

REGULAR MORLEY TETRAHEDRA FROM p -SECTING PLANES

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ABSTRACT. Morley's theorem says that the adjacent angle trisectors of a triangle meet in the vertices of an equilateral triangle. Tran carried the construction to a tetrahedron by trisecting its six dihedral angles and, more generally, by p -secting planes, with one parameter $e, f, g > 1$ for each pair of opposite edges; the resulting tetrahedron is denoted $\mathcal{M}_{e,f,g}(ABCD)$. Tran showed that a regular Morley tetrahedron does not force the original tetrahedron to be regular, observed that in his examples some pair of opposite edges has four equal cross edges, and conjectured that this happens whenever $\mathcal{M}_{e,f,g}(ABCD)$ is regular. We show that the conjecture is false. We place an isosceles tetrahedron in the box $[-a, a] \times [-b, b] \times [-c, c]$ and prove that $\mathcal{M}_{e,f,g}(ABCD)$ is regular exactly when its vertex A' differs from the centre of the box and lies on one of the four lines through the centre in the directions $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$. Every interior point of these lines other than the centre occurs for exactly one triple of parameters. When a, b and c are pairwise different, no pair of opposite edges has four equal cross edges, so each of these tetrahedra is a counterexample. Two examples are given with the parameters in closed form. Finally, for quadrisecting planes, $e = f = g = 4$, an isosceles tetrahedron has a regular Morley tetrahedron only if it is regular.

1. INTRODUCTION

Morley's theorem says that the adjacent angle trisectors of a triangle meet in the vertices of an equilateral triangle [3, 4]. Tran [5] carried the construction to a tetrahedron $ABCD$ by trisecting its six dihedral angles, and proved that the resulting Morley tetrahedron of an isosceles tetrahedron is again isosceles. In the same paper he replaced the trisecting planes by p -secting planes [5, Theorem 6]. We use the following form of the definition.

Definition 1.1. Let XY be an edge of a tetrahedron, let $\theta_{XY} \in (0, \pi)$ be its interior dihedral angle, let F be one of the two faces through XY , and let $p > 1$. Let F^+ be the half-plane bounded by the line XY that contains F , and let H be the half-plane bounded by XY that lies on the same side of the plane of F as the tetrahedron and makes the angle θ_{XY}/p with F^+ . The p -secting plane of XY towards F is the plane that contains H .

Since $0 < \theta_{XY}/p < \theta_{XY}$, the half-plane H lies inside the dihedral angle at XY . For $p = 3$ this is the trisecting plane of XY that is closer to F . For numbers $e, f, g > 1$ we attach the parameter e to the edges AB and CD , f to BC and AD , and g to AC and BD , as in [5, Theorem 6]. The tetrahedron

$$\mathcal{M}_{e,f,g}(ABCD) = A'B'C'D'$$

is defined as follows: A' is the common point of the three planes that are p -secting towards the face BCD through the edges CD , BD and BC , with p equal to the parameter of the edge, so $p = e$, $p = g$ and $p = f$ respectively; B' , C' and D' are obtained in the same way from the

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faces ACD , ABD and ABC . Proposition 2.3 below shows that the three planes always meet in exactly one point. For $e = f = g = 3$ this is Tran's Morley tetrahedron.¹

Tran [6] recently showed that both converse conjectures of [5] are false: there are tetrahedra T_1 (not isosceles) and T_2 (isosceles but not regular) whose Morley tetrahedra are regular. In both, some pair of opposite edges AB , CD has its four *cross edges* AC , AD , BC , BD of equal length. He conjectured that a regular Morley tetrahedron always forces this [6, Conjecture 8.1], proved it for every tetrahedron with a nontrivial symmetry [6, Corollary 11.6], and stated the analogue for p -secting planes.

Conjecture 1.2 (Tran [6, Conjecture 8.6]). *Let $e, f, g > 1$. If $\mathcal{M}_{e,f,g}(ABCD)$ is regular, then, after relabeling the vertices, $AC = AD = BC = BD$.*

Tran notes that his methods do not apply to parameters other than $e = f = g = 3$, and he lists the decision of Conjecture 1.2 among his open problems [6, Section 18.5 and Problem 8 of Section 19]. We show that the conjecture is false, already for isosceles tetrahedra, where it holds for $e = f = g = 3$ [6, Theorems 10.1 and 10.3].

A tetrahedron is *isosceles* if its opposite edges have equal lengths. Every isosceles tetrahedron is congruent to one with vertices

$$(1.1) \quad A = (a, b, c), \quad B = (a, -b, -c), \quad C = (-a, b, -c), \quad D = (-a, -b, c), \quad a, b, c > 0,$$

that is, to a tetrahedron inscribed in the box $[-a, a] \times [-b, b] \times [-c, c]$ [2, 1]; Lemma 3.1 below gives a short proof, with the labels of the vertices preserved. Its edge lengths are

$$(1.2) \quad AB = CD = 2\sqrt{b^2 + c^2}, \quad AC = BD = 2\sqrt{a^2 + c^2}, \quad AD = BC = 2\sqrt{a^2 + b^2}.$$

Let O be the origin, the centre of the box, and let $\ell_1, \ell_2, \ell_3, \ell_4$ be the lines through O in the directions $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$.

