

Unbounded strong-product ratios for the Grundy domination number

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Abstract

A sequence of vertices of a graph is legal when every vertex in it dominates some vertex that none of the earlier ones dominated, and the Grundy domination number of the graph is the length of its longest legal sequence. In the strong product of two graphs, two vertices are adjacent when each coordinate is equal or adjacent in its factor. It has been known since 2016 that the Grundy domination number of a strong product is at least the product of the Grundy domination numbers of the two factors, and it was conjectured, by Brešar, Bujtás, Gologranc, Klavžar, Košmrlj, Patkós, Tuza and Vizer, that equality always holds. We show that the conjecture is false, and false by an unbounded factor: for every number C there are graphs whose strong product has Grundy domination number more than C times the product of the two factors. The graphs are built on the five-element subsets of an m -element set, with adjacency decided by the size of the intersection; incidence matrices over the fields with two and three elements bound the two factors by m and $\binom{m}{2}$, while the diagonal of the product is an independent set of size $\binom{m}{5}$. We also compute the Grundy domination number of the graph on $\{0, \dots, N-1\}$ in which u and v are adjacent when $|u-v| \equiv 2 \pmod{3}$: it is $\lfloor (N+1)/3 \rfloor + 3$ for every $N \geq 6$. Its strong product with the octahedron has a legal sequence of length N , so the conjecture already fails for $N = 19$, on a graph with 114 vertices, and for every $N \geq 21$. Every statement is proved by hand.

Keywords. Grundy domination number; strong product; dominating sequence; incidence matrix.

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Note. The counterexample of this paper was first made public on 14 October 2026, as release v1.0.0 of the repository <https://github.com/creelie/grundy-strong-product>, archived on Zenodo as 10.5281/zenodo.23378021. An earlier and incomplete version of the paper appeared on Hexagon on 23 October 2026 as the preprint [hexagon:2610.00494](https://hexagonmath.org/2610.00494), available at <https://hexagonmath.org/2610.00494>. That version gave one counterexample, the strong product of the octahedron with the graph H_{21} on 21 vertices, and relied on a computer certificate for the Grundy domination number of H_{21} . The revisions that followed are archived on Zenodo as 10.5281/zenodo.23607502 and 10.5281/zenodo.23629621, in the repository https://github.com/creelie/Paper_1. This version is complete and self-contained: the Grundy domination number of H_N is computed by hand for every $N \geq 6$, the ratio is shown to be unbounded, the scope of each result is stated, no statement rests on a computer, and the only unrefereed source it depends on is the withdrawn proof claim that it mentions for the record. The source of this revision is in the repository named in the data availability statement at the end, archived on Zenodo as 10.5281/zenodo.23659882 (concept 10.5281/zenodo.23607651), together with programs that recompute the small cases.

1 Introduction

Let G be a finite simple graph and write $N[v]$ for the closed neighbourhood of a vertex v , the set consisting of v and its neighbours. A sequence $(v_1, \dots, v_k)$ of vertices of G is *legal* if every v_i dominates a vertex that none of its predecessors dominates, that is,

$$N[v_i] \setminus \bigcup_{j < i} N[v_j] \neq \emptyset \quad (1 \leq i \leq k);$$

a vertex in this difference is a *footprint* of v_i .¹ The *Grundy domination number* $\gamma_{\text{gr}}(G)$ is the length of a longest legal sequence; a longest legal sequence always dominates G , since otherwise an undominated vertex could be appended. The parameter was introduced by Brešar, Golovc, Milanič, Rall and Rizzi [7], who connected it with the domination game.

In the strong product $G \boxtimes H$ two distinct vertices (g, h) and (g', h') are adjacent when $g' \in N_G[g]$ and $h' \in N_H[h]$; equivalently $N_{G \boxtimes H}[(g, h)] = N_G[g] \times N_H[h]$. Taking a Grundy dominating sequence of G and one of H and listing their pairs in lexicographic order gives a legal sequence of $G \boxtimes H$, so

$$\gamma_{\text{gr}}(G \boxtimes H) \geq \gamma_{\text{gr}}(G) \gamma_{\text{gr}}(H) \quad (1.1)$$

for all G and H [6, Proposition 21].² In the same paper the authors conjectured that the bound is always attained.

