\ldots,e_s)}\|D_{A_{e_1}}\cdots D_{A_{e_s}}\Phi_j(c)\| \leq C_{M,s,a_{\mathrm{sp}}}(\epsilon_0/\epsilon_j)^3.$$

If $\mathtt{t}_j = \epsilon_j^{-1}\mathtt{t}$ is the connecting length in current-output lattice units, the same bounds hold with $e^{a_{\mathrm{sp}}\mathtt{t}_j}$ and constants $C_{M,s,a_{\mathrm{sp}}}$, since locality gives $\mathtt{t}_j \leq sC_M$. In the second bound any one input may be fixed, and multiplying by $(\epsilon_j/\epsilon_0)^3$ gives a bounded volume-normalized column sum in the sense of Definition 4.6. All orientations, repeated inputs, and periodic identifications are included. The constants may depend on M, s, a_{sp}, the radii of Lemma 8.9, and the ratio of the nested balls, but not on j, d, n, L, γ, or the holonomies of the background. The bounds hold for each fixed order s; no bound uniform in s, such as a factorial bound, and no factor ϵ_0^{3s} for individual coefficients of order s, is asserted.

Proof. The background configurations $U_0, \ldots, U_d$ are fixed and are not differentiated. We shrink the ball of fluctuations inside the ball of Lemma 8.10, so that the derivative bounds of Lemma 8.9 hold with a fixed margin.

Step 1: Exact incidence and the first derivative of one step. A coarse i-link has M^3 segment starts and M positively oriented links in each segment, and its scalar row sum is $M^4M^{-4} = 1$. For a fixed fine link $e = (z, z + e_i)$, the segment starts whose segments contain e are exactly $x = z - se_i$, $0 \leq s < M$. Each start lies in exactly one centered block, and the vertex of that block determines exactly one coarse link in direction i. There are exactly M occurrences, and the scalar column sum is $MM^{-4} = M^{-3}$. If several occurrences share a coefficient block, combining their orthogonal transports cannot increase the sum of the norms, so the color-block sums of K_j obey the same upper bounds. The count also holds across the periodic seams: the fine side of a single averaging step is at least $4M$; the M starts are distinct, and the centered blocks partition the vertices. No reversed link occurs in K_j; reversed links and mixed orientations in the other paths are covered by the bound on overlaps in Lemma 8.9.

By the proof of Lemma 8.10, we have $\|E_j\|_{\mathrm{row}} \le C_M(\delta_j + \epsilon_{j+1}\|A_j\|_\infty) = C_M e_j$. Let J_M be the largest number of coarse output stencils that contain a given fine input link, counting all orientations and repetitions. Each coefficient norm is bounded by the row sum of its output, so $\|E_j\|_{\mathrm{col}} \le J_M C_M e_j$. Since M is fixed, we replace the constant by $M^3 J_M C_M$ and write this as $\|E_j\|_{\mathrm{col}} \le M^{-3} C_M e_j$, which keeps the factor M^{-3} of the column sum of K_j. Adding the bounds for K_j proves the four one-step estimates. This also applies for complex A_j, since orthogonality was used only for K_j, whose transports are those of the background.

Step 2: Products and the ordered telescope. For counting norms, $\|(AB)_{xz}\| \le \sum_y \|A_{xy}\|\|B_{yz}\|$ gives submultiplicativity of row sums and of column sums. As in Step 3 of the proof of Lemma 8.10, we obtain

$$\|P_j\|_{\mathrm{row}} \le \prod_{\ell<j}(1 + C_M e_\ell) \le e^{C_M S_j}, \qquad \|P_j\|_{\mathrm{col}} \le M^{-3j} e^{C_M S_j},$$

and $M^{-3j} = (\epsilon_0/\epsilon_j)^3$. For the telescope, we replace the leftmost factor L_{j-1} by $K_{j-1} + E_{j-1}$ and keep the term with E_{j-1}; then we make the same replacement in the next factor of the remaining product of factors K, and so on. After the last replacement the remaining product is P_j^0, which gives the identity in the statement. Its summand with index ℓ has row norm at most $C_M e_\ell e^{C_M S_j}$. Its column norm contains exactly j factors M^{-3}, including the one in E_ℓ, and is at most $M^{-3j} C_M e_\ell e^{C_M S_j}$. Summing over ℓ proves the bounds for $P_j - P_j^0$; the error is proportional to S_j rather than to the number of levels. We also recall from the proof of Lemma 8.10 the uniform bound

$$S_j \le \frac{2\varepsilon_*}{M^2 - 1} + \frac{2a_* M}{M - 1}. \tag{8.9}$$

Step 3: Pointwise and ℓ^2 bounds. Each coefficient $D_{A_e}\Phi_d(c)$ is bounded by the counting column sum containing it, hence by $C_M\epsilon_0^3$, since $\epsilon_d = 1$. It vanishes unless e is within distance C_M of c, by the locality in Lemma 8.10. Multiplying by $e^{a_{\mathrm{sp}} C_M}$ proves the pointwise exponential bound for every fixed $a_{\mathrm{sp}} \ge 0$. For a block matrix P with row bound R and column bound C, the weighted Cauchy–Schwarz inequality at each output implies that

$$\sum_c \Big|\sum_e P_{ce} u_e\Big|^2 \le R \sum_{c,e} \|P_{ce}\| |u_e|^2 \le RC \sum_e |u_e|^2.$$

For $P = P_j$ we have $RC \le C_M(\epsilon_0/\epsilon_j)^3$. Multiplying the left side by ϵ_j^3, we find that it is at most $C_M\epsilon_0^3 \sum_e |u_e|^2$, as required.

Step 4: Absolute bounds for higher derivatives of one step. For $r \ge 2$, we apply the derivative bounds of Lemma 8.9 to $\Psi_{U_j,\epsilon_j}(A) = \epsilon_{j+1}^{-1} F_{U_j}(\epsilon_j A)$. Every partial derivative

of order r carries the factor $\epsilon_j^r/\epsilon_{j+1} = \epsilon_j^{r-1}/M$, which is at most ϵ_{j+1} since $\epsilon_j \le 1/M$. Summing over the at most q_M^r tuples of local inputs, we obtain, with $\Psi_j = \Psi_{U_j,\epsilon_j}$ evaluated at A_j,

$$\begin{aligned} \sup_c \sum_{b_1,\dots,b_r} \|D_{b_1}\cdots D_{b_r}\Psi_j(c)\| &\le C_{M,r}\epsilon_{j+1}, \\ \sup_{b_1} \sum_{c,b_2,\dots,b_r} \|D_{b_1}\cdots D_{b_r}\Psi_j(c)\| &\le M^{-3}C_{M,r}\epsilon_{j+1}. \end{aligned} \tag{8.10}$$

The second estimate uses the bound J_M on output stencils and absorbs $M^3 J_M$ into the constant, as in Step 1. These absolute sums use only that finitely many local variables enter, not directional Cauchy estimates. By symmetry of derivatives, any input may be the fixed one in the second line.

Step 5: Induction over the order. We first omit the spatial weights and set

$$R_s(j) = \sup_c \sum_{e_1,\dots,e_s} \|D_{e_1}\cdots D_{e_s}\Phi_j(c)\|, \qquad C_s(j) = \sup_{e_1} \sum_{c,e_2,\dots,e_s} \|D_{e_1}\cdots D_{e_s}\Phi_j(c)\|,$$

and $\widehat{C}_s(j) = (\epsilon_j/\epsilon_0)^3 C_s(j)$. Step 2 bounds R_1 and $\widehat{C}_1$. For $s \ge 2$ both vanish at $j = 0$, since $\Phi_0(A) = A$. By the set-partition form of the chain rule (Faà di Bruno's formula), the derivatives of $\Phi_{j+1} = \Psi_j \circ \Phi_j$ are

$$D^s\Phi_{j+1}[u_1,\dots,u_s] = \sum_{\pi} D^{|\pi|}\Psi_j(A_j)\Big[D^{|B|}\Phi_j(A)[u_\ell : \ell \in B] : B \in \pi\Big], \tag{8.11}$$

where π runs over the set partitions of $\{1,\dots,s\}$. The formula follows by induction on s: a further derivative either differentiates the outer tensor, creating a new block, or differentiates exactly one inner tensor, enlarging its block, and these alternatives produce each partition once.

For the sums with the output fixed, the partition with one block contributes at most $(1 + C_M e_j)R_s(j)$. Every other partition has at least two blocks, and all inner orders are smaller than s. Summing the inner coefficients at each fixed intermediate label uses $R_{|B|}(j)$, and summing the outer coefficients then uses the first line of (8.10). By induction on s we obtain, with a constant $B_{M,s}$ determined by the lower orders,

$$R_s(j+1) \le (1 + C_M e_j)R_s(j) + B_{M,s}\epsilon_{j+1}.$$

For the sums with one input fixed, fix e_1 and, in each partition, let B_* be the block containing it. We sum the inner tensor of B_* using $C_{|B_*|}(j)$ and every other inner tensor using $R_{|B|}(j)$. With the intermediate input of B_* fixed, the remaining outer sum uses the second line of (8.10). This is the rule for contractions with a fixed input in Lemma 5.3, applied with counting volumes. The partition with one block contributes at most $M^{-3}(1 +$

$C_M e_j) C_s(j)$, and, by the induction bounds for the lower orders, the other partitions contribute at most

$$M^{-3} B_{M,s} \epsilon_{j+1} (\epsilon_0/\epsilon_j)^3.$$

Multiplying the recurrence by $(\epsilon_{j+1}/\epsilon_0)^3 = M^3(\epsilon_j/\epsilon_0)^3$ cancels the factor M^{-3} exactly, and we find

$$\widehat{C}_s(j+1) \le (1 + C_M e_j)\widehat{C}_s(j) + B_{M,s}\epsilon_{j+1}.$$

This is why exactly one volume factor appears.

Both recurrences start at zero. Expanding the products of their first-order factors and using (8.9), we find that their solutions are bounded by

$$B_{M,s} \sum_{\ell<j} \epsilon_{\ell+1} \exp\Big(C_M \sum_{\ell<j} e_\ell\Big) \le B_{M,s} \frac{M}{M-1} \exp\Big[C_M\Big(\frac{2\varepsilon_*}{M^2-1} + \frac{2a_* M}{M-1}\Big)\Big].$$

There are finitely many partitions for each fixed s, so the induction closes at every fixed order and gives bounds for $R_s(j)$ and $\widehat{C}_s(j)$ that are uniform in j, d, and n. The same argument applies with any fixed input, either by selecting its block or by the symmetry of derivatives.

Step 6: Spatial weights. A nonzero derivative coefficient of $\Phi_j(c)$ has all its inputs within distance $C_M\epsilon_j$ of c, by the locality in Lemma 8.10. Joining each input to c by a shortest periodic path gives a connected network of length at most $sC_M\epsilon_j \le sC_M$, and the length of a shortest connecting network is no larger. Thus $\mathrm{t} \le sC_M\epsilon_j$ in final-coarse units, whereas $\mathrm{t}_j = \epsilon_j^{-1}\mathrm{t} \le sC_M$ in current-output units. Either weight $e^{a_{\mathrm{sp}}\mathrm{t}}$ or $e^{a_{\mathrm{sp}}\mathrm{t}_j}$ multiplies each sum by at most $e^{a_{\mathrm{sp}}sC_M}$, proving both versions of the weighted bounds. Repeated inputs are ordinary terms of these ordered sums, and all orientations and periodic identifications were included in the one-step bounds. Multiplying the column bound by $\epsilon_j^3/\epsilon_0^3$ gives the volume-normalized column sum of Definition 4.6. This completes the proof. □

PART III

Classical background fields

SECTION 9

Constraint coordinates and local gauge fixing

Fix an odd integer $M \geq 3$, integers $n \geq 4$ and $d \geq 0$, and set $\epsilon = M^{-d}$. Let b be a configuration close to the identity on the coarse lattice of side n. In this section we describe the configurations near the identity on the fine lattice of side $M^d n$ whose image under d steps of the gauge-corrected averaging of Lemma 8.10 is b, choose a gauge compatible with this constraint, and prove that the Wilson action is uniformly strictly convex in the remaining coordinates. The statements are deterministic and finite-dimensional, and all constants are independent of d and n. The existence of the constrained minimizer (the classical background field) is proved in Section 10 (Theorem 10.6); here we show only that a critical point in the chart is unique and is a strict minimum there.

We explain the construction of the coordinates. With background connection $U_0 = I$, write $\mathsf{F}_d(A)$ for the logarithm of the output of d gauge-corrected averaging steps applied to the fine field $e^{\epsilon A}$. By Lemmas 8.1 and 8.10, $\mathsf{F}_d(A) = Q_d A + \mathscr{E}_d^{\mathrm{av}}(A)$, where Q_d is the linear average of physical potentials and $\mathscr{E}_d^{\mathrm{av}}$ is quadratically small. If H_d is a linear right inverse of Q_d, then for a fine field w and a coarse field η,

$$\mathsf{F}_d(w - H_d\eta) = Q_d w - \eta + \mathscr{E}_d^{\mathrm{av}}(w - H_d\eta).$$

The constraint $\mathsf{F}_d(w - H_d\eta) = Q_d w$ reduces to the coarse fixed-point equation $\eta = \mathscr{E}_d^{\mathrm{av}}(w - H_d\eta)$, which we solve by a contraction; the fine parameter $(I - H_d Q_d)w$ is unchanged. For the contraction to be uniform in d, the right inverse must have exponentially decaying kernels, with bounds on the potential and its first differences. Further, the counting column sums of the derivatives of $\mathscr{E}_d^{\mathrm{av}}$, which are of size ϵ^3, must cancel the counting column sums of H_d, which are of size ϵ^{-3}. We take for H_d the minimizing linear extension of Bałaban [10, 11], translated to our centered averages.

The gauge condition is $\mathsf{P}_d \operatorname{div}_\epsilon A = 0$, where P_d is the orthogonal projection onto $\mathscr{L}_\epsilon \ker Q'_d$, $\mathscr{L}_\epsilon$ is the positive scalar Laplacian, and Q'_d is the scalar block average. At $(A, \lambda) = (0, 0)$, the derivatives with respect to the gauge parameter λ of the divergence map and of the logarithmic coarse gauge map are $-\mathscr{L}_\epsilon \lambda$ and $Q'_d \lambda$, respectively. Thus $\ker Q'_d$ consists of the infinitesimal parameters preserving the coarse gauge at the identity, and their divergence variations span $\mathscr{L}_\epsilon \ker Q'_d$. The corresponding nonlinear equations, with prescribed coarse gauge, are solved in Theorem 9.9. This projected condition is the one used by Bałaban [10, (1.39)–(1.44), pp. 24–25]; it is not the ordinary condition $\operatorname{div}_\epsilon A = 0$.

After fixing the lattices and norms in Section 9.1, we transfer Bałaban's linear estimates through finite covers (Section 9.2) and construct H_d (Theorem 9.3). We then construct the constraint coordinates (Theorem 9.6), prove the constrained Hessian bound (Theorem 9.7), solve the scalar and nonlinear gauge equations (Lemma 9.8 and Theorem 9.9), and give the trial configuration (Corollary 9.10); the plaquette estimates used throughout are collected in Lemma 9.4.

The inputs are the weighted product and inverse estimates of Lemma 5.3, the inverse branch of the averaging constraint and its domain in Lemma 6.3, the identification of the coordinates of a configuration with small curvature with that branch in Lemma 7.2, the linear identities and nonlinear estimates of Lemmas 8.1, 8.10, and 8.11, and Bałaban's estimates for the linear theory [10, 11, 13]. The results are used in Section 10 to construct the background field and prove its symmetries and the nested minimization identity for the effective classical action (Theorems 10.6 and 10.10 and Lemma 10.8), in Section 11, and in the quartic expansion of Theorem 36.2.

Our choice of gauge representatives and our nonlinear solution of the averaging constraint follow the background-field program of Bałaban [14, 16], in particular his strategy of fixing the gauge and linearizing the constraint [14, Theorem 2, p. 83], [16, Theorem 1, p. 279]. The linear estimates are taken from [10, 11, 13] after the identifications checked below. The nonlinear right inverse, the coordinates on the fibers, and the gauge fixing with prescribed coarse gauge are constructed here for the centered averaging map. For comparison, the axial gauge of Dimock [70] concerns an Abelian theory and does not supply these non-Abelian statements.

9.1 Lattices, scalar products, and norms

Fix an odd integer $M \geq 3$, a depth $d \geq 0$, and a coarse side $n \geq 4$. We set $m = M^d$ and $\epsilon = M^{-d}$, and we use the fine torus $\epsilon\mathbb{Z}^3/(n\mathbb{Z})^3$ and the coarse torus $\mathbb{Z}^3/(n\mathbb{Z})^3$; lengths are measured in units in which the coarse spacing is one. The vector average $Q_d = K_m/m$ and the scalar average $Q'_d = \mathsf{A}_m$ are those of Lemma 8.1. These definitions are used for every torus side that occurs, including the covers of Section 9.2.

The coefficient of a positive fine link $e = (x, x + \epsilon e_i)$ is located at x, and that of a fine plaquette at its lower vertex. Changing these locations within their cells changes each spatial estimate by a constant independent of d and n. Scalar, link, and plaquette scalar products on the fine torus are ϵ^3 times the counting sums, each positive orientation counted once, and on the coarse torus they are counting sums. We write

$$(\nabla_{\epsilon,i} f)(x) = \frac{f(x + \epsilon e_i) - f(x)}{\epsilon}, \qquad \operatorname{div}_\epsilon A(x) = \sum_i \frac{A_i(x - \epsilon e_i) - A_i(x)}{\epsilon},$$

$$(\mathrm{d}_\epsilon A)_{ij}(x) = \nabla_{\epsilon,i} A_j(x) - \nabla_{\epsilon,j} A_i(x), \qquad i < j,$$

and $\mathscr{L}_\epsilon = \operatorname{div}_\epsilon \nabla_\epsilon = -\Delta_\epsilon \geq 0$. Then $\operatorname{div}_\epsilon = \nabla_\epsilon^\sharp$, where $^\sharp$ always denotes the adjoint with respect to the scalar products just described; adjoints between the fine and the coarse torus contain the ratio of the cell volumes. These formulas fix the sign of the scalar Laplacian and of the linearized gauge action. We use the color norm of Section 4.1, and set

$$\|v\|_\epsilon^2 = \epsilon^3 \sum_e |v_e|^2, \qquad \|v\|_{H^1_\epsilon}^2 = \|v\|_\epsilon^2 + \sum_\mu \|\nabla_{\epsilon,\mu} v\|_\epsilon^2.$$

Let P_d be the orthogonal projection onto $\mathscr{L}_\epsilon \ker Q'_d$, and let

$$\mathcal{V}_d = \{v : Q_d v = 0,\ \mathsf{P}_d \operatorname{div}_\epsilon v = 0\}.$$

Unless stated otherwise, constants in this section depend only on M and on the following fixed choices. In an estimate on a ball of radius r inside a larger analytic ball of radius $R > r$, constants may depend on the chosen radii and on $R - r$. In a weighted composition estimate they may also depend on the input and output exponential decay rates and on their positive difference. This is the dependence described in item (2) of Definition 4.14. They do not depend on d, n, L, γ, or on the integer J that fixes the length of a renormalization group step. Additional parameters are listed where they occur.

Lemma 9.1. *Let K be a matrix from a field space on a lattice of spacing b_{in} to a field space on a lattice of spacing b_{out}, written in counting coordinates. If*

$$R = \sup_x \sum_y \|K(x,y)\|, \qquad C = \sup_y \frac{b_{\mathrm{out}}^3}{b_{\mathrm{in}}^3} \sum_x \|K(x,y)\|,$$

then the operator norm of K from ℓ^2 with weight b_{in}^3 to ℓ^2 with weight b_{out}^3 is at most $\sqrt{RC}$. The same holds for each component of a left gradient of K.

Proof. For each output, by the triangle inequality and the Cauchy–Schwarz inequality, we have

$$|(Ku)(x)|^2 \leq \Big(\sum_y \|K(x,y)\|\Big) \sum_y \|K(x,y)\| |u(y)|^2.$$

Multiplying by b_{out}^3, summing over x, and exchanging the finite sums, we find that the coefficient of $b_{\mathrm{in}}^3 |u(y)|^2$ is at most RC. The argument applies to color blocks, and a finite collection of gradient components changes only a dimensional constant. □

9.2 Transfer of periodic kernels through finite covers

Bałaban's linear estimates are proved for periodic lattices whose periods are compatible with an auxiliary system of blocks used in the proofs. A general side $n \geq 4$ need not have this property. We apply the estimates on a finite periodic cover whose side is a

suitable multiple of n, and transfer them to the torus of side n by the following lemma. The transfer uses uniqueness of the solutions and deck invariance, and it does not use any estimate for an inverse.

Lemma 9.2. *Fix an odd $M \geq 3$. Fix, as functions of M, integers R_0 and M_0, which are powers of M, and an integer K_{loc} such that $M_{\mathrm{loc}} = K_{\mathrm{loc}} R_0 M_0 \geq M_1(3, M)$, where $M_1(3, M)$ is the lower bound for the auxiliary localization block size in [13, Theorem 3.1, p. 397], and the subdivisions and enlarged localization cubes are those of Section 3C, pp. 408–409 of that paper. Let $k = 6M_{\mathrm{loc}}$. Then the following statements hold for all integers $n \geq 4$ and $d \geq 0$.*

1. Geometry. *The periodic lattice of side kn at mesh $\epsilon = M^{-d}$, called the cover, satisfies $M_{\mathrm{loc}} \mid kn$ and $kn \geq 24M_{\mathrm{loc}}$. At every mesh $\epsilon_j = M^{j-d}$, $0 \leq j \leq d$, the whole cover is tiled by the auxiliary localization blocks of [13, Section 3C], and its localization cubes, of side at most $18M_{\mathrm{loc}}$, have injective lifts to coordinate cubes. With the domain at every scale equal to the whole torus, the separation conditions of [11, (2.1)–(2.4), p. 224] are vacuous.*
2. Projection and constrained space. *Let π^* map fields on the torus of side n to fields on the cover by periodic repetition, and let the deck group $\mathcal{D} = (\mathbb{Z}/k\mathbb{Z})^3$ act on the cover by translations by multiples of n. The averages Q_d and Q'_d, the physical differences, and their adjoints commute with π^*, and*

$$\mathsf{P}_d^{(kn)} \pi^* = \pi^* \mathsf{P}_d^{(n)}, \qquad \mathcal{V}_d^{(kn)} \cap \operatorname{Fix} \mathcal{D} = \pi^* \mathcal{V}_d^{(n)}.$$

3. Kernels. *Let T be the solution operator of one of the following linear problems at the identity connection, on the torus of side $r \in \{n, kn\}$: the scalar problem $(\mathscr{L}_\epsilon + \alpha (Q'_d)^\sharp Q'_d)\lambda = f$ with $\alpha > 0$; the scalar system with projected Laplacian and prescribed scalar average; the minimization of $\|\mathrm{d}_\epsilon A\|_\epsilon^2$ under the constraints $Q_d A = u$ and $\mathsf{P}_d \operatorname{div}_\epsilon A = 0$; or a problem $A \in \mathcal{V}_d$, $\langle \mathrm{d}_\epsilon A, \mathrm{d}_\epsilon v \rangle_\epsilon = \langle f, v \rangle_\epsilon$ for $v \in \mathcal{V}_d$, under the assumption that the curl form is positive definite on $\mathcal{V}_d$. Suppose that the problem has a unique solution on both tori. Then $T^{(kn)} \pi^* = \pi^* T^{(n)}$, and their matrices in the coordinate bases satisfy*

$$T^{(n)}(e, f) = \sum_{h \in \mathcal{D}} T^{(kn)}(\widetilde{e}, \widetilde{f} + nh),$$

where $\widetilde{e}, \widetilde{f}$ are fixed lifts; any physical source volume is already contained in these matrices. The same formula holds for the first physical differences of the output. In particular, for every $b > 0$,

$$\sum_f e^{b \operatorname{dist}_n(e,f)} \|T^{(n)}(e, f)\| \leq \sum_{f'} e^{b \operatorname{dist}_{kn}(\widetilde{e}, f')} \|T^{(kn)}(\widetilde{e}, f')\|,$$

and a bound $Ce^{-b \operatorname{dist}_{kn}}$ for the response to a source supported in one coarse block descends to the torus of side n with C replaced by $k^3 C$.

In the applications, estimates proved on the cover are transferred to the torus of side n by part (3), uniformly in d; each application checks the constraints and the uniqueness required there.

Proof. The auxiliary data are fixed before n and d. The integers R_0 and M_0 are powers of M chosen large enough for the auxiliary construction, and $M_{\mathrm{loc}} = K_{\mathrm{loc}} R_0 M_0$ is an integer multiple with $M_{\mathrm{loc}} \geq M_1(3, M)$. This is the subdivision of [13, Section 3C, pp. 408–409]: an auxiliary localization block is a union of $R_0 M_0$-blocks, and a localization cube is a union of auxiliary localization blocks. The required lower bounds depend only on M and the dimension three, and these choices can be made as functions of M; none uses n, d, a coupling, or J.

Let $n_{\mathrm{cov}} = kn = 6M_{\mathrm{loc}} n$. It is a multiple of M_{loc} and $n_{\mathrm{cov}} \geq 24 M_{\mathrm{loc}}$. At the mesh $\epsilon_j = M^{j-d}$ the number of sites in each coordinate direction is $n_{\mathrm{cov}}/\epsilon_j = knM^{d-j}$, which is divisible by M_{loc} for $0 \leq j \leq d$. The whole torus at every scale has a periodic partition into auxiliary localization blocks, and the subdivision into $R_0 M_0$-blocks is periodic as well. We take the domain sequence $\Omega_0 = \Omega_1 = \cdots = \Omega_d$ equal to the whole fine torus, with $A_j = \varnothing$ for $j < d$ and A_d the whole coarse torus. This whole-torus specialization is allowed in [11, (2.1)–(2.4), p. 224], and each complement in the separation condition (2.2) there is empty. The enlarged localization cubes of the background-field construction have side $(2 + 2r) M_{\mathrm{loc}} \epsilon_j$ with $r \leq 8$ [13, Section 3C, pp. 408–409 and 415]. This side is at most $18 M_{\mathrm{loc}} < n_{\mathrm{cov}}$, and each cube has an injective lift to a coordinate cube of the cover. The tori of [10, pp. 17–18] have general rectangular periods $2L_\mu$, and we take $L_\mu = n_{\mathrm{cov}}/2 = 3M_{\mathrm{loc}} n$; that definition does not require the period to be a power of M. The auxiliary localization blocks are used only in the proofs of the cited estimates and do not enter the definitions of the averages, the projection, or the solution operators. A deck translation need not preserve the auxiliary partition; the invariance of the operators is proved next from the averaging partitions and their equations.

For each scalar, link, or plaquette space, $\pi^* u$ is the field u repeated in the k^3 translates of a fundamental cell of side n, and the image of π^* is $\operatorname{Fix} \mathcal{D}$. Since the fine meshes agree on both tori, we have

$$\langle \pi^* u, \pi^* v \rangle_{\epsilon, kn} = k^3 \langle u, v \rangle_{\epsilon, n}, \tag{9.1}$$

and the coarse counting products satisfy the same identity. Forward differences, their adjoints, and the curl are given by the same periodic finite stencils on both tori, and they commute with π^*. The formulas for Q_d and Q'_d in Lemma 8.1 use the same centered block and segment at each coarse vertex. A deck translation moves the fine lattice by nm sites and preserves the averaging partitions at every scale. Repeating an input repeats the outputs. The adjoints for the scalar products of Section 9.1 also commute with π^*: if $Q_d(c, e)$ denotes the matrix of Q_d in the coordinate bases, then

$$(Q_d^\sharp b)(e) = \epsilon^{-3} \sum_c Q_d(c, e)^* b(c), \qquad ((Q'_d)^\sharp b)(x) = b(Y) \quad \text{for } x \text{ in the block of } Y,$$

and the occurrences and volume factors are the same on the cover. The translation between the centered and the lower-corner averages commutes with deck translations and with π^* for the same reason.

For the projection, define $S^{(r)} = \mathscr{L}_\epsilon^{(r)} \ker Q_d'^{(r)}$ for $r = n, kn$, and $\mathcal{A}_\mathcal{D} = k^{-3} \sum_{h \in \mathcal{D}} \tau_{nh}$, where τ_{nh} is the translation by nh. The deck translations are unitary, preserve $S^{(kn)}$, and commute with $\mathscr{L}_\epsilon$ and Q_d'. First, $\pi^* S^{(n)} \subset S^{(kn)} \cap \operatorname{Fix} \mathcal{D}$. Conversely, let $f = \mathscr{L}_\epsilon^{(kn)} \lambda$ with $Q_d'^{(kn)} \lambda = 0$ and $\tau_{nh} f = f$ for all h. Then

$$Q_d'^{(kn)} \mathcal{A}_\mathcal{D} \lambda = 0, \qquad \mathscr{L}_\epsilon^{(kn)} \mathcal{A}_\mathcal{D} \lambda = f.$$

The averaged preimage is invariant, hence equal to $\pi^* \lambda_0$ for some λ_0, and since π^* is injective, we have $Q_d'^{(n)} \lambda_0 = 0$. We conclude that

$$S^{(kn)} \cap \operatorname{Fix} \mathcal{D} = \pi^* S^{(n)}. \tag{9.2}$$

Since $S^{(kn)}$ and its orthogonal complement are invariant under the deck group, the projection $\mathsf{P}_d^{(kn)}$ commutes with every deck translation. For a fine scalar u, $\mathsf{P}_d^{(kn)} \pi^* u$ lies in the left side of (9.2), and $\pi^* u - \mathsf{P}_d^{(kn)} \pi^* u$ is orthogonal to $\pi^* S^{(n)}$. By (9.1), the unique vector with these two properties is $\pi^* \mathsf{P}_d^{(n)} u$. This proves

$$\mathsf{P}_d^{(kn)} \pi^* = \pi^* \mathsf{P}_d^{(n)}. \tag{9.3}$$

Applying it to the divergence and using the commutation for Q_d, we obtain the identity for the invariant part of $\mathcal{V}_d$.

We next consider one of the problems in part (3). For the scalar problem with penalty $\alpha > 0$, the weak equation is

$$\langle \nabla_\epsilon \lambda, \nabla_\epsilon \phi \rangle_\epsilon + \alpha \langle Q_d' \lambda, Q_d' \phi \rangle = \langle f, \phi \rangle_\epsilon \quad \text{for every fine scalar } \phi,$$

and all its operators commute with the deck group. For the scalar system with prescribed scalar average and projected Laplacian, the same commutation follows from (9.3). For the minimizing extension, the affine constraints are $Q_d A = u$ and $\mathsf{P}_d \operatorname{div}_\epsilon A = 0$, and the stationarity equation is $\langle \mathrm{d}_\epsilon A, \mathrm{d}_\epsilon v \rangle_\epsilon = 0$ for every $v \in \mathcal{V}_d$. For the constrained curl problem, if the curl form is positive definite on the finite-dimensional space $\mathcal{V}_d$, its Gram matrix in an orthonormal basis of $\mathcal{V}_d$ is invertible, and the problem is uniquely solvable.

On the cover with repeated data, a deck translate of a solution solves the same equation with the same data. By uniqueness, the solution is invariant and equals $\pi^* u_0$ (or $\pi^* \lambda_0$). The identities above show that the descended field satisfies the constraints and the coarse data on the torus of side n. For every test field on the torus of side n, its repetition is a test field on the cover, and by (9.1) every quadratic and source pairing is multiplied by the same factor k^3. Canceling this factor, we obtain the variational equation on the torus of side n, and uniqueness identifies the solution operator applied to the data. This proves $T^{(kn)} \pi^* = \pi^* T^{(n)}$; for the scalar system the same argument uses the commutation

in its two equations directly. An invertible coarse Schur matrix formed from commuting factors also commutes with π^*, and applying it to both sides gives the corresponding identity for its inverse. When a scalar source is naturally restricted to $\operatorname{Ran} \mathsf{P}_d$, we use the source $\mathsf{P}_d f$ for an arbitrary fine field f. This fixes its matrix in the coordinate basis on the whole coordinate space, and (9.3) preserves this convention.

Finally we write the operators in the coordinate bases. For an input u on the torus of side n, expanding $(T^{(kn)}\pi^* u)(\widetilde{e})$ first over a fundamental set of lifts of the sources and then over their k^3 deck images, we obtain

$$\sum_f \Big(\sum_{h \in \mathcal{D}} T^{(kn)}(\widetilde{e}, \widetilde{f} + nh) \Big) u(f).$$

Comparing coefficients proves the formula for $T^{(n)}$. A physical source volume is part of the matrix coefficient and is the same on both tori. Physical first differences of the output commute with repetition and with finite sums, which gives their formula. For every rate $b > 0$,

$$\begin{aligned} \sum_f e^{b \operatorname{dist}_n(e,f)} \| T^{(n)}(e,f) \| &\le \sum_{f,h} e^{b \operatorname{dist}_{kn}(\widetilde{e}, \widetilde{f}+nh)} \| T^{(kn)}(\widetilde{e}, \widetilde{f} + nh) \| \\ &= \sum_{f'} e^{b \operatorname{dist}_{kn}(\widetilde{e}, f')} \| T^{(kn)}(\widetilde{e}, f') \|, \end{aligned} \tag{9.4}$$

since the lifts of the sources partition the cover and periodic distances can only decrease on passing to the smaller torus. A weighted row bound on the cover descends with the same constant. A source in one coarse block has k^3 images on the cover, and by the triangle inequality for their responses, we obtain the bound $Ck^3 e^{-b \operatorname{dist}_n(e, \text{source block})}$. These arguments apply also to the first output differences. All these factors depend only on the auxiliary data, hence only on M. Neither uniqueness nor this transfer produces an estimate for first differences; such an estimate must be proved for the operator on the cover. □

The specialization of Bałaban's linear estimates. We use the estimates for the Gaussian theory at the identity connection collected in Table 9.1. They are applied on the cover of Lemma 9.2 in the following specialization, which we check once here and use in the proofs of Theorem 9.3 and Lemma 9.8. The geometry is periodic, with one region, equal to the whole torus at every scale, so that the separation conditions are vacuous by part (1) of Lemma 9.2. The coarse spacing is one, the fine scalar products carry the volume ϵ^3, and the coarse ones are counting sums. The background is the identity connection. The gradient of the Wilson action at the background vanishes, the regularity conditions [13, (3.35)–(3.36), p. 396] hold with zero left sides, and the smallness requirement of [13, Theorem 3.1] holds for every sufficiently small positive regularity parameter. The

averages in these papers are the lower-corner averages, which are related to ours by the translation T_d of Theorem 9.3. The auxiliary localization size M_{loc} is fixed as a function of M; it is distinct from the blocking factor M. In the statement of [11, Corollary 2.8, p. 249], the inequality is printed as (1.151); we refer to it by that printed number.

The link with gauge-corrected averaging is the linearization [12, (124), p. 36], used to define the vector compositions and scalar averages in [13, (3.15) and (3.19), p. 393]. At the identity background all transports are the identity and the correction terms in (124) vanish. In three dimensions the surviving one-step vector average is

$$(Q(I)A)(c) = M^{-4} \sum_{x \in B(c_-)} A([x, x']),$$

where x' is the translate of x by the coarse link c and $A([x, x'])$ is the oriented sum of the fine-link coefficients along the straight path. Iteration gives the lower-corner vector average in (9.5) before translation. The background vector operator [13, (3.26), p. 395] likewise reduces at $U = I$ to the Gaussian operator of [10, (1.69), p. 29].

Object here	Corresponding statements in Bałaban's papers	Relation
Q_d, Q'_d	lower-corner vector averages [10, (1.11), p. 19; (1.18), p. 20]; scalar average: definition immediately after (1.20), p. 20	$Q_d = Q_d^{\mathrm{pub}} \mathsf{T}_d$
P_d	projection R onto the Laplacian image of the kernel of the scalar average [10, (1.39)–(1.44), pp. 24–25]; the source's $P = I - R$ is complementary	conjugation by T_d
$\mathscr{D}_d^0$	vector operator Δ_a with $a = 1$, defined in [10, (1.69), p. 29]; lower bound [10, Proposition 1.1, (1.89)–(1.90), p. 33]	conjugation by T_d
H_d	minimizing extension [10, (1.60)–(1.69), pp. 28–29; (1.103), p. 34]; kernel bounds [11, Proposition 2.7, (2.149); Corollary 2.8, (2.150) and the bound printed as (1.151), p. 249]	$H_d = \mathsf{T}_d^{-1} H_d^{\mathrm{pub}}$
G, T, P_d (scalar)	scalar algebra [10, (1.43)–(1.45), pp. 25–26], [11, (2.17), p. 225]; bounds [13, Theorem 3.1, (3.42), p. 397; (3.47), p. 398; Theorem 3.2, (3.48), p. 398; (3.49), p. 399]	conjugation by T_d; penalty converted

Table 9.1. Linear estimates of Bałaban used in this section, at the identity connection, and their relation to the operators defined here.

9.3 The constrained linear extension

Theorem 9.3. *Use the lattices, scalar products, and operators of Section 9.1. Let* $\mathsf{T}_d A(x) = A(x - \epsilon t_m)$, *where* $t_m = ((m-1)/2, (m-1)/2, (m-1)/2)$ *in fine lattice units, and let* Q_d^{pub} *and* $(Q'_d)^{\mathrm{pub}}$ *be the lower-corner physical vector and scalar averages at the identity connection: the vector averages are defined in [10, (1.11), p. 19; (1.18), p. 20], and the scalar average immediately after (1.20), p. 20 of the same paper. Then the following statements hold.*

1. $Q_d = Q_d^{\mathrm{pub}} \mathsf{T}_d$ *and* $Q'_d = (Q'_d)^{\mathrm{pub}} \mathsf{T}_d$.
2. *Set*
$$\mathscr{D}_d^0 = \mathrm{d}_\epsilon^\sharp \mathrm{d}_\epsilon + \nabla_\epsilon \mathsf{P}_d \operatorname{div}_\epsilon + Q_d^\sharp Q_d.$$
There exist $c_0, C, a_0 > 0$, *depending only on* M, *such that*
$$\mathscr{D}_d^0 \geq c_0 (I + \nabla_\epsilon^\sharp \nabla_\epsilon), \qquad \|\mathrm{d}_\epsilon v\|_\epsilon^2 \geq c_0 \|v\|_{H^1_\epsilon}^2 \quad (v \in \mathcal{V}_d),$$
where $\nabla_\epsilon^\sharp \nabla_\epsilon$ *acts componentwise.*
3. *The map* Q_d *is onto, and the operator*
$$H_d = (\mathscr{D}_d^0)^{-1} Q_d^\sharp \big[Q_d (\mathscr{D}_d^0)^{-1} Q_d^\sharp \big]^{-1}$$
satisfies
$$H_d = \mathsf{T}_d^{-1} H_d^{\mathrm{pub}}, \qquad Q_d H_d = I, \qquad \mathsf{P}_d \operatorname{div}_\epsilon H_d = 0,$$
where H_d^{pub} *is the extension of [10, (1.60)–(1.64), pp. 28–29; (1.103), p. 34]. For each coarse field* u, $H_d u$ *is the unique minimizer of* $\|\mathrm{d}_\epsilon A\|_\epsilon^2$ *among the fields* A *with* $Q_d A = u$ *and* $\mathsf{P}_d \operatorname{div}_\epsilon A = 0$; *in particular* $\langle \mathrm{d}_\epsilon H_d u, \mathrm{d}_\epsilon v \rangle_\epsilon = 0$ *for* $v \in \mathcal{V}_d$. *Constant fields are preserved:* $H_d u = u$ *for every constant coarse field* u.
4. *The kernel of* H_d *satisfies*
$$|H_d(e,c)| + |\nabla_\epsilon H_d(e,c)| \leq C e^{-a_0 \operatorname{dist}(e,c)},$$
and for every $0 < a < a_0$
$$\sup_e \sum_c e^{a \operatorname{dist}(e,c)} \big(|H_d(e,c)| + |\nabla_\epsilon H_d(e,c)| \big) \leq C_a,$$
$$\sup_c \epsilon^3 \sum_e e^{a \operatorname{dist}(e,c)} \big(|H_d(e,c)| + |\nabla_\epsilon H_d(e,c)| \big) \leq C_a.$$
It follows that $\|H_d u\|_\infty + \|\nabla_\epsilon H_d u\|_\infty \leq C \|u\|_\infty$ *and* $\|H_d u\|_{H^1_\epsilon} \leq C \|u\|_2$.

The constant C_a *depends only on* M *and* a. *The translation* T_d *relates the linear averages at the identity; it does not relate nonlinear block averages.*

The positivity in part (2) is a form estimate. The kernel bounds in part (4) are a separate regularity estimate, which we take from [11]; they do not follow from the form estimate.

Proof. We treat $d \geq 1$ first and verify the case $d = 0$ at the end without the cited kernel estimate.

Step 1: The averages. We write a fine point as $x = \epsilon r'$, and let $\mathsf{I}_m = \{-(m-1)/2, \dots, (m-1)/2\}^3$. By Lemma 8.1,

$$(Q_d A)_i(Y) = m^{-4} \sum_{r \in \mathsf{I}_m} \sum_{s=0}^{m-1} A_i\big(Y + \epsilon(r + s e_i)\big), \qquad (Q'_d f)(Y) = m^{-3} \sum_{r \in \mathsf{I}_m} f(Y + \epsilon r). \tag{9.5}$$

There are m^4 vector occurrences, and a constant physical potential is mapped to itself. The lower-corner formula in [10, (1.11), p. 19; (1.18), p. 20] has coefficient $m^{-4} = \epsilon^4$ and $r \in \{0, \dots, m-1\}^3$. Subtracting the integer vector t_m maps this box bijectively onto I_m, which proves part (1), including at the periodic seams.

Since fine translations are unitary for the physical scalar products and commute with the physical differences, we have

$$\mathsf{P}_d = \mathsf{T}_d^{-1} \mathsf{P}_d^{\mathrm{pub}} \mathsf{T}_d, \qquad \mathscr{D}_d^0 = \mathsf{T}_d^{-1} (\mathscr{D}_d^0)^{\mathrm{pub}} \mathsf{T}_d. \tag{9.6}$$

Indeed, T_d^{-1} maps $\mathscr{L}_\epsilon \ker(Q'_d)^{\mathrm{pub}}$ onto $\mathscr{L}_\epsilon \ker Q'_d$. Since T_d is unitary and commutes with $\mathscr{L}_\epsilon$, the first identity follows by orthogonality. Taking adjoints, we obtain $Q_d^\sharp = \mathsf{T}_d^{-1} (Q_d^{\mathrm{pub}})^\sharp$, and the second identity follows term by term.

Step 2: The cited estimates on the cover. We apply the estimates of Table 9.1 first on the cover of side kn of Lemma 9.2, in the specialization described after that lemma; the penalty in the vector operator is one. The projected gauge of [10, (1.39)–(1.44), pp. 24–25] projects onto the Laplacian image of the kernel of the scalar average. By [10, Proposition 1.1, (1.89)–(1.90), p. 33], the vector operator with penalty one is bounded below by $c_0(I + \nabla_\epsilon^\sharp \nabla_\epsilon)$. The extension of [10, (1.60)–(1.64), pp. 28–29; (1.103), p. 34] is a right inverse of the vector average, has zero projected divergence, minimizes the curl form under these two constraints, and equals

$$\big((\mathscr{D}_d^0)^{\mathrm{pub}}\big)^{-1} (Q_d^{\mathrm{pub}})^\sharp \big[Q_d^{\mathrm{pub}} \big((\mathscr{D}_d^0)^{\mathrm{pub}}\big)^{-1} (Q_d^{\mathrm{pub}})^\sharp \big]^{-1}.$$

These statements concern one real color, and we apply them to each of our three colors. No restriction to mean zero is made on coarse data; the constant fields are kept. Conjugation by T_d gives the same statements for the centered operators on the cover.

Step 3: Coercivity on the torus of side n. Before transferring any regularity estimate, we prove the finite-dimensional statements on the torus of side n. The commutation of P_d, of the differences, and of the averages and their adjoints with π^* (Lemma 9.2) implies that $\mathscr{D}_d^{0,(kn)}\pi^* = \pi^*\mathscr{D}_d^{0,(n)}$. For every fine vector field v on the torus of side n, we then have

$$\langle v, \mathscr{D}_d^{0,(n)} v\rangle_{\epsilon,n} = k^{-3}\langle \pi^* v, \mathscr{D}_d^{0,(kn)}\pi^* v\rangle_{\epsilon,kn} \geq c_0 \|v\|^2_{H^1_{\epsilon,n}}, \tag{9.7}$$

where we used (9.1) also for each first difference. This shows that $\mathscr{D}_d^{0,(n)}$ is positive definite. For $v \in \mathcal{V}_d^{(n)}$, the averaging and projected divergence terms vanish, which leaves the curl coercivity in part (2).

Step 4: Construction and identification of the extension. For a coarse field u on the torus of side n, let $A_{\mathrm{cov}} = H_d^{(kn)}\pi^* u$ be the centered cover extension. The cover minimizer is unique: the difference of two minimizers lies in $\mathcal{V}_d^{(kn)}$; subtracting their stationarity identities tested on this difference shows that its curl vanishes, and by the coercivity on the cover the difference vanishes. Deck translations preserve both constraints and the curl energy, and they fix the data. By uniqueness, A_{cov} is invariant and equals $\pi^* A_0$. By the commutation relations, we have $Q_d^{(n)} A_0 = u$ and $\mathsf{P}_d^{(n)} \operatorname{div}_\epsilon A_0 = 0$. For every $v \in \mathcal{V}_d^{(n)}$, the repetition $\pi^* v$ lies in the constrained space on the cover, and by stationarity there we have

$$0 = \langle \mathrm{d}_\epsilon A_{\mathrm{cov}}, \mathrm{d}_\epsilon \pi^* v\rangle_{\epsilon,kn} = k^3 \langle \mathrm{d}_\epsilon A_0, \mathrm{d}_\epsilon v\rangle_{\epsilon,n}.$$

The energy at $A_0 + v$ exceeds the energy at A_0 by $\|\mathrm{d}_\epsilon v\|^2_{\epsilon,n}$, and the coercivity of Step 3 shows that A_0 is the unique constrained minimizer on the torus of side n. We set $H_d^{(n)} u = A_0$; this also shows that $Q_d^{(n)}$ is onto.

The formula for H_d also descends. For $r = n, kn$, let $G_0^{(r)} = (\mathscr{D}_d^{0,(r)})^{-1}$ and $C_0^{(r)} = Q_d^{(r)} G_0^{(r)} (Q_d^{(r)})^\sharp$. Both are positive definite, the latter because the average is onto. The commutation relations and uniqueness of these finite-dimensional inverses imply that

$$G_0^{(kn)}\pi^* = \pi^* G_0^{(n)}, \qquad (C_0^{(kn)})^{-1}\pi^* = \pi^* (C_0^{(n)})^{-1};$$

for example, the first identity follows by applying $\mathscr{D}_d^{0,(kn)}$ to both sides, and the second uses in addition the commutation of Q_d and $Q_d^\sharp$ with π^* and the positivity of C_0. The formula on the cover identifies the descended operator with $G_0^{(n)} Q_d^\sharp (C_0^{(n)})^{-1}$, and translation identifies it with $\mathsf{T}_d^{-1} H_d^{\mathrm{pub}}$ on the torus of side n. This proves all identities of part (3) without using any regularity estimate. If u is constant, the constant fine field equal to u satisfies both constraints and has zero curl, and by uniqueness $H_d u = u$.

Step 5: Kernel bounds. We apply [11, Proposition 2.7, (2.149), and Corollary 2.8, (2.150) and the bound printed as (1.151), p. 249] on the cover. In this whole-torus specialization

the receiving and source regions both have coarse spacing one. The source integration factor in (2.150) and the scale factors for the potential and its first gradient in the bound printed as (1.151) are equal to one. We obtain

$$|H_d^{\mathrm{pub},(kn)}(e,c)| + |\nabla_\epsilon H_d^{\mathrm{pub},(kn)}(e,c)| \le C e^{-a_0 \operatorname{dist}_{kn}(e,c)},$$

including the constant fields. For any positive auxiliary vector penalty α, its contribution on the constraint set $Q_d A = u$ is the constant $\alpha \|u\|_2^2$, so the minimizing extension and the projected gauge subspace are unchanged [10, (1.91)–(1.103), pp. 33–34]. Since $|\epsilon t_m| \le \sqrt{3}/2$, the centered cover kernel and its first gradient satisfy the same bound after multiplying C by at most $e^{a_0\sqrt{3}/2}$.

The minimization property, the uniqueness, and both constraints were verified in Step 4, and part (3) of Lemma 9.2 applies to the extension. For its matrix in the coordinate bases and the first physical differences of its output,

$$\begin{aligned} H_d^{(n)}(e,c) &= \sum_{h \in \mathcal{D}} H_d^{(kn)}(\widetilde{e}, \widetilde{c} + nh), \\ \nabla_\epsilon H_d^{(n)}(e,c) &= \sum_{h \in \mathcal{D}} \nabla_\epsilon H_d^{(kn)}(\widetilde{e}, \widetilde{c} + nh). \end{aligned} \tag{9.8}$$

Since every distance on the cover is at least the corresponding distance on the torus of side n, the pointwise estimate of part (4) follows with at most an additional factor k^3, which depends only on M. Weighted rows also descend without this factor, by (9.4).

To control the fine column sums, note that for a periodic mesh $b \le 1$ with period at least four and every $\eta > 0$,

$$\sup_x b^3 \sum_y e^{-\eta \operatorname{dist}(x,y)} \le C_\eta. \tag{9.9}$$

Indeed, choose representatives of the mesh points in a fundamental cube centered at x and assign to them disjoint mesh cubes. Distances vary by at most $\sqrt{3}b$ on such a cube, so the sum is bounded by $e^{\eta\sqrt{3}}$ times the integral of $e^{-\eta|z|}$ over $\mathbb{R}^3$. The same argument with $b = 1$ bounds the coarse counting sums, and the fixed number of orientations changes only a dimensional constant. Applying (9.9) with $\eta = a_0 - a$ to the pointwise estimate, we obtain the weighted rows and the weighted columns with the factor ϵ^3. The rows imply the supremum estimate, and Lemma 9.1, applied also to the gradient, gives the H_ϵ^1 estimate.

Step 6: Depth zero. At $d = 0$ we have $Q_0 = H_0 = I$ and $\mathsf{P}_0 = 0$ directly. Also $\sum_\mu \|\nabla_{1,\mu} v\|_2^2 \le 12 \|v\|_2^2$, so after reducing c_0 to at most $1/13$ the inequality of part (2) holds at unit mesh. □

9.4 Plaquette expansions

The following elementary estimates for the logarithm of a plaquette transport are used for the curvature and action bounds in Sections 9.6, 9.8, and 9.9. They involve only the four link variables of the plaquette and no differences of the field.