Theorem 1.3. *Let $ABCD$ be the isosceles tetrahedron (1.1).*

- (a) *For all $e, f, g > 1$, the tetrahedron $\mathcal{M}_{e,f,g}(ABCD)$ is regular if and only if $A' \neq O$ and A' lies on $\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$.*
- (b) *Every point $X \neq O$ of $\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ in the interior of $ABCD$ is the vertex A' of $\mathcal{M}_{e,f,g}(ABCD)$ for exactly one triple (e, f, g) with $e, f, g > 1$.*

Distinct points X in part (b) give distinct triples, and the interior points of ℓ_1 already form an open segment through O (Section 3), so there are infinitely many such triples. Figure 1 shows one of the regular Morley tetrahedra.

Corollary 1.4. *Conjecture 1.2 is false. If a, b, c are pairwise different, the tetrahedron (1.1) has three different edge lengths, no pair of its opposite edges has four equal cross edges, and $\mathcal{M}_{e,f,g}(ABCD)$ is regular for infinitely many triples (e, f, g) .*

The parameters can be written down in closed form. Here is an example in which all three parameters lie strictly between 2 and 4, and one in which two of them are equal to 4.

Example 1.5. Take $(a, b, c) = (1, 2, 3)$, so that $AB = CD = 2\sqrt{13}$, $AC = BD = 2\sqrt{10}$ and $AD = BC = 2\sqrt{5}$, and put

$$e = \frac{\arccos(-23/49)}{\arctan(4\sqrt{13}/25)}, \quad f = \frac{\arccos(41/49)}{\arctan(20\sqrt{5}/297)}, \quad g = \frac{\arccos(31/49)}{\arctan(5\sqrt{10}/66)}.$$

¹Tran's paper [6] states the definition by saying that a p -secting plane makes the angle θ_{XY}/p with one of the two faces through XY . Remark 2.5 explains why the results below do not depend on which face the angle is measured from.

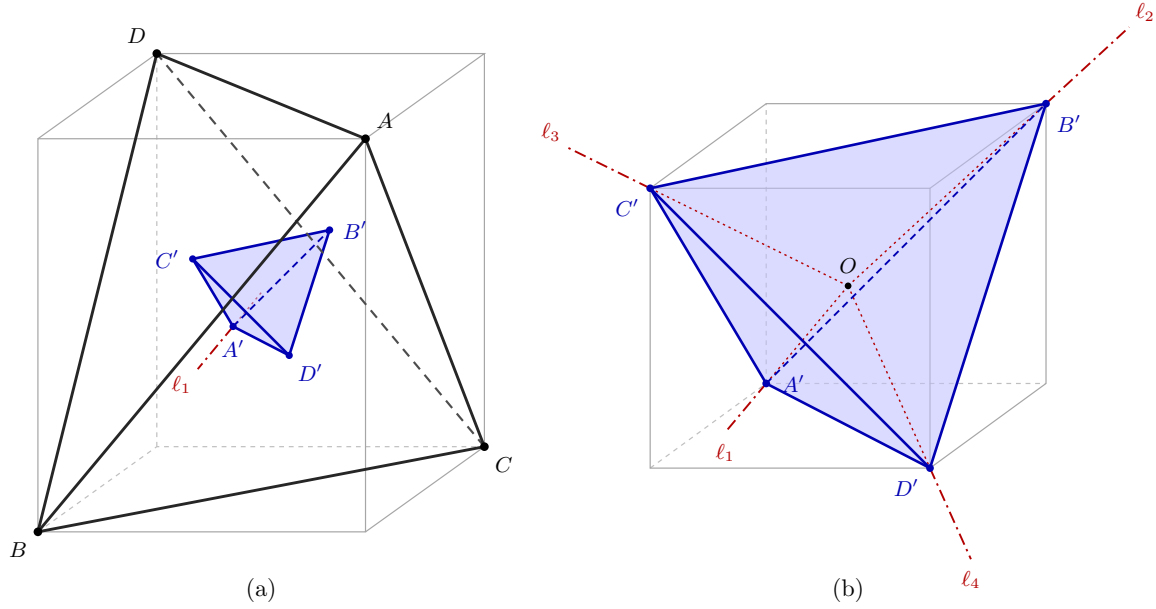


FIGURE 1. (a) An isosceles tetrahedron $ABCD$ in its box, and a regular tetrahedron $\mathcal{M}_{e,f,g}(ABCD) = A'B'C'D'$; its vertex A' lies on the line ℓ_1 . (b) The same Morley tetrahedron, enlarged: its vertices are alternate vertices of a cube centred at O with edges parallel to those of the box, and each of them lies on one of the lines $\ell_1, \dots, \ell_4$.

