

EXPLICIT STRING VACUA AND THE GEOMETRIC SELECTION CONJECTURE

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ABSTRACT. We combine explicit spectral calculations for the Dwork and weighted Fermat pencils with an audit of the Geometric Selection Conjecture. The residue calculation determines the character labels, cubic couplings and invariant-direction mass maps, while an index argument excludes three standard-embedding families on free quotients of thirteen hypergeometric complete-intersection types. For a nonstandard construction we fix one tetraquadric equation, prove smoothness and a free degree-four action, and calculate equivariant bundle cohomology by an exact determinant certificate. Its projected charged spectrum contains three families and one Higgs doublet pair. At the specified split-bundle point, however, an additional Abelian gauge factor survives, the perturbative Yukawa matrices fail the required rank test, and the tree-level volume remains flat. We then show that the first of these failures is forced: on every Calabi–Yau threefold with at most four Kähler parameters, every polystable sum of five line bundles with vanishing first Chern class has a charge matrix of rank at most three and keeps at least one Abelian gauge boson massless at its split locus. On the tetraquadric the polystable Kähler classes of the fixed bundle form exactly the volume ray. For the second failure we decide the perturbative selection rules on eleven split data, the fixed bundle and ten further line-bundle sums with three families on the same manifold, by a flow criterion for the singlet insertions: six of the eleven lose a Yukawa matrix identically and five are not excluded. We then follow the extension branch: a single non-split extension never removes the forced gauge boson, while for the fixed bundle and five of the ten further sums a polystable sum of two rank-two extensions and a line bundle, with the same Chern character, exists on one side of a stability wall arbitrarily close to the split point and carries no massless Abelian gauge boson; for three further sums no such bundle exists at any Kähler class, and for one it does not exist on the general tetraquadric. We fix the acceptance windows for the stabilization, Yukawa and supersymmetry breaking conditions, so that the conjecture becomes a definite statement, and show that the failures found do not depend on them. No vacuum satisfying all nine conditions is constructed; the Geometric Selection Conjecture remains unproved.

Note. An earlier and incomplete version of this work appeared in two parts on Hexagon, as 2610.00446v1 (the spectral calculations on the Dwork and weighted Fermat pencils) and 2610.00493v1 (the mirror clause), with the archived versions doi:10.5281/zenodo.23567115, doi:10.5281/zenodo.23646340 and doi:10.5281/zenodo.23647986. This version adds Section 16: the residual Abelian factor is forced for every rank-five split bundle on a threefold with at most four Kähler parameters, the polystable locus of the fixed bundle is the volume ray, the perturbative selection rules are decided on eleven split data, and the extension branch of the eleven data is decided for its two smallest shapes; and Definition 9.1, which fixes the acceptance windows of the three phenomenological conditions. The programs that check the finite arithmetic are in the repository `creelie/geometric-selection-conjecture` on GitHub, release v3.1.0, concept DOI 10.5281/zenodo.23567114, version DOI 10.5281/zenodo.23688897.

1. INTRODUCTION

The physical problem has two distinct levels. One can calculate a spectrum and cubic form on a specified Calabi–Yau family. One can also ask for a single compactification satisfying

Date: 27 October 2026.

2020 Mathematics Subject Classification. 14J32; 81T30; 14J33; 32G20.

Key words and phrases. heterotic compactification; Calabi–Yau threefold; Dwork pencil; line-bundle model.

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the gauge, matter, mass, stabilization and supersymmetry requirements C1–C9. The first calculation is not a solution to the second existence problem.

This paper brings together the spectral calculations on the Dwork and weighted Fermat pencils, the mirror restriction of the selector, and the construction criteria for the conjecture, in one narrative with one bibliography. The three strands are kept apart where they must be: the Dwork mass matrix below is not substituted for a Yukawa matrix of the tetraquadric bundle later in the paper.

The fixed-candidate calculation is Theorems 13.2 and 14.2: a single equation at the parameter $\theta = \pi$ gives a smooth free quotient, with a rank-five bundle of index -3 and explicitly projected charged spectrum. The use of a transcendental coefficient certifies avoidance of algebraic bad parameters; it does not stabilize that coefficient. The line-bundle model originates in [29]. The determinant and explicit seed identities make the chosen complex structure auditable without assuming it is generic. Section 16 then shows that the two failures found at that point are not accidents of the choice: the residual Abelian factor is forced for every sum of five line bundles on a threefold with at most four Kähler parameters (Theorem 16.1), and the perturbative selection rules, decided by a flow criterion, exclude six of eleven split data on the tetraquadric and leave five (Proposition 16.4). The same section follows the extension branch away from the split locus: one non-split extension never removes the forced gauge boson (Proposition 16.8), while a polystable sum of two rank-two extensions and a line bundle does so for the fixed datum and five of the ten further data, on one side of a stability wall through the polystable point, and is excluded for the other four at every Kähler class or on the general tetraquadric (Theorem 16.10). Definition 9.1 fixes the acceptance windows of C7–C9, so that the conjecture is the definite statement $\text{GSC}(W)$ of Conjecture 9.2; the negative results do not depend on the windows. Section 17 tests the fixed datum against every condition. It records both verified statements and failures.

For comparison, consider the Dwork family

$$X_\psi = \{\sum_{i=1}^5 x_i^5 - 5\psi \prod_{i=1}^5 x_i = 0\} \subset \mathbb{P}^4, \quad \psi^5 \neq 1. \quad (1)$$

The standard-embedding E_6 theory has 101 charged **27**'s, not three Standard Model families. In the next statements “generation” denotes one such E_6 multiplet. The group of diagonal phases is $\Gamma \cong (\mathbb{Z}/5)^3$; the notation is local to these pencil sections and will later be reused for the separate tetraquadric group.

Theorem 1.1 (Labels). *The hundred and one classes carry hundred and one distinct characters of Γ , each with multiplicity one, so every generation is labelled by its character. Of the hundred and twenty-five characters of Γ the twenty-four that do not occur are exactly the classes of the permutations of $(0, 1, 2, 3, 4)$. The permutation group S_5 of the coordinates permutes the labels in five orbits of sizes 1, 20, 20, 30 and 30, the five types of generation.*

Theorem 1.2 (Couplings). *Let $\psi^5 \neq 1$. The cubic form of the pencil, normalised so that the monomial $x_1^3 \cdots x_5^3$ has value one, takes on three generations **a**, **b**, **c** the value ψ^r when $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv (3 - r)\mathbf{1} \pmod{5}$ for some $r \in \{0, 1, 2, 3\}$, and zero otherwise. There are 1431 unordered triples with a nonzero coupling, 566, 425, 280 and 160 of them with $r = 0, 1, 2, 3$, and all 1431 are nonzero at every smooth member with $\psi \neq 0$.*

Theorem 1.3 (Mass map). *The invariant class $h = x_1 x_2 x_3 x_4 x_5$, the direction of the pencil itself, pairs each generation **b** with exactly one generation $\sigma(\mathbf{b})$, where σ is an involution of the labels with one fixed point. The coupling of h , **b** and $\sigma(\mathbf{b})$ equals 1 for the fifty-one labels with all entries at most two, ψ for the thirty labels of type $(3, 1, 1, 0, 0)$ and ψ^2 for the twenty labels of type $(3, 2, 0, 0, 0)$. Hence the holomorphic mass map induced by a vacuum value of the **27** labelled by h has full rank one hundred and one for every $\psi \neq 0$, $\psi^5 \neq 1$, its determinant is ψ^{70} , and at the Fermat point $\psi = 0$ its rank is fifty-one with the fifty generations containing an exponent three massless. The spectrum consists of one singlet and fifty doublets, and the Γ -invariant metric is diagonal in the labels, so the fifty doublet masses are, in units of the vacuum value, $|\psi|^{r(\mathbf{b})} (g_{\mathbf{b}} g_{\sigma(\mathbf{b})})^{-1/2}$ with g the Hodge norms of the classes.*

The same residue and character questions extend to three weighted families of Fermat type, the pencils

$$X_\psi = \left\{ \sum_{i=1}^5 x_i^{n_i} - d\psi x_1 x_2 x_3 x_4 x_5 = 0 \right\} \subset \mathbb{P}(w), \quad n_i = \frac{d}{w_i}, \quad (2)$$

of sextics in $\mathbb{P}(1, 1, 1, 1, 2)$, octics in $\mathbb{P}(1, 1, 1, 1, 4)$ and dectics in $\mathbb{P}(1, 1, 1, 2, 5)$, whose periods and mirror maps were computed by Morrison [17], Klemm and Theisen [18] and Font [19]. Here $d = \sum_i w_i$ is 6, 8 and 10, each member with $\psi^d \delta \neq 1$, where $\delta = \prod_i w_i^{w_i}$ is 4, 256 and 12 500, is smooth, and the group Γ of diagonal phases preserving the pencil has order 108, 128 and 100. The answer is different there.

Theorem 1.4 (Weighted pencils). *Let X_ψ be the sextic, octic or dectic pencil (2) with $\psi^d \delta \neq 1$. Then X_ψ is a smooth Calabi–Yau threefold with $h^{1,1} = 1$ and $h^{2,1} = 103, 149$ and 145, and the following hold.*

- (i) *The generations are the monomials $x^{\mathbf{b}}$ of weighted degree d with $0 \leq b_i \leq n_i - 2$, but they carry only 96, 122 and 100 distinct characters of Γ : 7, 27 and 42 classes of two or three generations share a character.*
- (ii) *The cubic form, normalised so that $\prod_i x_i^{n_i-2}$ has value one, takes on three generations $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the value $\psi^r \prod_i w_i^{k_i}$ when $a_i + b_i + c_i = n_i - 2 - r + n_i k_i$ for every i , with $r \in \{0, \dots, d-1\}$ and integers $k_i \geq 0$, and zero otherwise; for $\psi \neq 0$ every coupling allowed by Γ is nonzero.*
- (iii) *For $\psi \neq 0$ the mass map of the invariant direction has rank 99, 125 and 109, so that 4, 24 and 36 generations stay massless at every member; at $\psi = 0$ its rank is 44, 85 and 72.*
- (iv) *No member, and no quotient of a member by a group acting freely, carries three families with the standard embedding.*

These results are proved in Sections 2–8 and their appendices. They concern specified holomorphic maps and their stated metric normalization, not a completed phenomenological vacuum. The global family-index obstruction is strengthened in Section 12. The GSC construction problem is formulated in Section 9, and the one-candidate audit follows.

2. THE DICTIONARY

Throughout, X is a compact Kähler threefold with holonomy exactly $SU(3)$, so that X has a Ricci-flat Kähler metric g by Yau’s theorem, a nowhere vanishing holomorphic three-form Ω , and $h^{1,0} = h^{2,0} = 0$. The tangent bundle T_X with the Levi-Civita connection is a holomorphic $SU(3)$ -bundle with a Hermitian Yang–Mills connection; this is the standard embedding.

2.1. Spinors and the Dolbeault complex. On a Kähler manifold of complex dimension n with trivial canonical bundle the spinor bundle is $S = \bigoplus_q \Lambda^{0,q} T_X^*$, and the Dirac operator of the Levi-Civita connection is $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$, with the chirality grading given by the parity of q (Hitchin [10]). If V is a holomorphic Hermitian bundle with a connection whose curvature is of type $(1, 1)$, the Dirac operator twisted by V is $\sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*)$ on $\Lambda^{0,\bullet} \otimes V$, and its kernel is the harmonic representative of $\bigoplus_q H^{0,q}(X, V)$. This is the only input from spin geometry.

2.2. The charged spectrum.

Theorem 2.1 (Spectrum of the standard embedding). *Let X be as above with the standard embedding. The massless charged chiral multiplets of the $E_8 \times E_8$ heterotic string on $\mathbb{R}^{3,1} \times X$ in the observable sector transform under E_6 as*

$$h^{2,1}(X) \text{ copies of } \mathbf{27} \oplus h^{1,1}(X) \text{ copies of } \overline{\mathbf{27}},$$

with the $\mathbf{27}$ ’s in natural bijection with $H^1(X, T_X) \cong H^{2,1}(X)$ and the $\overline{\mathbf{27}}$ ’s with $H^1(X, T_X^*) \cong H^{1,1}(X)$. The net number of generations is $|h^{2,1} - h^{1,1}| = \frac{1}{2}|\chi(X)|$.

Proof. The ten-dimensional gaugino is a Majorana–Weyl spinor with values in the adjoint of E_8 . Under $E_8 \supset E_6 \times SU(3)$,

$$248 = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \overline{\mathbf{3}}),$$

and the standard embedding identifies the $SU(3)$ factor with the holonomy group acting on T_X and sends $\mathbf{3}$ to T_X , $\overline{\mathbf{3}}$ to T_X^* . A ten-dimensional Weyl spinor on $\mathbb{R}^{3,1} \times X$ decomposes as a four-dimensional Weyl spinor times a six-dimensional one of correlated chirality, so the four-dimensional massless chiral fermions in the $\mathbf{27}$ are the zero modes of the Dirac operator on X twisted by T_X , counted with their six-dimensional chirality; the $SU(3)$ holonomy makes the twisted operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ on $\Lambda^{0,\bullet} \otimes T_X$, and its chiral zero modes are $H^{0,1}(X, T_X) \oplus H^{0,3}(X, T_X)$ and $H^{0,0}(X, T_X) \oplus H^{0,2}(X, T_X)$ for the two chiralities. Since Ω trivialises $\Lambda^3 T_X^*$, contraction with Ω gives $H^{0,q}(X, T_X) \cong H^{2,q}(X)$ and $H^{0,q}(X, T_X^*) = H^{1,q}(X)$. With $h^{2,0} = h^{1,0} = 0$ and Serre duality, $H^{0,0}(T_X) = H^{2,0} = 0$, $H^{0,3}(T_X) = H^{2,3} = \overline{H^{1,0}}^* = 0$, $H^{0,2}(T_X) = H^{2,2} \cong \overline{H^{1,1}}$, and $H^{0,1}(T_X) = H^{2,1}$. A four-dimensional Weyl fermion in the $\mathbf{27}$ of one chirality and one in the $\overline{\mathbf{27}}$ of the opposite chirality are related by conjugation, so the chiral multiplets of definite four-dimensional chirality are $h^{2,1}$ copies of $\mathbf{27}$ from $H^{0,1}(T_X)$ and $h^{1,1}$ copies of $\overline{\mathbf{27}}$ from $H^{0,2}(T_X)$. The index of the twisted Dirac operator is $h^{2,1} - h^{1,1}$, which equals $-\chi(X)/2$ because $\chi = 2(h^{1,1} - h^{2,1})$ on a threefold with $h^{1,0} = h^{2,0} = 0$. $\square$

2.3. The cubic form. Let $V = H^1(X, T_X)$. The composition of cup product and contraction,

$$V \otimes V \otimes V \rightarrow H^3(X, \Lambda^3 T_X) = H^3(X, K_X^{-1}) \cong H^3(X, \mathcal{O}_X) \cong \mathbb{C},$$

where the last two isomorphisms use Ω , is a symmetric trilinear form Y on V , the cubic form of X . In terms of the variation of Hodge structure, with ∇ the Gauss–Manin connection and Ω extended holomorphically over the moduli space,

$$Y(a, b, c) = \int_X \Omega \wedge \nabla_a \nabla_b \nabla_c \Omega, \quad (3)$$

and this is the coupling of three $\mathbf{27}$'s [3]. Changing Ω to $f\Omega$ multiplies Y by f^2 , so only the cubic form up to a nowhere vanishing holomorphic function is intrinsic, and the physical couplings are Y divided by the square roots of the Hodge norms of the three classes and by $\langle \Omega, \overline{\Omega} \rangle$ [6, 7].

2.4. Mass terms. A vacuum value v of the scalar in the $\mathbf{27}$ labelled by a class $h \in V$ in a direction that breaks E_6 gives, through the superpotential $Y(h, \cdot, \cdot)$, a Dirac mass term $v Y(h, b, c)$ between the fermions labelled by b and c [2]. The physical mass matrix is

$$M_{bc} = v \frac{Y(h, b, c)}{\sqrt{g_b g_c}} \quad (4)$$

in an orthogonal basis of V , where $g_b = \langle \omega_b, \overline{\omega_b} \rangle$ is the Hodge norm of the harmonic representative, and the masses are its singular values. We call $Y(h, \cdot, \cdot)$ the mass map of h .

2.5. Hypersurfaces. For a smooth hypersurface $X = \{F = 0\} \subset \mathbb{P}^4$ of degree five let $J_F = (\partial_1 F, \dots, \partial_5 F)$ be the Jacobian ideal and $R = \mathbb{C}[x_1, \dots, x_5]/J_F$ the Jacobian ring, graded by degree. Griffiths' theorem [8] identifies $H_{\text{prim}}^{3-q,q}(X)$ with $R_{5(q+1)-5}$ through residues: for a homogeneous P of degree $5q$,

$$\omega_P = \text{Res}_X \frac{P \Omega_0}{F^{q+1}}, \quad \Omega_0 = \sum_i (-1)^i x_i dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_5,$$

so $H^{3,0} = R_0$, $H^{2,1} = R_5$, $H^{1,2} = R_{10}$, $H^{0,3} = R_{15}$, and every class of a quintic threefold is primitive in degree three. The ring R is a graded complete intersection with Hilbert series $(1 + t + t^2 + t^3)^5$, so $\dim R_5 = 101$, $\dim R_{15} = 1$, and R_{15} is the socle. Macaulay's theorem, as stated in [9], says that the Hessian $\text{Hess } F = \det(\partial_i \partial_j F)$ has nonzero image in R_{15} and

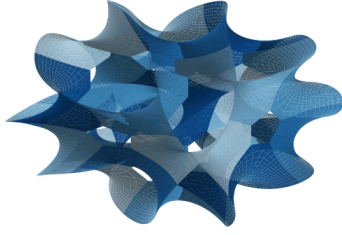
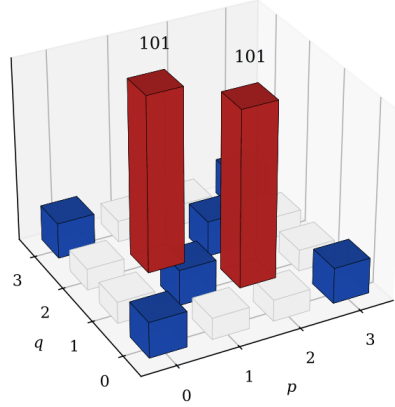
the curve $z_1^5 + z_2^5 = 1$ on the Fermat quinticHodge numbers $h^{p,q}$ (height $\sim \log$)

FIGURE 1. Left: the intersection of the Fermat quintic with the plane $x_3 = x_4 = 0$, which in an affine chart is the curve $z_1^5 + z_2^5 = 1$ in $\mathbb{C}^2$, drawn with the parametrisation of Hanson [11] and projected to three dimensions along $(\operatorname{Re} z_1, \operatorname{Re} z_2, \cos \alpha \operatorname{Im} z_1 + \sin \alpha \operatorname{Im} z_2)$. It is assembled from twenty-five patches, one for each pair of fifth roots of unity; Γ preserves the curve and acts on it through $\mu_5 \times \mu_5$, permuting the patches. Right: the Hodge numbers $h^{p,q}$ of a smooth quintic threefold, with height proportional to $1 + \log_2 h^{p,q}$; the six blue bars have height one, the two red bars are $h^{2,1} = h^{1,2} = 101$, and the grey cells are zero.

that the cubic form is multiplication in R followed by the residue functional $\lambda_F : R_{15} \rightarrow \mathbb{C}$, $\lambda_F(\operatorname{Hess} F) = 1$, up to a constant depending only on the normalisation of Ω :

$$Y(\omega_P, \omega_Q, \omega_S) = c \lambda_F(PQS). \quad (5)$$

Figure 1 shows a curve on the Fermat quintic and the Hodge numbers of a quintic threefold: by Theorem 2.1 the hundred and one classes of type $(2, 1)$ are the generations and the single class of type $(1, 1)$ is the antigration.

3. LABELS

Let $\zeta = e^{2\pi i/5}$ and let $K = \{a \in (\mathbb{Z}/5)^5 : \sum a_i = 0\}$ act on $\mathbb{P}^4$ by $x_i \mapsto \zeta^{a_i} x_i$. The diagonal subgroup D acts trivially on $\mathbb{P}^4$, and $\Gamma = K/D \cong (\mathbb{Z}/5)^3$ preserves $F_\psi = \sum x_i^5 - 5\psi \prod x_i$ for every ψ . The characters of Γ are the characters of K trivial on D , that is the classes

$$\hat{\Gamma} = \{\mathbf{b} \in (\mathbb{Z}/5)^5 : \sum b_i = 0\} / \langle \mathbf{1} \rangle, \quad |\hat{\Gamma}| = 125,$$

with $\mathbf{b}$ acting by $a \mapsto \zeta^{a \cdot \mathbf{b}}$. A monomial $x^{\mathbf{b}}$ transforms under $a \in K$ by $\zeta^{a \cdot \mathbf{b}}$, so it lies in the isotypic component of the class of $\mathbf{b}$.

Let B be the set of $\mathbf{b} \in \{0, 1, 2, 3\}^5$ with $|\mathbf{b}| = \sum b_i = 5$. The monomials $x^{\mathbf{b}}$, $\mathbf{b} \in B$, are a basis of R_5 for $\psi^5 \neq 1$, because the Jacobian ideal of F_ψ is generated by $x_i^4 - \psi \prod_{j \neq i} x_j$, and modulo these every monomial reduces to one with all exponents at most three; the reduced monomials of degree five are the 101 elements of B , which matches $\dim R_5$. The count is the number of compositions of five into five parts, $\binom{9}{4} = 126$, less the twenty with a part equal to four and the five with a part equal to five.

Proof of Theorem 1.1. Two elements $\mathbf{b}, \mathbf{b}' \in B$ define the same character when $\mathbf{b}' \equiv \mathbf{b} + k\mathbf{1} \pmod{5}$ for some k . Since the entries of $\mathbf{b}'$ lie in $\{0, \dots, 4\}$, $\mathbf{b}'$ is the vector of residues of $\mathbf{b} + k\mathbf{1}$ in that range: each b'_i is $b_i + k$ or $b_i + k - 5$, and $\sum b'_i = 5$. For $k = 1$ the residues are $b_i + 1 \leq 4$ unless $b_i = 4$, impossible, so all residues are $b_i + 1$, with sum $10 \neq 5$. For $k = 2$ the residues are $b_i + 2$ if $b_i \leq 2$ and $b_i - 3$ if $b_i = 3$; the sum is $15 - 5m$ where m is the number

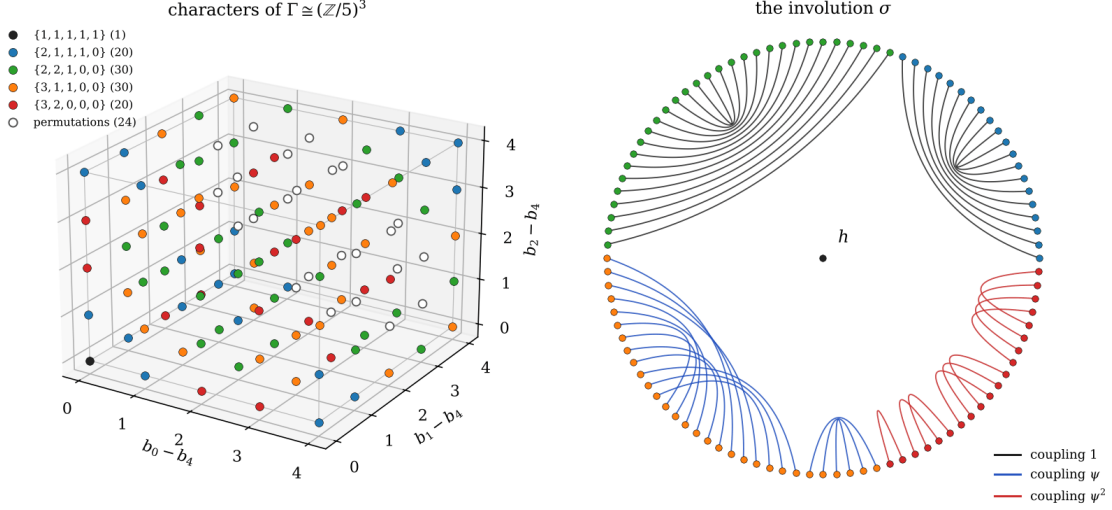


FIGURE 2. The hundred and one generations of the Dwork quintic as points of the character group of Γ . Left: $\widehat{\Gamma} \cong (\mathbb{Z}/5)^3$ drawn as a cube of 125 points, the class of $\mathbf{b}$ placed at $(b_0 - b_4, b_1 - b_4, b_2 - b_4) \bmod 5$; the 101 labels are filled and coloured by type (the five S_5 -orbits), and the 24 classes of permutations of $(0, 1, 2, 3, 4)$ are hollow. Right: the involution σ of Theorem 1.3, drawn as chords between paired labels, with the chord colour giving the power of ψ of the pairing: black for 1, blue for ψ , red for ψ^2 ; the fixed label is the invariant class h at the centre.

of threes, so $m = 2$, but two threes already sum to six. For $k = 3$ the residues are $b_i + 3$ if $b_i \leq 1$ and $b_i - 2$ if $b_i \geq 2$, sum $20 - 5m'$ with m' the number of entries at least two, so $m' = 3$, three entries at least two summing to at most five, impossible. For $k = 4$ the residues are $b_i + 4$ if $b_i = 0$ and $b_i - 1$ otherwise, sum $25 - 5m''$ with m'' the number of nonzero entries, so $m'' = 4$: four nonzero entries summing to five means $(2, 1, 1, 1, 0)$ up to order, and then $\mathbf{b}' = (1, 0, 0, 0, 4)$ has a four. So only $k = 0$ works and the 101 characters are distinct; being characters of an abelian group acting on R_5 , they are then the full decomposition, each isotypic component being one-dimensional.

The group $\widehat{\Gamma}$ has 125 elements and B realises 101 of them, so exactly 24 are missed. A permutation of $(0, 1, 2, 3, 4)$ has sum ten, so its class lies in $\widehat{\Gamma}$, and adding $\mathbf{1}$ to it gives another permutation of $(0, 1, 2, 3, 4)$ modulo five; the hundred and twenty permutations therefore fall into 24 classes of five. None of these classes meets B , because every representative with entries in $\{0, \dots, 4\}$ is a permutation of $(0, 1, 2, 3, 4)$ and contains a four. Hence the 24 missed classes are exactly the classes of the permutations.

Finally S_5 permutes the exponent vectors, and the orbits in B are the multisets of entries: $\{1, 1, 1, 1, 1\}$ (one element), $\{2, 1, 1, 1, 0\}$ (twenty), $\{3, 2, 0, 0, 0\}$ (twenty), $\{2, 2, 1, 0, 0\}$ (thirty) and $\{3, 1, 1, 0, 0\}$ (thirty). $\square$

Figure 2 shows the hundred and one labels.

4. THE SOCLE FUNCTIONAL

Fix ψ with $\psi^5 \neq 1$, write $R = R_{F_\psi}$ and $m_0 = x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$, the only monomial of degree fifteen with all exponents at most three. Define the sector of a monomial $x^{\mathbf{e}}$ of degree fifteen as follows: $\mathbf{e}$ is in sector $r \in \{0, 1, 2, 3, 4\}$ if $e_i \equiv 3 - r \pmod{5}$ for every i ; otherwise $\mathbf{e}$ is in no sector. Sector 4 is empty, because its exponents are at least four each.

Theorem 4.1 (Socle functional). *For $\psi^5 \neq 1$, $m_0 \neq 0$ in R_{15} , and the linear functional λ on R_{15} with $\lambda(m_0) = 1$ satisfies, for every monomial $x^{\mathbf{e}}$ of degree fifteen,*

$$\lambda(x^{\mathbf{e}}) = \begin{cases} \psi^r & \text{if } \mathbf{e} \text{ is in sector } r, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $\lambda(\text{Hess } F_\psi) = 3\,200\,000(1 - \psi^5)$.