Lemma 9.4. *There exist $\rho_0, C > 0$ such that the following holds for every $\epsilon \in (0,1]$ and every fine plaquette p. For a complex link field A with $\|A\|_\infty \le \rho_0$, let $X_p(A) = \operatorname{Log}((e^{\epsilon A})_p)$ be the logarithm of the transport of the configuration $e^{\epsilon A}$ around p. For a complex link field z, write $Z_p = \sum_{e\in\partial p}|z_e|$. Then*

$$|X_p(A) - \epsilon^2(\mathrm{d}_\epsilon A)_p| \le C\epsilon^2 \sum_{e\in\partial p}|A_e|^2, \tag{9.10}$$

$$|DX_p(A)[z] - \epsilon^2(\mathrm{d}_\epsilon z)_p| \le C\epsilon^2\|A\|_\infty Z_p, \tag{9.11}$$

$$|D^2X_p(A)[z,z]| \le C\epsilon^2 Z_p^2. \tag{9.12}$$

Moreover, in the color normalization of Section 4.1, the function $\varphi(X) = q(e^X)$ satisfies

$$D\varphi(X) = \tfrac{1}{4}X + O(|X|^3), \qquad D^2\varphi(X) = \tfrac{1}{4}I + O(|X|^2)$$

for small X, where $D\varphi$ is the gradient with respect to the complex-bilinear extension of the color inner product $\langle X, Y\rangle = -2\operatorname{tr}(XY)$, and $q(e^X) \le C|X|^2$ for real X in the identity chart.

Proof. Regard the four link variables of p as independent, with signs fixed by the orientation of the boundary, and write

$$X_p = \operatorname{Log}\bigl(e^{\epsilon A_1}e^{\epsilon A_2}e^{-\epsilon A_3}e^{-\epsilon A_4}\bigr).$$

This is a holomorphic function of four variables on a fixed product ball that does not depend on ϵ. Its Taylor term of degree one is $\epsilon(A_1 + A_2 - A_3 - A_4) = \epsilon^2(\mathrm{d}_\epsilon A)_p$, and every monomial of degree at least two contains at least two factors ϵ. Since the series in these four variables converges absolutely on a larger ball, its tail is bounded by $C\epsilon^2(\sum_e|A_e|)^2$, the first derivative of the tail by $C\epsilon^2(\sum_e|A_e|)Z_p$, and the second derivative of X_p by $C\epsilon^2 Z_p^2$. There are four links, which gives (9.10), (9.11), and (9.12). For complex X, set $s = -2\operatorname{tr}(X^2)$ and use the holomorphic series

$$\varphi(X) = \sum_{k\ge1}\frac{(-1)^{k+1}s^k}{2^{2k}(2k)!} = \frac{s}{8} + O(s^2).$$

Differentiating it with the complex-bilinear color pairing gives the asserted formulas for $D\varphi$ and $D^2\varphi$. The absolute values in the remainder estimates are the usual nonnegative norms, not the bilinear quadratic form s. The last bound holds since $1 - \cos t \le t^2/2$. □

9.5 Local coordinates on constraint fibers

We use the following form of the contraction principle several times.

Lemma 9.5. *Let* B *be a closed ball in a finite-dimensional complex normed space, let* $\mathcal{O}$ *be an open subset of a finite-dimensional complex space, and let* $\Phi(t,x)$ *be holomorphic on* $\mathcal{O}\times\mathcal{U}$ *for an open set* $\mathcal{U}\supset\mathsf{B}$*. Suppose that* $\Phi(t,\cdot)$ *maps* B *into itself and is Lipschitz on* B *with constant* $\kappa<1$*, uniformly in* $t\in\mathcal{O}$*. Then for every* $t\in\mathcal{O}$ *there is a unique fixed point* $x(t)\in\mathsf{B}$*, and* $t\mapsto x(t)$ *is holomorphic. If, in addition,* B *is invariant under complex conjugation and* $\Phi(\bar t,\bar x)=\overline{\Phi(t,x)}$ *in the real coordinates, then* $x(t)$ *is real for real* t*.*

Proof. We start the iteration $x_{k+1}(t)=\Phi(t,x_k(t))$ at a point $x_0\in\mathsf{B}$ that does not depend on t. Then $\|x_{k+1}-x_k\|\le\kappa^k\|x_1-x_0\|$, and the geometric sum shows that the iterates converge uniformly on $\mathcal{O}$ to a fixed point, which is unique in B since $\kappa<1$. Since the iterates are holomorphic and Cauchy's formula on coordinate polydiscs passes to their uniform limit, the fixed point is holomorphic. At a real parameter, the complex conjugate of the fixed point is another fixed point in B when B is invariant under conjugation and $\Phi(\bar t,\bar x)=\overline{\Phi(t,x)}$. By uniqueness, the fixed point is real. No constant in this argument depends on the dimension of the space. □

In the following theorem, w denotes a fine link field; it is unrelated to the normalized fluctuation variables of Section 12. For the sums of absolute derivative coefficients we use the conventions of Definition 4.6. For each fixed order, the row is the sum over all inputs with the output fixed, and the column is the sum over the output and the other inputs with one input fixed. Both are weighted by $e^{a\mathsf{t}}$, where t is the length of a shortest connecting network of the locations, measured in units in which the coarse spacing is one. A counting column sums with counting measure over fine outputs; a column with fine outputs summed with the weight ϵ^3 is volume-normalized.

Theorem 9.6. *Use the geometry and operators of Theorem 9.3, and let* $V_d(A)$ *be the gauge-corrected iteration of Lemma 8.10 with background connection* $U_0=I$*, applied to the fine field* $e^{\epsilon A}$*. Write*

$$\mathsf{F}_d(A)=\operatorname{Log}V_d(A)=Q_dA+\mathscr{E}_d^{\mathrm{av}}(A),\qquad \mathscr{E}_d^{\mathrm{av}}(0)=D\mathscr{E}_d^{\mathrm{av}}(0)=0.$$

There exist $r_*,\theta_*,C>0$*, independent of* d *and* n*, such that the following statements hold.*

1. The remainder. F_d *is holomorphic on* $\|A\|_\infty<r_*$*, and*

$$\|\mathscr{E}_d^{\mathrm{av}}(A)\|_\infty\le C\|A\|_\infty^2,\qquad \|D\mathscr{E}_d^{\mathrm{av}}(A)\|_{\mathrm{row}}\le C\|A\|_\infty,$$
$$\|D\mathscr{E}_d^{\mathrm{av}}(A)\|_{\text{counting col}}\le C\epsilon^3\|A\|_\infty.$$

These bounds, and the first-order bounds below, also hold with one fixed positive exponential weight in the distance, on fixed smaller domains.

2. An analytic right inverse. *The radii can be chosen so that* $2\|H_d\|_{\infty\to\infty}\theta_* < r_*$ *and so that, for every complex coarse field* θ *with* $\|\theta\|_\infty < \theta_*$*, the equation*

$$q_d(\theta) = \theta - \mathscr{E}_d^{\mathrm{av}}\big(H_d q_d(\theta)\big)$$

has a unique solution in the coarse ball of radius $2\theta_*$*, and* $\|q_d(\theta)\|_\infty \le 2\|\theta\|_\infty$*. The map* $A_{\mathrm{sec},d}(\theta) = H_d q_d(\theta)$ *is holomorphic, real for real* θ*, and satisfies*

$$\mathsf{F}_d(A_{\mathrm{sec},d}) = \theta, \qquad \|A_{\mathrm{sec},d}\|_\infty + \|\nabla_\epsilon A_{\mathrm{sec},d}\|_\infty \le C\|\theta\|_\infty, \qquad \|q_d(\theta)\|_2 \le C\|\theta\|_2.$$

For every fixed $s \ge 1$*, the sums of the absolute derivative coefficients of order* s *of* $A_{\mathrm{sec},d}$*, and of* $\nabla_\epsilon A_{\mathrm{sec},d}$*, with the output fixed or with any one coarse input fixed, are bounded by* C_s *with the weight* $e^{a_s \mathsf{t}}$ *for some* $a_s > 0$*; in the sums with a coarse input fixed, the fine outputs are summed with the weight* ϵ^3*. The constants* C_s *and* a_s *depend only on* s *and on the radii and decay rates, as in the convention of Section 9.1.*

3. Coordinates reducing the constraint to its linear part. *There exist* $r_{\mathrm{st}}, r_D > 0$ *with* $r_{\mathrm{st}} < r_*/4$ *such that, for every fine field* w *with* $\|w\|_\infty < r_{\mathrm{st}}$*, there is a unique coarse correction* $\mathscr{D}_d(w)$ *with* $\|\mathscr{D}_d(w)\|_\infty < r_D$ *satisfying*

$$\mathscr{D}_d(w) = \mathscr{E}_d^{\mathrm{av}}\big(w - H_d \mathscr{D}_d(w)\big).$$

It is holomorphic in w*, and* $\Xi_d(w) = w - H_d\mathscr{D}_d(w)$ *satisfies*

$$\mathsf{F}_d(\Xi_d(w)) = Q_d w.$$

Further, $\|\mathscr{D}_d(w)\|_\infty \le C\|w\|_\infty^2$*, the row norm of* $D\mathscr{D}_d(w)$ *is at most* $C\|w\|_\infty$*, and its counting column norm is at most* $C\epsilon^3\|w\|_\infty$*. For every fixed* $s \ge 2$*, the derivatives of order* s *of* $\mathscr{D}_d$ *have row sums at most* C_s *and counting column sums, with any one input fixed, at most* $C_s\epsilon^3$*, with a weight* $e^{a_s\mathsf{t}}$*,* $a_s > 0$*. Even if* w *has no control of its differences,*

$$\|\Xi_d(w) - w\|_\infty + \|\nabla_\epsilon(\Xi_d(w) - w)\|_\infty \le C\|w\|_\infty^2.$$

4. Coordinates on the fibers. *There exists* $0 < r_{\mathrm{fib}} < r_{\mathrm{st}}$*, independent of* d *and* n*, such that every fine field* A *with* $\|A\|_\infty < r_{\mathrm{fib}}$ *and* $\mathsf{F}_d(A) = \theta$*,* $\|\theta\|_\infty < \theta_*$*, can be written uniquely as*

$$A = \Xi_d(a + H_d\theta), \qquad a \in \ker Q_d,$$

namely with $a = (I - H_dQ_d)A$*, and then* $\|a + H_d\theta\|_\infty < r_{\mathrm{st}}$*. Conversely,* $\mathsf{F}_d(\Xi_d(a + H_d\theta)) = \theta$ *for every* $a \in \ker Q_d$ *with* $\|a + H_d\theta\|_\infty < r_{\mathrm{st}}$*. Moreover* $\mathsf{P}_d \operatorname{div}_\epsilon \Xi_d(a + H_d\theta) = \mathsf{P}_d \operatorname{div}_\epsilon a$*, so restricting* a *to* $\mathcal{V}_d$ *gives exactly the fields of this form that satisfy* $\mathsf{P}_d \operatorname{div}_\epsilon A = 0$*.*

The configuration with prescribed image under T_M^d obtained from $A_{\mathrm{sec},d}$, and its action, are described in Corollary 9.10.

The following list summarizes the roles of the radii in Theorem 9.6; it does not replace the inequalities in the statement.

Radius	Domain or role
r_*	Fine input ball on which F_d is holomorphic and the remainder bounds of part (1) hold.
θ_*, $2\theta_*$	Coarse target ball, and coarse ball in which $q_d(\theta)$ is the unique fixed point.
r_{st}, r_D	Fine parameter ball on which Ξ_d is defined, and coarse ball in which $\mathscr{D}_d(w)$ is the unique correction.
r_{fib}	Fine ball of solutions of $\mathsf{F}_d(A) = \theta$ covered by the coordinates of part (4); their parameters lie in the ball of radius r_{st}, but the image of that whole parameter ball is not claimed to lie in the ball of radius r_{fib}.

Proof. We first extract the quadratic remainder of F_d from Lemma 8.11, then solve the two coarse fixed-point equations by Lemma 9.5, and finally prove the bounds for derivatives of higher order by induction over the order.

Step 1: The remainder. We apply Lemma 8.10 with background connection $U_0 = I$, and denote the intermediate potentials by A_j. Since $\epsilon_d = 1$, the last potential is $\mathsf{F}_d(A)$. At the identity every background curvature vanishes, and the transported segment averages K_j are the averages K_M/M of physical potentials, whose product is Q_d by Lemma 8.1. Also

$$\sum_{j<d} \epsilon_{j+1} \|A_j\|_\infty \le 2\|A\|_\infty \frac{M}{M-1}.$$

Using the telescope of Lemma 8.11 with $j = d$, we obtain

$$\|D\mathsf{F}_d(A) - Q_d\|_{\mathrm{row}} \le C\|A\|_\infty, \qquad \|D\mathsf{F}_d(A) - Q_d\|_{\text{counting col}} \le C\epsilon^3\|A\|_\infty. \tag{9.13}$$

All terms have a dependence radius at most C_M, and a fixed exponential weight multiplies their coefficients by at most a fixed factor. We shrink the complex ball once and call its radius r_*. Integrating $D\mathsf{F}_d(tA) - Q_d$ against A for $0 \le t \le 1$ gives the quadratic bound for $\mathscr{E}_d^{\mathrm{av}} = \mathsf{F}_d - Q_d$ and its two vanishing Taylor coefficients. By Lemma 8.11, every derivative of fixed order of this remainder has a bounded row sum and a counting column sum with a fixed input at most $C_s\epsilon^3$; subtracting the linear part affects only the first order.

Step 2: The coarse remainder. Fix a spatial rate smaller than the rate of the kernel of H_d. Let B_H be a common upper bound for $\|H_d\|_{\infty\to\infty}$ and for the weighted row and volume-normalized column norms of H_d given by Theorem 9.3, and set $r(q) = \mathscr{E}_d^{\mathrm{av}}(H_d q)$. Composing, with a further fixed reduction of the rate if necessary, we obtain

$$\|r(q)\|_\infty \le C_r\|q\|_\infty^2, \qquad \|Dr(q)\|_{\mathrm{row}} + \|Dr(q)\|_{\mathrm{col}} \le C_r\|q\|_\infty, \tag{9.14}$$

where here and below the norms in derivative estimates may carry the spatial weight. The column composition is explicit: with $E = D\mathcal{O}_d^{\mathrm{av}}(H_d q)$,

$$\sum_c \|(EH_d)(c,b)\| \le \sum_e \Big(\sum_c \|E(c,e)\|\Big)\|H_d(e,b)\| \le C\epsilon^3 \|q\|_\infty \sum_e \|H_d(e,b)\| \le C\|q\|_\infty.$$

The row composition uses the two row bounds. In the weighted version, $\mathrm{dist}(c,b) \le \mathrm{dist}(c,e) + \mathrm{dist}(e,b)$, and the same calculation applies. No power of the fine volume remains.

Step 3: The analytic right inverse. We enlarge C_r so that it controls both weighted norms, and choose θ_* so small that

$$2B_H\theta_* < r_*, \qquad 2C_r\theta_* \le \tfrac{1}{4}.$$

The map $q \mapsto \theta - r(q)$ sends the closed ball $\|q\|_\infty \le 2\theta_*$ into itself, since $\|r(q)\|_\infty \le 4C_r\theta_*^2 \le \theta_*/2$, and it is a contraction with constant at most $1/4$. By Lemma 9.5, there exists a unique fixed point, which is holomorphic and real for real θ. Since $r(0) = 0$, the derivative bound implies that $\|q\|_\infty \le \|\theta\|_\infty + \|q\|_\infty/4$, which gives $\|q\|_\infty \le 2\|\theta\|_\infty$.

In both the weighted row norm and the weighted column norm on the coarse lattice,

$$U^{-1} = (I + Dr(q))^{-1} = \sum_{k\ge 0} (-Dr(q))^k, \qquad \|U^{-1}\| \le \tfrac{4}{3},$$

by submultiplicativity (Lemma 5.3). Differentiating the fixed-point equation, we find $D_\theta q = U^{-1}$. Lemma 9.1, applied with counting measure on both coarse lattices, bounds its ℓ^2 operator norm by $4/3$. Integrating along $t\theta$ with $q(0) = 0$, we obtain $\|q(\theta)\|_2 \le C\|\theta\|_2$. The identity $\mathsf{F}_d(A_{\mathrm{sec},d}) = Q_d H_d q + r(q) = \theta$ follows from $Q_d H_d = I$ and the fixed-point equation, and the bounds for $A_{\mathrm{sec},d}$ and its first differences follow from the row bounds of the extension.

Step 4: Derivatives of higher order of the right inverse. For each fixed order s, we compose a remainder tensor of order $p \le s$ with p copies of H_d. With the output fixed, all copies use rows. With a coarse input fixed, the copy on the branch of that input has counting column sum at most $C\epsilon^{-3}$, which is canceled by the counting column sum $C_p\epsilon^3$ of the remainder tensor with a fine input fixed. All other branches use rows. The union of the connecting networks of the components connects all external locations, and the weight of a shortest connecting network is bounded by the product of the weights of the components, as in Lemma 5.3. We obtain finite bounds for $D^p r$, for $1 \le p \le s$, with any index fixed, at a common positive rate. For $s \ge 2$, differentiating the equation for q, we obtain

$$U\, D^s q[h_1,\dots,h_s] = - \sum_{\substack{\pi \text{ a partition of } \{1,\dots,s\} \\ |\pi|\ge 2}} D^{|\pi|} r(q) \Big[D^{|B|} q[h_i : i \in B] : B \in \pi \Big], \tag{9.15}$$

and every block in the sum has size smaller than s. Starting from the bound for Dq, multiplication by U^{-1} and induction on s give the bounds with the output or any coarse input fixed. The number of partitions and the constants up to order s depend only on s and on the radii and decay rates fixed above. A final composition with H_d, or with $\nabla_\epsilon H_d$, proves the bounds for the section; with a coarse input fixed and fine outputs, this last column uses $\epsilon^3 \sum_e$.

Step 5: The coordinates Ξ_d. We now construct Ξ_d without any assumption on the differences of w. Let C_E dominate the quadratic and derivative constants of Step 1 on a smaller ball. We choose $r_{\mathrm{st}} < r_*/4$ and set $r_D = 8C_E r_{\mathrm{st}}^2$, decreasing r_{st} until

$$B_H r_D \le r_{\mathrm{st}}, \qquad 2C_E B_H r_{\mathrm{st}} \le \tfrac{1}{4}.$$

For $\|w\|_\infty < r_{\mathrm{st}}$ and $\|\mathscr{D}\|_\infty \le r_D$ we have $\|w - H_d\mathscr{D}\|_\infty \le 2r_{\mathrm{st}}$. Therefore the map $\mathscr{D} \mapsto \mathscr{E}_d^{\mathrm{av}}(w - H_d\mathscr{D})$ sends the closed ball of radius r_D into the concentric ball of radius $r_D/2$ and is a contraction with constant at most $1/4$. Lemma 9.5 constructs $\mathscr{D}_d(w)$, unique in that ball. Comparing its equation with the value at $\mathscr{D} = 0$, we find

$$\|\mathscr{D}_d(w)\|_\infty \le C_E\|w\|_\infty^2 + \tfrac{1}{4}\|\mathscr{D}_d(w)\|_\infty, \tag{9.16}$$

and absorbing the last term gives the quadratic bound. Using $Q_d H_d = I$ in the defining equation, we obtain

$$\mathsf{F}_d\big(w - H_d\mathscr{D}_d(w)\big) = Q_d w - \mathscr{D}_d(w) + \mathscr{E}_d^{\mathrm{av}}\big(w - H_d\mathscr{D}_d(w)\big) = Q_d w.$$

For the derivative bounds, write $X = \Xi_d(w)$, $E_1 = D\mathscr{E}_d^{\mathrm{av}}(X)$, and $W = I + E_1 H_d$. The coarse weighted row and column norms of $E_1 H_d$ are $O(\|w\|_\infty)$, and a Neumann series bounds W^{-1} uniformly. Differentiating, we obtain

$$D\mathscr{D}_d = W^{-1}E_1, \qquad D\Xi_d = I - H_d D\mathscr{D}_d. \tag{9.17}$$

The right factor E_1 has counting column sum at most $C\epsilon^3\|w\|_\infty$, which the first formula keeps, and its row sum is at most $C\|w\|_\infty$. The second formula has bounded rows and columns from fine fields to fine fields: in the column composition, the factor ϵ^{-3} from H_d cancels against the factor ϵ^3 from $D\mathscr{D}_d$. For $s \ge 2$,

$$W\,D^s\mathscr{D}_d[h_1,\dots,h_s] = \sum_{\substack{\pi \text{ a partition of } \{1,\dots,s\} \\ |\pi| \ge 2}} D^{|\pi|}\mathscr{E}_d^{\mathrm{av}}(X)\Big[D^{|B|}\Xi_d[h_i : i \in B] : B \in \pi\Big], \tag{9.18}$$

where $D^t\Xi_d = -H_d D^t\mathscr{D}_d$ for $t \ge 2$. By induction, these tensors from fine fields to fine fields have bounded rows and columns. With an external fine input fixed in a term of (9.18), we follow its branch: the tensor $D\Xi_d$ or $D^t\Xi_d$ on that branch uses its column, and the remainder tensor then uses its column with a fine input fixed, which supplies one factor ϵ^3. All other branches use rows. The coarse inverse W^{-1} preserves this factor. By the symmetry of derivatives any input may be the fixed one. This proves, by induction, the bounds C_s for rows and $C_s\epsilon^3$ for counting columns, with positive rates.

Step 6: Coordinates on the fibers. Let $r_{\mathrm{fib}} > 0$ be so small that

$$r_{\mathrm{fib}} < r_{\mathrm{st}}/2, \qquad r_{\mathrm{fib}} + B_H C_E r_{\mathrm{fib}}^2 < r_{\mathrm{st}}, \qquad C_E r_{\mathrm{fib}}^2 < r_D.$$

If $\mathsf{F}_d(A) = \theta$, set $a = (I - H_d Q_d)A$. Then $Q_d a = 0$ and

$$w = a + H_d\theta = A + H_d \mathscr{E}_d^{\mathrm{av}}(A).$$

By these inequalities, w lies in the ball of radius r_{st} and $\mathscr{E}_d^{\mathrm{av}}(A)$ in the ball of radius r_D in which the correction is unique. Since $w - H_d\mathscr{E}_d^{\mathrm{av}}(A) = A$, the field $\mathscr{E}_d^{\mathrm{av}}(A)$ solves the correction equation at w, and uniqueness implies that $\mathscr{D}_d(w) = \mathscr{E}_d^{\mathrm{av}}(A)$ and $\Xi_d(w) = A$. Conversely, $Q_d(a + H_d\theta) = \theta$, and the identity of part (3) yields the constraint for every such parameter. Applying $I - H_d Q_d$ to $\Xi_d(a + H_d\theta) = a + H_d\theta - H_d\mathscr{D}_d(a + H_d\theta)$ recovers a, which proves uniqueness of the parameter. Since $\mathsf{P}_d \operatorname{div}_\epsilon H_d = 0$,

$$\mathsf{P}_d \operatorname{div}_\epsilon \Xi_d(a + H_d\theta) = \mathsf{P}_d \operatorname{div}_\epsilon a,$$

which proves the statement about $\mathcal{V}_d$. Finally $\Xi_d(w) - w = -H_d\mathscr{D}_d(w)$. Using the row bounds of H_d and $\nabla_\epsilon H_d$ and (9.16), we obtain the bound for the correction even when w has no controlled differences. □

9.6 The constrained Wilson Hessian

We now show that the Wilson action, written in the coordinates $a \in \mathcal{V}_d$ of Theorem 9.6, has a Hessian comparable to the curl form. The domain is controlled by the supremum norms of the field and of its curl only, so that fields without control of all first differences are allowed. The Hessian of the composition contains, besides the Hessian of the action in the direction of the tangent map, the first variation of the action applied to the second derivative of the coordinate map; this second term must be kept.

Theorem 9.7. *In the setting of Theorem 9.6, let*

$$\mathscr{S}_\epsilon(A) = 4\epsilon^{-1}\sum_p q\big((e^{\epsilon A})_p\big), \qquad \mathcal{S}_{d,\theta}(a) = \mathscr{S}_\epsilon\big(\Xi_d(a + H_d\theta)\big).$$

There exist $r_0, C_*, C > 0$ such that the following holds with the constant c_0 of Theorem 9.3. Let $0 < r_{\mathrm{ph}} \le r_0$, let θ be a real coarse field with $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$, and let $a \in \mathcal{V}_d$ be real with

$$\|a\|_\infty + \|\mathrm{d}_\epsilon a\|_\infty < r_{\mathrm{ph}}.$$

1. *The field $A = \Xi_d(a + H_d\theta)$ satisfies $\|A\|_\infty + \|\mathrm{d}_\epsilon A\|_\infty \le C r_{\mathrm{ph}}$.*
2. *For every real $v \in \mathcal{V}_d$,*

$$\frac{c_0}{2}\|v\|_{H^1_\epsilon}^2 \le D_a^2\mathcal{S}_{d,\theta}(a)[v,v] \le C\|v\|_{H^1_\epsilon}^2.$$

3. *With $w = a + H_d\theta$ and $T = D\Xi_d(w) = I - H_d D\mathscr{D}_d(w)$, the Hessian is*

$$D^2_a \mathcal{S}_{d,\theta}(a)[v,v] = D^2\mathscr{S}_\epsilon(A)[Tv,Tv] - D\mathscr{S}_\epsilon(A)\big[H_d D^2\mathscr{D}_d(w)[v,v]\big].$$

4. *The following bounds hold for real directions v, z, where e_c denotes a unit coarse link-color vector:*

$$\big|D^2\mathscr{S}_\epsilon(A)[z,z] - \|\mathrm{d}_\epsilon z\|^2_\epsilon\big| \le C r_{\mathrm{ph}} \|z\|^2_{H^1_\epsilon},$$
$$\sup_c |D\mathscr{S}_\epsilon(A)[H_d e_c]| \le C r_{\mathrm{ph}}, \qquad \sum_c |D^2\mathscr{D}_{d,c}(w)[v,v]| \le C\|v\|^2_\epsilon.$$

The same conclusions hold, after a fixed adjustment of the constants, when the hypothesis on $\|\mathrm{d}_\epsilon a\|_\infty$ is replaced by the same hypothesis on $\|\nabla_\epsilon a\|_\infty$. The domain of real a described above is convex. If it contains a critical point $a_*(\theta)$ of $\mathcal{S}_{d,\theta}$ on $\mathcal{V}_d$, then a_* is the only critical point in the domain, and for every a in the domain

$$\mathcal{S}_{d,\theta}(a) \ge \mathcal{S}_{d,\theta}(a_*) + \frac{c_0}{4}\|a - a_*\|^2_{H^1_\epsilon}.$$

For the Wilson action at coarse coupling g_n, every energy and Hessian in these statements is multiplied by g_n^{-2}. The constants do not depend on r_{ph}, d, n, L, γ, g_n, or J.

The bound on $D\mathscr{S}_\epsilon(A)[H_d e_c]$ concerns the first variation tested against the extensions of unit coarse fields; no bound on the first variation at individual fine links is used.

Proof. We first bound the Hessian and the tested first variation of $\mathscr{S}_\epsilon$ near the identity, then control the two terms of the chain rule separately, and finally deduce the convexity statements.

Step 1: The Hessian of the Wilson action. Let A and z be real, suppose that $\|A\|_\infty + \|\mathrm{d}_\epsilon A\|_\infty \le \rho \le \rho_0$ with a fixed small ρ_0, and use the notation of Lemma 9.4. By (9.10) and the triangle inequality, $|X_p(A)| \le C\epsilon^2\rho$. Let $b_p = (\mathrm{d}_\epsilon z)_p$ and $E_p = DX_p(A)[z] - \epsilon^2 b_p$. The second derivative of the term of p in $\mathscr{S}_\epsilon$ is

$$\frac{4}{\epsilon}\Big(D^2\varphi(X_p)\big[\epsilon^2 b_p + E_p, \epsilon^2 b_p + E_p\big] + D\varphi(X_p)\big[D^2 X_p(A)[z,z]\big]\Big).$$

Since $D^2\varphi = I/4 + O(|X|^2)$, its leading term is exactly $\epsilon^3|b_p|^2$. By (9.11), the cross term and the square of E_p are bounded by $C\epsilon^3\rho(|b_p|^2 + Z_p^2)$, since $\rho \le \rho_0$. The error in $D^2\varphi$ is bounded by the same expression, since $\epsilon \le 1$ and $|X_p| \le C\epsilon^2\rho$. By (9.12) and $|D\varphi(X_p)| \le C|X_p|$, the last term, the first variation applied to the second variation of X_p, is bounded by $C\epsilon^3\rho Z_p^2$. Summing over plaquettes, using the fixed overlap of links and plaquettes, $\|\mathrm{d}_\epsilon z\|_\epsilon \le C\|\nabla_\epsilon z\|_\epsilon$, and $\epsilon^3\sum_p Z_p^2 \le C\|z\|^2_\epsilon$, we obtain

$$\big|D^2\mathscr{S}_\epsilon(A)[z,z] - \|\mathrm{d}_\epsilon z\|^2_\epsilon\big| \le C\rho\|z\|^2_{H^1_\epsilon}. \tag{9.19}$$

In particular $D^2\mathscr{S}_\epsilon(0)[z,z] = \|\mathrm{d}_\epsilon z\|^2_\epsilon$.

Step 2: The first variation tested against the extension. Let $z = H_d e_c$. Using the same first-variation formula, $|D\varphi(X_p)| \le C\epsilon^2\rho$, and (9.11), we obtain

$$|D\mathscr{S}_\epsilon(A)[H_d e_c]| \le C\rho\,\epsilon^3 \sum_p \Big(|(\mathrm{d}_\epsilon H_d e_c)_p| + \rho \sum_{e\in\partial p} |H_d(e,c)| \Big) \le C\rho.$$

In the last step we used the column bounds with the factor ϵ^3 for the kernels of H_d and $\nabla_\epsilon H_d$ in Theorem 9.3, and the fixed overlap of plaquettes. The bound is uniform in the coarse link c.

Step 3: The size of the chart. Let B_0 and B_1 bound the rows of the kernels of H_d and $\nabla_\epsilon H_d$. For $w = a + H_d\theta$, the hypotheses imply that

$$\|w\|_\infty \le r_{\mathrm{ph}} + B_0 r_{\mathrm{ph}}/C_*, \qquad \|\mathrm{d}_\epsilon w\|_\infty \le r_{\mathrm{ph}} + CB_1 r_{\mathrm{ph}}/C_*.$$

We take $C_* > 4(1 + B_0 + CB_1)$ and r_0 so that these fields lie in the ball of radius r_{st} of Theorem 9.6 and their images lie in the potential ball of Step 1. By the bound for $\Xi_d(w) - w$ in that theorem, we have

$$\|A\|_\infty + \|\mathrm{d}_\epsilon A\|_\infty \le C_1 r_{\mathrm{ph}} = \rho, \tag{9.20}$$

with constants fixed before r_{ph}. If the hypothesis is instead the bound on $\|\nabla_\epsilon a\|_\infty$, the inequality $\|\mathrm{d}_\epsilon a\|_\infty \le C\|\nabla_\epsilon a\|_\infty$ gives the same estimate after enlarging C_1 by this dimensional factor. We choose r_0 below with the enlarged constant; the Hessian bounds then hold under both hypotheses.

Step 4: The term with the tangent map. Let $D_1 = D\mathscr{D}_d(w)$ and $T = I - H_d D_1$. By Theorem 9.6, the row sum of D_1 is at most Cr_{ph} and its counting column sum is at most $C\epsilon^3 r_{\mathrm{ph}}$. By Lemma 9.1 and the H^1_ϵ bound for the extension, we have

$$\|D_1 v\|_2 \le C r_{\mathrm{ph}} \|v\|_\epsilon, \qquad \|Tv - v\|_{H^1_\epsilon} \le C r_{\mathrm{ph}} \|v\|_\epsilon. \tag{9.21}$$

Applying (9.19) to Tv and expanding the difference of the two curl squares, we obtain

$$\big| D^2\mathscr{S}_\epsilon(A)[Tv, Tv] - \|\mathrm{d}_\epsilon v\|^2_\epsilon \big| \le C_2 r_{\mathrm{ph}} \|v\|^2_{H^1_\epsilon}. \tag{9.22}$$

Step 5: The term with the second derivative of the coordinates. This term requires an ℓ^1 bound on a tensor. Let $B_{c;e,f}$ denote the color block of the second derivative of $\mathscr{D}_{d,c}$ at w with respect to the fine inputs e, f. By part (3) of Theorem 9.6 with $s = 2$,

$$\sup_e \sum_{c,f} \|B_{c;e,f}\| \le C\epsilon^3.$$

Since the tensor is symmetric in its two fine inputs, we have

$$\begin{aligned}\sum_c |D^2\mathscr{D}_{d,c}(w)[v,v]| &\le \sum_{c,e,f} \|B_{c;e,f}\||v_e||v_f| \le \frac{1}{2}\sum_{c,e,f}\|B_{c;e,f}\|\big(|v_e|^2+|v_f|^2\big)\\ &\le C\epsilon^3\sum_e |v_e|^2 = C\|v\|_\epsilon^2.\end{aligned} \tag{9.23}$$

Combining this with Step 2, we obtain

$$\big|D\mathscr{S}_\epsilon(A)\big[H_d D^2\mathscr{D}_d(w)[v,v]\big]\big| \le C_3 r_{\mathrm{ph}}\|v\|_\epsilon^2. \tag{9.24}$$

No norm of the first variation summed over the coarse lattice enters this pairing.

Step 6: The chain rule and the conclusion. With θ fixed, $D_a A = T$ and $D_a^2 A[v,v] = -H_d D^2\mathscr{D}_d(w)[v,v]$, and the second-order chain rule gives the formula of part (3). By (9.22) and (9.24), the Hessian differs from $\|\mathrm{d}_\epsilon v\|_\epsilon^2$ by at most $C_4 r_{\mathrm{ph}}\|v\|_{H^1_\epsilon}^2$. We shrink r_0 so that $C_4 r_0 \le c_0/2$. For $v \in \mathcal{V}_d$, the lower bound of part (2) follows from the coercivity of the curl in Theorem 9.3, and the componentwise bound $\|\mathrm{d}_\epsilon v\|_\epsilon \le C\|\nabla_\epsilon v\|_\epsilon$ implies the upper bound. This also proves the bounds of part (4), with $\rho = C_1 r_{\mathrm{ph}}$ absorbed into the constants.

The domain of a is the intersection of the linear space $\mathcal{V}_d$ with the open ball for the sum of a norm and a seminorm, which is convex. If a_* is a critical point in the domain and $h = a - a_*$, then the derivative at a_* vanishes on $h \in \mathcal{V}_d$, and by Taylor's formula along the segment we have

$$\mathcal{S}_{d,\theta}(a) - \mathcal{S}_{d,\theta}(a_*) = \int_0^1 (1-t) D_a^2\mathcal{S}_{d,\theta}(a_* + th)[h,h]\,dt \ge \frac{c_0}{4}\|h\|_{H^1_\epsilon}^2.$$

If there were two critical points, integrating the Hessian along the segment between them and pairing the difference of their derivatives with their difference would give a positive number and also zero, so they coincide. Finally, since the fine coupling satisfies $g_0^2 = g_n^2\epsilon$, the Wilson action at coarse coupling g_n equals $g_n^{-2}\mathscr{S}_\epsilon$, and energies and derivatives are multiplied by g_n^{-2}. □

9.7 A scalar constrained inverse including constant modes

Up to the sign of its first component, the linearization of the gauge-fixing problem of Section 9.8 is the scalar system $(\mathsf{P}_d\mathscr{L}_\epsilon\lambda, Q_d'\lambda) = (f,b)$. The following lemma inverts it, with estimates for λ, its first differences, and its Laplacian. Bounds on all second differences of λ are not needed and are not proved; the Laplacian is controlled through exact identities.

Lemma 9.8. *In the geometry of Theorem 9.3, let $Q = Q'_d$ and $\mathscr{L} = \mathscr{L}_\epsilon$, and define, with the adjoints of Section 9.1,*

$$G = (\mathscr{L} + Q^\sharp Q)^{-1}, \qquad T = QG^2Q^\sharp, \qquad \mathsf{P}_d = I - GQ^\sharp T^{-1}QG,$$
$$\mathscr{B} = G - G^2Q^\sharp T^{-1}QG, \qquad \mathscr{E} = G^2Q^\sharp T^{-1} + \mathscr{B}Q^\sharp.$$

Then the following statements hold.

1. *G is defined, T is positive definite, and the formula above for P_d is the orthogonal projection onto $\mathscr{L}\ker Q$.*
2. *We have*
$$Q\mathscr{B} = 0, \quad \mathscr{L}\mathscr{B} = \mathsf{P}_d, \quad Q\mathscr{E} = I, \quad \mathsf{P}_d\mathscr{L}\mathscr{E} = 0, \quad \mathscr{L}\mathscr{E} = GQ^\sharp T^{-1}(I - QGQ^\sharp).$$
3. *For every fine scalar field $f \in \operatorname{Ran}\mathsf{P}_d$ and every coarse scalar color field b, the unique solution of*
$$(\mathsf{P}_d\mathscr{L}\lambda, Q\lambda) = (f, b)$$
is $\lambda = \mathscr{B}f + \mathscr{E}b$, and in the norm
$$\|\lambda\|_{\mathsf{Y}} = \|\lambda\|_\infty + \|\nabla_\epsilon\lambda\|_\infty + \|\mathscr{L}\lambda\|_\infty$$
it satisfies $\|\lambda\|_{\mathsf{Y}} \le C(\|f\|_\infty + \|b\|_\infty)$.
4. *The maps $\mathscr{B}$ and $\mathscr{E}$, their left first differences, and $\mathscr{L}\mathscr{B}$ and $\mathscr{L}\mathscr{E}$ have absolute row sums bounded with a positive exponential weight in the distance between coarse blocks.*
5. *For a constant coarse field b, $\mathscr{E}b = b$; the constant scalar mode is fixed by the coarse datum, not removed.*

The constants depend only on M, the localization size M_{loc} of Lemma 9.2, and the decay rates and radii, as in the convention of Section 9.1; they do not depend on d, n, L, γ, or J.

The algebra in parts (1) and (2) is the scalar algebra of [10, (1.43)–(1.45), pp. 25–26] and the projection of [11, (2.17), p. 225], and the analytic input is [13, Theorems 3.1 and 3.2]; see Table 9.1. We give the finite-dimensional proof with our adjoints.

Proof. We first prove the finite-dimensional identities, then transfer the cited estimates for penalty α_d from the cover to the torus of side n, convert them to penalty one, and compose.

Step 1: Algebra. The centered scalar blocks partition the fine sites, and

$$(Qf)(Y) = \epsilon^3 \sum_{x \in \mathsf{B}_m(Y)} f(x), \qquad (Q^\sharp b)(x) = b(Y) \quad (x \in \mathsf{B}_m(Y)),$$

where $\mathsf{B}_m(Y)$ is the block of Y. Then $QQ^\sharp = I$, Q is onto, and $Q^\sharp$ is injective. If the nonnegative form of $\mathscr{L} + Q^\sharp Q$ vanishes at u, then $\nabla_\epsilon u = 0$ and $Qu = 0$; a periodic

function with zero gradient is constant, and its block average is that constant, so $u = 0$. This shows that G exists for all finite d and n. Further, $\langle b, Tb\rangle = \|GQ^\sharp b\|_\epsilon^2 > 0$ for $b \neq 0$, and T is positive definite.

Let $Y = GQ^\sharp$, whose adjoint is $Y^\sharp = QG$. Then $Y^\sharp Y = T$, and $YT^{-1}Y^\sharp$ is a self-adjoint idempotent with range $\operatorname{Ran} Y$. We identify the orthogonal complement of this range. For $u \in \ker Q$,

$$\langle \mathscr{L}u, GQ^\sharp b\rangle_\epsilon = \langle u, (I - Q^\sharp QG)Q^\sharp b\rangle_\epsilon = 0,$$

where we used $\mathscr{L}G = I - Q^\sharp QG$; both terms of the second expression vanish since $Qu = 0$. The restriction of $\mathscr{L}$ to $\ker Q$ is injective, since its null vectors would be constants with zero block average. Its image has codimension equal to the number of coarse scalar variables, which is the dimension of the range of the injective map Y. Together with the orthogonality just proved, this implies that $(\operatorname{Ran} Y)^\perp = \mathscr{L} \ker Q$, which proves part (1).

Multiplying in the stated order, we find $Q\mathscr{B} = QG - (QG^2Q^\sharp)T^{-1}QG = 0$, and hence

$$\mathscr{L}\mathscr{B} = (G^{-1} - Q^\sharp Q)\mathscr{B} = I - GQ^\sharp T^{-1}QG = \mathsf{P}_d. \tag{9.25}$$

Let $Z = G^2Q^\sharp T^{-1}$. Then $QZ = I$ and $\mathscr{L}Z = GQ^\sharp T^{-1} - Q^\sharp$, which gives $Q\mathscr{E} = I$ and

$$\mathscr{L}\mathscr{E} = GQ^\sharp T^{-1} - Q^\sharp + \mathsf{P}_d Q^\sharp = GQ^\sharp T^{-1}(I - QGQ^\sharp).$$

Since $\mathsf{P}_d GQ^\sharp = 0$, this implies that $\mathsf{P}_d \mathscr{L}\mathscr{E} = 0$, which completes part (2). In particular, $\lambda = \mathscr{B}f + \mathscr{E}b$ solves the two equations for every $f \in \operatorname{Ran} \mathsf{P}_d$. For a solution of the homogeneous system, $Q\lambda = 0$ implies $\mathscr{L}\lambda \in \operatorname{Ran} \mathsf{P}_d$, the projected equation gives $\mathscr{L}\lambda = 0$, and λ is a constant with zero block average, hence zero. This proves uniqueness. If b is constant, the constant fine field equal to b solves the system with $f = 0$, and by uniqueness $\mathscr{E}b = b$.

At $d = 0$, substituting $Q = I$, we find $T = G^2$, $\mathsf{P}_0 = 0$, $\mathscr{B} = 0$, and $\mathscr{E} = I$. The unit-mesh first difference and Laplacian have finite stencils with bounded absolute rows, and all estimates hold directly, with $f = 0$. In the rest of the proof we assume $d \geq 1$.

Step 2: The cited estimates and the penalty. The uniform estimates are [13, Theorem 3.1, the potential and first-gradient entries of (3.42), p. 397, and the global bounds (3.47), p. 398]. Further, [13, Theorem 3.2, (3.48), p. 398] gives the exponential decay of the kernel of the corresponding coarse inverse, which is $\widehat{T}^{-1}$ in the notation introduced below, and (3.49), p. 399, gives the kernel of the complementary projection. For a source supported in one coarse block of unit size, the two entries of (3.42) bound the potential and its first physical gradient in another coarse block by $Ce^{-a\,\mathrm{dist(blocks)}}$ times the supremum of the source, and the coarse inverse in (3.48) has the same exponential bound. All scale factors are one in this whole-torus specialization with unit coarse spacing. We apply these estimates on the cover of side kn of Lemma 9.2, in the specialization described after

that lemma. The local gauge transformations in [13, (3.35)–(3.36), p. 396] are the identity with potential zero, and the scalar average is the lower-corner average translated by T_d as in (9.5) and (9.6). Conjugation by T_d translates the scalar inverse and the kernels of the projection and preserves the coarse Schur operator. The fine measure is ϵ^3, and coarse kernels are summed with counting measure.

In the whole-torus convention of [13, (3.24), p. 394], the parameter $a = 1$ gives a coarse penalty coefficient α_d with

$$\alpha_0 = 1, \qquad \alpha_d = \frac{1 - M^{-2}}{1 - M^{-2d}} \quad (d \geq 1), \qquad 1 - M^{-2} \leq \alpha_d \leq 1.$$

Indeed, the recurrence $a_{j+1} = a_j/(M^{-2} + a_j)$ with $a_1 = 1$ is solved by taking reciprocals and summing the geometric progression. The operators to which the cited estimates apply directly are

$$\widehat{G} = (\mathscr{L} + \alpha_d Q^\sharp Q)^{-1}, \qquad \widehat{T} = Q\widehat{G}^2 Q^\sharp.$$

The estimates hold for $\widehat{G}^{(kn)}$ and $(\widehat{T}^{(kn)})^{-1}$ on the cover, uniformly in d and n. We next transfer them to the torus of side n and then convert to penalty one.

Step 3: Transfer to the torus of side n. For every $\alpha > 0$, on either torus, the form $\|\nabla_\epsilon \lambda\|_\epsilon^2 + \alpha \|Q\lambda\|_2^2$ is positive at $\lambda \neq 0$, since a field with zero gradient is constant and Q preserves constants. Hence $\mathscr{L} + \alpha Q^\sharp Q$ is invertible. The operators $\mathscr{L}$, Q, and $Q^\sharp$ commute with π^*, and α_d is the same on both tori. Applying the positive operator to both sides gives the first identity below; the second follows by composition, and the third by positivity and uniqueness of the coarse inverse:

$$\widehat{G}^{(kn)}\pi^* = \pi^*\widehat{G}^{(n)}, \qquad \widehat{T}^{(kn)}\pi^* = \pi^*\widehat{T}^{(n)}, \qquad (\widehat{T}^{(kn)})^{-1}\pi^* = \pi^*(\widehat{T}^{(n)})^{-1}. \tag{9.26}$$

Here $\langle b, \widehat{T}b\rangle = \|\widehat{G}Q^\sharp b\|_\epsilon^2 > 0$ for $b \neq 0$, since $Q^\sharp$ is injective. The nonlocal constraint is also the same on both tori: by Lemma 9.2, $\mathsf{P}_d^{(kn)}\pi^* = \pi^*\mathsf{P}_d^{(n)}$, where both are the projections onto the corresponding $\mathscr{L}\ker Q$. These checks justify the scalar applications of that lemma. Their matrices in the coordinate bases and the first output differences satisfy its formulas, weighted rows descend without the factor k^3, and an estimate for a source in one coarse block descends with at most the factor $k^3 = (6M_{\mathrm{loc}})^3$. We conclude that all cited estimates hold on the torus of side n.

At the identity connection these scalar kernels act separately and identically on the colors. At a fixed output site and output component, choosing the signs of a real source in one source block identifies the supremum operator norm with the absolute row sum of the kernel over that block. Hence the estimates for sources in one block bound the absolute row sums, also for each component of $\nabla_\epsilon \widehat{G}$. Summing the exponential block estimates over the coarse unit blocks and using (9.9), we obtain bounds for the rows of $\widehat{G}$, $\nabla_\epsilon \widehat{G}$, and $\widehat{T}^{-1}$, and for their weighted block rows at a smaller fixed rate. Both Q and $Q^\sharp$ have rows of norm one and bounded dependence radius.

Step 4: Conversion to penalty one. We set $\delta = 1 - \alpha_d$, $S = Q\widehat{G}Q^\sharp$, and $K = I + \delta S$. The coarse matrix S is self-adjoint and nonnegative. Its exponentially weighted rows are bounded by the composition bounds above, and so are its columns, by self-adjointness. Since $0 \le \delta \le M^{-2}$, we have $K \ge I$ and the weighted off-diagonal row and column sums of K are bounded. The coarse lattice has unit mesh and polynomial counting growth independent of n. By the inverse estimate of Lemma 5.3 with lower bound one, we obtain bounded weighted rows and columns of K^{-1} at a smaller positive rate, independent of d and n. By direct multiplication, we have

$$G = \widehat{G} - \delta\widehat{G}Q^\sharp K^{-1}Q\widehat{G}, \qquad GQ^\sharp = \widehat{G}Q^\sharp K^{-1}, \qquad T = K^{-1}\widehat{T}K^{-1}, \qquad T^{-1} = K\widehat{T}^{-1}K. \tag{9.27}$$

For example, multiplying the first expression by $\mathscr{L} + Q^\sharp Q = \widehat{G}^{-1} + \delta Q^\sharp Q$ leaves the extra terms $\delta Q^\sharp Q\widehat{G} - \delta Q^\sharp K K^{-1} Q\widehat{G} = 0$. Its product with $Q^\sharp$ is the second expression, since $I - \delta K^{-1}S = K^{-1}$. Taking the adjoint and forming $(GQ^\sharp)^\sharp(GQ^\sharp)$ gives the formula for T, and inverting it, we obtain the last one. These identities transfer the bounds for the potential and first gradient from $\widehat{G}$ to G, and the bounds for the coarse inverse from $\widehat{T}^{-1}$ to T^{-1}, with one further fixed reduction of the rate. A left gradient in the first identity acts only on the leftmost factor $\widehat{G}$.

Step 5: The solution maps and the Laplacian. In each term of $\mathscr{B}$ and $\mathscr{E}$, these operators are composed in the displayed order. Row sums are submultiplicative, and for a left gradient only the leftmost factor G is differentiated. Therefore $\mathscr{B}$, $\mathscr{E}$, and their left first differences have bounded rows and bounded exponentially weighted block rows. For example, the left gradient of the second term of $\mathscr{B}$ is $(\nabla_\epsilon G)GQ^\sharp T^{-1}QG$, all of whose factors have the required rows; the triangle inequality for block distances gives the weighted version. For the Laplacian we use (9.25) and the formula for $\mathscr{L}\mathscr{E}$ in part (2). The projection $\mathsf{P}_d = I - GQ^\sharp T^{-1}QG$ has a bounded weighted block row, including its identity part, and $\mathscr{L}\mathscr{E}$ is a composition of operators whose rows are already bounded. Adding the bounds for the potential, the first differences, and the Laplacian proves part (3), and the block rows prove part (4). No estimate for the first differences of the identity part of P_d applied to a general source is used. □

9.8 Local gauge fixing with prescribed coarse gauge

We now bring a small fine field into the chart of Theorem 9.7 by a small fine gauge transformation. The transformation is fixed by two conditions: the projected divergence of the transformed field vanishes, and the composed coarse gauge transformation of the gauge-corrected averaging, multiplied by the coarse values of the fine gauge transformation, is a prescribed coarse gauge transformation e^c. The divergence of a gauge-transformed field involves the Laplacian of the gauge parameter, and a direct estimate would require

bounds on all second differences of the parameter. We avoid this by an exact formula on each link in which the inverse lattice spacing cancels. We measure the gauge parameter in the norm $\|\cdot\|_{\mathsf{Y}}$ of Lemma 9.8, which controls the Laplacian but not the individual second differences. The map $\mathscr{H}_d$ below is not related to the linear extension H_d.

Theorem 9.9. *Use the geometry of Theorem 9.3 at the identity connection, and set*

$$\|A\|_{\mathsf{X}} = \|A\|_\infty + \|\mathrm{div}_\epsilon\, A\|_\infty, \qquad \|\lambda\|_{\mathsf{Y}} = \|\lambda\|_\infty + \|\nabla_\epsilon \lambda\|_\infty + \|\mathscr{L}_\epsilon \lambda\|_\infty.$$

Let $\mathsf{G}_d(A)$ be the composed gauge transformation of Lemma 8.10 with background connection $U_0 = I$, applied to the fine field $e^{\epsilon A}$. There exist $r_1, C > 0$, independent of d and n, such that the following statements hold.