Then $2 < e, f, g < 4$, and $\mathcal{M}_{e,f,g}(ABCD)$ is the regular tetrahedron with vertices

$$\frac{2}{7}(-1, -1, -1), \quad \frac{2}{7}(-1, 1, 1), \quad \frac{2}{7}(1, -1, 1), \quad \frac{2}{7}(1, 1, -1).$$

Example 1.6. Take $(a, b, c) = (4, 2, 1)$, so that $AB = CD = 2\sqrt{5}$, $AC = BD = 2\sqrt{17}$ and $AD = BC = 4\sqrt{5}$, and put $e = f = 4$ and

$$g = \frac{\arccos(13/21)}{\arctan\left(2\sqrt{17}/(17 + 35\sqrt{5/21})\right)}.$$

Then $2 < g < 4$, and $\mathcal{M}_{e,f,g}(ABCD)$ is the regular tetrahedron with vertices

$$s(-1, -1, -1), \quad s(-1, 1, 1), \quad s(1, -1, 1), \quad s(1, 1, -1), \quad \text{where } s = \frac{5}{2} - \frac{9}{2}\sqrt{5/21} > 0.$$

When the three parameters are equal, Theorem 1.3 still applies, but the three conditions on A' become harder to satisfy. For $e = f = g = 3$, Tran showed that an isosceles tetrahedron with a regular Morley tetrahedron is regular or similar to T_2 [6, Theorem 10.3]. For quadrisecting planes the second possibility disappears.

Theorem 1.7. *An isosceles tetrahedron $ABCD$ has a regular tetrahedron $\mathcal{M}_{4,4,4}(ABCD)$ if and only if $ABCD$ is regular.*

Conjecture 1.2 for $e = f = g$ remains open; for $e = f = g = 3$ it is Tran's Conjecture 8.1 in [6].

Section 2 describes the construction through the distances to the faces; Proposition 2.3 shows that A' runs once through the interior of the tetrahedron as (e, f, g) runs through

$(1, \infty)^3$. Section 3 proves Theorem 1.3, Corollary 1.4 and the two examples, and Section 4 proves Theorem 1.7.

2. THE CONSTRUCTION IN TERMS OF DISTANCES

Let $ABCD$ be a nondegenerate tetrahedron with volume V . For a vertex X let F_X be the opposite face, S_X its area, n_X its inward unit normal, and $d_X(P)$ the signed distance from a point P to the plane of F_X , positive on the side of X . Each d_X is an affine function. Splitting $ABCD$ into four tetrahedra with apex P gives

$$(2.1) \quad S_A d_A(P) + S_B d_B(P) + S_C d_C(P) + S_D d_D(P) = 3V$$

for P in the tetrahedron, and hence for every $P \in \mathbb{R}^3$, since both sides are affine. The interior of $ABCD$ is the set where all four distances are positive. The interior dihedral angle between the faces F_U and F_W , along the edge they share, satisfies $\cos \theta = -n_U \cdot n_W$.

For $\theta \in (0, \pi)$ and $p > 1$ put

$$R_p(\theta) = \frac{\sin((p-1)\theta/p)}{\sin(\theta/p)}.$$

Lemma 2.1. *Fix $\theta \in (0, \pi)$. Then $p \mapsto R_p(\theta)$ is a strictly increasing bijection from $(1, \infty)$ onto $(0, \infty)$, and $R_p(\theta) > 1$ if and only if $p > 2$. For $\rho > 0$, the unique p with $R_p(\theta) = \rho$ is $p = \theta/\varphi$, where $\varphi \in (0, \theta)$ is given by $\cot \varphi = (\rho + \cos \theta)/\sin \theta$.*

Proof. Write $\varphi = \theta/p$, which decreases from θ to 0 as p increases from 1 to ∞ . Then

$$R_p(\theta) = \frac{\sin(\theta - \varphi)}{\sin \varphi} = \sin \theta \cot \varphi - \cos \theta.$$

Since $\sin \theta > 0$ and $\cot$ decreases on $(0, \pi)$, this is a strictly decreasing function of φ ; it tends to 0 as $\varphi \rightarrow \theta$ and to ∞ as $\varphi \rightarrow 0$. This gives the bijection and the formula for p . Finally $\sin(\theta - \varphi) - \sin \varphi = 2 \cos(\theta/2) \sin(\theta/2 - \varphi)$, and $\cos(\theta/2) > 0$, so $R_p(\theta) > 1$ exactly when $\varphi < \theta/2$, that is, when $p > 2$. $\square$

