

Upper k -star-forming sets and k -independence in bipartite graphs

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Abstract

Sahbi conjectured that the k -independence number β_k and the upper k -star-forming number SF_k agree on bipartite graphs, and in particular on chain graphs, and asked whether $\beta_3 = \text{SF}_3$ on triangle-free graphs. For every $k \geq 3$ we give a connected bipartite graph on $3k + 3$ vertices with $\beta_k = 2k + 2$ and $\text{SF}_k \geq 3k$. On chain graphs, however, $\beta_k = \text{SF}_k$ for every k . We also show, answering another of his questions, that deciding whether a bipartite graph has a minimal 2-star-forming set with at least t vertices is NP-complete.

Keywords: k -independence, k -star-forming set, upper domination, bipartite graph, chain graph, NP-completeness, formal verification

2020 MSC: 05C69, 68Q17

1. Introduction

Let G be a finite simple graph and k a positive integer. A set I of vertices is k -independent if the induced subgraph $G[I]$ has maximum degree less than k , and $\beta_k(G)$ is the largest size of such a set; this notion goes back to Fink and Jacobson [7], see the survey [5] and [3]. Chellali and Favaron [4] called a set S k -star-forming if for every vertex $v \notin S$ the graph $G[S \cup \{v\}]$ contains a copy of the star $K_{1,k}$ that contains v , and they showed that $\beta_k(G) \leq \text{SF}_k(G)$, where $\text{SF}_k(G)$ is the largest size of a minimal k -star-forming set.¹ For $k = 1$

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¹A maximal k -independent set I is k -star-forming, and it is minimal with that property, since a vertex $s \in I$ has fewer than k neighbours in I and so lies in no copy of $K_{1,k}$ inside $G[I]$.

these are the independence number α and the upper domination number Γ , and the theorem of Cockayne, Favaron, Payan and Thomason [6] that $\Gamma = \alpha$ for bipartite graphs is the first case of the following conjecture of Sahbi.²

Question 1.1 ([13, Conjecture 4.1]). Is $\beta_k(G) = \text{SF}_k(G)$ for every bipartite graph G and every $k \geq 1$?

Sahbi proved the equality for complete bipartite graphs and every k , and for chain graphs when $k = 2$; he gave a triangle-free graph on ten vertices with $\beta_4 < \text{SF}_4$, and asked [13, Problem 7.3] whether $\beta_3 = \text{SF}_3$ for every triangle-free graph. The case $k = 2$ of the conjecture was refuted by a ten-vertex bipartite graph with $\beta_2 = 5$ and $\text{SF}_2 = 6$, posted on 9 October 2026 to a public repository of refutations [1].³ We show that the conjecture fails for every larger k as well.

Theorem 1.2. *For every $k \geq 3$ the connected bipartite graph G_k of Section 2, on $3k + 3$ vertices, satisfies*

$$\beta_k(G_k) = 2k + 2 \quad \text{and} \quad \text{SF}_k(G_k) \geq 3k.$$

In particular $\text{SF}_k(G_k) - \beta_k(G_k) \geq k - 2$, and the answer to Problem 7.3 of [13] is negative already for bipartite graphs: G_3 has twelve vertices and $\beta_3(G_3) = 8 < 9 \leq \text{SF}_3(G_3)$. Together with the example for $k = 2$, the equality $\beta_k = \text{SF}_k$ holds on all bipartite graphs only for $k = 1$, which answers Sahbi's Problem 9.4 for the class of bipartite graphs.

The equality does survive on the more rigid class of chain graphs, the bipartite graphs in which the neighbourhoods of the vertices on one side are nested. Sahbi proved $\beta_2 = \text{SF}_2$ for chain graphs and conjectured [13, Conjecture 6.4] that $\beta_k = \text{SF}_k$ for chain graphs and all $k \geq 3$; this is his Problem 9.1. We prove it.

Theorem 1.3. *For every chain graph G and every $k \geq 2$, $\beta_k(G) = \text{SF}_k(G)$.*

The proof in Section 4 shows that in a chain graph a minimal k -star-forming set has at most one vertex with k or more neighbours inside it, and

²Sahbi withdrew the preprint [13] on 12 October 2026 with the comment ‘‘Some conjectures in this paper have been disproved’’. We refer to its second version.