1. *On the complex balls $\|A\|_{\mathsf{X}} < 2r_1$ and $\|\lambda\|_{\mathsf{Y}} < 2r_1$, the maps*

$$A^{e^\lambda}_{xy} = \epsilon^{-1} \mathrm{Log}\big(e^{\lambda_x} e^{\epsilon A_{xy}} e^{-\lambda_y}\big), \qquad \mathscr{H}_d(A,\lambda) = \mathrm{Log}\big[\mathsf{G}_d(A^{e^\lambda})\, e^{\lambda|_{\mathrm{roots}}}\big]$$

   are holomorphic, with bounds on their operator derivatives of each fixed order on smaller balls. They satisfy

$$\begin{aligned} \mathrm{div}_\epsilon\, A^{e^\lambda} &= \mathrm{div}_\epsilon\, A - \mathscr{L}_\epsilon \lambda + \mathscr{R}_D(A,\lambda), \\ \mathscr{H}_d(A,\lambda) &= \mathrm{Log}\, \mathsf{G}_d(A) + Q_d' \lambda + \mathscr{R}_H(A,\lambda), \end{aligned}$$

   where both remainders vanish at $\lambda = 0$, and for $0 < r_{\mathrm{loc}} \le r_1$ and $\|A\|_{\mathsf{X}}, \|\lambda\|_{\mathsf{Y}}, \|\lambda'\|_{\mathsf{Y}} \le r_{\mathrm{loc}}$,

$$\|\mathscr{R}_D(A,\lambda) - \mathscr{R}_D(A,\lambda')\|_\infty + \|\mathscr{R}_H(A,\lambda) - \mathscr{R}_H(A,\lambda')\|_\infty \le C r_{\mathrm{loc}} \|\lambda - \lambda'\|_{\mathsf{Y}}.$$

   In particular $D_\lambda \mathscr{H}_d(0,0) = Q_d'$, including the constants and across the periodic seams. The order of the factors in $\mathscr{H}_d$ follows from $(U^k)^l = U^{lk}$.

2. *There exist $0 < r_{\mathsf{Y}}, r_{\mathsf{X}}, r_c < r_1$, independent of d and n, such that for every fine field A with $\|A\|_{\mathsf{X}} < r_{\mathsf{X}}$ and every coarse field c with $\|c\|_\infty < r_c$ there is a unique λ with $\|\lambda\|_{\mathsf{Y}} < r_{\mathsf{Y}}$ satisfying*

$$\mathsf{P}_d\, \mathrm{div}_\epsilon\, A^{e^\lambda} = 0, \qquad \mathsf{G}_d(A^{e^\lambda})\, e^{\lambda|_{\mathrm{roots}}} = e^c.$$

   This solution depends holomorphically on (A,c) on smaller balls, is real for real data, and satisfies $\|\lambda(A,c)\|_{\mathsf{Y}} \le C(\|A\|_{\mathsf{X}} + \|c\|_\infty)$. If A_* lies in these domains, $\mathsf{P}_d\, \mathrm{div}_\epsilon\, A_* = 0$, and $c = \mathrm{Log}\, \mathsf{G}_d(A_*)$, then $\lambda(A_*, c) = 0$.

3. *With $b = T_M^d(e^{\epsilon A})$, the gauge-corrected output of the transformed field is $V_d(A^{e^\lambda}) = b^{e^c}$.*

4. *For real data the transformation preserves the plaquette distances ρ_G, and*

$$\|A^{e^\lambda}\|_\infty + \|\mathrm{d}_\epsilon A^{e^\lambda}\|_\infty \le C\big(\|A\|_{\mathsf{X}} + \|c\|_\infty + \|\mathrm{d}_\epsilon A\|_\infty\big).$$

Further, for every fixed $r_{\mathrm{ph}} \le r_0$, if the supremum norms of A, of $\mathrm{div}_\epsilon A$, of $\mathrm{d}_\epsilon A$, and of c are smaller than r_{ph} divided by a sufficiently large constant, then $A' = A^{e^\lambda}$ lies in the domain of Theorem 9.7: $A' = \Xi_d(a + H_d\theta)$ with $a \in \mathcal{V}_d$, $\|a\|_\infty + \|\mathrm{d}_\epsilon a\|_\infty < r_{\mathrm{ph}}$, and $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$.

The constants depend only on M, the localization size M_{loc} of Lemma 9.2, and the radii of the balls in the statement and the larger balls containing them; they do not depend on d, n, L, γ, or J. A bound on all first differences of the output is not required.

We also summarize the remaining radii of Theorems 9.7 and 9.9.

Radius	Domain or role
r_0, C_*	Upper bound on the radius r_{ph} of the convex chart in Theorem 9.7, and the factor in the target condition $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$.
$2r_1$, r_{loc}	Complex input radii for the gauge maps $A \mapsto A^{e^\lambda}$ and $\mathscr{H}_d$ in the norms $\|\cdot\|_{\mathsf{X}}$ and $\|\cdot\|_{\mathsf{Y}}$, and radius $r_{\mathrm{loc}} \le r_1$ for the Lipschitz bounds of the remainders.
r_{X}, r_c, r_{Y}	Fine input radius in $\|\cdot\|_{\mathsf{X}}$, radius of the prescribed coarse gauge c, and radius of the ball in $\|\cdot\|_{\mathsf{Y}}$ in which the gauge parameter λ is unique.

Proof. We first derive an exact formula for the gauge transformation on one link, then expand the divergence and the coarse gauge to first order in λ, and finally solve the two equations by a contraction in the norm $\|\cdot\|_{\mathsf{Y}}$.

Step 1: The gauge transformation on one link. For a positive link $e = (x, x + \epsilon e_i)$ set

$$u = \lambda_x, \qquad v = \nabla_{\epsilon,i}\lambda(x), \qquad a = A_i(x), \qquad F_\epsilon(u, v, a) = \epsilon^{-1}\,\mathrm{Log}\big(e^u e^{\epsilon a} e^{-u-\epsilon v}\big).$$

The product inside the logarithm is the identity at $\epsilon = 0$. Its logarithm is a holomorphic function vanishing there, and the quotient by ϵ therefore has a removable analytic extension. Differentiating at $\epsilon = 0$, we obtain

$$F_\epsilon(u, v, a) = C(u)a - J(u)v + \epsilon R_\epsilon(u, v, a), \qquad C(u) = \mathrm{Ad}(e^u), \quad J(u)v = \int_0^1 \mathrm{Ad}(e^{su})v\,ds. \tag{9.28}$$

For example, the derivative of $e^{-u-\epsilon v}$ at $\epsilon = 0$, multiplied on the left by e^u, is $-J(u)v$, which fixes the sign and the order of the adjoints. On fixed small complex balls of (u, v, a) the products stay in the identity chart for $0 \le \epsilon \le 1$, and the analytic extension has bounded Taylor coefficients and derivatives, uniformly in ϵ. Its terms linear in (a, v) are exactly $C(u)a - J(u)v$, independently of ϵ, and R_ϵ starts at degree two in (a, v). Further, conjugation shows that $F_\epsilon(u, 0, a) = C(u)a$ exactly, and $R_\epsilon(u, 0, a) = 0$. Using its convergent Taylor series on a smaller ball, we obtain

$$|R_\epsilon(u, v, a)| \le C|v|\big(|a| + |v|\big), \tag{9.29}$$

with derivatives in (a, v) bounded by $C(|a| + |v|)$ and derivative in u bounded by $C|v|(|a| + |v|)$. In particular $\|A^{e^\lambda}\|_\infty \le C(\|A\|_\infty + \|\nabla_\epsilon \lambda\|_\infty)$.

Step 2: The divergence. We abbreviate $C_x = C(\lambda_x)$, $J_x = J(\lambda_x)$, $v_i(x) = \nabla_{\epsilon,i}\lambda(x)$, and $R_i(x) = R_\epsilon(\lambda_x, v_i(x), A_i(x))$. Substituting (9.28) into the backward divergence of Section 9.1 and freezing the coefficients at x, we obtain

$$\begin{aligned} \operatorname{div}_\epsilon A^{e^\lambda}(x) = {} & C_x \operatorname{div}_\epsilon A(x) - J_x \mathscr{L}_\epsilon \lambda(x) \\ & + \sum_i \frac{C_{x-\epsilon e_i} - C_x}{\epsilon} A_i(x - \epsilon e_i) - \sum_i \frac{J_{x-\epsilon e_i} - J_x}{\epsilon} v_i(x - \epsilon e_i) \\ & + \sum_i \big(R_i(x - \epsilon e_i) - R_i(x)\big). \end{aligned} \tag{9.30}$$

The last line has no factor ϵ^{-1}: the factor ϵ in (9.28) has canceled it. Its size is bounded by the two values in (9.29), and no difference of A is needed. For the differences of the coefficients, the integral formula along the segment from λ_x to $\lambda_{x-\epsilon e_i}$ gives a bound $C\|\nabla_\epsilon \lambda\|_\infty$; also $\|C_x - I\| + \|J_x - I\| \le C\|\lambda\|_\infty$. It follows that the difference between the first line of (9.30) and $\operatorname{div}_\epsilon A - \mathscr{L}_\epsilon \lambda$, together with the other two lines, defines a holomorphic remainder $\mathscr{R}_D$ that vanishes at $\lambda = 0$.

For the Lipschitz bound, take λ, λ' with $\|\lambda\|_{\mathsf{Y}}, \|\lambda'\|_{\mathsf{Y}} \le r_{\mathrm{loc}}$ and set $\delta\lambda = \lambda - \lambda'$. In $(C - I)\operatorname{div}_\epsilon A$, the change is at most $C\|\operatorname{div}_\epsilon A\|_\infty \|\delta\lambda\|_\infty$. In $(J - I)\mathscr{L}_\epsilon \lambda$, the product rule for differences bounds the change by

$$C\big(\|\lambda\|_\infty \|\mathscr{L}_\epsilon \delta\lambda\|_\infty + \|\mathscr{L}_\epsilon \lambda'\|_\infty \|\delta\lambda\|_\infty\big).$$

In the second line of (9.30), the same integral formula and the bounded second derivatives of C and J bound the changes of the difference quotients by $C(\|\nabla_\epsilon \delta\lambda\|_\infty + r_{\mathrm{loc}}\|\delta\lambda\|_\infty)$. Their other factors have norm at most r_{loc}, and changing v_i in the second sum gives one more product of the same type. The derivative bounds following (9.29) bound the change of the last line by $C r_{\mathrm{loc}} \|\delta\lambda\|_{\mathsf{Y}}$. These estimates prove the Lipschitz bound for $\mathscr{R}_D$, as well as holomorphy and bounds on the operator derivatives of fixed order of the divergence map from $\mathsf{X} \times \mathsf{Y}$ to the supremum norm on smaller balls. Only the scalar Laplacian, not the individual second differences, was used. Applying P_d preserves these estimates, since P_d has bounded rows by Lemma 9.8.

Step 3: The composed coarse gauge transformation. By Lemma 8.10, $\operatorname{Log} \mathsf{G}_d(A)$ is holomorphic and bounded on a fixed complex ball in the supremum norm, since the sizes of its factors are at most $C_M \epsilon_j \|A_j\|_\infty$, whose sum over j is a geometric sum. After a fixed decrease of r_1, the bound on A^{e^λ} in Step 1 puts A^{e^λ} inside that ball, and $\lambda|_{\mathrm{roots}}$ has supremum norm at most $\|\lambda\|_{\mathsf{Y}}$. The product defining $\mathscr{H}_d$ lies in a fixed identity chart and is a bounded holomorphic function. Its operator derivatives of fixed order are bounded

independently of the dimension. To see this, we restrict the bounded holomorphic map to $x + \sum_{i=1}^{s} z_i h_i$ with $\|h_i\| \le 1$ and $|z_i| < \delta/s$ at distance at least δ from the boundary of the ball on which the map is bounded. Cauchy's formula then bounds the mixed derivative by $C(s/\delta)^s$. This bounds an operator norm; it does not sum absolute coefficients.

We compute the derivative in λ at $(0,0)$. At $A = 0$, take the pure fine gauge transformation $\Omega_x = e^{t\lambda_x}$. At one level, each path transport is $\Omega_v \Omega_x^{-1}$, and the infinitesimal logarithm of the endpoint gauge transformation at the coarse vertex v is

$$-M^{-3} \sum_{x \in \mathsf{B}_M(v)} (\lambda_v - \lambda_x) = -\lambda_v + M^{-3} \sum_{x \in \mathsf{B}_M(v)} \lambda_x,$$

and the pure gauge transformation itself contributes $+\lambda_v$ at that vertex. Their sum is the centered scalar block average of λ. Iterating this calculation composes the scalar block averages, whose composition is Q'_d by Lemma 8.1. Every intermediate vertex of the paths cancels, also for paths across the periodic seams, and a constant is averaged to itself. For $d = 0$ the composed gauge transformation is the identity and the restriction to the roots is $Q'_0 = I$. In all cases

$$D_\lambda \mathscr{H}_d(0,0) = Q'_d. \tag{9.31}$$

The order of the factors follows from $(U^k)^l = U^{lk}$: the fine gauge transformation is applied first and $\mathsf{G}_d(A^{e^\lambda})$ second, and at the coarse vertices their product is $\mathsf{G}_d(A^{e^\lambda}) e^{\lambda|_{\mathrm{roots}}}$.

We define $\mathscr{R}_H = \mathscr{H}_d(A,\lambda) - \mathscr{H}_d(A,0) - Q'_d\lambda$, which vanishes at $\lambda = 0$; note $\mathscr{H}_d(A,0) = \operatorname{Log} \mathsf{G}_d(A)$. By the bounds on the second operator derivatives and (9.31), we have

$$\|D_\lambda \mathscr{H}_d(A,\lambda) - Q'_d\| \le C\big(\|A\|_{\mathsf{X}} + \|\lambda\|_{\mathsf{Y}}\big),$$

and integrating along the segment from λ' to λ gives the Lipschitz bound for $\mathscr{R}_H$. Together with Step 2 this proves part (1).

Step 4: Solving the two equations. On $\operatorname{Ran} \mathsf{P}_d \times \{\text{coarse scalar fields}\}$ we use the product supremum norm and define $\mathcal{L}\lambda = (-\mathsf{P}_d \mathscr{L}_\epsilon \lambda, Q'_d \lambda)$. By Lemma 9.8, its inverse is $\mathcal{L}^{-1}(f,b) = -\mathscr{B} f + \mathscr{E} b$, with norm at most C_I into Y. We set

$$\mathsf{d}(A,c) = \big(\mathsf{P}_d \operatorname{div}_\epsilon A,\ \operatorname{Log} \mathsf{G}_d(A) - c\big), \qquad R(A,\lambda) = \big(\mathsf{P}_d \mathscr{R}_D(A,\lambda),\ \mathscr{R}_H(A,\lambda)\big).$$

By part (1), the two equations of part (2) are equivalent to

$$\lambda = -\mathcal{L}^{-1}\big(\mathsf{d}(A,c) + R(A,\lambda)\big), \tag{9.32}$$

where the first component of the argument lies in $\operatorname{Ran} \mathsf{P}_d$, so that $\mathcal{L}^{-1}$ is applied within its domain. Let C_R dominate the Lipschitz constants of the remainders, including the row bound of P_d, and choose $0 < r_{\mathsf{Y}} < r_1/2$ with $C_I C_R r_{\mathsf{Y}} \le 1/4$. The composed gauge transformation is the identity at $A = 0$ and has a bounded first operator derivative,

and $\|\mathsf{d}(A,c)\| \le C_d(\|A\|_{\mathsf{X}} + \|c\|_\infty)$. Let $r_{\mathsf{X}}, r_c < r_{\mathsf{Y}}$ satisfy $C_I C_d(r_{\mathsf{X}} + r_c) < r_{\mathsf{Y}}/4$. Since $R(A,0) = 0$, the right side of (9.32) maps the closed ball of radius r_{Y} in Y into the concentric ball of radius $r_{\mathsf{Y}}/2$ and has Lipschitz constant at most $1/4$. By Lemma 9.5, the solution exists, is unique in the ball, depends holomorphically on the data on smaller balls, and is real for real data. Absorbing the Lipschitz term, we obtain

$$\|\lambda(A,c)\|_{\mathsf{Y}} \le \tfrac{4}{3} C_I C_d \big(\|A\|_{\mathsf{X}} + \|c\|_\infty\big).$$

If $\mathsf{P}_d \operatorname{div}_\epsilon A_* = 0$ and $c = \operatorname{Log} \mathsf{G}_d(A_*)$, then $\lambda = 0$ solves (9.32), and by uniqueness $\lambda(A_*, c) = 0$. The logarithm and the exponential are inverse to each other in the chart, so the logarithmic equation for the coarse gauge is equivalent to the group equation in part (2).

Step 5: The gauge-corrected output. By part (3) of Lemma 8.10 with background connection $U_0 = I$, we have

$$V_d(A^{e^\lambda}) = \big[T_M^d\big((e^{\epsilon A})^{e^\lambda}\big)\big]^{\mathsf{G}_d(A^{e^\lambda})} = \big[b^{e^\lambda|_{\mathrm{roots}}}\big]^{\mathsf{G}_d(A^{e^\lambda})} = b^{\mathsf{G}_d(A^{e^\lambda})e^\lambda|_{\mathrm{roots}}} = b^{e^c}.$$

The second equality is the gauge covariance of T_M, which follows from the cancellation of the gauge transformations at the interior vertices of the paths in its definition. The small relative logarithms in the complex charts transform analytically under conjugation, so the calculation also holds on the holomorphic domains; for real data it is the gauge covariance for $\mathrm{SU}(2)$-valued gauge transformations.

Step 6: Real data and the chart. Let the data be real and set $A' = A^{e^\lambda}$. Plaquette transports transform by conjugation, so $\rho_G((e^{\epsilon A'})_p) = \rho_G((e^{\epsilon A})_p)$. Since both potentials are small, their plaquette transports lie in the same real identity chart, where the logarithms are related by an adjoint at the base point of the plaquette. Hence $|X_p(A')| = |X_p(A)|$. Using (9.10) for A' with the reverse triangle inequality and for A with the triangle inequality, we obtain

$$\begin{aligned} \epsilon^2 |(\mathrm{d}_\epsilon A')_p| &\le |X_p(A')| + C\epsilon^2 \sum_{e \in \partial p} |A'_e|^2 = |X_p(A)| + C\epsilon^2 \sum_{e \in \partial p} |A'_e|^2 \\ &\le \epsilon^2 |(\mathrm{d}_\epsilon A)_p| + C\epsilon^2 \sum_{e \in \partial p} \big(|A_e|^2 + |A'_e|^2\big). \end{aligned} \tag{9.33}$$

Dividing by ϵ^2 and taking the supremum over plaquettes, we obtain

$$\|\mathrm{d}_\epsilon A'\|_\infty \le \|\mathrm{d}_\epsilon A\|_\infty + C\big(\|A\|_\infty^2 + \|A'\|_\infty^2\big).$$

The bound on A^{e^λ} from Step 1 and the bound on λ from Step 4 imply that $\|A'\|_\infty \le C(\|A\|_{\mathsf{X}} + \|c\|_\infty)$. Since these quantities lie in a fixed small ball, the squares can be absorbed into a linear bound, which proves the estimate of part (4).

To verify the statement about the chart, take the data so small that $\|A'\|_\infty < r_{\mathrm{fib}}$, and set

$$\theta = \mathsf{F}_d(A'), \qquad a = (I - H_d Q_d)A'.$$

Then $Q_d a = 0$, and $\mathsf{P}_d \operatorname{div}_\epsilon a = 0$ by the solved gauge equation and $\mathsf{P}_d \operatorname{div}_\epsilon H_d = 0$. By part (4) of Theorem 9.6, we have $A' = \Xi_d(a + H_d\theta)$. The row bound for the average and the quadratic bound for its remainder imply that $\|\theta\|_\infty \le C\|A'\|_\infty$, and by the row bounds of H_d and $\nabla_\epsilon H_d$, we have

$$\|a\|_\infty + \|\mathrm{d}_\epsilon a\|_\infty \le C\|A'\|_\infty + \|\mathrm{d}_\epsilon A'\|_\infty.$$

For any fixed $r_{\mathrm{ph}} \le r_0$, if the input potential, divergence, curl, and coarse gauge are smaller than r_{ph} divided by a sufficiently large constant, both conditions of Theorem 9.7 hold, including $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$. All radii depend only on M, the localization data, and the radii and decay rates fixed above. □

9.9 A trial configuration with prescribed coarse field

The analytic right inverse of Theorem 9.6 produces, after a gauge transformation, a fine configuration with prescribed image under T_M^d and Wilson action at most $C g_n^{-2}\|\theta\|_2^2$, where e^θ is the prescribed coarse field.

Corollary 9.10. *In the setting of Theorem 9.6, after a fixed decrease of θ_*, let θ be a real coarse field with $\|\theta\|_\infty < \theta_*$, let $A_{\mathrm{sec}} = A_{\mathrm{sec},d}(\theta)$, and let $U = e^{\epsilon A_{\mathrm{sec}}}$. Let k be any periodic $\mathrm{SU}(2)$-valued gauge transformation on the fine lattice whose values at the coarse vertices are those of the composed gauge transformation $\mathsf{G}_d(A_{\mathrm{sec}})$ of Lemma 8.10 with background connection $U_0 = I$, and set $U_{\mathrm{sec}} = U^k$. Then the following statements hold.*

1. *$T_M^d U_{\mathrm{sec}} = e^\theta$, the plaquette distances of U_{sec} equal those of U, and for every coarse coupling $g_n > 0$*
$$4 g_n^{-2}\epsilon^{-1}\sum_p q\big((U_{\mathrm{sec}})_p\big) \le C g_n^{-2}\|\theta\|_2^2.$$
2. *Let $\epsilon_j = M^{j-d}$ and $g_j = g_n\sqrt{\epsilon_j}$, so that the fine coupling is $g_0 = g_n\sqrt{\epsilon}$. For $0 \le j < d$, let z_j be the free logarithmic coordinates of $T_M^j U_{\mathrm{sec}}$ relative to $T_M^{j+1}U_{\mathrm{sec}}$ in the block axial gauge, given by the inverse branch of Lemma 6.3 for one averaging step. Then*
$$\|z_j/g_j\|_\infty \le C\frac{\|\theta\|_\infty}{g_n}\epsilon_j^{3/2} \qquad (0 \le j < d).$$
3. *Let $p_j > 0$ satisfy*
$$C\frac{\|\theta\|_\infty}{g_n}\epsilon_j^{3/2} \le p_j/2, \qquad g_j p_j < r_M \qquad (0 \le j < d),$$

where r_M is the radius of Lemma 6.3. Then the inverse branches are defined on the whole products $\prod_e\{|z_{j,e}| \le g_j p_j\}$, and the coordinates of U_{sec} lie in the interior of the product $\prod_{j,e}\{|z_{j,e}/g_j| \le p_j\}$, in its concentric half. For a common radius $p_j = p$, the condition $\|\theta\|_\infty \le c g_n p$ suffices, together with $\|\theta\|_\infty < \theta_*$ and $g_j p < r_M$.

4. *The infimum of the Wilson action $4g_n^{-2}\epsilon^{-1}\sum_p q(U_p)$ over any set of configurations that contains U_{sec} lies between 0 and $C g_n^{-2}\|\theta\|_2^2$.*

The gauge transformation k may be rough; the derivative bounds of Theorem 9.6 refer to A_{sec}, before the gauge transformation. The constants do not depend on d, n, L, γ, g_n, or J.

Proof. We first verify the image under T_M^d, then the bounds on the coordinates, and finally the bound on the action.

Step 1: The image under T_M^d. Since $\mathsf{F}_d(A_{\mathrm{sec}}) = \theta$, part (3) of Lemma 8.10 with background connection $U_0 = I$ reads

$$e^\theta = (T_M^d U)^{\mathsf{G}_d(A_{\mathrm{sec}})}.$$

One choice of k assigns to each fine vertex the value of this gauge transformation at the center of its block on the coarse lattice. This is a periodic fine gauge transformation with the required values. For any such k, the gauge covariance of T_M implies that $T_M^d(U^k) = (T_M^d U)^{k|_{\mathrm{roots}}} = e^\theta$, where $k|_{\mathrm{roots}}$ is the restriction to the coarse vertices.

Step 2: The choice of k does not affect the coordinates. In a block with center v, the block axial gauge uses the path transports $k_x^{\mathrm{ax}} = U(\sigma_{v,x})$. Under a real fine gauge transformation l, this transport becomes $l_v U(\sigma_{v,x}) l_x^{-1}$, and the two gauge-fixed fields differ by the gauge transformation equal to l_v on the whole block. Since the coarse-field lift of the target transforms by the same blockwise constant gauge transformation, the relative logarithms transform as $X_e \mapsto \mathrm{Ad}(l_v)X_e$, and their norms are unchanged. The same calculation applies at each level, with the restriction of l to the vertices of that level. All the coordinates z_j of U and U^k have the same norms.

Step 3: The coordinates. By (9.10) and the bound $\|A_{\mathrm{sec}}\|_\infty + \|\nabla_\epsilon A_{\mathrm{sec}}\|_\infty \le C\|\theta\|_\infty$,

$$\delta(U) \le C\epsilon^2\|\theta\|_\infty. \tag{9.34}$$

We apply part (2) of Lemma 8.10, now regarding the real configuration U as the background. Its curvature hypothesis holds with $\varepsilon = C\|\theta\|_\infty$ after a fixed decrease of θ_*, and it yields for the iterates

$$\delta(T_M^j U) \le C\epsilon_j^2\|\theta\|_\infty \qquad (0 \le j \le d).$$

We decrease θ_* once more so that all these curvatures lie below the threshold of Lemma 7.2. We use its clause for one averaging step under a curvature hypothesis at each $j < d$, not

its clause for J consecutive steps under a bound on the links. The lattice sides at this use are

$$n_j = M^{d-j} n = M n_{j+1}, \qquad n_{j+1} = M^{d-j-1} n \geq n \geq 4, \tag{9.35}$$

so the local patches have the contractible lifts required there. That lemma identifies the relative coordinates in the block axial gauge with the inverse branch for one averaging step and bounds their free components by $C\epsilon_j^2 \|\theta\|_\infty$. Dividing by $g_j = g_n \sqrt{\epsilon_j}$, we obtain part (2) for U, and Step 2 transfers it to U_{sec}.

The inequalities with $p_j/2$ imply strict membership in the product. For the statement about the whole product, the target $b = T_M^{j+1} U_{\mathrm{sec}}$ is SU(2)-valued, and Lemma 6.3 defines Γ_b at every free input with $\max_e |z_e| < r_M$, uniformly in the target. The condition $g_j p_j < r_M$ puts the closed product $\prod_e \{|z_e| \leq g_j p_j\}$ strictly inside this domain, and the coordinates of U_{sec} lie in the concentric half product. For a common radius we have $\epsilon_j^{3/2} \leq 1$, and choosing the constant c small enough gives the sufficient condition in part (3). At depth zero there are no coordinates to check.

Step 4: The action. On the ball of the section, Lemma 9.4 implies that

$$q(U_p) \leq C\epsilon^4 \Big(|(\mathrm{d}_\epsilon A)_p|^2 + \|A\|_\infty^2 \sum_{e \in \partial p} |A_e|^2 \Big), \qquad A = A_{\mathrm{sec}}. \tag{9.36}$$

Indeed, the linear term of $X_p(A)$ is exactly $\epsilon^2 (\mathrm{d}_\epsilon A)_p$, $q(e^X) \leq C|X|^2$, and the square of the remainder is bounded using $(\sum_{e \in \partial p} |A_e|^2)^2 \leq 4\|A\|_\infty^2 \sum_{e \in \partial p} |A_e|^2$. Each positive link belongs to a fixed number of plaquette boundaries, and $\|\mathrm{d}_\epsilon A\|_\epsilon \leq C \|\nabla_\epsilon A\|_\epsilon$. We obtain

$$4\epsilon^{-1} \sum_p q(U_p) \leq C\big(\|\nabla_\epsilon A\|_\epsilon^2 + \|A\|_\infty^2 \|A\|_\epsilon^2 \big) \leq C \|q_d(\theta)\|_2^2 \leq C\|\theta\|_2^2. \tag{9.37}$$

In the middle inequality we used the H_ϵ^1 bound for the extension and the supremum bound on the ball of the section, and in the last one part (2) of Theorem 9.6. Since gauge transformations preserve the group distance of each plaquette, and hence its value of q, the same bound holds for U_{sec}. Since the fine coupling satisfies $g_0^2 = g_n^2 \epsilon$, multiplication by g_n^{-2} gives part (1). Since the Wilson action of an SU(2)-valued configuration is nonnegative, every infimum over a set containing U_{sec} lies between zero and this bound, which proves part (4). □

SECTION 10

Classical background fields and analytic estimates

In this section we construct the classical background field on the periodic torus and prove the analytic estimates for it that the renormalization group analysis uses. Fix a coarse lattice of side $n \geq 4$ and a depth $d \geq 0$, and consider the fine lattice of side nM^d, so that d averaging steps lead from the fine lattice to the coarse one. For a coarse configuration e^θ close to the identity, we minimize the Wilson action, rescaled to coarse lattice units, in a local gauge-fixed constraint chart. A gauge transform of the resulting field is a local minimum of the action on the fiber $\{U : T_M^d U = e^\theta\}$, modulo gauge transformations. Following Bałaban [16], we call the minimizer the classical background field. Its action is the effective classical action $E_d(e^\theta)$; divided by the square of the coupling, it is the classical part of the effective densities of Sections 13 and 14.

We prove four results. Theorem 10.6 constructs the background field in a gauge-fixed chart and shows that it depends analytically on θ and that its response to a change of θ decays exponentially. It also proves the local minimality on the fiber $\{U : T_M^d U = e^\theta\}$ described above. Lemma 10.8 shows that E_d is invariant under small coarse gauge transformations, lattice translations, the spatial inversion, and constant color rotations. Theorem 10.10 proves the nested minimization identity

$$E_{d+1}(c) = M \min_{\|z\|_\infty \leq r_z} E_d(b(c,z)),$$

where $b(c,z)$, for $\|z\|_\infty \leq r_z$, parametrizes a local branch of the fiber $\{b : T_M b = c\}$ on the intermediate lattice, in block axial gauge. The minimizer $z_d^*(c)$ is an interior point, it is analytic in c, and the Hessian in z has a positive lower bound that does not depend on d. The factor M is the change of units in three dimensions: with $g_+^2 = Mg^2$ the identity reads $E_{d+1}(c)/g_+^2 = g^{-2} \min_z E_d(b(c,z))$. Finally, Theorem 10.11 proves factorial bounds, in weighted row and column norms of the derivative tensors, for E_d, the background field, its block averages at every intermediate scale (in explicitly given gauges), and its plaquette logarithms divided by the square of the mesh. In all these statements the radii, decay rates, and constants are the same for every d and n.

The background field and the conditional constrained minimizer z^* center the fluctuation coordinates of Section 12, and the constrained Hessian normalizes them. The uniform estimates are the classical input to the contraction estimates of Section 11, and they are used throughout the later parts of the paper, from the analysis on block regions in Section 18 to the weak-coupling analysis in Section 36.

The inputs are the block averaging map of Definition 4.3 and the coefficient norms of Definition 4.6, the estimates for weighted absolute Taylor coefficients in Lemma 5.3, and the effective Maxwell quadratic form of Theorem 8.3. From Section 9 we use the linear constrained extension (Theorem 9.3), the nonlinear constraint parametrization (Theorem 9.6), the convexity of the constrained action (Theorem 9.7), the gauge slice with prescribed coarse gauge (Theorem 9.9), and the transfer of kernels from periodic covers (Lemma 9.2). The only other input from the literature is Bałaban's estimate for the constrained Green operator at the identity background [13, Theorem 3.13, p. 426].

There are two main ideas. The first is to invert the constrained linear curl form from the supremum norm to the C^1 norm, using Bałaban's estimate, and to solve the Euler equation in the constraint coordinates by a contraction argument. The strict convexity proved in Theorem 9.7 then identifies the solution as the minimum. The second concerns the derivative bounds. At the linear level, one averaging step has row sums one and column sums M^{-3}. Its nonlinear part at mesh $\epsilon_j = M^{j-d}$ carries a factor ϵ_j, and these factors are summable over j. Composing nonnegative majorant series with these linear masses, we obtain bounds in which no radius and no decay rate is lost at any scale.

The construction follows Bałaban's treatment of the variational problem in [16]. There the minimizing configuration with prescribed averages is constructed in a projected Landau gauge [16, Theorem 1, p. 279], and its analytic dependence on the averages and the exponential localization of its derivative are studied in [16, Section G, pp. 305–309, and Proposition 9, p. 309]. Our averaging map, coordinate domains, and coefficient norms differ from those of [16], and we give the proofs in full. Uniform regularity and decay of lattice Green functions go back to [9].

After the notation of Section 10.1, the proof proceeds through the constrained Green operator (Section 10.2), the block maps and constraint coordinates (Section 10.3), the minimization (Sections 10.4 to 10.7), and the estimates at intermediate scales (Section 10.8).

10.1 Rescaled fields, scalar products, and norms

Units and constants. Throughout this section $M \geq 3$ is a fixed odd integer, $n \geq 4$ is the side of the coarse lattice, and $d \geq 0$ is the depth. We measure lengths in coarse lattice units, in which the coarse lattice spacing is one. The fine lattice then has side nM^d and mesh $\varepsilon_d = M^{-d}$, and all distances are periodic distances in these units. On the torus of side L the coarse spacing is $a_n = L/n$, and the physical length of the fine mesh is $a_n \varepsilon_d$. The coupling plays no role in this section; in particular ε_d is unrelated to g_n^2.

All constants, radii, and decay rates in this section depend only on M, on fixed larger complex and real coordinate domains and the ratios of their radii, and on finitely many spatial decay rates chosen in the proofs, as in item (2) of Definition 4.14. In particular, they do not depend on d, n, L, or γ. The integer J of a renormalization group step does

not enter the constructions of this section, and the constants are also independent of J. A constant that carries a derivative order s as an index may depend on s. In proofs, K denotes a constant of this kind whose value may change from line to line. The auxiliary localization size used in Lemma 10.2 is a fixed function of M.

We identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ as in Section 4. A color vector has its Euclidean norm; a multilinear color block T of order s has the induced norm $|T| = \sup_{|v_1|,\dots,|v_s|\le 1} |T[v_1,\dots,v_s]|$, with the corresponding absolute value for scalar output. Complex fields take values in the complexification of $\mathfrak{su}(2)$, and the scalar products below are extended bilinearly to them. For a field and its first differences in the three coordinate directions we use the maximum of the norms of the four components. This maximum and the C^1_ε norm below are equivalent up to a fixed factor, which enters only at the beginning and at the end of an estimate.

Fields and scalar products. The operators Q_d, $\mathsf{P}^{\mathrm{div}}_d$, H_d, and D_d of the following definition are constructed in Section 9; here we only name them.

Definition 10.1. Fix $n \ge 4$ and $d \ge 0$, and put $\varepsilon = \varepsilon_d = M^{-d}$. A rescaled lattice connection is a function A on the positive links of the fine lattice of side nM^d, with values in $\mathfrak{su}(2)$ or its complexification; it determines the configuration $e^{\varepsilon A}$, $(e^{\varepsilon A})_e = \exp(\varepsilon A_e)$. Let ∇_ε be the collection of all componentwise forward differences divided by ε, and let d_ε be the lattice exterior derivative (the curl) divided by ε. Set

$$\mathscr{S}_\varepsilon(A) = 4\varepsilon^{-1}\sum_p q((e^{\varepsilon A})_p), \qquad \langle A, B\rangle_\varepsilon = \varepsilon^3 \sum_e \langle A_e, B_e\rangle, \qquad \|A\|_{C^1_\varepsilon} = \|A\|_\infty + \|\nabla_\varepsilon A\|_\infty,$$

where p runs over the positive fine plaquettes and e over the positive fine links. We also write $\mathscr{S}_\varepsilon(U)$ for the same sum $4\varepsilon^{-1}\sum_p q(U_p)$ at a fine configuration U. On coarse fields we use the counting scalar product $\sum_c \langle \theta_c, \eta_c\rangle$, where c runs over the positive coarse links.

Let Q_d be the centered linear block average, $\mathsf{P}^{\mathrm{div}}_d$ the orthogonal projection onto $\mathscr{L}_\varepsilon \ker Q'_d$, where Q'_d is the scalar centered box average and $\mathscr{L}_\varepsilon$ the positive scalar Laplacian, and H_d the linear constrained extension, all as in Theorem 9.3, where they are denoted Q_d, P_d, and H_d. Thus

$$\mathsf{Q}_d\mathsf{H}_d = I, \qquad \mathsf{P}^{\mathrm{div}}_d \operatorname{div}_\varepsilon \mathsf{H}_d = 0.$$

Write

$$\mathcal{V}_d = \ker \mathsf{Q}_d \cap \ker(\mathsf{P}^{\mathrm{div}}_d \operatorname{div}_\varepsilon).$$

The constrained Green operator C_d is defined by the conditions $\mathsf{C}_d f \in \mathcal{V}_d$ and

$$\langle d_\varepsilon v, d_\varepsilon \mathsf{C}_d f\rangle_\varepsilon = \langle v, f\rangle_\varepsilon \qquad (v \in \mathcal{V}_d);$$

its existence is proved in Lemma 10.2. Let D_d be the coarse correction and $\mathsf{P}^{\mathrm{cl}}_d$ the nonlinear constraint parametrization of Theorem 9.6, where they are denoted $\mathscr{D}_d$ and Ξ_d:

$$\mathsf{P}^{\mathrm{cl}}_d(w) = w - \mathsf{H}_d\mathsf{D}_d(w), \qquad w = v + \mathsf{H}_d\theta, \quad v \in \mathcal{V}_d.$$

Here $\mathsf{D}_d(0) = D\mathsf{D}_d(0) = 0$, and the field $\mathsf{P}_d^{\mathrm{cl}}(w)$ has gauge-corrected block average e^θ and projected divergence $\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon \mathsf{P}_d^{\mathrm{cl}}(w) = 0$; the gauge-corrected block average is defined by the recursion (10.9) below. A superscript $\sharp$ denotes the adjoint with respect to the fine and coarse scalar products just defined, with the bilinear transpose on complexifications. In particular

$$((D\mathsf{D}_d(w))^\sharp u)_e = \varepsilon^{-3} \sum_c (D_{w_e} \mathsf{D}_{d,c}(w))^T u_c.$$

None of these objects removes the constant (harmonic) part of the coarse field θ.

The letter d denotes the depth, and d_ε the exterior derivative. The constrained Green operator is C_d, while in Section 10.7 the symbol $\mathsf{C}_{c'}$, indexed by a coarse link c', denotes the block averaging constraint of Definition 4.3. In Sections 10.3 and 10.8 we write $\epsilon_j = M^{j-d}$ for the mesh of the lattice reached after j averaging steps from the fine lattice; thus $\epsilon_0 = \varepsilon_d$ is the fine mesh, and the coarse mesh is $\epsilon_d = 1$. The variable w of Definition 10.1 is a fine field; it is not one of the normalized fluctuation variables of later sections, which do not occur here.

Coefficient norms. We use the weighted absolute coefficient norms of Definition 4.6. Let T be a tensor with one output index x and input indices $y_1, \dots, y_s$. Typically T is the derivative of order s, at a given point, of a map between spaces of fields, and a block index consists of a link of some lattice and, where relevant, a component type; scalar color indices are internal to the multilinear block and are not summed again in the row and column sums below. Each index has a spatial location. Let $\mathsf{t}(x; y_1, \dots, y_s)$ be the length, in coarse lattice units, of a shortest connected network containing the locations of all the indices; internal vertices are allowed. For $a \geq 0$ the output row and the counting input column of T are

$$\|T\|_{a,\mathrm{out}} = \sup_x \sum_{y_1,\dots,y_s} e^{a\mathsf{t}} |T(x; y_1, \dots, y_s)|,$$
$$\|T\|_{a,\mathrm{input,count}} = \sup_{y_1} \sum_x \sum_{y_2,\dots,y_s} e^{a\mathsf{t}} |T(x; y_1, \dots, y_s)|,$$

where $|\cdot|$ is the norm of the color block and $\mathsf{t} = \mathsf{t}(x; y_1, \dots, y_s)$. A derivative tensor is symmetric in its inputs, so fixing y_1 is the same as fixing any one input. If the output lives on a lattice of mesh a_{out} and the inputs on a lattice of mesh a_{in}, the volume-normalized input column of Definition 4.6 is $(a_{\mathrm{out}}/a_{\mathrm{in}})^3$ times the counting column. For a tensor with coarse inputs and outputs at mesh ε we write $\|T\|_{a,\mathrm{input};\varepsilon^3}$ for the volume-normalized column, which is ε^3 times the counting column. For a scalar function E of a coarse field, $\|D^s E\|_{a,\mathrm{root}}$ is the supremum over one argument of the weighted sum of $|D^s E|$ over the remaining arguments; by symmetry every argument may serve as the fixed one. For a matrix T, $\|T\|_{a,\mathrm{row+col}}$ is the sum of the output row and the counting input column. For matrices whose input and output indices have the same mesh, counting

and volume-normalized columns coincide. For these matrices, and with the same spatial weights, the sum used here lies between the maximum used in Sections 4, 14, and 31 and twice that maximum. The bounds imported from Theorem 8.3 use the same sum. The matrices measured in this sum in Lemma 10.9 and Theorem 10.10 have their input and output indices on one lattice, and their upper bounds therefore imply the corresponding maximum-norm bounds in Section 14 with unchanged constants. For matrices between different meshes, the column measures must first be related by the factor $(a_{\mathrm{out}}/a_{\mathrm{in}})^3$. We omit a from the notation when $a = 0$. These norms are sums of the norms of the Taylor coefficient blocks. When the scalar-index calculus of Lemma 5.3 is used, we expand each block in a fixed orthonormal color basis. At degree s, block-to-scalar and scalar-to-block bounds cost at most a fixed factor exponential in $s + 1$; these factors are absorbed once into the common nonlinear coefficient majorants and their radii. Exact scalar linear masses, such as those of the identity and averaging maps, are kept separately. No equivalence uniform in the degree is asserted. A bound on directional derivatives, or on the values of an analytic map, does not control them, which is why several proofs below work with majorant series for the coefficients.

For example, $D_\theta^2 A(\theta)$, for a map $\theta \mapsto A(\theta)$ from coarse fields to fine fields, has a fine output link e and coarse input links c_1, c_2. Its output row is $\sup_e \sum_{c_1,c_2} e^{a\mathsf{t}(e;c_1,c_2)} |D_\theta^2 A(\theta)(e;c_1,c_2)|$, and its volume-normalized input column is

$$\sup_{c_1} \varepsilon^3 \sum_e \sum_{c_2} e^{a\mathsf{t}(e;c_1,c_2)} |D_\theta^2 A(\theta)(e;c_1,c_2)|.$$

There are of order ε^{-3} fine links within distance one of c_1, and the factor ε^3 makes their contribution of order one. Since $\mathsf{t}(e;c_1,c_2)$ is at least each of the distances $\mathrm{dist}(e,c_1)$, $\mathrm{dist}(e,c_2)$, and $\mathrm{dist}(c_1,c_2)$, bounds on these norms give exponential decay in all three distances.

Normalizations. The following factors occur repeatedly.

1. The fine scalar product carries the factor ε^3 and the coarse one does not. For a linear map L from fine to coarse fields with kernel $L(c,e)$, the adjoint is $(L^\sharp u)_e = \varepsilon^{-3} \sum_c L(c,e)^T u_c$, and for a linear map L from coarse to fine fields, $(L^\sharp f)_c = \varepsilon^3 \sum_e L(e,c)^T f_e$.
2. By Theorem 9.3, the kernels of H_d and $\nabla_\varepsilon \mathsf{H}_d$ have bounded weighted rows and bounded volume-normalized columns

$$\sup_c \varepsilon^3 \sum_e e^{a\,\mathrm{dist}(e,c)} \big(|\mathsf{H}_d(e,c)| + |\nabla_\varepsilon \mathsf{H}_d(e,c)|\big) \le C_a;$$

   their counting columns are of order ε^{-3}.
3. The derivatives of D_d have bounded output rows, and their counting columns with a fine input fixed carry the factor ε^3 (Theorem 9.6 and Lemma 10.4 below). When H_d

and a derivative of D_d are contracted over a fine index, the factors ε^{-3} and ε^3 cancel. In the adjoint $(D\mathsf{D}_d)^\sharp$, the factor ε^{-3} of item (1) cancels the factor ε^3 of the column.

4. The fine lattice at depth d over a coarse lattice of side Mm is also the fine lattice at depth $d+1$ over the coarse lattice of side m. In the new coarse units the fine mesh is ε/M. The rescaled connection of a fixed configuration is multiplied by M, its first differences by M^2, and $\mathscr{S}_{\varepsilon/M} = M\mathscr{S}_\varepsilon$. With the couplings $g_+^2 = Mg^2$, the action divided by the square of the coupling is unchanged.

10.2 The constrained Green operator

We invert the Euler equation of the variational problem with the operator C_d of Definition 10.1. A bound on C_d from the supremum norm to the supremum norm of the field and its first differences, uniform in d and n, does not follow from the coercivity of the curl form on $\mathcal{V}_d$. We obtain it from Bałaban's estimate for the constrained Green operator. A unit block below is a cube of side one in coarse lattice units, and $\|u\|_{C^1_\varepsilon(x)}$ is the supremum of $|u|$ and $|\nabla_\varepsilon u|$ over the fine links in the unit block x.

Lemma 10.2. *The operator C_d of Definition 10.1 exists, is real, and is self-adjoint with respect to $\langle\cdot,\cdot\rangle_\varepsilon$. There exist $b, K > 0$ such that the counting kernels of C_d and of $\nabla_{\varepsilon_d}\mathsf{C}_d$ satisfy*

$$\sup_e \sum_f e^{b\,\mathrm{dist}(e,f)} \big(|\mathsf{C}_d(e,f)| + |\nabla_{\varepsilon_d}\mathsf{C}_d(e,f)|\big) \le K. \tag{10.1}$$

Further, there exist $a, C > 0$ such that, for every fine field f,

$$\|\mathsf{C}_d f\|_{C^1_{\varepsilon_d}} \le C\|f\|_\infty,$$

and, if f is supported in one unit block y, then for every unit block x

$$\|\mathsf{C}_d f\|_{C^1_{\varepsilon_d}(x)} \le Ce^{-a\,\mathrm{dist}(x,y)}\|f\|_\infty.$$

The counting kernel is defined by $(\mathsf{C}_d f)_e = \sum_f \mathsf{C}_d(e,f) f_f$; since the variational definition uses $\langle\cdot,\cdot\rangle_\varepsilon$, it already contains the factor ε^3 of the source. The first bound in the lemma follows from (10.1).

Proof of Lemma 10.2. Existence and self-adjointness are finite-dimensional linear algebra (Step 1). The estimates come from [13, Theorem 3.13, p. 426], which applies to periodic tori whose sides satisfy divisibility conditions relative to fixed auxiliary localization sizes. We first work on a finite periodic cover of side kn which satisfies these conditions (Step 2), identify C_d on the cover with Bałaban's constrained Green operator (Step 3), apply his estimate (Step 4), and convert it into absolute row bounds (Step 5). Finally we restrict the cover operator to fields invariant under the deck translations (Step 6) and transfer the estimates to side n (Step 7). Throughout, $\varepsilon = \varepsilon_d$.

Step 1: finite solvability. On a torus of any finite side $s \geq 4$, the curl coercivity of Theorem 9.3 makes the form $\langle d_\varepsilon u, d_\varepsilon v\rangle_\varepsilon$ positive definite on $\mathcal{V}_d^{(s)}$, the space $\mathcal{V}_d$ on that torus. Inverting this form after restricting the source pairing $v \mapsto \langle v, f\rangle_\varepsilon$ to $\mathcal{V}_d^{(s)}$, we obtain a unique operator $\mathsf{C}_d^{(s)}$ with the variational definition, and it is real and self-adjoint. If $d = 0$, then $\mathsf{Q}_0 = I$, $\mathcal{V}_0 = \{0\}$, and $\mathsf{C}_0 = 0$, and all assertions hold. We assume $d \geq 1$ from now on.

Step 2: a cover satisfying the conditions of the literature. Fix the admissible auxiliary integers $R_0, M_0, K_{\mathrm{loc}}$ of Lemma 9.2, and define

$$M_{\mathrm{loc}} = K_{\mathrm{loc}} R_0 M_0 \geq M_1(3, M), \qquad k = 6M_{\mathrm{loc}}, \qquad s = kn.$$

These integers depend only on M, not on n or d. At mesh ε the cover has fine side knM^d. The periodic torus of [10, Section A, pp. 17–18] has sides $2L_\mu$ with general side parameters; we take $2L_\mu = kn$ in coarse units, which is possible since k is even. For the scale domains of [11, (2.1)–(2.2), p. 224] we take every domain to be the whole cover. Since their complements are empty, the separation condition imposes no boundary restriction. At scale j the side of the cover, in the lattice units of that scale, is knM^{d-j}, which is divisible by M_{loc}. Both endpoints of the symmetric fundamental interval are boundaries of auxiliary blocks, since $knM^{d-j}/2 = 3M_{\mathrm{loc}} nM^{d-j}$. Hence the auxiliary blocks tile every scale domain.

The domain condition in [13, Section A, p. 393] is the same. The subdivisions there use the auxiliary sizes R_0, M_0, and $K_{\mathrm{loc}} R_0 M_0$. The localization construction of [13, Section C, pp. 408–409 and 415] uses enlarged localization cubes up to $\widetilde{\square}^8$ of side $18M_{\mathrm{loc}}$ in the units of their own scale. Their side in coarse units is at most $18M_{\mathrm{loc}}$. Since $kn \geq 24M_{\mathrm{loc}}$, these cubes have embedded lifts in the cover. This verifies the subdivision, divisibility, and size requirements and the empty boundary condition, which are the geometric conclusions recorded in Lemma 9.2. A lower bound on n alone would not give these requirements on the torus of side n; this is why we pass to the cover.

Step 3: identification of the operator on the cover. We use the identity background $U = I$, zero background force, and zero local background potentials. Both regularity conditions (3.35)–(3.36) of [13, pp. 396–397] then have vanishing left sides. The dimension is 3, the blocking factor is our odd M, and the auxiliary localization size is M_{loc}. The constrained Gaussian covariance is defined in [13, (3.148), p. 425] and represented in [13, (3.153), p. 426]; at this background the additional background quadratic terms vanish.

The constraints match ours. By Theorem 9.3, the centered translation T_d conjugates the published vector average to Q_d, that is, $\mathsf{Q}_d = Q_d^{\mathrm{pub}} \mathsf{T}_d$. The scalar centered box average Q'_d is obtained from the published scalar average by the same fine translation. Since the translation is orthogonal and commutes with the scalar Laplacian, it also conjugates the orthogonal projection onto $\mathscr{L}_\varepsilon \ker Q_d'^{\mathrm{pub}}$ to $\mathsf{P}_d^{\mathrm{div}}$. The two constraints become $\mathsf{Q}_d v = 0$

and $\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon v = 0$. The translation has bounded length in coarse units and preserves the fine scalar product. It changes localization distances and block assignments by at most a fixed collar, which is absorbed into the constants.

