

The Grundy domination number is not multiplicative on strong products

Deep Bhattacharjee

Submitted: TBD; Accepted: TBD; Published: TBD

©The author. Released under the CC BY license (International 4.0).

Abstract

The strong product of the octahedron with a graph H on 21 vertices of Grundy domination number 10 has Grundy domination number at least 21. This disproves the conjecture of Brešar, Bujtás, Golobranec, Klavžar, Košmrlj, Patkós, Tuza and Vizer that $\gamma_{\text{gr}}(G \boxtimes H) = \gamma_{\text{gr}}(G)\gamma_{\text{gr}}(H)$ for all graphs G and H .

Mathematics Subject Classifications: 05C69, 05C76

1 Introduction

Let G be a finite simple graph and write $N[v]$ for the closed neighbourhood of a vertex v . A sequence $(v_1, \dots, v_k)$ of vertices of G is *legal* if every v_i dominates a vertex not dominated by its predecessors, that is,

$$N[v_i] \setminus \bigcup_{j < i} N[v_j] \neq \emptyset \quad (1 \leq i \leq k);$$

a vertex in this difference is a *footprint* of v_i . The vertices of a legal sequence are distinct. The *Grundy domination number* $\gamma_{\text{gr}}(G)$ is the length of a longest legal sequence; a longest legal sequence always dominates G . The parameter was introduced by Brešar, Golobranec, Milanič, Rall and Rizzi [4], who connected it with the domination game.

In the strong product $G \boxtimes H$ two distinct vertices (g, h) and (g', h') are adjacent when $g' \in N_G[g]$ and $h' \in N_H[h]$; equivalently $N_{G \boxtimes H}[(g, h)] = N_G[g] \times N_H[h]$. Taking a Grundy dominating sequence of G and one of H and listing their pairs in lexicographic order gives a legal sequence of $G \boxtimes H$, so

$$\gamma_{\text{gr}}(G \boxtimes H) \geq \gamma_{\text{gr}}(G) \gamma_{\text{gr}}(H) \quad (1)$$

for all G and H [3, Proposition 21]. In the same paper the authors conjectured that the bound is always attained.

Formerly, Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, Odisha 751030, India (d.bhattacharjee@erl-forschung.de, itsdeep@live.com; ORCID 0000-0003-0466-750X).

Conjecture 1.1 ([3, Conjecture 22]). For all graphs G and H , $\gamma_{\text{gr}}(G \boxtimes H) = \gamma_{\text{gr}}(G) \gamma_{\text{gr}}(H)$.

The conjecture holds when G is a caterpillar [3] and, more generally, when G is a forest, by the work of Bell, Driscoll, Krop and Wolff [2]. A proof for all graphs was announced by Herrman and Smith [5] and later withdrawn.¹ We show that the conjecture is false.

For $N \geq 1$ let H_N be the graph on $\{0, 1, \dots, N-1\}$ in which u and v are adjacent when $|u - v| \equiv 2 \pmod{3}$. Let $K_{2,2,2}$ be the octahedron, the complete tripartite graph with three parts of size two.

Theorem 1.2. $\gamma_{\text{gr}}(K_{2,2,2}) = 2$, $\gamma_{\text{gr}}(H_{21}) = 10$ and $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_{21}) \geq 21$. In particular

$$\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_{21}) > \gamma_{\text{gr}}(K_{2,2,2}) \gamma_{\text{gr}}(H_{21}),$$

and Conjecture 1.1 fails.

The lower bound for the product is a short explicit argument valid for every N (Proposition 3.1), and so is $\gamma_{\text{gr}}(K_{2,2,2}) = 2$ (Lemma 2.1). The upper bound $\gamma_{\text{gr}}(H_{21}) \leq 10$ is the one step that rests on a finite computation: an exhaustive search through the sets dominated by legal sequences, which we have certified in the Lean 4 proof assistant [6] (Section 4). The same search gives a family of counterexamples.

Theorem 1.3. For every $N \geq 1$, $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) \geq N$. For $6 \leq N \leq 63$,

$$\gamma_{\text{gr}}(H_N) = \left\lfloor \frac{N+1}{3} \right\rfloor + 3,$$

so $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) > \gamma_{\text{gr}}(K_{2,2,2}) \gamma_{\text{gr}}(H_N)$ for $21 \leq N \leq 63$. For $N = 63$ the ratio $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) / (\gamma_{\text{gr}}(K_{2,2,2}) \gamma_{\text{gr}}(H_N))$ is at least $63/48 = 21/16$.