³We checked this example and found by exhaustive search that it is the only bipartite graph with at most ten vertices and $\beta_2 < \text{SF}_2$ (Remark 6.2).

that this vertex can be exchanged for a vertex outside the set. The graphs G_k show that some such rigidity is needed: in G_k all $k - 1$ hubs have $2k - 1$ neighbours in the set S of Lemma 3.1.

Sahbi also asked [13, Problem 9.3] for the complexity of computing SF_2 on bipartite graphs, and in particular whether it is NP-complete to decide, for a bipartite graph G and an integer t , whether G has a minimal 2-star-forming set with at least t vertices. It is. For a graph H let $H^\sharp$ be the graph obtained from H by replacing every edge with a path of length four and attaching a pendant vertex to every vertex of H (Section 5).

Theorem 1.4. *Let H be a graph with n vertices, m edges and independence number $\alpha(H)$. Then $H^\sharp$ is a bipartite graph on $2n + 3m$ vertices, and*

$$\beta_2(H^\sharp) = \text{SF}_2(H^\sharp) = n + 2m + \alpha(H).$$

Hence it is NP-complete to decide whether a bipartite graph has a minimal 2-star-forming set with at least t vertices, and this remains so for planar bipartite graphs.

The graphs $H^\sharp$ satisfy Sahbi's equality, so the hardness does not come from a gap between the two numbers. The same identity gives another proof that computing β_2 , the dissociation number, is NP-hard on bipartite graphs, a result of Yannakakis [14].

2. The graphs G_k

We write $N_S(v)$ for the set of neighbours of v in S and $d_S(v) = |N_S(v)|$. A copy of $K_{1,k}$ in $G[S \cup \{v\}]$ containing $v \notin S$ has v as its centre or as a leaf, which gives the following local form of the definition [13, Lemma 2.2].

Lemma 2.1. *A set S is k -star-forming if and only if every vertex $v \notin S$ satisfies $d_S(v) \geq k$ or has a neighbour $u \in S$ with $d_S(u) \geq k - 1$. Every superset of a k -star-forming set is k -star-forming, so a k -star-forming set S is minimal if and only if $S \setminus \{s\}$ is not k -star-forming for each $s \in S$.*

Fix $k \geq 2$. The graph G_k has the $3k + 3$ vertices

$$h_1, \dots, h_{k-1}, \quad u_1, \dots, u_{k-1}, \quad w_1, \dots, w_{k-1}, \quad x_0, p, c_1, c_2, q_1, q_2.$$

Write $H = \{h_1, \dots, h_{k-1}\}$ for the hubs, $X_1 = \{u_1, \dots, u_{k-1}\}$, $X_2 = \{w_1, \dots, w_{k-1}\}$ and $X = X_1 \cup X_2 \cup \{x_0\}$. The edges are

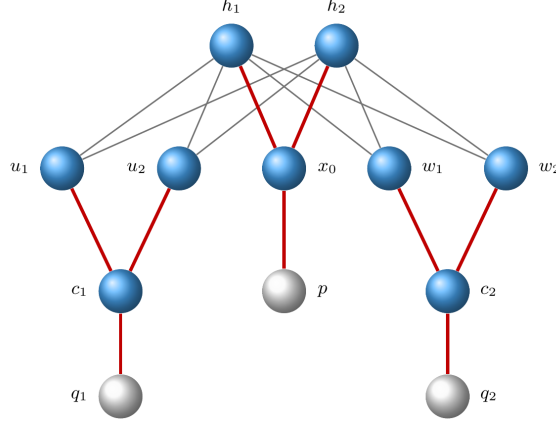


Figure 1: The graph G_3 . The nine shaded vertices form a minimal 3-star-forming set S , while every 3-independent set has at most eight vertices. The thick edges are the three stars $K_{1,3}$ that place p , q_1 and q_2 in stars over S .