Conjecture 1.1 ([6, Conjecture 22]). *For all graphs G and H , $\gamma_{\text{gr}}(G \boxtimes H) = \gamma_{\text{gr}}(G) \gamma_{\text{gr}}(H)$.*

The conjecture holds when G is a caterpillar [6] and, more generally, when G is a forest, by the work of Bell, Driscoll, Krop and Wolff [2]. A proof for all graphs was announced by Herrman and Smith [8] and later withdrawn.³ We show that the conjecture is false and that no constant multiple of its proposed value is an upper bound.

Theorem 1.2. *For every $C > 0$ there are nonempty finite graphs G, H such that*

$$\gamma_{\text{gr}}(G \boxtimes H) > C \gamma_{\text{gr}}(G) \gamma_{\text{gr}}(H).$$

¹A vertex may have several footprints; any one of them witnesses that the step is legal. The vertices of a legal sequence are distinct, since a repeated vertex dominates nothing new.

²If d_i footprints u_i in G and d'_j footprints u'_j in H , then (d_i, d'_j) footprints (u_i, u'_j) in the lexicographic order of the pairs, because an earlier pair (d_a, d'_b) has $a < i$, so $u_i \notin N[d_a]$, or $a = i$ and $b < j$, so $u'_j \notin N[d'_b]$.

³The second version of arXiv:2212.04565 (12 January 2023) carries the comment “There is an error. Some cases were not considered in the proof of Theorem 1.”

Section 2 proves this with two graphs on the five-element subsets of $\{1, \dots, m\}$. Each factor is bounded through an incidence matrix over a finite field whose nonzero entries are exactly the closed neighbourhoods, and the diagonal of the product is an independent set. The ratio grows like m^2 .

The smallest counterexamples we know come from a different construction, with the octahedron as first factor. For $N \geq 1$ let H_N be the graph on $\{0, 1, \dots, N-1\}$ in which u and v are adjacent when $|u - v| \equiv 2 \pmod{3}$, and let $K_{2,2,2}$ be the octahedron, the complete tripartite graph with three parts of size two. We compute the Grundy domination number of H_N for every order.

Theorem 1.3. *For every $N \geq 6$,*

$$\gamma_{\text{gr}}(H_N) = \left\lfloor \frac{N+1}{3} \right\rfloor + 3.$$

Moreover $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) \geq N$ for every $N \geq 1$. Thus Conjecture 1.1 fails for $N = 19$ and for every $N \geq 21$.

The proof is in Sections 3 to 5. After the first vertex of a legal sequence is chosen in a suitable circulant graph containing H_N , the undominated vertices form a half graph, and a count at the first selected edge, or an injection at the first selected vertex of the third residue class, gives the upper bound. Explicit sequences attain it. At $N = 19$ the factors have Grundy domination numbers 2 and 9, while their product has a legal sequence of length 19. This is a counterexample on 114 vertices. By Proposition 25 of [6], $\gamma_{\text{gr}}(G \boxtimes H) \leq |V(G)| \gamma_{\text{gr}}(H)$, so with the octahedron as first factor the ratio is at most 3; Theorem 1.2 must vary both factors.

The arguments are elementary: rank bounds for matrices with prescribed support, and counting in a half graph. Counts of this kind, attached to a finite structure or a group action, also appear in the author's earlier work [3, 4, 5].

2 Unbounded product ratios

To obtain unbounded ratios we vary both factors. The factor bounds come from incidence matrices over different fields, in the manner of the intersection-congruence arguments used by Alon [1, Section 3] for the Shannon capacity; the application here needs the nonzero entries to be exactly the closed neighbourhoods, so that every legal sequence, and not only every independent set, is bounded. No result about the Shannon capacity is used.

