
Finite Field Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

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Abstract: Hudzik and Landes [Math. Ann., 1992] derived a major generalization of the triangle inequality for two nonzero elements in normed linear spaces, which was extended to finitely many nonzero elements independently by Dragomir [Bull. Aust. Math. Soc., 2006] and by Kato, Saito and Tamura [Math. Inequal. Appl., 2007]. We derive a finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

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1. INTRODUCTION

Let $\mathcal{X}$ be a normed linear space (NLS) over scalar field. Definition of the norm has the triangle inequality

$$(1) \quad \|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathcal{X}.$$

In 1992, Hudzik and Landes derived a breakthrough generalization of Inequality (1) which is valid for any two nonzero elements in a NLS [2].

Theorem 1.1. [2] (**Hudzik-Landes Inequality**) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,

$$(2) \quad \|x + y\| \leq \|x\| + \|y\| - \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \min\{\|x\|, \|y\|\}.$$

In 2006, Maligranda independently derived Inequality (2) [4]. It is natural to look for a generalization of Inequality (2) to more than two non-zero vectors. This is done independently by Dragomir in 2006 [1] and by Kato, Saito and Tamura in 2007 [3].

Theorem 1.2. [1, 3] (**Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality**) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$, we have

$$\left\| \sum_{j=1}^n x_j \right\| \leq \sum_{j=1}^n \|x_j\| - \left(n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq k \leq n} \|x_k\|.$$

It is natural and important to ask what are finite field versions of Theorems 1.1 and 1.2? We answer the question by deriving finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality (Theorem 2.9).

2. FINITE FIELD HUDZIK-LANDES-DRAGOMIR-KATO-SAITO-TAMURA INEQUALITY

We begin by introducing the notion of sub-modulus field.

Definition 2.1. Let $\mathbb{F}$ be a (finite) field. A map $|\cdot| : \mathbb{F} \rightarrow [0, \infty)$ is said to be sub-modulus or sub-valued if the following conditions hold.

- (i) If $\lambda \in \mathbb{F}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| \leq |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{F}$.
- (iii) $|\lambda + \mu| \leq |\lambda| + |\mu|$ for all $\lambda, \mu \in \mathbb{F}$.

In this case, we say that $\mathbb{F}$ is a sub-modulus or sub-valued field.

Example 2.2. Let p be a prime, let $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$ be the standard field of integers modulo p . Given $n \in \mathbb{Z}_p$, we define

$$|n| := \text{unique natural number } a \text{ such that } 0 \leq a \leq p-1 \text{ and } n \equiv a \pmod{p}.$$

We next introduce the notion of sub-normed linear space.

Definition 2.3. Let $\mathcal{X}$ be a vector space over a sub-modulus field $\mathbb{F}$. A map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be sub-norm if the following conditions hold.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| \leq |\lambda|\|x\|$ for all $\lambda \in \mathbb{F}$, for all $x \in \mathcal{X}$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.

In this case, we say $\mathcal{X}$ is a sub-normed linear space.

Example 2.4. Let $p \in [1, \infty)$. Let $\mathbb{F}$ be a sub-modulus field. For $d \in \mathbb{N}$, let $\mathbb{F}^d$ be the standard vector space. We define

$$\|(a_j)_{j=1}^d\|_p := \left(\sum_{j=1}^d |a_j|^p \right)^{\frac{1}{p}}, \quad \forall (a_j)_{j=1}^d \in \mathbb{F}^d.$$

Then $\|\cdot\|_p$ is a sub-norm on $\mathbb{F}^d$.

Example 2.5. $p \in [1, \infty)$. Let $\mathbb{F}$ be a sub-modulus field. Define

$$c_{00}^p(\mathbb{N}, \mathbb{F}) := \{ \{a_n\}_{n=1}^\infty : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n \neq 0 \text{ only for finitely many } n \}.$$

We define

$$\|\{a_n\}_{n=1}^\infty\|_p := \left(\sum_{n=1}^\infty |a_n|^p \right)^{\frac{1}{p}}, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}^p(\mathbb{N}, \mathbb{F}).$$

Then $\|\cdot\|_p$ is a sub-norm on $c_{00}^p(\mathbb{N}, \mathbb{F})$.

Example 2.6. Let $\mathbb{F}$ be a sub-modulus field. Define

$$c_{00}(\mathbb{N}, \mathbb{F}) := \{ \{a_n\}_{n=1}^\infty : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n \neq 0 \text{ only for finitely many } n \}.$$

We define

$$\|\{a_n\}_{n=1}^\infty\|_{00} := \max_{n \in \mathbb{N}} |a_n|, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}(\mathbb{N}, \mathbb{F}).$$

Then $\|\cdot\|_{00}$ is a sub-norm on $c_{00}(\mathbb{N}, \mathbb{F})$.

We now derive finite field version of Inequality (2).