The penalty used in [13] does not change the operator on $\mathcal{V}_d$. For a penalty $\alpha > 0$, we define

$$\mathscr{D}^0_{d,\alpha} = d_\varepsilon^* d_\varepsilon + \nabla_\varepsilon \mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon + \alpha \mathsf{Q}_d^\sharp \mathsf{Q}_d.$$

For $u, v \in \mathcal{V}_d$ its pairing is $\langle d_\varepsilon v, d_\varepsilon u \rangle_\varepsilon$, since the last two terms vanish by the two constraints. In particular the covariance on $\mathcal{V}_d$ is the same whether the positive homogeneous penalty of [13], with its parameter equal to one, or the penalty $\alpha = 1$ is used. The Gaussian quadratic form is $\|d_\varepsilon v\|_\varepsilon^2/2$, with the source pairing $\langle v, f \rangle_\varepsilon$. No constant mode has to be removed separately: a constant field with zero block average is zero. By uniqueness of this finite variational problem, the translated published covariance on the cover equals $\widehat{\mathsf{C}}_d = \mathsf{C}_d^{(kn)}$.

Step 4: Bałaban's estimate on the cover. We use a special case. On a periodic torus satisfying the conditions verified in Step 2, at the identity background with zero background force and zero local background potentials, [13, Theorem 3.13, p. 426] transfers [13, Theorem 3.3, p. 399] to the constrained covariance (3.148). Its prerequisite coarse inverse estimate (3.132), assumed immediately before Theorem 3.13, follows here from [11, Proposition 2.7, (2.149), p. 249]: at this background the force-dependent quadratic corrections vanish, and the flat operator is identified with that of [11] immediately after [13, (3.26), p. 395]. We use the potential and first-difference entries of the localized estimate [13, (3.42), p. 397]; their global versions are [13, (3.47), p. 398]. The scale in these entries and in the coarse inverse bound is $M^d \varepsilon = 1$, so all scale factors equal one. With the hypotheses checked in Steps 2 and 3, the result on the cover is: there exist $K_0, b_0 > 0$ such that

$$\|\widehat{\mathsf{C}}_d f\|_{C^1_\varepsilon(\widehat{x})} \le K_0 e^{-b_0 \operatorname{dist}_{kn}(\widehat{x},\widehat{y})} \|f\|_\infty \quad \text{if } \operatorname{supp} f \text{ is contained in the unit block } \widehat{y}. \tag{10.2}$$

The entry for the covariant Laplacian, which is excepted in Theorem 3.13, is not used.

Step 5: absolute rows on the cover. Fix one scalar output component at a fine link $\widehat{e}$ and one source unit block, and let each scalar source coordinate in that block be the sign of the real kernel coefficient of that output. The supremum norm of this source is bounded by a fixed color factor, and its output at $\widehat{e}$ is the sum of the absolute values of the coefficients in the block. Applying (10.2) to this source, and repeating for the finitely many color components and first-difference outputs, we find that the sum of the absolute kernel coefficients over a unit block $\widehat{y}$ is at most $K_0' e^{-b_0 \operatorname{dist}_{kn}(\widehat{x},\widehat{y})}$, where $\widehat{x}$ is the unit block containing $\widehat{e}$. For $0 < b < b_0$, summing over source blocks, we obtain

$$\sup_{\widehat{e}} \sum_{\widehat{f}} e^{b \operatorname{dist}_{kn}(\widehat{e},\widehat{f})} \left(|\widehat{\mathsf{C}}_d(\widehat{e},\widehat{f})| + |\nabla_\varepsilon \widehat{\mathsf{C}}_d(\widehat{e},\widehat{f})| \right) \le K_1. \tag{10.3}$$

In the summation we used that the number of unit blocks in a shell of radius R and thickness one is at most $K(1+R)^2$, uniformly on periodic tori, and that the remaining exponential is summable. The counting kernel already contains the factor ε^3 of the source, and the choice of signs introduces no factor that counts the fine source coordinates.

Step 6: restriction to deck-invariant fields. Let π^* repeat fields from side n to side kn, and let $\mathcal{D} = (\mathbb{Z}/k\mathbb{Z})^3$ act on the cover by the translations by $n\eta$, $\eta \in \mathcal{D}$, in coarse units. These translations are integer multiples of every lattice spacing M^{j-d}, $0 \le j \le d$, so they preserve the centered blocks, their patterns at all scales, and the fine translation T_d. The vector and scalar averages, the differences, and the adjoints therefore intertwine π^*. Both the fine and the coarse scalar products satisfy

$$\langle \pi^* u, \pi^* v \rangle_{kn} = k^3 \langle u, v \rangle_n. \tag{10.4}$$

The scalar projection needs an additional argument, since it is not a local operator. We write $\mathcal{S}^{(s)} = \mathscr{L}_\varepsilon \ker Q_d'^{(s)}$; the cover space $\mathcal{S}^{(kn)}$ is invariant under $\mathcal{D}$. Let $g = \mathscr{L}_\varepsilon \lambda$ be an element of $\mathcal{S}^{(kn)}$ invariant under $\mathcal{D}$. The average $\overline{\lambda}$ of λ over $\mathcal{D}$ is of the form $\pi^* \lambda_0$, it still has zero scalar average, and $\mathscr{L}_\varepsilon \overline{\lambda} = g$. Since π^* is injective and $\pi^* Q_d'^{(n)} \lambda_0 = Q_d'^{(kn)} \pi^* \lambda_0 = 0$, we have $Q_d'^{(n)} \lambda_0 = 0$. Conversely, the repetition of every element of $\mathcal{S}^{(n)}$ is an invariant element of $\mathcal{S}^{(kn)}$. We conclude that

$$\mathcal{S}^{(kn)} \cap \operatorname{Fix} \mathcal{D} = \pi^* \mathcal{S}^{(n)}.$$

Let g_0 be an arbitrary scalar field on the torus of side n. Since the orthogonal projection onto $\mathcal{S}^{(kn)}$ commutes with the unitary action of $\mathcal{D}$, the projection of $\pi^* g_0$ is invariant. It is therefore $\pi^* h$ for some $h \in \mathcal{S}^{(n)}$, and $\pi^*(g_0 - h)$ is orthogonal to $\pi^* \mathcal{S}^{(n)}$. By (10.4), $g_0 - h$ is orthogonal to $\mathcal{S}^{(n)}$, that is, $h = \mathsf{P}_d^{\mathrm{div},(n)} g_0$. Combining this with the identities for the averages and the divergence, we obtain

$$\begin{aligned} \mathsf{P}_d^{\mathrm{div},(kn)} \pi^* &= \pi^* \mathsf{P}_d^{\mathrm{div},(n)}, \\ \mathcal{V}_d^{(kn)} \cap \operatorname{Fix} \mathcal{D} &= \pi^* \mathcal{V}_d^{(n)}. \end{aligned} \tag{10.5}$$

This verifies the nonlocal constraint in Lemma 9.2.

Let f be a field on the torus of side n, and let $\widehat{u} = \widehat{\mathsf{C}}_d \pi^* f$. A deck translation preserves $\mathcal{V}_d^{(kn)}$, the curl form, and the source $\pi^* f$, so it maps $\widehat{u}$ to a solution of the same variational problem on the cover. By uniqueness, $\widehat{u} = \pi^* u_0$ for some u_0, and (10.5) implies that $u_0 \in \mathcal{V}_d^{(n)}$. Testing the equation on the cover against $\pi^* v$, $v \in \mathcal{V}_d^{(n)}$, and using (10.4), we have

$$k^3 \langle d_\varepsilon v, d_\varepsilon u_0 \rangle_{\varepsilon,n} = k^3 \langle v, f \rangle_{\varepsilon,n}.$$

Canceling k^3, we see that u_0 solves the variational problem on the torus of side n. By uniqueness,

$$\widehat{\mathsf{C}}_d \pi^* = \pi^* \mathsf{C}_d^{(n)}. \tag{10.6}$$

This establishes the hypotheses of the constrained curl case of Lemma 9.2.

Evaluating (10.6) on a coordinate source, we obtain for any lifts $\widetilde{e}, \widetilde{f}$ the identity of counting kernels

$$\mathsf{C}_d^{(n)}(e,f) = \sum_{\eta \in \mathcal{D}} \widehat{\mathsf{C}}_d(\widetilde{e}, \widetilde{f} + n\eta). \tag{10.7}$$

It also holds after a first difference in the output: differences commute with repetition, also across periodic seams, where the neighboring output is taken at a neighboring lift. There is no factor k^{-3}, because repetition repeats a source rather than averaging it.

Step 7: the estimates on the torus of side n. For every image of the source, $\operatorname{dist}_n(e,f) \le \operatorname{dist}_{kn}(\widetilde{e}, \widetilde{f} + n\eta)$. By the triangle inequality in (10.7) and summation over f and η, the weighted row in (10.1) is bounded by the left side of (10.3), since the images $\widetilde{f} + n\eta$ run over the positive source links of the cover exactly once. The identity for first differences gives the same bound for $\nabla_\varepsilon \mathsf{C}_d^{(n)}$. This proves (10.1), without a multiplicity factor, and the bound from the supremum norm to C^1_ε follows.

If f is supported in one unit block y, then $\pi^* f$ is the sum of the sources in the k^3 image blocks of y. At a lift $\widetilde{x}$ of x, using (10.2) and $\operatorname{dist}_{kn}(\widetilde{x}, \widetilde{y} + n\eta) \ge \operatorname{dist}_n(x,y)$, we obtain

$$\|\mathsf{C}_d^{(n)} f\|_{C^1_\varepsilon(x)} \le K_0 \sum_{\eta \in \mathcal{D}} e^{-b_0 \operatorname{dist}_{kn}(\widetilde{x}, \widetilde{y}+n\eta)} \|f\|_\infty \le k^3 K_0 e^{-b_0 \operatorname{dist}_n(x,y)} \|f\|_\infty.$$

Since the factor k^3 is a fixed function of M, this is the bound for sources in one unit block, for every $n \ge 4$. Taking the smaller of the decay rates and the larger of the constants proves all assertions for real sources. Complex sources follow by splitting into real and imaginary parts. □

10.3 Analytic block maps and constraint coordinates

In this section we prove coefficient bounds for the composition of d gauge-corrected averaging steps and for the coarse correction D_d, with one radius and one decay rate for all derivative orders and all depths. These bounds strengthen the fixed-order estimates of Lemma 8.10 and Theorem 9.6 to convergent majorant series. We first isolate the composition argument, since it is used again in Section 10.8.

Let $d \ge 1$ and $\epsilon_j = M^{j-d}$. By level j we mean the lattice of mesh ϵ_j, that is, of side nM^{d-j}. A label at level j is a pair consisting of a positive link of this lattice and a component type from a fixed finite set (which may consist of one element); a field at level j is a function on the labels at level j with values in the complexification of $\mathfrak{su}(2)$, and $\|A\|_\infty$ is the maximum of $|A|$ over the labels. The norms below are those of Section 10.1, with the component type as part of the index.

Lemma 10.3. *Let $d \geq 1$, and for $0 \leq j < d$ let Φ_j be a holomorphic map from the complex fields A at level j with $\|A\|_\infty < \rho$ to the fields at level $j+1$. Let $K \geq 1$ and $0 < R < \rho/2$, and assume that for $0 \leq j < d$, for every complex field A at level j with $\|A\|_\infty \leq 2R$, and for every $s \geq 2$, the following hold:*

1. $\|\Phi_j(A)\|_\infty \leq \|A\|_\infty + K\epsilon_j\|A\|_\infty^2$;
2. $\|D\Phi_j(A)\|_{\text{out}} \leq 1 + K\epsilon_j R$ *and* $\|D\Phi_j(A)\|_{\text{input,count}} \leq M^{-3}(1 + K\epsilon_j R)$;
3. $\|D^s\Phi_j(A)\|_{\text{out}} \leq s!K^s\epsilon_j$ *and* $\|D^s\Phi_j(A)\|_{\text{input,count}} \leq M^{-3}s!K^s\epsilon_j$;
4. *each output label of Φ_j depends only on the input labels at distance at most $K\epsilon_j$ from it.*

Put $S = 1/(M-1)$ and assume that $8KSR < 1$. For a complex field A at level 0 with $\|A\|_\infty \leq R$ let $A_0 = A$ and $A_{j+1} = \Phi_j(A_j)$. Then A_j is defined and

$$\|A_j\|_\infty \leq \frac{\|A\|_\infty}{1 - KS\|A\|_\infty} \leq 2\|A\|_\infty \qquad (0 \leq j \leq d).$$

Further, for every $b \geq 0$ there exists K_1, depending only on K, M, R, and b, such that for all such A, all $0 \leq j \leq d$, and all $s \geq 1$, the derivatives of the map $A \mapsto A_j$ at A satisfy

$$\|D^s A_j\|_{b,\text{out}} \leq K_1^{s+1}s!, \qquad \|D^s A_j\|_{b,\text{input,count}} \leq (\epsilon_0/\epsilon_j)^3 K_1^{s+1}s!.$$

Each output label of $A \mapsto A_j$ depends only on the labels at level 0 at distance at most $K\epsilon_j/(M-1)$ from it.

Proof of Lemma 10.3. We begin with the values. Let $a_j = \|A_j\|_\infty$. If some a_j vanishes, then by hypothesis (1) every later a_l vanishes, and their bounds are immediate. We may therefore assume that the iterates under consideration are nonzero when we take reciprocals. As long as $a_j \leq 2R$, hypothesis (1) implies $a_{j+1} \leq a_j + K\epsilon_j a_j^2$, and hence, if $a_{j+1} > 0$,

$$\frac{1}{a_{j+1}} \geq \frac{1}{a_j(1 + K\epsilon_j a_j)} \geq \frac{1}{a_j} - K\epsilon_j.$$

Since $\sum_{l<d} \epsilon_l = (1 - M^{-d})/(M-1) \leq S$, by induction on j we have $1/a_j \geq 1/a_0 - KS$, that is, $a_j \leq a_0/(1 - KSa_0)$. As $KSa_0 \leq KSR < 1/8$, this is at most $2a_0 \leq 2R$, which keeps the induction going.

Next we bound the rows. Fix A with $\|A\|_\infty \leq R$, let $R_s(j)$ be the unweighted output row of $D^s A_j$ at A, and let $F_j(t) = \sum_{s \geq 1} R_s(j)t^s/s!$, a power series with nonnegative coefficients. Then $F_0(t) = t$, since $A_0 = A$. We write $F \preceq G$ if every coefficient of $G - F$ is nonnegative. By the chain rule, $D^s(\Phi_j \circ A_j)$ is the sum, over all partitions of $\{1, \dots, s\}$ into blocks $B_1, \dots, B_k$, of $D^k\Phi_j(A_j)[D^{|B_1|}A_j, \dots, D^{|B_k|}A_j]$. By Lemma 5.3, with output rows on all factors, the output row of each term is at most $\|D^k\Phi_j(A_j)\|_{\text{out}} \prod_i R_{|B_i|}(j)$. The

sum over partitions with k blocks has generating function $F_j^k/k!$. Since $\|A_j\|_\infty \le 2R$, hypotheses (2) and (3) give

$$F_{j+1} \preceq (1 + K\epsilon_j R)F_j + K^2\epsilon_j \frac{F_j^2}{1 - KF_j}, \tag{10.8}$$

where we used $\sum_{k\ge 2} K^k \epsilon_j F_j^k = K^2\epsilon_j F_j^2/(1 - KF_j)$. Let $P_j = \prod_{l<j}(1 + K\epsilon_l R)$, so that $P_j \le P = \exp(KRS)$, and let $f_j = F_j/P_j$. At a fixed $t > 0$ with $KF_j(t) \le 1/2$, (10.8) implies

$$f_{j+1}(t) \le f_j(t) + 2K^2 P \epsilon_j f_j(t)^2,$$

because $P_j^2/P_{j+1} \le P$. Fix $t_0 > 0$ with $4KPt_0 \le 1$ and $8K^2PSt_0 \le 1$. By the reciprocal inequality used for the values and induction on j, we have $1/f_j(t_0) \ge 1/t_0 - 2K^2PS \ge 3/(4t_0)$. In particular, $f_j(t_0) \le 2t_0$ and $F_j(t_0) \le 2Pt_0 \le 1/(2K)$, which keeps the induction going. Since the coefficients are nonnegative, $R_s(j) \le s!\, t_0^{-s} F_j(t_0) \le 2Pt_0\, s!\, t_0^{-s}$.

For the columns, let $C_s(j)$ be the unweighted counting column of $D^s A_j$ at A, with one input at level 0 fixed, and define $\widehat{C}_s(j) = (\epsilon_j/\epsilon_0)^3 C_s(j)$ and $H_j(t) = \sum_{s\ge 1} \widehat{C}_s(j) t^{s-1}/(s-1)!$. Then $H_0 = 1$. In the chain rule, the block containing the fixed input contributes the counting column of $D^{|B|}A_j$, the corresponding slot of $D^k\Phi_j$ contributes the counting column of $D^k\Phi_j$, and all other blocks contribute output rows; this is the contraction rule of Lemma 5.3 with the column on the branch containing the fixed index. The factor M^{-3} in hypotheses (2) and (3) cancels the change from $(\epsilon_j/\epsilon_0)^3$ to $(\epsilon_{j+1}/\epsilon_0)^3 = M^3(\epsilon_j/\epsilon_0)^3$. With $\psi(z) = Kz^2/(1 - Kz)$ we obtain

$$H_{j+1} \preceq \big[1 + K\epsilon_j R + K\epsilon_j \psi'(F_j)\big] H_j.$$

At t_0 we have $KF_j(t_0) \le 1/2$ and $\psi'(F_j(t_0)) \le 4$, and therefore $H_j(t_0) \le \exp(KS(R+4))$. Since the coefficients are nonnegative, $C_s(j) \le (\epsilon_0/\epsilon_j)^3 (s-1)!\, t_0^{1-s} H_j(t_0)$.

Finally we add the spatial weights. By hypothesis (4) and induction, each output label of A_j depends only on labels at level 0 at distance at most $\sum_{l<j} K\epsilon_l \le K\epsilon_j/(M-1)$. A nonzero coefficient of $D^s A_j$ therefore has connecting length $\mathrm{t} \le sK\epsilon_j \le sK$, and the weight $e^{b\mathrm{t}}$ is at most e^{bKs}. Combining this with the unweighted bounds and enlarging the constant, we obtain the claim. □

We now apply this lemma to the gauge-corrected block averages. They are defined by the explicit path products of the following lemma, which also gives the gauge transformation relating them to the iterates of T_M.

Lemma 10.4. *Let $d \geq 0$ and $\epsilon_j = M^{j-d}$. For a field A on the lattice of mesh ϵ, with $V = e^{\epsilon A}$, and for a vertex r of the next coarser lattice, set*

$$\begin{aligned} k_\epsilon(A)_r &= \exp\Big\{-M^{-3} \sum_{x \in \mathsf{B}_M(r)} \operatorname{Log} V(\sigma_{r,x})\Big\}, \\ (\mathcal{M}_\epsilon(V))_{rs} &= k_\epsilon(A)_r (T_M V)_{rs} k_\epsilon(A)_s^{-1}, \\ \Psi_\epsilon(A) &= (M\epsilon)^{-1} \operatorname{Log} \mathcal{M}_\epsilon(e^{\epsilon A}), \qquad A_{j+1} = \Psi_{\epsilon_j}(A_j), \quad A_0 = A. \end{aligned} \tag{10.9}$$

The block $\mathsf{B}_M(r)$, the paths $\sigma_{r,x}$, and the coarse link (r,s) are those of Definition 4.3, at the current fine mesh. This is the gauge-corrected averaging step of Lemma 8.10 at the identity background, and e^{A_d} is the gauge-corrected block average of $e^{\epsilon_0 A}$; in the notation of Theorem 9.6, $A_d = \mathsf{F}_d(A)$. There exist $R, R', b, K > 0$ such that the following hold for every complex fine field A with $\|A\|_\infty \leq R$.

1. *For $0 \leq j \leq d$ and $s \geq 1$, the map $A \mapsto A_j$ satisfies $\|D^s A_j\|_{b,\mathrm{out}} \leq K^{s+1} s!$ and $\|D^s A_j\|_{b,\mathrm{input,count}} \leq (\epsilon_0/\epsilon_j)^3 K^{s+1} s!$.*
2. *The nonlinear remainder $\mathscr{E}_d^{\mathrm{av}}(A) = A_d - \mathsf{Q}_d A$ of Theorem 9.6 has no constant or linear term. Its first derivative has output row at most $K\|A\|_\infty$ and counting input column at most $\epsilon_0^3 K \|A\|_\infty$, and its derivatives of order $s \geq 2$ have counting input columns at most $\epsilon_0^3 K^{s+1} s!$.*
3. *For the coarse correction D_d, which has fine inputs and coarse outputs, let $\|\cdot\|_{b,\mathrm{input,count}}$ be the counting sum over the coarse output and all remaining fine inputs with one fine input fixed. Since $\epsilon_0 = \varepsilon_d$, the volume-normalized column is ϵ_0^{-3} times this sum. For complex fine fields w with $\|w\|_\infty \leq R'$,*

$$\begin{aligned} \|D^s \mathsf{D}_d\|_{b,\mathrm{out}} &\leq K^s s!, \qquad & \|D^s \mathsf{D}_d\|_{b,\mathrm{input,count}} &\leq \epsilon_0^3 K^s s! \quad (s \geq 2), \\ \|D\mathsf{D}_d(w)\|_{b,\mathrm{out}} &\leq K\|w\|_\infty, & \|D\mathsf{D}_d(w)\|_{b,\mathrm{input,count}} &\leq \epsilon_0^3 K \|w\|_\infty. \end{aligned} \tag{10.10}$$

4. *There is a gauge transformation G_j at the vertices of level j, holomorphic in A, such that*

$$e^{\epsilon_j A_j} = (T_M^j e^{\epsilon_0 A})^{G_j}, \qquad G_0 = I, \quad G_{j+1} = k_{\epsilon_j}(A_j)\, G_j|_{\text{next roots}}, \tag{10.11}$$

where $G_j|_{\text{next roots}}$ is the restriction of G_j to the vertices of level $j+1$. Its final logarithm satisfies $\|\operatorname{Log} G_d\|_\infty \leq K\|A\|_\infty$.

For $d = 0$ the recursion is empty, $G_0 = I$, and $\mathsf{D}_0 = 0$.

We write $G_d(A)$ for the gauge transformation G_d of (10.11) determined by the fine field A; it is the composed gauge transformation denoted $\mathsf{G}_d(A)$ in Theorem 9.9. The definition (10.9) fixes the representatives used later. They are not the iterates of the block averaging map, to which they are related through (10.11).

Proof of Lemma 10.4. For $d = 0$ there is nothing to prove, so let $d \geq 1$. We first prove bounds for one step (Steps 1–3), then apply Lemma 10.3 (Step 4), identify the linear part of the composition and the remainder (Step 5), solve the equation for D_d (Step 6), and finally prove the gauge identity (Steps 7 and 8).

Step 1: local analyticity. Every expression in (10.9) for one output involves at most q_M distinct link variables and paths of length at most q_M, for an integer q_M depending only on M. We shrink a complex polydisc of link angles until every exponential, relative logarithm, and final logarithm in (10.9) is holomorphic there. Cauchy estimates on smaller closed polydiscs and the geometric series for the matrix logarithm give a bound depending only on q_M and the ratio of the radii. Cauchy's formula in these $3q_M$ scalar variables bounds the sum of the absolute values of the derivatives of order s by $s!K_M^s$. The factor from counting the local variables is exponential in s and does not depend on the size of the lattice.

Step 2: the linear part at zero. Before composing the steps, we compute the linear masses exactly. At zero, the derivative of $\operatorname{Log} T_M V$ is the average of the sums of A along the paths $F_{c,x}$. The derivative of the reference path enters each relative logarithm with a negative sign and cancels the derivative of the final reference factor. The derivative of k_ϵ cancels the averaged sums along the initial root paths, and the derivative of its inverse at the other endpoint cancels the averaged sums along the final root paths. After division by $M\epsilon$ there remains the positive operator

$$(QA)_{(r,r+Me_i)} = M^{-4} \sum_{x \in \mathsf{B}_M(r)} \sum_{l=0}^{M-1} A_i(x + le_i). \tag{10.12}$$

Its row mass is 1. A fixed positive fine link $(y, y + e_i)$ of orientation i occurs for the M starting points $x = y - le_i$, $0 \le l < M$, each of which lies in exactly one centered block. Its counting column mass is $M \cdot M^{-4} = M^{-3}$. The starting points are distinct on every input lattice of side at least $4M$, and the same count holds across periodic seams.

Step 3: bounds for one step. We write $\Psi_\epsilon(A) = (M\epsilon)^{-1} P(\epsilon A)$, where P is the fixed local analytic expression of Step 1. Its linear part is Q, and its derivatives of order $s \ge 2$ carry the factor ϵ^{s-1}/M. Together with Steps 1 and 2, this gives, on $\|A\|_\infty \le 2R$,

$$\begin{aligned} &\|D\Psi_\epsilon(A)\|_{\text{out}} \le 1 + K_M \epsilon R, \quad \|D\Psi_\epsilon(A)\|_{\text{input,count}} \le M^{-3}(1 + K_M \epsilon R), \\ &\|D^s\Psi_\epsilon(A)\|_{\text{out}} \le s! K_M^s \epsilon, \qquad \|D^s\Psi_\epsilon(A)\|_{\text{input,count}} \le M^{-3} s! K_M^s \epsilon \quad (s \ge 2). \end{aligned} \tag{10.13}$$

For the columns of the nonlinear part, we note that a fixed input lies in at most q'_M output stencils. Bounding each such coefficient by the row bound and absorbing $M^3 q'_M$ into K_M, we obtain the column bounds. This counting of incidences is applied only to the nonlinear part; the leading factor M^{-3} is the count of Step 2. Since Q has nonnegative scalar coefficients and row mass one, $\|QA\|_\infty \le \|A\|_\infty$. Using this, the Taylor series of $\Psi_\epsilon - Q$ at zero, and (10.13), we obtain, after enlarging K_M,

$$\|\Psi_\epsilon(A)\|_\infty \le \|A\|_\infty + K_M \epsilon \|A\|_\infty^2.$$

Each output of Ψ_ϵ depends only on the fine links in the two blocks of its coarse link, that is, on labels within distance $K_M \epsilon$.

Step 4: composition. By Step 3, the maps $\Phi_j = \Psi_{\epsilon_j}$ satisfy the hypotheses of Lemma 10.3 with $K = K_M$. We choose R so that $8K_M SR < 1$ and the polydisc of Step 1 contains the ball of radius $2R$. By the lemma, the recursion is defined on the common complex ball $\|A\|_\infty \le R$, with $\|A_j\|_\infty \le 2\|A\|_\infty$, and assertion (1) holds at a fixed exponent $b > 0$. Every composed expression depends only on initial inputs within distance $K_M\epsilon_j$ of its output. Its connecting length is therefore at most $K_M s\epsilon_j$ at order s, and the spatial weight does not reduce the rate at any scale or order.

Step 5: the linear map at level d and the remainder. At $j = d$ the linear part of $A \mapsto A_d$ is the product of the kernels (10.12), and it equals Q_d. The centered sums of the base-M digits give the centered box, and the segment sums give its convolution in the direction of the link. This is also the linear map of Theorem 9.6. Subtracting it, we obtain $\mathscr{E}_d^{\mathrm{av}}$, which has no constant or linear term. Its derivatives of order $s \ge 2$ are those of $A \mapsto A_d$, and assertion (1) at $j = d$, where $\epsilon_d = 1$, gives the factor ϵ_0^3 in their counting columns. Integrating the second derivative from zero, we find that the first derivative of $\mathscr{E}_d^{\mathrm{av}}$ has row $O(\|A\|_\infty)$ and counting input column $O(\epsilon_0^3\|A\|_\infty)$. This proves assertion (2).

Step 6: the coarse correction. By Theorem 9.6, D_d is the unique small solution of

$$\mathsf{D}_d(w) = \mathscr{E}_d^{\mathrm{av}}(w - \mathsf{H}_d\mathsf{D}_d(w)).$$

The rows of H_d and its volume-normalized columns are bounded by Theorem 9.3. When H_d is contracted with $\mathscr{E}_d^{\mathrm{av}}$ over a fine index, the counting column $O(\epsilon_0^{-3})$ of H_d cancels the counting input column $O(\epsilon_0^3)$ of $\mathscr{E}_d^{\mathrm{av}}$. In a tensor whose fixed input is a fine input w_e, the factor ϵ_0^3 of $\mathscr{E}_d^{\mathrm{av}}$ is kept instead. The weighted tensor composition inequality of Lemma 5.3 applies at a rate below the decay rate of H_d. It shows that all output rows and all columns with a fixed fine input are majorized by the solution of the scalar equation

$$x = K\frac{(t + Kx)^2}{1 - K(t + Kx)}.$$

This is the implicit-series assertion of Lemma 5.3, with zero linear coefficient in x, and it gives (10.10) at $w = 0$, including the factor ϵ_0^3 at a fixed fine input. To pass to a nonzero point w of a smaller ball of radius r, we translate the series: a series with coefficients bounded by K^{s+1} acquires the binomial sum $\sum_{l\ge 0}\binom{s+l}{l}(Kr)^l = (1 - Kr)^{-s-1}$. Contracting the additional slots with the fixed point w does not increase the connecting length of the external indices. One smaller ball and one rate therefore suffice for all the bounds in (10.10). By the uniqueness statement of Theorem 9.6, the solution of the majorized system is D_d. This proves assertion (3).

Step 7: gauge covariance of the block averaging map. For real configurations and compact gauge transformations, the gauge covariance below, including the Borel extension at

the logarithm singularity, is stated in Proposition 6.2(3). We recall the argument, since we also need its holomorphic version. Let ω be a periodic vertex gauge transformation and $P = e_1 \cdots e_l$ a path from r to s. When we multiply the transformation laws of the links, the adjacent factors $\omega_x^{-1}\omega_x$ cancel, and we obtain

$$(U^\omega)(P) = \omega_r U(P) \omega_s^{-1}.$$

This also covers reversed links, whose transports are the inverses of the transports of the corresponding positive links. Since both $F_{c,x}$ and C_c run from r to s, their relative product transforms by conjugation:

$$(U^\omega)(F_{c,x})(U^\omega)(C_c)^{-1} = \omega_r \big[U(F_{c,x})U(C_c)^{-1}\big]\omega_r^{-1}.$$

On the identity chart of the logarithm, conjugation commutes with Log, as one sees by conjugating each term of the power series of $\mathrm{Log}(I+X)$; it also commutes with the exponential. For compact real gauge transformations, the principal real logarithm away from $-I$ has the same property: the conjugated logarithm exponentiates to the conjugated group element and has the same norm, less than 2π, so it is the principal logarithm of that element. The mean logarithm in the definition of T_M is then conjugated at r, and the final reference factor transforms from r to s. This proves

$$T_M(U^\omega) = (T_M U)^{\omega|_{\mathrm{roots}}}, \tag{10.14}$$

where $\omega|_{\mathrm{roots}}$ is the restriction to the vertices of the coarser lattice. The exceptional case is preserved as well: a relative product equals $-I$ if and only if its conjugate does. In that case both sides use the reference path, for which covariance was just proved. The identity (10.14) therefore holds for the block averaging map with its Borel extension. For complex gauge transformations it holds on the domains where the logarithms before and after the transformation exist, and the power series argument gives its holomorphic version.

Step 8: the composed gauge transformation. Let $V_j = e^{\epsilon_j A_j}$, and suppose that $V_j = (T_M^j V_0)^{G_j}$, which holds for $j = 0$ with $G_0 = I$. By (10.14), $T_M V_j = (T_M^{j+1} V_0)^{G_j|_{\text{next roots}}}$. The definition (10.9) applies the gauge transformation $k_{\epsilon_j}(A_j)$ to this field. Since $(U^a)^b = U^{ba}$, the new gauge transformation is $G_{j+1} = k_{\epsilon_j}(A_j) G_j|_{\text{next roots}}$. This proves (10.11), with the displayed order of the factors, and its holomorphic continuation.

Each $\mathrm{Log}\, k_{\epsilon_j}(A_j)$ has norm at most $K_M \epsilon_j \|A_j\|_\infty \le 2K_M \epsilon_j \|A\|_\infty$, and the sum of these bounds is at most $K\|A\|_\infty$. After reducing R, products of factors this close to the identity stay in a fixed ball of the logarithm. The power series of the product and of its logarithm give the holomorphy and the bound on $\mathrm{Log}\, G_d$, uniformly in j, d, and n. □

10.4 The Euler equation in constraint coordinates

We now write the condition for a critical point of $v \mapsto \mathscr{S}_\varepsilon(\mathsf{P}_d^{\mathrm{cl}}(v + \mathsf{H}_d\theta))$, $v \in \mathcal{V}_d$, as a fixed point equation that can be solved by contraction.

We begin with a heuristic. If A is the restriction to the lattice of a smooth connection, then the logarithm of the plaquette variable is $\varepsilon^2(\partial_i A_j - \partial_j A_i + [A_i, A_j]) + O(\varepsilon^3)$. Since $q(e^X) = |X|^2/8 + O(|X|^4)$, the action $\mathscr{S}_\varepsilon(A)$ is a Riemann sum for one half of $\sum_{i<j} \int |F_{ij}|^2$, and its gradient should be $d_\varepsilon^* d_\varepsilon A$ plus terms of the form $A \nabla A$ and A^3. However, a direct expansion of the lattice gradient produces a quadratic term with the prefactor ε^{-1}, which is of size $\varepsilon^{-1}|A|^2$ unless it cancels. The first part of the following lemma shows that this term vanishes on fields that are constant in each direction, and hence is of order $\|A\|_\infty \|\nabla_\varepsilon A\|_\infty$. The second part records the gradient of the action in the constraint coordinates. Differentiating these coordinates produces the term $(D\mathsf{D}_d)^\sharp \mathsf{J}_d$ below, which must be kept.

Lemma 10.5. *There exist $r_0, a_0, C > 0$ with the following properties for every $d \geq 0$ and $n \geq 4$, with $\varepsilon = \varepsilon_d$.*

1. *Define the gradient $G_\varepsilon(A)$ of the action by $D\mathscr{S}_\varepsilon(A)[v] = \langle G_\varepsilon(A), v \rangle_\varepsilon$. Then*

$$G_\varepsilon(A) = d_\varepsilon^* d_\varepsilon A + N_\varepsilon(A),$$

where d_ε^* is the adjoint of d_ε with respect to the fine scalar products. On the complex C^1_ε ball of radius r_0 the map N_ε is holomorphic, and for $0 < r \leq r_0$,

$$\|N_\varepsilon(A)\|_\infty \leq C\|A\|^2_{C^1_\varepsilon}, \qquad \|N_\varepsilon(A) - N_\varepsilon(B)\|_\infty \leq Cr\|A - B\|_{C^1_\varepsilon} \quad (\|A\|_{C^1_\varepsilon}, \|B\|_{C^1_\varepsilon} \leq r).$$

2. *There is a holomorphic function $\widetilde{N}_\varepsilon(A, G)$ of a fine field A and a triple $G = (G_\mu)_{\mu=1,2,3}$ of independent fine fields, defined on a fixed complex ball, with $\widetilde{N}_\varepsilon(A, \nabla_\varepsilon A) = N_\varepsilon(A)$, where $G_{i,\mu}$ takes the place of $\nabla_{\varepsilon,\mu} A_i$. It vanishes to second order at the origin, and its derivatives of order $s \geq 2$ have output rows and counting input columns at most $C^{s+1} s!$ at the exponent a_0. Each output of $\widetilde{N}_\varepsilon$ depends only on inputs at distance at most $C_M \varepsilon$.*
3. *For $w = v + \mathsf{H}_d \theta$ with $v \in \mathcal{V}_d$, and $A = \mathsf{P}^{\mathrm{cl}}_d(w)$, put*

$$\Delta_d = \mathsf{H}_d^\sharp d_\varepsilon^* d_\varepsilon \mathsf{H}_d, \qquad \mathsf{J}_d(w) = \Delta_d[\theta - \mathsf{D}_d(w)] + \mathsf{H}_d^\sharp N_\varepsilon(A).$$

If $\|v\|_{C^1_\varepsilon} + \|\theta\|_\infty \leq r \leq r_0$, then

$$\|\mathsf{J}_d(w)\|_\infty \leq C(\|\theta\|_\infty + r^2), \qquad \|(D\mathsf{D}_d(w))^\sharp\|_{\infty\to\infty} \leq Cr.$$

The second bound follows from $\sup_e \sum_c e^{a_0 \operatorname{dist}(e,c)} |D_{w_e} \mathsf{D}_{d,c}(w)| \leq C\varepsilon^3 r$. At fixed θ, the maps $v \mapsto N_\varepsilon(A)$ and $v \mapsto (D\mathsf{D}_d(w))^\sharp \mathsf{J}_d(w)$ are Lipschitz from the C^1_ε norm to the supremum norm, with constant Cr.
4. *For every $u \in \mathcal{V}_d$,*

$$D_v \mathscr{S}_\varepsilon(\mathsf{P}^{\mathrm{cl}}_d(w))[u] = \langle d_\varepsilon u, d_\varepsilon v \rangle_\varepsilon + \langle u, N_\varepsilon(A) - (D\mathsf{D}_d(w))^\sharp \mathsf{J}_d(w) \rangle_\varepsilon.$$

Consequently $D_v \mathscr{S}_\varepsilon(\mathsf{P}^{\mathrm{cl}}_d(w))[u] = 0$ for all $u \in \mathcal{V}_d$ if and only if

$$v = -\mathsf{C}_d \big\{ N_\varepsilon(A) - (D\mathsf{D}_d(w))^\sharp \mathsf{J}_d(w) \big\}.$$

In the proof we show that $\mathsf{J}_d(w) = \mathsf{H}_d^\sharp G_\varepsilon(A)$: it is the gradient of the action pulled back to the coarse variables. The bounds of the lemma concern local rows and volume-normalized columns. The sum of $|\mathsf{J}_d(w)|$ over all coarse links is not bounded independently of the volume, and it is not used.

Proof of Lemma 10.5. Parts (1) and (2) are proved in Steps 1–4, part (3) in Steps 5 and 6, and part (4) in Step 7.

Step 1: expansion of the gradient. Fix a positive fine link. Differentiating the finitely many plaquette variables containing it, in the link angles $X_e = \varepsilon A_e$, we see that the gradient $G_\varepsilon(A)$ at that link is $4\varepsilon^{-3}$ times the angle derivative of $\sum_p q$. The common power series of these local expressions has no constant term. Its linear part is $d_\varepsilon^* d_\varepsilon A$, because the quadratic part of the action is $\varepsilon^3 \sum_p |d_\varepsilon A(p)|^2/2$, by $q(e^X) = |X|^2/8 + O(|X|^4)$. A term of degree k in A in the gradient carries the factor ε^{k-3}. After the linear part is subtracted, only the quadratic term can carry a negative power of ε; we write it as $\varepsilon^{-1} P_e(A)$, where P_e is a quadratic polynomial in the values of A near e.

Step 2: the quadratic term vanishes on constant fields. Let $A_i(x) = a_i$ on the whole lattice, for constants a_1, a_2, a_3. A plaquette variable is then the group commutator of $e^{\varepsilon a_i}$ and $e^{\varepsilon a_j}$. Its logarithm begins at degree two, so its value of q begins at degree four, and the cubic part of the action vanishes identically on such fields. At a constant field, translation invariance makes the gradient of the cubic part of the action the same at all links of a given orientation. Differentiating the vanishing cubic polynomial in a_i shows that the number of links of orientation i times this common coefficient is zero. We conclude that $P_e(a) = 0$ for every link e.

Step 3: the nonlinear bounds. Fix a vertex near e as base point, and replace each value $A_i(x)$ in the stencil of P_e by the value $a_i = A_i(\text{base point})$. The differences $A_i(x) - a_i$ are sums of at most K_M first differences times ε. Subtracting the two quadratic polynomials and using $P_e(a) = 0$, we obtain

$$|P_e(A)| \le K\varepsilon \|A\|_\infty \|\nabla_\varepsilon A\|_\infty,$$

and the differential of P_e at A in the direction v is bounded by

$$K\varepsilon\big(\|A\|_\infty \|\nabla_\varepsilon v\|_\infty + \|\nabla_\varepsilon A\|_\infty \|v\|_\infty\big).$$

These identities cancel the apparent inverse power of the mesh before any norm is taken. All terms of degree at least three in the gradient carry nonnegative powers of ε and are given by a uniformly convergent local series; their value and differential are bounded by $K\|A\|_\infty^3$ and $K\|A\|_\infty^2 \|v\|_\infty$. Integrating along the segment from B to A, we obtain the two inequalities of part (1).

Step 4: a version for independent value and difference variables. In the difference of the two quadratic polynomials used in Step 3, we change the arguments one at a time. Every resulting term contains a factor

$$A_i(x) - a_i = \varepsilon \sum_l \pm G_{i,\mu_l}(x_l),$$

a sum of first differences along a path from the base point to x. Treating the remaining local values of A and these variables G as independent, we obtain a polynomial extension of $\varepsilon^{-1}P_e$ which is quadratic in (A, G). Adding the terms of degree at least three, which are local analytic functions of A, defines $\widetilde{N}_\varepsilon(A, G)$. It agrees with $N_\varepsilon(A)$ when $G = \nabla_\varepsilon A$. Cauchy estimates on a fixed smaller polydisc in finitely many variables bound each absolute derivative tensor of order s by $K^{s+1}s!$. Since a fixed input belongs to a bounded number of stencils, this bound holds with any input fixed as well as with the output fixed. Since the stencils have radius $K_M\varepsilon$, a spatial weight contributes a factor at most $e^{a_0 K_M s}$. Since these arguments use polynomial identities and convergent power series, they hold on the complex balls. This proves part (2).

Step 5: the coarse gradient. We use H_d and its minimizing property from Theorem 9.3: $\langle d_\varepsilon \mathsf{H}_d \eta, d_\varepsilon v\rangle_\varepsilon = 0$ for every coarse field η and every $v \in \mathcal{V}_d$. The kernel of Δ_d is $\Delta_d(c, c') = \varepsilon^3 \sum_p (d_\varepsilon \mathsf{H}_d)(p, c)^T (d_\varepsilon \mathsf{H}_d)(p, c')$, the pairing with the fine measure of two curl kernels of H_d. Since each curl kernel is controlled by the bounds of Theorem 9.3 on first differences, Δ_d has bounded weighted rows and columns. The same volume-normalized column bound implies $\|\mathsf{H}_d^\sharp f\|_\infty \le K\|f\|_\infty$. Since $A = v + \mathsf{H}_d[\theta - \mathsf{D}_d(w)]$, the minimizing property gives the identity

$$\mathsf{H}_d^\sharp G_\varepsilon(A) = \Delta_d[\theta - \mathsf{D}_d(w)] + \mathsf{H}_d^\sharp N_\varepsilon(A) = \mathsf{J}_d(w).$$

The bounds on H_d and on D_d (Theorems 9.3 and 9.6) place A in a C^1_ε ball of radius Kr and give $\|\mathsf{D}_d(w)\|_\infty \le Kr^2$. Together with part (1), this proves $\|\mathsf{J}_d(w)\|_\infty \le K(\|\theta\|_\infty + r^2)$.

Step 6: the adjoint of the derivative of the correction. By (10.10), the counting column of $D\mathsf{D}_d(w)$ with a fine input fixed is at most $K\varepsilon^3 r$, and the same holds with the exponential weight. Taking the adjoint multiplies this column by exactly ε^{-3} (item (1) of the normalizations in Section 10.1), which proves $\|(D\mathsf{D}_d(w))^\sharp\|_{\infty\to\infty} \le Kr$, together with its weighted version. The column bound for the second derivative of D_d bounds the difference of these adjoints at w and $w' = v' + \mathsf{H}_d\theta$ by $K\|v - v'\|_{C^1_\varepsilon}$. At fixed θ, the parametrized fields $\mathsf{P}^{\mathrm{cl}}_d(w)$ and $\mathsf{P}^{\mathrm{cl}}_d(w')$ differ by at most $K\|v - v'\|_{C^1_\varepsilon}$ in C^1_ε, the nonlinear terms N_ε by at most $Kr\|v - v'\|_{C^1_\varepsilon}$ by part (1), and J_d by the same amount. By the product rule and $\|\theta\|_\infty \le r$, the difference of $(D\mathsf{D}_d)^\sharp \mathsf{J}_d$ at w and w' is at most $Kr\|v - v'\|_{C^1_\varepsilon}$. This proves part (3).

Step 7: the tested equation. For $u \in \mathcal{V}_d$, the derivative of $v \mapsto \mathsf{P}_d^{\mathrm{cl}}(v + \mathsf{H}_d\theta)$ in the direction u is $u - \mathsf{H}_d D\mathsf{D}_d(w)u$. Testing the derivative of the action against it and using the adjoints, we have

$$\langle u, G_\varepsilon(A)\rangle_\varepsilon - \sum_c \langle (D\mathsf{D}_d(w)u)_c, \mathsf{J}_d(w)_c\rangle = \langle d_\varepsilon u, d_\varepsilon v\rangle_\varepsilon + \langle u, N_\varepsilon(A) - (D\mathsf{D}_d(w))^\sharp \mathsf{J}_d(w)\rangle_\varepsilon.$$

Here we used $\langle u, d_\varepsilon^* d_\varepsilon A\rangle_\varepsilon = \langle d_\varepsilon u, d_\varepsilon v\rangle_\varepsilon$, which follows from the minimizing property of H_d; only the linear contribution of the extension is removed by this property. By the defining identity of C_d, the right side vanishes for every $u \in \mathcal{V}_d$ if and only if v satisfies the displayed fixed point equation; uniqueness follows from the coercivity of the curl form on $\mathcal{V}_d$. We decrease r_0 and a_0 to the minima of the finitely many radii and rates used above, and take C larger than the corresponding constants. □

10.5 The classical background field and local minimality

The two descriptions of the constraint. The renormalization group prescribes the block average $T_M^d U$ of a fine configuration. The coordinates of Section 9 prescribe its gauge-corrected block average instead. For a fine field A the gauge-corrected block average is $e^{\mathsf{F}_d(A)}$, where $\mathsf{F}_d(A)$ is the d-th iterate in (10.9). Taking $j = d$ in (10.11), we obtain

$$e^{\mathsf{F}_d(A)} = (T_M^d e^{\varepsilon A})^{G_d(A)}, \qquad \text{that is,} \qquad T_M^d e^{\varepsilon A} = \big(e^{\mathsf{F}_d(A)}\big)^{G_d(A)^{-1}}.$$

Thus a field A with gauge-corrected block average e^θ lies in the fiber of T_M^d over $(e^\theta)^{G_d(A)^{-1}}$. Let Ω be any fine gauge transformation whose values at the coarse vertices are $G_d(A)$. Applying (10.14) d times and using $(U^k)^l = U^{lk}$, we have

$$T_M^d\big((e^{\varepsilon A})^\Omega\big) = \big((e^\theta)^{G_d(A)^{-1}}\big)^{G_d(A)} = e^\theta, \qquad \mathscr{S}_\varepsilon\big((e^{\varepsilon A})^\Omega\big) = \mathscr{S}_\varepsilon(A).$$

We construct the background field in the chart $\{\mathsf{P}_d^{\mathrm{cl}}(v + \mathsf{H}_d\theta) : v \in \mathcal{V}_d\}$. All points of this chart have gauge-corrected block average e^θ and projected divergence zero, and we transfer the conclusions to the fiber $\{U : T_M^d U = e^\theta\}$ with such a gauge transformation. In the opposite direction, we use Theorem 9.9 to move a field near the fiber into the chart by a gauge transformation, in part (6) of the next theorem and in Lemma 10.7.

By Lemma 10.5, the critical points in the chart are the fixed points of the map

$$\mathcal{T}_\theta(v) = -\mathsf{C}_d\big\{N_\varepsilon(A) - (D\mathsf{D}_d(w))^\sharp \mathsf{J}_d(w)\big\}, \qquad w = v + \mathsf{H}_d\theta, \quad A = \mathsf{P}_d^{\mathrm{cl}}(w).$$

In the following theorem, r_{ph} and C_* denote the constants of Theorem 9.7, and $\mathcal{T}$ is called the Euler map.