Lemma 2.1. *Let G be a finite graph and let M be a matrix over a field $\mathbb{F}$, with rows and columns indexed by $V(G)$. Suppose that*

$$M_{uv} \neq 0 \iff v \in N_G[u].$$

Then $\gamma_{\text{gr}}(G) \leq \text{rank}_{\mathbb{F}} M$.

Proof. Let $(v_1, \dots, v_k)$ be legal and choose a footprint w_i for each v_i . The footprints are distinct, since w_j is not dominated by $v_1, \dots, v_{j-1}$ while $w_i \in N[v_i]$ for $i < j$. If $i < j$, then $w_j \notin N_G[v_i]$, so $M_{v_i w_j} = 0$, and each $M_{v_i w_i}$ is nonzero. The submatrix with rows $v_1, \dots, v_k$ and columns $w_1, \dots, w_k$, in this order, is lower triangular with nonzero diagonal, so its rank is k , and $k \leq \text{rank } M$. $\square$

The condition on M cannot be weakened to vanishing on non-edges: the footprint w_i need not equal v_i , so the entries M_{uv} with v a neighbour of u must be nonzero as well.⁴

⁴For an independent set the diagonal alone would do; this is the difference between bounding the independence number, as in the Shannon capacity arguments, and bounding the Grundy domination number.

Theorem 2.2. For $m \geq 5$, let A_m and B_m have vertex set $\binom{[m]}{5}$, the five-element subsets of $[m] = \{1, \dots, m\}$. Distinct sets S, T are adjacent in A_m when $|S \cap T|$ is odd, and in B_m when $|S \cap T| \equiv 2 \pmod{3}$. Then

$$\gamma_{\text{gr}}(A_m) \leq m, \quad \gamma_{\text{gr}}(B_m) \leq \binom{m}{2}, \quad \gamma_{\text{gr}}(A_m \boxtimes B_m) \geq \binom{m}{5}.$$

In particular,

$$\frac{\gamma_{\text{gr}}(A_m \boxtimes B_m)}{\gamma_{\text{gr}}(A_m)\gamma_{\text{gr}}(B_m)} \geq \frac{(m-2)(m-3)(m-4)}{60m}, \quad (2.1)$$

so the ratios are unbounded.

Proof. Let U be the incidence matrix between five-element subsets of $[m]$ and elements of $[m]$, with $U_{Sx} = 1$ when $x \in S$. Over $\mathbb{F}_2$,

$$(UU^T)_{ST} = |S \cap T| \pmod{2}.$$

Since $|S| = 5$ is odd, the diagonal entries are nonzero, and an off-diagonal entry is nonzero exactly when $|S \cap T|$ is odd, that is, when S and T are adjacent in A_m . So the nonzero entries of UU^T are exactly the closed neighbourhoods of A_m . Its rank is at most the number m of columns of U , and Lemma 2.1 gives the first bound.

Let V be the incidence matrix between five-element subsets and two-element subsets of $[m]$, with $V_{SP} = 1$ when $P \subseteq S$. Over $\mathbb{F}_3$,

$$(VV^T)_{ST} = \binom{|S \cap T|}{2} \pmod{3},$$

the number of two-element subsets of $S \cap T$. For intersection sizes $t = 0, 1, 2, 3, 4, 5$ the values $\binom{t}{2} = 0, 0, 1, 3, 6, 10$ reduce to $0, 0, 1, 0, 0, 1$ modulo 3. So the entry is nonzero exactly when $S = T$ or $|S \cap T| = 2$, which are precisely the closed neighbourhoods of B_m . The rank is at most $\binom{m}{2}$, proving the second bound.