Proof. The relations $x_i^4 \equiv \psi \prod_{j \neq i} x_j$ in R give, for every exponent vector $\mathbf{e}$ with $e_i \geq 4$,

$$x^{\mathbf{e}} \equiv \psi x^{\mathbf{e}'}, \quad \mathbf{e}' = \mathbf{e} + \mathbf{1} - 5\mathbf{e}_i, \quad (6)$$

where $\mathbf{e}_i$ is the i -th unit vector. If $\mathbf{e}$ is in sector r then $\mathbf{e}'$ is in sector $r - 1$: its i -th entry is $e_i - 4 \equiv 3 - (r - 1)$ and the other entries are $e_j + 1 \equiv 3 - (r - 1)$. The sectors are listed by hand. In sector 0 the entries are congruent to 3 and sum to fifteen, so they are all equal to three and $\mathbf{e} = (3, 3, 3, 3, 3)$. In sector 1 the entries are congruent to two, so each is 2, 7 or 12; a twelve would leave three for four entries of at least two, so the entries are 2 or 7, and a total of fifteen forces exactly one seven: $\mathbf{e}$ is a permutation of $(7, 2, 2, 2, 2)$, and (6) applied to the seven gives ψm_0 . In sector 2 the entries are 1, 6 or 11, and the total forces two sixes and three ones (an eleven would leave four for the other four entries, which are congruent to one and so are all one, a total of fifteen; but then $\mathbf{e} = (11, 1, 1, 1, 1)$, and (6) applied to the eleven gives $(7, 2, 2, 2, 2)$, in sector 1 as well). In sector 3 the entries are multiples of five, so $\mathbf{e}$ is a permutation of $(15, 0, 0, 0, 0)$, $(10, 5, 0, 0, 0)$ or $(5, 5, 5, 0, 0)$, and one reduction at an entry of at least five lands in sector 2. By induction on r , every monomial in sector r satisfies $x^{\mathbf{e}} \equiv \psi^r m_0$ in R .¹

If $\mathbf{e}$ is in no sector, the monomial $x^{\mathbf{e}}$ has a nontrivial character of Γ : its character is the class of $\mathbf{e}$ modulo $\mathbf{1}$, which is trivial exactly when $\mathbf{e} \equiv c\mathbf{1}$ for some c , that is, when $\mathbf{e}$ is in a sector. The ideal J_{F_ψ} is Γ -stable, so R is graded by the characters of Γ and $x^{\mathbf{e}}$ lies in a nontrivial isotypic component of R_{15} . Since R_{15} is one-dimensional it carries a single character, and m_0 carries the trivial one; so if $m_0 \neq 0$, every nontrivial component vanishes and $x^{\mathbf{e}} \equiv 0$.

It remains to show that $m_0 \neq 0$. The Hessian $\text{Hess } F_\psi$ has degree fifteen and is Γ -invariant, so its monomials in no sector vanish in R and the others are multiples of m_0 : $\text{Hess } F_\psi \equiv H(\psi) m_0$, with $H(\psi)$ the sum over the sector monomials of the Hessian of their coefficients times ψ^r . By Macaulay's theorem $\text{Hess } F_\psi \neq 0$ in R_{15} , so $m_0 \neq 0$ and $H(\psi) \neq 0$. The Hessian matrix has diagonal entries $20x_i^3$ and off-diagonal entries $-5\psi \prod_{k \neq i, j} x_k$, each of degree three; an off-diagonal entry omits exactly the two coordinates of its row and column. The determinant is the sum over permutations π of the sign of π times $\prod_i (\text{entry } (i, \pi(i)))$, and the monomial of π depends only on its cycle type: a fixed point a contributes x_a^3 from the diagonal and x_a from every off-diagonal factor, and a moved coordinate m contributes x_m from every off-diagonal factor whose row and column both differ from m . Table 1 lists the seven cycle types; the derivation of each row is in Appendix A. Summing the last column,

$$\begin{aligned} H(\psi) &= 3\,200\,000 - (2\,000\,000 + 1\,000\,000 + 375\,000 \\ &\quad + 75\,000 - 187\,500 - 62\,500)\psi^5 \\ &= 3\,200\,000(1 - \psi^5). \end{aligned}$$

□

Remark 4.2 (Normalisation). With the residue normalisation $\lambda_F(\text{Hess } F) = 1$ of (5), the cubic form of the pencil is $\lambda_{F_\psi} = \lambda/(3\,200\,000(1 - \psi^5))$. For $\mathbf{a} = \mathbf{b} = \mathbf{c} = h$ the monomial is $x^{\mathbf{1}} = m_0$, so $Y(h, h, h)$ is a constant times $1/(1 - \psi^5)$, which is the coupling $5/(1 - \psi^5)$ of [5] in the gauge in which Ω is the residue of Ω_0/F_ψ ; the factor five is the constant c of

¹The reduction (6) uses only the relation $\partial_i F_\psi / 5 = x_i^4 - \psi \prod_{j \neq i} x_j$. It lowers the i -th exponent by four and raises each of the other four by one, so it preserves the degree and moves the class of $\mathbf{e}$ modulo $\mathbf{1}$ by $-5\mathbf{e}_i + \mathbf{1} \equiv \mathbf{1}$; this is why the sectors, which are the classes $(3 - r)\mathbf{1}$, are permuted cyclically and why each reduction costs exactly one factor of ψ .

TABLE 1. The Hessian of F_ψ in the socle, by cycle type of the permutation. Each row gives the number of permutations, their sign, the monomial they contribute, its sector r , and the total contribution to $H(\psi)$ after the reduction $x^{\mathbf{e}} \equiv \psi^r m_0$.

type	count	sign	monomial	r	contribution
1^5	1	+	$(3, 3, 3, 3, 3)$	0	3 200 000
$2 1^3$	10	−	$(5, 5, 5, 0, 0)$	3	$-2\,000\,000 \psi^5$
$3 1^2$	20	+	$(6, 6, 1, 1, 1)$	2	$-1\,000\,000 \psi^5$
$4 1$	30	−	$(7, 2, 2, 2, 2)$	1	$-375\,000 \psi^5$
5	24	+	$(3, 3, 3, 3, 3)$	0	$-75\,000 \psi^5$
$2^2 1$	15	+	$(7, 2, 2, 2, 2)$	1	$187\,500 \psi^5$
$3 2$	20	−	$(3, 3, 3, 3, 3)$	0	$62\,500 \psi^5$

(5) in that gauge. All ratios of couplings at fixed ψ are independent of these constants, and Theorem 1.2 is stated for λ .

5. COUPLINGS

Proof of Theorem 1.2. By (5) and Theorem 4.1, $Y(x^{\mathbf{a}}, x^{\mathbf{b}}, x^{\mathbf{c}}) = \lambda(x^{\mathbf{a}+\mathbf{b}+\mathbf{c}})$ up to the normalisation of Remark 4.2, and $\mathbf{a} + \mathbf{b} + \mathbf{c}$ has degree fifteen; the value is ψ^r if $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv (3-r)\mathbf{1}$ and zero otherwise. For $\psi \neq 0$ every sector value is nonzero. The count of the unordered triples is in Section 5.1. $\square$

5.1. The count. A triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ of elements of B , with repetitions allowed, has a nonzero coupling exactly when $\mathbf{a} + \mathbf{b} + \mathbf{c} \equiv c\mathbf{1} \pmod{5}$ for some c , and then $c \equiv 3-r$ with r the sector. We count ordered triples first. The congruence says that the character of $\mathbf{c}$ is the inverse of the character of $\mathbf{a} + \mathbf{b}$ in $\widehat{\Gamma}$. Every character occurs in B at most once (Theorem 1.1), so for each ordered pair $(\mathbf{a}, \mathbf{b})$ there is at most one $\mathbf{c}$, and there is one exactly when the class of $-(\mathbf{a} + \mathbf{b})$ is one of the hundred and one realised classes, that is, when it is not the class of a permutation of $(0, 1, 2, 3, 4)$. The negative of a permutation of $(0, 1, 2, 3, 4)$ is, modulo five, a permutation of $(0, 1, 2, 3, 4)$ again, so this fails exactly when $\mathbf{a} + \mathbf{b}$ has five distinct residues modulo five.

Excluded pairs. The entries of $\mathbf{a} + \mathbf{b}$ lie in $\{0, \dots, 6\}$ and sum to ten. Five distinct residues are the residues $0, 1, 2, 3, 4$, whose sum is already ten, so no entry can exceed four and $\mathbf{a} + \mathbf{b}$ is a permutation of $(0, 1, 2, 3, 4)$. Fix the permutation, so that the coordinate k of $\mathbf{a} + \mathbf{b}$ equals k for $k = 0, \dots, 4$ after renaming. Write p_k for the part given to $\mathbf{a}$ in coordinate k ; then $0 \leq p_k \leq \min(k, 3)$ and $k - p_k \leq 3$, so $p_0 = 0$, $p_1 \in \{0, 1\}$, $p_2 \in \{0, 1, 2\}$, $p_3 \in \{0, 1, 2, 3\}$ and $p_4 \in \{1, 2, 3\}$, and $|\mathbf{a}| = 5$ means $p_1 + p_2 + p_3 + p_4 = 5$. The number of solutions is the coefficient of t^5 in

$$(1+t)(1+t+t^2)(1+t+t^2+t^3)(t+t^2+t^3),$$

which is 16 .² With 120 permutations there are 1920 excluded ordered pairs, so the number of ordered triples with a nonzero coupling is $101^2 - 1920 = 8281$.

Repetitions. An ordered triple $(\mathbf{a}, \mathbf{a}, \mathbf{a})$ has a nonzero coupling when $3\mathbf{a} \equiv c\mathbf{1}$, that is, when all entries of $\mathbf{a}$ are congruent modulo five; entries in $\{0, 1, 2, 3\}$ summing to five then force $\mathbf{a} = \mathbf{1}$. For a triple $(\mathbf{a}, \mathbf{a}, \mathbf{c})$ the partner $\mathbf{c}$ exists unless $2\mathbf{a}$ has five distinct residues; the residues of $2a_i$ for $a_i = 0, 1, 2, 3$ are $0, 2, 4, 1$, and five distinct residues would need five distinct entries of $\mathbf{a}$ in a set of four values, which is impossible. So every $\mathbf{a}$ has a partner $\mathbf{c}$, and $\mathbf{c} = \mathbf{a}$ only for $\mathbf{a} = \mathbf{1}$. There are therefore 100 unordered triples of the form $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$

²Expand $(1+t)(1+t+t^2) = 1+2t+2t^2+t^3$ and $(1+t+t^2+t^3)(t+t^2+t^3) = t+2t^2+3t^3+3t^4+2t^5+t^6$. The coefficient of t^5 in the product is $1 \cdot 2 + 2 \cdot 3 + 2 \cdot 3 + 1 \cdot 2 = 16$, pairing the terms of degree 0, 1, 2, 3 of the first factor with those of degree 5, 4, 3, 2 of the second.

with $\mathbf{c} \neq \mathbf{a}$, giving 300 ordered triples, one triple $\{\mathbf{1}, \mathbf{1}, \mathbf{1}\}$, and $8281 - 300 - 1 = 7980$ ordered triples with three distinct entries. The number of unordered triples is

$$\frac{7980}{6} + 100 + 1 = 1330 + 101 = 1431.$$

Sectors. In sector r the sum $\mathbf{e} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ is one of the vectors listed in the proof of Theorem 4.1 with entries at most nine: $(3, 3, 3, 3, 3)$ for $r = 0$, the five permutations of $(7, 2, 2, 2, 2)$ for $r = 1$, the ten permutations of $(6, 6, 1, 1, 1)$ for $r = 2$ and the ten permutations of $(5, 5, 5, 0, 0)$ for $r = 3$. The number $O(\mathbf{e})$ of ordered triples with sum $\mathbf{e}$ is the coefficient of $x^5 y^5$ in $\prod_{i=1}^5 P_{e_i}(x, y)$, where $P_e(x, y) = \sum x^a y^b$ over the pairs $0 \leq a, b \leq 3$ with $0 \leq e - a - b \leq 3$. These coefficients are

$$\begin{aligned} O(3, 3, 3, 3, 3) &= 3391, & O(7, 2, 2, 2, 2) &= 480, \\ O(6, 6, 1, 1, 1) &= 162, & O(5, 5, 5, 0, 0) &= 87, \end{aligned}$$

so the ordered counts by sector are 3391, 2400, 1620 and 870, with total 8281. The hundred unordered triples $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$ with $\mathbf{c} \neq \mathbf{a}$ split as 0, 50, 20 and 30 by sector (the sector of $2\mathbf{a} + \mathbf{c}$), and $\{\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ lies in sector 0, so the unordered counts are

$$N_r = \frac{O_r - 3D_r - T_r}{6} + D_r + T_r = 566, 425, 280, 160,$$

with O_r, D_r, T_r the ordered, repeated and constant counts of sector r . The coefficient extractions are written out in Appendix B.

6. THE MASS MAP

Proof of Theorem 1.3. For $\mathbf{b} \in B$ the coupling $\lambda(x^{\mathbf{b}+\mathbf{c}+\mathbf{1}})$ is nonzero exactly when $\mathbf{b} + \mathbf{c} + \mathbf{1} \equiv (3-r)\mathbf{1}$, that is $\mathbf{c} \equiv (2-r)\mathbf{1} - \mathbf{b} \pmod{5}$, with $\mathbf{c} \in B$. For each r the residues of $(2-r)\mathbf{1} - \mathbf{b}$ are determined, and $\mathbf{c}$ exists in B exactly when those residues, taken in $\{0, 1, 2, 3\}$ (a residue 4 excludes), sum to five. Go through the types. If all $b_i \leq 2$ then $r = 0$ gives $\mathbf{c} = 2\mathbf{1} - \mathbf{b} \in \{0, 1, 2\}^5$ of sum five: this is the partner. For $r = 1$ the residues of $\mathbf{1} - \mathbf{b}$ are 1, 0, 4 for $b_i = 0, 1, 2$, so a two in $\mathbf{b}$ excludes, and $\mathbf{b} \in \{0, 1\}^5$ of sum five means $\mathbf{b} = \mathbf{1}$, whose $r = 1$ residues are all zero, sum zero: excluded. For $r = 2$ the residues of $-\mathbf{b}$ are 0, 4, 3: a one excludes, and $\mathbf{b} \in \{0, 2\}^5$ has even sum: excluded. For $r = 3$ the residues of $-\mathbf{1} - \mathbf{b}$ are 4, 3, 2: a zero excludes, so $\mathbf{b} \in \{1, 2\}^5$ of sum five means $\mathbf{b} = \mathbf{1}$, residues all 3, sum fifteen: excluded. So a label with entries at most two has exactly one partner, $2\mathbf{1} - \mathbf{b}$, with coupling 1; this covers the fifty-one labels of the types $\{1, 1, 1, 1, 1\}$, $\{2, 1, 1, 1, 0\}$ and $\{2, 2, 1, 0, 0\}$, and $\sigma(\mathbf{b}) = 2\mathbf{1} - \mathbf{b}$ is of the same type, with the single fixed point $\mathbf{b} = \mathbf{1}$.

If $\mathbf{b}$ has type $\{3, 1, 1, 0, 0\}$: $r = 0$ gives residue 4 at the three, excluded; $r = 1$ gives residues 3, 0, 0, 1, 1, sum five, so $\mathbf{c}$ is the label with 3 where $\mathbf{b}$ has 3, zeros where $\mathbf{b}$ has ones and ones where $\mathbf{b}$ has zeros, of the same type, with coupling ψ ; $r = 2$ gives residues 2, 4, $\dots$, excluded by the ones; $r = 3$ gives residues 1, 3, 3, 4, 4, excluded. If $\mathbf{b}$ has type $\{3, 2, 0, 0, 0\}$: $r = 0$ excluded by the three; $r = 1$ gives residues 3, 4, $\dots$, excluded by the two; $r = 2$ gives residues 2, 3, 0, 0, 0, sum five, so $\mathbf{c}$ swaps the three and the two of $\mathbf{b}$, of the same type, coupling ψ^2 ; $r = 3$ gives residues 1, 2, 4, 4, 4, excluded. This proves that every label has exactly one partner, that σ is an involution preserving type with the one fixed point $\mathbf{1}$, and the three values of the coupling. The matrix $M = (\lambda(x^{\mathbf{b}+\mathbf{c}+\mathbf{1}}))_{\mathbf{b}, \mathbf{c}}$ is therefore a monomial matrix with entries 1 (fifty-one times), ψ (thirty times) and ψ^2 (twenty times), so its determinant is $\psi^{30+40} = \psi^{70}$, the sign of σ , a product of fifty transpositions, being +1, and its rank is 101 for $\psi \neq 0$. At $\psi = 0$ the rows of the fifty labels containing a three vanish and the remaining fifty-one rows are a permutation matrix on the fifty-one labels with entries at most two: rank fifty-one.

The Γ -invariant Hermitian metric on $H^{2,1}$, the Hodge metric, pairs the isotypic components of distinct characters to zero because a unitary representation of a finite abelian group is an orthogonal sum of its isotypic components; by Theorem 1.1 the components are the lines spanned by the $x^{\mathbf{b}}$, so the metric is diagonal with entries $g_{\mathbf{b}}$. The mass matrix (4) is then v times a monomial matrix with entries $\psi^{r(\mathbf{b})}(g_{\mathbf{b}}g_{\sigma\mathbf{b}})^{-1/2}$ at $(\mathbf{b}, \sigma\mathbf{b})$. The singular values of a

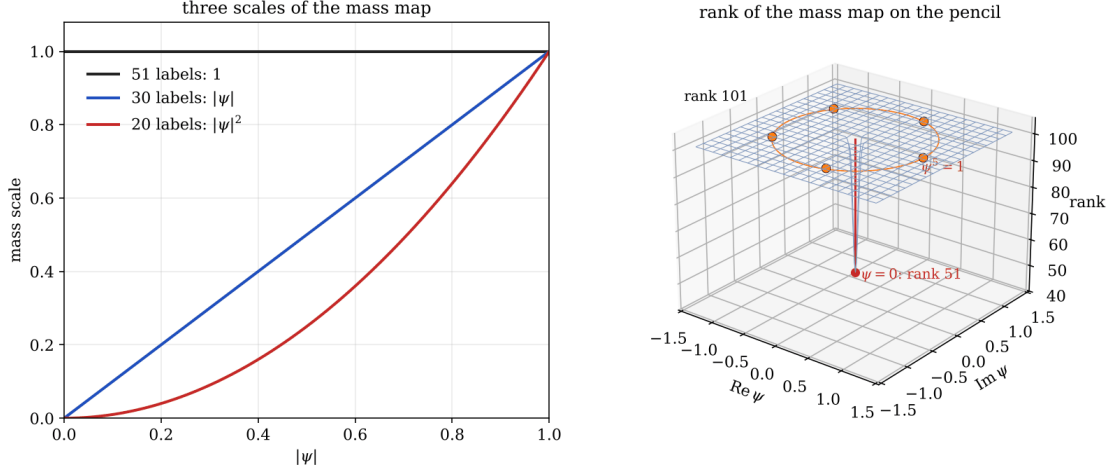


FIGURE 3. The mass spectrum of Theorem 1.3 along the pencil. Left: the three mass scales 1, $|\psi|$ and $|\psi|^2$ against $|\psi|$ for $0 \leq |\psi| < 1$, with the multiplicities of the labels (fifty-one, thirty, twenty) that carry them; the singlet h sits in the first group. Right: the rank of the mass map over the ψ -plane, fifty-one at the Fermat point and a hundred and one elsewhere (the drop at $\psi = 0$ is drawn with a small width); the conifold points $\psi^5 = 1$, where the Jacobian ring degenerates, are marked on the unit circle.

symmetric monomial matrix with an involution are the absolute values of its entries, each two-cycle giving a doubly degenerate pair and the fixed point a singlet; this is the stated spectrum. $\square$

Remark 6.1 (Degenerations). The determinant $\pm\psi^{70}$ shows that the mass map of the invariant direction degenerates on the pencil only at the Fermat point, where the group of automorphisms of X_ψ jumps from $\Gamma \rtimes S_5$ to the full group of order $5^4 \cdot 120$ of the Fermat quintic. The fifty massless generations at $\psi = 0$ are the labels with an exponent three. The larger group acts on a monomial through its full exponent vector, not only through its class modulo $\mathbf{1}$, so at $\psi = 0$ a coupling of h , $\mathbf{b}$ and $\mathbf{c}$ survives only if $\mathbf{1} + \mathbf{b} + \mathbf{c} = (3, 3, 3, 3, 3)$; for a label with an exponent three the required partner $\mathbf{21} - \mathbf{b}$ has a negative entry. At the conifold points $\psi^5 = 1$ the matrix M has nonzero determinant, but the normalisation of Remark 4.2 has a pole: the holomorphically normalized residue expression has a pole. A statement about a physical coupling also requires its matter and Hodge metrics; a pole of that expression alone is not a proof of a universal physical divergence. See [5] for the period behavior.

7. THE FERMAT POINT

At $\psi = 0$ the Hodge norms are explicit.

Proposition 7.1 (Hodge norms at the Fermat point). *Let X_0 be the Fermat quintic and $\omega_{\mathbf{b}}$ the class of $x^{\mathbf{b}}$, $\mathbf{b} \in B$. Write $k_i = b_i + 1$ and $\gamma(t) = \Gamma(t)/\Gamma(1-t)$. There is a constant $C > 0$ independent of $\mathbf{b}$ such that*

$$g_{\mathbf{b}} = \langle \omega_{\mathbf{b}}, \overline{\omega_{\mathbf{b}}} \rangle = C \prod_{i=1}^5 \gamma(k_i/5). \quad (7)$$

Proof. The full group μ_5^5/μ_5 of diagonal phases acts on X_0 , and $\omega_{\mathbf{b}}$ spans the eigenline of the character $\mathbf{k} = (k_i)$, with $1 \leq k_i \leq 4$ and $\sum k_i = 10$. Eigenlines of distinct characters are orthogonal for the Hodge metric, so $g_{\mathbf{b}}$ is the only entry of the metric on that line, and it is computed from the periods of $\omega_{\mathbf{b}}$ by the second Riemann bilinear relation: if P is an integral cycle and $P_{\mathbf{k}}$ its component in the eigenspace of $\mathbf{k}$ in $H_3(X_0, \mathbb{C})$, then $g_{\mathbf{b}} = |\int_P \omega_{\mathbf{b}}|^2 / |P_{\mathbf{k}} \cdot P_{-\mathbf{k}}|$

TABLE 2. Hodge norms and masses at the Fermat point, from (7) and Theorem 1.3. Norms are in units of C and masses in units of v/C ; the types with an exponent three are massless at $\psi = 0$.

type	count	$g_{\mathbf{b}}/C$	value	mass
$\{1, 1, 1, 1, 1\}$	1	$\gamma(2/5)^5$	7.33179	0.136392
$\{2, 1, 1, 1, 0\}$	20	$\gamma(2/5)^2\gamma(1/5)$	8.74859	0.114304
$\{2, 2, 1, 0, 0\}$	30	$\gamma(1/5)^2/\gamma(2/5)$	10.4392	0.095793
$\{3, 1, 1, 0, 0\}$	30	$\gamma(2/5)^2\gamma(1/5)$	8.74859	0
$\{3, 2, 0, 0, 0\}$	20	$\gamma(1/5)^2/\gamma(2/5)$	10.4392	0

up to a factor depending only on the Hodge type. For the cycle of Pham, the join of the five sets of fifth roots of unity, the period is a product of beta integrals, and the intersection number is a product of factors $1 - \zeta^{k_i}$; combining them with $\Gamma(t)\Gamma(1-t) = \pi/\sin \pi t$ gives (7). This is the computation of Candelas [4] on the Fermat quintic and, in the language of the Gepner model, of Distler and Greene [13]; the steps are written out in Appendix C. $\square$

The proposition fixes the ratios of the Hodge norms inside $H^{2,1}$, which is all that the masses of Theorem 1.3 need. Since $\gamma(1-t) = 1/\gamma(t)$, every norm is a monomial in the two numbers $\gamma(1/5) = 3.94325$ and $\gamma(2/5) = 1.48951$. For the singlet, $\mathbf{k} = (2, 2, 2, 2, 2)$ and $g_{\mathbf{1}} = C\gamma(2/5)^5$; for the type $\{2, 1, 1, 1, 0\}$, $\mathbf{k}$ is a permutation of $(3, 2, 2, 2, 1)$ and $g_{\mathbf{b}} = C\gamma(2/5)^2\gamma(1/5)$; for the type $\{2, 2, 1, 0, 0\}$, $\mathbf{k}$ is a permutation of $(3, 3, 2, 1, 1)$ and $g_{\mathbf{b}} = C\gamma(1/5)^2/\gamma(2/5)$. The ratio of the second to the first is $\Gamma(3/5)^3\Gamma(1/5)/\Gamma(2/5)^3\Gamma(4/5) = 1.19324$, the square of the constant $\kappa = 1.09236$ in which Distler and Greene [13] expressed the normalised couplings of the Gepner model (see [12, eq. (4.3)]). The norm of Ω , the class of 1 with $\mathbf{k} = (1, 1, 1, 1, 1)$, has the same shape with a different constant, and the Weil–Petersson metric of the pencil at the Fermat point, computed from the periods by Candelas, de la Ossa, Green and Parkes [5], is

$$G_{\psi\bar{\psi}}(0) = 25 \frac{\Gamma(2/5)^5 \Gamma(4/5)^5}{\Gamma(1/5)^5 \Gamma(3/5)^5} = 0.19226,$$

the value that Butbaia and collaborators [12] reproduce with numerical Ricci-flat metrics.³

Inserting (7) into Theorem 1.3 gives the fifty-one nonzero masses at the Fermat point. Each label of the types $\{2, 1, 1, 1, 0\}$ and $\{2, 2, 1, 0, 0\}$ is paired with a label of the same type and the same multiset of k_i , so $g_{\mathbf{b}} = g_{\sigma\mathbf{b}}$ and its mass is $v/g_{\mathbf{b}}$; the singlet has mass $v/g_{\mathbf{1}}$. Table 2 lists the values. In units of the singlet mass the two doublet masses are 0.83805 (twenty labels, ten doublets) and 0.70234 (thirty labels, fifteen doublets). The fifty generations with an exponent three are massless there, and acquire masses proportional to $|\psi|$ (thirty) and $|\psi|^2$ (twenty) as the modulus moves off the Fermat point.

8. WEIGHTED FERMAT PENCILS

8.1. The pencils. In this section $(d; w)$ is one of $(6; 1, 1, 1, 1, 2)$, $(8; 1, 1, 1, 1, 4)$ and $(10; 1, 1, 1, 2, 5)$, so that $d = \sum w_i$ and $n_i = d/w_i$ is an integer: $n = (6, 6, 6, 6, 3)$, $(8, 8, 8, 8, 2)$ and $(10, 10, 10, 5, 2)$. The polynomial $F_{\psi} = \sum_i x_i^{n_i} - d\psi P$, $P = x_1x_2x_3x_4x_5$, is weighted homogeneous of degree d for $\deg x_i = w_i$, and $X_{\psi} \subset \mathbb{P}(w)$ is its zero locus (2). We write

$$\delta = \prod_i w_i^{w_i} \in \{4, 256, 12\,500\},$$

³In [12] the value is given for the mirror quintic, whose complex structure moduli space is the ψ -line of the Dwork pencil divided by the symmetry $\psi \mapsto \zeta\psi$. The Weil–Petersson metric is $-\partial\bar{\partial} \log \int \Omega \wedge \bar{\Omega}$, and passing to the quotient by Γ multiplies $\int \Omega \wedge \bar{\Omega}$ by the constant $1/125$, which does not change the metric; the two numbers are therefore the same. Their machine-learned harmonic representatives give 0.1906 against the exact 0.1923.