Theorem 10.6. *There exist $r_v, r_\theta, a, C > 0$ such that the following hold for every $d \geq 0$ and $n \geq 4$, with $\varepsilon = \varepsilon_d$.*

1. *For every complex coarse field θ with $\|\theta\|_\infty < r_\theta$, the Euler map $\mathcal{T}_\theta$ maps the closed complex ball $\{v \in \mathcal{V}_d : \|v\|_{C^1_\varepsilon} \le r_v\}$ strictly into itself and is a contraction with factor at most $1/2$. Its unique fixed point $v^*_d(\theta)$ is holomorphic in θ and real for real θ, and*

$$\|v^*_d(\theta)\|_{C^1_\varepsilon} \le C\|\theta\|^2_\infty, \qquad A_d(\theta) = \mathsf{P}^{\mathrm{cl}}_d(v^*_d(\theta) + \mathsf{H}_d\theta),$$
$$\|A_d(\theta)\|_{C^1_\varepsilon} \le C\|\theta\|_\infty, \qquad DA_d(0) = \mathsf{H}_d.$$

   We call $A_d(\theta)$ the classical background field with coarse field e^θ.
2. *For real θ with $\|\theta\|_\infty < r_\theta$, the point $v^*_d(\theta)$ is the unique minimum of $v \mapsto \mathscr{S}_\varepsilon(\mathsf{P}^{\mathrm{cl}}_d(v + \mathsf{H}_d\theta))$ on the closed ball $\{v \in \mathcal{V}_d : \|v\|_{C^1_\varepsilon} \le r_v\}$. It is also the unique minimum of this function on the convex set $\{v \in \mathcal{V}_d : \|v\|_\infty + \|d_\varepsilon v\|_\infty < r_{\mathrm{ph}}\}$, for every r_{ph} allowed in Theorem 9.7 with $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$.*
3. *For a fine output link e and a coarse input link c, and $\|\theta\|_\infty < r_\theta$,*

$$|D_{\theta_c} A_d(\theta)_e| + |\nabla_\varepsilon D_{\theta_c} A_d(\theta)_e| \le Ce^{-a\,\mathrm{dist}(e,c)}.$$

4. *For every integer $s \ge 1$ there exist $a_s, C_s > 0$, depending on s, such that on $\|\theta\|_\infty < r_\theta/2$,*

$$\|D^s_\theta(A_d, \nabla_\varepsilon A_d)\|_{a_s,\mathrm{out}} + \|D^s_\theta(A_d, \nabla_\varepsilon A_d)\|_{a_s,\mathrm{input};\varepsilon^3} \le C_s.$$

5. *Define the effective classical action by*

$$E_d(e^\theta) = \mathscr{S}_\varepsilon(A_d(\theta)), \qquad E_0 = S_W.$$

   It is holomorphic on a common ball, nonnegative for real θ, and $E_d(I) = DE_d(I) = 0$. With $w = v^*_d(\theta) + \mathsf{H}_d\theta$, its gradient in θ is

$$\nabla_\theta E_d = (I - D\mathsf{D}_d(w)\mathsf{H}_d)^T \mathsf{J}_d(w).$$

   For each $s \ge 2$ there exist $a_s, C_s > 0$ such that $\|D^s E_d\|_{a_s,\mathrm{root}} \le C_s$. For $g > 0$ let $W_{d,g}(B) = g^{-2}E_d(e^{gB})$. If $\|gB\|_\infty < r_\theta/2$, then

$$\|D^s_B W_{d,g}\|_{a_s,\mathrm{root}} \le C_s g^{s-2} \quad (s \ge 2), \qquad \|D_B W_{d,g}\|_\infty \le C\|B\|_\infty.$$

6. *Let θ be real with $\|\theta\|_\infty < r_\theta$, and let Ω_* be a fine gauge transformation whose values at the coarse vertices are $G_d(A_d(\theta))$. Then $U_* = (e^{\varepsilon A_d(\theta)})^{\Omega_*}$ satisfies $T^d_M U_* = e^\theta$, and U_* is a local minimum of $\mathscr{S}_\varepsilon$ on the fiber $\{U : T^d_M U = e^\theta\}$ modulo fine gauge transformations. More precisely, there is a neighborhood of U_* in this fiber on which $\mathscr{S}_\varepsilon(U) \ge \mathscr{S}_\varepsilon(U_*)$, with equality only if $U = U^h_*$ for a fine gauge transformation h whose values at the coarse vertices leave e^θ fixed. Every tangent vector to the fiber at U_* has zero first variation. The neighborhood may depend on the fine lattice; the radii and constants in (1)–(5) do not.*

7. *The constrained Green operator C_d satisfies the bounds of Lemma 10.2: $\|\mathsf{C}_d f\|_{C^1_\varepsilon} \le C\|f\|_\infty$, and $\|\mathsf{C}_d f\|_{C^1_\varepsilon(x)} \le Ce^{-a\,\mathrm{dist}(x,y)}\|f\|_\infty$ if f is supported in the unit block y. They give weighted absolute rows for C_d and its first differences, and they are the specialization of [13, Theorem 3.13, p. 426] described in the proof of Lemma 10.2.*

The fixed-order bounds in (4) and (5) follow from the factorial bounds (10.17) and (10.20) proved below, which are also collected in Theorem 10.11. The minimality statements in (2) and (6) concern real coarse fields; for complex θ, the field $A_d(\theta)$ and the value $E_d(e^\theta)$ are holomorphic continuations of the critical point and the critical value.

Proof of Theorem 10.6. Step 1 constructs the fixed point by contraction. Steps 2 and 3 prove factorial coefficient bounds for A_d, which give (3) and (4). Step 4 proves (2). Steps 5 and 6 treat the effective classical action, and Step 7 proves the local minimality on the fiber. Part (7) is Lemma 10.2, applied on the torus of side n. Since the geometry of the cover, both constraints, and the variational identity were checked there, no divisibility condition or additional lower bound on n enters.

Step 1: contraction. Let K be larger than the constants of the extension H_d, of the correction D_d, of the nonlinear remainder of the action gradient in Lemma 10.5, and of the Green operator in Lemma 10.2, on fixed larger domains. By Lemmas 10.2 and 10.5, for $\|v\|_{C^1_\varepsilon}, \|v'\|_{C^1_\varepsilon} \le r_v$ and $\|\theta\|_\infty \le r_\theta$,

$$\begin{aligned} \|\mathcal{T}_\theta(v)\|_{C^1_\varepsilon} &\le K(\|v\|_{C^1_\varepsilon} + \|\theta\|_\infty)^2, \\ \|\mathcal{T}_\theta(v) - \mathcal{T}_\theta(v')\|_{C^1_\varepsilon} &\le K(r_v + r_\theta)\|v - v'\|_{C^1_\varepsilon}. \end{aligned} \tag{10.15}$$

Fix $r_v > 0$ with $16Kr_v \le 1$ and so small that the closed C^1_ε ball of radius r_v in $\mathcal{V}_d$ lies inside all the larger domains. We also require that $3r_v$ is an allowed value of r_{ph} in Theorem 9.7. Since $\|d_\varepsilon v\|_\infty \le 2\|\nabla_\varepsilon v\|_\infty$, the closed C^1_ε ball of radius r_v is then contained in the convex domain $\{v \in \mathcal{V}_d : \|v\|_\infty + \|d_\varepsilon v\|_\infty < 3r_v\}$ of that theorem. Next fix $0 < r_\theta \le r_v$ so that $\mathsf{H}_d\theta$ and the correction lie in their domains with fixed margins, uniformly in d and n, and so that $r_\theta < 3r_v/C_*$ and $6Kr_\theta \le C_*$. Then (10.15) implies that $\|\mathcal{T}_\theta(v)\|_{C^1_\varepsilon} \le 4Kr_v^2 \le r_v/4$, and it gives a contraction factor at most $2Kr_v \le 1/8$. Iteration from zero converges, since successive differences are bounded by a geometric series. The limit lies in the closed linear subspace $\mathcal{V}_d$, and it is the unique fixed point $v_d^*(\theta)$ in the closed ball. Comparing the fixed point with the first iterate $\mathcal{T}_\theta(0)$, we obtain

$$\|v_d^*(\theta)\|_{C^1_\varepsilon} \le \tfrac{1}{2}\|v_d^*(\theta)\|_{C^1_\varepsilon} + K\|\theta\|_\infty^2.$$

This proves the quadratic bound $\|v_d^*(\theta)\|_{C^1_\varepsilon} \le 2K\|\theta\|_\infty^2$. Since $A_d(\theta) = v_d^*(\theta) + \mathsf{H}_d\theta - \mathsf{H}_d\mathsf{D}_d(w)$ and $\|\mathsf{D}_d(w)\|_\infty \le K\|w\|_\infty^2$, it also implies the bound on A_d and $DA_d(0) = \mathsf{H}_d$. Since the equation is preserved by complex conjugation of all fields, uniqueness shows that $v_d^*(\theta)$ is real for real θ. Holomorphy follows from Step 2.

Step 2: factorial bounds for the background field. We prove analyticity and locality through the coefficients. Let $p = (v, V_1, V_2, V_3)$ be independent variables for a fine field and its first differences in the three directions. In the nonlinear expressions we replace A and its differences by

$$\Big(v + \mathsf{H}_d[\theta - \mathsf{D}_d(w)], \quad V_\mu + \nabla_{\varepsilon,\mu}\mathsf{H}_d[\theta - \mathsf{D}_d(w)]\Big), \qquad w = v + \mathsf{H}_d\theta, \tag{10.16}$$

and we use the extension $\widetilde{N}_\varepsilon$ of Lemma 10.5. If f denotes the negative of the resulting nonlinear constrained first variation, we define the output to be the pair $(\mathsf{C}_d f, \nabla_\varepsilon \mathsf{C}_d f)$. This is an analytic map on a product ball. Its output is a field together with its own first differences, its field component lies in $\mathcal{V}_d$, and when $V_\mu = \nabla_{\varepsilon,\mu} v$ it coincides with the Euler map.

Every factor has a majorant series with nonnegative coefficients at one rate $b > 0$: (10.10) for the correction, part (2) of Lemma 10.5 for the nonlinear remainder of the action gradient, (10.1) for the pair $(\mathsf{C}_d, \nabla_\varepsilon \mathsf{C}_d)$, and the rows and volume-normalized columns of Theorem 9.3 for H_d and $\nabla_\varepsilon \mathsf{H}_d$. The kernel Δ_d is the contraction of two curl kernels of H_d with the fine measure, as in Step 5 of the proof of Lemma 10.5, and $\mathsf{H}_d^\sharp$ carries the factor ε^3 in its sum. The only apparent inverse power of the volume is in

$$((D\mathsf{D}_d(w))^\sharp u)_e = \varepsilon^{-3} \sum_c (D_{w_e} \mathsf{D}_{d,c}(w))^T u_c.$$

An output row of this expression fixes a fine input w_e of the tensor of D_d, and the column bound in (10.10) supplies the factor ε^3 that cancels ε^{-3}. Taking s further derivatives produces a factor at most $(s+1)!K^{s+1}$; after division by $s!$, the additional factor $s + 1 \le 2^{s+1}$ changes only the constant.

Here is an explicit scalar majorant for the composition. Choose $C \ge 1$ so that $q(z) = C^3 z^2/(1 - Cz)$ majorizes both D_d and $\widetilde{N}_\varepsilon$ in the normalized coefficient rows just described, including the ε^{-3}-normalized columns of D_d, and so that C bounds all the linear operators in this composition. For scalar variables t, x corresponding to θ, p, put

$$u = x + Ct, \qquad a = x + C\big[t + q(u)\big], \qquad F(t,x) = C\Big\{q(a) + Cq'(u)\big[t + q(u) + q(a)\big]\Big\}.$$

The series u majorizes w, a majorizes the pair in (10.16), and $C[t + q(u) + q(a)]$ majorizes J_d. The marked derivative $q'(u)$ majorizes $(D\mathsf{D}_d(w))^\sharp$, with the volume cancellation already checked above. Thus F majorizes the pair Euler map coefficientwise, and it has no terms of total degree less than two. Choose $R > 0$ so that these substitutions converge at (R, R) and set $B = F(R, R)$. If $F(t,x) = \sum_{i+j\ge 2} f_{ij} t^i x^j$, its nonnegative coefficients satisfy $f_{ij} \le BR^{-i-j}$. Hence, for $K \ge \max\{1, R^{-1}, B/R^2\}$ and $i + j \ge 2$,

$$f_{ij} \le BR^{-i-j} \le K^{i+j-1} \le \binom{i+j}{i} K^{i+j-1} = [t^i x^j] \frac{K(t+x)^2}{1 - K(t+x)}.$$

The output-row part of Lemma 5.3 therefore majorizes the implicit solution by the solution of $x = K(t+x)^2/(1-K(t+x))$ on a fixed small disc. The comparison can also be read off from Picard iteration: the coefficient of degree s of the solution depends only on coefficients of lower degree, and the nonnegative scalar recursion bounds all of them. The convergent series defines a holomorphic solution on every finite lattice, and the contraction of Step 1 identifies it with v_d^*. Composing with the majorants of H_d and D_d, we obtain, at $\theta = 0$ and on a smaller common ball of coarse fields,

$$\|D_\theta^s(A_d, \nabla_\varepsilon A_d)\|_{b,\mathrm{out}} \le K^{s+1} s! \qquad (s \ge 1). \tag{10.17}$$

To pass from $\theta = 0$ to the smaller ball, we use the binomial sum of Step 6 of the proof of Lemma 10.4, at the same rate.

Step 3: from rows to columns. The rows also give the volume-normalized columns. Fix a coarse input c_1 and $0 < b' < b$. For a coefficient $T_s(e; c_1, \dots, c_s)$ of order s, the connecting length satisfies $\mathsf{t} \ge \mathrm{dist}(e, c_1)$. Therefore

$$\begin{aligned} &\varepsilon^3 \sum_e \sum_{c_2,\dots,c_s} e^{b'\mathsf{t}} |T_s(e; c_1, \dots, c_s)| \\ &\qquad \le K^{s+1} s!\, \varepsilon^3 \sum_e e^{-(b-b')\,\mathrm{dist}(e,c_1)} \le K_b K^{s+1} s!. \end{aligned} \tag{10.18}$$

In the first inequality we used $e^{b'\mathsf{t}} \le e^{b\mathsf{t}} e^{-(b-b')\,\mathrm{dist}(e,c_1)}$ and (10.17). The last inequality follows by summing over fine cubes in annuli around c_1: ε^3 times the number of fine links within distance R is at most $K(1+R)^3$, on every periodic torus and for every mesh at most one. By symmetry any c_i may be the fixed input. Taking $s = 1$ in (10.17) gives part (3). Taking $a_s = b'$ and enlarging C_s, we obtain part (4), with no loss of exponent depending on the order.

Step 4: minimality in the chart. Let θ be real. The fixed point $v_d^*(\theta)$ is an interior point of the convex domain $\{v \in \mathcal{V}_d : \|v\|_\infty + \|d_\varepsilon v\|_\infty < 3r_v\}$, and its tested Euler equation vanishes by Lemma 10.5. The formula for the Hessian and the stability inequality of Theorem 9.7 then show that $v_d^*(\theta)$ is the unique critical point and the unique minimum throughout this domain, and in particular on the closed C_ε^1 ball of radius r_v. Now let r_{ph} be any value allowed in Theorem 9.7 with $\|\theta\|_\infty < r_{\mathrm{ph}}/C_*$. By Step 1, $\|v_d^*\|_\infty + \|d_\varepsilon v_d^*\|_\infty \le 3\|v_d^*\|_{C_\varepsilon^1} \le 6K\|\theta\|_\infty^2$, and since $6K\|\theta\|_\infty \le C_*$ this is at most $C_*\|\theta\|_\infty < r_{\mathrm{ph}}$. Thus $v_d^*(\theta)$ is an interior point of the larger domain as well, all its constrained first variations vanish, and the same stability inequality gives uniqueness and minimality there. This proves part (2).

Step 5: the effective classical action. The finite sum defining $\mathscr{S}_\varepsilon$ is holomorphic on the common ball and nonnegative for real fields. At $\theta = 0$ we have $A_d(0) = 0$ and $\mathsf{J}_d = 0$; in particular $E_d(I) = 0$, and $DE_d(I) = 0$ once the gradient formula is proved. We

differentiate at fixed v before using stationarity. The derivative in θ of the parametrized field $v + \mathsf{H}_d\theta - \mathsf{H}_d\mathsf{D}_d(v + \mathsf{H}_d\theta)$ is $\mathsf{H}_d(I - D\mathsf{D}_d(w)\mathsf{H}_d)$. The contribution of Dv_d^* is a variation in $\mathcal{V}_d$, and its derivative of the action vanishes because v_d^* is critical. Using $\mathsf{J}_d(w) = \mathsf{H}_d^\sharp G_\varepsilon(A)$, we obtain

$$\nabla_\theta E_d = (I - D\mathsf{D}_d(w)\mathsf{H}_d)^T \mathsf{J}_d(w), \tag{10.19}$$

where T is the transpose for the counting scalar product on coarse fields. Note that this formula keeps the contribution of the constraint parametrization, and that it gives the derivatives of E_d without estimating the size of the total action.

Step 6: coefficient bounds for the action. We now derive coefficient bounds at one rate from (10.19). We write $(\nabla_\theta E_d)_c = \mathsf{J}_d(w)_c - \mathsf{T}_c$, where

$$\mathsf{T}_c = \sum_{c'} \sum_e \mathsf{H}_d(e,c)^T \big(D_{w_e}\mathsf{D}_{d,c'}(w)\big)^T \mathsf{J}_d(w)_{c'}, \qquad w = v_d^*(\theta) + \mathsf{H}_d\theta,$$

with c, c' coarse links and e a fine link. In the term $\mathsf{H}_d^\sharp N_\varepsilon$ of J_d, a fixed coarse index c sees the volume-normalized column of H_d, which is bounded. In T_c the fine index e is summed between H_d and $D\mathsf{D}_d$, and the factors ε^{-3} and ε^3 cancel. We carry out the bookkeeping for the first derivative, which shows all the mechanisms. The derivative of T_c in θ_{c_1} is $\mathsf{T}_c^{(1)}(c_1) + \mathsf{T}_c^{(2)}(c_1)$, where

$$\mathsf{T}_c^{(1)}(c_1) = \sum_{c',e} \mathsf{H}_d(e,c)^T \big(D_{w_e}\mathsf{D}_{d,c'}(w)\big)^T D_{\theta_{c_1}}\big[\mathsf{J}_d(w)_{c'}\big],$$
$$\mathsf{T}_c^{(2)}(c_1) = \sum_{c',e,e'} \mathsf{H}_d(e,c)^T \big(D^2_{w_e w_{e'}}\mathsf{D}_{d,c'}(w)\, D_{\theta_{c_1}} w_{e'}\big)^T \mathsf{J}_d(w)_{c'}.$$

Let $b'' > 0$ be smaller than all rates used, and write $\omega(x,y) = e^{b''\,\mathrm{dist}(x,y)}$. By the triangle inequality, $e^{b''\mathrm{t}(c;c_1)} = \omega(c,c_1)$ is at most the product of the weights ω along the chain c, e, c', c_1, or c, e, e', c_1. We first consider the output row, with c fixed. For $\mathsf{T}^{(1)}$ we sum in the order c_1, c', e:

$$\sum_{c_1} \omega(c,c_1)|\mathsf{T}_c^{(1)}(c_1)|$$
$$\leq \sum_e \omega(c,e)|\mathsf{H}_d(e,c)| \sum_{c'} \omega(e,c')|D_{w_e}\mathsf{D}_{d,c'}(w)| \sup_{c'} \sum_{c_1} \omega(c',c_1)|D_{\theta_{c_1}}\mathsf{J}_d(w)_{c'}|.$$

The first sum is a counting column of H_d, of order ε^{-3}. For each fixed fine e, the second sum is a counting column of $D\mathsf{D}_d$ with the fine input fixed, of order $\varepsilon^3 r$ by (10.10). The last factor is a row of the coarse tensor $D_\theta \mathsf{J}_d$, which is bounded by Step 2. The product is of order r, with no power of ε. For $\mathsf{T}^{(2)}$ we bound $|\mathsf{J}_d(w)_{c'}| \leq \|\mathsf{J}_d(w)\|_\infty$ and sum in the order c_1, (c', e'), e: the counting column of H_d at c, of order ε^{-3}, multiplies the counting column of $D^2\mathsf{D}_d$ with the fine input e fixed, of order ε^3, and the row of $D_\theta w = Dv_d^* + \mathsf{H}_d$,

which is bounded by Step 2. Next we consider the input column, with c_1 fixed. For $\mathsf{T}^{(1)}$ the fixed input lies in $D_\theta \mathsf{J}_d$. We use its column $\sum_{c'} \omega(c', c_1)|D_{\theta_{c_1}} \mathsf{J}_d(w)_{c'}|$, and then the rows $\sup_{c'} \sum_e \omega(e, c')|D_{w_e} \mathsf{D}_{d,c'}(w)| \le Kr$ and $\sup_e \sum_c \omega(c, e)|\mathsf{H}_d(e, c)| \le K$; no power of ε occurs. For $\mathsf{T}^{(2)}$ the fixed input lies in $D_\theta w$, whose counting column $\sum_{e'} \omega(e', c_1)|D_{\theta_{c_1}} w_{e'}|$ is of order ε^{-3}; it multiplies the counting column of $D^2\mathsf{D}_d$ with the fine input e' fixed, of order ε^3, the row of H_d, and $\|\mathsf{J}_d(w)\|_\infty$.

The derivatives of higher order are bounded in the same way. In each term of the chain rule, the branch containing the fixed input, whether it lies in w or in J_d, uses the input column of that factor, and every other branch uses its output row; these are the contraction rules of Lemma 5.3. Every contraction over a fine index pairs a counting column of order ε^{-3}, of H_d or of $D_\theta w$, with a counting column of order ε^3 of a derivative of D_d. Together with (10.17), this gives, on a fixed smaller ball and at the rate b'',

$$\|D_\theta^s \nabla_\theta E_d\|_{b'',\mathrm{root}} \le K^{s+1} s! \qquad (s \ge 1), \tag{10.20}$$

where the norm is the maximum of the output row and the input columns. Since the derivatives of a scalar function are symmetric, every argument may be the fixed one. This proves the bounds $\|D^s E_d\|_{a_s,\mathrm{root}} \le C_s$ of part (5). Integrating the row of the Hessian along $t\theta$, $0 \le t \le 1$, and using $\nabla_\theta E_d(0) = 0$, we obtain $\|\nabla_\theta E_d\|_\infty \le K\|\theta\|_\infty$. The substitution $\theta = gB$ contributes exactly the factor g^{s-2} at order s of $g^{-2}E_d(e^{gB})$, and its first derivative is $g^{-1}\nabla_\theta E_d(gB)$, which gives $\|D_B W_{d,g}\|_\infty \le K\|B\|_\infty$.

Step 7: local minimality on the fiber. We now fix the final value of r_θ, reducing the radius of Step 1 so that its closed ball lies strictly inside the common balls of (10.17) and (10.20). We also require, uniformly in d and n, that $\|A_d(\theta)\|_{\mathsf{X}} < r_{\mathsf{X}}/2$, $\|\mathrm{Log}\, G_d(A_d(\theta))\|_\infty < r_c/2$, and the smallness conditions in part (4) of Theorem 9.9 hold with a fixed margin. The bounds on A_d, its first differences, and $\mathrm{Log}\, G_d$ tend to zero with $\|\theta\|_\infty$, so this is a finite uniform reduction using only previously fixed constants. All preceding conclusions hold on this final ball.

Let θ be real, and write $A_* = A_d(\theta)$, $G_* = G_d(A_*)$, and $c_* = \mathrm{Log}\, G_*$. Let Ω_* be a periodic fine gauge transformation with values G_* at the coarse vertices. By the discussion at the beginning of this section, $U_* = (e^{\varepsilon A_*})^{\Omega_*}$ satisfies $T_M^d U_* = e^\theta$. On the fixed finite lattice, let U be a configuration in this fiber close to U_*, and write $U^{\Omega_*^{-1}} = e^{\varepsilon A}$. Since $A = A_*$ for $U = U_*$, continuity in finitely many variables shows that A lies in the ball of inputs of Theorem 9.9 if U is close enough to U_*. Its block average is $T_M^d e^{\varepsilon A} = (e^\theta)^{G_*^{-1}}$.

We apply Theorem 9.9 to A with prescribed coarse datum $c = c_*$. For $A = A_*$ the gauge transformation of that theorem is the identity, since $\mathsf{P}_d^{\mathrm{div}} \mathrm{div}_\varepsilon A_* = 0$ and $c_* = \mathrm{Log}\, G_d(A_*)$. The transformed field A' has projected divergence zero and gauge-corrected block average

$$\left[(e^\theta)^{G_*^{-1}}\right]^{G_*} = e^\theta,$$

where the order of the factors comes from $(U^k)^l = U^{lk}$. By that theorem $\|A'\|_\infty + \|d_\varepsilon A'\|_\infty$ is small, and the parametrization of all nearby solutions of the constraint in Theorem 9.6 gives

$$v' = (I - \mathsf{H}_d \mathsf{Q}_d) A' \in \mathcal{V}_d, \qquad A' = \mathsf{P}_d^{\mathrm{cl}}(v' + \mathsf{H}_d \theta).$$

By continuity and the supremum and curl bounds for $\mathsf{H}_d \mathsf{Q}_d A'$, we see that v' lies in the larger convex domain of Step 4. By the stability inequality of Theorem 9.7, we have

$$\mathscr{S}_\varepsilon(U) - \mathscr{S}_\varepsilon(U_*) \geq \frac{c_0}{4} \|v' - v_d^*(\theta)\|_{H_\varepsilon^1}^2,$$

where we used that all the gauge transformations involved preserve the action $\mathscr{S}_\varepsilon$. Equality forces $v' = v_d^*(\theta)$, hence $A' = A_*$. Undoing the two fine gauge transformations shows that U and U_* lie on the same fine gauge orbit. Since both configurations lie in the fiber over e^θ, the values of the relating gauge transformation at the coarse vertices leave e^θ fixed. The construction applies to all points of a sufficiently small neighborhood of U_* in the fiber on the given finite lattice. Applying the inequality along a differentiable curve in the fiber through U_*, for both signs of the parameter, shows that every tangent vector to the fiber has zero first variation. This proves part (6) and completes the proof. □

10.6 Gauge extensions and symmetries of the effective action

In this section we prove that E_d is invariant under small coarse gauge transformations, the lattice symmetries, and constant color rotations. A symmetry applied to the background field moves it out of the gauge slice in which it was constructed. To compare, we bring the transformed field back into the slice with Theorem 9.9 and use the uniqueness of critical points in the convex domain of Theorem 9.7. The following lemma isolates this argument; it is used here and again in the proof of the nested minimization identity.

Lemma 10.7. *There exists $r_{\mathrm{tr}} > 0$ such that the following holds for every $d \geq 0$ and $n \geq 4$, with $\varepsilon = \varepsilon_d$. Let $U = e^{\varepsilon A}$ be a real fine configuration with $\|A\|_{C_\varepsilon^1} < r_{\mathrm{tr}}$, and let $b = T_M^d U$ satisfy $\|\operatorname{Log} b\|_\infty < r_{\mathrm{tr}}$. Assume that U is a critical point of $\mathscr{S}_\varepsilon$ on the fiber $\{U' : T_M^d U' = b\}$, that is, $(d/dt)\mathscr{S}_\varepsilon(U(t))|_{t=0} = 0$ for every differentiable curve $t \mapsto U(t)$ in this fiber with $U(0) = U$. Then, with $\theta = \operatorname{Log} b$, there is a real fine gauge transformation h such that $U^h = e^{\varepsilon A_d(\theta)}$. In particular $\mathscr{S}_\varepsilon(U) = E_d(b)$.*

Proof of Lemma 10.7. If $d = 0$, the fiber is the single point $U = b$, so $A = \theta$; since $\mathcal{V}_0 = \{0\}$, $\mathsf{H}_0 = I$, and $\mathsf{D}_0 = 0$, we have $A_0(\theta) = \theta$, and $h = I$ works. Let $d \geq 1$. We choose r_{tr} so small that the following three steps are possible.

Step 1: moving U into the gauge slice. We have $\|A\|_\infty + \|\operatorname{div}_\varepsilon A\|_\infty \leq 3\|A\|_{C_\varepsilon^1}$ and $\|d_\varepsilon A\|_\infty \leq 2\|\nabla_\varepsilon A\|_\infty$. For small r_{tr}, Theorem 9.9 therefore applies to A with prescribed

coarse datum 0. It gives a real fine scalar field λ such that the field $A' = A^{e^\lambda}$, defined by $e^{\varepsilon A'} = U^{e^\lambda}$, satisfies

$$\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon A' = 0, \qquad G_d(A')\, e^{\lambda|_{\mathrm{roots}}} = I,$$

and has gauge-corrected block average b and $\|A'\|_\infty + \|d_\varepsilon A'\|_\infty \le K r_{\mathrm{tr}}$. By the parametrization of all nearby solutions of the constraint in Theorem 9.6, $A' = \mathsf{P}_d^{\mathrm{cl}}(a' + \mathsf{H}_d\theta)$ with $a' = (I - \mathsf{H}_d\mathsf{Q}_d)A' \in \ker \mathsf{Q}_d$. Since $\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon A' = 0$ and $\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon \mathsf{H}_d = 0$, also $\mathsf{P}_d^{\mathrm{div}} \operatorname{div}_\varepsilon a' = 0$, that is, $a' \in \mathcal{V}_d$. The bounds on H_d give $\|a'\|_\infty + \|d_\varepsilon a'\|_\infty \le K r_{\mathrm{tr}}$.

Step 2: criticality in the chart. Let $u \in \mathcal{V}_d$, and for small real t let $a(t) = a' + tu$ and $A(t) = \mathsf{P}_d^{\mathrm{cl}}(a(t) + \mathsf{H}_d\theta)$. Since the gauge-corrected block average of $A(t)$ is $e^\theta = b$ for all t, the relation at the beginning of Section 10.5 gives $T_M^d e^{\varepsilon A(t)} = b^{G_d(A(t))^{-1}}$. For a fine vertex x let $r(x)$ be the coarse vertex whose centered block at depth d contains x, and let $\Omega(t)_x = G_d(A(t))_{r(x)}$. Then $\Omega(t)$ has the values $G_d(A(t))$ at the coarse vertices, and it depends differentiably on t because G_d is holomorphic. The curve $U(t) = (e^{\varepsilon A(t)})^{\Omega(t)}$ lies in the fiber over b, and $\mathscr{S}_\varepsilon(U(t)) = \mathscr{S}_\varepsilon(A(t))$. At $t = 0$ we have $e^{\varepsilon A'} = U^{e^\lambda}$, so $U(0) = U^{h_0}$ with $h_0 = \Omega(0)e^\lambda$, by $(U^k)^l = U^{lk}$. The values of h_0 at the coarse vertices are $G_d(A')e^{\lambda|_{\mathrm{roots}}} = I$. Since a fine gauge transformation equal to the identity at the coarse vertices maps the fiber over b onto itself and preserves $\mathscr{S}_\varepsilon$, the configuration $U(0)$ is critical on the fiber, as U is. It follows that

$$\frac{d}{dt}\mathscr{S}_\varepsilon(A(t))\Big|_{t=0} = \frac{d}{dt}\mathscr{S}_\varepsilon(U(t))\Big|_{t=0} = 0.$$

Since $u \in \mathcal{V}_d$ was arbitrary, a' is a critical point of $a \mapsto \mathscr{S}_\varepsilon(\mathsf{P}_d^{\mathrm{cl}}(a + \mathsf{H}_d\theta))$ on $\mathcal{V}_d$.

Step 3: identification. Fix an allowed value r_{ph} of Theorem 9.7, and choose r_{tr} so that $r_{\mathrm{tr}} < r_\theta$, $K r_{\mathrm{tr}} < r_{\mathrm{ph}}$, and $r_{\mathrm{tr}} < r_{\mathrm{ph}}/C_*$. Then a' is a critical point in the interior of the convex domain $\{a \in \mathcal{V}_d : \|a\|_\infty + \|d_\varepsilon a\|_\infty < r_{\mathrm{ph}}\}$. By Theorem 10.6 and Lemma 10.5, $v_d^*(\theta)$ is also an interior critical point of this domain. By the uniqueness of interior critical points in Theorem 9.7, $a' = v_d^*(\theta)$. Hence $A' = A_d(\theta)$, that is, $U^{e^\lambda} = e^{\varepsilon A_d(\theta)}$, and $\mathscr{S}_\varepsilon(U) = \mathscr{S}_\varepsilon(A_d(\theta)) = E_d(b)$, since gauge transformations preserve the action. □

The next lemma constructs a smooth extension of coarse gauge transformations to the fine lattice by cardinal interpolation, and proves the symmetries of E_d. The first differences of the extension are of the size of the coarse gauge transformation, not of its size divided by the mesh. For a coarse vertex field κ and a point x of the torus we write $\sum_c \chi_c(x)\kappa_c$, where c runs over the coarse vertices.

Lemma 10.8. *There exist smooth periodic functions χ_c on the torus, indexed by the coarse vertices c, such that $\sum_c \chi_c = 1$, $\chi_c(c') = \delta_{cc'}$ for all coarse vertices c', the supports of the χ_c have diameter at most $2\sqrt{3}$, and*

$$\sup_x \sum_c |\partial^j \chi_c(x)| \le C_j \qquad (0 \le j \le 3);$$

their restrictions to the fine vertices satisfy $\varepsilon_d^3 \sum_x |\partial^j \chi_c(x)| \le C_j$. *There exist* $r_{\mathrm{lift}}, r_{\mathrm{sym}}, \kappa_{\mathrm{sym}}, C > 0$ *with* $2Cr_{\mathrm{sym}} < \min(r_{\mathrm{lift}}, \kappa_{\mathrm{sym}})$, *where* C *also bounds the constants of Theorem 10.6 and the constant in the bound on* $\operatorname{Log} G_d$ *in Lemma 10.4, such that the following hold for every* $d \ge 0$ *and* $n \ge 4$, *with* $\varepsilon = \varepsilon_d$.

1. *Let* $k_c = e^{\kappa_c}$ *be a coarse gauge transformation with* $\|\kappa\|_\infty < \kappa_{\mathrm{sym}}$, *and let* $\Omega(x) = \exp(\sum_c \chi_c(x)\kappa_c)$ *at the fine vertices* x. *Then* Ω *equals* k *at the coarse vertices, and for fine fields* A *with* $\|A\|_{C^1_\varepsilon} < r_{\mathrm{lift}}$,

$$\|\varepsilon^{-1} \operatorname{Log}[(e^{\varepsilon A})^\Omega]\|_{C^1_\varepsilon} \le C(\|A\|_{C^1_\varepsilon} + \|\kappa\|_\infty).$$

   In particular, for $\|\theta\|_\infty < r_{\mathrm{sym}}$ *the gauge orbit of* $e^{\varepsilon A_d(\theta)}$ *contains a representative* U, *analytic in* θ, *with* $T_M^d U = e^\theta$ *and* $\|\varepsilon^{-1} \operatorname{Log} U\|_{C^1_\varepsilon} \le C\|\theta\|_\infty$.
2. *On common smaller complex domains,*

$$E_d(b^k) = E_d(b), \qquad E_d(\sigma b) = E_d(b), \qquad E_d(b^u) = E_d(b) \quad (u \in \mathrm{SU}(2) \text{ constant}),$$

   where b *and* b^k *lie in the chart* $\{\|\operatorname{Log} b\|_\infty < r_{\mathrm{sym}}\}$, $\|\operatorname{Log} k\|_\infty < \kappa_{\mathrm{sym}}$, *and* σ *is a translation of the coarse lattice or the spatial inversion* $x \mapsto -x$, *which maps links to reversed links and inverts their variables. Differentiating the first identity, we obtain the infinitesimal gauge-invariance identities* $(d/dt)E_d(b^{e^{t\xi}})|_{t=0} = 0$ *for every field* ξ *on the coarse vertices with values in* $\mathfrak{su}(2)$.

The representative in part (1) is used in the proof of Theorem 10.10. Part (2) concerns small gauge transformations only. Gauge transformations far from the identity are treated in Section 11.

Proof of Lemma 10.8. We construct the cardinal functions (Step 1), prove the bound on the gauge transform (Steps 2 and 3), and then prove the three identities (Steps 4–6). Step 7 fixes the radii.

Step 1: cardinal functions. Let

$$\eta(t) = \begin{cases} \exp\bigl(1 - (1-t^2)^{-1}\bigr), & |t| < 1, \\ 0, & |t| \ge 1, \end{cases}$$

and

$$D(t) = \sum_{l \in \mathbb{Z}} \eta(t-l), \qquad \phi_l(t) = \eta(t-l)/D(t).$$

The function D is positive and periodic, so its minimum on $[0,1]$ is positive. The derivatives of each fixed order of η are bounded and vanish at the endpoints of its support. At most two translates of η are nonzero at a noninteger point and only one at an integer point, and the sums of their derivatives are bounded in terms of the order.

It follows that $\sum_l \phi_l = 1$ and $\phi_l(k) = \delta_{lk}$ for integers k. Taking tensor products of the ϕ_l in the three coordinates and periodizing with period n, we obtain functions χ_c whose supports have diameter at most $2\sqrt{3}$ in the metric of the torus. Since $n \geq 4$, the periodization keeps the finite overlap and the values at the coarse vertices. The bounds on the sums of derivatives follow from the product rule. For fixed c, the support of χ_c contains at most $K\varepsilon^{-3}$ fine vertices and the derivatives of χ_c are bounded, which proves the bounds with the factor ε^3. Finite differences divided by ε satisfy the same bounds, since they are averages of the corresponding derivatives over intervals of length ε.

Step 2: the gauge transform has a bounded rescaled connection. Let $\lambda(x) = \sum_c \chi_c(x)\kappa_c$, so that $\Omega = e^{\lambda}$. By Step 1, the value of λ and its first two differences divided by ε are bounded by $K\|\kappa\|_\infty$. For a fine link xy we write $u = \lambda_x$, $v = (\lambda_y - \lambda_x)/\varepsilon$, and $a = A_{xy}$. The analytic function of finitely many variables

$$\varepsilon^{-1} \operatorname{Log}\bigl(e^u e^{\varepsilon a} e^{-u-\varepsilon v}\bigr) = \operatorname{Ad}(e^u)a - \Bigl(\int_0^1 \operatorname{Ad}(e^{tu})\,dt\Bigr)v + \varepsilon R_\varepsilon(u,v,a)$$

has a removable singularity at $\varepsilon = 0$, uniformly on a fixed small complex ball of (u,v,a). Indeed, the product of the three exponentials is the identity at $\varepsilon = 0$, and its first derivative in ε at $\varepsilon = 0$ gives the two displayed principal terms. The remainder R_ε and its derivatives are bounded by the convergent series in finitely many variables on a smaller ball. The difference of the first principal term involves $\nabla_\varepsilon A$ and $\nabla_\varepsilon \lambda$, and the difference of the second involves the first two differences of λ. For the last term,

$$|\nabla_\varepsilon(\varepsilon R)_e| \leq |R_{e+\varepsilon e_\mu}| + |R_e|,$$

with no inverse power of the mesh. Since the remainder vanishes at $(a,v) = 0$, after shrinking the domain its value is at most $K(|a| + |v|)$. This proves the bound of part (1), on a radius r_{lift} independent of the fine mesh.

Step 3: a representative with block average e^θ. By Lemma 10.4 and part (1) of Theorem 10.6, $\|\operatorname{Log} G_d(A_d(\theta))\|_\infty \leq K\|\theta\|_\infty$. We apply the cardinal extension of Step 2 with these values at the coarse vertices. By the discussion at the beginning of Section 10.5, the resulting field U satisfies $T_M^d U = e^\theta$, and Step 2 implies that $\|\varepsilon^{-1}\operatorname{Log} U\|_{C^1_\varepsilon} \leq K\|\theta\|_\infty$. The formulas in finitely many variables and the holomorphic maps already constructed show that U is analytic in θ. The radii and the bound are the same for all d and n. This proves part (1).

Step 4: gauge invariance. We first let b and k be real with $\|\operatorname{Log} b\|_\infty$ and $\|\operatorname{Log} k\|_\infty$ small, and write $b = e^\theta$ and $G = G_d(A_d(\theta))$. Using the local logarithm of kG, we apply the cardinal extension of Step 2 to the coarse gauge transformation kG, and we apply the resulting fine gauge transformation to $e^{\varepsilon A_d(\theta)}$. Since $T_M^d e^{\varepsilon A_d(\theta)} = b^{G^{-1}}$, the transformed

field U_k satisfies $T_M^d U_k = (b^{G^{-1}})^{kG} = b^k$. By Step 2 its rescaled connection has C^1_ε norm at most $K(\|\theta\|_\infty + \|\mathrm{Log}\, k\|_\infty)$. The field U_k is obtained from the configuration U_* of part (6) of Theorem 10.6 by a fine gauge transformation, which maps the fiber over b onto the fiber over b^k and preserves $\mathscr{S}_\varepsilon$. Since U_* is critical on its fiber, U_k is critical on the fiber over b^k. By Lemma 10.7, $\mathscr{S}_\varepsilon(U_k) = E_d(b^k)$. Since all the transformations used preserve the action, we conclude that $E_d(b^k) = \mathscr{S}_\varepsilon(e^{\varepsilon A_d(\theta)}) = E_d(b)$.

Step 5: translations, inversion, and constant color rotations. Translations of the coarse lattice preserve the centered blocks and their coordinate-ordered paths. They lift to translations of the fine lattice and commute with the block averaging map. For the inversion, let $c = (r, r + Me_i)$, let $c' = (-r - Me_i, -r)$, and let $y = -x - Me_i$. The inversion maps the paths as follows: $\sigma_{r,x} \mapsto \sigma_{-r,-x}$, $F_{c,x} \mapsto F^{-1}_{c',y}$, and $C_c \mapsto C^{-1}_{c'}$. For group elements F, C in the domain of the logarithm,

$$\mathrm{Log}(F^{-1}C) = -\,\mathrm{Ad}(C^{-1})\,\mathrm{Log}(FC^{-1}).$$

Inserting this identity into the exponential average defining T_M, we find that the averaged reversed link is the inverse of the averaged link. The same holds for the Borel extension when a ratio equals $-I$, although that case does not occur in the present chart. The inversion therefore also commutes with the block averaging map. A translation preserves the C^1_ε norm of the rescaled connection; the inversion replaces each component by its negative at a translated link and also preserves that norm. Let σ be one of these maps and let U be the representative of part (1) for b, which is critical on its fiber by part (6) of Theorem 10.6. Then σU has block average σb and the same action, it is critical on the fiber over σb, and its rescaled connection has the same small norm. By Lemma 10.7, $E_d(\sigma b) = \mathscr{S}_\varepsilon(\sigma U) = \mathscr{S}_\varepsilon(U) = E_d(b)$. This proves the spatial identities on a common smaller domain.

A constant $u \in \mathrm{SU}(2)$ acts by the orthogonal color rotation $\mathrm{Ad}(u)$ on every path product, every linear operator, the constraints, and the action. It preserves all the domains above. By uniqueness of the fixed point, $v_d^*(\mathrm{Ad}(u)\theta) = \mathrm{Ad}(u)v_d^*(\theta)$ and $A_d(\mathrm{Ad}(u)\theta) = \mathrm{Ad}(u)A_d(\theta)$, which gives the identity $E_d(b^u) = E_d(b)$ for every such u.

Step 6: complex continuation. For translations, the inversion, and each fixed constant u, both sides of the identity are holomorphic functions of the coarse logarithm. Since they agree on a real open ball, every coefficient of the Taylor series of their difference vanishes. For the gauge identity, choose a connected joint complex polydisc centered at $(\mathrm{Log}\, b, \mathrm{Log}\, k) = (0, 0)$ on which both $\mathrm{Log}\, b$ and $\mathrm{Log}(b^k)$ remain in the larger action chart of radius r_θ. The local word bounds give such a polydisc uniformly in d, n. The identity on its real open part from Step 4 implies the holomorphic identity on the polydisc, which we then restrict to the stated common smaller domains.

Step 7: the radii and the infinitesimal identities. Let C be a constant larger than the constants for the minimum, for the gauge transformation G_d, and for the extension. First fix r_{lift} and then κ_{sym} so small that the cardinal lift and the product $kG_d(A_d(\theta))$ stay in their larger domains and $C\kappa_{\mathrm{sym}} < \frac{1}{4}\min\{r_{\mathrm{lift}}, r_{\mathrm{tr}}, r_\theta\}$. Next fix r_{sym} inside the connected product of Step 6 and so small that $2Cr_{\mathrm{sym}} < \min(r_{\mathrm{lift}}, \kappa_{\mathrm{sym}})$ and all the requirements of Lemma 10.7 and of Steps 4 and 5 hold. This order gives the dependence of the constants stated in Section 10.1. Differentiating the gauge identity along $k_x(t) = e^{t\xi_x}$ gives the infinitesimal identity for every coarse vertex field ξ; for t in a small interval about zero, both b and $b^{k(t)}$ stay in the chart. □

10.7 One averaging step and nested minimization

We now consider one averaging step, from a lattice of side Mm to the lattice of side $m \geq 4$. We minimize E_d over the configurations on the lattice of side Mm whose block average is a given coarse field c. We first recall the local coordinates on the fiber $\{b : T_M b = c\}$ and prove the bounds we need for them; then we prove the nested minimization identity.

Coordinates for one averaging step. The coordinates are those of Definition 4.3 and Lemma 6.3; the eliminated edges are denoted $e^*_{c'}$ here. In this paragraph c denotes a coarse field and c' a positive coarse link. The coarse-field lift $U_0(c)$ is the identity on the internal links of the blocks and equals $c_{c'}$ on the interface links of the coarse link c'; thus $T_M U_0(c) = c$. In each block, the paths $\sigma_{r,x}$ from the block center r form a rooted spanning tree, and we let $\mathcal{F}$ be the set of positive fine links whose unoriented edges belong to these trees. For the coarse link $c' = (r, r + Me_i)$, the eliminated edge is $e^*_{c'} = (r + R_M e_i, r + (R_M + 1)e_i)$, where $M = 2R_M + 1$. We write a fine configuration as $e^X U_0(c)$, and we let

$$\mathsf{C}_{c'}(X) = \operatorname{Log}\big((T_M(e^X U_0(c)))_{c'} c_{c'}^{-1}\big)$$

be the block averaging constraint at c'. The free logarithmic coordinates z are the values X_e on the positive fine links $e \notin \mathcal{F}$ that are not eliminated edges, and $X_e = 0$ for $e \in \mathcal{F}$. The local constraint coordinates $X = \Gamma_c(z)$ solve the constraints $\mathsf{C}_{c'}(X) = 0$ for the eliminated-edge variables $X_{e^*_{c'}}$, with $\Gamma_c(z)_e = z_e$ on the free links. We set

$$b(c,z) = e^{\Gamma_c(z)} U_0(c), \qquad B(c,z) = \operatorname{Log} b(c,z), \qquad \mathsf{E}_{\mathrm{f}} = D_z B(I,0).$$

Let f be the number of free coordinates, each a group variable, let $c_{\mathrm{Haar}} = 1/(16\pi^2)$ and $j_H(X) = \operatorname{sinc}^2(|X|/2)$ be as in (4.2), and define the Haar disintegration density

$$\mathcal{J}_c(z) = c_{\mathrm{Haar}}^f \frac{\prod_{e \notin \mathcal{F}} j_H(\Gamma_c(z)_e)}{\prod_{c'} \det D_{X_{e^*_{c'}}} \mathsf{C}_{c'}(\Gamma_c(z))},$$

where e runs over the positive fine links and c' over the positive coarse links. In complex formulas $|X|^2$ denotes the bilinear sum of the squares of the three coordinates of X. This is the density of Lemma 6.3; the numerator contains the Haar factors of the eliminated edges, which lie outside $\mathcal{F}$. We recall its derivation in the proof below, since the estimates use its explicit form.

Lemma 10.9. *There exist $R'_M, b, K_M > 0$ such that the following hold for every $m \geq 4$.*

1. *On the complex product $\|\theta\|_\infty < R'_M$, $\|z\|_\infty < R'_M$, with $c = e^\theta$, the maps $\Gamma_c(z)$ and $B(c,z)$ are holomorphic, their absolute derivative tensors have factorial bounds and finite stencils, and*

$$\|B(c,z)\|_\infty \leq K_M(\|\theta\|_\infty + \|z\|_\infty), \qquad \|D_z B(c,z) - \mathsf{E}_{\mathrm{f}}\|_{b,\mathrm{row+col}} \leq K_M(\|\theta\|_\infty + \|z\|_\infty).$$

2. *On a smaller real product, the constraint Jacobian determinants $\det D_{X_{e^*_{c'}}} \mathsf{C}_{c'}(\Gamma_c(z))$ are bounded below by a fixed positive constant, and*

$$\|D_z^2 \log \mathcal{J}_c(z)\|_{b,\mathrm{row+col}} \leq K_M.$$

3. *Let $t \mapsto b(t)$ be a differentiable curve in the fiber $\{T_M b = c\}$, and let $k(t)$, equal to the identity at the coarse vertices, be the block axial gauge transformation of $b(t)$, so that $b(t)^{k(t)} = b(c, z(t))$, with $z(t)$ in the domain of (1). Put $k_0 = k(0)$ and $z_0 = z(0)$, and let $\mathcal{G}_h(U) = U^h$. Then*

$$b'(0) = D\mathcal{G}_{k_0^{-1}}\big|_{b(c,z_0)} D_z b(c,z_0) z'(0) + \frac{d}{dt}\Big[b(c,z_0)^{k(t)^{-1}}\Big]\Big|_{t=0}.$$

The second term is the tangent vector at $b(0)$ of a curve of fine gauge transformations along the block trees which are the identity at the coarse vertices. If $b(0) = b(c,z_0)$, then $k_0 = I$, and the first term is $D_z b(c,z_0) z'(0)$.

Proof of Lemma 10.9. We construct the block axial gauge (Step 1), solve the constraint for the eliminated edges (Steps 2 and 3), derive the density and bound its Hessian (Steps 4 and 5), and prove the decomposition of tangent vectors (Step 6). The density formula is the tree-gauge specialization of the disintegration of Haar measure for this averaging constraint.

Step 1: the block axial gauge. We gauge the links of $\mathcal{F}$ to the identity with $k_r = I$ at each block center r. On a tree edge directed from a parent x to a child y we prescribe

$$k_y = k_x U_{xy}, \qquad k_x = U(\sigma_{r,x}). \tag{10.21}$$

If this direction reverses a positive link, U_{xy} is the inverse of the variable of that link, as in the path convention. Then $k_x U_{xy} k_y^{-1} = I$ on every tree edge. By recursion along the

tree, we see that this gauge transformation is unique given the values at the roots and that each k_x is an analytic path product. By (10.14), a gauge transformation equal to the identity at the coarse vertices preserves the coarse field. In these coordinates each straight reference path $C_{c'}$ has only one link that is not gauged to the identity, the eliminated edge $e^*_{c'}$.

Step 2: the derivative in the eliminated edge. We note that in these coordinates the constraint at c' contains, among all the eliminated variables, only its own. Indeed, the root paths are identities, and a straight middle segment crosses only the interface from its starting block to the neighboring block in direction i. Exactly M of the M^3 segments cross $e^*_{c'}$: their transverse offset is zero, and the longitudinal start has M choices. At $z = 0$, let only $X = X_{e^*_{c'}}$ vary. Then the M contours through $e^*_{c'}$ have transport $e^X c_{c'}$, all other contours have transport $c_{c'}$, and the reference path has transport $e^X c_{c'}$. The relative logarithms are 0 for the first M contours and $-X$ for the others, so their mean is $-(1 - M^{-2})X$, and the output is exactly

$$\exp\bigl(-(1 - M^{-2})X\bigr) e^X c_{c'} = e^{M^{-2}X} c_{c'}.$$

Thus the derivative of the constraint in the eliminated edge is $M^{-2} I_3$ at $z = 0$, for every sufficiently small c, not only for $c = I$.

Step 3: solving the constraints. Each constraint involves a bounded number of variables, and a fixed free link meets a bounded number of constraints, both depending only on M. The derivatives of these path products on a fixed smaller polydisc are bounded by $s! K_M^s$. We consider the equation

$$X_{e^*_{c'}} = X_{e^*_{c'}} - M^2 \mathsf{C}_{c'}(X, z).$$

By Step 2, the derivative of its right side in $X_{e^*_{c'}}$ vanishes at $(X, z) = (0, 0)$. On a fixed small product this derivative has norm less than $1/4$, and the constant term is $O_M(\|z\|_\infty)$. Hence the right side maps a ball of radius $K_M \|z\|_\infty$ strictly into itself, and iteration gives the unique small holomorphic solution. Majorizing by the scalar quadratic implicit series of Lemma 5.3, we obtain factorial bounds on its derivatives. Since the equations are independent in their eliminated unknowns, the solution on the whole lattice has the same bounds and a fixed stencil, with no factor depending on the dimension. At $z = 0$ the solution vanishes identically. Multiplying by $U_0(c)$ and taking the local logarithm, we obtain the bounds on B and on $D_z B - \mathsf{E}_\mathrm{f}$ from Taylor's formula. Since a stencil has bounded diameter, a fixed spatial weight changes only the constant. A fixed input enters a bounded number of outputs, which gives the column bounds. This proves part (1).