- $h_i x$ for every hub h_i and every $x \in X$;
- $x_0 p$, $c_1 q_1$ and $c_2 q_2$;
- $c_1 u_j$ and $c_2 w_j$ for $1 \leq j \leq k - 1$.

So G_k has $(k - 1)(2k - 1) + 2(k - 1) + 3 = 2k^2 - k + 2$ edges. It is connected, and it is bipartite with parts $H \cup \{p, c_1, c_2\}$ and $X \cup \{q_1, q_2\}$. Figure 1 shows G_3 , which has 12 vertices and 17 edges.

The vertices x_0 , c_1 and c_2 each see exactly $k - 1$ vertices of $H \cup X$: the hubs, the block X_1 and the block X_2 , respectively. Each of them also carries a pendant vertex, and the three pendant vertices p , q_1 , q_2 are left outside the set S below. Each pendant vertex is a leaf of a star whose centre and other leaves are forced into S , and this is what makes S minimal although S is far from k -independent: a hub has $2k - 1$ neighbours in S .

3. Proof of Theorem 1.2

Lemma 3.1. *For $k \geq 2$ the set $S = V(G_k) \setminus \{p, q_1, q_2\}$ is a minimal k -star-forming set of G_k with $|S| = 3k$.*

Proof. Clearly $|S| = 3(k - 1) + 3 = 3k$. In S we have $N_S(x_0) = H$, $N_S(c_1) = X_1$ and $N_S(c_2) = X_2$, so $d_S(x_0) = d_S(c_1) = d_S(c_2) = k - 1$. The only vertices

outside S are p, q_1, q_2 , and their neighbours x_0, c_1, c_2 lie in S ; by Lemma 2.1, S is k -star-forming.

For minimality we use Lemma 2.1 again. The pendant vertices have degree $1 < k$, so after the deletion of a vertex s from S a pendant vertex is still in a star only if its unique neighbour remains in $S \setminus \{s\}$ with at least $k - 1$ neighbours there. If s is a hub, then x_0 keeps only $k - 2$ neighbours, so p lies in no star. If $s = x_0$, then p has no neighbour left. Likewise deleting a vertex of X_1 , or c_1 , leaves q_1 outside every star, and deleting a vertex of X_2 , or c_2 , does the same to q_2 . These cases cover all of S . $\square$

Lemma 3.2. *For $k \geq 2$, $\beta_k(G_k) = 2k + 2$.*

Proof. The set $I_0 = X \cup \{p, q_1, q_2\}$ induces the single edge x_0p together with isolated vertices, so it is k -independent, and $|I_0| = 2(k - 1) + 1 + 3 = 2k + 2$.

Conversely, let I be k -independent and put $t = |I \cap H|$ and $s = |I \cap X|$. If $t = 0$, then I lies in $G_k - H$, which is the disjoint union of the edge x_0p and the two stars with centres c_1, c_2 and leaf sets $X_1 \cup \{q_1\}$, $X_2 \cup \{q_2\}$, each a copy of $K_{1,k}$. A k -independent set meets $K_{1,k}$ in at most k vertices, since if it contains the centre it must omit a leaf; hence $|I| \leq 2 + k + k$.

Suppose $t \geq 1$. A hub in I is adjacent to every vertex of $I \cap X$, so $s \leq k - 1$. Besides H and X , at most the five vertices p, c_1, c_2, q_1, q_2 remain, so $|I| \leq t + s + 5$, which is at most $2k + 2$ when $t \leq k - 2$. Finally let $t = k - 1$. Every $x \in I \cap X$ is adjacent to all $k - 1$ hubs, all of which lie in I , so its remaining neighbour among p, c_1, c_2 (namely p for x_0 , c_1 on X_1 and c_2 on X_2) is not in I . If $s \geq 1$ this removes at least one of p, c_1, c_2 , and $|I| \leq (k - 1) + (k - 1) + 2 + 2 = 2k + 2$. If $s = 0$, then $|I| \leq (k - 1) + 5 \leq 2k + 2$. $\square$

Proof of Theorem 1.2. By Lemma 3.1, $\text{SF}_k(G_k) \geq |S| = 3k$, and by Lemma 3.2, $\beta_k(G_k) = 2k + 2$, which is smaller than $3k$ once $k \geq 3$. $\square$

4. Chain graphs

Let G be a chain graph with parts A and B , so that the neighbourhoods $N(a)$, $a \in A$, form a chain under inclusion. Then the neighbourhoods $N(b)$, $b \in B$, form a chain as well.⁴ Two vertices on the same side therefore satisfy

⁴If $a \in N(b) \setminus N(b')$ and $a' \in N(b') \setminus N(b)$, then $b \in N(a) \setminus N(a')$ and $b' \in N(a') \setminus N(a)$, so $N(a)$ and $N(a')$ are incomparable.