Lemma 2.2. *Let XY be an edge, and let U and W be the other two vertices, so that $XY = F_U \cap F_W$. For $p > 1$, the p -secting plane of XY towards F_U is*

$$\{P \in \mathbb{R}^3 : d_W(P) = R_p(\theta_{XY}) d_U(P)\}.$$

Proof. Let Π be the p -secting plane and $\theta = \theta_{XY}$. The line XY cuts Π into two half-planes, and one of them lies inside the dihedral angle at XY . A point P of that half-plane at distance r from the line XY has $d_U(P) = r \sin(\theta/p)$ and $d_W(P) = r \sin(\theta - \theta/p)$, because the half-plane makes the angles θ/p and $\theta - \theta/p$ with F_U and F_W (Figure 2). So the affine function $d_W - R_p(\theta) d_U$ vanishes on this half-plane and on the line XY . It is not constant, because n_U and n_W are linearly independent, so its zero set is a plane, and this plane is Π . $\square$

For the vertex A' the three edges are $CD = F_A \cap F_B$, $BD = F_A \cap F_C$ and $BC = F_A \cap F_D$, with parameters e , g and f . By Lemma 2.2, A' is the common point of the three planes

$$(2.2) \quad d_B = R_e(\theta_{CD}) d_A, \quad d_C = R_g(\theta_{BD}) d_A, \quad d_D = R_f(\theta_{BC}) d_A,$$

and B' , C' , D' are described in the same way.

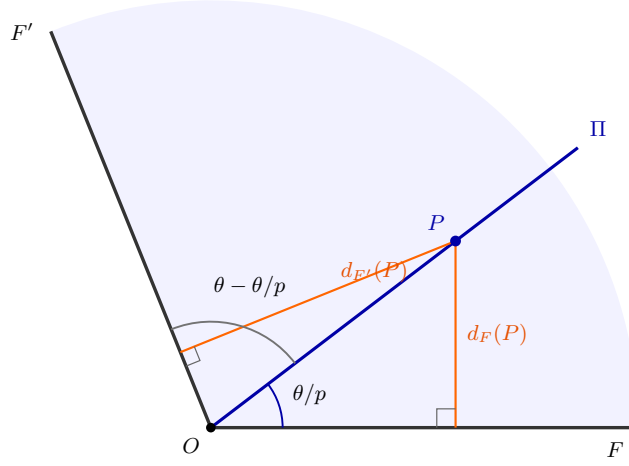


FIGURE 2. A section perpendicular to an edge XY , which appears as the point O ; the faces F and F' through XY appear as two rays. The p -secting plane Π of XY towards F makes the angle θ/p with F , and each of its points P satisfies $d_{F'}(P) = R_p(\theta) d_F(P)$.

Proposition 2.3. *For all $e, f, g > 1$ the three planes (2.2) meet in exactly one point A' , and A' lies in the interior of $ABCD$. The map $(e, f, g) \mapsto A'$ is a bijection from $(1, \infty)^3$ onto the interior of $ABCD$. Its inverse sends an interior point P to the triple determined by*

$$(2.3) \quad R_e(\theta_{CD}) = \frac{d_B(P)}{d_A(P)}, \quad R_g(\theta_{BD}) = \frac{d_C(P)}{d_A(P)}, \quad R_f(\theta_{BC}) = \frac{d_D(P)}{d_A(P)}.$$

The same holds for B' , C' and D' .

Proof. The four normals $n_A, \dots, n_D$ span $\mathbb{R}^3$, so $P \mapsto (d_A(P), d_B(P), d_C(P), d_D(P))$ is an injective affine map from $\mathbb{R}^3$ to $\mathbb{R}^4$. By (2.1) its image lies in the hyperplane $H = \{t \in \mathbb{R}^4 : S_A t_A + S_B t_B + S_C t_C + S_D t_D = 3V\}$, and since both are three-dimensional, the image is all of H . Write $\rho_e = R_e(\theta_{CD})$, $\rho_g = R_g(\theta_{BD})$ and $\rho_f = R_f(\theta_{BC})$; these are positive by Lemma 2.1. A point lies on the three planes (2.2) exactly when its distance vector has the form $(\tau, \rho_e \tau, \rho_g \tau, \rho_f \tau)$, and such a vector lies in H for exactly one τ , namely

$$\tau = \frac{3V}{S_A + \rho_e S_B + \rho_g S_C + \rho_f S_D} > 0.$$

So the planes meet in exactly one point, and all its distances to the faces are positive. Conversely, for an interior point P the three ratios in (2.3) are positive, and by Lemma 2.1 each of them equals R_p of the corresponding dihedral angle for exactly one $p > 1$. With these parameters P lies on the three planes (2.2), so $P = A'$; and no other triple gives P , again by Lemma 2.1. $\square$

Two vertices of $\mathcal{M}_{e,f,g}(ABCD)$ can coincide only if a parameter equals 2. For instance, a point $P = A' = B'$ satisfies $d_B(P) = R_e(\theta_{CD}) d_A(P)$ and, since B' lies on the e -secting plane of CD towards F_B , also $d_A(P) = R_e(\theta_{CD}) d_B(P)$; so $R_e(\theta_{CD}) = 1$ and $e = 2$ by Lemma 2.1.