Step 4: the density. To derive the density, we first eliminate the gauge transformations along the trees recursively, by left and right translations of the Haar probability measure; their integral is one. With the remaining links fixed, we change each eliminated link to the

output of its constraint; the Jacobian of this change is the displayed 3×3 determinant. The remaining exponential coordinates lie outside $\mathcal{F}$ and contribute their Haar factors to the numerator of $\mathcal{J}_c$, including those of the eliminated edges. Since these changes of variables are triangular, a derivative in an earlier coordinate contributes no further determinant. More explicitly, we fix c as a parameter before the change of variables. A nearby output is $e^{\varphi}c$, and the Haar probability measure in this coordinate is $c_{\mathrm{Haar}}^{q} \prod_{c'} j_H(\varphi_{c'})\, d\varphi$, where q is the number of positive coarse links. On the fiber $\varphi = 0$ these factors j_H equal one. There are $f + q$ input links outside $\mathcal{F}$, so dividing their Haar constants by those of the output leaves c_{Haar}^{f}. Right translation by the fixed configuration $U_0(c)$ preserves the Haar measure of the inputs. This gives the displayed density at each fixed coarse field. Letting this parameter vary afterwards introduces no further Jacobian and no additional Haar factor for the coarse field.

Step 5: the Hessian of the logarithm of the density. We shrink the product until the Haar factors are bounded away from zero and the derivatives of the constraints in the eliminated edges differ from $M^{-2} I_3$ by less than $M^{-2}/4$. Then the real determinants are positive, and their logarithms and the logarithms of the Haar factors have bounded local analytic derivatives. The function $\log \mathcal{J}_c$ is the sum of these local terms and of the constant $f \log c_{\mathrm{Haar}}$. Each differentiated row meets a bounded number of terms, and each term involves a bounded number of coordinates. This proves the bounds on the rows and columns of the Hessian in part (2); the undifferentiated density depends on the volume.

Step 6: tangent vectors to the fiber. For a curve $b(t)$ in the fiber as in part (3), we apply to each $b(t)$ its unique block axial gauge transformation $k(t)$, which is the identity at the coarse vertices and depends differentiably on t, since each $k_x(t)$ is a path product. The gauged curve has the same block average c, and its free coordinates are some $z(t)$. By uniqueness of the solution of the constraints, all its remaining coordinates are those of $\Gamma_c(z(t))$, that is, $b(t)^{k(t)} = b(c, z(t))$. Hence $b(t) = \mathcal{G}_{k(t)^{-1}}(b(c, z(t)))$. Differentiating at $t = 0$ by the product rule, we obtain the displayed formula: the first term comes from the derivative of $b(c, z(t))$ at the fixed gauge transformation k_0^{-1}, and the second from the derivative of $k(t)^{-1}$ at the fixed configuration $b(c, z_0)$. If $b(0) = b(c, z_0)$, the uniqueness of the block axial gauge transformation gives $k_0 = I$. The argument is local and requires no global choice on the compact fiber. □

In the following theorem, E_{d_0} is the effective classical action of depth d_0 over the lattice of side Mm, in the coarse lattice units of that lattice, and E_{d_0+1} is the effective classical action of depth $d_0 + 1$ over the lattice of side m. The coordinates z are the free logarithmic coordinates of Lemma 10.9; no coupling has been inserted.

Theorem 10.10. *There exist $r_c, r_z, m_0, C_0, a, C, g_* > 0$ such that the following hold for every $d_0 \geq 0$ and every $m \geq 4$. Let*

$$F_{d_0}(c,z) = E_{d_0}(b(c,z)), \qquad K^0_{d_0} = \mathsf{E}_{\mathrm{f}}^T D^2 E_{d_0}(I) \mathsf{E}_{\mathrm{f}}.$$

Here $D^2 E_{d_0}(I)$ is the effective Maxwell quadratic form of Theorem 8.3; its gauge and constant null directions are kept.

1. *On the real space of free coordinates,*

$$m_0 I \leq K^0_{d_0} \leq C_0 I, \qquad \|K^0_{d_0}\|_{a,\mathrm{row+col}} + \|(K^0_{d_0})^{-1}\|_{a,\mathrm{row+col}} \leq C_0.$$

2. *On the real product $\|\mathrm{Log}\, c\|_\infty < r_c$, $\|z\|_\infty \leq r_z$, the configuration $b(c,z)$ lies in the domain of E_{d_0} and in the domain of positivity of the Haar disintegration density with fixed margins. The Hessian*

$$D_z^2 F_{d_0} = (D_z B)^T D^2 E_{d_0}(B) D_z B + D E_{d_0}(B) D_z^2 B, \qquad B = B(c,z),$$

satisfies

$$\|D_z^2 F_{d_0} - K^0_{d_0}\|_{a,\mathrm{row+col}} \leq C(\|\mathrm{Log}\, c\|_\infty + \|z\|_\infty), \qquad D_z^2 F_{d_0} \geq \tfrac{1}{2} m_0 I.$$

3. *For real c with $\|\mathrm{Log}\, c\|_\infty < r_c$ there is a unique minimizer $z^*_{d_0}(c)$ of $z \mapsto F_{d_0}(c,z)$ on the closed real ball $\|z\|_\infty \leq r_z$. It is an interior point, it depends real analytically on c, and it has a holomorphic continuation to the complex ball $\|\mathrm{Log}\, c\|_\infty < r_c$. Further,*

$$D_z F_{d_0}(c, z^*_{d_0}(c)) = 0, \qquad \|z^*_{d_0}(c)\|_\infty \leq C\|\mathrm{Log}\, c\|_\infty < r_z/8,$$

$$E_{d_0+1}(c) = M \min_{\|z\|_\infty \leq r_z} F_{d_0}(c,z) = M F_{d_0}(c, z^*_{d_0}(c)).$$

   We call $z^*_{d_0}(c)$ the conditional constrained minimizer.

4. *There exists $0 < r_c' \leq r_c$, independent of d_0 and m, such that for every integer $s \geq 1$ there exist $a_s, C_s > 0$, depending on s but independent of d_0 and m, for which the following bound holds on the complex ball $\|\theta\|_\infty < r_c'$:*

$$\|D_\theta^s z^*_{d_0}(e^\theta)\|_{a_s,\mathrm{out}} + \|D_\theta^s z^*_{d_0}(e^\theta)\|_{a_s,\mathrm{input,count}} \leq C_s,$$

   where the outputs are the free coordinates, the inputs are the coarse links, and distances are measured in the coarse lattice units of the lattice of side m. The volume-normalized column differs from the counting column by the fixed factor M^{-3}.

5. *The identity $E_{d_0+1}(c) = M F_{d_0}(c, z^*_{d_0}(c))$ holds for complex c in the ball of (3), with the holomorphic continuation of $z^*_{d_0}$. For real c and couplings $g_+^2 = M g^2$, the identity in (3) reads*

$$\frac{E_{d_0+1}(c)}{g_+^2} = \frac{1}{g^2} \min_{\|z\|_\infty \leq r_z} F_{d_0}(c,z).$$

6. *Let c be real with $\|\operatorname{Log} c\|_\infty < r_c$, let $p \geq 1$, and let $0 < g \leq g_*$ with $gp < r_z/4$. Then the function*
$$y \longmapsto \frac{1}{g^2}\big[F_{d_0}(c, z^*_{d_0} + gy) - F_{d_0}(c, z^*_{d_0})\big] - \log \frac{\mathcal{J}_c(z^*_{d_0} + gy)}{\mathcal{J}_I(0)}$$
of real y has Hessian at least $m_0 I/4$ on the product $\prod_e \{|y_e| \leq p\}$.

The theorem applies at every depth with the same constants. It can be nested whenever the intermediate fields remain in its input domain; for any fixed number of steps this follows by reducing the initial coarse-field radius, as in Step 1 of the proof of Theorem 12.13. The minimization statements and the lower bounds on Hessians concern real coarse fields and real free coordinates. The identity in (5) for complex c and the bounds in (4) concern the holomorphic continuations of the critical point and the critical value. In (6) the denominator $\mathcal{J}_I(0)$ is a constant, so the value of the numerator at the shifted point is kept.

Proof of Theorem 10.10. We identify the Hessian of E_{d_0} at the identity (Step 1), obtain the bounds on $K^0_{d_0}$ (Step 2), and bound the Hessian away from the identity (Step 3). We then choose the radii (Step 4), construct the minimizer (Step 5), identify the nested minimum with the background field of depth $d_0 + 1$ (Steps 6–8), and prove part (6) (Step 9).

Step 1: the Hessian of the action at the identity. Differentiating the action twice at $\theta = 0$ in Theorem 10.6, using $DA_{d_0}(0) = \mathsf{H}_{d_0}$ and the vanishing of the gradient of $\mathscr{S}_\varepsilon$ at 0, we obtain
$$D^2 E_{d_0}(I) = \mathsf{H}^\sharp_{d_0} d^*_\varepsilon d_\varepsilon \mathsf{H}_{d_0}. \tag{10.22}$$
We show that this is the effective Maxwell quadratic form of Theorem 8.3, which is defined with the linearization of $T^{d_0}_M$ rather than with the gauge-corrected average. Let $\ell(a)$ be the derivative at $a = 0$ of $\operatorname{Log} T^{d_0}_M e^{\varepsilon a}$, and let $\gamma(a)$ be the derivative at $a = 0$ of $\operatorname{Log} G_{d_0}(a)$, a field on the coarse vertices. For a coarse vertex field κ we write $(\nabla^{\mathrm{c}}\kappa)_{(r,s)} = \kappa_s - \kappa_r$. Linearizing (10.11) at $j = d_0$ and (10.14), we obtain, for fine fields a and fine scalar fields λ,
$$\mathsf{Q}_{d_0} a = \ell(a) - \nabla^{\mathrm{c}}\gamma(a), \qquad \ell(\nabla_\varepsilon \lambda) = \nabla^{\mathrm{c}}(\lambda|_{\mathrm{roots}}).$$
Thus the linearized block average and Q_{d_0} differ by a coarse gradient, and a fine gradient changes the linearized block average by a coarse gradient and has zero curl. Let a have $\ell(a) = \eta$. Linearizing the gauge slice of Theorem 9.9 with prescribed coarse datum zero, we obtain a fine scalar field λ such that $a' = a - \nabla_\varepsilon \lambda$ has $\mathsf{P}^{\mathrm{div}}_{d_0} \operatorname{div}_\varepsilon a' = 0$ and $\gamma(a') + \lambda|_{\mathrm{roots}} = 0$. Then $d_\varepsilon a' = d_\varepsilon a$ and $\mathsf{Q}_{d_0} a' = \eta - \nabla^{\mathrm{c}}(\lambda|_{\mathrm{roots}}) - \nabla^{\mathrm{c}}\gamma(a') = \eta$. Conversely, let a' have $\mathsf{Q}_{d_0} a' = \eta$, and let λ' be a fine scalar field with $\lambda'|_{\mathrm{roots}} = \gamma(a')$, for instance the linearization of the cardinal extension of Lemma 10.8. Then $a'' = a' - \nabla_\varepsilon \lambda'$ has $d_\varepsilon a'' = d_\varepsilon a'$ and $\ell(a'') = \eta$. We conclude that the minimum of $\|d_\varepsilon a\|^2_\varepsilon$ over the fields with $\ell(a) = \eta$ equals its minimum over the fields with $\mathsf{Q}_{d_0} a = \eta$ and $\mathsf{P}^{\mathrm{div}}_{d_0} \operatorname{div}_\varepsilon a = 0$.

Every field of the latter kind is $\mathsf{H}_{d_0}\eta + v$ with $v \in \mathcal{V}_{d_0}$, and the minimizing property of H_{d_0} shows that the minimum equals the right side of (10.22) evaluated at η. This identifies $D^2E_{d_0}(I)$ with the effective Maxwell quadratic form of Theorem 8.3, including its gauge and constant null directions. The function $W_{d_0,g}(B) = g^{-2}E_{d_0}(e^{gB})$ has the same Hessian at $B = 0$, since the two factors g from the derivatives cancel g^{-2}.

Step 2: the constrained Hessian at the identity. We apply the statement of Theorem 8.3 about the next constrained fluctuation space, with the lattice of side Mm as the coarse lattice of that theorem, the next lattice of side m, depth d_0, and the embedding $E_0 = D_z\Gamma_I(0)$ of Lemma 6.5, which equals E_f because $U_0(I) = I$. That theorem gives $m_0, C_0, a_0 > 0$, independent of d_0 and m, for the lower bound, the upper bound, and the weighted bounds on $K^0_{d_0}$ and on its inverse. The inverse is asserted only on the next constrained fluctuation space, on which the vanishing of the linearized next average removes the constant fluctuation. If distances are measured in the coarse units of the lattice of side m rather than in those of the lattice of side Mm, they are divided by M, and we decrease the exponent once. This proves part (1).

Step 3: the Hessian away from the identity. We write $t = \operatorname{Log} c$, $H_0 = D^2E_{d_0}(I)$, and $B = B(c, z)$. The Hessian in the coordinates z is

$$D_z^2F_{d_0} = (D_zB)^T D^2E_{d_0}(B)D_zB + DE_{d_0}(B)D_z^2B. \tag{10.23}$$

By the bounds on the third derivatives and on the gradient in Theorem 10.6, we have

$$\|D^2E_{d_0}(B) - H_0\|_{a,\mathrm{row+col}} \le K\|B\|_\infty, \qquad \|DE_{d_0}(B)\|_\infty \le K\|B\|_\infty.$$

For the second term in (10.23), we fix a free coordinate and use the column of D_z^2B with this input fixed; the remaining input and the output are summed against the bound on the components of $DE_{d_0}(B)$. This term is $O(\|B\|_\infty)$ in both the row and the column norm. For the first term we expand around $\mathsf{E}_\mathrm{f}^T H_0 \mathsf{E}_\mathrm{f}$, using the bound on $D_zB - \mathsf{E}_\mathrm{f}$ in Lemma 10.9 and the displayed bound on $D^2E_{d_0}(B) - H_0$. Using the contraction rule of Lemma 5.3 and $\|B\|_\infty \le K(\|t\|_\infty + \|z\|_\infty)$, we obtain

$$\|D_z^2F_{d_0} - K^0_{d_0}\|_{a,\mathrm{row+col}} \le K(\|t\|_\infty + \|z\|_\infty). \tag{10.24}$$

Every sum here has one index fixed; no sum of $|DE_{d_0}|$ over all links, and no factor of the volume, occurs.

Step 4: the radii. We take a complex radius for the coordinates of Lemma 10.9 that leaves fixed margins for five requirements: the relative logarithms, the solution of the constraints, the positivity of the real density, the larger domain of E_{d_0}, and the chart $\{\|\operatorname{Log} b\|_\infty < r_\mathrm{sym}\}$ of Lemma 10.8. Let C_inv bound the weighted norm of the inverse of

$K^0_{d_0}$. Fix $r_z > 0$ so small that the entire product and its image $b(c,z)$ lie strictly inside the corresponding domains and

$$2Kr_z \le m_0/2, \qquad 2C_{\mathrm{inv}}Kr_z \le 1/2.$$

Next we take $r_c \le r_z$ inside the domains for the coarse field. Since the weighted bound (10.24) also bounds the Euclidean operator norm, the real Hessian is at least $m_0 I/2$ on the entire product. This proves part (2). The radii use only the bounds and margins on these fixed larger domains, which do not depend on d_0, and not a recursively reduced domain of E_{d_0}; the same radii work for every d_0.

Step 5: the conditional constrained minimizer. At $t = 0$ and $z = 0$ the gradient in z vanishes. Integrating its mixed derivative in the coarse field, we obtain $\|D_z F_{d_0}(c,0)\|_\infty \le K_1\|t\|_\infty$. The equation for the minimizer is the fixed point equation

$$z = -(K^0_{d_0})^{-1}\big[D_z F_{d_0}(c,z) - K^0_{d_0} z\big].$$

By (10.24), the right side has Lipschitz constant at most $1/2$ on the complex ball of radius r_z. If we reduce r_c so that $2C_{\mathrm{inv}}K_1 r_c < r_z/8$, the right side maps this ball strictly into itself, and its fixed point satisfies

$$\|z^*_{d_0}(c)\|_\infty \le 2C_{\mathrm{inv}}K_1\|t\|_\infty < r_z/8.$$

The action-gradient tensors have one common convergent majorant at one spatial rate by (10.20). The local analytic coordinates B of Lemma 10.9 have geometric normalized coefficient bounds, obtained on a fixed larger finite stencil from Lemma 6.3. Composing these majorants with the weighted inverse $(K^0_{d_0})^{-1}$ gives the common scalar majorant required by the implicit-series clause of Lemma 5.3. Its output rows give holomorphy and the derivative bounds; for an input column we follow the fixed input through the inverse and the mixed derivatives of $D_z F_{d_0}$, using rows on the other branches. Thus the bounds hold at one smaller rate, which may be used for every a_s in part (4). Choose this common majorant compatible with the fixed complex product, increasing its coefficients if necessary so that its implicit-series unknown ball lies inside $\|z\|_\infty < r_z$, and let $t_0 > 0$ be its input radius. The majorized solution solves the same fixed point equation and lies in the complex ball used above, so uniqueness identifies it with the already constructed minimizer. We take $r'_c = \frac{1}{2}\min\{r_c, t_0\}$, where r_c denotes its final value after the reductions in this step and in Step 7. The implicit-series clause of Lemma 5.3 includes the translation to backgrounds in such a fixed smaller ball, so the coefficient bounds of each fixed order hold on $\|\theta\|_\infty < r'_c$. This proves part (4). Strict convexity in the real variables shows that this interior critical point is the unique minimum on the closed real ball $\|z\|_\infty \le r_z$.

Step 6: the nested minimizer is critical on the longer fiber. We now identify the minimum with the effective classical action of depth $d_0 + 1$. Let $b_* = b(c, z^*_{d_0}(c))$, so that $\|\operatorname{Log} b_*\|_\infty \le K\|t\|_\infty$. For coarse fields b near b_*, let $s_{d_0}(b)$ be the representative of part (1) of Lemma 10.8: it lies in the gauge orbit of the background field of depth d_0, it depends analytically on b, its rescaled connection is small in C^1_ε, and $T^{d_0}_M s_{d_0}(b) = b$. Its action is $E_{d_0}(b)$. The fine configuration $U_* = s_{d_0}(b_*)$ satisfies

$$T^{d_0}_M U_* = b_*, \qquad T^{d_0+1}_M U_* = c.$$

Let $\dot U$ be a tangent vector at U_* to the fiber $\{T^{d_0+1}_M U = c\}$, and let $\dot b = D(T^{d_0}_M)\dot U$; then $DT_M\,\dot b = 0$. The vector $\dot U - Ds_{d_0}(b_*)\dot b$ is tangent to the fiber $\{T^{d_0}_M U = b_*\}$. By part (6) of Theorem 10.6, the first variation of the action in this direction vanishes. Indeed, U_* is of the form considered there: it is obtained from the background field with coarse field b_* by the cardinal extension of Step 3 of the proof of Lemma 10.8, whose values at the coarse vertices are $G_{d_0}(A_{d_0}(\operatorname{Log} b_*))$. The remaining first variation, in the direction $Ds_{d_0}(b_*)\dot b$, equals $DE_{d_0}(b_*)\dot b$, in the coarse units of the lattice of side Mm, since the action on the analytic family s_{d_0} equals E_{d_0}. Since $b_* = b(c, z^*_{d_0})$ is already in block axial gauge, the fixed transport in part (3) of Lemma 10.9 is the identity. That part then shows that $\dot b$ is the sum of a tangent vector $D_z b(c, z^*_{d_0})\dot z$ of the free coordinates and of a tangent vector of gauge transformations along the block trees that are the identity at the coarse vertices. The first is annihilated by $D_z F_{d_0}(c, z^*_{d_0}) = 0$, and the second by the infinitesimal gauge-invariance identities of Lemma 10.8. This shows that U_* is critical on the fiber $\{T^{d_0+1}_M U = c\}$.

Step 7: identification with the background field of depth $d_0 + 1$. In the coarse units of the lattice of side m, the same fine lattice has mesh ε/M, where $\varepsilon = M^{-d_0}$. By item (4) of the normalizations in Section 10.1, the rescaled connection of U_* is multiplied by M and its first differences by M^2, so its $C^1_{\varepsilon/M}$ norm is at most $K_M\|t\|_\infty$. We decrease r_c once more so that $K_M r_c < r_{\mathrm{tr}}$ and $r_c < r_{\mathrm{tr}}$, the radius of Lemma 10.7. That lemma, at depth $d_0 + 1$, shows that U_* lies in the gauge orbit of the background field of depth $d_0 + 1$ with coarse field c, and that $\mathscr{S}_{\varepsilon/M}(U_*) = E_{d_0+1}(c)$.

Step 8: the change of units. By item (4) of the normalizations, we have $\mathscr{S}_{\varepsilon/M} = M\mathscr{S}_\varepsilon$ for the same fine configuration, and therefore

$$E_{d_0+1}(c) = \mathscr{S}_{\varepsilon/M}(U_*) = M\mathscr{S}_\varepsilon(U_*) = ME_{d_0}(b_*) = M \min_{\|z\|_\infty \le r_z} F_{d_0}(c, z).$$

This proves part (3) on the real ball of coarse fields. The value $MF_{d_0}(c, z^*_{d_0}(c))$ at the holomorphic minimizer and the effective classical action $E_{d_0+1}(c)$ are both holomorphic, and their difference has vanishing real Taylor coefficients on a real open ball. Its convergent complex Taylor series therefore vanishes as well, which proves the first statement of part (5). The second statement follows by dividing by $g^2_+ = Mg^2$.

Step 9: the Hessian with the density. Let c and y be real. Under $gp < r_z/4$ we have $\|z^*_{d_0} + gy\|_\infty < 3r_z/8$. Two derivatives in y cancel the prefactor g^{-2} of the classical part, while two derivatives of the logarithm of the density give $g^2 D_z^2 \log \mathcal{J}_c$. By part (2) and Lemma 10.9, the Hessian of the function in part (6) is

$$D_z^2 F_{d_0}(c, z^*_{d_0} + gy) - g^2 D_z^2 \log \mathcal{J}_c(z^*_{d_0} + gy) \geq \tfrac{1}{2} m_0 I - K_M g^2 I.$$

Taking $g_* \leq 1$ with $K_M g_*^2 \leq m_0/4$, we obtain part (6) on the entire convex product. All the choices above have the dependence stated in Section 10.1. □

10.8 Analytic estimates at intermediate scales

The last theorem collects the factorial bounds proved so far and extends them to the block averages of the background field at every intermediate scale and to their curvature. At the intermediate scales we use the gauge-corrected averages of (10.9) as representatives; they differ from the iterates of T_M by the explicit gauge transformations G_j of (10.11). As in Section 10.3, $\epsilon_j = \varepsilon_{d,j} = M^{j-d}$.

Theorem 10.11. *There exist $r_*, a_*, C_* > 0$ such that the following hold for every $d \geq 0$, $n \geq 4$, $0 \leq j \leq d$, and every real or complex coarse field θ with $\|\theta\|_\infty < r_*$. All connecting lengths are measured in coarse lattice units.*

1. *For $s \geq 2$,*
$$\|D_\theta^s E_d(e^\theta)\|_{a_*,\mathrm{root}} \leq C_*^{s+1} s!, \qquad \|DE_d(e^\theta)\|_\infty \leq C_* \|\theta\|_\infty.$$
2. *Put $\varepsilon_{d,j} = M^{j-d}$, and let $A_{d,j}(\theta)$ be the j-th iterate of* (10.9) *started at $A_{d,0}(\theta) = A_d(\theta)$. Then $e^{\varepsilon_{d,j} A_{d,j}(\theta)}$ is a representative of the gauge orbit of $T_M^j(e^{\varepsilon_d A_d(\theta)})$, related to it by the gauge transformation G_j of* (10.11)*, and for every $s \geq 1$,*
$$\|D_\theta^s(A_{d,j}, \nabla_{\varepsilon_{d,j}} A_{d,j})\|_{a_*,\mathrm{out}} + \|D_\theta^s(A_{d,j}, \nabla_{\varepsilon_{d,j}} A_{d,j})\|_{a_*,\mathrm{input};\varepsilon^3_{d,j}} \leq C_*^{s+1} s!.$$
In the second norm one coarse input is fixed, and the intermediate outputs and the remaining inputs are summed, with the output measure $\varepsilon^3_{d,j}$.
3. *For the coarse correction D_d we use the counting input columns $\|\cdot\|_{a_*,\mathrm{input,count}}$ of Lemma 10.4; multiplying them by ε_d^{-3} gives the volume-normalized columns. Then*
$$\|D_w^s \mathsf{D}_d\|_{a_*,\mathrm{out}} \leq C_*^s s!, \qquad \|D_w^s \mathsf{D}_d\|_{a_*,\mathrm{input,count}} \leq \varepsilon_d^3 C_*^s s! \quad (s \geq 2),$$
and, on a fixed smaller ball of fine fields w, the first derivative satisfies the corresponding bounds $C_*\|w\|_\infty$ and $\varepsilon_d^3 C_* \|w\|_\infty$.
4. *For a plaquette f of the lattice of mesh $\varepsilon_{d,j}$ put*
$$\mathcal{F}_{d,j,f}(\theta) = \varepsilon_{d,j}^{-2} \operatorname{Log}\big[(e^{\varepsilon_{d,j} A_{d,j}(\theta)})_f\big].$$

It vanishes at $\theta = 0$, and it satisfies the bound of (2) for every $s \geq 1$: the sum of its output row and its volume-normalized input column with output measure $\varepsilon_{d,j}^3$ is at most $C_*^{s+1} s!$. The underlying local fact is that the function

$$\varepsilon^{-2} \operatorname{Log}\bigl(e^{\varepsilon a} e^{\varepsilon b + \varepsilon^2 v} e^{-\varepsilon a - \varepsilon^2 u} e^{-\varepsilon b}\bigr)$$

of independent variables (a, b, u, v) in a small ball extends holomorphically to $\varepsilon = 0$, with value $v - u + [a, b]$ there, uniformly for $0 \leq \varepsilon \leq 1$.

5. *In the coupling-rescaled variables $\theta = g_+ B$, with $g_j = \sqrt{\varepsilon_{d,j}}\, g_+$, let*

$$X_1(B) = \varepsilon_{d,j} A_{d,j}(g_+ B)/g_j, \qquad X_2(B) = \operatorname{Log}\bigl[(e^{\varepsilon_{d,j} A_{d,j}(g_+ B)})_f\bigr]/g_j$$

be the normalized connection and curvature. Then, for every $s \geq 1$,

$$\|D_B^s X_1\|_{a_*,\mathrm{out}} + \|D_B^s X_1\|_{a_*,\mathrm{input};\varepsilon_{d,j}^3} \leq C_*^{s+1} s!\, \sqrt{\varepsilon_{d,j}}\, g_+^{s-1},$$
$$\|D_B^s X_2\|_{a_*,\mathrm{out}} + \|D_B^s X_2\|_{a_*,\mathrm{input};\varepsilon_{d,j}^3} \leq C_*^{s+1} s!\, \varepsilon_{d,j}^{3/2} g_+^{s-1}.$$

The columns use the output measure $\varepsilon_{d,j}^3$, as in (2).

If the connecting lengths are replaced by the lengths of spanning trees on the external indices, the bounds hold with a_* replaced by $a_*/2$. No reduction of the rate or of the radius depends on the order s.

The bounds in (2)–(5) concern the representatives $e^{\varepsilon_{d,j} A_{d,j}}$ on the stated domain of fields and first differences; in other gauges the iterates $T_M^j(e^{\varepsilon_d A_d(\theta)})$ need not satisfy them.

Proof of Theorem 10.11. Parts (1) and (3), and part (2) at $j = 0$, follow from the earlier coefficient bounds (10.20), (10.17), (10.18), and (10.10). It remains to prove the estimates at intermediate scales and for the curvature. We first construct a map on independent field and difference variables that describes one averaging step (Step 1). We check its mesh powers (Step 2) and its domain (Step 3), apply Lemma 10.3 (Step 4), compose with the background field (Step 5), treat the curvature (Steps 6 and 7), and collect the constants (Step 8). For $d = 0$ only $j = 0$ occurs, so let $d \geq 1$.

Step 1: one step on field and difference variables. We consider one step at mesh $\epsilon \leq 1/M$. A shift of the coarser lattice by one coarse spacing $M\epsilon$ is a shift of the fine lattice by M sites, and it preserves the centered partition and every ordered path in (10.9). Let $S_{M,\mu}$ be this shift of the fine lattice and S'_μ the shift by one spacing of the coarser lattice, in direction μ. Then

$$S'_\mu \Psi_\epsilon(A) = \Psi_\epsilon(S_{M,\mu} A),$$

also at periodic seams. For a triple $G = (G_\mu)$ of fine fields, we define

$$(T_\mu G_\mu)_i(x) = M^{-1} \sum_{l=0}^{M-1} G_{i,\mu}(x + l\epsilon e_\mu).$$

This average of shifts on the fine lattice has counting row and column mass one. If $G = \nabla_\epsilon A$, telescoping gives $S_{M,\mu} A - A = M\epsilon\, T_\mu G_\mu$, and by the fundamental theorem of calculus in finitely many variables we have

$$\nabla_{M\epsilon,\mu} \Psi_\epsilon(A) = \int_0^1 D\Psi_\epsilon\big(A + tM\epsilon\, T_\mu G_\mu\big)\,[T_\mu G_\mu]\, dt. \tag{10.25}$$

We define the right side for independent (A, G) and let $\mathcal{P}_\epsilon$ be the map whose outputs are $\Psi_\epsilon(A)$ and these three expressions. It is holomorphic on a common small ball of (A, G): since $M\epsilon \le 1$, the auxiliary arguments $A + tM\epsilon T_\mu G_\mu$ stay in the domain of the local expressions. By (10.25), on the fields with $G = \nabla_\epsilon A$ its outputs are the next pair $(\Psi_\epsilon(A), \nabla_{M\epsilon} \Psi_\epsilon(A))$.

The linear part of $\mathcal{P}_\epsilon$ at zero is

$$\operatorname{diag}(Q, QT_1, QT_2, QT_3),$$

with Q from (10.12). For each of the four component types, its row mass is 1 and its counting column mass is M^{-3}; for the latter, we sum first over the outputs of Q, whose column mass is M^{-3}, and then use the column mass one of T_μ. We keep the four component types as separate labels, so that no factor comparing norms of different components enters at any scale.

Step 2: the mesh powers. We record the powers of the mesh in the nonlinear part, rather than differentiating a bound on values. With the fixed local expression P of the proof of Lemma 10.4, the integrand in (10.25) is

$$M^{-1} DP\big(\epsilon A + tM\epsilon^2 T_\mu G_\mu\big)\,[T_\mu G_\mu]. \tag{10.26}$$

The constant coefficient $DP(0)$ contributes the linear term $QT_\mu G_\mu$, and every other term contains ϵA or $M\epsilon^2 T_\mu G_\mu$. On a ball $\|A\|_\infty \le a$, $\|G\|_\infty \le b$, the nonlinear part is bounded by $K_M(\epsilon ab + \epsilon^2 b^2)$, with first derivatives $O_M(\epsilon b)$ in A and $O_M(\epsilon a + \epsilon^2 b)$ in G. We consider a derivative of order s of the integrand, with p derivatives in A and q in G. If $q \ge 1$ and one of the derivatives in G falls on the external factor $T_\mu G_\mu$, the explicit factor is $\epsilon^p (M\epsilon^2)^{q-1}$. If none falls on it, the factor is $\epsilon^p (M\epsilon^2)^q$ times an undifferentiated G. When $p = 0$ and $q = 1$, the contribution from $DP(0)$ is the linear operator separated above; the remaining terms contain ϵA or $M\epsilon^2 T_\mu G_\mu$. When $q = 0$ and $p \ge 2$, the factor is ϵ^p times G. For the field output the factor at order s is ϵ^{s-1}/M. Using Cauchy estimates in finitely many variables on the fixed polydisc, we obtain, for $s \ge 2$,

$$\begin{aligned} &\|D\mathcal{P}_\epsilon\|_{\text{out}} \le 1 + K_M\epsilon(a+b), \quad &&\|D\mathcal{P}_\epsilon\|_{\text{input,count}} \le M^{-3}[1 + K_M\epsilon(a+b)], \\ &\|D^s\mathcal{P}_\epsilon\|_{\text{out}} \le s!K_M^s\epsilon, &&\|D^s\mathcal{P}_\epsilon\|_{\text{input,count}} \le M^{-3} s!K_M^s\epsilon. \end{aligned} \tag{10.27}$$

For the columns, the nonlinear terms have a fixed enlarged stencil of inputs, which includes the M shifts in T_μ. A fixed input lies in at most q''_M such stencils; this count

enters as a factor only in the nonlinear part, while the linear masses are those computed in Step 1. Integration in t preserves all the bounds.

Step 3: a common domain under iteration. Let u_j be the maximum of the norms of the field and difference variables after j steps. By the bound on the field in Step 3 of the proof of Lemma 10.4 and the bound on the nonlinear part of the difference outputs in Step 2, we have $u_{j+1} \le u_j + K_M \epsilon_j u_j^2$, after enlarging K_M. As in Lemma 10.3, with $\sum_j \epsilon_j < 1/(M-1)$, the radius stays below twice its initial value if that value is small enough. On the fields with $G = \nabla_\epsilon A$, the argument $A + tM\epsilon T_\mu G_\mu$ in (10.25) is a convex combination of A and its shift. This shows directly that the difference variables after step $j+1$ have norm at most $(1 + K_M \epsilon_j \|A_j\|_\infty)$ times their norm after step j. This common domain justifies treating A and G as independent variables in (10.27).

Step 4: composition of the steps. By Steps 2 and 3, the maps $\mathcal{P}_{\epsilon_j}$ satisfy the hypotheses of Lemma 10.3, with the nonlinear coefficient ϵ_j and the linear column factor M^{-3} as before. The labels consist of a link and one of the four component types, and the joint radius takes the place of R. Each output depends only on inputs within distance $K_M \epsilon_j$. Applying the lemma, we obtain, for the composition of j steps,

$$R_s(j) \le K^{s+1} s!, \qquad C_s(j) \le (\epsilon_0/\epsilon_j)^3 K^{s+1} s!, \tag{10.28}$$

where $R_s(j)$ and $C_s(j)$ are the output row and the counting input column of the derivative of order s. This holds for all orders and all $0 \le j \le d$. Since the enlarged stencils have total radius at most $K_M \epsilon_j$, a nonzero tensor has connecting length at most $K_M s \epsilon_j$, and a fixed spatial weight only changes K^s.

Step 5: composition with the background field. We start the recursion from the pair $(A_d(\theta), \nabla_{\epsilon_0} A_d(\theta))$. Its value is $O(\|\theta\|_\infty)$, and it has the majorant (10.17). We take the radius of θ so small that this pair lies strictly inside the common domain of Step 3. Composing the two majorant series, we obtain factorial output rows at one fixed smaller radius and one rate. For a column with a coarse input fixed, we follow the branch of the composition containing that input. The initial tensor has counting column $O(\epsilon_0^{-3})$, by (10.18), and the composed map has counting column $O((\epsilon_0/\epsilon_j)^3)$ by (10.28); their product is $O(\epsilon_j^{-3})$, and all other branches use output rows. The output measure ϵ_j^3 cancels the factor ϵ_j^{-3}. By the contraction rule of Lemma 5.3, the same bound holds for every fixed coarse input. The field outputs of the composition are the fields $A_{d,j}(\theta)$ of part (2), and their gauge transformation is the G_j of (10.11), evaluated at the initial field $A_d(\theta)$. These fields represent the gauge orbits of $T_M^j(e^{\varepsilon_d A_d(\theta)})$, including for $j = 0$ and $j = d$. For complex θ the gauge transformations are the analytic continuations of these path products, with values in $\mathrm{SL}(2,\mathbb{C})$; for real θ they are unitary. This proves part (2).

Step 6: the local curvature formula. It remains to treat the curvature and its factor ϵ^{-2}. At a plaquette in the directions μ, ν based at x, let

$$a = A_\mu(x), \qquad b = A_\nu(x), \qquad u = \nabla_{\epsilon,\nu} A_\mu(x), \qquad v = \nabla_{\epsilon,\mu} A_\nu(x).$$

The plaquette variable is

$$P_\epsilon = e^{\epsilon a} e^{\epsilon b + \epsilon^2 v} e^{-\epsilon a - \epsilon^2 u} e^{-\epsilon b}.$$

We treat a, b, u, v as independent and fix a small complex ball of these variables on which this product has a logarithm for all complex $|\epsilon| \le 2$. The constant and the linear terms in ϵ of $P_\epsilon - I$ vanish identically. Expanding the four factors to second order, we obtain

$$\operatorname{Log} P_\epsilon = \epsilon^2 (v - u + [a, b]) + O(\epsilon^3).$$

The quotient $\epsilon^{-2} \operatorname{Log} P_\epsilon$ therefore has a removable singularity at $\epsilon = 0$, with the value stated in part (4). Cauchy's formula on the disc $|\epsilon| \le 2$, or the Taylor series beginning at degree two, bounds the quotient uniformly for $|\epsilon| \le 1$. Cauchy's formula in the finitely many variables (a, b, u, v) on a strictly smaller ball then bounds its coefficients by $K^{s+1} s!$, uniformly for $0 \le \epsilon \le 1$. This division by ϵ^2 uses no second difference of the field.

Step 7: composition with the intermediate pair. We compose this local quotient with the majorant of Step 5 for the intermediate pair. Each plaquette involves a fixed number of field and difference entries, at distance $O(\epsilon_j)$, and each entry occurs in a fixed number of plaquettes. The composition rule gives the factorial row for the curvature. With a coarse input fixed, the branch containing it uses the volume-normalized column of the pair, with the factor ϵ_j^3, and the other branches use rows. Hence the plaquette output has the same volume-normalized column, with no inverse power of the mesh. Its value at $\theta = 0$ is zero, because all fields and differences vanish there. This proves part (4).

Step 8: constants and coupling-rescaled variables. We take a_* smaller than the finitely many rates used for the extension H_d, the Green operator, the correction, the background field, the envelope formula for the action, and the passage from rows to columns. We take r_* inside all the finitely many common balls, and we increase C_* to dominate their constants and the fixed counts of incidences and comparisons of norms. These choices are independent of d, n, and j. With them, parts (1) and (3) are (10.20), the bound $\|\nabla_\theta E_d\|_\infty \le K \|\theta\|_\infty$ of the proof of Theorem 10.6, and (10.10). Since $\epsilon_j = \varepsilon_{d,j}$ and $g_j = \sqrt{\epsilon_j}\, g_+$,

$$\frac{\epsilon_j A_{d,j}(g_+ B)}{g_j} = \sqrt{\epsilon_j}\, g_+^{-1} A_{d,j}(g_+ B), \qquad \frac{\operatorname{Log}[(e^{\epsilon_j A_{d,j}(g_+ B)})_f]}{g_j} = \epsilon_j^{3/2} g_+^{-1} \mathcal{F}_{d,j,f}(g_+ B).$$

Each of the s derivatives in B produces a factor g_+, which gives the powers g_+^{s-1} in part (5), with the same row and column conventions. Finally, by Lemma 5.3, the length

of a spanning tree on the external indices is at most twice the connecting length, and replacing a_* by $a_*/2$ proves the last assertion. □

SECTION 11

Classical continuum limits and contraction estimates

This section proves four deterministic properties of the effective classical actions E_d and of the classical background fields constructed in Section 10. They are the classical inputs to the fluctuation integration in Part IV and to the large-field analysis in Part VI.

1. Section 11.2 fixes the coarse lattice and the coarse field and lets the number d of averaging steps tend to infinity, so that the fine lattice spacing M^{-d} tends to zero. We prove that a summably corrected version of E_d is nondecreasing in d, that E_d converges, that E_d is bounded below by a fixed multiple of the Wilson action, and that, at a fixed coarse lattice, E_d converges in the weighted analytic coefficient norm. We also prove that the reconstruction maps through finitely many averaging steps converge, together with all the matrices and densities entering them. This is Theorem 11.2; it is used in Sections 14, 15, and 22.
2. Section 11.3 proves that two small representatives of the same periodic gauge orbit have the same effective classical action, even when the gauge transformation relating them has no small logarithm. This implies that E_d extends to a real analytic function on the gauge-invariant small-field domain D_ρ (Theorem 11.6). This is used in Sections 13, 14, and 19.
3. Section 11.4 constructs, for one renormalization group step of J averaging steps, a representative of the gauge orbit of the classical background field that satisfies the block averaging constraint exactly and whose potential and first differences have coefficient bounds uniform in J. For each fixed J it identifies this representative, after an explicit gauge transformation, with the zero-fluctuation reconstruction (Lemma 11.9). This is used in Sections 13, 15, 23, and 31.
4. Section 11.5 proves that evaluating an effective interaction R at the classical background field of one renormalization group step contracts its weighted analytic norm by a factor $O(M^{-J})$, at the same analytic radius and the same spatial decay rate, provided that R is invariant under lattice translations, constant gauge transformations, spatial inversion, and infinitesimal gauge transformations (Theorem 11.10). This contraction allows us to choose J in Theorem 13.5.

The inputs are the results of Section 10 on the classical background field, its symmetries, the nested minimization identity for E_d, and its derivative estimates (Theorems 10.6, 10.10, and 10.11 and Lemma 10.8); the gauge slice with prescribed coarse gauge of Theorem 9.9;

the finite-torus estimate for gauge-invariant quadratic forms of Lemma 8.6; and the one-step energy estimate of Theorem 7.3 and Lemma 7.2.

The classical background fields are constrained minimizers in the sense of Bałaban's variational construction [16, Theorem 1, p. 279], which rests on his averaging operations and gauge-fixing conditions [12, 14]. The constraints, coordinate domains, and coefficient norms used here are those of Sections 9 and 10, and the results of this section are proved here.

11.1 Setting and conventions

Units and norms. Fix an odd integer $M \geq 3$. Throughout this section we measure lengths in units of the coarse lattice, which has integer side $n \geq 4$ (or $m \geq 4$ when we consider one renormalization group step) and spacing one. For $d \geq 0$ the fine lattice has side nM^d and spacing

$$\varepsilon_d = M^{-d}$$

in these units; its physical spacing on the torus $\mathbb{T}_L^3$ is $a_n \varepsilon_d$. For a configuration $U = e^{\varepsilon A}$ on a lattice of spacing ε we write

$$\mathscr{S}_\varepsilon(U) = 4\varepsilon^{-1} \sum_p q(U_p), \qquad \|A\|_{C^1_\varepsilon} = \|A\|_\infty + \|\nabla_\varepsilon A\|_\infty,$$

where ∇_ε is the collection of all componentwise forward differences divided by ε, as in Definition 10.1. We also write $\mathscr{S}_\varepsilon(A)$ for $\mathscr{S}_\varepsilon(e^{\varepsilon A})$. All spatial statements in this section concern complete periodic tori. In analytic coefficient norms, a and b denote spatial decay rates. We use ε_d, t, δ, and δ_l for dimensionless lattice spacings and a_n for the physical coarse spacing.

Tensor norms, their Taylor factorials, and their volume normalizations are those of Definition 4.6. For a derivative tensor of a scalar function, $\|\cdot\|_{a,\mathrm{root}}$ denotes the coefficient bound with each external index fixed in turn: it fixes any one index and sums the absolute values of the coefficients over the other indices, with weight $e^{a\mathsf{t}}$, where t is the length of a shortest network connecting the locations of all indices (internal vertices allowed). Let F be an analytic field of a coarse variable θ, with values in the link variables of a lattice of spacing ε, and let $v = \varepsilon^3$. The output-row norm $\|D_\theta^s F\|_{b,\mathrm{out}}$ fixes one output index and sums the absolute values of the coefficients over all coarse input indices, with weight $e^{b\mathsf{t}}$. The volume-normalized column norm $\|D_\theta^s F\|_{b,\mathrm{input};v}$ fixes one coarse input index, sums over the output index and the remaining inputs with the same weight, and multiplies the result by the output volume v.

Constants. Unless stated otherwise, a constant in this section depends only on M and on the fixed larger domains, spatial decay rates, and coefficient bounds of Sections 9 and 10, with the dependence described in item (2) of Definition 4.14. Such a constant does not

depend on the coarse side, on the depth d, on the coarse field, on L, on γ, or on J. We identify explicitly the constants that may depend on J, on a finite lattice, or on a fixed number of averaging steps.

Real and complex arguments. Variational comparisons and lower bounds on Hessians are applied only at real coarse fields and real free coordinates. At complex arguments the effective classical action and the conditional constrained minimizers denote the holomorphic continuations of the critical values and critical points from real data, as in Theorems 10.6 and 10.10.

The effective classical action and the gauge orbit of the classical background field. We recall the objects of Section 10. For a small real coarse logarithm θ, Theorem 10.6 constructs the classical background field $A_d(\theta)$ on the lattice of spacing ε_d. It is the unique minimizer of $\mathscr{S}_{\varepsilon_d}$ among the fields whose gauge-corrected block average (see below) equals e^θ, in a convex coordinate domain inside the gauge slice $\mathsf{P}_d \operatorname{div}_{\varepsilon_d} A = 0$, where P_d is the projection of Theorem 9.9. The effective classical action is its value,

$$E_d(e^\theta) = \mathscr{S}_{\varepsilon_d}(A_d(\theta)), \qquad E_0 = S_W.$$

We distinguish three objects.

1. The value $E_d(c)$, a function of the coarse field c. It is the minimum on the locally unique branch constructed in Sections 9 and 10; it is not asserted to be the infimum of $\mathscr{S}_{\varepsilon_d}$ over the whole fiber $\{U : T_M^d U = c\}$.
2. The gauge orbit of the classical background field, that is, the orbit of $e^{\varepsilon_d A_d(\theta)}$ under periodic gauge transformations of the fine lattice. By gauge invariance of the Wilson action, every point of this orbit has action $E_d(e^\theta)$.
3. Representatives at the fine and intermediate resolutions. On the fine lattice, the slice representative $e^{\varepsilon_d A_d(\theta)}$ and the representative with $T_M^d U = e^\theta$ from Lemma 10.8 belong to the orbit in item (2). For each j, the field $e^{\varepsilon_{d,j} A_{d,j}(\theta)}$, with $\varepsilon_{d,j} = M^{j-d}$, represents the gauge orbit of $T_M^j(e^{\varepsilon_d A_d(\theta)})$ on its intermediate lattice, as in Theorem 10.11. The representative $C_{d_0,J}(\theta)$ and its orbit on the fine lattice of one renormalization group step are defined in Section 11.4.

We prove convergence of the values and of specified finite reconstruction maps. We do not prove convergence of the representatives in a fixed gauge as $d \to \infty$, and we do not identify a continuum variational problem.

Block averages and gauge-corrected block averages. Two maps from fine to coarse configurations appear below. The first is the iterated block averaging map T_M^d. The second is the gauge-corrected block average of Lemma 8.10, which at each averaging step multiplies the output of T_M by explicit vertex gauge factors at the coarse vertices. Let

$V_d(U)$ denote the gauge-corrected block average of a configuration U after d steps, and let $\mathsf{G}_d(U)$ denote the accumulated gauge transformation at the coarse vertices. Then

$$V_d(U) = \big(T_M^d U\big)^{\mathsf{G}_d(U)}, \tag{11.1}$$

for configurations in the domain of Lemma 8.10. This follows from the gauge covariance of T_M (Section 6), induction over the d steps, and the composition rule $(U^k)^l = U^{lk}$ of (4.1); for the representatives of Theorem 10.11 we verify it explicitly in (11.12) below. The classical background field is defined through V_d, but the block averaging constraint in the renormalization group is $T_M^d U = c$.

The gauge slice of Theorem 9.9 connects the two. Given a small field A on the lattice of spacing ε_d and a small prescribed coarse gauge logarithm ζ, it produces a gauge transformation e^λ such that A^{e^λ} satisfies $\mathsf{P}_d \operatorname{div}_{\varepsilon_d} A^{e^\lambda} = 0$ and

$$V_d\big(e^{\varepsilon_d A^{e^\lambda}}\big) = b^{e^\zeta}, \qquad b = T_M^d\big(e^{\varepsilon_d A}\big).$$

With prescribed coarse gauge logarithm $\zeta = 0$, the gauge-corrected block average of the transformed field is exactly the block average b of the original field. Since gauge transformations preserve the Wilson action, this converts a configuration U with $T_M^d U = b$ into a field in the gauge slice with gauge-corrected block average b and the same action. We use this in the comparison of Section 11.2, the transfer of criticality in Section 11.3, and the identification in Section 11.4. We also use that the transformed field has small potential and curl (Theorem 9.9). It therefore lies in the larger domain of Theorem 9.7, on which the background field is still the unique minimum (Theorem 10.6). We do not need an estimate on all first differences of the output.

Majorant series. For power series with nonnegative coefficients we write $P \preccurlyeq Q$ if every coefficient of P is at most the corresponding coefficient of Q. For an analytic field F of the coarse variable θ, a decay rate b, and an output volume v, we define

$$\begin{aligned} \mathscr{R}_b(F;u) &= \sum_{s\ge 1} \frac{u^s}{s!} \|D_\theta^s F(0)\|_{b,\mathrm{out}}, \\ \mathscr{C}_{b,v}(F;u) &= \sum_{s\ge 1} \frac{u^{s-1}}{(s-1)!} \|D_\theta^s F(0)\|_{b,\mathrm{input};v}. \end{aligned} \tag{11.2}$$

For a group-valued F, only $F - F(0)$ enters these series, so the identity matrix does not contribute to a series vanishing at $u = 0$. All comparisons of series below are coefficientwise comparisons of absolute tensor sums, not comparisons of values. The following lemma gives the counting rule for compositions of such series.

Lemma 11.1. *Let $k \geq 1$, let T be a symmetric k-linear map between finite-dimensional spaces, let F be holomorphic near $0 \in \mathbb{C}^D$ with $F(0) = 0$, and put $G(\theta) = T[F(\theta), \dots, F(\theta)]/k!$. For $s \geq 1$ and $h_1, \dots, h_s \in \mathbb{C}^D$,*

$$\frac{1}{s!} D^s G(0)[h_1, \dots, h_s] = \frac{1}{k!} \cdot \frac{1}{s!} \sum_{(S_1, \dots, S_k)} T\big[D^{|S_1|} F(0)[h_{S_1}], \dots, D^{|S_k|} F(0)[h_{S_k}]\big], \tag{11.3}$$

where the sum runs over the ordered partitions of $\{1, \dots, s\}$ into k nonempty sets and h_S is the list of the h_i with $i \in S$. For given sizes $s_1 + \cdots + s_k = s$ there are $s! / \prod_i s_i!$ such partitions.