$N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$, and $N(u) \subseteq N(v)$ implies $d_S(u) \leq d_S(v)$ for every set S . We say that S covers a vertex $v \notin S$ if v satisfies the condition of Lemma 2.1. Deleting a vertex s from S changes d_S only on the neighbours of s , which lie on the other side.

The exchange step works in every graph.

Lemma 4.1. *Let S be a minimal k -star-forming set of a graph G , $k \geq 1$, such that at most one vertex $v \in S$ has $d_S(v) \geq k$. Then $\beta_k(G) \geq |S|$.*

Proof. If no vertex of S has $d_S(v) \geq k$, then S itself is k -independent. Otherwise let h be the vertex with $d_S(h) \geq k$. By Lemma 2.1 some vertex $z \notin S \setminus \{h\}$ is not covered by $S \setminus \{h\}$: it has at most $k - 1$ neighbours in $S \setminus \{h\}$, and each of these has at most $k - 2$ neighbours in $S \setminus \{h\}$. As h has $d_S(h) \geq k$ neighbours in $S \setminus \{h\}$, we have $z \neq h$, so $z \notin S$. In $I = (S \setminus \{h\}) \cup \{z\}$ the vertex z has at most $k - 1$ neighbours, each neighbour of z has at most $(k - 2) + 1$, and every other vertex v has at most $d_S(v) \leq k - 1$, since $v \neq h$. Hence I is k -independent, and $|I| = |S|$. $\square$

Lemma 4.2. *Let G be a chain graph, $k \geq 2$, and S a minimal k -star-forming set. If $h \in S$ has $d_S(h) \geq k$, then $d_S(y) \leq k - 2$ for every $y \in S$ on the other side.*

Proof. We first show that every neighbour y of h in S has a neighbour $w \notin S$ such that $d_S(u) \leq k - 2$ for every neighbour $u \neq y$ of w in S . Some vertex $w \notin S \setminus \{y\}$ is not covered by $S \setminus \{y\}$. It is not y , since h is a neighbour of y with $d_S(h) - 1 \geq k - 1$ neighbours in $S \setminus \{y\}$; so $w \notin S$, and S covers w . Suppose w lies on the side of y . Then w is not adjacent to y , so $d_S(w) = d_{S \setminus \{y\}}(w) < k$, and w has a neighbour $u \in S$ with $d_S(u) \geq k - 1$, and u lies on the side of h . If $N(u) \subseteq N(h)$, then h is a neighbour of w with at least $k - 1$ neighbours in $S \setminus \{y\}$; if $N(h) \subseteq N(u)$, then $d_S(u) \geq d_S(h) \geq k$ and u keeps at least $k - 1$ neighbours in $S \setminus \{y\}$. Either way $S \setminus \{y\}$ covers w , which is impossible. So w lies on the side of h , and its neighbours, lying on the side of y , have the same degree in $S \setminus \{y\}$ as in S . If w were not adjacent to y , it would also keep its own degree, and $S \setminus \{y\}$ would cover w as S does. Hence w is adjacent to y , and since $S \setminus \{y\}$ does not cover w , every neighbour $u \neq y$ of w in S has $d_S(u) \leq k - 2$.