TABLE 3. The four pencils of Fermat type. The rows give the degree and weights, the constant $\delta = \prod w_i^{w_i}$ of the discriminant $\psi^d \delta = 1$, the intersection numbers of the hyperplane class L , the Euler characteristic and $h^{2,1}$, the order of Γ , the number of distinct characters carried by the generations, the number of classes of two or three generations sharing a character, the number of characters of Γ carried by no generation, and the rank of the mass map of the invariant direction for $\psi \neq 0$ and at $\psi = 0$, with the number of generations it leaves massless.

	quintic	sextic	octic	dectic
$(d; w)$	(5; 1, 1, 1, 1, 1)	(6; 1, 1, 1, 1, 2)	(8; 1, 1, 1, 1, 4)	(10; 1, 1, 1, 2, 5)
δ	1	4	256	12 500
$L^3, c_2 \cdot L$	5, 50	3, 42	2, 44	1, 34
$\chi, h^{2,1}$	-200, 101	-204, 103	-296, 149	-288, 145
$ \Gamma $	125	108	128	100
distinct characters	101	96	122	100
shared classes	0	7	27	42
characters not carried	24	12	6	0
rank for $\psi \neq 0$ (massless)	101 (0)	99 (4)	125 (24)	109 (36)
rank at $\psi = 0$ (massless)	51 (50)	44 (59)	85 (64)	72 (73)

and use the letters i, j, k, l for distinct indices among the coordinates of weight one. Table 3 collects the numbers proved in this section, with the quintic for comparison.⁴

Lemma 8.1. *If $\psi^d \delta \neq 1$, then X_ψ is quasi-smooth, it misses the singular points of $\mathbb{P}(w)$, and it is a smooth Calabi–Yau threefold with $h^{1,1} = 1$. Its hyperplane class $L = c_1(\mathcal{O}_X(1))$ generates $H^2(X, \mathbb{Z})$ modulo torsion, and*

$$L^3 = \frac{d}{\prod_i w_i}, \quad c_2 \cdot L = \sigma_2(w) L^3, \quad (8)$$

$$\chi = (\sigma_3(w) - d \sigma_2(w)) L^3,$$

with σ_k the elementary symmetric functions of the weights. If $\psi^d \delta = 1$, then X_ψ is singular.

Proof. The Euler relations $x_i \partial_i F_\psi = n_i x_i^{n_i} - d \psi P$ show that at a common zero of the partial derivatives $x_i^{n_i} = w_i \psi P$ for every i . If $P = 0$ this forces $x = 0$. Otherwise raise the i -th relation to the power w_i and multiply: since $\prod_i x_i^{n_i w_i} = P^d$, this gives $P^d = \psi^d \delta P^d$, so $\psi^d \delta = 1$. Hence X_ψ is quasi-smooth when $\psi^d \delta \neq 1$. Conversely, at $\psi = 2^{-1/3}$, $\psi = 1/2$ and $\psi = 2^{-1/5} 5^{-1/2}$ the points $(1, 1, 1, 1, 2^{1/3})$, $(1, 1, 1, 1, 2)$ and $(1, 1, 1, 2^{1/5}, 5^{1/2})$ satisfy $x_i^{n_i} = w_i \psi P = w_i$, so they are singular points of the sextic, octic and dectic; the other roots of $\psi^d \delta = 1$ are $\zeta \psi$ with $\zeta^d = 1$, and $x_1 \mapsto \zeta^{-1} x_1$ maps $X_{\zeta \psi}$ to X_ψ because $n_1 = d$.

The singular points of $\mathbb{P}(w)$ are $(0:0:0:0:1)$ for the sextic and the octic, with isotropy μ_2 and μ_4 , and $(0:0:0:1:0)$ and $(0:0:0:0:1)$ for the dectic, with isotropy μ_2 and μ_5 . Each monomial $x_i^{n_i}$ occurs in F_ψ with coefficient one, so F_ψ does not vanish at a coordinate point and X_ψ misses them all; a quasi-smooth hypersurface in the smooth locus of $\mathbb{P}(w)$ is smooth. By adjunction $K_X = \mathcal{O}_X(d - \sum w_i) = \mathcal{O}_X$, and the Lefschetz theorem for quasi-smooth hypersurfaces [21] gives $h^{1,0} = h^{2,0} = 0$ and $h^{1,1} = 1$.

The tangent bundle of $\mathbb{P}(w)$ on its smooth locus has total Chern class $\prod_i (1 + w_i H)$ by the weighted Euler sequence, and the normal bundle of X is $\mathcal{O}_X(d)$, so $c(T_X) = \prod_i (1 + w_i L)(1 +$

⁴The three pencils are the weighted members of the family of one-parameter models whose periods and mirror maps were computed by Morrison [17], Klemm and Theisen [18] and Font [19]. In those papers the coefficient of P is normalised differently and the natural coordinate on the moduli space is ψ^d up to a constant; the discriminant $\psi^d \delta = 1$ below is the conifold locus of those papers, and the Fermat point $\psi = 0$ is the Gepner point.

$dL)^{-1}$. Expanding with $\sigma_1(w) = d$ gives $c_1 = 0$, $c_2 = \sigma_2 L^2$ and $c_3 = (\sigma_3 - d\sigma_2)L^3$, and $L^3 = d \int_{\mathbb{P}(w)} H^4 = d / \prod w_i$. For the three pencils $(\sigma_2, \sigma_3) = (14, 16)$, $(22, 28)$ and $(34, 52)$, which gives Table 3. If $L = mL'$ with L' integral, then m^3 divides $L^3 \in \{3, 2, 1\}$, which is cube-free, so $m = \pm 1$; as $H^2(X, \mathbb{Q})$ is spanned by L , the class L generates $H^2(X, \mathbb{Z})$ modulo torsion. $\square$

8.2. Labels and characters. The Jacobian ring $R = \mathbb{C}[x_1, \dots, x_5]/(\partial_1 F_\psi, \dots, \partial_5 F_\psi)$ is graded by $\deg x_i = w_i$. Quasi-smoothness says that the partial derivatives, of degrees $d - w_i$, form a regular sequence, so R is a graded complete intersection with Hilbert series $\prod_i (1 - t^{d-w_i})/(1 - t^{w_i})$: it has dimension $\prod_i (n_i - 1)$, which is 1250, 2401 and 2916, its top degree is $\sum_i (d - 2w_i) = 3d$, and R_{3d} is one-dimensional and spanned by $\text{Hess } F_\psi$ by Macaulay's theorem, which holds for any regular sequence of weighted homogeneous polynomials. Griffiths' description of the Hodge structure extends to quasi-smooth hypersurfaces in weighted projective space, as shown by Steenbrink [20] and Dolgachev [21]: with

$$\Omega_0 = \sum_i (-1)^i w_i x_i dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_5,$$

the residue along X of $P \Omega_0 / F_\psi^{q+1}$, for P of degree qd , identifies R_{qd} with $H^{3-q,q}(X)$, so that $H^{2,1} = R_d$ and R_{3d} is the socle. The infinitesimal variation of the Hodge structure is again multiplication in R , and the cubic form is (5) with λ a functional on R_{3d} ; the derivation in [9] uses only the residue map and the residue pairing of R , both of which are available in the weighted setting.

Let B be the set of $\mathbf{b} \in \mathbb{Z}^5$ with $0 \leq b_i \leq n_i - 2$ and $\sum_i w_i b_i = d$. We call its elements the labels.

Lemma 8.2. *Let $\psi^d \delta \neq 1$. Every monomial $x^\mathbf{e}$ of degree d satisfies $x^\mathbf{e} \equiv c \psi^r x^\mathbf{b}$ in R with $\mathbf{b} \in B$, c a product of weights, and $r \leq 1$ for the sextic and $r \leq 2$ for the octic and the dectic. The monomials $x^\mathbf{b}$, $\mathbf{b} \in B$, are a basis of $R_d = H^{2,1}$, so $|B| = h^{2,1}$.*

Proof. The relations $\partial_i F_\psi / n_i = 0$ read

$$x_i^{n_i-1} \equiv w_i \psi \prod_{j \neq i} x_j. \quad (9)$$

In a monomial of degree d at most one exponent satisfies $e_i \geq n_i - 1$: two such exponents would carry weight at least $2d - w_i - w_j > d$. If $e_i \geq n_i - 1$, then $e_i \in \{n_i - 1, n_i\}$, and (9) replaces $x^\mathbf{e}$ by $w_i \psi x^{\mathbf{e}'}$ with $e'_i = e_i - n_i + 1$ and $e'_j = e_j + 1$ for $j \neq i$. If $e_i = n_i$, then $x^\mathbf{e} = x_i^{n_i}$ and $\mathbf{e}' = \mathbf{1}$. If $e_i = n_i - 1$, then $e'_i = 0$, the other coordinates carry weight w_i , and $e'_j \geq n_j - 1$ forces $w_j e_j \geq d - 2w_j$, hence $w_i + 2w_j \geq d$. For the sextic no pair $i \neq j$ has $w_i + 2w_j \geq 6$, and $\mathbf{1} \in B$ because every $n_j \geq 3$; so one step suffices. For the octic and the dectic the only j with $w_i + 2w_j \geq d$ for some $i \neq j$ is $j = 5$, where $w_5 = d/2$ and $n_5 = 2$; in that case, and in the case $\mathbf{e}' = \mathbf{1}$, the exponent of x_5 is one and a second step at x_5 gives an exponent vector with $e''_5 = 0$ and the other entries raised by one. A third step would need $w_5 + 2w_k \geq d$, that is $w_k \geq d/4$, for some $k \leq 4$, which fails. So at most two steps reach a vector with all $e_i \leq n_i - 2$, which is a label.

At $\psi = 0$ the ring is $\bigotimes_i \mathbb{C}[x_i]/(x_i^{n_i-1})$, whose monomials with $e_i \leq n_i - 2$ are a basis; so $|B| = \dim R_d$. The Hilbert series does not depend on ψ , and the labels span R_d for every ψ by the first part, so they are a basis. $\square$

The symmetry group. Let K be the group of $\varepsilon \in \prod_i \mu_{n_i}$ with $\prod_i \varepsilon_i = 1$, acting by $x_i \mapsto \varepsilon_i x_i$. It preserves F_ψ for every ψ . The elements $(\lambda^{w_1}, \dots, \lambda^{w_5})$ with $\lambda^d = 1$ lie in K and act trivially on $\mathbb{P}(w)$, and they are the only such elements; they form a subgroup isomorphic to μ_d because the weights are coprime. Let $\Gamma = K/\mu_d$. Since $\text{lcm}(n_i) = d$, the product map $\prod_i \mu_{n_i} \rightarrow \mu_d$ is onto and $|\Gamma| = \prod_i n_i / d^2$, which is 108, 128 and 100. A character of $\prod_i \mu_{n_i}$ is a vector $\mathbf{b} \in \bigoplus_i \mathbb{Z}/n_i$, acting by $\varepsilon \mapsto \prod_i \varepsilon_i^{b_i}$; it is trivial on K exactly when it is a multiple of $\mathbf{1}$, and

it is trivial on μ_d exactly when $\sum_i w_i b_i \equiv 0 \pmod{d}$, a condition that is well defined because $w_i n_i = d$. Hence

$$\widehat{\Gamma} = \left\{ \mathbf{b} \in \bigoplus_i \mathbb{Z}/n_i : \sum_i w_i b_i \equiv 0 \pmod{d} \right\} / \langle \mathbf{1} \rangle.$$

A monomial $x^{\mathbf{e}}$ of degree divisible by d transforms by the class $[\mathbf{e}]$ of its exponent vector, and Ω_0 transforms by $\prod_i \varepsilon_i = 1$, so the isomorphisms $R_{qd} \cong H^{3-q,q}$ are Γ -equivariant and R is graded by $\widehat{\Gamma}$. The Hessian is invariant, since $\partial_i \partial_j F_\psi$ transforms by $\varepsilon_i^{-1} \varepsilon_j^{-1}$ and the determinant by $\prod_i \varepsilon_i^{-2} = 1$; so R_{3d} carries the trivial character. Permutations of coordinates of equal weight also preserve F_ψ and permute the labels.

Lemma 8.3 (Wrapping). *Two labels $\mathbf{b} \neq \mathbf{b}'$ carry the same character if and only if there is a shift $s \in \{1, \dots, d-1\}$ such that, with $a_i = \lfloor (b_i + s)/n_i \rfloor$,*

$$\begin{aligned} b'_i &= b_i + s - n_i a_i, & \sum_i a_i &= s, \\ b_i + s &\not\equiv n_i - 1 \pmod{n_i} & \text{for all } i. \end{aligned}$$

The shift from $\mathbf{b}'$ back to $\mathbf{b}$ is then $d - s$.

Proof. Both labels satisfy $\sum w_i b_i = d \equiv 0$, so they define the same character exactly when $\mathbf{b}' \equiv \mathbf{b} + s\mathbf{1}$ modulo $(n_1, \dots, n_5)$ for some s modulo d . As $0 \leq b'_i \leq n_i - 2$, the entry b'_i is the residue of $b_i + s$ in $\{0, \dots, n_i - 1\}$, which must differ from $n_i - 1$. Then $\sum_i w_i b'_i = \sum_i w_i b_i + sd - d \sum_i a_i$, which equals d exactly when $\sum_i a_i = s$. $\square$

Proposition 8.4. *Let $\psi^d \delta \neq 1$. The classes of labels that carry the same character are exactly the following, and every other label carries a character of its own.*

- (a) *Sextic:* the 4 pairs $\{x_i^4 x_5, x_j^2 x_k^2 x_l^2\}$ at shift 2 and the 3 pairs $\{x_i^3 x_j^3, x_k^3 x_l^3\}$ at shift 3; 96 distinct characters.
- (b) *Octic:* the 12 pairs $\{x_i^6 x_j^2, x_j^4 x_k^2 x_l^2\}$ and the 12 pairs $\{x_i^6 x_j x_k, x_j^3 x_k^3 x_l^2\}$ at shift 2, and the 3 pairs $\{x_i^4 x_j^4, x_k^4 x_l^4\}$ at shift 4; 122 distinct characters.
- (c) *Dectic:* the three triples $\{x_i^8 x_4, x_j^2 x_k^2 x_4^3, x_l^2 x_j^4 x_k^4\}$; at shift 2 the 12 pairs $\{x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^3, x_1^{c_1+2} x_2^{c_2+2} x_3^{c_3+2}\}$ with $c_1 + c_2 + c_3 = 4$ and $\mathbf{c}$ not a permutation of $(2, 2, 0)$, the 6 pairs $\{x_i^8 x_j^2, x_j^4 x_k^2 x_4^2\}$ and the 3 pairs $\{x_i^8 x_j x_k, x_j^3 x_k^3 x_4^2\}$; at shift 4 the 6 pairs $\{x_i^6 x_j^2 x_4, x_j^6 x_k^4\}$, the 3 pairs $\{x_i^6 x_j x_k x_4, x_j^5 x_k^5\}$, the 6 pairs $\{x_i^7 x_j x_4, x_i x_j^5 x_k^4\}$ and the 3 pairs $\{x_i^6 x_4^2, x_j^4 x_k^4 x_4\}$; 100 distinct characters, so every character of Γ is carried by some generation.

The characters carried by no label are the 12 classes of $(\pi(0, 1, 3, 4), 2)$ for the sextic and the 6 classes of $(\pi(0, 2, 4, 6), 1)$ for the octic, π running over the permutations of four entries. The set of characters carried by labels is closed under $\chi \mapsto -\chi$.

Proof. We use Lemma 8.3 and treat the shifts $s \leq d/2$; the others are their inverses. For the sextic, $b_i + s \leq 9 < 12$ for $i \leq 4$, so $a_i = 1$ exactly when $b_i \geq 6 - s$, and the residue $n_i - 1 = 5$ is excluded when $b_i = 5 - s$; for x_5 , $b_5 \leq 1$ and $a_5 = \lfloor (b_5 + s)/3 \rfloor$, with $b_5 + s \equiv 2 \pmod{3}$ excluded. At $s = 1$ we have $a_5 = 0$ and need one $b_i \geq 5$, which is impossible. At $s = 2$ the value $b_5 = 0$ is excluded, so $b_5 = 1$ and $a_5 = 1$; we need exactly one $b_i \geq 4$ among the first four, which sum to four, so $\mathbf{b} = (4, 0, 0, 0, 1)$ up to order, and its image is $(0, 2, 2, 2, 0)$. At $s = 3$ we have $a_5 = 1$ and need two entries $b_i \geq 3$ among the first four and none equal to two; two entries of at least three already have weight six, so $\mathbf{b} = (3, 3, 0, 0, 0)$ up to order, with image $(0, 0, 3, 3, 0)$. No label occurs at two different shifts, so the classes are the pairs listed, 14 labels in 7 classes and $103 - 7 = 96$ characters. The same analysis for the octic and the dectic is carried out in Appendix D. It uses two facts: for these pencils $n_5 = 2$ and $b_5 = 0$, so $a_5 = s/2$ and an odd shift is excluded by the residue of x_5 ; and the bounds $b_i \leq n_i - 2$ make each a_i either 0 or 1, except $a_4 \in \{0, 1, 2\}$ for the dectic.

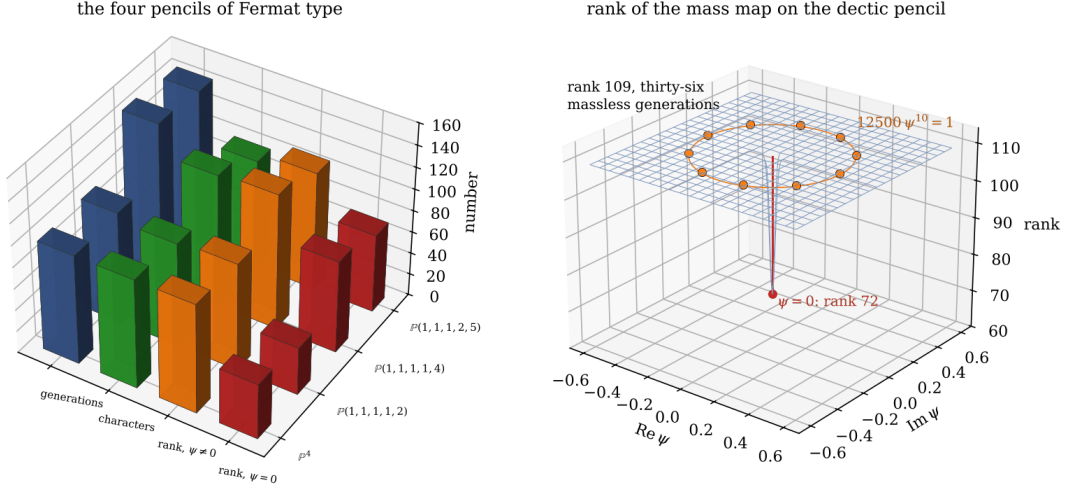


FIGURE 4. Left: for the quintic in $\mathbb{P}^4$ and the sextic, octic and dectic pencils in $\mathbb{P}(1, 1, 1, 1, 2)$, $\mathbb{P}(1, 1, 1, 1, 4)$ and $\mathbb{P}(1, 1, 1, 2, 5)$, the number of generations (101, 103, 149, 145), the number of distinct characters they carry (101, 96, 122, 100), and the rank of the mass map of the invariant direction for $\psi \neq 0$ (101, 99, 125, 109) and at $\psi = 0$ (51, 44, 85, 72). Only on the quintic do the first three bars agree. Right: the rank of the mass map over the ψ -plane of the dectic pencil, 109 off the Fermat point and 72 at $\psi = 0$ (the drop is drawn with a small width); the ten points of the circle $12500\psi^{10} = 1$ are the singular members.

The characters not carried by a label are $108 - 96 = 12$, $128 - 122 = 6$ and $100 - 100 = 0$ in number. For the sextic, adding $s\mathbf{1}$ to $(\pi(0, 1, 3, 4), 2)$ gives a vector with an entry 5 for $s \not\equiv 0 \pmod{3}$, and the vector $(\pi'(0, 1, 3, 4), 2)$ for $s = 3$, where the last entry is $n_5 - 1$. So every representative of these classes has an entry $n_i - 1$, and none is a label; each class contains two of the 24 vectors $(\pi(0, 1, 3, 4), 2)$, which gives 12 classes, and these are all the missing characters. For the octic, adding $s\mathbf{1}$ to $(\pi(0, 2, 4, 6), 1)$ gives an entry 7 when s is odd and the last entry $1 = n_5 - 1$ when s is even; each class contains four of the 24 vectors, which gives 6 classes. Finally $-(\pi(0, 1, 3, 4), 2) = (\pi(0, 5, 3, 2), 1)$, which becomes $(\pi(1, 0, 4, 3), 2)$ after adding $\mathbf{1}$, and $-(\pi(0, 2, 4, 6), 1) = (\pi(0, 6, 4, 2), 1)$; so the missing characters are closed under negation, and so are the carried ones. $\square$

Figure 4 compares the four pencils.

8.3. The socle functional. Let $m_0 = x^{n-2\mathbf{1}}$, the monomial with exponents $n_i - 2$; it has degree $\sum_i w_i(n_i - 2) = 3d$. A monomial $x^{\mathbf{e}}$ of degree $3d$ is in sector $r \in \{0, \dots, d-1\}$ if $e_i \equiv n_i - 2 - r \pmod{n_i}$ for every i . We then write

$$e_i = n_i - 2 - r + n_i k_i; \quad (10)$$

each k_i is a nonnegative integer, since $k_i \leq -1$ would give $e_i \leq -2 - r$, and $\sum_i k_i = r$, by comparing degrees: $\sum_i w_i e_i = 5d - 2d - rd + d \sum_i k_i = 3d$.

Theorem 8.5 (Weighted socle functional). *Let $\psi^d \delta \neq 1$. Then m_0 spans R_{3d} , and the functional λ on R_{3d} with $\lambda(m_0) = 1$ satisfies, for every monomial $x^{\mathbf{e}}$ of degree $3d$,*

$$\lambda(x^{\mathbf{e}}) = \begin{cases} \psi^r \prod_i w_i^{k_i} & \text{if } \mathbf{e} \text{ is in sector } r, \text{ with } k \text{ as in (10),} \\ 0 & \text{otherwise.} \end{cases}$$

A monomial of degree $3d$ is in a sector if and only if its character is trivial.

Proof. The character of $x^{\mathbf{e}}$ is trivial when $\mathbf{e} \equiv c\mathbf{1}$ modulo $(n_1, \dots, n_5)$ for some integer c ; taking $r \equiv -2 - c \pmod{d}$ and using $n_i \mid d$ this is the condition $e_i \equiv n_i - 2 - r$, so the monomials with trivial character are those in a sector. Since R_{3d} is one-dimensional with trivial character, every monomial with a nontrivial character vanishes in R .

Let $\mathbf{e}$ be in sector $r \geq 1$. We claim that some i has $n_i k_i \geq r + 1$. Otherwise $k_i \leq r/n_i = rw_i/d$ for every i , so $r = \sum_i k_i \leq r \sum_i w_i/d = r$, with equality only if $n_i k_i = r$ for every i ; then every n_i divides r , and so does their least common multiple d , which is impossible for $1 \leq r < d$. For such an i we have $e_i \geq n_i - 1$, and (9) gives $x^{\mathbf{e}} \equiv w_i \psi x^{\mathbf{e}'}$ with $e'_i = e_i - n_i + 1$ and $e'_j = e_j + 1$ for $j \neq i$; by (10) $\mathbf{e}'$ is in sector $r - 1$ with $k'_i = k_i - 1$ and the other k_j unchanged. After r steps we reach sector 0, where all k_i vanish and $\mathbf{e} = \mathbf{n} - 2\mathbf{1}$. Collecting the factors, $x^{\mathbf{e}} \equiv \psi^r \prod_i w_i^{k_i} m_0$. Finally, R_{3d} is spanned by monomials, each of which is zero or a multiple of m_0 , and $R_{3d} \neq 0$; so m_0 spans R_{3d} . $\square$

Corollary 8.6. *Let $\psi^d \neq 1$. The cubic form takes on three labels $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the value $\lambda(x^{\mathbf{a}+\mathbf{b}+\mathbf{c}})$, up to the normalisation of λ : this is $\psi^r \prod_i w_i^{k_i}$ when $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is in sector r , and zero when the character of $x^{\mathbf{a}+\mathbf{b}+\mathbf{c}}$ is nontrivial. In particular, for $\psi \neq 0$ every coupling allowed by Γ is nonzero. This is Theorem 1.4(ii).*

8.4. The mass map. The only label with trivial character is $h = \mathbf{1}$ for the sextic and $h = (2, 2, 2, 2, 0)$ for the octic and the dectic: a label $\mathbf{b} \equiv c\mathbf{1}$ has $b_i = c$ for the coordinates of weight one, and the condition on the weight leaves only these. So the invariant part of R_d is spanned by x^h , the invariant direction. For $\psi \neq 0$ it contains the tangent vector $\partial_\psi F_\psi = -dP$ of the pencil: for the sextic $P = x^h$, and for the octic and the dectic $P = x^{\mathbf{1}} \equiv w_5 \psi x^h$ by (9) at x_5 , whose exponent one equals $n_5 - 1$. By Corollary 8.6 the mass map of the invariant direction in the basis of labels is

$$M_{\mathbf{bc}} = \lambda(x^{h+\mathbf{b}+\mathbf{c}}), \quad \mathbf{b}, \mathbf{c} \in B.$$

Since h has trivial character, $M_{\mathbf{bc}} \neq 0$ requires $[\mathbf{c}] = -[\mathbf{b}]$. For a character χ let B_χ be the set of labels carrying it; then M is the direct sum of the blocks on $B_\chi \cup B_{-\chi}$, one for each pair $\{\chi, -\chi\}$ of characters carried by labels (Proposition 8.4 shows that $-\chi$ is carried whenever χ is).

Lemma 8.7. *Let $\psi \neq 0$ and $\psi^d \delta \neq 1$. Every entry of a block is $\psi^r \prod_i w_i^{k_i} \neq 0$. For $\mathbf{b}, \mathbf{b}' \in B_\chi$ and $\mathbf{c}, \mathbf{c}' \in B_{-\chi}$ let r, r', ρ, ρ' be the sectors of $h + \mathbf{b} + \mathbf{c}$, $h + \mathbf{b}' + \mathbf{c}'$, $h + \mathbf{b} + \mathbf{c}'$ and $h + \mathbf{b}' + \mathbf{c}$, and $\Delta = r + r' - \rho - \rho'$. Then $\Delta \in \{-d, 0, d\}$, and*

$$\begin{aligned} M_{\mathbf{bc}} M_{\mathbf{b}'\mathbf{c}'} - M_{\mathbf{bc}'} M_{\mathbf{b}'\mathbf{c}} \\ = M_{\mathbf{bc}'} M_{\mathbf{b}'\mathbf{c}} ((\psi^d \delta)^{\Delta/d} - 1), \end{aligned}$$

which vanishes exactly when $\Delta = 0$.