The product vertices

$$\mathcal{D}_m = \{(S, S) : S \in \binom{[m]}{5}\}$$

form an independent set of $A_m \boxtimes B_m$: for $S \neq T$, adjacency of (S, S) and (T, T) would need S and T adjacent in both factors, which would require $|S \cap T|$ to be 1 or 3 and also 2. Listing the vertices of $\mathcal{D}_m$ in any order gives a legal sequence, each vertex footprinting itself, because no earlier vertex of the list dominates it. This proves the lower bound $\binom{m}{5}$ for the product.

The three bounds give (2.1), since

$$\begin{aligned} & \binom{m}{5} / \left(m \binom{m}{2} \right) \\ &= \frac{m(m-1)(m-2)(m-3)(m-4)}{120} \cdot \frac{2}{m^2(m-1)} \\ &= \frac{(m-2)(m-3)(m-4)}{60m}, \end{aligned}$$

and the right side tends to infinity with m . □

Proof of Theorem 1.2. Take $G = A_m$ and $H = B_m$ with m so large that $(m-2)(m-3)(m-4) > 60Cm$. □

For example, $m = 13$ gives $\gamma_{\text{gr}}(A_{13} \boxtimes B_{13}) \geq 1287$, whereas $\gamma_{\text{gr}}(A_{13})\gamma_{\text{gr}}(B_{13}) \leq 13 \cdot 78 = 1014$. The two factor bounds use different fields; no matrix product over a common field is involved, and nothing is claimed about the exact values of $\gamma_{\text{gr}}(A_m)$ and $\gamma_{\text{gr}}(B_m)$.

3 The octahedron and the graphs H_N

We write the vertices of the octahedron as $g_0, g_1, g_2, x_0, x_1, x_2$, where x_t is the antipode of g_t ; the three antipodal pairs are the only non-adjacent pairs (Figure 1). Thus $N[v]$ is everything except the antipode of v .

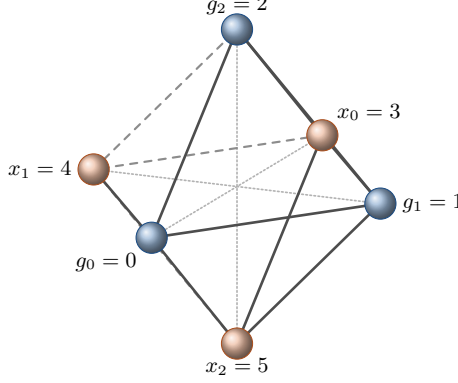


Figure 1: The octahedron $K_{2,2,2}$. Each vertex is adjacent to all others except its antipode; the dotted segments join the antipodal pairs $\{g_t, x_t\}$.

Lemma 3.1. $\gamma_{\text{gr}}(K_{2,2,2}) = 2$.

Proof. The sequence (g_0, x_0) is legal: x_0 footprints itself. Conversely, let (u, v) be legal. If v is the antipode of u , then $N[u] \cup N[v]$ is the whole vertex set; otherwise $N[u]$ misses only the antipode of u , which lies in $N[v]$. Either way nothing is left for a third vertex to footprint. $\square$

In H_N we write $a_i = 3i$, $b_i = 3i + 1$ and $c_i = 3i + 2$. Two vertices of the same residue class modulo 3 are never adjacent, and for vertices of different classes

$$a_i \sim c_j \iff j \geq i, \quad b_i \sim a_j \iff j > i, \quad c_i \sim b_j \iff j > i, \quad (3.1)$$

because $c_j - a_i = 3(j - i) + 2$, $a_j - b_i = 3(j - i) - 1$ and $b_j - c_i = 3(j - i) - 1$, and a difference $3k - 1$ with $k \leq 0$ has absolute value $3|k| + 1 \equiv 1 \pmod{3}$.

4 A long legal sequence in the product

The construction uses one property of H_N : with the colouring $v \mapsto v \bmod 3$, every edge uv with $u < v$ satisfies $u \equiv v + 1 \pmod{3}$, since $v - u \equiv 2$. We state it for any graph with this property.