Now h has two neighbours $a \neq b$ in S , and we may assume $N(a) \subseteq N(b)$. Let w be the vertex just found for a . Then w is adjacent to b , so $d_S(b) \leq k - 2$. Let $y \in S$ lie on the side of a and b . If $N(y) \subseteq N(b)$, then $d_S(y) \leq d_S(b) \leq$

$k - 2$. If $N(b) \subseteq N(y)$, then w is adjacent to y , so $d_S(y) \leq k - 2$ when $y \neq a$, while $d_S(a) \leq d_S(b)$. $\square$

Lemma 4.3. *Let G be a chain graph, $k \geq 2$, and S a minimal k -star-forming set. At most one vertex $v \in S$ has $d_S(v) \geq k$.*

Proof. Suppose that $h \neq t$ are two such vertices. By Lemma 4.2 they lie on the same side, and we may assume $N(t) \subseteq N(h)$. Some vertex $z \notin S \setminus \{t\}$ is not covered by $S \setminus \{t\}$, and $z \neq t$ since $d_{S \setminus \{t\}}(t) = d_S(t) \geq k$. So $z \notin S$, and S covers z . If z lies on the side of t , then z is not adjacent to t , so $d_S(z) = d_{S \setminus \{t\}}(z) < k$, and z has a neighbour $u \in S$ with $d_S(u) \geq k - 1$; but u lies on the other side from h , and $d_S(u) \leq k - 2$ by Lemma 4.2. If z lies on the other side, then z is not adjacent to h , which has $d_S(h) \geq k$ neighbours in $S \setminus \{t\}$; hence z is not adjacent to t either, as $N(t) \subseteq N(h)$. Then neither z nor any of its neighbours, which lie on the side of t , changes its degree when t is deleted, and $S \setminus \{t\}$ covers z as S does. Both cases are impossible. $\square$

Proof of Theorem 1.3. We have $\beta_k(G) \leq \text{SF}_k(G)$ by [4]. Conversely, every minimal k -star-forming set S satisfies $|S| \leq \beta_k(G)$ by Lemmas 4.3 and 4.1. $\square$

For $k = 1$ the equality holds on all bipartite graphs [6]; the argument above needs $k \geq 2$, since in Lemma 4.2 the vertex h must have two neighbours in S .

5. Complexity

Let H be a graph with vertex set V , n vertices and m edges, and write each edge as an ordered pair $e = (u_e, v_e)$ of its ends, in either order. The graph $H^\sharp$ has the vertices of H , a new vertex v' for every $v \in V$ and three new vertices a_e, b_e, c_e for every edge e . Its edges are vv' for $v \in V$ and the four edges of the path $u_e a_e b_e c_e v_e$ for every edge e of H . So $H^\sharp$ has $2n + 3m$ vertices; it is bipartite with parts $V \cup \{b_e\}$ and $\{v'\} \cup \{a_e, c_e\}$, and it is planar when H is. Figure 2 shows $K_3^\sharp$.

Lemma 5.1. *For every independent set I of H the set*

$$J(I) = I \cup \{v' : v \in V\} \cup \{b_e : e \in E(H)\} \cup \{a_e : u_e \notin I\} \cup \{c_e : u_e \in I\}$$

is 2-independent in $H^\sharp$ and has $n + 2m + |I|$ vertices.

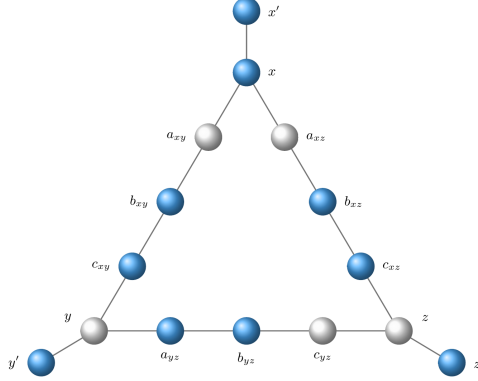


Figure 2: The graph $K_3^\sharp$, with the edges ordered as (x, y) , (y, z) , (x, z) . The ten shaded vertices form the set $J(\{x\})$ of Lemma 5.1. It is a largest 2-independent set and a largest minimal 2-star-forming set, and $10 = n + 2m + \alpha(K_3)$.