The symmetries we need permute the vertices by elements of the Klein group

$$K = \{\text{id}, (AB)(CD), (AC)(BD), (AD)(BC)\}.$$

Every element of K maps each of the three pairs of opposite edges to itself; for instance $(AB)(CD)$ maps AB and CD to themselves, interchanges AC and BD , and interchanges AD and BC . So it preserves the parameter attached to every edge.

Lemma 2.4. *Let T_1 be a second tetrahedron, let π be a permutation of the labels A, B, C, D , and let h be an isometry of $\mathbb{R}^3$ that maps each vertex X of $ABCD$ to the vertex of T_1 with label $\pi(X)$. Let $e_1, f_1, g_1 > 1$ be parameters for T_1 such that, for all vertices $Y \neq Z$, the edge of T_1 with labels $\pi(Y)\pi(Z)$ carries the same parameter as YZ . Then h maps the vertex X' of $\mathcal{M}_{e,f,g}(ABCD)$ to the vertex of $\mathcal{M}_{e_1,f_1,g_1}(T_1)$ with label $\pi(X)$.*

Proof. Write d_X^1 for the signed distances of T_1 . The isometry h maps the plane of F_X to the plane of the face of T_1 opposite $\pi(X)$, and the side of X to the side of $\pi(X)$, so $d_{\pi(X)}^1(h(P)) = d_X(P)$ for every point P . It maps the inward normals of $ABCD$ to those of T_1 , so it also preserves the dihedral angles: the angle of T_1 between the faces opposite $\pi(U)$ and $\pi(W)$ equals θ at $F_U \cap F_W$. By Lemma 2.2, X' satisfies $d_Y(X') = R_p(\theta) d_X(X')$ for the three vertices $Y \neq X$, where θ is the angle at $F_X \cap F_Y$ and p is the parameter of that edge. Hence $h(X')$ satisfies $d_{\pi(Y)}^1(h(X')) = R_p(\theta) d_{\pi(X)}^1(h(X'))$, with the same p and the same angle. By Lemma 2.2 and Proposition 2.3 for T_1 , $h(X')$ is the vertex of $\mathcal{M}_{e_1,f_1,g_1}(T_1)$ with label $\pi(X)$. $\square$

We use Lemma 2.4 in two ways: with $T_1 = ABCD$, the same parameters and $\pi \in K$; and with π the identity, when h moves a tetrahedron into a standard position.

Remark 2.5. If the angle θ_{XY}/p in Definition 1.1 is measured from the other face through XY , the plane used for a given parameter p is the one we use for $p/(p-1)$. The map $p \mapsto p/(p-1)$ is an involution of $(1, \infty)$, so the family of tetrahedra $\mathcal{M}_{e,f,g}(ABCD)$ is the same, with the parameters renamed. If instead one always takes the p -secting plane closer to the face, only parameters $p \geq 2$ occur. In Examples 1.5 and 1.6 all parameters exceed 2, so these examples disprove Conjecture 1.2 under each of these readings.

3. ISOSCELES TETRAHEDRA

Lemma 3.1. *Let $ABCD$ be an isosceles tetrahedron. There are $a, b, c > 0$ and an isometry h of $\mathbb{R}^3$ such that $h(A), h(B), h(C), h(D)$ are the points (1.1), in this order.*

Proof. Put $O = \frac{1}{4}(A + B + C + D)$ and

$$u_1 = \frac{1}{4}(A + B - C - D), \quad u_2 = \frac{1}{4}(A - B + C - D), \quad u_3 = \frac{1}{4}(A - B - C + D).$$

Then $A = O + u_1 + u_2 + u_3$, $B = O + u_1 - u_2 - u_3$, $C = O - u_1 + u_2 - u_3$ and $D = O - u_1 - u_2 + u_3$. Since $A - B = 2(u_2 + u_3)$ and $C - D = 2(u_2 - u_3)$, we get $|AB|^2 - |CD|^2 = 16 u_2 \cdot u_3$; in the same way $|AC|^2 - |BD|^2 = 16 u_1 \cdot u_3$ and $|AD|^2 - |BC|^2 = 16 u_1 \cdot u_2$. So u_1, u_2, u_3 are pairwise orthogonal. A direct expansion gives $(B - A) \cdot ((C - A) \times (D - A)) = -16 u_1 \cdot (u_2 \times u_3)$, which is not zero, so none of the u_i vanishes. With $a = |u_1|$, $b = |u_2|$, $c = |u_3|$, the vectors $u_1/a, u_2/b, u_3/c$ form an orthonormal basis, and the isometry $h(P) = ((P - O) \cdot u_1/a, (P - O) \cdot u_2/b, (P - O) \cdot u_3/c)$ sends A to (a, b, c) , B to $(a, -b, -c)$, C to $(-a, b, -c)$ and D to $(-a, -b, c)$. $\square$