Proof. Since $F(0) = 0$, inserting the Taylor series of F in each argument of T, we find that the homogeneous part of degree s of G at h is

$$\frac{1}{k!} \sum_{\substack{s_1 + \cdots + s_k = s \\ s_i \geq 1}} T\Big[\frac{D^{s_1} F(0)[h^{\otimes s_1}]}{s_1!}, \dots, \frac{D^{s_k} F(0)[h^{\otimes s_k}]}{s_k!}\Big].$$

The right side of (11.3) is symmetric in $h_1, \dots, h_s$, and at $h_1 = \cdots = h_s = h$ it equals this expression, because the partitions with sizes $(s_1, \dots, s_k)$ number $s! / \prod_i s_i!$. A symmetric multilinear form is determined by its values on the diagonal, which proves (11.3). □

We use Lemma 11.1 as follows. In (11.3), the factor $1/k!$ cancels the reorderings of the k slots of T. After division by $s!$, the partitions with sizes $(s_1, \dots, s_k)$ carry exactly the weights $\prod_i 1/s_i!$ of the Taylor coefficients of the arguments. Taking absolute values before summing over labels, we bound the absolute order-s coefficients of G, divided by $s!$, by the products of the absolute order-s_i coefficients of F, divided by $s_i!$, summed over $s_1 + \cdots + s_k = s$, with the weight $1/k!$ and no further factorial. This also holds when some labels coincide. Consider an absolute weighted sum over labels in which the vertex T contributes a factor τ and each argument contributes the coefficients of a series $f(u)$ with nonnegative coefficients. Then the series of G is dominated by $\tau f(u)^k / k!$. If one label is fixed, it lies in exactly one of the k groups, and a coefficient of order j containing the fixed label has denominator $(j-1)!$ rather than $j!$. Summing over the k choices replaces one factor f by its termwise derivative f', which gives $\tau f(u)^{k-1} f'(u) / (k-1)!$. Any external index may serve as the fixed one. The spatial weights are handled separately in each application: the network connecting the external indices of a composition is contained in the union of the networks of T and of the arguments, so the weight $e^{b\mathrm{t}}$ of the composition is at most the product of the weights of the factors.

11.2 Classical actions at a fixed coarse lattice

Theorem 11.2. *Fix an odd integer $M \geq 3$. There exist $r_{\mathrm{CK}}, a_0, C > 0$, depending only on M and the fixed domains, and in particular independent of the coarse side $n \geq 4$ and of $d \geq 0$, with the following properties.*

1. *Let $0 < r \le r_{\mathrm{CK}}$ and let $c = e^{\theta}$ be a real coarse field with $\|\theta\|_\infty < r$. Then, with both actions evaluated at the same coarse field on the same coarse lattice,*

$$E_d(c) \le (1 + CrM^{-2(d+1)})E_{d+1}(c) \qquad (d \ge 0).$$

For this r, set $P_0 = 1$, $P_d = \prod_{\ell=0}^{d-1}(1 + CrM^{-2(\ell+1)})$, and $P_\infty = \lim_d P_d$. Define the uniform constant

$$\kappa_* = \Big[\prod_{\ell\ge 0}(1 + Cr_{\mathrm{CK}}M^{-2(\ell+1)})\Big]^{-1} \ge \exp\big(-Cr_{\mathrm{CK}}/(M^2-1)\big) > 0.$$

Then $P_d E_d(c)$ is nondecreasing in d, $0 \le E_d(c) \le Cn^3\|\theta\|_\infty^2$, the limit $E_\infty(c) = \lim_{d\to\infty} E_d(c)$ exists, and

$$E_d(c) \ge P_d^{-1}E_0(c) \ge \kappa_* E_0(c), \qquad E_\infty(c) \ge (P_d/P_\infty)E_d(c),$$
$$E_d(c) - E_\infty(c) \le \frac{Cr}{M^2-1}M^{-2d}E_d(c).$$

In particular these inequalities apply to $c = U^k$ whenever U is a coarse configuration and k is a fixed periodic gauge transformation with U^k in the stated domain; the identification of the values for different choices of k is Theorem 11.6.

2. *For each fixed n, the whole sequence E_d converges locally uniformly and holomorphically on the common complex coarse-field polydisc of Theorem 10.6 to a holomorphic function E_∞. For $0 < a \le a_0$ and every sufficiently small fixed $\rho > 0$ inside that polydisc,*

$$\mathcal{B}_{\rho,a}(E_d - E_\infty) \longrightarrow 0 \qquad (d \to \infty).$$

Every derivative of fixed order converges on strictly smaller neighborhoods. The radii, the coefficient bounds, and κ_* are chosen uniformly in n; the convergence modulus is asserted only at fixed n.

3. *Fix two lattices of sides $M^s m$ and m, with $1 \le s < \infty$, and fix the couplings. Consider the reconstruction maps through s averaging steps in the free logarithmic coordinates, built from E_{d+i}, $0 \le i < s$, as in Theorem 10.10. As $d \to \infty$ they converge, and so do the conditional constrained minimizers, the constrained Hessians, their positive square roots and inverse square roots, every Haar disintegration density, and the whole triangular reconstruction map. The convergence holds with every derivative of fixed order in the coarse and free variables, uniformly on every compact set of real coarse fields and real free coordinates that lies in the domains of Theorem 10.10 with a fixed strict margin, and on a complex neighborhood of each such set. On these fixed finite lattices it also holds in an absolute coefficient norm with a smaller radius. The constants and convergence moduli in this item may depend on the two lattices, on s, on the couplings, and on the margins.*

The proof uses two lemmas. The first controls one application of T_M; the second is a form of Montel's theorem with a statement about coefficient norms.

Lemma 11.3. *There exist $b_M > 0$ and $C_M < \infty$, depending only on M and on fixed margins of the identity chart of the logarithm, with the following properties. Let U be a configuration on a periodic lattice of side Mn, with $n \geq 4$, such that $\eta = \max_p \rho_G(U_p) \leq b_M$. Then every based loop holonomy entering the definition of the plaquettes of $T_M U$ lies in a fixed strict identity chart of the logarithm, and*

$$\sum_P q((T_M U)_P) \leq M(1 + C_M \eta) \sum_p q(U_p). \tag{11.4}$$

If, in addition, the spacing of this lattice is $t \leq M^{-1}$ in coarse units, $U = e^{tA}$, and $\|A\|_{C^1_t} \leq R \leq b_M$, then

$$T_M U = e^{MtB}, \qquad \|B\|_{C^1_{Mt}} \leq C_M R, \qquad \max_p \rho_G(U_p) \leq C_M t^2 (R + R^2). \tag{11.5}$$

All constants are independent of the spacing and of the side.

Proof. Let $b_M \leq \min(\delta_*(M), \delta_M)$, where $\delta_*(M)$ and δ_M are the thresholds of Theorem 7.3 and Lemma 7.2, reduced further for the fixed margins of the logarithm if needed. Theorem 7.3, applied to the averaging step from side Mn to side n with maximal plaquette distance η, implies (11.4). The part of Lemma 7.2 that assumes only a bound on plaquette distances places every based loop holonomy in a fixed strict identity chart, without a bound on the link variables themselves. Its constants depend only on M and the fixed margins, and not on t, n, or any coupling.

For the bound on the potential, consider the averaging word (4.3) with every fine link variable written as e^{ta_e}. Its identity logarithm vanishes at $t = 0$ for all small a. Division by Mt is removable, since after expanding the absolutely convergent product and logarithm series, every nonconstant monomial contains at least one factor t. We obtain a local analytic function of the finitely many a_e, with first derivative bounded uniformly for $0 \leq t \leq M^{-1}$ and value zero at $a = 0$. Its value is bounded by $C_M R$. To compare the words at two neighboring coarse vertices, we translate every input edge by Mt. By telescoping M fine differences, each input changes by at most $Mt\|\nabla_t A\|_\infty$. Integrating the derivative of the local function along the segment between the two lists of inputs gives the bound on $\nabla_{Mt} B$ in (11.5). Finally, in a fine plaquette the linear sum of the four exponents tA is bounded by $Ct^2\|\nabla_t A\|_\infty$, and the remainder of the product series is at most $Ct^2\|A\|_\infty^2$. The logarithm series preserves this bound, which gives the last assertion of (11.5). □

Lemma 11.4. *Fix $D \geq 1$, and let (f_j) be holomorphic functions on a fixed open polydisc in $\mathbb{C}^D$, uniformly bounded on every strictly smaller closed polydisc.*

1. *Every sequence of the f_j has a subsequence converging locally uniformly to a holomorphic function, together with all derivatives of fixed order.*
2. *If (f_j) converges pointwise on a real open subpolydisc, then the whole sequence converges locally uniformly on the polydisc.*
3. *Under the convergence hypothesis of item (2), let f be the locally uniform limit. Assume in addition that $Df_j(0) = 0$ for every j. If also $\|D^k f_j(0)\|_{a,\mathrm{root}} \le C^{k+1}k!$ for all $k \ge 2$ and all j, then for $C\rho < 1$ we have $\mathcal{B}_{\rho,a}(f_j - f) \to 0$.*

Here D and the finite spatial weights are fixed; only the coefficient bound C may be uniform over a family of dimensions.

Proof. Consider a polydisc of radius R on which $|f_j| \le B_R$. By the iterated one-variable Cauchy integral for the Taylor coefficient c_α, we have $|c_\alpha| \le B_R R^{-|\alpha|}$. (The Cauchy formula follows by integrating the power series on a smaller circle, where it converges uniformly, and then enlarging the circle inside the domain of holomorphy.) Fix an increasing sequence of radii approaching the outer radius. By successive selection over the countably many multi-indices, the bounded complex coefficients have a common convergent subsequence. On a smaller radius $r < R$, the tails of the series are bounded uniformly by $B_R \sum_{k>K} \binom{D+k-1}{k}(r/R)^k$, which tends to zero as $K \to \infty$. This implies that the subsequence converges locally uniformly to the corresponding power series. Multiplying the coefficients by a fixed power of k keeps these tails summable, which gives the convergence of all derivatives of fixed order. This proves (1).

For (2), two subsequential limits agree on the real open subpolydisc. Fix all but one coordinate at real values there. A holomorphic function of one variable that vanishes on a real interval vanishes on its disc: if it had a first nonzero Taylor coefficient, factoring out the corresponding power would make its zeros isolated. Repeating this for each coordinate, we find that the two limits agree on a complex subpolydisc. By continuation along line segments through overlapping Taylor discs, they agree on the whole polydisc. Suppose that the whole sequence fails to converge uniformly on some compact subset. Then a subsequence stays at a fixed positive distance from any limit there. By (1), it has a further locally convergent subsequence, whose limit is the common limit just identified. This contradiction proves (2).

For (3), derivative convergence gives $Df(0) = 0$, so $\mathcal{B}_{\rho,a}(f_j - f)$ is defined. At fixed D, convergence of the finitely many entries of order k implies convergence of every absolute sum at that order with a fixed index, and the limit inherits the assumed bound. The term of order k in $\mathcal{B}_{\rho,a}(f_j - f)$ is therefore at most $2C^{k+1}k!\rho^{k-2}/k! \le 2C^3(C\rho)^{k-2}$. Its tail is a geometric series, uniformly in j, while every finite initial sum converges. This proves (3). For derivatives at other centers on a strictly smaller neighborhood, we use the Cauchy integrals about those centers on a larger compact neighborhood; by the uniform convergence just proved, these integrals converge uniformly. □

Proof of Theorem 11.2. The proof has four steps: the comparison of consecutive depths, the monotone limit and its consequences, the analytic convergence at fixed n, and the convergence of the finite reconstruction maps.

Step 1: comparison of consecutive depths. Let θ be real and lie in a ball of coarse logarithms inside the common domains of Theorems 10.6, 10.10, and 10.11 and Lemma 10.8 and of the gauge slice of Theorem 9.9. By Lemma 10.8, the gauge orbit of the classical background field of depth $d+1$ has a real representative

$$U = e^{tA^{\mathrm{ord}}}, \qquad t = M^{-(d+1)}, \qquad T_M^{d+1}U = c, \qquad \|A^{\mathrm{ord}}\|_{C_t^1} \le C\|\theta\|_\infty,$$

with action $\mathscr{S}_t(U) = E_{d+1}(c)$. Let $b = T_M U$, a configuration on the lattice of spacing $Mt = \varepsilon_d$. Since the fine side is $nM^{d+1} = M(nM^d)$, the map $U \mapsto b$ is one averaging step with integer coarse side $nM^d \ge 4$, and $T_M^d b = T_M^{d+1}U = c$. If $C\|\theta\|_\infty \le b_M$, which holds after reducing r_{CK}, then Lemma 11.3 implies that $b = e^{\varepsilon_d B}$ with $\|B\|_{C^1_{\varepsilon_d}} \le C_M\|\theta\|_\infty$. The divergence and the curl of B are also bounded by this number. Let $r_{\mathrm{CK}} > 0$ be so small that these three bounds meet the hypotheses of Theorem 9.9 with prescribed coarse gauge logarithm zero, and that the output of the slice lies strictly inside the larger domain of Theorem 9.7 on which Theorem 10.6 applies. Each of these requirements divides a fixed positive radius by a fixed constant, so r_{CK} is independent of d and n. By the discussion in Section 11.1, the slice output has gauge-corrected block average exactly c and the same action as b. By the minimizing property of the background field in that larger domain, we have

$$E_d(c) \le \mathscr{S}_{\varepsilon_d}(b).$$

For $d = 0$ this is an equality, since $b = c$ and $E_0 = S_W$. No stationarity of b is needed.

Let $\|\theta\|_\infty < r \le r_{\mathrm{CK}}$. By the last bound of (11.5), we have $\eta = \max_p \rho_G(U_p) \le Crt^2$. Inserting this into (11.4), with the normalization of $\mathscr{S}$, we obtain

$$\mathscr{S}_{Mt}(T_M U) \le \frac{4}{Mt} M(1 + C\eta) \sum_p q(U_p) \le (1 + CrM^{-2(d+1)}) E_{d+1}(c).$$

Combining the last two displays, we obtain the comparison in item (1).

Step 2: the monotone limit. We integrate the bound $\|DE_d(e^\theta)\|_\infty \le C_*\|\theta\|_\infty$ of Theorem 10.11 along the segment $u\theta$, $0 \le u \le 1$, and use $E_d(I) = 0$ (Theorem 10.6). Since there are $3n^3$ link variables, each with a fixed number of color components, on both the real and the complex balls of coarse logarithms we have

$$|E_d(e^\theta)| \le Cn^3\|\theta\|_\infty^2. \tag{11.6}$$

Nonnegativity on real data is part of Theorem 10.6. Since $P_{d+1} = P_d(1 + CrM^{-2(d+1)})$, the comparison of Step 1 is equivalent to $P_d E_d \le P_{d+1} E_{d+1}$. Further, using $\log(1+x) \le x$ and $\sum_{\ell \ge d} M^{-2(\ell+1)} = M^{-2d}/(M^2-1)$, we have

$$1 \le P_d \le P_\infty \le \exp\Big(\frac{Cr}{M^2-1}\Big), \qquad 0 \le \log(P_\infty/P_d) \le \frac{CrM^{-2d}}{M^2-1}.$$

The sequence $P_d E_d(c)$ is nondecreasing and, by (11.6) and $P_d \le P_\infty$, bounded. It therefore has a limit, and dividing by $P_d \to P_\infty$ shows that $E_\infty(c) = \lim_d E_d(c)$ exists. Since $r \le r_{\mathrm{CK}}$, we have $P_\infty^{-1} \ge \kappa_*$. The inequality $P_d E_d \ge P_0 E_0$ implies $E_d \ge P_d^{-1} E_0 \ge P_\infty^{-1} E_0 \ge \kappa_* E_0$, and $P_\infty E_\infty = \lim_j P_j E_j \ge P_d E_d$ implies $E_\infty \ge (P_d/P_\infty) E_d$. We conclude that

$$E_d - E_\infty \le \big(1 - P_d/P_\infty\big) E_d \le \log(P_\infty/P_d)\, E_d \le \frac{Cr}{M^2-1} M^{-2d} E_d,$$

where we used $1 - e^{-u} \le u$ for $u \ge 0$. This is the one-sided error bound in item (1). We obtain no bound of the same order for $E_\infty - E_d$.

Step 3: analytic convergence at fixed n. Fix n. The coarse field has finitely many coordinates, and (11.6) bounds the holomorphic functions E_d uniformly on the common complex polydisc of Theorem 10.11. By Step 2, they converge on real points. By part (2) of Lemma 11.4, the whole sequence converges locally uniformly and holomorphically. Let $a_0 \le a_*$, and let the norm radius ρ satisfy $C_* \rho < 1/2$ and lie strictly inside that polydisc, where a_*, C_* are the constants of Theorem 10.11. Since $\|D^k E_d\|_{a_*,\mathrm{root}} \le C_*^{k+1} k!$ for $k \ge 2$, part (3) of the same lemma implies that $\mathcal{B}_{\rho,a}(E_d - E_\infty) \to 0$ for $0 < a \le a_0$. Cauchy integrals on larger compact neighborhoods give the convergence of derivatives of fixed order on strictly smaller neighborhoods. The radii, the coefficient bound, and κ_* were chosen before n was fixed; only the subsequence argument and the finite list of coefficients use the fixed value of n.

The convergence also holds throughout the common coarse-field polydisc of Theorem 10.6, which may be larger than the all-root coefficient domain just used. Indeed, on every strictly smaller polydisc that theorem gives $\|A_d\|_{C^1_{\tilde\varepsilon_d}} \le C_R$, uniformly in d and n. The complex plaquette logarithm is then $O(\varepsilon_d^2)$: the linear plaquette sum is a first difference times ε_d, and every nonlinear product has at least two factors ε_d. Its holomorphic Wilson contribution is $O(\varepsilon_d^4)$, since the action has no constant or linear term in this logarithm. There are $O(n^3 \varepsilon_d^{-3})$ plaquettes and the action prefactor is $4\varepsilon_d^{-1}$, so

$$|E_d(e^\theta)| \le C_R' n^3$$

on each such polydisc, uniformly in d. At fixed n, Lemma 11.4(1)–(2) therefore applies on this larger connected polydisc. Every subsequential holomorphic limit agrees on the smaller real open set already treated, so uniqueness identifies the limits throughout the

larger domain. Cauchy estimates give derivative convergence on its compact subsets. The stronger absolute-coefficient convergence assertion above remains at its original smaller radius.

Step 4: finite reconstruction maps. Fix the two lattices, the number s of averaging steps, and the couplings. For the i-th averaging step write

$$F_{d+i}(c,z) = E_{d+i}(b(c,z)), \qquad D_z F_{d+i}(c, z^*_{d+i}(c)) = 0,$$
$$K_{d+i}(c) = D_z^2 F_{d+i}(c, z^*_{d+i}(c)),$$

where $b(c,z)$ is the configuration of Theorem 10.10 with coarse field c and free logarithmic coordinates z. At real c, the conditional constrained minimizers $z^*_{d+i}(c)$ are real minima; at complex c, they are the holomorphic continuations solving the displayed critical equation. The map $b(c,z)$ does not depend on d, and Theorem 10.10 provides its coordinate neighborhoods and the uniform lower bound on the real Hessian. Since E_{d+i} converges on each fixed lattice by Step 3, F_{d+i} and all its derivatives of fixed order converge on every compact complex product with a strict margin. On real products the Hessian is at least $m_0 I/2$. We subtract the critical equations at depths $d+i$ and $d'+i$, and integrate the Hessian of $F_{d'+i}$ along the segment joining the two minimizers, which lies in the convex ball of free coordinates. We obtain

$$|z^*_{d+i}(c) - z^*_{d'+i}(c)|_2 \le \frac{2}{m_0} \sup_z |D_z F_{d+i}(c,z) - D_z F_{d'+i}(c,z)|_2.$$

This shows that the conditional constrained minimizers are uniformly Cauchy on compact sets of real coarse fields.

For complex coarse fields, we freeze the Hessian K at a real minimizer, so that $\|K^{-1}\|_2 \le 2/m_0$. On a sufficiently small complex neighborhood in the coarse and free variables, the uniform analytic bounds on third and mixed derivatives give $\|I - K^{-1} D_z^2 F_{d+i}\|_2 \le 1/2$. These bounds are uniform in d; their conversion between finite-dimensional norms may depend on the fixed lattice. The map $z \mapsto z - K^{-1} D_z F_{d+i}(c,z)$ is therefore a contraction there and maps a smaller closed ball into itself. Its iterates are holomorphic and converge geometrically, and they define a holomorphic critical point on a common neighborhood. For two late depths, we may use the matrix K of the first depth for both equations, since their Hessians converge. If Φ_d and $\Phi_{d'}$ denote the two contraction maps, then

$$|z_d - z_{d'}| \le |\Phi_d(z_d) - \Phi_d(z_{d'})| + |\Phi_d(z_{d'}) - \Phi_{d'}(z_{d'})| \le \frac{1}{2}|z_d - z_{d'}| + \sup |\Phi_d - \Phi_{d'}|,$$

which implies that the difference of the fixed points is at most twice the uniform difference of the maps. A finite cover of the compact set of real coarse fields gives common complex neighborhoods, and on overlaps the branches agree by uniqueness of the fixed point. This

proves locally uniform convergence of the complex minimizers, and Cauchy integrals give the convergence of each derivative of fixed order. The constrained Hessians K_{d+i} then converge with their derivatives. Here the chain rule reads

$$D_z^2 F = (D_z B)^{\mathsf{T}} D^2 E(B) D_z B + DE(B) D_z^2 B, \qquad B = \operatorname{Log} b,$$

and the second term is kept throughout.

The positive square root and its inverse are holomorphic operations on the common matrix neighborhood in use. For a real symmetric $K \geq m_0 I/2$,

$$K^{-1/2} = \frac{2}{\pi} \int_0^\infty (K + t^2 I)^{-1}\, dt, \qquad K^{1/2} = K K^{-1/2}, \tag{11.7}$$

as one checks by diagonalizing K and using $\int_0^\infty (\lambda + t^2)^{-1} dt = \pi/(2\sqrt{\lambda})$. On a complex neighborhood on which the Hermitian part of K is at least $m_0 I/4$, the resolvent norm is at most $(m_0/4 + t^2)^{-1}$. Its Neumann series and the differentiated series converge on smaller neighborhoods, and every derivative is bounded by a finite product of these integrable resolvent bounds. Integrating over $[0, T]$ and using the uniform integrable tails, we obtain the convergence in (11.7) with every derivative of fixed order.

The reconstruction through the s averaging steps has the explicit triangular form

$$z_i = z_{d+i}^*(c_i) + g_i K_{d+i}(c_i)^{-1/2} w_i, \qquad c_{i-1} = b(c_i, z_i),$$

with the indexing adjusted to the input and output sides of each step. Starting at the coarsest field, we prove by finite induction over the s steps the convergence of each z_i and c_{i-1} and of every joint derivative of fixed order in the coarse and free variables. At each step, the strict margin keeps the argument in a compact subset of the domain of the next map. The Haar disintegration density of one step is the product of the Haar factors $\operatorname{sinc}^2(|\Gamma_e|/2)$ over the non-tree edges, divided by the constraint determinant at the eliminated edges, times the constant Haar normalization ((4.2) and the formula following Definition 4.3). For complex arguments, the Haar factor is defined by its even power series in the bilinear expression $\Gamma_e \cdot \Gamma_e$, whose restriction to real arguments is the real formula; no Hermitian modulus is continued. All the arguments just shown to converge enter these fixed functions, and the fixed margin keeps the denominators uniformly away from zero. These factors and their derivatives therefore converge. The triangular factors $g_i^{3f_i} \det K_{d+i}^{-1/2}$ converge as well.

Finally, on these fixed finite lattices the finitely composed maps have a common holomorphic bound on a strictly larger product. The Cauchy estimate in the proof of Lemma 11.4 then gives a common absolute majorant on a smaller product. The spatial weight and the volume factor are finite here: the weight of a coefficient of degree k is at most $e^{aD_0 k}$, where D_0 is the diameter of the fixed finite set of indices. Let the radius of the coefficient norm be smaller than the larger radius times e^{-aD_0}. Then the tail of degree

k is a polynomial in k times a fixed geometric sequence, and the tails are uniformly small, while every finite list of coefficients converges. This proves convergence in an absolute coefficient norm with a smaller radius. All constants in this last step may depend on the fixed lattices, on s, on the couplings, and on the margins, as asserted. □

Remark 11.5. Item (3) of Theorem 11.2 asserts complex convergence on a neighborhood of each compact set of real data; this is what the proof constructs, and it is what is used in Section 15. If, on a polydisc centered at a real point of such a set, all the maps in item (3) are holomorphic and bounded uniformly in d, then part (2) of Lemma 11.4 extends the locally uniform convergence to that polydisc.

11.3 Gauge invariance on overlapping charts

The classical background field exists only in a chart around the identity, but the renormalization group needs a well-defined value of E_d on the gauge-invariant small-field domain $D_\rho = \mathcal{G}B_\rho$. The representatives in B_ρ of a configuration in D_ρ are related by periodic gauge transformations that need not have a small global logarithm.

Theorem 11.6. *There exists $\rho_{\mathrm{ov}} > 0$, depending only on M and the fixed domains, with the following properties. Let $n \geq 4$, $d \geq 0$, and $0 < \rho \leq \rho_{\mathrm{ov}}$, and let k be a real periodic gauge transformation of the coarse lattice such that c and c^k both lie in B_ρ. Then*

$$E_d(c^k) = E_d(c).$$

No smallness of a global logarithm of k is assumed. Moreover the following statements hold.

1. *At adjacent coarse vertices x, y, $\rho_G(k_x k_y^{-1}) < 2\rho$.*
2. *There is a smooth periodic map $\mathscr{K}$ from the torus (in coarse units) to $\mathrm{SU}(2)$, obtained in each unit cell by normalizing a nonnegative cardinal average of the values of k at the corners of the cell, regarded as unit quaternions, such that*

$$\mathscr{K} = k \text{ at the coarse vertices}, \qquad \|D\mathscr{K}\|_\infty + \|D^2\mathscr{K}\|_\infty \leq C\rho.$$

   The map $k \mapsto \mathscr{K}$ commutes with left and right multiplication by constant elements of $\mathrm{SU}(2)$, with translations of the coarse lattice, and with spatial inversion. Near every real k satisfying the hypotheses with a fixed strict margin, $\mathscr{K}$ has a holomorphic continuation in the values of k whose absolute Taylor coefficients of order s are bounded by $C^{s+1}s!$, together with the corresponding volume-normalized column sums; these constants depend on the margin but not on d or n. On the fine lattice of spacing ε, the gauge transformation by $\mathscr{K}$ sends a small potential A to a potential whose C^1_ε norm is at most $C(\|A\|_{C^1_\varepsilon} + \rho)$, uniformly in ε.
3. *Consequently the function $E_{d,\mathrm{sat}}(U) = E_d(U^k)$, where k is any periodic gauge transformation with $U^k \in B_\rho$, is well defined and real analytic on D_ρ, and its local holomorphic expressions*

agree on every strict real overlap. If, in addition, $\rho \le r_{\mathrm{CK}}$, then $E_{d,\mathrm{sat}} \ge \kappa_* E_0$ on D_ρ, with κ_* as in Theorem 11.2. The same identities and the same extension hold for E_∞ at each fixed coarse side n.

Remark 11.7. The following picture describes the interpolation in item (2). In a unit cell of the coarse lattice, the values of k at the eight corners are unit quaternions that lie pairwise within angle 3ρ of each other, although they may be far from 1. Their weighted average $m(x)$ lies in the convex hull of these eight points, hence at distance more than $1/2$ from the origin, and $\mathscr{K}(x)$ is its radial projection back to the unit sphere $\mathrm{SU}(2) \cong S^3$. The holomorphic continuation in item (2) concerns the dependence on the corner values; the weights are smooth, but not analytic, functions of the position.

Proof of Theorem 11.6. The proof has seven steps. Steps 1–5 construct the interpolation $\mathscr{K}$ and prove the estimates in items (1) and (2). Step 6 proves $E_d(c^k) = E_d(c)$, and Step 7 proves item (3).

Step 1: neighboring values of k. Let $c' = c^k$. On a coarse edge (x, y) we have $c'_{xy} = k_x c_{xy} k_y^{-1}$, so

$$k_x k_y^{-1} = c'_{xy}\big(k_y c_{xy}^{-1} k_y^{-1}\big), \qquad \rho_G(k_x k_y^{-1}) \le \rho_G(c'_{xy}) + \rho_G(c_{xy}) < 2\rho,$$

where we used that ρ_G is invariant under conjugation and satisfies the triangle inequality. The eight values of k at the corners of any unit cell of the coarse lattice are therefore pairwise within 6ρ. This estimate concerns only differences and does not restrict a logarithm of k.

Step 2: the interpolation. Let

$$b(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0, \end{cases} \qquad f(t) = \frac{b(t)}{b(t) + b(1-t)} \quad (0 \le t \le 1).$$

The function f is smooth, flat at both endpoints, takes values in $[0,1]$, and satisfies $f(1-t) = 1 - f(t)$. In a cell $x_0 + [0,1]^3$, we write $x = x_0 + t$ and, for $v \in \{0,1\}^3$, define

$$W_v(t) = \prod_{i=1}^{3} \big[v_i f(t_i) + (1 - v_i)(1 - f(t_i))\big], \quad m(x) = \sum_v W_v(t) k_{x_0+v}, \quad \mathscr{K}(x) = \frac{m(x)}{|m(x)|}. \tag{11.8}$$

Here elements of $\mathrm{SU}(2)$ are represented by unit quaternions in $\mathbb{R}^4$. The weights W_v are nonnegative, sum to one, and are cardinal at the corners. With our normalization of $\mathfrak{su}(2)$, the angle between two unit quaternions is half the group distance between the corresponding elements. Fix one corner value k_0. By Step 1, every corner value lies within angle 3ρ of k_0, so

$$\langle m(x), k_0 \rangle \ge \cos(3\rho) > 1/2$$

after one fixed reduction of ρ. In particular, $|m(x)| > 1/2$. On a face of the cell, the weights of the corners off that face vanish, and the formulas of the two adjacent cells reduce to the same four-corner formula. Since f is flat at the endpoints, all normal derivatives of every order match across the face. We conclude that (11.8) defines a smooth periodic map, also across the periodic seams.

Step 3: derivative bounds and symmetries. For every spatial derivative of positive order j we have $\sum_v D^j W_v = 0$, so

$$D^j m = \sum_v D^j W_v \, (k_{x_0+v} - k_0).$$

The derivatives of the weights of each fixed order are bounded independently of the cell and of the side, and $|k_{x_0+v} - k_0| \le C\rho$ by Step 1. For $j = 1, 2$, the last display is therefore bounded by $C\rho$. The normalization $m \mapsto m/|m|$ has bounded derivatives on $|m| > 1/2$, and the second derivative of $\mathscr{K}$ contains, besides $D^2 m$, only products of two first derivatives of m. This proves

$$\|D\mathscr{K}\|_\infty + \|D^2\mathscr{K}\|_\infty \le C\rho. \tag{11.9}$$

Since left and right quaternion multiplications by fixed elements of $\mathrm{SU}(2)$ are orthogonal linear maps of $\mathbb{R}^4$, they commute with the average and its normalization. Integer translations commute with the cell formula. Spatial inversion reverses each coordinate of the cell and exchanges v with $1 - v$; the identity $f(1-t) = 1 - f(t)$ then gives the asserted equivariance. We do not assert a multiplication law for the interpolation of two variable gauge transformations.

Step 4: holomorphic continuation. We regard a quaternion as a 2×2 matrix. On real inputs,

$$\mathscr{K} = m(\det m)^{-1/2}, \qquad \det m = |m|^2 > 1/4.$$

We use exponential coordinates for the eight corner values about their fixed real values. By the equivariance under left multiplication, multiplying all eight on the left by the inverse of one real corner value puts them in a common fixed neighborhood of the identity. For sufficiently small complex increments of the coordinates, $\det m$ stays in a disc around its positive real value that avoids zero. The absolutely convergent binomial series then continues the branch of the square root that is positive on real data. On a smaller ball of increments, the exponential and binomial series give an absolute majorant $C/(1 - Ct)$ for this function and for each of its first two spatial derivatives. Since there are only eight inputs, the absolute tensors of order s are bounded by $C^{s+1} s!$. The constants depend only on the fixed margins of the determinant and of the coordinates. Since an input vertex belongs to at most eight cells, each of bounded diameter, the tensors have finite range and satisfy the weighted bounds at every fixed decay rate. For a fine output lattice of

spacing ε, the column sums with a fixed input index are bounded, because ε^3 times the number of fine vertices in these cells is at most a fixed constant. The same conclusions hold for fixed spatial differences, by integrating the corresponding spatial derivatives of the cell formula. Strict margins allow a common choice of all these neighborhoods. This proves the holomorphic and coefficient assertions of item (2), uniformly in d and n.

Step 5: the effect on a fine potential. For a fine edge in direction i, with $y = x + \varepsilon e_i$, we set

$$V_i(x) = \varepsilon^{-1} \operatorname{Log}\big(\mathscr{K}(x)^{-1}\mathscr{K}(y)\big).$$

By (11.9), the ratio $\mathscr{K}(x)^{-1}\mathscr{K}(y)$ has group distance at most $C\varepsilon\rho$ from the identity, so this logarithm is taken on the identity branch. To see that the factor ε^{-1} is harmless, let $R(x,s) = \mathscr{K}(x)^{-1}\mathscr{K}(x + s\varepsilon e_i)$; then

$$V_i(x) = \int_0^1 D \operatorname{Log}_{R(x,s)}\big[\mathscr{K}(x)^{-1}\partial_i \mathscr{K}(x + s\varepsilon e_i)\big]\, ds.$$

All derivatives of this local logarithm are bounded. Differentiating in x uses only the first and second derivatives in (11.9): differentiating R contributes a factor at most $C\rho$, and differentiating the other factor contributes at most $C(\rho + \rho^2)$. Using the mean value integral in x for the lattice differences, we have $\|V\|_\infty + \|\nabla_\varepsilon V\|_\infty \le C\rho$, uniformly in ε.

Since $\mathscr{K}(y) = \mathscr{K}(x)e^{\varepsilon V_i(x)}$, the transformed link variable $\mathscr{K}(x)e^{\varepsilon A_i(x)}\mathscr{K}(y)^{-1}$ has logarithmic potential

$$\varepsilon^{-1} \operatorname{Log}[(e^{\varepsilon A})^{\mathscr{K}}_{xy}] = \operatorname{Ad}(\mathscr{K}(x))\big(A_i(x) - V_i(x) + \varepsilon \mathcal{R}_\varepsilon(A_i(x), V_i(x))\big). \tag{11.10}$$

Here the remainder is defined by the product and logarithm series for $\varepsilon^{-1}\operatorname{Log}(e^{\varepsilon A}e^{-\varepsilon V})$ after subtracting $A - V$. Since all remaining monomials contain at least one factor ε, the series and its first derivatives are bounded on a fixed small ball of (A, V), for $0 \le \varepsilon \le 1$. The real factor $\operatorname{Ad}(\mathscr{K}(x))$ is an isometry, and its first difference is bounded by $C\rho$. Taking a first difference in (11.10), by integrating the derivative of the local function between the neighboring arguments, we obtain

$$\big\|\varepsilon^{-1} \operatorname{Log}[(e^{\varepsilon A})^{\mathscr{K}}]\big\|_{C^1_\varepsilon} \le C(\|A\|_{C^1_\varepsilon} + \rho) \tag{11.11}$$

on a fixed small ball of potentials. The constants do not depend on the distance of $\mathscr{K}$ from the identity.

Step 6: identification of the minima. Let $U^{\mathrm{ord}}(c)$ be the real representative of the gauge orbit of the classical background field with $T^d_M U^{\mathrm{ord}}(c) = c$ from Lemma 10.8. Its potential has $C^1_{\varepsilon_d}$ norm at most $C\rho$, its action is $E_d(c)$, and by Theorem 10.6 every tangent vector to the fiber $\{U : T^d_M U = c\}$ at $U^{\mathrm{ord}}(c)$ has zero first variation. The restriction of $\mathscr{K}$ to the fine vertices takes the values k at the coarse vertices. The block averaging map is gauge

covariant: the vertex factors cancel between adjacent edges of each defining path, and the logarithm of each based loop holonomy is conjugated at its basepoint. Hence

$$T_M^d\big[(U^{\mathrm{ord}}(c))^{\mathscr{K}}\big] = c^k = c'.$$

A fixed real gauge transformation preserves the Wilson action and maps the fiber over c diffeomorphically onto the fiber over c'. It therefore preserves constrained criticality. By (11.11), the potential, the divergence, and the curl of $(U^{\mathrm{ord}}(c))^{\mathscr{K}}$ are bounded by $C'\rho$. Let ρ_{ov} be smaller than the fixed radii of Theorems 9.7 and 9.9 divided by C', and smaller than the thresholds used in Steps 2–5. We also impose $\rho_{\mathrm{ov}} < \frac{1}{2}\min\{r_{\mathrm{sym}}, r_\theta, r_*\}$, the fixed representative and action-chart radii of Lemma 10.8 and Theorems 10.6 and 10.11. Then the gauge slice of Theorem 9.9, with prescribed coarse gauge logarithm zero, sends $(U^{\mathrm{ord}}(c))^{\mathscr{K}}$ into the larger domain of Theorem 9.7, with gauge-corrected block average exactly c' (Section 11.1).

We verify that the resulting field is critical in that domain, rather than identifying the two constraint maps. Consider a curve $\gamma(t)$ through it in the gauge slice, with fixed gauge-corrected block average c'. By (11.1), its block average is $T_M^d\gamma(t) = (c')^{\mathsf{G}(t)^{-1}}$, where $\mathsf{G}(t)$ is the accumulated gauge transformation of the gauge-corrected average along the curve. Since all defining words have strict margins in the identity chart of the logarithm, $\mathsf{G}(t)$ is differentiable. We interpolate $\mathsf{G}(t)$ smoothly to the fine vertices with the cardinal interpolation of Lemma 10.8, and apply the resulting gauge transformation to $\gamma(t)$. The new curve has block average fixed at c' and the same action as $\gamma(t)$, and its value at $t = 0$ lies on the critical gauge orbit of $(U^{\mathrm{ord}}(c))^{\mathscr{K}}$. The constrained criticality on the fiber over c' therefore kills its derivative at $t = 0$. The additional term coming from the field-dependent gauge transformation is a gauge tangent and has zero first variation as well. Hence the curve γ has zero first variation of the action. By the existence of the interior critical point in Theorem 10.6 and the final uniqueness clause of Theorem 9.7, the resulting field is the classical background field for c'. Since all transformations used preserve the action, we conclude that $E_d(c') = E_d(c)$. For $d = 0$, this identity follows directly by conjugating the plaquette variables.

Step 7: extension to D_ρ. If $U^k, U^l \in B_\rho$, then $U^l = (U^k)^{lk^{-1}}$, and Step 6 gives the same value in both charts. This proves that $E_{d,\mathrm{sat}}$ is well defined on D_ρ. Each fixed k defines an open set on which $U \mapsto E_d(U^k)$ is real analytic. On a strict real overlap, two such local holomorphic expressions agree on a real open set, and the coordinatewise uniqueness argument in the proof of part (2) of Lemma 11.4 identifies their germs. They therefore patch to a real analytic function. The lower bound follows from Theorem 11.2:

$$E_{d,\mathrm{sat}}(U) = E_d(U^k) \geq \kappa_* E_0(U^k) = \kappa_* E_0(U),$$

since the Wilson action E_0 is gauge invariant. For a fixed coarse side, passing to the limit $d \to \infty$ in the two chart values by Theorem 11.2, we find the same identities and the same extension for E_∞. □

11.4 Gauge alignment of the successive constrained minimizers

In this section, we consider one renormalization group step. Let $J \geq 1$, $d_0 \geq 0$, and $m \geq 4$, and let $\delta = M^{-J}$. The step goes from the fine lattice of side $m/\delta = mM^J$, of spacing δ in coarse units, to the coarse lattice of side m, through the intermediate lattices of spacings $\delta_l = M^{l-J}$, $0 \leq l \leq J$. The effective classical action on the fine lattice of the step is E_{d_0}.

At each step, the renormalization group needs a representative of the gauge orbit of the classical background field that satisfies the block averaging constraint $T_M^J(\cdot) = e^\theta$ exactly. Indeed, the small-field event and the reconstruction maps are defined through T_M, not through the gauge-corrected average. Its coefficient bounds must be uniform in J, since J is chosen only after these bounds are known. The identification with the reconstruction maps of Sections 12 and 13 is needed only for fixed J.

The classical background field of the step. Let θ be a small coarse logarithm. We apply Theorem 10.11 with coarse side $n = m$, depth $d = d_0 + J$, and intermediate index $j = d_0 + l$, $0 \leq l \leq J$. The intermediate spacing is then exactly $M^{j-d} = \delta_l$, and the fine side is $mM^J = m/\delta$. Since these are complete periodic tori of side at least four, Theorem 10.11 applies in its form for arbitrary sides. We denote the resulting representatives by

$$D_l(\theta) = e^{\delta_l A_l(\theta)}, \qquad 0 \leq l \leq J,$$

so that $D_J = e^\theta$ and $D_0 = e^{\delta A_0(\theta)}$ is a configuration on the fine lattice of the step. For real θ, these are gauge-corrected block averages of the classical background field of depth $d_0 + J$; on the common complex ball, they are holomorphic continuations. We refer to the gauge orbit of $D_0(\theta)$ as the gauge orbit of the classical background field of the step. By Theorem 10.11, the pairs $(A_l, \nabla_{\delta_l} A_l)$ have one common factorial majorant, for output-row norms and for column norms with output volume δ_l^3, independent of d_0, J, l, and m.

The next lemma identifies this orbit, for each fixed J, with the configuration obtained by successive conditional constrained minimization. For a configuration V on the fine lattice of the step in the domain of Lemma 10.8 (applied with depth d_0 and coarse side m/δ), let $U_{d_0}^{\mathrm{ord}}(V)$ denote the representative of the gauge orbit of the classical background field of depth d_0 with $T_M^{d_0} U_{d_0}^{\mathrm{ord}}(V) = V$.

Lemma 11.8. *Let $J \geq 1$, $d_0 \geq 0$, $m \geq 4$, and $\delta = M^{-J}$.*

1. *For $0 \leq j \leq J$ define functions Φ_j of configurations c_j on the lattice of spacing δ_j by $\Phi_0 = E_{d_0}$ and $\Phi_{j+1}(c_{j+1}) = \Phi_j\big(b(c_{j+1}, z^*)\big)$, where $b(c, z)$ is the configuration of Theorem 10.10 for*

the averaging step from spacing δ_j to δ_{j+1} and z^* is the conditional constrained minimizer of $z \mapsto \Phi_j(b(c_{j+1}, z))$. Then $\Phi_j = M^{-j} E_{d_0+j}$, wherever the successive minimizers lie in the domains of Theorem 10.10. For real data this is an identity between minima on the branch of Theorem 10.10; it extends holomorphically.

2. *Let $\widehat{c}_0(\theta)$ be the configuration on the fine lattice of the step obtained by successive conditional constrained minimization, starting from e^θ, that is, the zero-fluctuation reconstruction before gauge transformation, and let $\widehat{U}(\theta) = U_{d_0}^{\mathrm{ord}}(\widehat{c}_0(\theta))$. For each fixed J there is $r'_J > 0$, depending on J and the fixed domains but not on d_0, m, L, or γ, such that the following holds for real $\|\theta\|_\infty < r'_J$. The configuration $\widehat{c}_0(\theta)$ is a critical point of E_{d_0} on the fiber $\{V : T_M^J V = e^\theta\}$. The configuration $\widehat{U}(\theta)$ satisfies $T_M^{d_0+J}\widehat{U} = e^\theta$, every tangent vector to the fiber $\{U : T_M^{d_0+J} U = e^\theta\}$ at $\widehat{U}(\theta)$ has zero first variation of the Wilson action, its action in coarse units equals $E_{d_0+J}(e^\theta)$, and it lies in the gauge orbit of the classical background field of depth $d_0 + J$ at e^θ. Finally, $\widehat{c}_0(\theta)$ and $D_0(\theta)$ are gauge equivalent, and $E_{d_0}(\widehat{c}_0(\theta)) = E_{d_0}(D_0(\theta))$.*

Proof. For (1), by Theorem 10.10 we have $E_{d+1}(c) = M \min_z E_d(b(c, z))$ on the branch constructed there, for every depth d. If $\Phi_j = M^{-j} E_{d_0+j}$, then

$$\Phi_{j+1}(c_{j+1}) = M^{-j} \min_z E_{d_0+j}(b(c_{j+1}, z)) = M^{-j-1} E_{d_0+j+1}(c_{j+1}),$$

which proves (1) by induction; the factor M^{-j} comes from applying Theorem 10.10 j times. The minima are taken for real coarse fields and real free coordinates, and the critical-value identities extend holomorphically, as stated in Theorem 10.10.

For (2), we first show that $\widehat{c}_0$ is critical on its fiber. The tangent space at a configuration on a lattice of one averaging step decomposes according to the coordinates of Theorem 10.10. These coordinates gauge the configuration by the transports along the root paths of the block trees, take the remaining free logarithmic coordinates, and solve the eliminated edges by the inverse branch. Differentiating them, we split a tangent vector into a tangent to the gauge action of the block trees, a derivative of $b(c, z)$ in the free coordinates z, and a derivative through the next coarser field c. Consider a tangent vector to the fiber $\{T_M^J V = e^\theta\}$ at $\widehat{c}_0$. At the first averaging step, the gauge part has zero derivative of E_{d_0} by the local gauge invariance of Lemma 10.8. The free part is killed by the critical equation of the conditional constrained minimizer. The remaining derivative is the derivative of the partially minimized value $\Phi_1 = M^{-1} E_{d_0+1}$ in the next coarser field. Repeating this at each of the J averaging steps, and using that the coarsest field e^θ is fixed, we find that the derivative vanishes. This proves the first assertion. The same induction, preceded by one more step, applies to $\widehat{U}$. A variation of $\widehat{U}$ at fixed coarse field splits into the variation of $U_{d_0}^{\mathrm{ord}}$ induced by the variation of its field on the fine lattice of the step, and a variation that keeps that field fixed. The latter has zero derivative of the action by Theorem 10.6. The former has derivative DE_{d_0} applied to the induced variation,

up to the fixed scale factor, and this vanishes by the first assertion. We conclude that every tangent to the fiber $\{T_M^{d_0+J}U = e^\theta\}$ at $\widehat{U}$ has zero first variation.

Changing from units in which the fine lattice of the step has spacing one to coarse units multiplies potentials by δ^{-1} and first differences by δ^{-2}, which are fixed factors for fixed J. The potential of $\widehat{U}(\theta)$ therefore has C^1 norm at most $C_J\|\theta\|_\infty$ in coarse units. By (1), its action in coarse units is M^J times $\Phi_J = M^{-J}E_{d_0+J}$, that is, $E_{d_0+J}(e^\theta)$. Let r'_J be so small that the gauge slice of Theorem 9.9, with prescribed coarse gauge logarithm zero, places $\widehat{U}(\theta)$ in the larger domain of Theorem 9.7. The argument of Step 6 in the proof of Theorem 11.6 transfers the constrained criticality to that domain. The uniqueness in Theorem 10.6 then identifies the gauge orbit of $\widehat{U}(\theta)$ with that of the classical background field of depth $d_0 + J$.

Finally, we write $\widehat{U} = (e^{\varepsilon A_{d_0+J}})^g$ for a fine gauge transformation g, where $\varepsilon = M^{-d_0-J}$. By gauge covariance of T_M, $\widehat{c}_0 = T_M^{d_0}\widehat{U}$ is the gauge transform of $T_M^{d_0}e^{\varepsilon A_{d_0+J}}$ by the restriction of g, and by (11.1) the latter is gauge equivalent to D_0. Thus $\widehat{c}_0(\theta)$ and $D_0(\theta)$ are gauge equivalent, and after reducing r'_J both lie in B_ρ with $\rho \le \rho_{\mathrm{ov}}$. Theorem 11.6 then implies that $E_{d_0}(\widehat{c}_0) = E_{d_0}(D_0)$. □

We write $\mathscr{C}_{\mathrm{raw}\,d_0,J}(\theta, 0) = \widehat{c}_0(\theta)$ for the zero-fluctuation reconstruction of the step before gauge transformation. It is built from the successive conditional constrained minimizers of Theorem 10.10 and their normalization by the positive inverse square roots of the constrained Hessians, with every normalized fluctuation variable equal to zero.

Lemma 11.9. *Let $J \ge 1$, $d_0 \ge 0$, and $m \ge 4$, and put $\delta = M^{-J}$.*

1. *On a common complex ball $\|\theta\|_\infty < r_{\mathrm{cen}}$ there is an analytic representative*

$$C_{d_0,J}(\theta) = e^{\delta\overline{A}_{d_0,J}(\theta)}, \qquad T_M^J C_{d_0,J}(\theta) = e^\theta, \qquad \overline{A}_{d_0,J}(0) = 0,$$

   of the gauge orbit of the classical background field of the step.
2. *There exist $r_{\mathrm{cen}}, a_{\mathrm{cen}}, C_{\mathrm{cen}} > 0$, depending only on M and the fixed domains and independent of J, d_0, and m, such that for every integer $s \ge 1$ and every $\|\theta\|_\infty < r_{\mathrm{cen}}$,*

$$\|D_\theta^s(\overline{A}_{d_0,J}, \nabla_\delta\overline{A}_{d_0,J})\|_{a_{\mathrm{cen}},\mathrm{out}} + \|D_\theta^s(\overline{A}_{d_0,J}, \nabla_\delta\overline{A}_{d_0,J})\|_{a_{\mathrm{cen}},\mathrm{input};\delta^3} \le C_{\mathrm{cen}}^{s+1}s!.$$

   In particular $\|\overline{A}_{d_0,J}(\theta)\|_{C^1_\delta} \le C_{\mathrm{cen}}\|\theta\|_\infty$.
3. *The representative is $C_{d_0,J} = D_0^L$, where L is the smooth interpolation, by the cardinal weights of Lemma 10.8, of the accumulated gauge transformation G at the coarse vertices that relates the gauge-corrected and the ordinary block averages of D_0; at the coarse vertices $L = G$ (not G^{-1}), the multiplication order being forced by $(U^k)^l = U^{lk}$. The real gauge path connecting D_0 to $C_{d_0,J}$ has logarithms of the fine link variables bounded by $C\delta\|\theta\|_\infty$.*

4. *After one fixed reduction of r_{cen} with $Cr_{\mathrm{cen}} < 1/2$,*

$$\mathscr{R}_{a_{\mathrm{cen}}}\big((\overline{A}_{d_0,J}, \nabla_\delta \overline{A}_{d_0,J}); u\big) \preccurlyeq Z(u) = \frac{Cu}{1-Cu}, \qquad \mathscr{C}_{a_{\mathrm{cen}},\delta^3}\big((\overline{A}_{d_0,J}, \nabla_\delta \overline{A}_{d_0,J}); u\big) \preccurlyeq Z'(u),$$

on $0 \le u < r_{\mathrm{cen}}$, with C and r_{cen} independent of J, d_0, and m. The accumulated gauge transformation G, its inverse, and its interpolation L have bounds of the same type, uniform in J. This item gives no bound uniform in J for the full reconstruction in coarse and free coordinates or for the gauge transformation F_J of item (6), and no column bound for arbitrary fine inputs to these gauge transformations.