Proof. The size is clear, since $J(I)$ contains exactly one of a_e and c_e for each edge e . We check that every vertex of $J(I)$ has at most one neighbour in $J(I)$. A vertex v' has only one neighbour, and b_e has exactly one of its neighbours a_e, c_e in $J(I)$. Let $v \in I$. For an edge e with $u_e = v$ the set $J(I)$ contains c_e and not a_e , and for an edge e with $v_e = v$ we have $u_e \notin I$, because I is independent, so $J(I)$ contains a_e and not c_e ; hence v' is the only neighbour of v in $J(I)$. If $a_e \in J(I)$, then $u_e \notin I$, so b_e is its only neighbour in $J(I)$. If $c_e \in J(I)$, then $u_e \in I$, so $v_e \notin I$, and again b_e is its only neighbour in $J(I)$. $\square$

For the upper bound we use Lemma 2.1 with $k = 2$: a set S covers $w \notin S$ if $d_S(w) \geq 2$ or w has a neighbour $x \in S$ with $d_S(x) \geq 1$. Let $s \in S$. If a vertex $w \notin S$ is neither adjacent to s nor at distance two from it, then $d_{S \setminus \{s\}}(w) = d_S(w)$, and every neighbour x of w satisfies $x \neq s$ and $d_{S \setminus \{s\}}(x) = d_S(x)$, so $S \setminus \{s\}$ covers w whenever S does. Hence, to see that $S \setminus \{s\}$ is still 2-star-forming, we only have to check s and the vertices outside S within distance two of s .

Lemma 5.2. *Let S be a minimal 2-star-forming set of $H^\sharp$, and let Y be the set of vertices $v \in V$ with $v \in S$ and $v' \in S$.*

- (a) *Every $v \in V$ satisfies $v \in S$ or $v' \in S$.*
- (b) *For no edge e do all three of a_e, b_e, c_e lie in S .*

(c) If both ends of an edge e lie in Y , then at most one of a_e, b_e, c_e lies in S .

Proof. (a) The vertex v' has the single neighbour v , so if $v' \notin S$ then S covers v' only through $v \in S$.

(b) Suppose that $a_e, b_e, c_e \in S$, and put $S' = S \setminus \{b_e\}$. The vertex b_e has its two neighbours a_e, c_e in S' . The vertices outside S at distance two from b_e are among u_e and v_e . If $u_e \notin S$, then $u'_e \in S$ by (a), so u_e has the two neighbours u'_e and a_e in S' , and likewise for v_e . So S' is 2-star-forming, against the minimality of S .

(c) Let $u_e, v_e \in Y$ and suppose that two of a_e, b_e, c_e lie in S ; by (b) the third does not. Since $u'_e \in S$ and $v'_e \in S$, the vertices u_e and v_e keep at least one neighbour in S after the deletion of any of a_e, b_e, c_e , so every vertex outside S that is adjacent to u_e or v_e stays covered. If $b_e \notin S$, put $S' = S \setminus \{a_e\}$: then a_e is covered through u_e , and b_e through c_e , whose neighbour v_e is in S' . If $c_e \notin S$, put $S' = S \setminus \{a_e\}$: then a_e is covered through u_e , and c_e has the two neighbours b_e and v_e in S' . The case $a_e \notin S$ is symmetric to the previous one, with $S' = S \setminus \{c_e\}$. In each case the vertices outside S within distance two of the deleted vertex are covered by S' , so S' is 2-star-forming, against the minimality of S . $\square$

Proof of Theorem 1.4. The order and the bipartition of $H^\sharp$ were noted above. By Lemma 5.1 applied to a largest independent set, $\beta_2(H^\sharp) \geq n + 2m + \alpha(H)$, and $\beta_2 \leq \text{SF}_2$ by [4]. Now let S be a minimal 2-star-forming set of $H^\sharp$ and Y as in Lemma 5.2. By part (a) the set S contains both of the vertices v, v' if $v \in Y$ and one of them otherwise, and by parts (b) and (c) it contains at most two of a_e, b_e, c_e , and at most one if both ends of e lie in Y . Writing $e(Y)$ for the number of edges of H with both ends in Y , we get

$$|S| \leq n + |Y| + 2m - e(Y).$$

The graph $H[Y]$ has at least $|Y| - e(Y)$ components, and one vertex from each component forms an independent set, so $|Y| - e(Y) \leq \alpha(H)$. Hence $|S| \leq n + 2m + \alpha(H)$, and the three numbers are equal.