Proof. The first statement is Theorem 8.5, since each $h + \mathbf{b} + \mathbf{c}$ in a block has trivial character. The two sums $(h + \mathbf{b} + \mathbf{c}) + (h + \mathbf{b}' + \mathbf{c}')$ and $(h + \mathbf{b} + \mathbf{c}') + (h + \mathbf{b}' + \mathbf{c})$ are equal, so by (10), with k, k', κ, κ' the vectors of the four monomials, $n_i(k_i + k'_i - \kappa_i - \kappa'_i) = \Delta$ for every i . Hence every n_i divides Δ , so d does, and $|\Delta| < 2d$ because the sectors lie in $\{0, \dots, d-1\}$. The quotient of the two products is $\psi^\Delta \prod_i w_i^{\Delta w_i/d} = (\psi^d \delta)^{\Delta/d}$. $\square$

Proposition 8.8. *Let $\psi \neq 0$ and $\psi^d \delta \neq 1$. The blocks of the mass map that involve labels sharing a character are those of Table 4. The mass map has rank 99, 125 and 109 for the sextic, octic and dectic, its kernel has dimension 4, 24 and 36 and lies in the span of the labels that share a character, and the rank is the same at every member with $\psi \neq 0$ and $\psi^d \delta \neq 1$. At $\psi = 0$ the mass map is the matrix of the partial involution $\mathbf{b} \mapsto \mathbf{n} - 2\mathbf{1} - h - \mathbf{b}$ on the labels with $\mathbf{b} \leq \mathbf{n} - 2\mathbf{1} - h$, of rank 44, 85 and 72.*

Proof. The classes of Proposition 8.4 and their negatives give the rows of Table 4: for one representative of each orbit the character $-\chi$ is computed from Lemma 8.3, and the entries

TABLE 4. The blocks of the mass map of the invariant direction that involve labels sharing a character. Each row gives the labels of B_χ and of $B_{-\chi}$ in the notation of Proposition 8.4 (one representative of the orbit under permutations of the coordinates of weight one), the number of such blocks, the entries $M_{\mathbf{bc}}$ with $\mathbf{b} \in B_\chi$ in the order listed, and the number of generations each block leaves massless for $\psi \neq 0$; “self” means $-\chi = \chi$. All other blocks are a single label with $\chi = -\chi$ or a pair of single labels, and have full rank.

pencil	B_χ	$B_{-\chi}$	entries	blocks	massless
sextic	$x_i^4 x_5, x_j^2 x_k^2 x_l^2$	$x_i^3 x_j x_k x_l$	$2\psi^2; 1$	4	1
	$x_i^3 x_j^3, x_k^3 x_l^3$	self	$(2\psi^3, 1; 1, 2\psi^3)$	3	0
octic	$x_i^6 x_j x_k, x_j^3 x_k^3 x_l^2$	$x_i^4 x_j x_k x_l^2$	$4\psi^2; 1$	12	1
	$x_i^6 x_j^2, x_j^4 x_k^2 x_l^2$	$x_j^6 x_i^2, x_i^4 x_k^2 x_l^2$	$(16\psi^4, 4\psi^2; 4\psi^2, 1)$	6	2
	$x_i^4 x_j^4, x_k^4 x_l^4$	self	$(16\psi^4, 1; 1, 16\psi^4)$	3	0
dectic	$x_i^2 x_j x_k x_4^3, x_i^4 x_j^3 x_k^3$	$x_i^2 x_j^3 x_k^3 x_4$	$10\psi^2; 1$	3	1
	$x_i^3 x_j x_4^3, x_i^5 x_j^3 x_k^2$	$x_i x_j^3 x_k^4 x_4$	$10\psi^2; 1$	6	1
	$x_i^8 x_j x_k, x_j^3 x_k^3 x_4^2$	$x_i^4 x_j x_k x_4^2$	$50\psi^4; 10\psi^2$	3	1
	$x_i^7 x_j x_4, x_i x_j^5 x_k^4$	$x_i^5 x_j x_k^2 x_4$	$50\psi^4; 1$	6	1
	$x_i^8 x_4, x_j^2 x_k^2 x_4^3, x_i^2 x_j^4 x_k^4$	$x_i^4 x_j^2 x_k^2 x_4$	$50\psi^4; 10\psi^2; 1$	3	2
	$x_i^4 x_4^3, x_i^6 x_j^2 x_k^2$	$x_i^6 x_4^2, x_j^4 x_k^4 x_4$	$(500\psi^6, 10\psi^2; 50\psi^4, 1)$	3	2
	$x_i^8 x_j^2, x_j^4 x_k^2 x_4^2$	$x_j^8 x_i^2, x_i^4 x_k^2 x_4^2$	$(250\psi^6, 50\psi^4; 50\psi^4, 10\psi^2)$	3	2
	$x_i^6 x_j^2 x_4, x_j^6 x_k^4$	$x_i^6 x_k^2 x_4, x_k^6 x_j^4$	$(50\psi^4, 1; 1, 250\psi^6)$	3	0
	$x_i^6 x_j x_k x_4, x_j^5 x_k^5$	self	$(50\psi^4, 1; 1, 250\psi^6)$	3	0

from Theorem 8.5; Appendix E lists the sectors and the vectors k . For example, for the sextic block with $i = 1$, $h + (3, 1, 1, 1, 0) + (4, 0, 0, 0, 1) = (8, 2, 2, 2, 2)$ is in sector 2 with $k = (1, 0, 0, 0, 1)$, which gives $2\psi^2$, and $h + (3, 1, 1, 1, 0) + (0, 2, 2, 2, 0) = (4, 4, 4, 4, 1) = \mathbf{n} - 21$ gives 1. A block in which one of $B_\chi, B_{-\chi}$ is a single label has the form $\begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$ with A a row or a column of nonzero entries, so it has rank two and leaves $|B_\chi| + |B_{-\chi}| - 2$ generations massless, in the span of the class of two or three labels. For a block with 2×2 matrix A , Lemma 8.7 decides the rank: $\Delta = 0$ for the second octic row and the sixth and seventh dectic rows, for instance $6 + 0 - 2 - 4 = 0$ in the sixth, so A has rank one and two generations on each side of the block are massless; $\Delta = \pm d$ for the eighth dectic row and the three self-dual rows, whose determinants are $4\psi^6 - 1$, $256\psi^8 - 1$ and $12500\psi^{10} - 1$ up to nonzero monomials, so these blocks have full rank. A single label with $\chi = -\chi$, and a pair of single labels, give a block of full rank by the first part of Lemma 8.7. Adding the last column of Table 4: $4 \cdot 1 = 4$, $12 \cdot 1 + 6 \cdot 2 = 24$ and $3 + 6 + 3 + 6 + 3 \cdot 2 + 3 \cdot 2 + 3 \cdot 2 = 36$. None of the entries or determinants depends on ψ beyond these monomials and the factor $\psi^d \delta - 1$, so the rank is constant on the stated set.

At $\psi = 0$ every sector $r \geq 1$ gives zero, so $M_{\mathbf{bc}} \neq 0$ exactly when $h + \mathbf{b} + \mathbf{c} = \mathbf{n} - 21$. The vector $\mathbf{c} = \mathbf{n} - 21 - h - \mathbf{b}$ has degree $3d - 2d = d$ and entries at most $n_i - 2$, so it is a label exactly when $\mathbf{b} \leq \mathbf{n} - 21 - h$. Here $\mathbf{n} - 21 - h = (3, 3, 3, 3, 0)$, $(4, 4, 4, 4, 0)$ and $(6, 6, 6, 1, 0)$, and the labels below these vectors are counted by the coefficient of t^6 in $(1 + t + t^2 + t^3)^4$, which is 44, of t^8 in $(1 + t + \dots + t^4)^4$, which is 85, and by $36 + 36 = 72$ solutions of $b_1 + b_2 + b_3 + 2b_4 = 10$ with $b_i \leq 6$ and $b_4 \leq 1$. $\square$

8.5. Three families.

Proposition 8.9. *Let X be a member of the sextic, octic or dectic pencil with $\psi^d \delta \neq 1$, and H a finite group acting freely on X . With the standard embedding, X/H does not have three net families.*

Proof. By Theorem 2.1 the number of net families on $Y = X/H$ is $|\chi(Y)|/2 = |\chi(X)|/(2|H|)$, and three families need $|H| = |\chi(X)|/6$, that is 34 for the sextic and 48 for the dextic; for the octic $296/6$ is not an integer. Suppose H has this order. The projection $\pi: X \rightarrow Y$ is étale of degree $|H|$, $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)/|H| = 0$ and $c_1(Y)$ is torsion. As $H^2(X, \mathbb{Q}) = \mathbb{Q}L$ and L generates $H^2(X, \mathbb{Z})$ modulo torsion (Lemma 8.1), every line bundle D on Y has $\pi^*D \equiv mL$ numerically for an integer m , and these m form a subgroup of $\mathbb{Z}$. It contains $|H|$: the line bundle $\bigotimes_{g \in H} g^*\mathcal{O}_X(1)$ is linearised by permuting its factors, so it descends to Y , and it is numerically $|H|L$ because H acts trivially on $H^2(X, \mathbb{Q})$. Let $m_0 > 0$ generate the subgroup and D realise it. Then m_0 divides $|H|$, and

$$D^3 = \frac{m_0^3 L^3}{|H|}, \quad c_2(Y) \cdot D = \frac{m_0 c_2 \cdot L}{|H|},$$

$$\chi(Y, D) = \frac{D^3}{6} + \frac{c_2(Y) \cdot D}{12},$$

the last by Riemann–Roch with $c_1(Y) \equiv 0$ and $\chi(\mathcal{O}_Y) = 0$. All three numbers are integers. For the sextic, $D^3 = 3m_0^3/34$ forces $34 \mid m_0$, so $m_0 = 34$, $D^3 = 3468$, $c_2(Y) \cdot D = 42$ and $\chi(Y, D) = 578 + 7/2$. For the dextic, $D^3 = m_0^3/48$ forces $12 \mid m_0$ and $c_2(Y) \cdot D = 17m_0/24$ forces $24 \mid m_0$, so $m_0 \in \{24, 48\}$, with $\chi(Y, D) = 48 + 17/12$ and $384 + 17/6$. None is an integer, a contradiction. $\square$

Proof of Theorem 1.4. Smoothness and $h^{2,1}$ are Lemma 8.1, part (i) is Proposition 8.4, part (ii) is Corollary 8.6, part (iii) is Proposition 8.8, and part (iv) is Proposition 8.9 together with $|\chi(X)|/2 \in \{102, 148, 144\}$ for X itself. $\square$

Remark 8.10 (Why the quintic is different). On the quintic all n_i equal five, so a shift moves every exponent by the same amount and the wrapping count $\sum_i a_i = s$ of Lemma 8.3 has no solution: this is the proof of Theorem 1.1. When the n_i differ, a wrap at a coordinate with large n_i can be balanced by wraps at the coordinate with $n_5 = 3$ or $n_5 = 2$, and the shared classes of Proposition 8.4 are exactly these balances. The mass map then has blocks larger than 1×1 whose entries, all nonzero, satisfy $\Delta = 0$, and these blocks carry the massless generations. Among the four pencils, the quintic is therefore the only one on which the characters of the symmetry group label the generations and the invariant deformation gives every generation a mass.

9. THE SIMULTANEOUS CONSTRUCTION PROBLEM

The GSC formulation in [27, 28] concerns a landscape $\mathcal{L} = \coprod_{\tau} M_{\tau}$ of compactification data and an involutive isometry μ defined on a union D of components. A desired function $S: \mathcal{L} \rightarrow [0, \infty)$ is smooth and proper on each component, has a unique global minimum at a viable point, and is invariant under μ wherever μ is defined. Its viability requirements originate in [26]. We use the following short restatement, without supplying new numerical meanings for the physical acceptance conditions.

C1	A slope-polystable bundle with $c_1 = 0$ and an effective anomaly class.
C2	Three net families, with the quotient index treated consistently.
C3	Wilson-line reduction of the visible gauge group to the Standard Model.
C4	Three quark–lepton families and no chiral exotics after projection.
C5	Allowed sufficiently heavy masses for vectorlike exotics.
C6	A Higgs pair with doublet–triplet splitting.
C7	Stabilization of the relevant moduli in a controlled effective regime.
C8	Acceptable physical Yukawa rank and hierarchy after normalization.
C9	Phenomenologically consistent supersymmetry breaking, if present.

These are nine simultaneous predicates on one complete datum, not nine alternative cases. Let A_i be the locus satisfying C_i . The existence problem is $\bigcap_{i=1}^9 A_i \neq \emptyset$. Nonemptiness of separate A_i , even of all pairs among three loci, need not imply a common point: $\{0, 1\}$,

$\{1, 2\}$ and $\{0, 2\}$ have pairwise intersections but empty triple intersection. Results at different bundles or different moduli values cannot be assembled into a single vacuum without proving their compatibility.

For a complete datum one must specify the threefold, visible and hidden bundles, polarization, free action and equivariant structures when used, Wilson line, effective Kähler potential and superpotential, and the domain in which omitted corrections are controlled. C5 and C7–C9 also need explicit acceptance predicates. Without them the conjecture is not a mathematical statement, since a threshold “fixed by comparison with measurement” is not a predicate on a datum. We fix one choice. The numbers are conventions, and a different choice gives a different conjecture; the failures proved in this paper for C7 and C8 hold for every choice, as Remark 9.3 records.

Definition 9.1 (The acceptance windows W). A datum satisfies the three conditions with the windows W when:

C7_W at the vacuum every modulus, that is every Kähler, complex structure, bundle and dilaton field of the four-dimensional theory, has a canonically normalized mass of at least $m_0 = 3 \times 10^4$ GeV; no exactly flat direction is allowed. The value is the lower bound for moduli that decay before nucleosynthesis [39, 40].

C8_W at the scale $\mu = 2 \times 10^{16}$ GeV the up-type, down-type and charged-lepton Yukawa matrices have rank three, and their singular-value ratios lie within a factor three of the values

$$\begin{aligned} \frac{m_u}{m_t} &= 5.3 \times 10^{-6}, & \frac{m_c}{m_t} &= 2.6 \times 10^{-3}, & \frac{m_d}{m_b} &= 8.9 \times 10^{-4}, \\ \frac{m_s}{m_b} &= 1.6 \times 10^{-2}, & \frac{m_e}{m_\tau} &= 2.8 \times 10^{-4}, & \frac{m_\mu}{m_\tau} &= 5.9 \times 10^{-2}, \end{aligned}$$

the ratios of the running masses at that scale in the minimal supersymmetric standard model with $\tan \beta = 10$ [38, Table V]; the Cabibbo element satisfies $0.15 \leq |V_{us}| \leq 0.30$. The factor three covers the dependence on $\tan \beta$ between the two cases tabulated in [38] and the threshold corrections at the matching scales.

C9_W supersymmetry is broken, some F-term being nonzero at the vacuum, and the gravitino mass lies in $[10^3, 10^6]$ GeV, the window around m_0 in which gravity-mediated soft masses lie between the weak scale and the moduli scale [40].

Conjecture 9.2 (GSC(W)). There is a complete datum, in the sense of the list above, satisfying C1–C6 and C7_W–C9_W.

Remark 9.3 (Dependence on the windows). Only a positive certificate depends on W . The negative results of this paper do not: a flat direction of the classical potential, as on the volume ray of the fixed datum in Section 15, violates C7_W for every $m_0 > 0$; a Yukawa matrix of rank at most two, or a vanishing one, violates C8_W for every window around any nonzero ratios; and the selection rules of Proposition 16.4 are statements about vanishing matrices. Where the paper writes C7, C8 or C9 below, the conditions are those of Definition 9.1.

10. A MARKED-POINT SELECTION CONSTRUCTION

The functional conditions have a clean differential-topological answer. They do not manufacture the point that is to be selected.

Theorem 10.1 (Selection with a supplied point). *Let M be a countable disjoint union of second-countable smooth Riemannian manifolds. Let μ be an involutive isometry on a union D of components. Given $p \in M$, the following are equivalent:*

- (i) $p \notin D$, or $\mu p = p$.
- (ii) *There is a nonnegative smooth globally proper function S on M with unique zero p and $S \circ \mu = S$ on D .*

The function can be chosen to have Hessian $2g_p$ at p .

Proof. An invariant function with unique zero must fix that zero under every symmetry defined there. This proves necessity.

Every second-countable smooth manifold has a nonnegative smooth proper exhaustion ρ . One direct construction takes a compact exhaustion $K_j \subset \text{int } K_{j+1}$, a locally finite partition of unity subordinate to the compact annular cover, and assigns successively increasing positive coefficients to its terms; outside K_j all nonzero coefficients can be arranged to be at least j . Its sublevel sets are closed and contained in some K_j , hence compact. These standard smooth-manifold constructions apply separately to the components; see [31].

On a component preserved by μ , replace ρ by $\rho + \rho \circ \mu$. On a pair of exchanged components choose ρ on one and transport it to the other. The resulting function is invariant and proper on each component. Enumerate the component orbits; outside the orbit of the component of p , add distinct positive integer offsets tending to infinity. Each orbit contains at most two components.

Choose a normal ball $B_r(p)$ with compact closure, and write $f(x) = d(p, x)^2$ there. If $p \in D$ and is fixed, this ball and f are invariant. Choose a smooth function $\theta : [0, \infty) \rightarrow [0, 1]$ equal to one for $t \leq r^2/4$ and zero for $t \geq r^2/2$. On the component of p set

$$S = \theta(f)f + (1 - \theta(f))(1 + \rho),$$

extending the expression by $1 + \rho$ outside the ball. This is smooth because $\theta(f)$ vanishes near its boundary. Its only zero is p : away from p every nonzero term in this convex combination is positive. It equals f near p , so its Hessian there is $2g_p$.

On the remaining components use their exhaustion plus the positive orbit offset. For a fixed sublevel only finitely many offsets occur. On the distinguished component a sublevel is a closed subset of $\bar{B}_r(p) \cup \{\rho \leq t - 1\}$; on all other relevant components it is compact by properness of ρ . Thus every sublevel of S is compact in the disjoint union. Invariance follows from the construction. $\square$

Remark 10.2 (What the point supplies). The construction uses the marked point p , exhaustions and cutoffs in addition to the metric. It proves the smooth-function assertion with a supplied compatible point. It does not prove that an unmarked metric determines p or that p satisfies C1–C9.

Proposition 10.3 (An intrinsic selector must respect all isometries). *Suppose a rule assigns S_g to an unmarked Riemannian manifold naturally under isometries. If S_g has a unique minimum p , every isometry fixes p . In particular there is no such uniquely minimizing rule on a positive-dimensional homogeneous Riemannian manifold.*

Proof. Naturality for an isometry h says $S_g(hx) = S_g(x)$. Hence hp is another minimum and equals p . Transitivity on a manifold with more than one point is incompatible with a common fixed point. $\square$

This is an obstruction to a particular notion of metric-only selection, not a proof that the actual heterotic moduli metric is homogeneous. It also shows why invariance under an additional compact group cannot be discarded merely because the partial mirror map is undefined at p : the additional group must still fix a unique selected point.

11. INDEX CONSTRAINTS AND THE MIRROR RESTRICTION

Proposition 11.1 (Index and finite descent). *Let X be a Calabi–Yau threefold and V a holomorphic bundle with $c_1(V) = 0$. Then*

$$\chi(X, V) = \frac{1}{2} \int_X c_3(V).$$

If $\pi : X \rightarrow Y$ is finite étale of degree d and $V = \pi^*V_Y$, then $\chi(X, V) = d\chi(Y, V_Y)$. Thus the three-family index condition downstairs requires $|\int_X c_3(V)| = 6d$.

Proof. In Hirzebruch–Riemann–Roch, the degree-six term of $\mathrm{ch}(V)\mathrm{td}(X)$ is $\mathrm{ch}_3(V) + c_1(V)c_2(X)/12 + \mathrm{rk}(V)c_1(X)c_2(X)/24$. The last two terms vanish and $\mathrm{ch}_3(V) = c_3(V)/2$. For an étale map the tangent bundle and Todd class pull back, and integration of a pulled-back top class multiplies by d . $\square$

For a quintic, expansion of $(1+H)^5/(1+5H)$ gives $c_3(T_X) = -40H^3$ and $\int_X H^3 = 5$, hence $\int c_3(T_X) = -200$. Consequently no standard-embedding free quotient has index of absolute value three, because $100 = 3d$ has no integer solution. This restriction is independent of the quotient group classification. It does not apply to arbitrary nonstandard bundles.

Under the standard-type partial mirror convention of [28], a fixed point has $h^{1,1} = h^{2,1}$ and hence $\chi(X) = 0$. The bundle on that component has $\int c_3(V) = \chi(X)$, so Proposition 11.1 gives zero net families. This proves that a selected viable point cannot be a mirror-fixed point in that specified domain. It does not exclude nonstandard data outside the domain, nor establish a mirror map on all $(0, 2)$ compactifications.

11.1. The Dwork pencil.

Proposition 11.2. (i) *For $\psi \neq 0$ and $\psi^5 \neq 1$, the mass map of the invariant direction h on X_ψ has rank 101, its entries in the basis of labels are 1, ψ and ψ^2 , on 51, 30 and 20 labels, and the masses of Theorem 1.3 are $|\psi|^{r(\mathbf{b})}$ times the Hodge factor $(g_{\mathbf{b}}g_{\sigma\mathbf{b}})^{-1/2}$.*

(ii) *Let X be a smooth quintic threefold and H a finite group acting freely on X . The standard embedding on X/H has $100/|H|$ net families. In particular condition (C2) fails on X/H for every such H , and no vacuum of the form $(X/H, T_{X/H})$ is viable.*

(iii) *On the sextic, octic and dextric pencils of Section 8, with $\psi \neq 0$ and $\psi^d \neq 1$, the mass map of the invariant direction has a kernel of dimension 4, 24 and 36, and condition (C2) fails on every member and on every quotient of a member by a group acting freely.*

Proof. Part (i) is Theorem 1.3. For (ii), a smooth quintic has $h^{1,1} = 1$ and $h^{2,1} = 101$, so $\chi(X) = -200$. A free action multiplies the Euler characteristic of the quotient by $1/|H|$, so $\chi(X/H) = -200/|H|$, and by Theorem 2.1 the net number of families is $|\chi(X/H)|/2 = 100/|H|$. Wilson lines on X/H break the gauge group but do not change this index. The equation $100/|H| = 3$ has no solution in integers, so (C2) fails. Part (iii) is Propositions 8.8 and 8.9. $\square$

12. FREE QUOTIENTS OF THE ONE-PARAMETER FAMILIES

The four pencils are four of the fourteen one-parameter families of Calabi–Yau threefolds with $h^{1,1} = 1$ whose Picard–Fuchs equations are hypergeometric, listed by Doran and Morgan [24]. Thirteen of them are complete intersections $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(w)$ of degrees $d_1, \dots, d_c$ with $\sum_j d_j = \sum_i w_i$, whose general member is quasi-smooth and misses the singular points of $\mathbb{P}(w)$; they are the rows of Table 5. The fourteenth, $X_{2,12} \subset \mathbb{P}(1, 1, 1, 1, 4, 6)$, is not of this kind: its quadric involves only the four coordinates of weight one, so it is not quasi-smooth at the point of $\mathbb{P}(4, 6)$ where the equation of degree twelve vanishes, and we leave it aside. For the thirteen, the count of families of Proposition 11.2 becomes a statement about every free quotient, through one arithmetic invariant.

Lemma 12.1 (Primes of a free action). *Let X be a compact Kähler threefold with $K_X \cong \mathcal{O}_X$, $h^{1,0} = h^{2,0} = 0$, $b_2 = 1$ and $\pi_1(X) = 0$, let L be an ample line bundle on X , and let g be an automorphism of prime order p without fixed points. Then p divides $\chi(X, kL)$ for every $k \geq 1$.*

Proof. Since X is simply connected, $H^2(X, \mathbb{Z})$ has no torsion, and since $h^{0,1} = h^{0,2} = 0$ the first Chern class identifies $\mathrm{Pic} X$ with $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Let A be its ample generator, so that $L = A^a$ with $a \geq 1$. The bundle g^*A is again an ample generator, so $g^*A \cong A$. The obstruction to lifting the action of the cyclic group $G = \langle g \rangle$ to A lies in $H^2(G, \mathbb{C}^*)$, which vanishes for a cyclic group, so A and all its powers are G -linearised and G acts on $H^q(X, A^m)$.

TABLE 5. The thirteen one-parameter complete intersections. The columns give the type, the weights, the intersection numbers, the Euler characteristic, the values $\chi(X, kL) = k^3 L^3/6 + k c_2 \cdot L/12$ for $k = 1, 2, 3$, their greatest common divisor g_X , and the order $|\chi|/6$ that a free action giving three families would need.

type	w	L^3	$c_2 \cdot L$	χ	$\chi(kL)$	g_X	$ \chi /6$
X_5	1^5	5	50	-200	5, 15, 35	5	—
X_6	$1^4, 2$	3	42	-204	4, 11, 24	1	34
X_8	$1^4, 4$	2	44	-296	4, 10, 20	2	—
X_{10}	$1^3, 2, 5$	1	34	-288	3, 7, 13	1	48
$X_{3,3}$	1^6	9	54	-144	6, 21, 54	3	24
$X_{4,2}$	1^6	8	56	-176	6, 20, 50	2	—
$X_{3,2,2}$	1^7	12	60	-144	7, 26, 69	1	24
$X_{2,2,2,2}$	1^8	16	64	-128	8, 32, 88	8	—
$X_{4,4}$	$1^4, 2^2$	4	40	-144	4, 12, 28	4	24
$X_{6,6}$	$1^2, 2^2, 3^2$	1	22	-120	2, 5, 10	1	20
$X_{4,3}$	$1^5, 2$	6	48	-156	5, 16, 39	1	26
$X_{6,4}$	$1^3, 2^2, 3$	2	32	-156	3, 8, 17	1	26
$X_{6,2}$	$1^5, 3$	4	52	-256	5, 14, 31	1	—

For $m \geq 1$ these groups vanish in positive degree by Kodaira's theorem, as $A^m = K_X \otimes A^m$ with A^m ample. For $1 \leq j < p$ the automorphism g^j has no fixed points, and the holomorphic Lefschetz formula of Atiyah and Bott [22] gives $\sum_q (-1)^q \text{tr}(g^j | H^q(X, A^m)) = 0$, that is, $\text{tr}(g^j | H^0(X, A^m)) = 0$. A representation of $\mathbb{Z}/p$ whose character vanishes off the identity is a multiple of the regular representation, so p divides $h^0(X, A^m) = \chi(X, A^m)$. Taking $m = ka$ gives the statement. $\square$

Proposition 12.2. *Let $X \subset \mathbb{P}(w)$ be a quasi-smooth complete intersection of one of the thirteen types of Table 5 that misses the singular points of $\mathbb{P}(w)$, and let H be a finite group acting freely on X . Then every prime divisor of $|H|$ divides g_X . Hence H is trivial for X_6 , X_{10} , $X_{3,2,2}$, $X_{6,6}$, $X_{4,3}$, $X_{6,4}$ and $X_{6,2}$; it is a 5-group for X_5 , a 3-group for $X_{3,3}$, and a 2-group for X_8 , $X_{4,2}$, $X_{2,2,2,2}$ and $X_{4,4}$. With the standard embedding, no free quotient of X carries three net families.*

Proof. As in Lemma 8.1, X is a smooth threefold with $K_X \cong \mathcal{O}_X(\sum_j d_j - \sum_i w_i) = \mathcal{O}_X$, and the Lefschetz theorem for quasi-smooth complete intersections [21] gives $h^{1,0} = h^{2,0} = 0$ and $b_2 = 1$. For π_1 , let $C \subset \mathbb{C}^N$ be the affine cone over X , the zero set of the c weighted homogeneous equations. Quasi-smoothness says that C is a complete intersection of dimension four with an isolated singular point at the origin, so by Hamm's theorem [23] its link is 2-connected; the weighted action of $\mathbb{R}_{>0}$ meets every sphere about the origin once, so $C \setminus \{0\}$ is homotopy equivalent to the link and is simply connected. As X misses the points with nontrivial isotropy, $\mathbb{C}^*$ acts freely on $C \setminus \{0\}$ with quotient X , and the exact sequence of homotopy groups of this fibration gives $\pi_1(X) = 0$.

Let p be a prime divisor of $|H|$ and $g \in H$ an element of order p . It has no fixed points, so by Lemma 12.1 with $L = \mathcal{O}_X(1)$, p divides $\chi(X, kL)$ for all $k \geq 1$. By Riemann–Roch, with $c_1(X) = 0$ and $\chi(\mathcal{O}_X) = 0$, $\chi(X, kL) = k^3 L^3/6 + k c_2 \cdot L/12$, a polynomial that takes integer values and vanishes at $k = 0$; writing it as $\alpha_1 \binom{k}{1} + \alpha_2 \binom{k}{2} + \alpha_3 \binom{k}{3}$ with integers α_i , its values at all $k \geq 1$ have the same greatest common divisor as $\alpha_1, \alpha_2, \alpha_3$, which is that of its values at $k = 1, 2, 3$, the number g_X . The intersection numbers are computed as in Lemma 8.1, from $c(T_X) = \prod_i (1 + w_i L) \prod_j (1 + d_j L)^{-1}$ and $L^3 = \prod_j d_j / \prod_i w_i$, and give the table.