From now on let $ABCD$ be the tetrahedron (1.1), and write $\alpha = 1/a$, $\beta = 1/b$, $\gamma = 1/c$ and $N = (\alpha^2 + \beta^2 + \gamma^2)^{1/2}$. For a point $P = (x, y, z)$ put

$$\begin{aligned} L_A(P) &= 1 + \alpha x + \beta y + \gamma z, & L_B(P) &= 1 + \alpha x - \beta y - \gamma z, \\ L_C(P) &= 1 - \alpha x + \beta y - \gamma z, & L_D(P) &= 1 - \alpha x - \beta y + \gamma z. \end{aligned}$$

Each L_X vanishes at the three vertices other than X and equals 4 at X ; hence $d_X = L_X/N$, and the inward normals are $n_A = (\alpha, \beta, \gamma)/N$, $n_B = (\alpha, -\beta, -\gamma)/N$, $n_C = (-\alpha, \beta, -\gamma)/N$, $n_D = (-\alpha, -\beta, \gamma)/N$. The dihedral angles at the edges of the face F_A are therefore given by

$$(3.1) \quad \cos \theta_{CD} = \frac{-\alpha^2 + \beta^2 + \gamma^2}{N^2}, \quad \cos \theta_{BD} = \frac{\alpha^2 - \beta^2 + \gamma^2}{N^2}, \quad \cos \theta_{BC} = \frac{\alpha^2 + \beta^2 - \gamma^2}{N^2}.$$

The three half-turns about the coordinate axes,

$$h_1(x, y, z) = (x, -y, -z), \quad h_2(x, y, z) = (-x, y, -z), \quad h_3(x, y, z) = (-x, -y, z),$$

map the tetrahedron to itself and induce the permutations $(AB)(CD)$, $(AC)(BD)$ and $(AD)(BC)$.

Proof of Theorem 1.3. (a) Let $A' = (x, y, z)$. Lemma 2.4, applied to h_1 , h_2 and h_3 , gives

$$B' = (x, -y, -z), \quad C' = (-x, y, -z), \quad D' = (-x, -y, z).$$

Hence

$$|A'B'| = |C'D'| = 2\sqrt{y^2 + z^2}, \quad |A'C'| = |B'D'| = 2\sqrt{x^2 + z^2}, \quad |A'D'| = |B'C'| = 2\sqrt{x^2 + y^2}.$$

If $A' = O$, the four points coincide. Otherwise the six distances are equal if and only if $x^2 = y^2 = z^2$, that is, if and only if A' lies on one of the lines ℓ_i ; and then A' , B' , C' , D' are alternate vertices of the cube $[-|x|, |x|]^3$, which form a regular tetrahedron.

(b) This is Proposition 2.3 applied to the interior point X . $\square$

For $0 < t < 1/(\alpha + \beta + \gamma)$ the point $-t(1, 1, 1)$ of ℓ_1 is interior: $L_A(-t, -t, -t) = 1 - t(\alpha + \beta + \gamma) > 0$, and L_B, L_C, L_D exceed L_A there by $2t(\beta + \gamma)$, $2t(\alpha + \gamma)$ and $2t(\alpha + \beta)$. By part (b) each of these points is the vertex A' of a regular Morley tetrahedron, and distinct points give distinct triples because the map of Proposition 2.3 is injective.

Proof of Corollary 1.4. If a, b, c are pairwise different, so are $b^2 + c^2$, $a^2 + c^2$ and $a^2 + b^2$, and by (1.2) the tetrahedron has three different edge lengths, each shared by a pair of opposite edges. Under any labeling, the four cross edges of a pair of opposite edges are the two other pairs, so they have two different lengths. On the other hand, by Theorem 1.3 and the paragraph above, $\mathcal{M}_{e,f,g}(ABCD)$ is regular for the infinitely many triples obtained from the points $-t(1, 1, 1)$ with $0 < t < 1/(\alpha + \beta + \gamma)$. $\square$

For a point X on ℓ_1 the ratios in (2.3) take a simple form. If $X = -t(1, 1, 1)$ with $t > 0$, then

$$\begin{aligned} L_A(X) &= 1 - t(\alpha + \beta + \gamma), & L_B(X) - L_A(X) &= 2t(\beta + \gamma), \\ L_C(X) - L_A(X) &= 2t(\alpha + \gamma), & L_D(X) - L_A(X) &= 2t(\alpha + \beta), \end{aligned}$$

so that, for $0 < t < 1/(\alpha + \beta + \gamma)$ and with $\lambda = t/L_A(X) > 0$,

$$(3.2) \quad \frac{d_B(X)}{d_A(X)} = 1 + 2(\beta + \gamma)\lambda, \quad \frac{d_C(X)}{d_A(X)} = 1 + 2(\alpha + \gamma)\lambda, \quad \frac{d_D(X)}{d_A(X)} = 1 + 2(\alpha + \beta)\lambda.$$

For $p = 4$ the identity $\sin 3u = \sin u (1 + 2 \cos 2u)$ gives

$$(3.3) \quad R_4(\theta) = 1 + 2 \cos(\theta/2), \quad \text{so} \quad R_4(\theta) - 1 = \sqrt{2(1 + \cos \theta)}.$$

Since $p \mapsto R_p(\theta)$ is increasing (Lemma 2.1), a parameter p with $R_p(\theta) = \rho$ satisfies $p < 4$ as soon as $(\rho - 1)^2 < 2(1 + \cos \theta)$, and $p > 2$ as soon as $\rho > 1$.