The second statement of Theorem 1.3 is computational (Section 5); the first is Proposition 3.1. If the formula for $\gamma_{\text{gr}}(H_N)$ holds for all N , the ratio approaches $3/2$. By Proposition 25 of [3], $\gamma_{\text{gr}}(G \boxtimes H) \leq |V(G)| \gamma_{\text{gr}}(H)$, so for $G = K_{2,2,2}$ the ratio can never exceed 3.

2 The graphs

We write the vertices of the octahedron as $g_0, g_1, g_2, x_0, x_1, x_2$ (in this order, $0, \dots, 5$), where x_t is the antipode of g_t ; the three antipodal pairs are the only non-adjacent pairs (Figure 1). Thus $N[v]$ is everything except the antipode of v .

Lemma 2.1. $\gamma_{\text{gr}}(K_{2,2,2}) = 2$.

Proof. The sequence (g_0, x_0) is legal: x_0 footprints itself. Conversely, let (u, v) be legal. If v is the antipode of u , then $N[u] \cup N[v]$ is the whole vertex set; otherwise $N[u]$ misses only the antipode of u , which lies in $N[v]$. Either way nothing is left for a third vertex to footprint. $\square$

¹The second version of arXiv:2212.04565 (12 January 2023) carries the comment “There is an error. Some cases were not considered in the proof of Theorem 1.”

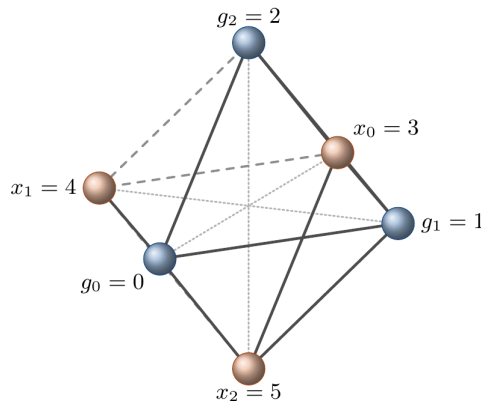


Figure 1: The octahedron $K_{2,2,2}$. Each vertex is adjacent to all others except its antipode; the dotted segments join the antipodal pairs $\{g_t, x_t\}$.

In H_N we write $a_i = 3i$, $b_i = 3i + 1$ and $c_i = 3i + 2$. Two vertices of the same residue class are never adjacent, and for vertices of different classes,

$$a_i \sim c_j \iff j \geq i, \quad b_i \sim a_j \iff j > i, \quad c_i \sim b_j \iff j > i, \quad (2)$$

because $c_j - a_i = 3(j - i) + 2$, $a_j - b_i = 3(j - i) - 1$ and $b_j - c_i = 3(j - i) - 1$. When $N = 3n$ the graph has a circular description (Figure 2). Recall that the circular clique $K_{p/q}$ has vertex set $\mathbb{Z}_p$, two vertices being adjacent when their cyclic distance is at least q .

Proposition 2.2. *Let $n \geq 2$ and place the vertices of H_{3n} on a cycle in the order*

$$a_0, \dots, a_{n-1}, c_0, \dots, c_{n-1}, b_0, \dots, b_{n-1}.$$

Two vertices are adjacent in H_{3n} if and only if their cyclic distance is at least $n + 1$, or they are a_i and c_i for some i . Thus H_{3n} is the circular clique $K_{3n/(n+1)}$ together with the perfect matching $\{a_i c_i\}$.

Proof. Give a_i , c_i and b_i the positions i , $n + i$ and $2n + i$ modulo $3n$. Vertices of the same class are at cyclic distance at most $n - 1$ and are not adjacent. For two vertices of different classes, name them so that the second follows the first in the cyclic order $A \rightarrow C \rightarrow B \rightarrow A$ of the blocks: the pairs are (a_i, c_j) , (c_i, b_j) and (b_i, a_j) . In each case the second position minus the first is $n + (j - i)$ modulo $3n$, with $|j - i| \leq n - 1$, so the cyclic distance is at least $n + 1$ when $j > i$, exactly n when $j = i$, and at most $n - 1$ when $j < i$. Comparing with (2): the pair is adjacent exactly when $j > i$, except that $a_i \sim c_i$. $\square$

3 A long legal sequence in the product

The construction uses one property of H_N : with the colouring $v \mapsto v \bmod 3$, every edge uv with $u < v$ satisfies $u \equiv v + 1 \pmod{3}$. We state it for any graph with this property.