Proposition 4.1. *Let H be a graph with vertices $v_0, \dots, v_{N-1}$ and a map $t: V(H) \rightarrow \mathbb{Z}_3$ such that $t(v_i) \equiv t(v_j) + 1 \pmod{3}$ for every edge $v_i v_j$ with $i < j$. Then the sequence*

$$s_j = (g_{t(v_j)}, v_j), \quad j = 0, 1, \dots, N-1,$$

is legal in $K_{2,2,2} \boxtimes H$, and $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H) \geq N$. In particular $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) \geq N$ for every $N \geq 1$, with $t(v) = v \bmod 3$.

Proof. We show that s_j footprints $f_j = (x_{t(v_j)+1}, v_j)$, the indices of g and x being taken modulo 3. First, $f_j \in N[s_j]$: the vertex $x_{t(v_j)+1}$ is not the antipode $x_{t(v_j)}$ of $g_{t(v_j)}$, so it lies in $N[g_{t(v_j)}]$, and $v_j \in N[v_j]$. Now let $i < j$. If $f_j \in N[s_i]$, then $x_{t(v_j)+1} \in N[g_{t(v_i)}]$, which means $t(v_i) \neq t(v_j) + 1$, and $v_j \in N[v_i]$, which for $i \neq j$ means that $v_i v_j$ is an edge, hence $t(v_i) = t(v_j) + 1$. The two conclusions contradict each other, so $f_j \notin N[s_i]$. $\square$

Figure 2 shows the sequence for H_{21} . Every column of the product contains exactly one vertex of the sequence, while a sequence of the form used for (1.1) would use only $\gamma_{\text{gr}}(H_{21}) = 10$ columns. The rotation of the three colours along the sequence is what lets each new vertex reach a footprint in a layer x_t that none of the earlier vertices covers in that column.

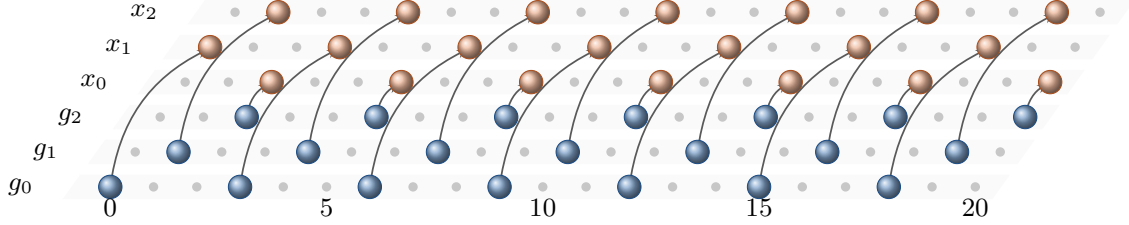


Figure 2: $K_{2,2,2} \boxtimes H_{21}$ as six stacked copies of H_{21} , one for each vertex of the octahedron. The 21 blue vertices $(g_{j \bmod 3}, j)$ form a legal sequence; the arrow from each of them ends at its footprint $(x_{(j+1) \bmod 3}, j)$.

5 The Grundy domination number of H_N

We prove the formula of Theorem 1.3 for every $N \geq 6$. A legal sequence in an induced subgraph remains legal in the larger graph, with the same footprints, since closed neighbourhoods in the subgraph are the intersections of those in the larger graph with the subgraph. We use this observation for both bounds.

Lemma 5.1. *For every integer $q \geq 1$, $\gamma_{\text{gr}}(H_{3q+1}) \leq q + 3$.*

Proof. Put $n = 3q + 1$. For $1 \leq d < n$, the conditions $d \equiv 2 \pmod{3}$ and $n - d \equiv 2 \pmod{3}$ are equivalent, because $n \equiv 1$. So u and v are adjacent in H_n exactly when $(u - v) \bmod n$ lies in the set of residues congruent to 2 modulo 3, and translation modulo n preserves adjacency.⁵ A legal sequence stays legal under a graph automorphism, so we may translate a nonempty legal sequence so that its first vertex is 0.