By Theorem 2.1 the quotient X/H carries $|\chi(X)|/(2|H|)$ net families, so three families need $|H| = |\chi(X)|/6$. This is not an integer for X_5 , X_8 , $X_{4,2}$, $X_{2,2,2,2}$ and $X_{6,2}$. In the remaining eight rows it is 34, 48, 24, 24, 24, 20, 26 or 26, and each of these has a prime factor not dividing g_X : 17, 2, 2, 2, 3, 2 and 2 (twice). $\square$

The proposition contains Theorem 1.4(iv) and Proposition 11.2(ii) and strengthens them: the sextic and dectic pencils admit no free action at all, and on a quintic every free action is by a 5-group, so that only the orders 5 and 25 divide $|\chi| = 200$ and a free quotient of a quintic has 20 or 4 families; the quotient by $\mathbb{Z}/5 \times \mathbb{Z}/5$ with four families is already in [1]. For the five complete intersections in ordinary projective space the free actions were classified by Braun [25] with a computer search; the proposition uses only the holomorphic Lefschetz formula and Riemann–Roch.⁵

13. ONE FIXED NONSTANDARD THREEFOLD

We now replace a generic existence assertion by a single specified complex structure. The line-bundle data are the example of Buchbinder, Constantin and Lukas [29, Section 5.1]; neither the bundle nor its phenomenological interpretation is claimed here as a new discovery. The purpose of fixing the equation is to make every subsequent calculation refer to the same geometric datum.

In $A = (\mathbb{P}^1)^4$, use coordinates $[x_i : y_i]$, $1 \leq i \leq 4$. For $S \subset \{1, 2, 3, 4\}$ put

$$m_S = \prod_{i \in S} y_i^2 \prod_{i \notin S} x_i^2, \quad (c_0, c_1, c_2, c_3, c_4) = (1, 2, 5, 2, 1), \quad F_0 = \sum_S c_{|S|} m_S.$$

Write $r_i = x_i y_i$ and $e_i = x_i^2 + y_i^2$. In the order (12), (13), (14), (23), (24), (34) let $(a_{ij}) = (1, 2, 3, 5, 7, 11)$, and define

$$G = \sum_{i < j} a_{ij} r_i r_j e_k e_l + 13 r_1 r_2 r_3 r_4, \quad F_\theta = F_0 + \theta G, \quad X = X_\pi = \{F_\pi = 0\}, \quad (11)$$

where $\{k, l\}$ is the complement of $\{i, j\}$. Here π is the usual transcendental real number, not a freely varying modulus. This is one projective complex threefold with completely specified coefficients. We use transcendence only to avoid the finitely many algebraic bad parameters of a rational pencil; it supplies no dynamical mechanism selecting this complex structure.

Lemma 13.1 (Smooth seed). *The hypersurface $F_0 = 0$ is smooth over $\mathbb{C}$.*

Proof. On a coordinate stratum set the remaining nonzero affine coordinates $t_i = y_i/x_i$ and $u_i = t_i^2$. If exactly k coordinates are fixed at infinity and $4 - k - r$ at zero, the restricted polynomial is

$$f_{k,r}(u) = \sum_{j=0}^r c_{k+j} e_j(u_1, \dots, u_r), \quad 0 \leq k, \quad k + r \leq 4,$$

with e_j the elementary symmetric polynomial. At a singular point the restricted polynomial and all its derivatives in the r free variables vanish. Indeed $\partial/\partial t_i = 2t_i \partial/\partial u_i$ and $t_i \neq 0$. It suffices to exclude an affine singularity of each $f_{k,r}$, even allowing zero values for its free variables.

For $r = 0$ the constant is nonzero. For $r = 1$ its linear coefficient is nonzero. For $r = 2$, the three determinants $c_k c_{k+2} - c_{k+1}^2$ are 1, -21 , 1, so the bilinear polynomials have no singular zero.

For $r = 3, k = 0$, subtraction of two partial derivatives gives $(u_j - u_i)(5 + 2u_l)$. If two coordinates differ, the third is $-5/2$, and the partial derivative in one of the differing coordinates is $-21/2$, a contradiction. Hence all three coordinates equal s . For $r = 3, k = 1$

⁵On a quintic the group $\mathbb{Z}/5 \times \mathbb{Z}/5$ of [1], generated by the cyclic permutation of the coordinates and the diagonal phases $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$, acts on the five coordinates through a central extension of order 125, the Heisenberg group, and $H^0(X, \mathcal{O}_X(1)) = \mathbb{C}^5$ is its irreducible representation of dimension five; each cyclic subgroup of order five acts on it by the regular representation, as Lemma 12.1 requires.

the same argument uses $(u_j - u_i)(2 + u_l)$; the forced value -2 makes another partial derivative equal to 1. The two diagonal polynomials are

$$q_{0,3} = 1 + 6s + 15s^2 + 2s^3, \quad q_{1,3} = 2 + 15s + 6s^2 + s^3.$$

Both are relatively prime to their derivatives, as the explicit Bézout identities in Appendix F show.

For $r = 4$ write $f = f_{0,4}$ and $Q(x, y) = 5 + 2(x + y) + xy = (x + 2)(y + 2) + 1$. Direct differentiation gives

$$f_i - f_j = (u_j - u_i)Q(u_k, u_l).$$

Classify the multiplicities of four coordinates. If all are equal, the polynomial is $q_{0,4} = 1 + 8s + 30s^2 + 8s^3 + s^4$, again coprime to its derivative. For multiplicities $(3, 1)$, say (x, x, x, y) with $x \neq y$, the equations force $Q(x, x) = x^2 + 4x + 5 = 0$. A derivative in a repeated coordinate is $2 + 10x + 2x^2 + yQ(x, x)$, which reduces to $2(x - 4)$ modulo $Q(x, x)$. It cannot vanish since $Q(4, 4) = 37$.

For multiplicities $(2, 1, 1)$ the equations force $Q(x, x) = Q(x, z) = 0$ with $x \neq z$. Their difference gives $(x - z)(x + 2) = 0$, impossible since $Q(-2, -2) = 1$. If all coordinates are distinct, $Q(x, y) = Q(x, z) = 0$ forces $x = -2$ and then $Q(x, y) = 1$, also impossible.

For multiplicities $(2, 2)$, say (x, x, y, y) with $x \neq y$, we have $Q(x, y) = 0$, so $y \neq -2$ and $x = -(2y + 5)/(y + 2)$. After substitution, one partial derivative and f have respective numerators

$$g = y^2 - 8y - 21, \quad h = y^4 - 16y^3 - 66y^2 - 16y + 89.$$

Their denominators are $y + 2$ and $(y + 2)^2$. The Bézout identity in Appendix F proves they cannot both vanish. These cases exhaust the strata and partitions. $\square$

Let a act in all four factors by $[x : y] \mapsto [x : -y]$, and let b act by $[x : y] \mapsto [y : x]$. They commute in PGL_2 and generate $\Gamma \cong (\mathbb{Z}/2)^2$. Both F_0 and G are invariant.

Theorem 13.2 (The fixed quotient). *The threefold X_π is smooth and Γ acts freely, preserving its holomorphic three-form. Thus $Y = X_\pi/\Gamma$ is a smooth projective Calabi–Yau threefold. Its Euler characteristic is -32 , and its Hodge numbers are $(h^{1,1}, h^{2,1}) = (4, 20)$.*

Proof. The locus of singular fibres of the projective pencil (11) is a closed algebraic subset of the parameter line, defined over $\mathbb{Q}$: project the closed relative singular locus from $A \times \mathbb{A}^1$ using properness. It omits zero by Lemma 13.1, so it is finite and all its points are algebraic over $\mathbb{Q}$. Transcendence of π therefore proves smoothness at the specified parameter, without a numerical sampling assertion.

Each of a, b, ab has sixteen ambient fixed points. They have coordinates respectively in $\{0, \infty\}$, $\{1, -1\}$ and $\{i, -i\}$ in each factor. At these points F_0 is respectively one of 1, 2, 5, or 48, or 16, after the indicated homogeneous normalizations. All are nonzero. The values of G there are algebraic. Thus $F_0 + \pi G$ is nonzero as well, either because $G = 0$ or because otherwise its vanishing would make π algebraic. Hence the action is free.

Adjunction gives $K_X = \mathcal{O}_X$. Either generator has determinant -1 on each of four coordinate pairs, with product 1. Since the defining polynomial is invariant, the residue holomorphic three-form is invariant and descends. Lefschetz and descent give $h^{1,0}(Y) = h^{2,0}(Y) = 0$. The ambient Chern formula gives $c_3(T_X) = -16 \sum_{i < j < k} J_i J_j J_k$, whose integral is -128 ; division by four gives $\chi(Y) = -32$. The four ambient hyperplane classes generate $H^2(X, \mathbb{Q})$ and are invariant, so $h^{1,1}(Y) = 4$; $\chi(Y) = 2(h^{1,1} - h^{2,1})$ then gives $h^{2,1}(Y) = 20$. $\square$

14. THE BUNDLE, ITS SPECTRUM AND THE FIXED WILSON LINE

On this X take $V = \bigoplus_{a=1}^5 L_a$, with $L_a = \mathcal{O}_X(k_a)$ and columns

$$K = (k_a^i) = \begin{pmatrix} -1 & -1 & 0 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 \\ 1 & 2 & 0 & -1 & -2 \end{pmatrix}. \quad (12)$$

Fix the Kähler class to be $J = 100(J_1 + J_2 + J_3 + J_4)$. The number 100 fixes a scale in chosen units; it is not a proved error bound for string corrections. Use the untwisted tensor lifts of a, b on all summands and zero bundle-deformation and charged-singlet expectation values. These choices remain fixed in the audit below.

Proposition 14.1 (Chern classes and slopes). *The bundle has $c_1(V) = 0$, is slope-polystable at J , and has*

$$\left(\int_X c_2(V) J_i \right) = (24, 8, 20, 12), \quad \int_X c_3(V) = -24, \quad \chi(X, V) = -12.$$

The cover anomaly class is the effective class $6J_1J_3 + 2J_1J_4$. The bundle descends to V_Y on Y , with $\chi(Y, V_Y) = -3$.

Proof. In the ambient ring $J_i^2 = 0$, the hypersurface class is $2 \sum_i J_i$. Consequently $\int_X J_i J_j J_k = 2$ for distinct indices and zero otherwise, and $c_2(T_X) = 4 \sum_{i < j} J_i J_j$, pairing to 24 with each J_i . The rows and columns of (12) sum to zero. Thus $c_1(V) = 0$ and every summand has slope $12 \cdot 100^2 \sum_i k_a^i = 0$. Line bundles are slope-stable; their equal-slope sum is polystable. For $l_a = \sum_i k_a^i J_i$ use $c_2(V) = -\frac{1}{2} \sum_a l_a^2$ and $c_3(V) = \frac{1}{3} \sum_a l_a^3$. Alternatively the line-bundle indices

$$\chi(\mathcal{O}_X(k)) = 2 \sum_{i < j < l} k_i k_j k_l + 2 \sum_i k_i$$

are $0, -8, 0, 0, -4$. These give the stated numbers. The anomaly pairs as $(0, 16, 4, 12)$, as does $6J_1J_3 + 2J_1J_4$. Lefschetz and Poincaré duality identify $H^4(X, \mathbb{Z})$ with the integral dual of the ambient H^2 basis, so these equal pairings identify the classes. The indicated intersections of distinct hyperplane divisors are effective.

The lifts of a, b anticommute on each $\mathcal{O}(1)$ factor. On $\mathcal{O}(k_a)$ their commutator is $(-1)^{\sum_i k_a^i} = 1$; their squares are one. These are commuting equivariant structures. Their product determinant on V is trivial since $\sum_a k_a = 0$. The invariant polarization descends as a real Kähler class, and polystability descends. Finite *étale* descent gives $\chi(Y, V_Y) = \chi(X, V)/4 = -3$. $\square$

Effectiveness of the anomaly on the cover proves rational effectiveness downstairs by pushforward and division by four. It does not by itself certify a quantized hidden sector or the integral torsion part of the anomaly on Y . Neither has been supplied here. This qualification is part of C1, not a correction to the proved three-family index C2.

14.1. An exact cohomology calculation. Let R_Γ denote the regular representation and let $\mathbf{1}$ be the trivial character. A character is specified by its values on (a, b) , ordered $(++)$, $(+-)$, $(-+)$, $(--)$.

Theorem 14.2 (Cohomology at the specified complex structure). *For the fixed choices above,*

$$H^1(X, V) = 3R_\Gamma, \quad H^2(X, V) = 0, \\ H^1(X, \Lambda^2 V) = 3R_\Gamma + (R_\Gamma - \mathbf{1}), \quad H^2(X, \Lambda^2 V) = R_\Gamma - \mathbf{1}.$$

Here $R_\Gamma - \mathbf{1}$ is the actual sum of the three nontrivial characters. The only nonzero individual dimensions are

$$h^1(L_2) = 8, \quad h^1(L_5) = 4, \\ h^1(L_2 L_4) = 4, \quad h^1(L_4 L_5) = 8, \\ h^1(L_2 L_5) = h^2(L_2 L_5) = 3.$$

Proof. For any multidegree k , use the equivariant Koszul sequence

$$0 \longrightarrow \mathcal{O}_A(k - \mathbf{2}) \xrightarrow{F_\pi} \mathcal{O}_A(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow 0.$$

On $\mathbb{P}^1$, $H^0(\mathcal{O}(d))$ has dimension $d + 1$ for $d \geq 0$; $H^1(\mathcal{O}(d))$ has dimension $-d - 1$ for $d \leq -2$; all cohomology vanishes for $d = -1$. Künneth therefore leaves at most one nonzero ambient

degree for each k . For L_2 , the source has H^2 of dimension 8 and the target vanishes; for L_5 , the target has H^1 of dimension 4 and the source vanishes. The other three summands are acyclic. For L_2L_4 the target has H^1 of dimension 4, source zero; for L_4L_5 the source has H^2 of dimension 8, target zero. For $L_2L_5 = \mathcal{O}_X(0, -2, 2, 0)$ the target has H^1 of dimension 3 and the source has H^3 of dimension 3, giving the two groups shown. Six other pair products are acyclic without a rank calculation.

The remaining product is $L_1L_2 = \mathcal{O}_X(-2, -3, 2, 3)$. Both ambient groups are H^2 of dimension 24, and the relevant multiplication matrix is $M(\theta) = M_0 + \theta M_G$. In monomial bases described in Appendix F, exact integer arithmetic gives

$$\det M(1) = 393460125696 \neq 0. \quad (13)$$

This is a certificate that $\det M(\theta) \in \mathbb{Z}[\theta]$ is nonzero. Since π is transcendental, $M(\pi)$ is invertible. Hence L_1L_2 is acyclic. No assertion that X_1 is smooth is needed for this determinant argument.

The character calculation is equally explicit. On a factor represent $H^0(\mathcal{O}(d))$ by $x^{d-v}y^v$, $0 \leq v \leq d$, and $H^1(\mathcal{O}(d))$ by the Čech Laurent monomials with $d+1 \leq v \leq -1$. The generator a multiplies the monomial by $(-1)^v$; b sends v to $d-v$, with a minus sign for an H^1 factor from interchanging the two Čech charts. Tensor these rules. The groups of dimensions 8, 4, 4, 8 above give respectively $2R_\Gamma, R_\Gamma, R_\Gamma, 2R_\Gamma$. Both groups for L_2L_5 have character multiplicities $(0, 1, 1, 1)$. The multiplication matrix is equivariant, so its isomorphism contributes no further kernel or cokernel characters. This proves every asserted representation, not just its Euler characteristic. $\square$

14.2. Projection of the charged matter. Fix

$$\rho(a) = \text{diag}(1, 1, 1, -1, -1) \in SU(5), \quad \rho(b) = I.$$

The centralizer inside $SU(5)$ is $S(U(3) \times U(2))$. Taking Γ invariants after tensoring with the Wilson representation, each component of the three regular representations occurs three times. The extra $R_\Gamma - \mathbf{1}$ in $H^1(\Lambda^2 V)$ contributes one weak doublet and no color triplet: doublets carry character $(-, +)$ and triplets $(+, +)$. Its dual in H^2 contributes the conjugate doublet and no triplet. The perturbative observable charged spectrum is therefore three $10 + \bar{5}$ families and one Higgs doublet pair, with no additional massless charged exotics. In this spectrum the mass requirement for additional vectorlike charged exotics has no fields to which it must be applied. This is a spectrum calculation, not a calculation of singlet masses, hidden matter or Higgs potential parameters.

14.3. A residual gauge factor at this same point. The preceding centralizer computation concerns only the $SU(5)$ factor. The split structure group $S(U(1)^5)$ has four associated visible Abelian gauge fields. For the standard axionic mass map at the Abelian locus, the charges are the columns of K . This matrix has rank 3, and

$$q = (-4, 1, 6, -4, 1), \quad Kq = 0, \quad \sum_a q_a = 0. \quad (14)$$

Thus one traceless Abelian combination is not removed by that map. A universal-axion row of the form $\beta_i k_a^i$ also annihilates q . With our zero charged-singlet expectation values there is no Higgs mechanism removing it. Its charges on the families are 1 on the 10's from L_2, L_5 and -3 on the $\bar{5}$'s from L_2L_4, L_4L_5 ; the Higgs charges are ± 2 . These are nonzero charges, so this is not a redundant generator. In the specified perturbative model, C3 interpreted as precisely the Standard Model visible group therefore fails.

15. THE YUKAWA AND STABILIZATION TESTS AT THAT POINT

This section keeps (11), (12), J , the Wilson line and all singlet expectation values fixed. In particular we do not import a different model's successful flavour fit.

Proposition 15.1 (A perturbative rank obstruction). *At the split-bundle point, the renormalizable perturbative up-type Yukawa matrix has rank at most 2, and the down-type matrix is zero. These statements hold after any invertible matter normalization.*

Proof. Let e_a be the standard basis of the five charge coordinates, modulo the common diagonal. There are two 10 families of charge e_2 and one of charge e_5 . The up-type Higgs has charge $-e_2 - e_5$. Consequently its allowed up-type couplings have the form

$$Y_u = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & B \\ A & B & 0 \end{pmatrix},$$

which has rank at most two regardless of the values of A, B . The $\bar{5}$ families have charges $e_2 + e_4$ or $e_4 + e_5$, and the down-type Higgs has charge $e_2 + e_5$. Every proposed $10 \bar{5} H_d$ coupling therefore has total charge with first and third components zero, nonnegative other components and total sum five. It cannot be a multiple of $(1, 1, 1, 1, 1)$; it is forbidden by the perturbative $S(U(1)^5)$ selection rule. Massive Abelian gauge fields still impose this perturbative selection rule. Hence $Y_d = 0$. Invertible left and right normalization preserves rank. $\square$

Nonzero singlet expectation values, a non-Abelian bundle deformation, or computed nonperturbative contributions could change these matrices. None is part of the fixed point just audited. Their possible existence cannot turn its rank-deficient matrices into a verified C8 certificate.

For a leading-order unfluxed Calabi–Yau description, take vanishing classical superpotential on the holomorphic locus and vanishing charged expectation values. The slopes remain zero along $J = \lambda \sum_i J_i$ for every $\lambda > 0$. Thus its D-term potential vanishes on that ray and its F-term potential also vanishes when $W = 0$. The overall volume is a flat direction. Under the requirement that the relevant moduli be stabilized, C7 fails in this specified approximation. There is also no verified supersymmetry-breaking sector: the classical F and D auxiliary fields vanish there. This cannot certify C9 if nonzero supersymmetry breaking is required. Corrections and a hidden sector require a joint construction and estimates on this same datum.

15.1. A targeted deformation beyond the split locus. The charge calculation indicates a concrete next construction, rather than a change of title. Consider the extensions

$$0 \longrightarrow L_2 \longrightarrow E \longrightarrow L_1 \oplus L_3 \longrightarrow 0, \quad V' = E \oplus L_4 \oplus L_5.$$

Their parameters lie in $H^1(L_2 L_1^{-1}) \oplus H^1(L_2 L_3^{-1})$. The first group has dimension 12, directly from ambient H^1 ; the second has dimension 20, from ambient H^2 in the Koszul source. They each contain invariant classes (three and five respectively) under the same lifts, so nonzero equivariant extensions exist. Their extension arrows have no composable pair, and the extensions are honest holomorphic bundles; no unsolved Maurer–Cartan equation is being assumed here.

The charges of these singlets under (14) are 5 and -5 . Their nonzero expectation values can therefore Higgs the residual Abelian factor at the level of the charge matrix. Moreover the exact sequences for E , $\Lambda^2 E$ and $E \otimes L_i$ show that the charged cohomology dimensions above persist for these extensions: $L_1, L_3, L_1 L_3, L_1 L_2, L_2 L_3$ and the indicated products with L_4, L_5 are acyclic except for the subbundles already counted. The equivariant isomorphisms follow from the same long exact sequences.

This does not give a replacement nine-condition certificate. At the original polarization all three L_1, L_2, L_3 have slope zero. A nontrivial extension then has the proper equal-slope subbundle L_2 and is not slope-stable; one cannot simply retain the original polystability proof. Let $\kappa_i = \int_X J_i J^2$. Requiring L_4 and L_5 to stay of slope zero gives $\kappa_3 = \kappa_4$ and $\kappa_1 + \kappa_2 = 2\kappa_4$. At a nearby class with $\kappa = (c + s, c - s, c, c)$, the slopes of L_1, L_3, L_2 are $-s, -s, 2s$. The sign $s < 0$ makes the obvious extension subbundles have negative slope. Proving stability against

every saturated subsheaf, fixing a polarization in that chamber, computing the resulting Yukawa operators, and stabilizing all these deformations remain separate obligations. We have not performed them. This branch is a construction route, not a second candidate silently substituted into the first audit.

15.2. A stronger obstruction for analytic singlet insertions. The extension route above does not, by itself, repair the missing down-type matrix. In fact the charge obstruction persists for a larger specified class of local operators.

Proposition 15.2 (Analytic singlet obstruction). *Consider a perturbative superpotential holomorphic near the split point in the singlet coordinates $H^1(X, \text{End } V)$ and neutral complex-structure coordinates, with no additional charged spurions. Then every Taylor coefficient of each down-type Yukawa entry vanishes. This statement holds on the local equivariant deformation chart and on any holomorphic branch through it on which this expansion applies.*

Proof. A charged singlet in $H^1(L_a L_b^{-1})$ has charge $e_a - e_b$. All groups with target $a = 1$ or $a = 3$ and $b \neq a$ have $H^1 = 0$. Here are their full cohomology dimensions (h^0, h^1, h^2, h^3) :

(a, b)	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(3, 1)$	$(3, 2)$	$(3, 4)$	$(3, 5)$
h^2	12	0	0	12	0	20	0	4

Every other cohomology degree in this table is zero. These facts follow from the ambient Koszul sequence without any multiplication-rank assumption: a degree -1 kills the relevant ambient group and the remaining nonzero source or target lies in degree two or three. Thus every available charged singlet has nonpositive first and third charge components. Diagonal bundle and complex-structure coordinates are neutral.

The charge Q of a proposed $10\bar{5} H_d$ operator has $Q_1 = Q_3 = 0$ and $\sum_a Q_a = 5$. Adding any nonnegative-power monomial in the available singlets gives a charge with first and third entries at most zero and still total sum five. A vector of total sum five is zero in the quotient by the diagonal charge exactly when it equals $(1, 1, 1, 1, 1)$. This is impossible here. Gauge equivariance therefore forces each coefficient in the holomorphic Taylor series to vanish. The same identity holds after restricting to any branch admitting that series. $\square$

This is an all-orders statement about the indicated *analytic perturbative singlet expansion*, not about every string contribution. Nonperturbative factors with axionic charges, a singular change of effective description, or a different topological bundle are outside its hypotheses. A new construction using them must calculate their actual coefficients and stabilize the resulting point. In particular the existence of an extension bundle alone cannot close C8 here.

16. THE SPLIT BRANCH ON THE TETRAQUADRIC

The ledger of the next section records two failures at the fixed datum that do not depend on the complex structure: the residual Abelian factor of Section 14.3 and the Yukawa deficiency of Proposition 15.1. This section asks whether another choice of five line bundles on the same manifold could avoid them while staying at the split locus. The first failure cannot be avoided by any sum of line bundles on any Calabi–Yau threefold with at most four Kähler parameters; the second is a property of the charge matrix, and the selection rules decide it in finitely many steps for each datum.

16.1. The residual Abelian factor is forced. Let X be a Calabi–Yau threefold, $h = h^{1,1}(X)$, and let $J_1, \dots, J_h$ be divisor classes spanning $H^{1,1}(X, \mathbb{Q})$. A sum of line bundles $V = \bigoplus_{a=1}^n L_a$ with $c_1(L_a) = \sum_i k_a^i J_i$ has the integer charge matrix $K = (k_a^i) \in \mathbb{Z}^{h \times n}$, whose columns are the vectors k_a . For a Kähler class J write $\kappa_i(J) = \int_X J_i J^2$.

Theorem 16.1 (The residual factor). *Suppose $c_1(V) = 0$ and V is slope-polystable with respect to J . Then*

- (i) $\text{rank } K \leq h - 1$;

- (ii) the traceless kernel $\{q \in \mathbb{Q}^n : Kq = 0, \sum_a q_a = 0\}$ has dimension at least $n - h$;
- (iii) at the split locus, with vanishing expectation values of all charged singlets, exactly $n - 1 - \text{rank } K \geq n - h$ of the $n - 1$ traceless Abelian gauge bosons of the structure group $S(U(1)^n)$ are massless.

In particular, for $n = 5$ and $h \leq 4$ at least one Abelian gauge boson beyond the Standard Model stays massless at every split point of every rank-five sum of line bundles on X , and condition C3, read as the visible gauge group being exactly the Standard Model group, fails there.

Proof. The slope of L_a is $\mu_J(L_a) = \int_X c_1(L_a)J^2 = \sum_i k_a^i \kappa_i$. A polystable sum has all summands of slope $\mu_J(V)$, and $\mu_J(V) = \int_X c_1(V)J^2 = 0$. Thus $K^T \kappa = 0$. The vector κ is nonzero, because $\sum_i t_i \kappa_i = \int_X J^3 > 0$ when $J = \sum_i t_i J_i$. Hence the rows of K satisfy one nontrivial linear relation and $\text{rank } K \leq h - 1$, which is (i). Since $c_1(V) = \sum_a c_1(L_a) = 0$, the vector $\mathbf{1} = (1, \dots, 1)$ lies in $\ker K$, which has dimension $n - \text{rank } K \geq n - h + 1$; the hyperplane $\sum_a q_a = 0$ does not contain $\mathbf{1}$, so it meets $\ker K$ in a space of dimension at least $n - h$, which is (ii).

For (iii) we use the Green–Schwarz masses of the Abelian gauge bosons as computed by Anderson, Gray, Lukas and Palti [33, Section 2.7]. The axions of the h Kähler moduli and the universal axion shift under the $U(1)$ gauge transformations with coefficients k_a^i and $\beta_i k_a^i$ respectively, and the mass matrix of the gauge bosons A_a in the basis of the n summands is $M_{ab} = k_a^T \tilde{G} k_b$, where $\tilde{G} = G + \beta \beta^T / (4s^2)$, G is the positive definite metric on the Kähler moduli and s the dilaton. The matrix $\tilde{G}$ is positive definite, so the quadratic form $q \mapsto q^T M q = (Kq)^T \tilde{G} (Kq)$ vanishes exactly on $\ker K$: the combination $\sum_a q_a A_a$ is massless if and only if $Kq = 0$. On the traceless vectors, which parametrize the gauge bosons of $S(U(1)^n)$, the massless combinations form the space of (ii), of dimension $n - 1 - \text{rank } K$. This is the count of [33, eq. (2.31)], and (i) gives the lower bound $n - h$, which is the statement of [33, eq. (2.32)] that fewer than five Kähler moduli leave at least one $U(1)$ massless. A nonzero traceless q in $\ker K$ acts nontrivially on some summand, so the surviving gauge boson is not a redundant generator. $\square$

Remark 16.2 (Reading the fixed datum). For the matrix (12), $\text{rank } K = 3 = h - 1$, the largest value Theorem 16.1 allows, and $\ker K$ is spanned by $\mathbf{1}$ and the vector q of (14). The residual factor of Section 14.3 is therefore not a defect of the choice (12); it is the minimum that the tetraquadric, with $h^{1,1} = 4$, imposes on every rank-five split datum. The polarization locus is tied to the same number. The constraint $K^T \kappa = 0$ confines κ to a space of dimension $h - \text{rank } K = 1$, the line through $(1, 1, 1, 1)$, and for $J = \sum_i t_i J_i$ with all $t_i > 0$ the intersection numbers of Proposition 14.1 give $\kappa_i = 4e_2(t_j : j \neq i) = 4(e_2(t) - t_i(e_1(t) - t_i))$, where e_1, e_2 are the elementary symmetric functions. All four κ_i are equal exactly when $t_i(e_1(t) - t_i)$ takes one value, so each t_i is a root of $x(e_1 - x) = c$, hence $t_i \in \{c', e_1 - c'\}$ for one c' ; if m of the coordinates equal c' and $4 - m$ equal $e_1 - c'$, summing gives $(m - 3)e_1 = (2m - 4)c'$, which has a solution with all $t_i > 0$ only for $m = 0$ or $m = 4$, that is, all coordinates equal. Thus the ray $J = \lambda \sum_i J_i$ of Section 15 is the whole locus of Kähler classes at which (12) is polystable, and the flat classical direction of C7 and the massless generator of C3 are two readings of the one equality $4 - \text{rank } K = 1$. For $n = h + 1$ the two counts $n - 1 - \text{rank } K$ and $h - \text{rank } K$ agree for every split datum.