5. *For each fixed J, the representative $C_{d_0,J}$, the gauge transformation L and its inverse, and the gauge transformation F_J of item (6) have weighted absolute coefficient bounds of factorial type, with every index fixed in turn, on a fixed larger domain of radius $r_J > 0$, with constants C_J. A positive spatial decay rate for these bounds can be chosen before J; r_J and C_J may depend on J and on the chosen analytic domains, but not on d_0, m, L, or γ. These constants are not used to choose J.*

6. *For the same fixed J, after reducing r_J only, let $F_J(\theta)$ be the gauge transformation obtained from $C_{d_0,J}(\theta)$ by imposing, in the order $j = J-1, J-2, \dots, 0$, the block axial gauge of the level-j blocks of the field after the preceding gauge transformations, each factor being extended constantly through the finer block partitions. With $(U^k)^l = U^{lk}$ its ordered product is $F_J = K_0 K_1 \cdots K_{J-1}$, and*

$$F_J = I \text{ at the coarse vertices}, \qquad F_J(0) = I, \qquad C_{d_0,J}(\theta)^{F_J(\theta)} = \mathscr{C}_{\mathrm{raw}\, d_0,J}(\theta, 0).$$

This identity holds on the complex ball of radius r_J, with strict margins in all intermediate charts. The gauge transformations F_J and F_J^{-1} depend only on θ and satisfy the bounds of item (5), uniformly in d_0, m, L, and γ; no bound on F_J uniform in J is asserted.

The bounds of item (2) hold at every order s; the reconstruction maps of Section 15 use only finitely many orders of such uniform bounds. Table 11.1 lists the order in which the constants of this section enter the choice of parameters.

Proof of Lemma 11.9. Steps 1–6 construct $C_{d_0,J}$ and prove items (1)–(4) with constants independent of J. Step 7 constructs F_J and proves item (5), and Step 8 proves the identity in item (6).

Step 1: conventions. We use the representatives D_l described before Lemma 11.8, with their common factorial majorant from Theorem 10.11. Fix $0 < a_{\mathrm{cen}} < a_*/32$. The larger multiples of this rate used below give spatial decay rates strictly larger than the working rate, which we use to sum columns. This choice does not depend on J, d_0, or m.

Constants	Dependence and role
a_*, C_*, r_*, m_0; r_{CK}, ρ_{ov}	Fixed geometry, decay rates, and classical coefficient bounds (Theorems 10.10, 10.11, 11.2, and 11.6); independent of J.
$a_{\mathrm{cen}}, r_{\mathrm{cen}}, C_{\mathrm{cen}}$ and Z	Items (2)–(4) of Lemma 11.9; independent of J, d_0, and m.
a, $\rho_0(a), \delta_0(a), C_P(a)$	Theorem 11.10; for the application to $C_{d_0,J}$, $0 < a < a_{\mathrm{cen}}/8$, $2\rho_0 < \min(r_{\mathrm{cen}}, r_*)$, and δ_0 chosen so that the image condition holds for both D_0 and $C_{d_0,J}$; independent of J, d_0, m, and of the later choice of $\rho \le \rho_0$.
J	Chosen after the preceding constants, for instance so that $C_P(a)M^{-J} < 1/4$ (Theorem 13.5; see also Theorem 23.2).
r_J, C_J	Items (5)–(6) of Lemma 11.9; depend on J but not on d_0, m, L, or γ; never used to choose J.
Later radii and coupling bounds	Chosen in Parts IV and VI, after J.

Table 11.1. Order of the constants in Section 11.

Step 2: the accumulated gauge transformation. We bound the gauge transformation relating the ordinary and the gauge-corrected block averages, rather than replacing the block averaging constraint by the gauge-corrected one. At level l, the vertex factor of the gauge-corrected average at a block center r is

$$H_l(r) = \exp\Big\{-M^{-3} \sum_{x \in \mathsf{B}_M(r)} \operatorname{Log} D_l(\sigma_{r,x})\Big\},$$

the identity-background factor of Lemma 8.10; by its definition, $D_{l+1} = (T_M D_l)^{H_l}$. By gauge covariance of T_M, the accumulated gauge transformations G_l satisfy

$$G_0 = I, \qquad G_{l+1} = H_l\,(G_l|_{\text{next coarser vertices}}), \qquad D_l = (T_M^l D_0)^{G_l};$$

the last identity follows by induction from $(U^k)^l = U^{lk}$. In particular, with $G = G_J$,

$$e^{\theta} = (T_M^J D_0)^G, \qquad T_M^J D_0 = (e^{\theta})^{G^{-1}}. \tag{11.12}$$

We now bound the coefficients of $\kappa = \operatorname{Log} G$ by a series that does not depend on J. Each path in H_l has at most C_M edges with variables $e^{\delta_l A_{l,e}}$. Expanding the product, the logarithm, and the exponential, we obtain the coefficientwise bound

$$H_l - I \preccurlyeq \delta_l \beta(Z(t)), \qquad \beta(z) \le \frac{Cz}{1 - Cz}, \quad \beta(0) = 0, \tag{11.13}$$

where $Z(t) = Ct/(1 - Ct)$ is the majorant of the pairs $(A_l, \nabla_{\delta_l} A_l)$ and $\preccurlyeq$ refers to absolute output-row sums at a specified smaller decay rate. The factor δ_l appears because every

nonconstant monomial of the local word contains at least one exponent $\delta_l A_l$, and $\delta_l^k \le \delta_l$. The finitely many path variables contribute a fixed factor per degree. The paths lie in a collar of radius $C_M \delta_l$, and joining it to the output vertex contributes another fixed factor per degree, absorbed in C. The absolute exponential, logarithm, and product series converge for one small fixed t. This proves (11.13) as an estimate on coefficients.

Restriction to coarser vertices cannot increase an output row. Using $\sum_{l=0}^{J-1} \delta_l < 1/(M-1)$, we obtain

$$G - I \preccurlyeq \prod_{l=0}^{J-1} (1 + \delta_l \beta(Z)) - 1 \preccurlyeq \exp\Big(\frac{\beta(Z)}{M-1}\Big) - 1,$$

and we denote the right side by $Q(t)$. Fix a radius of coarse logarithms such that $Q(t) < 1/2$. The logarithm series then bounds the absolute row series of κ by $\sum_{k\ge 1} Q(t)^k/k$. This series converges, vanishes at zero, and is dominated coefficientwise by $Ct/(1-Ct)$ after enlarging C. Indeed, the coefficient of degree s of a series with nonnegative coefficients that converges at $t_0 > 0$ is at most its value at t_0 times t_0^{-s}. The radius and the constants are independent of J, d_0, and m. The inverse needs no implicit differentiation: since $\det G = 1$, for 2×2 matrices

$$G^{-1} = \operatorname{adj} G, \qquad \operatorname{adj}(I + X) - I = (\operatorname{tr} X) I - X. \tag{11.14}$$

This is linear in X, with absolute rows at most a fixed multiple of those of its argument, and on the common branch $\operatorname{Log} G^{-1} = -\kappa$. Enlarging the constant of the rational majorant, we bound κ, $G - I$, and $G^{-1} - I$ simultaneously, at every order, before J is chosen.

Step 3: from rows to columns. We work with row bounds at twice the decay rate used in the conclusion. For an output vertex x and a fixed coarse input u, the length of a shortest connecting network is at least $\operatorname{dist}(x, u)$. We remove the sum over that input from an absolute output row and use half of the rate. Then the remaining weighted sum of order s is at most $C^{s+1} s! e^{-a \operatorname{dist}(x,u)}$. For the meshes that occur here,

$$\sup_u \sup_{\substack{\varepsilon = M^{-j} \\ j \in \mathbb{Z}_{\ge 0}}} \varepsilon^3 \sum_{x \in \varepsilon \mathbb{Z}^3/(m\mathbb{Z})^3} e^{-a \operatorname{dist}(x,u)} \le C_a, \tag{11.15}$$

where m is an integer, $m/\varepsilon = mM^j \in \mathbb{N}$, and u is a vertex of the fine lattice; this covers the coarse spacing 1 and every intermediate spacing δ_l. To prove (11.15), we take the shortest periodic representatives and bound the torus sum by the sum over the infinite lattice. By equivalence of the fixed lattice metrics, this reduces to a product of three one-dimensional sums. For $b > 0$, by monotonicity of e^{-bt} we have $\varepsilon \sum_{k\in\mathbb{Z}} e^{-b\varepsilon|k|} \le 1 + 2\int_0^\infty e^{-bt} dt = 1 + 2/b$, uniformly in the side. Hence κ has bounded counting columns with coarse outputs, and the same argument gives volume-normalized columns for outputs on a lattice of spacing ε. The input in this argument is the coarse logarithm θ, through the representatives D_l.

Step 4: the interpolated gauge transformation and the exact constraint. With the smooth periodic cardinal weights χ_r of Lemma 10.8, we define on the fine vertices

$$\lambda_x(\theta) = \sum_r \chi_r(x)\kappa_r(\theta), \qquad L_x(\theta) = e^{\lambda_x(\theta)}, \qquad C_{d_0,J}(\theta) = D_0(\theta)^{L(\theta)}. \tag{11.16}$$

Since $L = G$ at the coarse vertices, gauge covariance of T_M^J and (11.12) give $T_M^J C_{d_0,J} = ((e^\theta)^{G^{-1}})^G = e^\theta$ exactly. This is why we interpolate G, and not G^{-1}. The interpolation need not be the identity at the coarse vertices, since it corrects the block average; the gauge transformation F_J constructed in Step 7 will be the identity there.

The weights and their first two spatial derivatives have bounded absolute rows and volume-normalized columns, with a fixed collar of support. Integrating these derivatives, we obtain the same bounds for the first and second divided differences of their restrictions to the lattice. By Steps 2 and 3, λ, $\nabla_\delta\lambda$, and all second divided differences of λ have factorial rows and columns in θ, uniformly in J. If $Y(u)$ is their common row majorant, the exponential series bounds $L - I$ by $e^{Y(u)} - 1$. Since $\det L = 1$, (11.14) applies to L and bounds $L^{-1} - I$. Differentiating the exponential series spatially once or twice inserts a first difference of λ, or a second difference and products of first differences, which have convergent majorant series of the same type. Row bounds at a spatial decay rate strictly larger than the working rate, together with (11.15), give their columns with one fixed coarse input and output volume δ^3. For G, the output measure is counting measure on the coarse lattice. This proves the bounds of item (4) for G, its inverse, and its interpolation, all with the coarse logarithm θ as input.

Step 5: removal of the inverse mesh. We remove the inverse mesh in the gauge word before taking spatial or θ derivatives. For $u = \lambda_x$, $v = (\lambda_{x+\delta e_i} - \lambda_x)/\delta$, and $a = A_{0,i}(x)$, we define

$$F_\delta(u, v, a) = \delta^{-1} \operatorname{Log}(e^u e^{\delta a} e^{-u-\delta v}). \tag{11.17}$$

The product equals I at $\delta = 0$, for all small u, v, a. The absolutely convergent exponential and logarithm series are jointly analytic in (δ, u, v, a) on a small ball, also for $|\delta| \le 2$, and the zero at $\delta = 0$ makes the division removable. After division, the absolute series have a common majorant for $0 \le \delta \le 1$. Differentiating at $\delta = 0$, we find

$$F_0(u, v, a) = \operatorname{Ad}(e^u)a - \int_0^1 \operatorname{Ad}(e^{su})v\, ds, \tag{11.18}$$

as follows from $De^X[Y] = \int_0^1 e^{(1-s)X} Y e^{sX} ds$. Thus $F_\delta = F_0 + \delta R_\delta$ with a uniformly convergent absolute analytic remainder. We define

$$\overline{A}_{d_0,J,i}(x) = F_\delta(\lambda_x, \nabla_{\delta,i}\lambda_x, A_{0,i}(x)).$$

Since $L_x e^{\delta A_{0,i}(x)} L^{-1}_{x+\delta e_i} = e^{\lambda_x} e^{\delta A_{0,i}(x)} e^{-\lambda_x - \delta\nabla_{\delta,i}\lambda_x}$, this is the logarithmic potential of $C_{d_0,J}$ in (11.16). Its first divided difference is the integral of the derivative of F_δ along the segment

between the neighboring lists of arguments, whose differences are $\nabla_\delta\lambda$, $\nabla_\delta^2\lambda$, and $\nabla_\delta A_0$. All of these have the majorants established above. Composing majorant series with the counting of Lemma 11.1, we obtain factorial rows for $(\overline{A}, \nabla_\delta\overline{A})$ at one fixed smaller decay rate. Here the union of the local collars and of the input networks connects every external index. Applying (11.15) to rows at a strictly larger rate gives the columns with output volume δ^3. All decay rates are chosen before J, and no rate is reduced from one averaging step to the next.

Step 6: one majorant for all bounds. We work first at twice the final rate a_{cen}. Let $Y(u) = C_0u/(1 - C_0u)$ dominate the absolute rows of $(A_0, \nabla_\delta A_0, \lambda, \nabla_\delta\lambda, \nabla_\delta^2\lambda)$, where C_0 is increased to account for the finitely many types of components. For a suitable fixed c, the series of F_δ and of its first derivative in its local arguments are bounded by $cz/(1 - cz)$ and $c/(1 - cz)^2$. Using the integral representation of the spatial difference, we obtain

$$\mathscr{R}_{2a_{\mathrm{cen}}}((\overline{A}, \nabla_\delta\overline{A}); u) \preccurlyeq P(u), \qquad P(u) = \frac{cY(u)}{1 - cY(u)} + \frac{cY(u)}{(1 - cY(u))^2}, \tag{11.19}$$

by multiplication of absolute Taylor series and Lemma 11.1. In the second term, the last factor Y accounts for the spatial difference of a local argument. The majorant is independent of δ and of J. At order s, since the rows are bounded at a larger rate than the columns, (11.15) bounds the column by $C_{a_{\mathrm{cen}}}$ times the row; in the notation (11.2),

$$\mathscr{C}_{a_{\mathrm{cen}},\delta^3}((\overline{A}, \nabla_\delta\overline{A}); u) \preccurlyeq C_{a_{\mathrm{cen}}}P'(u).$$

The same conversion applies to the series for G, G^{-1}, and L. We add all these row majorants, together with P and the fixed constants of the column conversions, into one series $S(u) = \sum_{s\geq 1} S_s u^s$ with nonnegative coefficients, convergent at some $u_* > 0$ independent of J, d_0, and m. Then $S_s \leq S(u_*)u_*^{-s}$. Choosing $C \geq \max\{1, u_*^{-1}\max\{1, S(u_*)\}\}$, we have $S_s \leq C^s$ for every $s \geq 1$, and hence

$$\begin{aligned} \mathscr{R}_{a_{\mathrm{cen}}}((\overline{A}, \nabla_\delta\overline{A}); u) &\preccurlyeq Z(u) = \frac{Cu}{1 - Cu}, \\ \mathscr{C}_{a_{\mathrm{cen}},\delta^3}((\overline{A}, \nabla_\delta\overline{A}); u) &\preccurlyeq Z'(u). \end{aligned} \tag{11.20}$$

The same Z and Z' dominate the nonconstant parts of the series of G, G^{-1}, and L, with output volume 1 for coarse outputs and δ^3 for fine outputs. The final Z may be larger than the majorant used in (11.13); all earlier estimates remain valid on the smaller common ball.

With this rate, let $r_{\mathrm{cen}} > 0$ be such that the series just used converge on a ball strictly larger than $\|\theta\|_\infty < r_{\mathrm{cen}}$ and $Cr_{\mathrm{cen}} < 1/2$. These constants are fixed before J. The bounds also hold at every point of the ball, not only at its center. Indeed, if a tensor of order k

at zero is bounded by $C^{k+1}k!$, then translating the argument by θ bounds the tensor of order s at θ by

$$\sum_{l\geq 0} \frac{\|\theta\|_\infty^l}{l!} C^{s+l+1}(s+l)! = \frac{C^{s+1}s!}{(1-C\|\theta\|_\infty)^{s+1}}.$$

Since contracting l inputs only shortens the external network, the same inequality holds for each coefficient bound with a fixed external index and for each volume-normalized column. A fixed reduction of the radius bounds the right side by $C_{\mathrm{cen}}^{s+1}s!$ for every $s \geq 1$, with no loss in the radius or the decay rate that depends on s. Enlarging C_{cen}, we also bound the first derivative itself. Since the pair $(\overline{A}, \nabla_\delta \overline{A})$ and the nonconstant parts of the gauge transformations vanish at $\theta = 0$, integrating the first derivative, we find $\|\overline{A}\|_{C^1_\delta} \leq C_{\mathrm{cen}}\|\theta\|_\infty$. Replacing λ by $t\lambda$ in (11.17), uniformly for $0 \leq t \leq 1$, we find that the real gauge path from D_0 to $C_{d_0,J}$ has fine logarithms at most $C\delta\|\theta\|_\infty$. This completes the proof of items (1)–(4); item (1) refers to this gauge orbit because $C_{d_0,J}$ is a gauge transform of D_0.

Step 7: the block axial gauge transformation for fixed J. For a configuration V on the fine lattice of one averaging step, we define on each block with center r

$$f(V)_x = V(\sigma_{r,x}), \qquad f(V)_r = I.$$

For each edge (x, y) of the block tree, oriented away from the center, the root path gives $f(V)_y = f(V)_x V_{xy}$, and hence $(V^{f(V)})_{xy} = I$. Starting from $V = C_{d_0,J}(\theta)$, we perform the following finite procedure for $j = J-1, J-2, \ldots, 0$. We compute $V_j = T_M^j V$ for the field V after the preceding gauge transformations, form $f(V_j)$, extend this gauge transformation constantly through the j finer block partitions to the fine vertices, and apply the resulting gauge transformation K_j to V. The factor K_j is the identity at the vertices of level $j+1$. By gauge covariance, it therefore leaves unchanged every coarser iterated block average and each block axial gauge condition already imposed there. The resulting gauge transformation is

$$F_J = K_0 K_1 \cdots K_{J-1}, \qquad F_J = I \text{ at the coarse vertices.} \tag{11.21}$$

Each factor is evaluated on the field transformed by the preceding factors, and the order of the product is forced by $(U^k)^l = U^{lk}$.

For fixed J, these operations are finite products and the local logarithms of the block averaging map. At $\theta = 0$, every intermediate field and every factor is the identity. Finite-word absolute series and the uniform series of $C_{d_0,J}$ therefore give a radius $r_J > 0$ with strict margins in all intermediate charts. The lengths of the words depend on M and J, not on d_0 or m, and all factorial row constants may depend on J. The collars are bounded by a fixed multiple of $\sum_{j<J} M^{j-J} < 1/(M-1)$; extension through finer block partitions enlarges them only by the same geometric sum. The union of networks preserves a

chosen positive decay rate at every composition, and the number of variables per coarse unit cell is finite for fixed J. Row bounds at a spatial decay rate strictly larger than the working rate and (11.15) give the volume-normalized columns. Since inversion of a matrix of determinant one is a fixed polynomial in its entries, the same statements hold for F_J^{-1}, for L, and for L^{-1}. This proves item (5).

Step 8: identification with the zero-fluctuation reconstruction. The stationarity and convexity arguments below use real coarse fields and real free coordinates; the final identity is then continued holomorphically. Consider the free logarithmic coordinates $z = (z_0, \dots, z_{J-1})$ of the J averaging steps in the nested coordinates of Theorem 10.10. Consider also the function of z obtained by composing E_{d_0} with the reconstruction of the fine field of the step from θ and z. By part (1) of Lemma 11.8, partial minimization over the first j steps gives $M^{-j}E_{d_0+j}$. At $\theta = 0$, let δz_j be a free tangent vector of step j, let δc_{j+1} be the induced tangent vector of the next coarser field, and set

$$\eta_j = \delta z_j - Dz^*_{d_0+j}(I)[\delta c_{j+1}].$$

We complete the square at each step. The critical equation of the conditional minimizer removes the mixed term, and differentiating the identity of part (1) gives the remaining term. We obtain the Hessian of this function at $\theta = 0$, $z = 0$:

$$\sum_{j=0}^{J-1} M^{-j}\langle \eta_j, K^0_{d_0+j}\eta_j\rangle, \qquad \delta c_J = 0, \tag{11.22}$$

where $K^0_{d_0+j}$ are the constrained Hessians of Theorem 10.10 at the identity. For a quadratic form $\langle z, Kz\rangle/2 + \langle z, Bc\rangle + \langle c, Dc\rangle/2$, the derivative of the minimizer is $-K^{-1}B$. Substituting $z = \eta - K^{-1}Bc$ leaves $\langle \eta, K\eta\rangle/2 + \langle c, (D - B^{\mathsf{T}}K^{-1}B)c\rangle/2$, where the last matrix, the Schur complement, is the Hessian of the partially minimized value. Iterating gives (11.22). In particular, the blocks are $M^{-j}K^0_{d_0+j}$, not $K^0_{d_0+j}$.

The map from (δz_j) to (η_j) is triangular with identity diagonal. Its inverse is given by backward substitution and involves only finitely many compositions of the mixed derivatives of the minimizers and of the geometric one-step maps. By their row and column bounds in Theorems 10.10 and 10.11, the rows and columns of the inverse are bounded by C_J. Its ℓ^2 norm is then at most C_J, since by the Cauchy–Schwarz inequality

$$\|Tv\|_2^2 \le \Big(\sup_x \sum_y |T_{xy}|\Big)\Big(\sup_y \sum_x |T_{xy}|\Big)\|v\|_2^2.$$

Since $K^0_{d_0+j} \ge m_0 I$, the Hessian (11.22) is bounded below by $m_0 M^{-(J-1)}C_J^{-2}I$. By the bounds on third derivatives for fixed J, with each external index fixed in turn, the Hessian at (θ, z) differs from this one by at most $C_J(\|\theta\|_\infty + \max_j \|z_j\|_\infty)$ in row and

column norm. We reduce the radius for this fixed J so that this is less than half the lower bound. Then the function is strictly convex on a smaller real convex ball of coordinates, uniformly in d_0 and m.

The successive conditional constrained minimizers give a critical point of this function. Indeed, at the first minimizer the derivative in its free variables vanishes. The derivative in the remaining variables is the derivative of $M^{-1}E_{d_0+1}$, whose derivative in the next free variables vanishes at its minimizer, and so on through the J steps. This critical point corresponds to $\widehat{c}_0(\theta) = \mathscr{C}_{\mathrm{raw}\, d_0,J}(\theta, 0)$. On the other hand, by part (2) of Lemma 11.8, $C_{d_0,J}(\theta)$ is a gauge transform of $\widehat{c}_0(\theta)$, by a gauge transformation whose restriction to the coarse vertices fixes e^θ. Since $\widehat{c}_0(\theta)$ is critical for E_{d_0} on the fiber $\{T_M^J V = e^\theta\}$ and E_{d_0} is gauge invariant on D_ρ (Theorem 11.6), so is $C_{d_0,J}(\theta)$. The gauge transformation $F_J(\theta)$ is the identity at the coarse vertices and preserves this criticality. The field $C_{d_0,J}(\theta)^{F_J(\theta)}$ satisfies the block axial gauge condition at every level. It is therefore the reconstruction of the step from θ and some free coordinates z, and it is a critical point of the function above. It lies in the ball of strict convexity when r_J is small, because it equals the identity at $\theta = 0$ and its coordinates have rows bounded for fixed J. By strict convexity, it is the unique critical point, so

$$C_{d_0,J}(\theta)^{F_J(\theta)} = \widehat{c}_0(\theta) = \mathscr{C}_{\mathrm{raw}\, d_0,J}(\theta, 0)$$

for real θ. The identity extends to the complex ball by the coordinatewise uniqueness argument of Lemma 11.4. Thus F_J^{-1} is the gauge transformation, depending only on θ and equal to the identity at the coarse vertices, that takes the zero-fluctuation reconstruction to the representative $C_{d_0,J}$. The gauge transformations F_J, F_J^{-1} have the bounds of item (5). We shrink r_J inside a larger domain on which these bounds hold, so that they remain available later. Their constants depend only on M, J, and the chosen domains and rates, never on d_0, m, L, or γ. The constants r_{cen}, a_{cen}, and C_{cen} were fixed without them, so the constants of item (5) are not used to choose J. □

11.5 Contraction of the interaction under the classical background field

In the renormalization group step of Section 13, the effective interaction R of the fine lattice is evaluated at the classical background field of the step. We show that this evaluation contracts the weighted analytic norm by a factor $O(\delta)$, $\delta = M^{-J}$, at the same radius and the same decay rate. A term of degree k in the link logarithms ξ of the fine lattice, evaluated at $\xi = \delta A$ with A of order one, gains δ^k. The sum over the position of a fixed fine index, measured against the coarse lattice, contributes a factor δ^{-3}. Terms of degree at least four therefore gain $\delta^{k-3} \le \delta$. The quadratic and cubic terms need additional cancellations. Gauge invariance and spatial inversion make the quadratic term a quadratic form in first differences of ξ, up to an exponentially small finite-period

remainder. Each first difference of δA gains an extra factor δ. Spatial inversion and translation invariance make the first derivative of the cubic term vanish at fields that are constant in each orientation, which yields one first difference. This use of gauge invariance is standard in renormalization group analyses of gauge theories; we prove the absolute kernel factorization, the cubic cancellation, and the composition estimate at unchanged radius below.

We use the following notation. On the fine lattice of side $q = m/\delta$, in units in which its spacing is one, ∇ denotes the collection of all componentwise first differences of link fields, $(\nabla v)_{\ell,i}(x) = v_i(x + e_\ell) - v_i(x)$ for each direction ℓ and each orientation and color component i. Its adjoint is ∇^*. We write grad for the lattice gradient from vertex fields to link fields, $(\operatorname{grad}\lambda)_{xy} = \lambda_y - \lambda_x$, and I for spatial inversion acting on link fields, $(\mathsf{I}v)_i(x) = -v_i(-x - e_i)$, as in Lemma 8.6. For a link logarithm ξ and a vertex field λ, let $\mathcal{D}_\xi\lambda$ be the infinitesimal gauge variation, that is, the derivative of $t \mapsto \operatorname{Log}\big((e^{\xi})^{e^{t\lambda}}\big)$ at $t = 0$; then $\mathcal{D}_0\lambda = -\operatorname{grad}\lambda$. In this section the letter q denotes the side of the fine lattice; the plaquette energy $q(U)$ does not occur. For a translation-invariant kernel K, $\|K\|_{b,\mathrm{row+col}}$ denotes the sum of its weighted absolute row and column norms at rate b, using the periodic Euclidean distance between initial vertices, consistently with Definition 4.6. The sum convention agrees with Lemma 8.6; that lemma uses the periodic ℓ^1 distance instead, so its rate must be converted when we apply it. These kernels act on a single lattice, so their columns carry no volume factor. For the same distance, the sum lies between the maximum of the row and column norms, used in Sections 4, 14, and 31, and twice that maximum; in particular, the upper bounds on the sum proved below also hold for the maximum.

Theorem 11.10. *Let a_* be the decay rate of Theorem 10.11 and fix $0 < a < a_*/8$. There exist $\rho_0(a), \delta_0(a), C_P(a) > 0$, depending on a and on the fixed classical data (including M), and independent of the later choice of $0 < \rho \le \rho_0$, of J, of d_0, and of the side of the lattice, with the following property. Let $\delta = M^{-J} \le \delta_0$, let $m \ge 4$ be the side of the coarse lattice, so that the fine lattice of the renormalization group step has side m/δ. Let R be a scalar function, holomorphic in the logarithms ξ of the fine link variables on $\|\xi\|_\infty < \rho$, such that*

$$DR(I) = 0, \qquad \mathcal{B}_{\rho,a}(R) \le \epsilon,$$

and such that R is invariant under translations of the fine lattice, under constant gauge transformations, and under spatial inversion, and satisfies the infinitesimal gauge-invariance identity $DR(\xi)[\mathcal{D}_\xi\lambda] = 0$ for all ξ in its domain and all vertex fields λ. These hypotheses concern the function R as a whole, not the individual terms of an expansion of R.

1. *Set $w_q = (1+q)^3 e^{-c_a q}$, where $c_a > 0$ is fixed using part of the spatial decay rate. Then, exactly on the finite torus,*

$$D^2R(I) = \nabla^* W \nabla + \mathsf{E}_{\mathrm{per}},$$
$$\|W\|_{a/2,\mathrm{row+col}} \le C_a \epsilon, \qquad \|\mathsf{E}_{\mathrm{per}}\|_{a/2,\mathrm{row+col}} \le C_a \epsilon w_{m/\delta}.$$

2. *Let R_3 be the cubic term of the Taylor expansion of R at I. The first derivative of R_3 vanishes at every field that is constant in each orientation and color component, and it has the exact representation through first differences along coordinate paths given in* (11.26) *and* (11.27) *below.*
3. *Let $A_0 = A_{d_0+J,d_0}$ as in Section 11.4, and let $e^{\delta A_0(\theta)}$ be the representative $D_0(\theta)$ of the gauge orbit of the classical background field of the step from Section 11.4, with the bounds of Theorem 10.11 on the output rows and volume-normalized columns of $(A_0, \nabla_\delta A_0)$. Assume that the image of $\|\theta\|_\infty \le 2\rho$, measured by absolute Taylor series, lies in the ball of radius $\rho/2$ of fine link logarithms; a sufficient condition is $\delta Z(2\rho) \le \rho/2$, where $Z(u) \le Cu/(1-Cu)$ is the majorant series of $(A_0, \nabla_\delta A_0)$. Set $\mathcal{P}_\delta R(\theta) = R(e^{\delta A_0(\theta)})$. Then*

$$\mathcal{B}_{\rho,a}(\mathcal{P}_\delta R) \le C_a[\delta + \delta^{-1} w_{m/\delta}] \mathcal{B}_{\rho,a}(R) \le C_P(a)\delta\, \mathcal{B}_{\rho,a}(R).$$

   The radius ρ and the rate a are the same on both sides; in the norm of R distances are measured in units of the fine lattice, and in the norm of $\mathcal{P}_\delta R$ in units of the coarse lattice. The constant $C_P(a)$ includes the bound on the finite-period remainder, uniformly for $m \ge 4$.
4. *More generally, let $e^{\delta \widetilde{A}(\theta)}$ be another representative of the gauge orbit of the classical background field of the step (for instance one satisfying $T_M^J(\cdot) = e^\theta$ exactly), holomorphic on $\|\theta\|_\infty < \widetilde{r}$, with $\widetilde{A}(0) = 0$, and suppose that for some rate $b > 8a$ the output rows and the volume-normalized columns, with one fixed coarse input and output volume δ^3, of the Taylor coefficients at $\theta = 0$ of $(\widetilde{A}, \nabla_\delta \widetilde{A})$ are dominated at rate b by a series $\widetilde{Z}(u) \le \widetilde{C}u/(1-\widetilde{C}u)$ and by its derivative $\widetilde{Z}'$, where $\widetilde{r}$ and $\widetilde{C}$ do not depend on J, d_0, or m. Then the conclusion of item (3) holds with $e^{\delta\widetilde{A}}$ in place of $e^{\delta A_0}$, provided $2\rho_0 < \widetilde{r}$ and the image condition holds with $\widetilde{Z}$ in place of Z. The constants ρ_0, δ_0, and C_P then depend also on b, $\widetilde{C}$, and $\widetilde{r}$, and they remain independent of J, d_0, m, and of the later choice of $\rho \le \rho_0$. No conclusion is asserted at output rates $a \ge b/8$. The equality of $\mathcal{P}_\delta R$ with the evaluation of R at a specified reconstruction in block axial gauge coordinates requires, in addition, either a real gauge path inside the domain of R connecting the two representatives, or the invariance of R under the gauge transformations between overlapping charts.*

We record the application to the representative $C_{d_0,J}$ of Lemma 11.9. By items (1), (2), and (4) of that lemma, $C_{d_0,J}$ satisfies the hypotheses of item (4) with $b = a_{\mathrm{cen}}$, $\widetilde{Z} = Z$, and $\widetilde{r} = r_{\mathrm{cen}}$. For $C_{d_0,J}$, the theorem therefore gives the estimate of item (3) at every output rate $0 < a < a_{\mathrm{cen}}/8$, and at no larger rate. Since $a_{\mathrm{cen}} < a_*/32$, such an a also satisfies $a < a_*/8$, and the estimate for D_0 holds at the same rate. To treat both representatives

with one set of constants, let $0 < a < a_{\mathrm{cen}}/8$ and $2\rho_0 < \min(r_{\mathrm{cen}}, r_*)$, where r_* is the radius of Theorem 10.11. We use one enlarged majorant $Cu/(1-Cu)$ at rate a_{cen} for both pairs; the bounds of Theorem 10.11 at rate a_* imply the same bounds at the smaller rate a_{cen}. Finally, δ_0 is chosen so that the image condition holds for both pairs for every $\rho \le \rho_0$. Steps 3–9 of the proof then give constants independent of J, d_0, and m for both representatives, before J is chosen. This is the form in which the theorem is used in Theorem 13.5. The proof of item (1) rests on the following kernel factorization, which strengthens the quadratic-form bound of Lemma 8.6 to a factorization with exponentially decaying kernels.

Lemma 11.11. *Fix $a > 0$ and a finite number of color components. Let $q \ge 4$, and let H be a real self-adjoint translation-invariant kernel on the link fields of the periodic lattice of side q such that*

$$H\,\mathrm{grad} = 0 = \mathrm{grad}^{\mathsf{T}}\,H, \qquad H\mathsf{I} = \mathsf{I}H, \qquad \|H\|_{a,\mathrm{row+col}} \le b.$$

Then there exist translation-invariant kernels W and E such that, exactly on the torus of side q,

$$H = \nabla^* W \nabla + \mathsf{E}, \qquad \|W\|_{a/2,\mathrm{row+col}} \le C_a b, \qquad \|\mathsf{E}\|_{a/2,\mathrm{row+col}} \le C_a b w_q.$$

The differences act on every orientation and color component, and the constants depend only on a, the fixed dimension, and the norm conventions, not on q, H, or b. The conclusion also holds for a complex symmetric kernel whose real and imaginary parts satisfy these hypotheses.

Proof. A bound on the quadratic form would not suffice here, since we need a factorization of the kernel. We write $h_{ij}(r)$ for the translation kernel, with color indices suppressed, and attach each positive link to its lower endpoint. The midpoint displacement of this entry is $p = r + (e_j - e_i)/2$. We lift it from the torus by its shortest representatives, splitting tied representatives equally, and use opposite lifts for the entry related by inversion. Since spatial inversion gives $h_{ij}(r) = h_{ij}(-r + e_i - e_j)$, this midpoint interpolation is even. It preserves every allowed Fourier sample: two lifts differ by qz with $z \in \mathbb{Z}^3$, and $e^{\mathrm{i}k\cdot qz} = 1$ at the allowed frequencies k. The lifted integer coefficients, attached to lower endpoints, have absolute exponentially weighted sum at most $C_a b$, since moving an endpoint to a midpoint changes distances by a fixed amount. We denote by $H(k)$ the trigonometric interpolation of the lower-endpoint coefficients and by $\widetilde{H}(k)$ that of the midpoint coefficients. Entrywise,

$$\widetilde{H}_{ij}(k) = e^{\mathrm{i}k\cdot(e_j-e_i)/2} H_{ij}(k), \qquad D_\ell \widetilde{H}(0) = 0,$$

so $D_\ell H_{ij}(0) = -\mathrm{i}(e_j - e_i)_\ell H_{ij}(0)/2$. Write d_2 and d_1 for the periodic Euclidean and ℓ^1 distances. Since $d_2 \le d_1 \le \sqrt{3} d_2$, the hypothesis at Euclidean rate a implies the same kernel bound at ℓ^1 rate $a/\sqrt{3}$. Applying the estimate on the Fourier block at zero frequency in part (1) of Lemma 8.6 at that rate, with $c_a = c(a/\sqrt{3}) > 0$ in w_q, we obtain

$$\|H(0)\| + \sum_\ell \|D_\ell H(0)\| \le C_a b w_q. \tag{11.23}$$

Since the symmetric choice of lifts preserves self-adjointness, $H(k)$ and its derivatives are Hermitian at real k. We set

$$E(k) = H(0) + \sum_{\ell=1}^{3} \sin(k_\ell) D_\ell H(0).$$

This is a Hermitian trigonometric polynomial of range one with the same value and first derivatives at zero as H, and its kernel norm is at most $C_a b w_q$. Let $h'(r)$ be the finitely supported lifted coefficients of $H - E$. They satisfy the two exact moment cancellations $\sum_r h'(r) = 0$ and $\sum_r r_\ell h'(r) = 0$.

Next, we divide by two differences, with an absolute bound. Let $t_i = e^{\mathrm{i}k_i}$ and $z_i = t_i - 1$. For each integer m, we write $t^m - 1 = z S_m(t)$ and $t^m - 1 - mz = z^2 B_m(t)$. For $m \geq 0$,

$$S_m(t) = \sum_{j=0}^{m-1} t^j, \qquad B_m(t) = \sum_{j=0}^{m-2} (m-1-j) t^j,$$

with empty sums equal to zero, and for $m = -p < 0$,

$$S_{-p}(t) = -\sum_{j=1}^{p} t^{-j}, \qquad B_{-p}(t) = \sum_{j=1}^{p} (p-j+1) t^{-j}.$$

Multiplying by $t - 1$, respectively $(t-1)^2$, we verify all four identities term by term. The sums of the absolute coefficients are at most $|m|$ and $C(1+|m|)^2$. We now telescope $t^r - 1 = \sum_i (t_i^{r_i} - 1) \prod_{j<i} t_j^{r_j}$. Substituting $t_i^{r_i} - 1 = r_i z_i + z_i^2 B_{r_i}(t_i)$ and telescoping each $\prod_{j<i} t_j^{r_j} - 1$ in the same way, we obtain

$$t^r - 1 - \sum_i r_i z_i = \sum_{i,j} z_i z_j P_{r,ij}(t). \tag{11.24}$$

The sum of the absolute coefficients of all $P_{r,ij}$ is at most $C(1+|r|)^2$: the pure terms contribute $\sum_i C(1+|r_i|)^2$ and the mixed terms $\sum_{j<i} |r_i||r_j|$. Their supports lie in the coordinate box from 0 to r, enlarged by a fixed distance, so their displacements are at most $|r| + C$ in the usual lattice metrics, with a fixed comparison of metrics absorbed in the constants.

By the moment cancellations, we may apply (11.24) termwise in $\sum_r h'(r) t^r$. The unused half of the exponential weight absorbs the factor $(1+|r|)^2$, since $\sup_{u \geq 0} (1+u)^2 e^{-au/2} < \infty$. Hence the coefficients in $H - E = \sum_{i,j} z_i L_{ij} z_j$ have weighted absolute sum at most $C_a b$ at rate $a/2$. Since $z_i = -t_i(t_i^{-1} - 1)$, setting $W_{ij} = -t_i L_{ij}$ gives $H - E = \nabla^* W \nabla$. Hermitian symmetrization of W preserves this identity and its bound. We periodize these kernels. Their Fourier samples, and hence their kernels, agree exactly with those of the original operator on the torus. Periodization does not increase the periodic weighted

absolute sums, because each distance on the torus is at most the distance of its lift. Taking E to be the periodized kernel of E proves the lemma. For a complex kernel, we apply the argument to the real and imaginary parts, each of norm at most b, and add the two factorizations. □

Proof of Theorem 11.10. We first prove the two cancellations, items (1) and (2), then the composition estimate of item (3), and finally item (4). Throughout, the norm of R is taken in units of the fine lattice, and $q = m/\delta$.

Step 1: the quadratic term. We subtract the constant $R(I)$, which does not enter $\mathcal{B}_{\rho,a}$, and write $T_k = D^k R(I)$ in the logarithmic coordinates of the fine lattice. By the definition of $\mathcal{B}_{\rho,a}$, we have

$$\frac{1}{k!}\|T_k\|_{a,\mathrm{root}} \le \epsilon\rho^{2-k} \qquad (k \ge 2). \tag{11.25}$$

We differentiate the identity $DR(\xi)[\mathcal{D}_{\tilde\xi}\lambda] = 0$ in ξ at $\xi = 0$. The term containing the derivative of $\mathcal{D}_{\tilde\xi}\lambda$ is multiplied by $DR(0) = 0$, and we are left with $T_2\,\mathrm{grad} = 0 = \mathrm{grad}^{\mathsf{T}}\,T_2$. Translation invariance and invariance under spatial inversion give the remaining hypotheses of Lemma 11.11, with $b \le C\epsilon$. Since both norms use the periodic Euclidean distance, the constant accounts only for the color components and for adding the row and column bounds. For a complex-valued R, the real and imaginary parts of the coefficients satisfy the same linear identities. Lemma 11.11 then implies the factorization and the remainder bound in item (1). Its constant c_a is fixed in that proof from the rate $a/\sqrt{3}$ in Lemma 8.6, and is independent of all later radii.

Step 2: the cubic term. Let $R_3(\xi) = T_3[\xi,\xi,\xi]/6$. By linearity of I and of the translations, R_3 has the same invariances as R. On a field a that is constant in each orientation and color component, spatial inversion acts as $\mathsf{I}a = -a$. Hence $R_3(a) = R_3(-a) = -R_3(a)$, and $R_3(a) = 0$. By translation invariance, the first derivative of R_3 at such a field is the same at every link of a given orientation and color component. Differentiating the vanishing restriction with respect to one constant component gives q^3 times this common value, which therefore vanishes. We conclude that $DR_3(a) = 0$ at every link. For a link e with lower endpoint x_e, let $a_i = \xi_i(x_e)$ and let a be the corresponding constant field. Subtracting $DR_3(a) = 0$, we obtain componentwise the exact identity

$$(DR_3(\xi))_e = \frac{1}{2}\sum_{f,g}(T_3)_{efg}(\xi_f\xi_g - a_{i_f}a_{i_g}), \tag{11.26}$$

where i_f is the orientation of the link f and the products denote the bilinear color contractions with the tensor. We expand the bracket as $(\xi_f - a_{i_f})\xi_g + a_{i_f}(\xi_g - a_{i_g})$. For a coordinate path $x_0 = x_e, x_1, \dots, x_l = x_f$ on the fine lattice,

$$\xi_f - a_{i_f} = \sum_{j=0}^{l-1}(\xi_{i_f}(x_{j+1}) - \xi_{i_f}(x_j)). \tag{11.27}$$

We use shortest coordinate displacements on the torus, with one fixed choice in case of ties. The path and its reverse stay within a fixed multiple of the length of a shortest network connecting the three links. Together with Euler's identity $3R_3 = \sum_e \xi_e (DR_3)_e$, (11.26) and (11.27) are the representation asserted in item (2).

Step 3: majorants of the background field and power counting. We write $A = A_0$. For real θ, this is the potential of the representative of item (3); for complex θ, we use its holomorphic continuation and make no variational comparison. By Theorem 10.11, the Taylor tensors of $(A, \nabla_\delta A)$ have output rows, and columns with one fixed coarse input and output volume δ^3, that are dominated at a decay rate larger than the one used here by one series

$$Z(t) \le \frac{Ct}{1 - Ct}, \qquad A(0) = \nabla_\delta A(0) = 0. \tag{11.28}$$

We enlarge C to cover both members of the pair and the finitely many orientation and color types. The derivative $Z'(t)$ bounds the series with one fixed coarse input, because a Taylor coefficient of order s then has denominator $(s-1)!$ instead of $s!$.

We expand each vertex $T_k[\xi, \dots, \xi]/k!$, with $\xi = \delta A(\theta)$, and replace each argument by its Taylor series in θ. Lemma 11.1 implies that the resulting absolute coefficients are controlled by products of the argument series, with no factorials beyond those of the Taylor coefficients, for any fixed external index. For the spatial and volume bookkeeping, at a vertex we fix the relative positions of its fine indices and sum the remaining position with the fixed-index coefficient bound of T_k. We use output rows for every argument except the one containing the fixed coarse input; for that argument, its volume-normalized column bounds the sum over the translated fine index by $\delta^{-3} Z'(t)$. Since the union of the network of the coefficient and the networks of the arguments connects all external coarse indices, the external weight is at most the product of the weights. A fine displacement has length δ times its length in fine units. Let δ_0 be smaller than a fixed geometric constant, so that half of the fine weight remaining after a polynomial or path moment controls the same output rate a. The networks of the arguments need only the rate a, which is smaller than the rate a_* of their bounds, since $a < a_*/8$. Any enlargement caused by coordinate paths is absorbed by decreasing this fixed δ_0, independently of the order. In particular, no decay rate depends on the number of Taylor inputs. Table 11.2 summarizes the resulting powers of δ, which are verified in Steps 4–6.

Step 4: the quadratic contribution. A first difference of $\xi = \delta A$ over one fine step, in fine units, is exactly δ^2 times a component of $\nabla_\delta A$, translated and with a sign if the step is negative. The two difference factors in $\nabla^* W \nabla$ therefore supply δ^4, and the single sum over a fine index supplies δ^{-3}. We find that the quadratic contribution to the series with one fixed coarse input is at most $C_a \epsilon \delta Z(t) Z'(t)$. Since the finite-period remainder has two undifferentiated factors δA, it contributes $C_a \epsilon \delta^{-1} w_q Z(t) Z'(t)$.

Contribution	Field factors	Sum over a fine index	Result
Quadratic term, two differences	$\delta^2 \cdot \delta^2$	δ^{-3}	δ
Cubic term, one difference	$\delta^2 \cdot \delta \cdot \delta$	δ^{-3}	δ
Degree $k \geq 4$	δ^k	δ^{-3}	δ^{k-3}
Finite-period remainder	$\delta \cdot \delta \cdot w_{m/\delta}$	δ^{-3}	$\delta^{-1} w_{m/\delta} = O(\delta)$

Table 11.2. Powers of $\delta = M^{-J}$ in the proof of Theorem 11.10.

Step 5: the cubic contribution. In the representation of Step 2, the factor given by (11.27) supplies $\delta^2 \nabla_\delta A$, and the two other field factors supply δ^2. The counting column again supplies δ^{-3}. The length of the path and the number of its vertices are absorbed by a polynomial moment of the unused fine weight, since $l \leq C(1 + \mathrm{t})$. If the fixed coarse derivative falls on one of the undifferentiated factors, we use the column of that factor. If it falls on the difference factor, we first fix the relative fine configuration and the position along the path, and then translate the summed fine index to that vertex of the path. This translation is a bijection of the periodic lattice. Translation invariance of T_3 leaves the coefficient unchanged, and the same column with volume δ^3 controls the sum. By (11.25), we find the bound $C_a \epsilon \delta \rho^{-1} Z(t)^2 Z'(t)$, for every choice of the fixed external input.

Step 6: terms of degree at least four. No cancellation is needed. The k field factors and the one counting column give exactly δ^{k-3}. The vertex is bounded by (11.25), and the fixed derivative can fall on any of the k slots. Thus these terms are bounded by

$$C_a \epsilon \delta \rho^{-2} Z(t)^3 Z'(t) \sum_{k \geq 4} k \big(\delta Z(t)/\rho\big)^{k-4}.$$

Combining Steps 4–6, we find that the majorant $\mathcal{D}(t)$ of the series of the composed function with one fixed coarse input satisfies

$$\mathcal{D}(t) \leq C_a \epsilon \delta \Big\{ Z Z' + \rho^{-1} Z^2 Z' + \rho^{-2} Z^3 Z' \sum_{k \geq 4} k (\delta Z/\rho)^{k-4} \Big\} + C_a \epsilon \delta^{-1} w_q Z Z', \tag{11.29}$$

where all series are evaluated at t. We used only symmetries of the tensors T_k, never symmetries of an individual term of an expansion of R.

Step 7: summation. Let $\rho_0(a) > 0$ lie inside the domain of Theorem 10.11 and satisfy $Z(2\rho) \leq C_1 \rho$ and $Z'(\rho) \leq C_1$ for $0 < \rho \leq \rho_0$. Let $\delta_0(a)$ be smaller than the geometric threshold of Step 3 and satisfy $\delta_0 C_1 \leq 1/2$. Through the sufficient condition $\delta Z(2\rho) \leq \rho/2$, these choices give the image condition of item (3) simultaneously for all later $\rho \leq \rho_0$. At $t = \rho$, put $x = \delta Z(\rho)/\rho \leq 1/2$. The series in (11.29) has the uniform numerical bound

$$\sum_{k \geq 4} k x^{k-4} = \frac{4 - 3x}{(1-x)^2} \leq 10 \qquad (0 \leq x \leq 1/2).$$

Since $DR(I) = 0$ and $A(0) = 0$, we also have $D\mathcal{P}_\delta R(0) = 0$. Since a coefficient of order s with one fixed input has denominator $(s-1)!$, the series $\rho^{-1}\mathcal{D}(\rho)$ dominates $\sum_{s\geq 2} \rho^{s-2}\|D^s\mathcal{P}_\delta R(0)\|_{a,\mathrm{root}}/s!$. Using $Z(\rho) \leq C_1\rho$ in each term of (11.29), we obtain

$$\mathcal{B}_{\rho,a}(\mathcal{P}_\delta R) \leq C_a(\delta + \delta^{-1}w_{m/\delta})\epsilon.$$

These constants do not depend on the later reduction of ρ. Replacing ϵ by $\mathcal{B}_{\rho,a}(R)$ gives the first inequality of item (3); the case $\mathcal{B}_{\rho,a}(R) = 0$ follows by continuity.

Step 8: the finite-period remainder. Since $q = m/\delta \geq 4/\delta$,

$$\delta^{-2}w_q \leq \frac{1}{16}q^2(1+q)^3e^{-c_aq} \leq C_a.$$

The last supremum is finite, for example because $q^je^{-c_aq} \leq j!c_a^{-j}$ by the exponential series. Thus $\delta^{-1}w_q \leq C_a\delta$, uniformly for $m \geq 4$, and we enlarge $C_P(a)$ to include this constant. This proves the second inequality of item (3).

Step 9: other representatives. About the representative, Steps 3–8 used only its domain of holomorphy, its centering $A(0) = 0$, the output rows and volume-normalized columns of $(A, \nabla_\delta A)$, their majorant Z, and the fact that these bounds hold at a rate larger than $8a$. Repeating Steps 3–8 with $\widetilde{A}$, $\widetilde{Z}$, $\widetilde{r}$, and the rate $b > 8a$ in place of A, Z, r_*, and a_*, we obtain the estimate in item (4), with constants depending on these data and independent of J, d_0, and m. To compare with the evaluation of R at a specified reconstruction, we integrate the infinitesimal gauge-invariance identity along a real gauge path inside the domain of R. Even if the path is only continuous, at each of its points the strict margin of the domain allows all sufficiently small exponential gauge increments. Integrating the identity along each such increment, we find that R is constant on that neighborhood in the gauge variable. Continuity of the path and a finite subcover of its parameter interval identify the values at the two endpoints, and their holomorphic continuations agree by the uniqueness argument of Lemma 11.4. Alternatively, invariance of R under the gauge transformations between overlapping charts identifies the values at the two real endpoints directly. Neither conclusion follows from the bound on the image alone, which concerns only $e^{\delta A}$. This proves item (4). □

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