The vertex 0 dominates itself and the vertices $C_k = 3k + 2$, $0 \leq k < q$. The undominated vertices are

$$X_i = 3(i + 1), \quad Y_j = 3j + 1 \quad (0 \leq i, j < q).$$

There are no edges within either class, and

$$X_i \sim Y_j \iff i \geq j, \tag{5.1}$$

because $X_i - Y_j = 3(i - j) + 2$ when $i \geq j$, and $Y_j - X_i = 3(j - i - 1) + 1$ when $j > i$. Thus the undominated vertices induce a half graph with the perfect matching $X_i Y_i$ (Figure 3).

Follow the sequence from its second vertex on until either a vertex C_k is selected or the selected X - and Y -vertices contain an edge for the first time. Before either event, let I and J be the sets of selected indices on the two sides, and put $r = |I|$, $s = |J|$. By (5.1), the absence of edges means

$$i < j \quad (i \in I, j \in J). \tag{5.2}$$

We treat the three possibilities in turn.

⁵For $u > v$ the difference $u - v = d$ and the residue $(v - u) \bmod n = n - d$ are both or neither congruent to 2, so the adjacency condition $|u - v| \equiv 2$ is the same as the circulant condition.

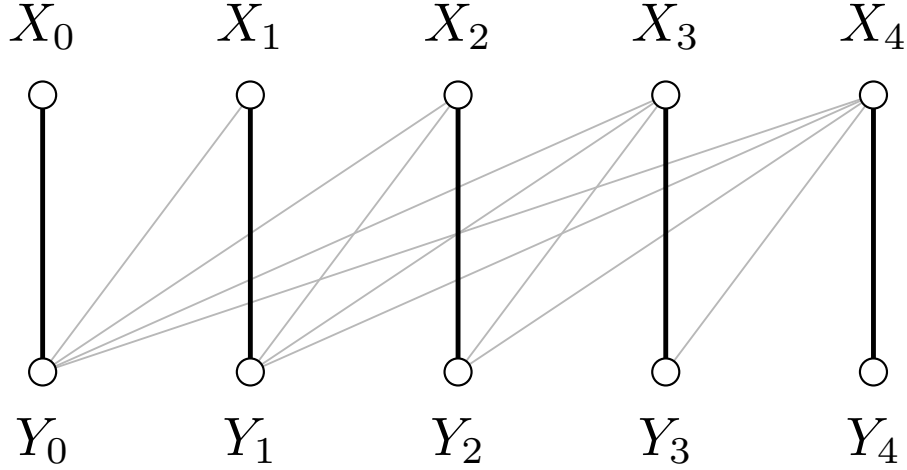


Figure 3: The undominated graph after selecting 0 in H_{3q+1} , drawn for $q = 5$. The matching edges $X_i Y_i$ are emphasised. Before the first selected edge, every selected X -index is smaller than every selected Y -index.

A new X_i creates an edge. Then $J \neq \emptyset$ and $b = \min J \leq i$. All earlier X -indices are less than b by (5.2), so i is the largest selected X -index. After this step, an X -vertex is dominated if it was selected or if it is adjacent to a selected Y_j , that is, if its index is at least b ; so the undominated X -indices are $\{0, \dots, b-1\} \setminus I$. A Y -vertex is dominated if it was selected or is adjacent to a selected X -vertex, and the selected X -vertex with the largest index is X_i ; so the undominated Y -indices are those greater than i that are not in J . Writing $s_0 = |J \cap \{b, \dots, i\}|$ for the number of selected Y -indices at most i , the number of undominated vertices is

$$(b-r) + (q-1-i) - (s-s_0).$$

The number of vertices selected after 0, which is $r+s+1$, plus the number still undominated, is

$$r+s+1+b-r+q-1-i-s+s_0 = q+b-i+s_0 \leq q+1,$$

because $s_0 \leq i-b+1$. Each later legal step dominates at least one vertex that was undominated, so the number of later steps is at most the number of undominated vertices. The full sequence, including its first vertex 0, has length at most $1+(q+1) = q+2$.