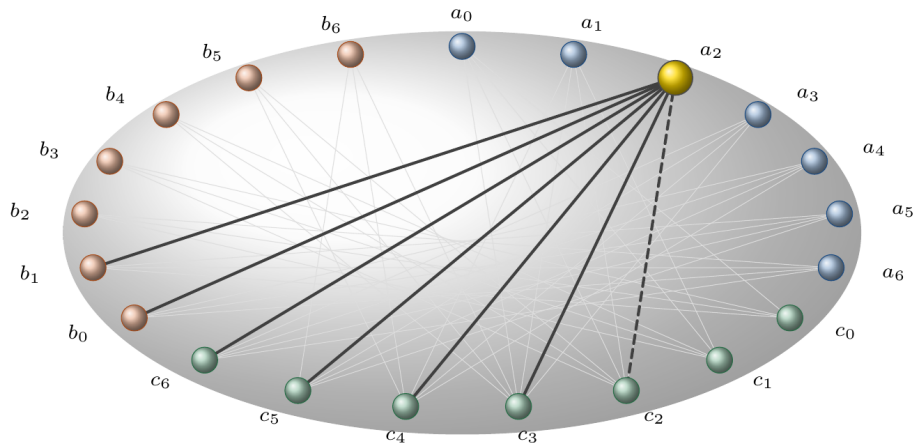


Figure 2: H_{21} on a ring in the order of Proposition 2.2; all edges are drawn faintly. The closed neighbourhood of a_2 is a_2 together with the arc $c_3, \dots, c_6, b_0, b_1$ of vertices at cyclic distance at least 8 and its partner c_2 (dashed).

Proposition 3.1. *Let H be a graph with vertices $v_0, \dots, v_{N-1}$ and a map $t: V(H) \rightarrow \mathbb{Z}_3$ such that $t(v_i) \equiv t(v_j) + 1 \pmod{3}$ for every edge $v_i v_j$ with $i < j$. Then the sequence*

$$s_j = (g_{t(v_j)}, v_j), \quad j = 0, 1, \dots, N-1,$$

is legal in $K_{2,2,2} \boxtimes H$, and $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H) \geq N$. In particular $\gamma_{\text{gr}}(K_{2,2,2} \boxtimes H_N) \geq N$ for every $N \geq 1$, with $t(v) = v \bmod 3$.

Proof. We show that s_j footprints $f_j = (x_{t(v_j)+1}, v_j)$, indices of g and x taken modulo 3. First, $f_j \in N[s_j]$: the vertex $x_{t(v_j)+1}$ is not the antipode $x_{t(v_j)}$ of $g_{t(v_j)}$, so it lies in $N[g_{t(v_j)}]$, and $v_j \in N[v_j]$. Now let $i < j$. If $f_j \in N[s_i]$, then $x_{t(v_j)+1} \in N[g_{t(v_i)}]$, which means $t(v_i) \neq t(v_j) + 1$, and $v_j \in N[v_i]$, which for $i \neq j$ means that $v_i v_j$ is an edge, hence $t(v_i) = t(v_j) + 1$. The two conclusions contradict each other, so $f_j \notin N[s_i]$. $\square$

Figure 3 shows the sequence for H_{21} . Every column of the product contains exactly one vertex of the sequence, while a Grundy dominating sequence of the form used for (1) would use only $\gamma_{\text{gr}}(H_{21}) = 10$ columns. The rotation of the three colours along the sequence is what lets each new vertex reach a footprint in a layer x_t that none of the earlier vertices covers in that column.

4 The Grundy domination number of H_{21}

Lemma 4.1. $\gamma_{\text{gr}}(H_{21}) \geq 10$.

Proof. The sequence $(a_0, a_1, a_2, a_3, a_4, b_6, c_4, c_0, a_5, b_0)$, that is $(0, 3, 6, 9, 12, 19, 14, 2, 15, 1)$, is legal and dominating. By (2), a_0 dominates $c_0, \dots, c_6$; each a_i with $1 \leq i \leq 4$ footprints itself and b_{i-1} ; then b_6 footprints itself, c_4 footprints b_5 , c_0 footprints b_4 , a_5 footprints itself and b_0 footprints a_6 . $\square$