16.2. Selection rules on eleven split data. Let $X \subset (\mathbb{P}^1)^4$ be any smooth tetraquadric and $V = \bigoplus_{a=1}^5 \mathcal{O}_X(k_a)$ with $c_1(V) = 0$. Under the structure group $S(U(1)^5)$ the charged fields carry the charges of Section 14.3, in the coordinates $e_1, \dots, e_5$ modulo the diagonal $\mathbf{1}$, as listed in Table 6. A term of the perturbative superpotential is a product of charged fields times a monomial in the singlets, and it is gauge invariant exactly when its total charge is a multiple of $\mathbf{1}$. For the down-type coupling $\mathbf{10}_a \mathbf{\bar{5}}_{bc} \mathbf{\bar{5}}_{de}$ the charge $Q = e_a + e_b + e_c + e_d + e_e$ has coordinate sum five, so invariance after insertion of singlets requires $\mathbf{1} - Q$ to be a nonnegative integer combination of the available singlet charges; for the up-type coupling $\mathbf{10}_a \mathbf{10}_b \mathbf{5}_{de}$ the

TABLE 6. The charged fields of a rank-five split datum under $S(U(1)^5)$.

field	cohomology	charge	representation
family 10	$H^1(L_a)$	e_a	10 of $SU(5)$
family or Higgs 5	$H^1(L_a L_b), a < b$	$e_a + e_b$	5
Higgs 5	$H^2(L_a L_b), a < b$	$-e_a - e_b$	5
singlet	$H^1(L_a L_b^{-1}), a \neq b$	$e_a - e_b$	1

same is required of $e_d + e_e - e_a - e_b$. Which groups are nonzero depends on X , but on every smooth tetraquadric the restriction sequence $0 \rightarrow \mathcal{O}_A(k - 2\mathbf{1}) \rightarrow \mathcal{O}_A(k) \rightarrow \mathcal{O}_X(k) \rightarrow 0$ gives

$$h^i(X, \mathcal{O}_X(k)) \leq h^i(A, \mathcal{O}_A(k)) + h^{i+1}(A, \mathcal{O}_A(k - 2\mathbf{1})), \quad (15)$$

with the ambient groups given by the Künneth formula on $A = (\mathbb{P}^1)^4$. We call a field *possibly present* when the bound (15) for its group is positive. A coupling forbidden with every possibly present singlet is forbidden on every smooth tetraquadric in the class; a coupling allowed by the charges is only not excluded by them.

Lemma 16.3 (Flow criterion). *Let E be a set of ordered pairs (p, q) of distinct elements of $N = \{1, \dots, n\}$ and $v \in \mathbb{Z}^n$ with $\sum_p v_p = 0$. The following are equivalent.*

- (a) $v = \sum_{(p,q) \in E} m_{pq}(e_p - e_q)$ with integers $m_{pq} \geq 0$.
- (b) For every $B \subset N$ that no pair of E leaves, that is, $p \in B$ and $(p, q) \in E$ imply $q \in B$, one has $\sum_{p \in B} v_p \leq 0$.

Proof. A pair (p, q) contributes $e_p - e_q$, whose sum over B is 1 if $p \in B \not\subset q$, -1 if $q \in B \not\subset p$ and 0 otherwise. When no pair leaves B the first case does not occur, so (a) gives $\sum_B v \leq 0$, which is (b).

Conversely, form the network with a source s , a sink t , the vertices of N , arcs $s \rightarrow p$ of capacity v_p for $v_p > 0$, arcs $q \rightarrow t$ of capacity $-v_q$ for $v_q < 0$, and arcs $p \rightarrow q$ of unbounded capacity for $(p, q) \in E$. Let $S \ni s, T \ni t$ be a cut of finite capacity and $B = S \cap N$. No pair of E leaves B , since such an arc would cross the cut with unbounded capacity. The capacity of the cut is

$$\sum_{p \notin B, v_p > 0} v_p + \sum_{q \in B, v_q < 0} (-v_q) = \sum_{v_p > 0} v_p - \sum_{p \in B} v_p \geq \sum_{v_p > 0} v_p$$

by (b). The cut $S = \{s\}$ has capacity $\sum_{v_p > 0} v_p$, so this is the minimum, and the maximal flow theorem of Ford and Fulkerson [34] gives an integral flow of that value, because all capacities are integers. It saturates every arc at s and at t . Let m_{pq} be the flow on the arc $p \rightarrow q$. Conservation at $p \in N$ says that $\sum_q m_{pq} - \sum_q m_{qp}$ equals the inflow from s minus the outflow to t , which is v_p . This is (a). $\square$

The criterion is decided by inspecting the 2^5 subsets; the cases below were also checked by a maximal flow computation.⁶

Proposition 16.4 (Eleven split data). *Consider the matrix (12) and the ten rank-five sums of line bundles with entries in $[-2, 2]$ listed in Table 7, which are the classes, up to permutation of the four factors of $(\mathbb{P}^1)^4$, found by the scan of [41] to satisfy $c_1(V) = 0$, the anomaly inequality, the index $-3|G|$ for a free group G of the stated order, a Kähler class with all slopes zero, and the upstairs spectrum tests of Section 14. With all possibly present singlets inserted to every order:*

- (i) for (12) every down-type coupling vanishes on every smooth tetraquadric, while the up-type couplings $\mathbf{10}_2 \mathbf{10}_5 \mathbf{5}_{25}$ and, with one singlet of charge $e_5 - e_2$, $\mathbf{10}_2 \mathbf{10}_2 \mathbf{5}_{25}$ and $\mathbf{10}_5 \mathbf{10}_5 \mathbf{5}_{25}$ are allowed by the charges;

⁶The program `code/yukawa_charges.py` of the repository named in the note under the abstract lists, for each datum, the possibly present fields by (15), decides the two couplings by a maximal flow and prints either a closed set B violating (b) or an explicit decomposition (a). Its output is the file `yukawa_charges_results.json`.

TABLE 7. The eleven split data. Rows of K are the vectors k_a ; the **10** column lists the summands L_a with $H^1(L_a)$ possibly present; “down” and “up” give a closed set B of Lemma 16.3 forbidding every coupling of that type, or one allowed coupling with the singlet insertions it needs ($p \rightarrow q$ denotes a singlet of charge $e_p - e_q$).

datum	rows k_a	10	down	up
(12)	$(-1, 0, 0, 1), (-1, -3, 2, 2), (0, 1, -1, 0), (1, 1, -1, -1), (1, 1, 0, -2)$	L_2, L_5	$B = \{1\}, B = \{3\}$	$\mathbf{10}_2 \mathbf{10}_5 \mathbf{5}_{25}$
1 ($ G = 2$)	$(-2, 1, 1, -1), (-1, -1, 1, 1), (-1, -1, 1, 1), (2, 0, -2, 1), (2, 1, -1, -2)$	L_4	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{12}; 3 \rightarrow 1, 5 \rightarrow 2$	$\mathbf{10}_4 \mathbf{10}_4 \mathbf{5}_{12}; 2 \times (5 \rightarrow 4), 1 \rightarrow 5, 2 \rightarrow 5$
2 ($ G = 4$)	$(-2, 1, 0, -1), (-1, -1, 1, 1), (-1, -1, 1, 1), (2, 0, -2, 1), (2, 1, 0, -2)$	L_4, L_5	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{12}; 3 \rightarrow 1, 5 \rightarrow 2$	$\mathbf{10}_4 \mathbf{10}_4 \mathbf{5}_{15}; 2 \times (5 \rightarrow 4), 1 \rightarrow 5$
3 ($ G = 2$)	$(-1, 0, 0, 1), (-1, 0, 0, 1), (-1, 0, 0, 1), (1, -2, 1, -1), (2, 2, -1, -2)$	L_5	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{34}; \text{none}$	$B = \{4\}$
4 ($ G = 2$)	$(-1, -1, 1, 0), (-1, 0, 0, 1), (-1, 0, 0, 1), (1, -1, 0, 0), (2, 2, -1, -2)$	L_5	$\mathbf{10}_5 \mathbf{5}_{14} \mathbf{5}_{33}; \text{none}$	$B = \{2, 4\}$
5 ($ G = 2$)	$(-1, 0, 1, 0), (-1, 0, 1, 0), (0, -2, 0, 1), (1, 1, -2, 0), (1, 1, 0, -1)$	L_3, L_4	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{15}; \text{none}$	$B = \{1\}$
6 ($ G = 4$)	$(-1, 0, 1, 0), (-1, 1, 0, 0), (0, -2, 1, 1), (1, 0, -2, 1), (1, 1, 0, -2)$	L_3, L_4, L_5	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{12}; 4 \rightarrow 1, 5 \rightarrow 2$	$\mathbf{10}_3 \mathbf{10}_3 \mathbf{5}_{25}; 2 \times (4 \rightarrow 3), 2 \times (5 \rightarrow 4), 2 \rightarrow 5$
7 ($ G = 4$)	$(-1, 0, 0, 1), (-1, 0, 0, 1), (0, -2, 1, 1), (1, 1, -2, -1), (1, 1, 1, -2)$	L_3, L_5	$\mathbf{10}_5 \mathbf{5}_{12} \mathbf{5}_{12}; 5 \rightarrow 1, 4 \rightarrow 5, 5 \rightarrow 2$	$\mathbf{10}_3 \mathbf{10}_3 \mathbf{5}_{12}; 2 \times (5 \rightarrow 3), 2 \times (4 \rightarrow 5), 1 \rightarrow 4, 2 \rightarrow 4$
8 ($ G = 4$)	$(-1, -1, 1, 1), (-1, 1, 0, 0), (0, -2, 1, 1), (1, 1, -2, 0), (1, 1, 0, -2)$	L_3, L_4, L_5	$\mathbf{10}_3 \mathbf{5}_{12} \mathbf{5}_{12}; 4 \rightarrow 1, 5 \rightarrow 2$	$\mathbf{10}_4 \mathbf{10}_4 \mathbf{5}_{24}; 2 \rightarrow 4$
9 ($ G = 4$)	$(-1, 0, 1, 0), (-1, 1, -2, 0), (0, -1, 0, 1), (0, -1, 0, 1), (2, 1, 1, -2)$	L_5	$B = \{1\}$	$B = \{1, 2, 3\}$
10 ($ G = 4$)	$(-1, 0, 1, 0), (-1, 1, -2, 0), (0, -1, 0, 1), (1, -1, -1, 1), (1, 1, 2, -2)$	L_5	$B = \{1\}$	$B = \{1, 2\}$

- (ii) for the classes 9 and 10 every up-type and every down-type coupling vanishes; for the classes 3, 4 and 5 every up-type coupling vanishes;
- (iii) for the classes 1, 2, 6, 7 and 8 both couplings are allowed by the charges, with the singlet insertions of Table 7.

Hence six of the eleven split data fail C8 to all orders of the perturbative singlet expansion on every smooth tetraquadric, and the five others are not excluded by the selection rules.

Proof. For each datum the possibly present fields follow from (15) and the Künneth formula $h^0(\mathbb{P}^1, \mathcal{O}(m)) = m + 1$ for $m \geq 0$, $h^1(\mathbb{P}^1, \mathcal{O}(m)) = -m - 1$ for $m \leq -2$, all other groups zero. For (12) the possibly present **10**'s lie in $H^1(L_2)$ and $H^1(L_5)$, the **5**'s in H^1 of $L_1 L_2$, $L_2 L_4$, $L_2 L_5$, $L_4 L_5$, the **5**'s in H^2 of $L_1 L_2$, $L_2 L_5$, and the singlets have charges $e_p - e_q$ with $p \in \{2, 4, 5\}$ and (p, q) one of $(2, 1), (2, 3), (2, 4), (2, 5), (4, 2), (5, 1), (5, 2), (5, 3)$. No singlet has $p = 1$ or $p = 3$, so the sets $B = \{1\}$ and $B = \{3\}$ are left by no pair. For a down-type coupling $v = \mathbf{1} - Q$ has $v_3 = 1 - Q_3 = 1$, since no charged field carries the coordinate e_3 ; Lemma 16.3(b) fails at $B = \{3\}$ and the coupling is forbidden. This recovers Proposition 15.2 for every smooth tetraquadric with the matrix (12). For the up-type couplings $Q = e_a + e_b - e_2 - e_5$ with $a, b \in \{2, 5\}$; $Q = 0$ for $\{a, b\} = \{2, 5\}$, and for $a = b = 2$ the singlet $(5, 2)$ gives $e_5 - e_2 = -Q$, for $a = b = 5$ the singlet $(2, 5)$. This is (i).

For (ii) and (iii) the same two steps are carried out for each row of Table 7. Where the table gives a set B , the reader checks that no possibly present singlet (p, q) has $p \in B$, $q \notin B$, and that $\sum_B v > 0$ for every choice of the charged fields: for the classes 9 and 10 the summand L_1 carries no **10** and lies in no possibly present **5**, so $v_1 = 1$ for every down-type coupling, while for the up-type couplings B contains the coordinates of every possibly present **5** and no **10**, so $\sum_B v = 2 > 0$; for the classes 3, 4, 5 the set B contains a coordinate of every possibly present **5** and misses the coordinates of the **10**'s. Where the table gives a coupling with insertions, the listed singlets are possibly present and their charges add to $\mathbf{1} - Q$, respectively $-Q$, which is (a) of the lemma. For the classes 3, 4 and 5 the down-type coupling listed has $Q = \mathbf{1}$ and needs no insertion. $\square$

Remark 16.5 (What the proposition does and does not say). The proposition decides a necessary condition. For the six excluded data the conclusion is unconditional within the perturbative singlet expansion: no complex structure, no polarization and no singlet expectation values restore the missing matrices, since the bound (15) holds on every smooth member and Lemma 16.3(b) is violated. For the five remaining data nothing is asserted beyond the absence of this obstruction. Their down-type couplings arise only at second or third order in the singlets, the singlet groups may vanish on a given member, the free action and its projection of the spectrum have not been computed, and the stability of the bundle after the singlets acquire expectation values, which is the same extension problem as in Section 15, is open. By Theorem 16.1 every one of the eleven data keeps a massless

Abelian gauge boson at its split point, so the five classes are candidates for the extension branch, not for the split locus. The next subsection decides, for each of the eleven data, whether the smallest non-Abelian deformations of the split bundle remove that gauge boson.

16.3. The extension branch. Throughout this subsection $X \subset (\mathbb{P}^1)^4$ is a smooth tetraquadric, $V = \bigoplus_{a=1}^5 L_a$ is one of the eleven data of Table 7 other than the class 3, so that $\text{rank } K = 3$, and $u \in \mathbb{Z}^5$ is a generator of the traceless kernel $\{q : Kq = 0, \sum_a q_a = 0\}$, which has dimension one by Theorem 16.1(ii); the vectors u are listed in Table 8. For a Kähler class $J = \sum_i t_i J_i$, $t_i > 0$, we write $\mu_a(J) = \mu_J(L_a) = \sum_i k_a^i \kappa_i(J)$ and $\mu(J) = (\mu_1, \dots, \mu_5) = K\kappa(J)$. The vector $\mu(J)$ lies in the column space of K , which has dimension three and is contained in the hyperplane $\sum_a \mu_a = 0$.

The bundles that replace V when the charged singlets of Section 15 acquire expectation values are the polystable bundles $E = \bigoplus_j E_j$ of slope zero whose stable summands E_j are iterated extensions of the line bundles L_a , $a \in B_j$, for a partition $\{B_j\}$ of $\{1, \dots, 5\}$; they have the Chern character of V , and we call the partition the *shape* of E . The split bundle has the shape with five blocks. The dictionary between such bundles and the four-dimensional theory is the one of the stability walls of Anderson, Gray, Lukas and Ovrut [35, 36]: an extension class in $H^1(L_p L_q^{-1}) = \text{Ext}^1(L_q, L_p)$ is the expectation value of a singlet of charge $e_p - e_q$, the slopes of the summands are the D-terms of the Abelian factors, and a non-split extension exists as a polystable bundle exactly where the subbundle has the smaller slope.

Lemma 16.6 (The Abelian factor on a branch). *Let $E = \bigoplus_{j=1}^m E_j$ be polystable of slope zero with the shape $\{B_j\}$. The structure group of E reduces to $S(\prod_j U(|B_j|))$, whose centre is parametrized by the traceless vectors $q \in \mathbb{Q}^5$ constant on every block. The Abelian gauge boson of such a generator q is massless at E if and only if $Kq = 0$. Consequently E keeps a massless Abelian gauge boson beyond the $SU(5)$ factor exactly when u is constant on every block of its shape.*

Proof. Each E_j is a holomorphic bundle of rank $|B_j|$ with a Hermitian Yang–Mills connection, so the connection on E has holonomy in $\prod_j U(|B_j|)$, and in the subgroup with determinant one because $c_1(E) = c_1(V) = 0$ and the connection is traceless. The centre of this group consists of the diagonal matrices with one scalar per block, that is the vectors q constant on the blocks, traceless for the determinant condition. The Green–Schwarz masses of these generators are computed as in the proof of Theorem 16.1: the axions shift under the generator q by the first Chern class $\sum_j q_{B_j} c_1(E_j) = \sum_a q_a c_1(L_a)$, whose coordinates are Kq , because the first Chern class of an iterated extension is the sum of those of its quotients. Hence the mass of q is $(Kq)^T \tilde{G}(Kq)$ with $\tilde{G}$ positive definite [33, Section 2.7], and q is massless exactly when $Kq = 0$. The traceless solutions of $Kq = 0$ are the multiples of u , so a massless generator exists if and only if u itself is constant on the blocks. The same conclusion follows from the Higgs mechanism of [36]: the extension class of a block is a singlet of charge $e_p - e_q$ with p, q in that block, and its expectation value gives a mass to every Abelian generator q with $q_p \neq q_q$. $\square$

Lemma 16.7 (One side of a wall). *Let Y be a smooth projective threefold, L, L' line bundles on Y and J_0 an ample class with $\mu_{J_0}(L) = \mu_{J_0}(L')$. Let $0 \rightarrow L \rightarrow E \rightarrow L' \rightarrow 0$ be a non-split extension. There is a neighbourhood U of J_0 in the ample cone such that E is μ_J -stable for every $J \in U$ with $\mu_J(L) < \mu_J(L')$.*

Proof. A saturated subsheaf $M \subset E$ of rank one is reflexive, hence a line bundle, as Y is smooth [37, Section 1.1]; it suffices to show $\mu_J(M) < \mu_J(E) = \frac{1}{2}(\mu_J(L) + \mu_J(L'))$ for every such M . If the composite $M \rightarrow L'$ vanishes, then $M \subset L$, so $M = L(-D)$ with D effective and $\mu_J(M) \leq \mu_J(L) < \mu_J(E)$. Otherwise $M \rightarrow L'$ is injective, $M = L'(-D)$ with D effective, and $D \neq 0$, since $D = 0$ would split the extension. Put $\delta_J = \frac{1}{2}(\mu_J(L') - \mu_J(L))$, so that $\mu_J(M) < \mu_J(E)$ means $J^2 \cdot D > \delta_J$.

Choose a rational ample class H with $J_0 - H$ ample. By Grothendieck's lemma [37, Lemma 1.7.9] the saturated subsheaves $M \subset E$ with $\mu_H(M) \geq \mu_H(L') - 1$, that is with $H^2 \cdot D \leq 1$, form a bounded family, so only finitely many classes $D_1, \dots, D_m \in H^2(Y, \mathbb{Z})$ occur among them. Each D_i is effective and nonzero, so $J_0^2 \cdot D_i > 0$, and $c_1 = \min_i J_0^2 \cdot D_i > 0$. Let U be a neighbourhood of J_0 in the ample cone on which $J - H$ is ample, $J^2 \cdot D_i > c_1/2$ for all i and $\delta_J < \min(c_1/2, 1)$; it exists by continuity, since $\delta_{J_0} = 0$. For $J \in U$ and M in the bounded family, $J^2 \cdot D > c_1/2 > \delta_J$. For M outside it, $H^2 \cdot D > 1$, and writing $J = H + (J - H)$ with both terms ample gives $J^2 \cdot D \geq H^2 \cdot D > 1 > \delta_J$, because $H \cdot (J - H) \cdot D$ and $(J - H)^2 \cdot D$ are nonnegative on the effective divisor D . In both cases $\mu_J(M) < \mu_J(E)$. $\square$

Proposition 16.8 (One block does not suffice). *No polystable bundle of slope zero whose shape has exactly one block of size two, $E = E_1 \oplus L_r \oplus L_s \oplus L_o$ with E_1 a non-split extension of L_q by L_p , removes the massless generator u : either no such bundle is polystable for any Kähler class, or $u_p = u_q$ and u stays massless by Lemma 16.6.*

Proof. Polystability of slope zero requires $\mu_r = \mu_s = \mu_o = 0$, that is $\kappa(J) \perp k_r, k_s, k_o$. If these three rows are linearly independent, the solutions form the line $\{K^T \kappa = 0\}$, since that line has dimension $4 - \text{rank } K = 1$ and is contained in them; then $\mu_p = \mu_q = 0$ as well, and E_1 has the subbundle L_p of the same slope, so it is not stable, and the shape does not occur. If the three rows are dependent, some nonzero $v \in \ker K$ vanishes at p and q . Now $\ker K = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}u$, so $v = \alpha\mathbf{1} + \beta u$ with $\alpha + \beta u_p = \alpha + \beta u_q = 0$ and $\beta \neq 0$, whence $u_p = u_q$. $\square$

Two blocks of size two can do what one cannot. For a shape $\{p, q\}, \{r, s\}, \{o\}$ let

$$W = \{\mu \in \text{Im } K : \mu_o = 0, \mu_p + \mu_q = 0\},$$

the slopes allowed to a polystable bundle of that shape (the third condition $\mu_r + \mu_s = 0$ follows from $\sum_a \mu_a = 0$). When W is a line $\mathbb{Q}w$, the slopes at every admissible J are $\mu(J) = f(J)w$ for a scalar $f(J) \neq 0$, so the sign of f decides which member of each pair is the subbundle: the one with $f w_a < 0$. We normalise w so that $f > 0$ gives the orientation shown in the tables, and we say that a shape is *realised* at J when the two extension groups $H^1(L_{\text{sub}} L_{\text{quot}}^{-1})$ of that orientation are nonzero.

Lemma 16.9 (A vanishing on the general tetraquadric). *For the class 1 of Table 7, $h^1(X, L_2 L_5^{-1}) = h^1(X, L_5 L_2^{-1}) = 0$ for every tetraquadric X in a nonempty Zariski open set of smooth members, in particular for the general one.*

Proof. Here $k_2 - k_5 = (-3, -2, 2, 3)$ and $k_5 - k_2 = (3, 2, -2, -3)$. For $k = (-3, -2, 2, 3)$ the Künneth formula gives $h^1(A, \mathcal{O}_A(k)) = 0$ and $h^2(A, \mathcal{O}_A(k)) = h^2(A, \mathcal{O}_A(k - 21)) = 2 \cdot 1 \cdot 3 \cdot 4 = 24$, both groups being $H^1 \otimes H^1 \otimes H^0 \otimes H^0$ in the four factors, so the restriction sequence gives $h^1(X, \mathcal{O}_X(k)) = \dim \ker(s: H^2(A, \mathcal{O}_A(k - 21)) \rightarrow H^2(A, \mathcal{O}_A(k)))$, where s is the equation of X . In the Čech description $H^1(\mathbb{P}^1, \mathcal{O}(m))$ has the basis of the monomials $x^a y^b$ with $a, b \leq -1$, $a + b = m$, and $H^0(\mathbb{P}^1, \mathcal{O}(m))$ that of the monomials with $a, b \geq 0$; a polynomial acts by multiplication, and a product whose exponents leave the range is zero. For the quadric

$$\begin{aligned} s_0 = & x_1^2 x_2^2 y_3^2 x_4^2 + y_1^2 y_2^2 x_3^2 x_4^2 + y_1^2 x_2^2 x_3^2 y_4^2 + x_1^2 y_2^2 y_3^2 y_4^2 \\ & + y_1^2 (x_2 y_2) (x_3 y_3) y_4^2 + x_1^2 (x_2 y_2) (x_3 y_3) x_4^2 \end{aligned}$$

each of the 24 monomials of the source is carried by exactly one of the six terms to a nonzero monomial of the target, and the 24 images are distinct, so the map is a bijection of the two bases; the same holds for $k = (3, 2, -2, -3)$ with the roles of the factors exchanged. Injectivity of a linear map is an open condition on s , and the smooth members form a nonempty open set, so both hold on a nonempty open set.⁷ $\square$

⁷The program `code/extension_branch.py` of the repository verifies the two permutation matrices and lists, for every datum and every shape with two blocks of size two, the space W , the orientations and the exact extension dimensions used below; its output is the file `extension_branch_results.json`.

TABLE 8. The realised shapes of Theorem 16.10(ii): the two extensions, the line bundle left alone, the vector w spanning W (normalised so that $f > 0$ makes the stated subbundles of negative slope), the dimensions of $H^1(L_{\text{sub}}L_{\text{quot}}^{-1})$ for the two blocks, the generator u and the coordinates t of $J_0 = \sum_i t_i J_i$. For the class 2, $r = 1.42764 \dots$ is the root of $9r^4 - 36r^2 + 14r + 16$ in $(1.4, 1.5)$, $t_1 = (18r^3 + 27r^2 - 39r - 28)/5 = 4.7454 \dots$ and $t_2 = (9r^3 + 6r^2 - 22r - 4)/10 = 0.3008 \dots$

datum	blocks	w	h^1	u	t
(12)	$L_2 \subset E_1 \rightarrow L_1, L_5 \subset E_2 \rightarrow L_3, L_4$	$(1, -1, 1, 0, -1)$	12, 4	$(4, -1, -6, 4, -1)$	$(1, 1, 1, 1)$
2	$L_4 \subset E_1 \rightarrow L_2, L_5 \subset E_2 \rightarrow L_3, L_1$	$(0, 1, 1, -1, -1)$	16, 16	$(0, -1, 1, 0, 0)$	$(t_1, t_2, r, 1)$
6	$L_3 \subset E_1 \rightarrow L_2, L_5 \subset E_2 \rightarrow L_4, L_1$	$(0, 1, -1, 1, -1)$	16, 12	$(14, -11, -6, 4, -1)$	$(1, 1, 1, 1)$
7	$L_3 \subset E_1 \rightarrow L_1, L_5 \subset E_2 \rightarrow L_2, L_4$	$(1, 1, -1, 0, -1)$	4, 24	$(1, -1, 0, 0, 0)$	$(1, \frac{5+\sqrt{105}}{8}, \frac{7+\sqrt{105}}{4}, 1)$
8	$L_4 \subset E_1 \rightarrow L_1, L_5 \subset E_2 \rightarrow L_2, L_3$	$(1, 1, 0, -1, -1)$	8, 3	$(6, -4, -4, 1, 1)$	$(1, 1, 1, 1)$
10	$L_5 \subset E_1 \rightarrow L_1, L_4 \subset E_2 \rightarrow L_2, L_3$	$(1, 1, 0, -1, -1)$	12, 12	$(4, -1, -6, 4, -1)$	$(2 + \sqrt{7}, 1, 2 + \sqrt{7}, 1)$

Theorem 16.10 (Two blocks). *Let V be one of the eleven data of Table 7 other than the class 3.*

- (i) *For the classes 4, 5 and 9 there is no Kähler class J and no polystable bundle of slope zero of a shape with two blocks of size two on any smooth tetraquadric. For the class 1 the same holds on every tetraquadric of Lemma 16.9.*
- (ii) *For the datum (12) and the classes 2, 6, 7, 8 and 10 there are a shape, listed in Table 8, and a Kähler class J_0 at which V is polystable, such that the locus $S = \{J : \mu_o(J) = 0, \mu_p(J) + \mu_q(J) = 0\}$ is a smooth surface near J_0 on which $\mu = fw$ with f changing sign along every curve through J_0 transverse to $\{f = 0\}$, and such that for every $J \in S$ with $f(J) > 0$ close enough to J_0 and every pair of nonzero classes in the two extension groups of the table, which are nonzero on every smooth tetraquadric, the bundle $E_1 \oplus E_2 \oplus L_o$ is polystable of slope zero, has the Chern character of V , and has structure group $S(U(2) \times U(2) \times U(1))$ with no massless Abelian gauge boson beyond $SU(5)$.*

Proof. (i) A polystable bundle of slope zero and shape $\{p, q\}, \{r, s\}, \{o\}$ has $\mu(J) \in W$. For each of the four data, Table 9 lists the shapes with $W \neq 0$, and in every one of them W is a line; in the other shapes all five slopes vanish and the extensions have a subbundle of equal slope, so they are not stable. On a listed shape $\mu(J) = fw$ with $f \neq 0$, and for each sign of f the subbundle of a block is its member of negative slope. The table lists for each sign a group $H^1(L_{\text{sub}}L_{\text{quot}}^{-1})$ that vanishes, on every smooth tetraquadric when its bound (15) is zero, which is checked with the Künneth formula as in the proof of Proposition 16.4, and on the tetraquadrics of Lemma 16.9 in the four cases marked with an asterisk, which are the groups $H^1(L_2L_5^{-1})$ and $H^1(L_5L_2^{-1})$ of the class 1 and their copies with $L_3 \cong L_2$. The corresponding block has no non-split extension, so the shape does not occur.