The decision problem lies in NP, since by Lemma 2.1 one checks in polynomial time that a given set S is 2-star-forming and that $S \setminus \{s\}$ is not, for each $s \in S$. For hardness, H has an independent set with at least t vertices if and only if $H^\sharp$ has a minimal 2-star-forming set with at least $n + 2m + t$ vertices, and $H^\sharp$ is built from H in polynomial time. Node Cover, and with

it Independent Set, is NP-complete [9], also on planar graphs [8], and $H^\sharp$ is planar when H is. $\square$

For $k \geq 3$ we have not determined the complexity of computing SF_k on bipartite graphs.

6. Remarks

Remark 6.1. For $k = 2$ the two numbers in Lemmas 3.1 and 3.2 coincide, and indeed $\beta_2(G_2) = \text{SF}_2(G_2) = 6$. An exhaustive computation over all subsets shows that $\text{SF}_k(G_k) = 3k$ exactly for $3 \leq k \leq 7$, so in this range $\text{SF}_k(G_k)/\beta_k(G_k) = 3k/(2k+2)$, which tends to $3/2$.

Remark 6.2. We computed β_k and SF_k for every bipartite graph with at most 12 vertices, generated with nauty [11], and $2 \leq k \leq 7$. The smallest bipartite graphs with $\beta_k < \text{SF}_k$ have 10 vertices for $k = 2$ (the graph of [1], which is unique), 12 vertices for $k = 3$ (there are 51 such graphs, G_3 among them), 10 vertices for $k = 4$ (a single graph) and 12 vertices for $k = 5$; for $k = 6, 7$ there is none with at most 12 vertices. Among triangle-free graphs the smallest ones with $\beta_3 < \text{SF}_3$ have 10 vertices, and there are two of them. We also found no tree with at most 16 vertices and $\beta_k < \text{SF}_k$ for $2 \leq k \leq 6$, and, in accordance with Theorem 1.3, no such chain graph with at most 16 vertices for $3 \leq k \leq 8$. These computations are not part of the proofs.

Remark 6.3. We also checked Theorem 1.4 by computer in two independent ways: by exhaustive search over all vertex subsets of $H^\sharp$ for every connected graph H with $2n + 3m \leq 26$, which also confirms parts (a)–(c) of Lemma 5.2 for every minimal 2-star-forming set, and with a SAT solver for every connected graph H with at most 7 vertices.

Formal verification

Theorems 1.2, 1.3 and 1.4 are formalised in Lean 4 [12] with Mathlib [10]. The definitions follow [13]: a set is k -star-forming if every outside vertex v lies in a copy of $K_{1,k}$ inside $G[S \cup \{v\}]$, minimality is required against every proper subset, and β_k and SF_k are the largest sizes of k -independent and of minimal k -star-forming sets. For every $k \geq 3$ the formal statement gives a 2-colouring of G_k together with $\beta_k(G_k) = 2k + 2$ and $\text{SF}_k(G_k) \geq 3k$. Theorem 1.3 is proved for every finite graph with a bipartition in which the neighbourhoods on one side are nested, and the inequality $\beta_k \leq \text{SF}_k$ of [4]

is proved along the way. The identity of Theorem 1.4 is proved for every finite graph H whose vertices are linearly ordered, the order fixing the ends $u_e < v_e$, together with a 2-colouring of $H^\#$; here $\alpha(H)$ is the independence number of Mathlib. The NP-completeness, which follows from the identity by the argument above, is not formalised. The proofs are checked by the kernel and use only the standard axioms of Lean. The development and the search programs are archived in [2].

Declaration of competing interest

The author has no competing interests to declare.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability

The Lean 4 development, the C and Python checks and the search programs are available at <https://github.com/creelie/Conjecture-002> and on Zenodo, [doi:10.5281/zenodo.23360032](https://doi.org/10.5281/zenodo.23360032).

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) to find the construction and the reduction, to check them in C, Python and Lean, and to draft the text. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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