A new Y_j creates an edge. The map $X_i \mapsto Y_{q-1-i}$, $Y_j \mapsto X_{q-1-j}$ exchanges the two sides and preserves (5.1), since $q-1-j \geq q-1-i$ exactly when $i \geq j$. It carries this case to the previous one, so again the sequence has length at most $q+2$.

A vertex C_k is selected first. By (3.1), the neighbours of $C_k = c_k$ among the X - and Y -vertices are the $X_i = a_{i+1}$ with $i < k$ and the $Y_j = b_j$ with $j > k$. After this step every vertex still undominated belongs to

$$W_k = \{X_i : k \leq i < q\} \cup \{Y_j : 0 \leq j \leq k\}, \quad |W_k| = q+1.$$

At least $r+s$ vertices of W_k were dominated already before C_k was selected. To see this, map each earlier selection by

$$X_i \mapsto \begin{cases} X_i, & i \geq k, \\ Y_i, & i < k, \end{cases} \quad Y_j \mapsto \begin{cases} Y_j, & j \leq k, \\ X_j, & j > k. \end{cases}$$

Each image lies in W_k and is either the selected vertex itself or its partner in the matching, which the selected vertex dominates by (5.1). Images of distinct selections on the same side are distinct. An image shared by a selection X_i and a selection Y_j would force $i = j$, with $i \in I$ and $j \in J$, contrary to (5.2). So the map is injective, at most $q + 1 - r - s$ vertices remain undominated after C_k , and the sequence has length at most

$$1 + (r + s) + 1 + (q + 1 - r - s) = q + 3.$$

Neither event occurs. Then the vertices selected after 0 form an independent set of the half graph, which has the perfect matching $X_i Y_i$, so there are at most q of them and the sequence has length at most $q + 1$.

In every case the length is at most $q + 3$. $\square$

Lemma 5.2. *If $N \geq 6$ and $q = \lfloor (N + 1)/3 \rfloor$, then $\gamma_{\text{gr}}(H_N) \geq q + 3$.*

Proof. For $q = 2$, the sequence $(1, 3, 5, 0, 2)$ in H_6 has the successive footprints $1, 5, 0, 2, 4$ and remains legal in H_7 . For $q = 3$, the sequence $(0, 7, 2, 4, 5, 1)$ in H_8 has the successive footprints $0, 7, 4, 6, 3, 1$ and remains legal in H_9 and H_{10} .⁶

For $q \geq 4$ it suffices to work in H_{3q-1} , which is an induced subgraph of H_N because $3q - 1 \leq N$. Its vertices are a_i, b_i for $0 \leq i \leq q - 1$ and c_i for $0 \leq i \leq q - 2$. Take the sequence

$$a_0, a_1, \dots, a_{q-3}, b_{q-1}, c_{q-3}, c_0, a_{q-2}, b_0. \quad (5.3)$$

The first $q - 2$ vertices footprint themselves, since vertices of one class are never adjacent. The last five have the successive footprints

$$b_{q-1}, b_{q-2}, b_{q-3}, a_{q-2}, a_{q-1},$$

by (3.1): b_{q-1} is adjacent to no a_i with $i \leq q - 3$, so it is undominated and footprints itself; b_{q-2} is adjacent to c_{q-3} and not to any earlier vertex of the sequence; b_{q-3} is adjacent to c_0 , because $q - 3 > 0$, and not to $a_0, \dots, a_{q-3}, b_{q-1}$ or c_{q-3} ; neither a_{q-2} nor a_{q-1} is adjacent to b_{q-1}, c_{q-3} or c_0 , so a_{q-2} footprints itself and b_0 footprints a_{q-1} . The sequence has length $(q - 2) + 5 = q + 3$. $\square$

Proof of Theorem 1.3. Put $q = \lfloor (N + 1)/3 \rfloor$. Then $3q \leq N + 1$, so $N \leq 3q + 1$ and H_N is an induced subgraph of H_{3q+1} ; Lemma 5.1 gives $\gamma_{\text{gr}}(H_N) \leq q + 3$, and Lemma 5.2 gives equality. The product bound is Proposition 4.1.