Proof of Example 1.5. Here $(\alpha, \beta, \gamma) = (1, \frac{1}{2}, \frac{1}{3})$ and $N^2 = \frac{49}{36}$, so (3.1) gives $\cos \theta_{CD} = -\frac{23}{49}$, $\cos \theta_{BD} = \frac{31}{49}$ and $\cos \theta_{BC} = \frac{41}{49}$, with sines $\frac{12\sqrt{13}}{49}$, $\frac{12\sqrt{10}}{49}$ and $\frac{12\sqrt{5}}{49}$. Take $X = -\frac{2}{7}(1, 1, 1)$. Then $L_A(X) = \frac{10}{21} > 0$, so X is interior, $\lambda = \frac{3}{5}$, and (3.2) gives the ratios $2, \frac{13}{5}$ and $\frac{14}{5}$. By Lemma 2.1 the parameters are θ/φ with $\tan \varphi = \sin \theta/(\rho + \cos \theta)$:

$$\begin{aligned}\tan \varphi_{CD} &= \frac{12\sqrt{13}/49}{2 - \frac{23}{49}} = \frac{4\sqrt{13}}{25}, & \tan \varphi_{BD} &= \frac{12\sqrt{10}/49}{\frac{13}{5} + \frac{31}{49}} = \frac{5\sqrt{10}}{66}, \\ \tan \varphi_{BC} &= \frac{12\sqrt{5}/49}{\frac{14}{5} + \frac{41}{49}} = \frac{20\sqrt{5}}{297}.\end{aligned}$$

These are the stated values of e, g and f ; the three angles φ lie in $(0, \pi/2)$, so the arctangents are the right branches. The three ratios exceed 1, and $2(1 + \cos \theta)$ equals $\frac{52}{49}, \frac{160}{49}$ and $\frac{180}{49}$ for the edges CD, BD and BC , which exceed $1^2, (\frac{8}{5})^2$ and $(\frac{9}{5})^2$; so $2 < e, f, g < 4$. By Proposition 2.3, $A' = X$, and by Theorem 1.3 and its proof the other vertices are $h_1(X), h_2(X)$ and $h_3(X)$. $\square$

Proof of Example 1.6. Here $(\alpha, \beta, \gamma) = (\frac{1}{4}, \frac{1}{2}, 1)$ and $N^2 = \frac{21}{16}$, so $\cos \theta_{CD} = \frac{19}{21}$, $\cos \theta_{BD} = \frac{13}{21}$ and $\cos \theta_{BC} = -\frac{11}{21}$. By (3.3), $R_4(\theta_{CD}) = 1 + 4\sqrt{5/21}$ and $R_4(\theta_{BC}) = 1 + 2\sqrt{5/21}$. By (3.2), with $\beta + \gamma = \frac{3}{2}$ and $\alpha + \beta = \frac{3}{4}$, both equations $e = 4$ and $f = 4$ say $\lambda = \frac{4}{3}\sqrt{5/21}$. For this λ the point $X = -t(1, 1, 1)$ with $t = \lambda/(1 + \frac{7}{4}\lambda) = \frac{5}{2} - \frac{9}{2}\sqrt{5/21}$ is interior, and the third ratio is $1 + \frac{5}{2}\lambda = 1 + \frac{10}{3}\sqrt{5/21}$. With $\sin \theta_{BD} = \frac{4\sqrt{17}}{21}$, Lemma 2.1 gives

$$\tan \varphi_{BD} = \frac{4\sqrt{17}/21}{1 + \frac{10}{3}\sqrt{5/21} + \frac{13}{21}} = \frac{2\sqrt{17}}{17 + 35\sqrt{5/21}},$$

which is the stated value of g . The third ratio exceeds 1, and $(\frac{10}{3})^2 \cdot \frac{5}{21} = \frac{500}{189} < \frac{612}{189} = 2(1 + \cos \theta_{BD})$, so $2 < g < 4$. Finally $s > 0$ because $\frac{5}{21} < \frac{25}{81}$. As before, $A' = X$ and the Morley tetrahedron is regular. $\square$

4. QUADRISECTING PLANES

Proof of Theorem 1.7. By Lemma 3.1 there is an isometry h that maps A, B, C, D to the points (1.1), in this order. By Lemma 2.4, with π the identity and all parameters equal to 4, h maps $\mathcal{M}_{4,4,4}(ABCD)$ onto $\mathcal{M}_{4,4,4}$ of the image, and an isometry preserves regularity. So we may assume that $ABCD$ is the tetrahedron (1.1). By (1.2) it is regular if and only if $a = b = c$, that is, $\alpha = \beta = \gamma$.