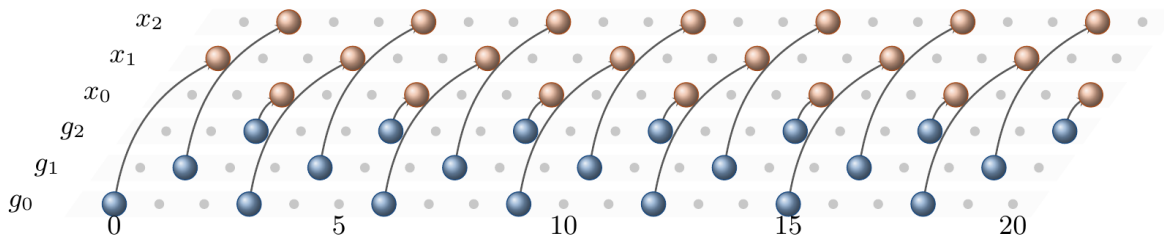


Figure 3: $K_{2,2,2} \boxtimes H_{21}$ as six stacked copies of H_{21} , one for each vertex of the octahedron. The 21 blue vertices $(g_{j \bmod 3}, j)$ form a legal sequence; the arrow from each of them ends at its footprint $(x_{(j+1) \bmod 3}, j)$.

For the upper bound we use the sets of dominated vertices. For a legal sequence $S = (v_1, \dots, v_k)$ put $D(S) = N[v_1] \cup \dots \cup N[v_k]$. Define families $\mathcal{L}_0, \mathcal{L}_1, \dots$ of vertex sets by $\mathcal{L}_0 = \{\emptyset\}$ and

$$\mathcal{L}_{k+1} = \left\{ D \cup N[v] : D \in \mathcal{L}_k, v \in V(G), N[v] \not\subseteq D \right\}.$$

Lemma 4.2. *If S is a legal sequence of length k , then $D(S) \in \mathcal{L}_k$. Hence $\gamma_{\text{gr}}(G)$ is the largest k with $\mathcal{L}_k \neq \emptyset$.*

Proof. Induction on k : if $(v_1, \dots, v_{k+1})$ is legal, then $D = D(v_1, \dots, v_k) \in \mathcal{L}_k$ and $N[v_{k+1}] \not\subseteq D$, so $D \cup N[v_{k+1}] \in \mathcal{L}_{k+1}$. Conversely, every member of $\mathcal{L}_k$ is $D(S)$ for a legal sequence S of length k , by the same induction. $\square$

The families can be large in general, but for H_{21} they are small: by (2), the vertices of a class dominated by the vertices of another class form an initial or a final segment of that class.² Their sizes are

$$|\mathcal{L}_0|, \dots, |\mathcal{L}_{10}| = 1, 21, 210, 1023, 1482, 1302, 848, 419, 150, 22, 1,$$

and $\mathcal{L}_{11} = \emptyset$; the only member of $\mathcal{L}_{10}$ is the whole vertex set. With Lemma 4.2 this gives the following.

Proposition 4.3. $\gamma_{\text{gr}}(H_{21}) \leq 10$.

We have verified Proposition 4.3 in three independent ways. In Lean 4, a certificate lists the 5479 sets of the families $\mathcal{L}_0, \dots, \mathcal{L}_{10}$ as bit masks stored in binary search trees, and the kernel checks by evaluation that $\emptyset \in \mathcal{L}_0$, that $D \cup N[v] \in \mathcal{L}_{k+1}$ for every $D \in \mathcal{L}_k$ and every v with $N[v] \not\subseteq D$, and that no member of $\mathcal{L}_{10}$ has a proper extension. A formal version of Lemma 4.2, proved in Lean for every graph and every such certificate, then bounds the length of every legal sequence of H_{21} by 10. The Lean statement uses the definitions of legal sequences, of H_{21} , of $K_{2,2,2}$ and of the strong product given in this paper, and the main theorem

²For instance, the vertices c_j dominated by the chosen vertices of the class a are those with $j \geq \min\{i : a_i \text{ chosen}\}$.

strong_product_counterexample: there is a dominating legal sequence of length 21 in $K_{2,2,2} \boxtimes H_{21}$, every legal sequence of $K_{2,2,2}$ has length at most 2, every legal sequence of H_{21} has length at most 10, and $2 \cdot 10 < 21$

depends only on the standard axioms **propext** and **Quot.sound**; no **native_decide** is used. Independently, a C program computes the families by breadth-first search, and a Python program finds $\gamma_{\text{gr}}(H_{21}) = 10$ by a memoised depth-first search over dominated sets; both use only the definition of H_{21} .

Proof of Theorem 1.2. Lemma 2.1, Lemma 4.1 with Proposition 4.3, and Proposition 3.1 with $N = 21$. $\square$

5 The family H_N

The same breadth-first search, run for $6 \leq N \leq 63$, gives the values in Table 1; for $N \leq 24$ they agree with the depth-first search. With Proposition 3.1 this proves Theorem 1.3.³ The largest search, for $N = 63$, visits 193 131 321 dominated sets and takes about five minutes on one core. The upper bound for $N = 21$ is the only one we have checked in Lean.