Theorem 2.7. (*Finite Field Hudzik-Landes Inequality*) *Let $\mathcal{X}$ be a sub-normed linear space over $\mathbb{F}$. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{F}$ it holds*

$$\|x + y\| \leq \min \left\{ \|x\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right), \|y\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\| \right) \right\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{F}$. Then

$$\begin{aligned} \|x + y\| &= \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) + \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \\ &\leq \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| + \left\| \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \\ &\leq \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| 1 - \frac{\|x\|}{\|y\|} \right| \|y\| \\ &\leq \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \|x\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \end{aligned}$$

and

$$\begin{aligned} \|x + y\| &= \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x + \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \\ &\leq \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x \right\| + \left\| \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \\ &\leq \left| 1 - \frac{\|y\|}{\|x\|} \right| \|x\| + \|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \\ &\leq \|y\| \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\| + \|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|. \end{aligned}$$

Therefore

$$(3) \quad \|x + y\| \leq \|x\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right)$$

and

$$(4) \quad \|x + y\| \leq \|y\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\| \right).$$

Inequalities (3) and (4) give

$$\|x + y\| \leq \min \left\{ \|x\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right), \|y\| \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| + \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\| \right) \right\}.$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathbb{F}$ in the previous theorem. The reason is that, since the norm is a real number, we generally do not have a guarantee that it belongs to the given sub-modulus field.

Example 2.8. Consider $\mathbb{Z}_3$. Let $x = 1, y = 2$. Then $|x + y| = |1 + 2| = |3| = 0$ and $|x| + |y| = 1 + 2 = 3$. We now find

$$\begin{aligned}
 ||x|| \left(\left| \frac{x}{|x|} + \frac{y}{|y|} \right| + \left| \frac{1}{|x|} - \frac{1}{|y|} \right| |y| \right) &= ||1|| \left(\left| \frac{1}{|1|} + \frac{2}{|2|} \right| + \left| \frac{1}{|1|} - \frac{1}{|2|} \right| |2| \right) \\
 &= |1| \left(\left| \frac{1}{1} + \frac{2}{2} \right| + \left| \frac{1}{1} - \frac{1}{2} \right| 2 \right) \\
 &= |1 + 1| + |1 - 2| 2 \\
 &= |2| + |1 + 1| 2 \\
 &= 2 + |2| 2 \\
 &= 2 + 2 \cdot 2 \\
 &= 2 + 1 = 0
 \end{aligned}$$

and

$$\begin{aligned}
 ||y|| \left(\left| \frac{x}{|x|} + \frac{y}{|y|} \right| + \left| \frac{1}{|x|} - \frac{1}{|y|} \right| |x| \right) &= ||2|| \left(\left| \frac{1}{|1|} + \frac{2}{|2|} \right| + \left| \frac{1}{|2|} - \frac{1}{|1|} \right| |1| \right) \\
 &= |2| \left(\left| \frac{1}{1} + \frac{2}{2} \right| + \left| \frac{1}{2} - \frac{1}{1} \right| 1 \right) \\
 &= 2(|1 + 1| + |2 - 1|) \\
 &= 2(|2| + |1|) \\
 &= 2(2 + 1) = 0.
 \end{aligned}$$

Therefore

$$0 = |x + y| = \min\{0, 0\} < |x| + |y| = 3.$$

Now we derive finite field version of Theorem 1.2.

Theorem 2.9. (Finite Field Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality) Let $\mathcal{X}$ be a sub-normed linear space over $\mathbb{F}$ and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{F}$ it holds

$$\begin{aligned}
 \left\| \sum_{j=1}^n x_j \right\| &\leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \sum_{1 \leq j \leq n, j \neq k} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right) \right\} \\
 &\leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + (n-1) \max_{1 \leq j \leq n, j \neq k} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right) \right\}.
 \end{aligned}$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{F}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned}
 \left\| \sum_{j=1}^n x_j \right\| &= \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} + \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \\
 &\leq \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} \right\| + \left\| \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \\
 &\leq \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} \right\| + \left\| \|x_k\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
 &\leq \|x_k\| \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \|x_k\| \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
 &= \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right) \\
 &= \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \left\| \sum_{1 \leq j \leq n, j \neq k} \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right) \\
 &\leq \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \sum_{1 \leq j \leq n, j \neq k} \left\| \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right) \\
 &\leq \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \sum_{1 \leq j \leq n, j \neq k} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right).
 \end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\begin{aligned}
 \left\| \sum_{j=1}^n x_j \right\| &\leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + \sum_{1 \leq j \leq n, j \neq k} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right) \right\} \\
 &\leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \left(\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| + (n-1) \max_{1 \leq j \leq n, j \neq k} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right) \right\}.
 \end{aligned}$$

□

3. CONCLUSIONS

- (1) In 1992, Hudzik and Landes improved triangle inequality for two nonzero elements in normed linear spaces [2].
- (2) In 2006, Dragomir extended Hudzik-Landes inequality for more than two nonzero vectors [1].
- (3) In 2007, Kato, Saito and Tamura extended Hudzik-Landes inequality without knowing the work of Dragomir [3].
- (4) In this article, we derived finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

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