Suppose first that $\alpha = \beta = \gamma$. By (3.1) the edges CD, BD and BC have the same dihedral angle θ , and since $d_X = L_X/N$, (2.2) says that $L_B = L_C = L_D = R_4(\theta) L_A$ at $A' = (x, y, z)$. Now $L_B - L_C = 2\alpha(x - y)$ and $L_C - L_D = 2\alpha(y - z)$, so $x = y = z$ and A' lies on ℓ_1 . If A' were O , then $L_A(A') = L_B(A') = 1$ would give $R_4(\theta) = 1$, while $R_4(\theta) > 1$ by Lemma 2.1. So $A' \neq O$, and $\mathcal{M}_{4,4,4}(ABCD)$ is regular by Theorem 1.3.

Conversely, suppose that $\mathcal{M}_{4,4,4}(ABCD)$ is regular. By Theorem 1.3, $A' = X$ is a point $t\delta \neq O$, where δ is one of $\delta_1 = (1, 1, 1), \delta_2 = (1, -1, -1), \delta_3 = (-1, 1, -1), \delta_4 = (-1, -1, 1)$. Put $\mu = t/L_A(X) \neq 0$ and

$$q_{CD} = \beta^2 + \gamma^2, \quad q_{BD} = \alpha^2 + \gamma^2, \quad q_{BC} = \alpha^2 + \beta^2.$$

By (3.1), $1 + \cos \theta_{CD} = 2q_{CD}/N^2$, and similarly for the other two edges. A direct computation from the formulas for L_X gives

$$\frac{d_B(X)}{d_A(X)} - 1 = -2\mu w_{CD}, \quad \frac{d_C(X)}{d_A(X)} - 1 = -2\mu w_{BD}, \quad \frac{d_D(X)}{d_A(X)} - 1 = -2\mu w_{BC},$$

where (w_{CD}, w_{BD}, w_{BC}) equals

$$(\beta + \gamma, \alpha + \gamma, \alpha + \beta), \quad (-\beta - \gamma, \alpha - \gamma, \alpha - \beta), \quad (\beta - \gamma, -\alpha - \gamma, \beta - \alpha), \quad (\gamma - \beta, \gamma - \alpha, -\alpha - \beta)$$

for $\delta = \delta_1, \delta_2, \delta_3, \delta_4$ respectively. Since all parameters equal 4, (2.3) and (3.3) give, for each of the three edges E of F_A ,

$$4\mu^2 w_E^2 = 2(1 + \cos \theta_E) = \frac{4q_E}{N^2}, \quad \text{so} \quad \frac{q_E}{w_E^2} = \mu^2 N^2.$$

In particular $w_E \neq 0$, since $q_E > 0$, and the three quotients q_E/w_E^2 are equal.

For $\delta = \delta_2$ this is impossible: $q_{CD}/w_{CD}^2 = (\beta^2 + \gamma^2)/(\beta + \gamma)^2 < 1$, while $q_{BD}/w_{BD}^2 = (\alpha^2 + \gamma^2)/(\alpha - \gamma)^2 > 1$. In the same way the edges BD and CD rule out δ_3 , and the edges BC and CD rule out δ_4 . So $\delta = \delta_1$, and

$$\frac{\beta^2 + \gamma^2}{(\beta + \gamma)^2} = \frac{\alpha^2 + \gamma^2}{(\alpha + \gamma)^2} = \frac{\alpha^2 + \beta^2}{(\alpha + \beta)^2}.$$

Since $(u^2 + v^2)/(u + v)^2 = 1 - 2r(u, v)$ with $r(u, v) = uv/(u + v)^2$, this says $r(\gamma, \beta) = r(\gamma, \alpha)$ and $r(\alpha, \gamma) = r(\alpha, \beta)$. For positive u, v, w ,

$$r(u, v) - r(u, w) = \frac{u(v - w)(u^2 - vw)}{(u + v)^2(u + w)^2},$$

so the two equations say

$$(\beta - \alpha)(\gamma^2 - \alpha\beta) = 0 \quad \text{and} \quad (\gamma - \beta)(\alpha^2 - \beta\gamma) = 0.$$

Suppose $\alpha \neq \beta$. Then $\gamma^2 = \alpha\beta$. If $\gamma = \beta$, this gives $\beta^2 = \alpha\beta$, so $\alpha = \beta$; otherwise $\alpha^2 = \beta\gamma$, and dividing $\gamma^2 = \alpha\beta$ by $\alpha^2 = \beta\gamma$ gives $\gamma^3 = \alpha^3$, so $\gamma = \alpha$ and then $\alpha^2 = \alpha\beta$, again $\alpha = \beta$. Both cases contradict $\alpha \neq \beta$. Hence $\alpha = \beta$, and the second equation becomes $-\beta(\gamma - \beta)^2 = 0$, so $\gamma = \beta$. Thus $a = b = c$, and $ABCD$ is regular. $\square$

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