Table 1: $\gamma_{\text{gr}}(H_N) = \lfloor (N+1)/3 \rfloor + 3$ for $6 \leq N \leq 63$, by exhaustive search. The conjecture fails for $K_{2,2,2} \boxtimes H_N$ when $N > 2\gamma_{\text{gr}}(H_N)$, that is, for $N \geq 21$.

N	6–7	8–10	11–13	14–16	17–19	20–22	23–25	26–28	29–31	...	59–61	62–63
$\gamma_{\text{gr}}(H_N)$	5	6	7	8	9	10	11	12	13	...	23	24

Question 5.1. Is $\gamma_{\text{gr}}(H_N) = \lfloor (N+1)/3 \rfloor + 3$ for every $N \geq 6$? An affirmative answer would give $\sup_{G,H} \gamma_{\text{gr}}(G \boxtimes H) / (\gamma_{\text{gr}}(G)\gamma_{\text{gr}}(H)) \geq 3/2$.

Question 5.2. Is $\gamma_{\text{gr}}(G \boxtimes H) / (\gamma_{\text{gr}}(G)\gamma_{\text{gr}}(H))$ bounded over all graphs G and H ? Which graphs G satisfy Conjecture 1.1 for every H ? By [2] all forests do, and Theorem 1.2 shows that $K_{2,2,2}$ does not.

Remark 5.3. Deleting the matching $\{a_i c_i\}$ from H_{3n} leaves the circular clique $K_{3n/(n+1)}$ (Proposition 2.2), a circulant graph, and it still satisfies the hypothesis of Proposition 3.1. Our search gives $\gamma_{\text{gr}}(K_{3n/(n+1)}) = n + 4$ for $3 \leq n \leq 11$, one more than $\gamma_{\text{gr}}(H_{3n})$. Hence $K_{2,2,2} \boxtimes K_{27/10}$, $K_{2,2,2} \boxtimes K_{30/11}$ and $K_{2,2,2} \boxtimes K_{33/12}$ are counterexamples as well, with a vertex-transitive second factor; the matching only makes the counterexamples smaller.

Computations and formal verification

The Lean development, the C and Python programs, their outputs and the scripts that rebuild this paper are in the archive [1]. The Lean files (Lean 4.34.1, no external libraries) contain 25 theorems; all of them depend only on the standard axioms.

³Grundy domination numbers add over disjoint unions, and $G \boxtimes (H \cup H') = (G \boxtimes H) \cup (G \boxtimes H')$, so disjoint copies of H_{21} already give infinitely many counterexamples, all with ratio 21/20.

Use of generative AI

The author used Claude (Anthropic) to check the results by computer and in Lean and to help draft the text; the author verified every step and is responsible for the content.

References

- [1] D. Bhattacharjee, *The Grundy domination number is not multiplicative on strong products: Lean 4 development, programs and data*, version 1.0.0, Zenodo, 2026. [doi:10.5281/zenodo.23378021](https://doi.org/10.5281/zenodo.23378021); <https://github.com/creelie/grundy-strong-product>
- [2] K. Bell, K. Driscoll, E. Krop and K. Wolff, *Grundy domination of forests and the strong product conjecture*, Electron. J. Combin. **28**(2) (2021), #P2.12. [doi:10.37236/9507](https://doi.org/10.37236/9507)
- [3] B. Brešar, Cs. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, B. Patkós, Zs. Tuza and M. Vizer, *Dominating sequences in grid-like and toroidal graphs*, Electron. J. Combin. **23**(4) (2016), #P4.34. [doi:10.37236/6269](https://doi.org/10.37236/6269)
- [4] B. Brešar, T. Gologranc, M. Milanič, D. F. Rall and R. Rizzi, *Dominating sequences in graphs*, Discrete Math. **336** (2014), 22–36. [doi:10.1016/j.disc.2014.07.016](https://doi.org/10.1016/j.disc.2014.07.016)
- [5] R. Herrman and S. G. Z. Smith, *A proof of the Grundy domination strong product conjecture*, arXiv:2212.04565, 2022 (withdrawn). <https://arxiv.org/abs/2212.04565>
- [6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Comput. Sci., Springer, Cham, 2021, pp. 625–635. [doi:10.1007/978-3-030-79876-5_37](https://doi.org/10.1007/978-3-030-79876-5_37)