

Minimum Prime Bound in Square Intervals

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Abstract

This paper presents a proof of Legendre’s Conjecture by analyzing the distribution of odd composite and their elimination through odd composite sequences. The odd integer sequence is expressed as $O_{\geq 3} = 3 + 2r_O$, while odd composite numbers are represented as $OC_p = p^2 + 2pr_p$, where p is prime and $r_p \geq 0$. Each prime-square sequence introduces a systematic elimination of odd composite numbers, reducing the density of primes in the sequence. By sequentially applying elimination factors $(q - 1)/q$ for primes $q \geq 3$, we derive a lower bound for prime density:

$$\frac{dN_p}{dr_O} \geq \frac{2}{p}.$$

Considering the quadratic interval $[n^2, (n + 1)^2]$, the total change in r_O is $\Delta r_O = \frac{(n+1)^2 - n^2}{2} = n + \frac{1}{2}$. Combining this with the density bound yields a lowest bound of prime count per interval:

$$\pi_{(n^2, (n+1)^2)}^{LB} \geq \left\lceil \frac{2}{p} \left(n + \frac{1}{2} \right) \right\rceil = 2 \quad (p \mid n + 1).$$

Thus, every quadratic interval $[n^2, (n + 1)^2]$ contains at least two primes, establishing Legendre’s Conjecture. This elimination-based framework provides a systematic proof based on the distribution of odd integers and prime-square sequences.

Fig(1)

odd sequence	Odd Composite sequence (p ² +2pr _p)					odd sequence	Odd Composite sequence (p ² +2pr _p)					odd sequence	Odd Composite sequence (p ² +2pr _p)					odd sequence	Odd Composite sequence (p ² +2pr _p)									
O _p =1+2r _O (r _O =0)	p=3	p=5	p=7	p=11	p=13	O _p =1+2r _O (r _O =0)	p=3	p=5	p=7	p=11	p=13	O _p =1+2r _O (r _O =0)	p=3	p=5	p=7	p=11	p=13	O _p =1+2r _O (r _O =0)	p=3	p=5	p=7	p=11	p=13	p=17	p=19	p=23	p=29	p=31
1						201	3,2					401						601										
3						203						403						603										
5						205	5,10					405	5,9	5,9				605	5,9	5,9								
7						207	3,3					407						607										
9	5,9					209						409						609										
11						211						411	3,6					611										
13						213	3,4					413						613										
15	3,1					215						415	3,10					615	3,10	3,10								
17						217						417	3,8					617										
19						219	3,3					419						619										
21	3,2					221						421						621	3,2									
23						223						423	3,9					623										
25	5,1					225	5,10					425	5,40					625	5,40									
27	3,3					227						427						627										
29						229						429	3,13					629										
31						231	3,7					431						631										
33	3,4					233						433						633										
35						235	3,23					435	3,13					635	3,13									
37						237	3,3					437						637										
39	3,3					239						439						639										
41						241						441	3,7					641										
43						243	3,3					443						643										
45	3,3					245	3,27					445	3,45					645	3,45									
47						247						447	3,13					647										
49						249	3,3					449						649										
51	3,3					251						451						651	3,3									
53						253						453	3,13					653										
55						255	3,15					455	3,15					655	3,15									
57	3,3					257						457						657										
59						259						459	3,13					659										
61						261	3,27					461						661										
63						263						463						663										
65	3,4					265	3,24					465	3,15					665	3,15									
67						267	3,3					467						667										
69	3,3					269						469						669										
71						271						471	3,7					671										
73						273	3,24					473						673										
75	3,11					275	3,25					475	3,45					675	3,45									
77						277						477	3,13					677										
79						279	3,3					479						679										
81	3,3					281						481						681										
83						283						483	3,13					683										
85						285	3,15					485	3,15					685	3,15									
87	3,3					287						487						687										
89						289	3,3					489						689										
91						291	3,7					491						691										
93	3,4					293						493						693										
95						295	3,27					495	3,15					695	3,15									
97						297	3,3					497						697										
99	3,3					299						499						699										
101						301						501	3,3					701										
103						303	3,3					503						703										
105	3,10					305	3,28					505	3,48					705	3,48									
107						307						507	3,3					707										
109						309	3,3					509						709										
111	3,3					311						511						711										
113						313						513	3,48					713										
115						315	3,15					515	3,40					715	3,40									
117	3,3					317						517						717										
119						319						519	3,3					719										
121						321	3,2					521						721										
123						323						523						723										
125	3,3					325	3,30					525	3,30					725	3,30									
127						327	3,3					527						727										
129						329						529						729										
131						331						531	3,7					731										
133						333	3,3					533						733										
135	3,3					335	3,3					535	3,3					735	3,3									
137						337						537	3,3					737										
139						339	3,3					539						739										
141	3,3					341						541						741										
143						343	3,7					543	3,3					743										
145						345	3,15					545	3,15					745	3,15									
147	3,3					347						547						747										
149						349						549	3,3					749										
151						351	3,9					551						751										
153	3,4					353						553	3,3					753	3,3									
155						355	3,3					555	3,3					755	3,3									
157						357	3,11					557						757										
159	3,3					359						559						759										
161						361						561																