(ii) Fix a row of Table 8 and write $\{p, q\} = \{\text{sub}_1, \text{quot}_1\}$, $\{r, s\}$ for the second block. The slopes vanish at J_0 : for $t = (1, 1, 1, 1)$ every κ_i equals 12 and every row of K has coordinate sum zero; for the other three J_0 the five polynomials $\mu_a(t)$ vanish at the listed algebraic point, which is verified by exact arithmetic. The surface S is smooth at J_0 when the differentials of μ_o and $\mu_p + \mu_q$ are independent there. For $i \neq j$ one has $\partial \kappa_i / \partial t_j = 4(e_1(t) - t_i - t_j)$ and $\partial \kappa_i / \partial t_i = 0$, so at $t = (1, 1, 1, 1)$, using the zero coordinate sum of the rows, $d\mu_a = -8 \sum_j k_a^j dt_j$: the two differentials are -8 times the rows k_o and $k_p + k_q$, which are independent for the three symmetric data (for (12), $(1, 1, -1, -1)$ and $(-2, -3, 2, 3)$). On S the slopes lie in $W = \mathbb{Q}w$, so $\mu = fw$ with $f = \mu_q/w_q$; its differential along S at J_0 is nonzero, for (12) because k_1 is not in the span of k_4 and $k_1 + k_2$, and for the other data by the same exact computation. Hence f takes both signs on S arbitrarily close to J_0 .

TABLE 9. Theorem 16.10(i): every shape with two blocks of size two whose space W is nonzero, for the four excluded data; in all other shapes $W = 0$. In each case W is the line through w , and for each sign of f a group $H^1(L_{\text{sub}}L_{\text{quot}}^{-1})$ is listed whose Koszul bound (15) is zero, or, marked with an asterisk, which vanishes by Lemma 16.9.

datum	shape	w	vanishing group	vanishing group
1	$\{L_2, L_4\}, \{L_3, L_5\}, L_1$	$(0, 1, 1, -1, -1)$	$f > 0: L_5L_3^{-1*}$	$f < 0: L_2L_4^{-1}$
1	$\{L_2, L_5\}, \{L_3, L_4\}, L_1$	$(0, 1, 1, -1, -1)$	$f > 0: L_5L_2^{-1*}$	$f < 0: L_3L_4^{-1}$
1	$\{L_1, L_2\}, \{L_3, L_5\}, L_4$	$(1, -1, -1, 0, 1)$	$f > 0: L_3L_5^{-1*}$	$f < 0: L_1L_2^{-1}$
1	$\{L_1, L_3\}, \{L_2, L_5\}, L_4$	$(1, -1, -1, 0, 1)$	$f > 0: L_2L_5^{-1*}$	$f < 0: L_1L_3^{-1}$
1	$\{L_1, L_2\}, \{L_3, L_4\}, L_5$	$(-1, 1, 1, -1, 0)$	$f > 0: L_1L_2^{-1}$	$f < 0: L_3L_4^{-1}$
1	$\{L_1, L_3\}, \{L_2, L_4\}, L_5$	$(-1, 1, 1, -1, 0)$	$f > 0: L_1L_3^{-1}$	$f < 0: L_2L_4^{-1}$
4	$\{L_2, L_4\}, \{L_3, L_5\}, L_1$	$(0, -1, -1, 1, 1)$	$f > 0: L_3L_5^{-1}$	$f < 0: L_4L_2^{-1}$
4	$\{L_2, L_5\}, \{L_3, L_4\}, L_1$	$(0, -1, -1, 1, 1)$	$f > 0: L_2L_5^{-1}$	$f < 0: L_4L_3^{-1}$
4	$\{L_1, L_2\}, \{L_3, L_5\}, L_4$	$(1, -1, -1, 0, 1)$	$f > 0: L_2L_1^{-1}$	$f < 0: L_1L_2^{-1}$
4	$\{L_1, L_3\}, \{L_2, L_5\}, L_4$	$(1, -1, -1, 0, 1)$	$f > 0: L_3L_1^{-1}$	$f < 0: L_1L_3^{-1}$
4	$\{L_1, L_2\}, \{L_3, L_4\}, L_5$	$(1, -1, -1, 1, 0)$	$f > 0: L_2L_1^{-1}$	$f < 0: L_1L_2^{-1}$
4	$\{L_1, L_3\}, \{L_2, L_4\}, L_5$	$(1, -1, -1, 1, 0)$	$f > 0: L_3L_1^{-1}$	$f < 0: L_1L_3^{-1}$
5	$\{L_1, L_4\}, \{L_2, L_5\}, L_3$	$(-1, -1, 0, 1, 1)$	$f > 0: L_1L_4^{-1}$	$f < 0: L_5L_2^{-1}$
5	$\{L_1, L_5\}, \{L_2, L_4\}, L_3$	$(-1, -1, 0, 1, 1)$	$f > 0: L_1L_5^{-1}$	$f < 0: L_5L_1^{-1}$
5	$\{L_1, L_3\}, \{L_2, L_5\}, L_4$	$(-1, -1, 1, 0, 1)$	$f > 0: L_1L_3^{-1}$	$f < 0: L_3L_1^{-1}$
5	$\{L_1, L_5\}, \{L_2, L_3\}, L_4$	$(-1, -1, 1, 0, 1)$	$f > 0: L_1L_5^{-1}$	$f < 0: L_5L_1^{-1}$
5	$\{L_1, L_3\}, \{L_2, L_4\}, L_5$	$(1, 1, -1, -1, 0)$	$f > 0: L_3L_1^{-1}$	$f < 0: L_1L_3^{-1}$
5	$\{L_1, L_4\}, \{L_2, L_3\}, L_5$	$(1, 1, -1, -1, 0)$	$f > 0: L_3L_2^{-1}$	$f < 0: L_1L_4^{-1}$
9	$\{L_2, L_3\}, \{L_4, L_5\}, L_1$	$(0, 1, -1, -1, 1)$	$f > 0: L_4L_5^{-1}$	$f < 0: L_2L_3^{-1}$
9	$\{L_2, L_4\}, \{L_3, L_5\}, L_1$	$(0, 1, -1, -1, 1)$	$f > 0: L_3L_5^{-1}$	$f < 0: L_2L_4^{-1}$
9	$\{L_1, L_3\}, \{L_4, L_5\}, L_2$	$(1, 0, -1, -1, 1)$	$f > 0: L_3L_1^{-1}$	$f < 0: L_1L_3^{-1}$
9	$\{L_1, L_4\}, \{L_3, L_5\}, L_2$	$(1, 0, -1, -1, 1)$	$f > 0: L_4L_1^{-1}$	$f < 0: L_1L_4^{-1}$
9	$\{L_1, L_3\}, \{L_2, L_4\}, L_5$	$(1, 1, -1, -1, 0)$	$f > 0: L_3L_1^{-1}$	$f < 0: L_1L_3^{-1}$
9	$\{L_1, L_4\}, \{L_2, L_3\}, L_5$	$(1, 1, -1, -1, 0)$	$f > 0: L_4L_1^{-1}$	$f < 0: L_1L_4^{-1}$

For $J \in S$ with $f(J) > 0$ the slopes of the subbundles are $fw_{\text{sub}} < 0$ and those of the quotients $fw_{\text{quot}} > 0$, while $\mu_o = 0$ and $\mu_{\text{sub}} + \mu_{\text{quot}} = 0$ on each block. The two extension groups are nonzero on every smooth tetraquadric, because for each the bound (15) is attained: the Künneth groups $H^1(A, \mathcal{O}_A(k-21))$ and $H^2(A, \mathcal{O}_A(k))$ vanish, so the restriction sequence gives $h^1(X, \mathcal{O}_X(k)) = h^1(A, \mathcal{O}_A(k)) + h^2(A, \mathcal{O}_A(k-21))$, which is the number in the table. Lemma 16.7, applied to each block at J_0 , gives a neighbourhood U of J_0 on which both E_1 and E_2 are stable for every J with $f(J) > 0$; their slopes and that of L_o are zero, so $E = E_1 \oplus E_2 \oplus L_o$ is polystable of slope zero for $J \in S \cap U$, $f(J) > 0$. Its Chern character is that of V . The determinants $c_1(E_1) = c_1(L_p) + c_1(L_q)$ and $c_1(E_2)$ differ in every row, so the two blocks are not isomorphic and the structure group is $S(U(2) \times U(2) \times U(1))$. Finally u is not constant on the blocks: in every row $u_{\text{sub}} \neq u_{\text{quot}}$ for both blocks, as Table 8 shows, so by Lemma 16.6 no Abelian gauge boson beyond $SU(5)$ is massless at E . $\square$

Remark 16.11 (What the branch does and does not give). Theorem 16.10(ii) removes the obstruction of Theorem 16.1 for six of the eleven data, including the fixed datum, at the cost of leaving the split locus: the polystable Kähler classes of E form a half of the surface S , of dimension two, in place of the ray of Remark 16.2, and the extension classes are further moduli. Everything is stated on the cover X . For the quotient X/Γ the Kähler class and the two extension classes have to be Γ -invariant, which requires the equivariant structures on the line bundles and the invariant parts of the two groups; for the fixed datum the group $H^1(L_2L_1^{-1})$ of dimension 12 has invariant classes by Section 15, and the invariant part of

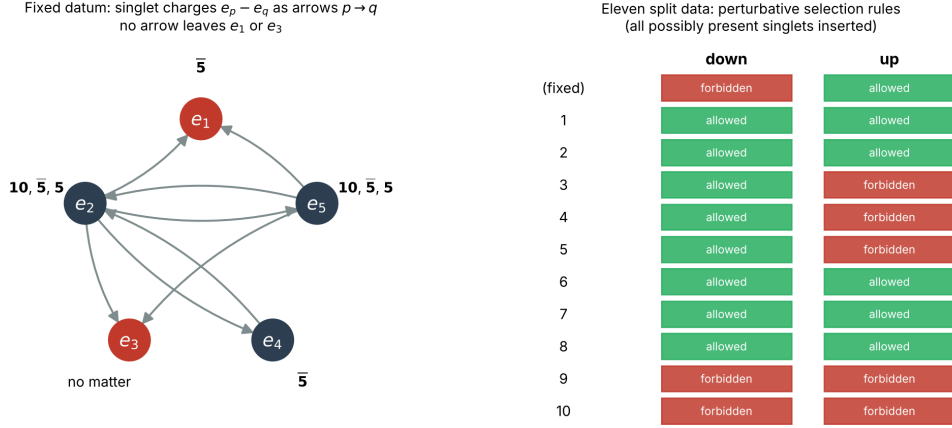


FIGURE 5. Left: the singlet charges $e_p - e_q$ of the fixed datum drawn as arrows $p \rightarrow q$ between the five line bundles, with the charged matter each bundle carries; no arrow leaves e_1 or e_3 , which is the certificate of Proposition 16.4 for the vanishing of Y_d . Right: the outcome of the selection rules for the up-type and down-type matrices on the eleven split data of Table 7, with every possibly present singlet inserted.

$H^1(L_5 L_3^{-1})$ has not been computed. The charged spectrum and the Yukawa couplings change along the branch and have not been recomputed; condition C7 is not addressed, since S and the extension classes are flat directions of the D-terms; and shapes with a block of size three or more, which by Lemma 16.6 remove u exactly when u is not constant on the large block, are not treated. Proposition 16.8 and part (i) close the smallest shapes for the classes 1, 4, 5 and 9; for these the Abelian factor can only disappear on a shape with a block of size at least three.

17. NINE-CONDITION LEDGER FOR THE ONE FIXED DATUM

The table refers only to X_π/Γ , the descended split bundle, $J = 100 \sum J_i$, the specified Wilson line and zero singlet expectation values. The perturbative effective-theory assumptions are stated in Section 15. “Verified” below always carries this stated mathematical or perturbative scope. Table 10 is the result. Thus the fixed datum is *not* a viable nine-condition vacuum. The construction proves explicit geometry and charged-spectrum statements, and it proves definite failures of some tests. It neither proves nor disproves existence of a different vacuum passing all nine.

18. AN ESTIMATE FOR COUPLING THE STABILIZATION SECTORS

Independent stabilization calculations cannot be joined without bounding their mixed derivatives. The following elementary estimate states one usable quantitative requirement.

Proposition 18.1 (Mixed-sector Hessian bound). *Let the real symmetric Hessian have blocks $H = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, with $A \succeq \alpha I$, $C \succeq \gamma I$ and $\|B\|_{\text{op}} \leq \beta$, where $\alpha, \gamma > 0$. If $\beta^2 < \alpha\gamma$, then*

$$H \succeq \frac{\alpha + \gamma - \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2} I > 0.$$

Proof. For vectors v, w , the quadratic form is bounded below by $\alpha\|v\|^2 - 2\beta\|v\|\|w\| + \gamma\|w\|^2$. The least eigenvalue of the two-by-two coefficient matrix is the displayed expression; its positivity is equivalent to the stated determinant inequality and positive trace. $\square$

This is a sufficient criterion with inputs to be measured or proved on a compactification chart. It does not provide those inputs for X_π .

TABLE 10. The nine conditions at the fixed datum.

	Status	Certificate or failed test
C1	Partial	Polystability, $c_1 = 0$ and cover anomaly effectivity proved; integral quotient anomaly and quantized hidden completion not certified.
C2	Verified	Descended bundle index is exactly -3 .
C3	Fails at split point	The $SU(5)$ centralizer is the SM subgroup, but the additional generator q in (14) survives; forced for every rank-five split datum by Theorem 16.1. On the cover it disappears on the two-block extension branch of Theorem 16.10, whose descent to X_π/Γ and spectrum are not certified.
C4	Verified spectrum	Equivariant cohomology gives three charged families and no additional chiral charged exotics.
C5	Vacant charged-exotic test	No additional massless charged vectorlike exotics occur in this spectrum. Neutral and hidden masses not asserted.
C6	Verified spectrum	One Higgs doublet pair and no massless Higgs triplets after the fixed Wilson projection.
C7	Fails at tree level	The volume ray is flat in the stated classical F- and D-term potential, so the volume modulus is massless there and $C7_W$ fails for every $m_0 > 0$.
C8	Fails perturbatively	rank $Y_u \leq 2$ and $Y_d = 0$ at the fixed zero-singlet point; normalization cannot repair rank, and no singlet insertion restores Y_d (Proposition 16.4); $C8_W$ fails for every window.
C9	Not certified	No supersymmetry-breaking sector has been constructed; at the fixed point all F- and D-terms vanish, so the first clause of $C9_W$ is not met there.

19. WHY HOLOMORPHIC MASSES DO NOT CLOSE C8

After fixing conventions and the Higgs factor, a physical Yukawa matrix has the form

$$Y_{\text{phys}} = c Z_L^{-T/2} Y_{\text{hol}} Z_R^{-1/2}, \quad c > 0, \quad (16)$$

where the matter metrics are positive definite. Their invertibility preserves rank but does not preserve singular-value ratios.

Proposition 19.1 (A normalization counterexample). *For $0 < \epsilon < 1$, take*

$$Y_{\text{hol}} = \text{diag}(\epsilon^8, \epsilon^4, 1), \quad Z_L = I, \quad Z_R = \text{diag}(\epsilon^{16}, \epsilon^8, 1), \quad c = 1.$$

All metrics are positive definite and $Y_{\text{phys}} = I$. Thus a full-rank holomorphic matrix with separated scales does not prove a physical hierarchy.

Proof. The positive inverse square root of Z_R is $\text{diag}(\epsilon^{-8}, \epsilon^{-4}, 1)$. Multiplication in (16) gives the identity. $\square$

The example does not claim that these metrics occur in the tetraquadric model. It refutes the inference based on a holomorphic matrix alone. It does not refute a hierarchy proved with regular nondegenerate metrics at the limiting point, as the following bound makes precise.

Proposition 19.2 (Bounded normalization preserves orders). *Suppose $mI \preceq Z_L, Z_R \preceq MI$ with $0 < m \leq M$ and $0 < c_- \leq c \leq c_+$. For every singular value, ordered consistently,*

$$\frac{c_-}{M} \sigma_i(Y_{\text{hol}}) \leq \sigma_i(Y_{\text{phys}}) \leq \frac{c_+}{m} \sigma_i(Y_{\text{hol}}).$$

If these bounds are uniform as a parameter tends to zero, any nonzero power-law orders of the holomorphic singular values are preserved.

Proof. For invertible A, B , the min–max characterization gives $\sigma_i(AYB) \leq \|A\|_{\text{op}}\|B\|_{\text{op}}\sigma_i(Y)$. Apply this also to $Y = A^{-1}(AYB)B^{-1}$ to obtain the reverse bound. The operator norms of the inverse square roots are at most $m^{-1/2}$ and those of their inverses at most $M^{1/2}$; transpose has the same singular values. Multiplication by c yields the result. Uniform positive constants do not change a power-law order. $\square$

Continuity and positive definiteness at a regular limiting point give such bounds locally. The preceding counterexample loses that lower bound as $\epsilon \rightarrow 0$. Conversely, an asymptotic order calculation alone does not fix the observed mass ratios or the point selected by stabilization. Published physical-flavour constructions are available [30]; that paper leaves stabilization and supersymmetry breaking untreated. This is a limitation of the result cited, not an assertion about every heterotic model in the literature.

20. A QUANTITATIVE SIMULTANEOUS-PERSISTENCE CRITERION

The following theorem identifies bounds that would make a local construction robust. All its assumptions must be checked on the same compactification chart and in the same normalization.

Theorem 20.1 (A certified interior minimum). *Let $B = \overline{B}_r(u_0) \subset \mathbb{R}^n$ and let V_0, Δ be C^2 on a neighborhood of B . Suppose on B that*

$$D^2V_0 \succeq \lambda I, \quad \|D^2\Delta\|_{\text{op}} \leq b < \lambda, \quad \|\nabla V_0(u_0)\| \leq \eta, \quad \|\nabla \Delta(u_0)\| \leq a.$$

Put $\kappa = \lambda - b$ and assume $\eta + a < \kappa r$. Then $V = V_0 + \Delta$ has exactly one critical point u_* in B . It is an interior minimum, $D^2V(u_*) \succeq \kappa I$, and

$$\|u_* - u_0\| \leq \delta := \frac{\eta + a}{\kappa} < r. \quad (17)$$

Proof. The Hessian of V is bounded below by κI . Integrating along a segment yields strong monotonicity:

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq \kappa \|x - y\|^2.$$

For x on the boundary and $v = x - u_0$ this gives $\nabla V(x) \cdot v \geq \kappa r^2 - (\eta + a)r > 0$. The inward directional derivative is negative, so a minimum on the compact ball cannot occur at its boundary. A minimum exists and is therefore critical in the interior. Strong monotonicity excludes two critical points. Apply it with $x = u_*$, $y = u_0$ and use $\nabla V(u_*) = 0$ to obtain $\kappa \|u_* - u_0\|^2 \leq (\eta + a)\|u_* - u_0\|$, which proves (17). $\square$

Proposition 20.2 (Preserving masses and inequalities). *In Theorem 20.1, suppose Y is a physical 3×3 Yukawa matrix with $\|Y(u) - Y(u_0)\|_{\text{op}} \leq L\|u - u_0\|$ on B . If s_i are its singular values at u_0 , then at u_* they differ from s_i by at most $e = L\delta$. In particular $s_{\min} > e$ certifies full rank. For $s_j > e$,*

$$\frac{\max(0, s_i - e)}{s_j + e} \leq \frac{s_i(u_*)}{s_j(u_*)} \leq \frac{s_i + e}{s_j - e}.$$

If a real acceptance function q is L_q -Lipschitz and has required lower bound q_{req} , then $q(u_0) > q_{\text{req}} + L_q\delta$ certifies $q(u_*) > q_{\text{req}}$.

Proof. The min–max formula for singular values and $|||Av|| - ||Bv||| \leq \|(A - B)v\|$ on unit vectors give $|\sigma_i(A) - \sigma_i(B)| \leq \|A - B\|_{\text{op}}$ by applying the formula in both directions. The ratio bounds follow by bounding numerator and denominator separately. The last assertion is the Lipschitz inequality. $\square$

Thus a controlled ball preserving the discrete spectrum and satisfying strict mass, hierarchy and consistency margins would turn an approximate minimum into a simultaneous certificate. If the real kinetic matrix obeys $0 < G(u_*) \preceq \Lambda I$, the physical scalar mass matrix $G(u_*)^{-1/2}D^2V(u_*)G(u_*)^{-1/2}$ has eigenvalues at least κ/Λ . This follows by applying the Hessian lower bound to $G^{-1/2}v$. At a critical point the connection correction to the Hessian

vanishes. Thus kinetic normalization must also be bounded when interpreting the Hessian physically. The bounds must include all claimed perturbative and nonperturbative corrections, the normalization metrics, the physical moduli coordinates, and any constraints or gauge quotients. A Hessian estimate for a freely invented scalar function is not a Hessian estimate for the compactification potential.

21. AN EXPLICIT LOCAL EFFECTIVE-THEORY CONSTRUCTION AND ITS LIMIT

It is useful to separate a solvable analytic problem from the missing string realization. Consider a classical global $N = 1$ effective theory with fields $S, Z_1, \dots, Z_m$, constants $a > 0$, $F \neq 0$, and $m_j \neq 0$:

$$K = |S|^2 - a|S|^4 + \sum_j |Z_j|^2, \quad W = FS + \frac{1}{2} \sum_j m_j Z_j^2.$$

We work on $|S|^2 < 1/(4a)$, where its Kähler metric is positive.

Proposition 21.1 (A stabilized local model). *For this specified classical global effective theory,*

$$V = \frac{|F|^2}{1 - 4a|S|^2} + \sum_j |m_j|^2 |Z_j|^2$$

has a unique minimum at the origin of the stated domain. Supersymmetry is broken there. In canonical real coordinates its Hessian has diagonal entries $4a|F|^2$, $4a|F|^2$ and $|m_j|^2$, $|m_j|^2$ for each j .

Proof. $K_{S\bar{S}} = 1 - 4a|S|^2$ and $K_{Z_i\bar{Z}_j} = \delta_{ij}$, so the global-supersymmetry potential $K^{A\bar{B}}W_A\bar{W}_B$ is the displayed expression. Its first term is strictly increasing with $|S|^2$ and the remaining terms are positive quadratics. The auxiliary field in the S direction is nonzero at zero because $W_S = F$. Expanding with $S = (x + iy)/\sqrt{2}$ gives $V = |F|^2 + 2a|F|^2(x^2 + y^2) + \sum_j |m_j|^2(u_j^2 + v_j^2)/2 + O(|S|^4)$, which proves the Hessian assertion. $\square$

This proposition supplies no Calabi–Yau bundle realizing its K and W . It omits supergravity, flux quantization, the anomaly constraints and the coupled visible sector. Consequently it verifies neither C7 nor C9 for the geometric example. Its role is to make the missing realization problem explicit rather than hide it inside a choice of coefficients.

Proposition 21.2 (A tree-level obstruction). *Let a supergravity model have*

$$K = K_0(z, \bar{z}) - \sum_{i=1}^r n_i \log(T_i + \bar{T}_i), \quad n_i > 0, \quad \sum_i n_i = 3, \quad W = W(z).$$

At a point with $D_z W = 0$, its F -term potential vanishes for all T_i in this product chart. It does not stabilize these Kähler directions.

Proof. The T -block is diagonal, $K_i = -n_i/(T_i + \bar{T}_i)$ and $K_{i\bar{i}} = n_i/(T_i + \bar{T}_i)^2$. Hence $\sum_i K^{i\bar{i}} K_i K_{\bar{i}} = \sum_i n_i = 3$. Since $D_i W = K_i W$ and the z block contributes zero, these terms cancel the $-3|W|^2$ in $V_F = e^K (K^{A\bar{B}} D_A W \bar{D}_B \bar{W} - 3|W|^2)$. Thus there are flat T directions. Additional D terms or corrections are not covered by this assertion and would need their own calculation. $\square$

22. CONCLUSION AND VERIFICATION SCOPE

The algebraic spectral calculation and the existence of a viable vacuum are separate statements. The mirror argument restricts a selector on its specified domain; it does not supply a point outside that domain. The explicit tetraquadric construction supplies a free quotient, index and projected charged spectrum at one fixed complex structure. Its residual gauge generator and deficient perturbative Yukawa matrices are exact obstructions to using that split point as the sought vacuum. The extension branch changes the bundle and

therefore requires its own stability, operator and potential calculations before replacing the audit. Section 16 shows that this branch is the only one left on the tetraquadric: no sum of five line bundles on a threefold with at most four Kähler parameters passes the gauge condition at its split locus, and on the tetraquadric the selection rules already remove six of the eleven three-family split data from the Yukawa condition. On the branch itself a single extension never removes the forced gauge boson, while two rank-two extensions do so for the fixed datum and for five of the ten further data, on the cover and on one side of a stability wall, and cannot do so for the other four. What the branch has not been shown to do is to descend to the quotient, to keep three families with the required Yukawa matrices, and to lift the flat directions of $C7_W$; the four classes closed at the two smallest shapes could only pass C3 on a shape with a block of rank three or more. With the windows of Definition 9.1 the conjecture is a definite statement, and every failure recorded here is independent of them.

The remaining task is to specify a single quantized compactification with its hidden sector and controlled effective potential, and to verify all nine predicates at its stabilized point. The estimates in this paper state sufficient analytic conditions for parts of that task. Their inputs have not been established for the fixed example. Consequently this manuscript does not prove the full Geometric Selection Conjecture.

DATA AND DECLARATIONS

The spectral sections and their five appendices present the work of the two authors on the Dwork and weighted Fermat pencils [27]; the mirror argument is that of [28] under its standard-type partial-mirror convention; the marked-point and effective-theory criteria reorganize [32]. The bundle example is the published one of [29], and the ten further line-bundle sums of Section 16.2 were produced by the scan of [41].

Python programs recompute the smooth-seed polynomial identities, the fixed-pencil determinant and character cohomology, the Dwork label, coupling and mass-map counts, the thirteen-row Chern and index arithmetic, the selection rules of Proposition 16.4 with their flow certificates, and the slope spaces, orientations, extension dimensions and polystable points of Theorem 16.10 with the permutation matrices of Lemma 16.9. Each program emits a machine-readable result with its scope. They are in the repository `creelie/geometric-selection-conjecture` on GitHub (release v3.1.0, concept DOI 10.5281/zenodo.23567114). Finite algebra checks do not certify string corrections, stabilize moduli, or supply a proof of all nine conditions; no statement of the paper rests on them alone.