At $N = 19$ the formula gives $\gamma_{\text{gr}}(H_{19}) = 9$, and $19 > 2 \cdot 9$. For $N \geq 21$ write $N = 3q - 1$, $3q$ or $3q + 1$; the differences $N - 2(q + 3)$ are $q - 7$, $q - 6$ and $q - 5$, and each is positive in its range ($q \geq 8$, $q \geq 7$ and $q \geq 7$ respectively). Hence $N > 2\gamma_{\text{gr}}(H_N) = \gamma_{\text{gr}}(K_{2,2,2})\gamma_{\text{gr}}(H_N)$ for $N = 19$ and every $N \geq 21$, while $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) \geq N$. $\square$

Example 5.3. For $N = 19$ the sequence (5.3) with $q = 6$, in $H_{17} \subseteq H_{19}$, is $(0, 3, 6, 9, 16, 11, 2, 12, 1)$, of length $9 = \gamma_{\text{gr}}(H_{19})$, and $K_{2,2,2} \boxtimes H_{19}$ has 114 vertices and a legal sequence of length 19. For $N = 21$, with $q = 7$, the sequence is $(0, 3, 6, 9, 12, 19, 14, 2, 15, 1)$, of length $10 = \gamma_{\text{gr}}(H_{21})$; this is the counterexample on 126 vertices with which the paper began. For $N = 20$ the formula gives $\gamma_{\text{gr}}(H_{20}) = 10$ and $20 = 2 \cdot 10$, so Theorem 1.3 says nothing about $N = 20$.

⁶Each footprint can be checked from (3.1). In H_6 , for instance, $1 = b_0$ dominates b_0 and $a_1 = 3$; then $3 = a_1$ dominates a_1 , $c_1 = 5$ and b_0 , so 5 is new; $5 = c_1$ dominates nothing new except itself, already dominated, so one checks instead that $c_1 \sim b_j$ needs $j > 1$, which does not exist, and $c_1 \sim a_i$ needs $i \leq 1$, both already dominated; hence the third footprint is $0 = a_0$, which c_1 dominates because $a_0 \sim c_1$. The listed footprints are $1, 5, 0, 2, 4$ in this order: the first vertex 1 footprints itself, 3 footprints 5, 5 footprints 0, 0 footprints 2, and 2 footprints 4.

Remark 5.4. Everything above is proved in full. Theorem 1.2 rests on Lemma 2.1 and the two incidence matrices; Theorem 1.3 on the half-graph count of Lemma 5.1, the explicit sequences of Lemma 5.2 and Proposition 4.1. The only results quoted from elsewhere are the inequality (1.1) and the bound $\gamma_{\text{gr}}(G \boxtimes H) \leq |V(G)|\gamma_{\text{gr}}(H)$, both from [6] with proofs of a few lines, and neither is needed for the counterexamples themselves. Nothing is claimed about the exact values of $\gamma_{\text{gr}}(A_m)$, $\gamma_{\text{gr}}(B_m)$ or $\gamma_{\text{gr}}(A_m \boxtimes B_m)$ beyond the three bounds, nor about the smallest possible counterexample.

6 A question

Question 6.1. Which graphs G satisfy Conjecture 1.1 for every H ? All forests do [2], whereas the octahedron does not.

Declarations

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Conflict of interest. None.

Data availability. https://github.com/creelie/Paper_1 (release v2.0.0, archived on Zenodo as 10.5281/zenodo.23659882) holds the source of this paper and programs that recompute $\gamma_{\text{gr}}(H_N)$ for small N and check the sequences of Section 5; the paper does not depend on them.

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