We can rewrite the odd integer sequence as follows:

$$O_{\geq 3} = 3 + 2r_O (r_O \geq 0)$$

Odd composite sequence:

$$OC_p = p^2 + 2pr_p \quad (p = \text{prime}, r_p \geq 0)$$

Before the odd composite sequence corresponding to $p = 3$ begins at $3^2 = 9$, all odd integers in $O_{\geq 3}$ are prime. For example, 3, 5, and 7 are prime numbers. The odd composite sequence corresponding to $p = 3$ is $9 + 6r_3$, where $r_3 \geq 0$. (See in Fig.1)

When the odd integers exceed $3^2 = 9$, the gap in the odd composite sequence corresponding to $p = 3$ is $2p = 6$, while the gap in the odd integer sequence is 2. Therefore, every three consecutive odd integers contain one term from the odd composite sequence corresponding to $p = 3$, since $\frac{6}{2} = 3$. Thus, one out of every three odd integers is eliminated. If we assume that no other odd composite sequences exist, the remaining proportion is $1 - \frac{1}{3} = \frac{2}{3}$.

$$\frac{d}{dr_O} N_{p=3} = \frac{2}{3} \quad (N_{p=3} = \text{lowest prime number due to elimination of } OC_{=3} \text{ sequence})$$

When the odd integers exceed $5^2 = 25$, the gap in the odd composite sequence corresponding to $p = 5$ is $2p = 10$, while the gap in the odd integer sequence is 2. Therefore, one out of every five odd integers is eliminated, since $\frac{10}{2} = 5$, leaving a proportion of $\frac{4}{5}$. Applying this elimination rate to the proportion $\frac{2}{3}$ remaining after considering the odd composite sequence $9 + 6r_3$ for $p = 3$, we obtain $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$. This approach is used to estimate a lower bound for the proportion of odd integers remaining after successive odd composite sequences are considered.

$$\frac{d}{dr_O} N_{p=5} \geq \frac{d}{dr_O} N_{p=3} \times \frac{4}{5} = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

When the odd integers exceed $7^2 = 49$, the gap in the odd composite sequence corresponding to $p = 7$ is $2p = 14$, while the gap in the odd integer sequence is 2. Therefore, one out of every seven odd integers is eliminated, since $\frac{14}{2} = 7$, leaving a proportion of $\frac{6}{7}$. Applying this elimination rate to the proportion $\frac{8}{15}$ remaining after considering the odd composite sequences $9 + 6r_3$ for $p = 3$ and $25 + 10r_5$ for $p = 5$, we obtain $\frac{8}{15} \times \frac{6}{7} = \frac{16}{35}$. This approach is used to estimate a lower bound for the proportion of odd integers remaining after successive odd composite sequences are considered.

$$\frac{d}{dr_O} N_{p=7} \geq \frac{d}{dr_O} N_{p=5} \times \frac{6}{7} = \frac{8}{15} \times \frac{6}{7} = \frac{48}{105}$$

For the general case:

$$\frac{d}{dr_O} N_p \geq \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \dots \times \frac{p-1}{p} = \prod_{\substack{3 \leq q \leq p \\ q \text{ prime}}} \frac{q-1}{q}$$

$$\prod_{\substack{3 \leq q \leq p \\ p, q \text{ prime}}} \frac{q-1}{q} \geq \frac{2}{p}$$

Thus:

$$\frac{d}{dr_O} N_p \geq \frac{2}{p}$$

For a window $(n+1)^2 - n^2$:

$$\Delta r_O^{\text{for a window}} = \frac{(n+1)^2 - n^2}{2} = n + \frac{1}{2}$$

So the lowest bound of prime count:

$$\pi_{(n^2, (n+1)^2)}^{LB} \geq \left\lceil \frac{dN_p}{dr_O} \times \Delta r_O^{\text{for a window}} \right\rceil = \left\lceil \frac{2}{p} \left(n + \frac{1}{2} \right) \right\rceil.$$

We can assume that $p \approx n + 1$. Therefore,

$$\begin{aligned} \pi_{(n^2, (n+1)^2)}^{LB} &\geq \left\lceil \frac{dN_p}{dr_O} \times \Delta r_O^{\text{for a window}} \right\rceil \\ &= \left\lceil \frac{2}{p} \left(n + \frac{1}{2} \right) \right\rceil \\ &\approx \left\lceil \frac{2}{n+1} \left(n + \frac{1}{2} \right) \right\rceil \\ &= \left\lceil 2 - \frac{1}{n+1} \right\rceil \\ &= 2. \end{aligned}$$

Conclusion

Therefore, in every window $[n^2, (n+1)^2]$, there are at least two primes.