Funding. This research received no external funding.

Conflicts of interest. The authors declare no conflict of interest.

APPENDIX A. THE HESSIAN IN THE SOCLE

The Hessian matrix of F_ψ has entries

$$H_{ii} = 20x_i^3, \quad H_{ij} = -5\psi \prod_{k \neq i, j} x_k \quad (i \neq j).$$

For a permutation π of $\{1, \dots, 5\}$ with m moved points and $f = 5 - m$ fixed points, the product $\prod_i H_{i\pi(i)}$ is $20^f (-5\psi)^m$ times a monomial whose exponents are read off as follows. A fixed coordinate a receives 3 from its diagonal entry and 1 from each of the m off-diagonal entries, since an off-diagonal entry $(i, \pi(i))$ omits only the moved coordinates i and $\pi(i)$. A moved coordinate c is omitted by exactly two off-diagonal entries, those with $i = c$ and with $\pi(i) = c$, which are different because $\pi(c) \neq c$; it receives 1 from each of the other $m - 2$. Hence the monomial of π is

$$\prod_{a \text{ fixed}} x_a^{3+m} \prod_{c \text{ moved}} x_c^{m-2},$$

of degree $f(3+m) + m(m-2) = 15$, and it depends only on the number of moved points. The five values of m give:

- $m = 0$: the monomial $m_0 = (3, 3, 3, 3, 3)$, sector 0;
- $m = 2$: $(5, 5, 5, 0, 0)$ up to order, sector 3, so $\lambda = \psi^3$;
- $m = 3$: $(6, 6, 1, 1, 1)$ up to order, sector 2, so $\lambda = \psi^2$;
- $m = 4$: $(7, 2, 2, 2, 2)$ up to order, sector 1, so $\lambda = \psi$;
- $m = 5$: $(3, 3, 3, 3, 3)$, sector 0.

The contribution of a permutation to $H(\psi) = \lambda(\text{Hess } F_\psi)$ is $\text{sgn}(\pi) 20^f (-5\psi)^m \psi^{r(m)}$ with $r(0) = r(5) = 0$, $r(2) = 3$, $r(3) = 2$, $r(4) = 1$, so $m + r(m) = 5$ for every $m \geq 2$ and every non-identity permutation contributes a multiple of ψ^5 . The cycle types are

$$\begin{aligned}
1^5 &: 1 \text{ permutation, } m = 0, +20^5 = 3\,200\,000; \\
2\,1^3 &: 10, \text{ odd, } m = 2: 10 \cdot (-1) \cdot 20^3 \cdot 25 = -2\,000\,000; \\
3\,1^2 &: 20, \text{ even, } m = 3: 20 \cdot 20^2 \cdot (-125) = -1\,000\,000; \\
4\,1 &: 30, \text{ odd, } m = 4: 30 \cdot (-1) \cdot 20 \cdot 625 = -375\,000; \\
2^2\,1 &: 15, \text{ even, } m = 4: 15 \cdot 20 \cdot 625 = 187\,500; \\
5 &: 24, \text{ even, } m = 5: 24 \cdot (-3125) = -75\,000; \\
3\,2 &: 20, \text{ odd, } m = 5: 20 \cdot (-1) \cdot (-3125) = 62\,500,
\end{aligned}$$

where the last six numbers are the coefficients of ψ^5 . The counts $1+10+20+30+15+24+20$ add up to 120. The coefficients of ψ^5 add up to $-3\,200\,000$, which gives $H(\psi) = 3\,200\,000(1 - \psi^5)$.

APPENDIX B. THE COUPLING COUNT

B.1. Ordered triples by sector. For $\mathbf{e}$ of degree fifteen, $O(\mathbf{e})$ is the number of ordered triples $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in B^3$ with $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{e}$. We count it by the third label: for each $\mathbf{c} \in B$ with $\mathbf{c} \leq \mathbf{e}$, put $\mathbf{s} = \mathbf{e} - \mathbf{c}$, a vector of degree ten, and count the $\mathbf{a} \in B$ with $\mathbf{s} - \mathbf{a} \in B$, that is, with

$$\max(0, s_i - 3) \leq a_i \leq \min(3, s_i) \quad \text{for all } i, \quad \sum a_i = 5.$$

This number $N(\mathbf{s})$ is the coefficient of t^5 in $\prod_i (t^{\max(0, s_i - 3)} + \dots + t^{\min(3, s_i)})$. Write $[n] = 1 + t + \dots + t^{n-1}$.

Sector 0, $\mathbf{e} = (3, 3, 3, 3, 3)$. Here $\mathbf{s} = 3\mathbf{1} - \mathbf{c}$ has entries in $\{0, \dots, 3\}$, so $N(\mathbf{s})$ is the coefficient of t^5 in $\prod_i [s_i + 1]$. The multiset of entries of $\mathbf{s}$ is the complement of the type of $\mathbf{c}$:

type of $\mathbf{c}$	entries of $\mathbf{s}$	$\#\mathbf{c}$	$N(\mathbf{s})$
$\{3, 2, 0, 0, 0\}$	$\{3, 3, 3, 1, 0\}$	20	24
$\{3, 1, 1, 0, 0\}$	$\{3, 3, 2, 2, 0\}$	30	28
$\{2, 2, 1, 0, 0\}$	$\{3, 3, 2, 1, 1\}$	30	38
$\{2, 1, 1, 1, 0\}$	$\{3, 2, 2, 2, 1\}$	20	44
$\{1, 1, 1, 1, 1\}$	$\{2, 2, 2, 2, 2\}$	1	51

The values of N are central coefficients. For $\{3, 3, 3, 1, 0\}$, $[4]^3 = 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + \dots$ and the factor $[2]$ gives $12 + 12 = 24$. For $\{3, 3, 2, 2, 0\}$, $[4]^2 = (1, 2, 3, 4, 3, 2, 1)$ and $[3]^2 = (1, 2, 3, 2, 1)$, so the coefficient is $2 + 6 + 12 + 6 + 2 = 28$. For $\{3, 3, 2, 1, 1\}$, $[3][2]^2 = (1, 3, 4, 3, 1)$ and the coefficient is $2 + 9 + 16 + 9 + 2 = 38$. For $\{3, 2, 2, 2, 1\}$, $[3]^3 = (1, 3, 6, 7, 6, 3, 1)$ and $[4][2] = (1, 2, 2, 2, 1)$, so the coefficient is $3 + 12 + 14 + 12 + 3 = 44$. For $\{2, 2, 2, 2, 2\}$ the coefficient of t^5 in $(1 + t + t^2)^5$ is 51. Hence

$$O(3, 3, 3, 3, 3) = 20 \cdot 24 + 30 \cdot 28 + 30 \cdot 38 + 20 \cdot 44 + 51 = 3391.$$

(The table for $\mathbf{e} = (3, 3, 3, 3, 3)$ also shows that every $\mathbf{c} \in B$ occurs, since $\mathbf{s} = 3\mathbf{1} - \mathbf{c}$ always admits some $\mathbf{a}$.)

Sector 1, $\mathbf{e} = (7, 2, 2, 2, 2)$. Since $a_1 + b_1 \leq 6$, $c_1 \geq 1$, and $c_i \leq 2$ for $i \geq 2$. With $\mathbf{s} = \mathbf{e} - \mathbf{c}$, the entry $s_1 = 7 - c_1$ forces $a_1 \geq 4 - c_1$, and the remaining entries $s_i = 2 - c_i \leq 2$ impose only $a_i \leq s_i$. Grouping $\mathbf{c}$ by c_1 and by the multiset of $(c_2, \dots, c_5)$:

c_1	$(c_2, \dots, c_5)$	$\#\mathbf{c}$	$N(\mathbf{s})$	total
1	$\{1, 1, 1, 1\}$	1	6	6
1	$\{2, 1, 1, 0\}$	12	4	48
1	$\{2, 2, 0, 0\}$	6	3	18
2	$\{1, 1, 1, 0\}$	4	14	56
2	$\{2, 1, 0, 0\}$	12	10	120
3	$\{1, 1, 0, 0\}$	6	26	156
3	$\{2, 0, 0, 0\}$	4	19	76

For instance, when $c_1 = 3$ and $(c_2, \dots, c_5) = (2, 0, 0, 0)$, a_1 ranges over $\{1, 2, 3\}$ and the other entries satisfy $a_i \leq (0, 2, 2, 2)$ with sum $5 - a_1$; the coefficients of t^4, t^3, t^2 in $[3]^3$ are 6, 7, 6, which add to 19. The total is $O(7, 2, 2, 2, 2) = 480$.

Sector 2, $\mathbf{e} = (6, 6, 1, 1, 1)$. Here $c_3, c_4, c_5 \leq 1$, and the same grouping gives

$\{c_1, c_2\}$	(c_3, c_4, c_5)	$\#\mathbf{c}$	$N(\mathbf{s})$	total
$\{1, 1\}$	$\{1, 1, 1\}$	1	2	2
$\{2, 0\}$	$\{1, 1, 1\}$	2	1	2
$\{2, 1\}$	$\{1, 1, 0\}$	6	4	24
$\{2, 2\}$	$\{1, 0, 0\}$	3	10	30
$\{3, 0\}$	$\{1, 1, 0\}$	6	2	12
$\{3, 1\}$	$\{1, 0, 0\}$	6	8	48
$\{3, 2\}$	$\{0, 0, 0\}$	2	22	44

and $O(6, 6, 1, 1, 1) = 162$.

Sector 3, $\mathbf{e} = (5, 5, 5, 0, 0)$. Here $c_4 = c_5 = 0$ and $c_1 + c_2 + c_3 = 5$. For $\{c_1, c_2, c_3\} = \{2, 2, 1\}$ the entries of $\mathbf{s}$ are $(3, 3, 4)$, the bounds are $a_i \leq 3$ on the first two and $1 \leq a_3 \leq 3$, and the coefficient of t^5 in $[4]^2(t + t^2 + t^3)$ is $3 + 4 + 3 = 10$; for $\{3, 1, 1\}$ the entries are $(2, 4, 4)$ and the coefficient of t^5 in $[3](t + t^2 + t^3)^2$ is $2 + 3 + 2 = 7$; for $\{3, 2, 0\}$ the entries are $(2, 3, 5)$ and the coefficient of t^5 in $[3][4](t^2 + t^3)$ is $3 + 3 = 6$. With three, three and six choices of $\mathbf{c}$, $O(5, 5, 5, 0, 0) = 30 + 21 + 36 = 87$.

The ordered counts by sector are $O_0 = 3391$, $O_1 = 5 \cdot 480 = 2400$, $O_2 = 10 \cdot 162 = 1620$ and $O_3 = 10 \cdot 87 = 870$, with total 8281 in agreement with Section 5.1.

B.2. Repeated labels. For $\mathbf{a} \in B$ the partner $\mathbf{c}$ with $2\mathbf{a} + \mathbf{c}$ in a sector is found type by type, as in the proof of Theorem 1.3. For $\mathbf{a} = \mathbf{1}$, $\mathbf{c} = \mathbf{1}$ in sector 0. For $\mathbf{a}$ of type $\{2, 1, 1, 1, 0\}$, sector 1 gives $c_i \equiv 2 - 2a_i$, that is $\mathbf{c}$ with a three where $\mathbf{a}$ has its two, a two where $\mathbf{a}$ has its zero and zeros elsewhere, of sum five. For type $\{3, 1, 1, 0, 0\}$, sector 1 gives a one where $\mathbf{a}$ has its three, zeros where it has ones and twos where it has zeros. For type $\{3, 2, 0, 0, 0\}$, sector 2 gives zero, two and ones. For type $\{2, 2, 1, 0, 0\}$, sector 3 gives ones at the twos, a three at the one and zeros elsewhere. In each case the other sectors produce a residue four or a total different from five. So the hundred triples $\{\mathbf{a}, \mathbf{a}, \mathbf{c}\}$ with $\mathbf{c} \neq \mathbf{a}$ lie 20 + 30 = 50 in sector 1, 20 in sector 2 and 30 in sector 3, and none in sector 0, which is the list $D_r = 0, 50, 20, 30$ used in Section 5.1.

APPENDIX C. HODGE NORMS AT THE FERMAT POINT

This appendix proves Proposition 7.1. Write $G = \mu_5^5 / \mu_5$ for the group of diagonal phases of $\mathbb{P}^4$, which acts on X_0 . For $\mathbf{k} \in \{1, 2, 3, 4\}^5$ with $\sum k_i \equiv 0 \pmod{5}$ let $\chi_{\mathbf{k}}(a) = \zeta^{a \cdot \mathbf{k}}$.

C.1. Eigenlines. The character $\chi_{\mathbf{k}}$ occurs in $H^3(X_0, \mathbb{C})$ exactly once, and these are all the characters that occur: the number of such $\mathbf{k}$ is $\frac{1}{5} \sum_{j=0}^4 (\sum_{k=1}^4 \zeta^{jk})^5 = \frac{1}{5}(4^5 - 4) = 204 = b_3(X_0)$, and each occurs at least once because the residue $\text{Res}(x^{\mathbf{k}-\mathbf{1}} \Omega_0 / F^{q+1})$, $5(q+1) = \sum k_i$,

is a nonzero class of character $\chi_{\mathbf{k}}$ (Griffiths' theorem and the fact that $x^{\mathbf{k}-1}$, with exponents at most three, is nonzero in the Jacobian ring $\mathbb{C}[x]/(x_1^4, \dots, x_5^4)$ of the Fermat quintic). In particular $\omega_{\mathbf{b}}$ spans the line of $\mathbf{k} = \mathbf{b} + \mathbf{1}$.

C.2. The bilinear relation. Let Q be the intersection form on $H_3(X_0, \mathbb{Z})$, extended to complex coefficients, and identify H^3 with H_3 by Poincaré duality D , so that $\int_c \omega = Q(c, D\omega)$ and $\int_{X_0} \omega \wedge \eta = Q(D\omega, D\eta)$. Since Q is G -invariant, it pairs the eigenline L of χ only with the eigenline L' of $\bar{\chi}$. Let e be an integral cycle and u its component in L' ; its component in L is $\bar{u}$, because e is real. If $D\omega_{\mathbf{b}} = \alpha \bar{u}$ with $\bar{u} \neq 0$, then

$$\int_e \omega_{\mathbf{b}} = Q(e, \alpha \bar{u}) = \alpha Q(u, \bar{u}), \quad \int_{X_0} \omega_{\mathbf{b}} \wedge \bar{\omega}_{\mathbf{b}} = |\alpha|^2 Q(\bar{u}, u),$$

and therefore

$$g_{\mathbf{b}} = \left| \int_{X_0} \omega_{\mathbf{b}} \wedge \bar{\omega}_{\mathbf{b}} \right| = \frac{|\int_e \omega_{\mathbf{b}}|^2}{|Q(u, \bar{u})|}. \quad (18)$$

It remains to compute both sides for one cycle.

C.3. The cycle of Pham. On the affine chart $x_5 = 1$ put $y_i = x_i/x_5$; the open part of X_0 is $U = \{f = 0\}$, $f = y_1^5 + y_2^5 + y_3^5 + y_4^5 + 1$, and μ_5^4 acts by $y_i \mapsto \zeta^{a_i} y_i$. A character $\mathbf{k}$ of G restricts to the character (k_1, k_2, k_3, k_4) of μ_5^4 , and $k_5 \equiv -(k_1 + k_2 + k_3 + k_4)$. Pham [14] showed that U retracts onto the join of four copies of the set of fifth roots of -1 , and that $H_3(U, \mathbb{Z})$ is generated over $\mathbb{Z}[\mu_5^4]$ by the cycle

$$e = (1 - T_1)(1 - T_2)(1 - T_3)(1 - T_4) \Delta,$$

where Δ is the real simplex $\{y_i = \epsilon v_i, v_i \geq 0, \sum v_i^5 = 1\}$ with $\epsilon^5 = -1$, and T_i multiplies y_i by ζ .

The period. By Griffiths' residue formula, $\int_e \omega_{\mathbf{b}} = \frac{1}{2\pi i} \int_{\tau(e)} x^{\mathbf{b}} \Omega_0 / F^2$ with τ the tube map. On the chart this is the integral of $y^{\mathbf{b}'} dy / f^2$, $\mathbf{b}' = (b_1, \dots, b_4)$. Put $f_s = \sum y_i^5 + s$; then $y^{\mathbf{b}'} dy / f_s^2 = -\partial_s (y^{\mathbf{b}'} dy / f_s)$, and $\frac{1}{2\pi i} \int_{\tau(e)} y^{\mathbf{b}'} dy / f_s = \int_{e_s} y^{\mathbf{b}'} dy / df_s$ with the Gelfand–Leray form. The substitution $y = s^{1/5} u$ shows that the last integral is $s^A P$ with $A = \frac{1}{5} \sum_{i \leq 4} k_i - 1$ and P its value at $s = 1$. Differentiating at $s = 1$,

$$\int_e \omega_{\mathbf{b}} = -A P.$$

On Δ , with $t_i = v_i^5$, the Gelfand–Leray form is $5^{-4} \prod_i t_i^{1/5-1} dt$ on the simplex $\sum t_i = 1$, and $y^{\mathbf{b}'} = \epsilon^{|\mathbf{b}'|} \prod t_i^{b_i/5}$; Dirichlet's integral gives $\int_{\Delta} = \epsilon^{|\mathbf{b}'|+4} 5^{-4} \prod_{i \leq 4} \Gamma(k_i/5) / \Gamma(\sum_{i \leq 4} k_i/5)$, and each factor $1 - T_i$ multiplies it by $1 - \zeta^{k_i}$. Since $\sum_{i \leq 4} k_i = 10 - k_5$, $A = 1 - k_5/5$ and $\Gamma(A+1) = A \Gamma(A)$, so

$$\left| \int_e \omega_{\mathbf{b}} \right| = 5^{-4} \prod_{i=1}^4 |1 - \zeta^{k_i}| \frac{\prod_{i=1}^4 \Gamma(k_i/5)}{\Gamma(1 - k_5/5)}. \quad (19)$$

The intersection number. For one variable, the reduced homology of the fibre $\{y^5 = -1\}$ is the augmentation ideal I of $\mathbb{Z}[t]/(t^5 - 1)$, the monodromy is multiplication by t , and the variation operator is multiplication by $t - 1$. The Seifert form $L(x, z) = \langle x, \text{Var}^{-1} z \rangle$ is therefore, in terms of the values $\hat{x}(k) = x(\zeta^k)$,

$$L(x, z) = \frac{1}{5} \sum_{k=1}^4 \frac{\hat{x}(k) \hat{z}(-k)}{\zeta^{-k} - 1}.$$

By the theorem of Sebastiani and Thom [15], in the form of Sakamoto [16] for Seifert forms, the Seifert form of $y_1^5 + \dots + y_4^5$ is, up to sign, the tensor product of four copies, and the

intersection form on the three-dimensional fibre is $Q = -L + L^T$. On the eigenline of $(k_1, \dots, k_4)$ this gives

$$Q(x, z) = \pm \frac{1}{5^4} \hat{x}(\mathbf{k}) \hat{z}(-\mathbf{k}) \left(\prod_{i \leq 4} \frac{1}{\zeta^{k_i} - 1} - \prod_{i \leq 4} \frac{1}{\zeta^{-k_i} - 1} \right).$$

With four factors, $\prod(\zeta^{k_i} - 1) = \zeta^{\sum k_i} \prod(\zeta^{-k_i} - 1)$, so the bracket has absolute value

$$\frac{|1 - \zeta^{k_1+k_2+k_3+k_4}|}{\prod_{i \leq 4} |1 - \zeta^{k_i}|^2} = \frac{|1 - \zeta^{k_5}|}{\prod_{i \leq 4} |1 - \zeta^{k_i}|^2}.$$

The cycle e has $\hat{e}(\mathbf{k}) = \prod_{i \leq 4} (1 - \zeta^{k_i})$ up to a unimodular factor, so its component u satisfies

$$|Q(u, \bar{u})| = 5^{-4} \prod_{i=1}^5 |1 - \zeta^{k_i}|. \quad (20)$$

This is nonzero, so $\bar{u} \neq 0$, as needed in (18).

C.4. Conclusion of the proof. Inserting (19) and (20) in (18),

$$g_{\mathbf{b}} = 5^{-4} \prod_{i=1}^4 |1 - \zeta^{k_i}| \Gamma(k_i/5)^2 \cdot \frac{1}{|1 - \zeta^{k_5}| \Gamma(1 - k_5/5)^2}.$$

With $|1 - \zeta^k| = 2 \sin(\pi k/5)$ and $\Gamma(t)\Gamma(1-t) = \pi/\sin \pi t$,

$$2 \sin(\pi t) \Gamma(t)^2 = 2\pi \gamma(t), \quad \frac{1}{2 \sin(\pi t) \Gamma(1-t)^2} = \frac{\gamma(t)}{2\pi},$$

so $g_{\mathbf{b}} = 5^{-4} (2\pi)^3 \prod_{i=1}^5 \gamma(k_i/5)$, which is (7) with $C = (2\pi)^3/5^4$. The class of Ω , with $\mathbf{k} = \mathbf{1}$, has no derivative in the period computation, so $|\int_e \Omega| = |P|$ with $\Gamma(\sum_{i \leq 4} k_i/5) = \Gamma(4/5) = \Gamma(1 - k_5/5)$, and the same formula holds: $\|\Omega\|^2 = C\gamma(1/5)^5$. Since $\partial_\psi \text{Res}(\bar{\Omega}_0/F_\psi) = \text{Res}(5x_1 \cdots x_5 \Omega_0/F_\psi^2) = 5\omega_1$ and ω_1 is of type $(2, 1)$ at $\psi = 0$ (its character differs from that of Ω), the Weil–Petersson metric at the Fermat point is

$$\begin{aligned} G_{\psi\bar{\psi}}(0) &= \frac{25 g_{\mathbf{1}}}{\|\Omega\|^2} = 25 \frac{\gamma(2/5)^5}{\gamma(1/5)^5} \\ &= 25 \frac{\Gamma(2/5)^5 \Gamma(4/5)^5}{\Gamma(1/5)^5 \Gamma(3/5)^5} = 0.19226. \end{aligned}$$

C.5. Numerical values. With $\Gamma(1/5) = 4.59084$, $\Gamma(2/5) = 2.21816$, $\Gamma(3/5) = 1.48919$ and $\Gamma(4/5) = 1.16423$, $\gamma(1/5) = 3.94325$ and $\gamma(2/5) = 1.48951$; $\gamma(3/5)$ and $\gamma(4/5)$ are their inverses. The norms of the five types follow by multiplying the five factors, and the masses of Table 2 are their inverses.

APPENDIX D. SHARED CHARACTERS ON THE OCTIC AND THE DECTIC

This appendix completes the proof of Proposition 8.4. By Lemma 8.3 it suffices to find, for each shift $s \leq d/2$, the labels $\mathbf{b}$ with $b_i + s \not\equiv n_i - 1 \pmod{n_i}$ for every i and $\sum_i a_i = s$, and their images $\mathbf{b}'$; the shift $d - s$ only reverses these pairs, so two labels carry the same character exactly when they are joined by a chain of the pairs found. For both pencils $n_5 = 2$ and $b_5 = 0$, so the condition at x_5 excludes every odd shift, and for an even shift $a_5 = s/2$. In this appendix we drop the last coordinate, which is zero, and write labels as vectors of length four.

TABLE 11. The entries of the blocks of Table 4, for the representative with $(i, j, k, l) = (1, 2, 3, 4)$. Exponent vectors are written in full; the column r is the sector of $h + \mathbf{b} + \mathbf{c}$ and k is the vector of (10).

pencil	row	$\mathbf{b}$	$\mathbf{c}$	$h + \mathbf{b} + \mathbf{c}$	r	k	entry
sextic	1	(4, 0, 0, 0, 1)	(3, 1, 1, 1, 0)	(8, 2, 2, 2, 2)	2	(1, 0, 0, 0, 1)	$2\psi^2$
	1	(0, 2, 2, 2, 0)	(3, 1, 1, 1, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(3, 3, 0, 0, 0)	(3, 3, 0, 0, 0)	(7, 7, 1, 1, 1)	3	(1, 1, 0, 0, 1)	$2\psi^3$
	2	(3, 3, 0, 0, 0)	(0, 0, 3, 3, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(0, 0, 3, 3, 0)	(3, 3, 0, 0, 0)	(4, 4, 4, 4, 1)	0	(0, 0, 0, 0, 0)	1
	2	(0, 0, 3, 3, 0)	(0, 0, 3, 3, 0)	(1, 1, 7, 7, 1)	3	(0, 0, 1, 1, 1)	$2\psi^3$
octic	1	(6, 1, 1, 0, 0)	(4, 1, 1, 2, 0)	(12, 4, 4, 4, 0)	2	(1, 0, 0, 0, 1)	$4\psi^2$
	1	(0, 3, 3, 2, 0)	(4, 1, 1, 2, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	2	(6, 2, 0, 0, 0)	(2, 6, 0, 0, 0)	(10, 10, 2, 2, 0)	4	(1, 1, 0, 0, 2)	$16\psi^4$
	2	(6, 2, 0, 0, 0)	(4, 0, 2, 2, 0)	(12, 4, 4, 4, 0)	2	(1, 0, 0, 0, 1)	$4\psi^2$
	2	(0, 4, 2, 2, 0)	(2, 6, 0, 0, 0)	(4, 12, 4, 4, 0)	2	(0, 1, 0, 0, 1)	$4\psi^2$
	2	(0, 4, 2, 2, 0)	(4, 0, 2, 2, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(4, 4, 0, 0, 0)	(4, 4, 0, 0, 0)	(10, 10, 2, 2, 0)	4	(1, 1, 0, 0, 2)	$16\psi^4$
	3	(4, 4, 0, 0, 0)	(0, 0, 4, 4, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(0, 0, 4, 4, 0)	(4, 4, 0, 0, 0)	(6, 6, 6, 6, 0)	0	(0, 0, 0, 0, 0)	1
	3	(0, 0, 4, 4, 0)	(0, 0, 4, 4, 0)	(2, 2, 10, 10, 0)	4	(0, 0, 1, 1, 2)	$16\psi^4$
dectic	1	(2, 1, 1, 3, 0)	(2, 3, 3, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	1	(4, 3, 3, 0, 0)	(2, 3, 3, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	2	(3, 1, 0, 3, 0)	(1, 3, 4, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	2	(5, 3, 2, 0, 0)	(1, 3, 4, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	3	(8, 1, 1, 0, 0)	(4, 1, 1, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	3	(0, 3, 3, 2, 0)	(4, 1, 1, 2, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	4	(7, 1, 0, 1, 0)	(5, 1, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	4	(1, 5, 4, 0, 0)	(5, 1, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	5	(8, 0, 0, 1, 0)	(4, 2, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	5	(0, 2, 2, 3, 0)	(4, 2, 2, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	5	(2, 4, 4, 0, 0)	(4, 2, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	6	(4, 0, 0, 3, 0)	(6, 0, 0, 2, 0)	(12, 2, 2, 7, 0)	6	(1, 0, 0, 2, 3)	$500\psi^6$
	6	(4, 0, 0, 3, 0)	(0, 4, 4, 1, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	6	(6, 2, 2, 0, 0)	(6, 0, 0, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	6	(6, 2, 2, 0, 0)	(0, 4, 4, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	7	(8, 2, 0, 0, 0)	(2, 8, 0, 0, 0)	(12, 12, 2, 2, 0)	6	(1, 1, 0, 1, 3)	$250\psi^6$
	7	(8, 2, 0, 0, 0)	(4, 0, 2, 2, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	7	(0, 4, 2, 2, 0)	(2, 8, 0, 0, 0)	(4, 14, 4, 4, 0)	4	(0, 1, 0, 1, 2)	$50\psi^4$
	7	(0, 4, 2, 2, 0)	(4, 0, 2, 2, 0)	(6, 6, 6, 6, 0)	2	(0, 0, 0, 1, 1)	$10\psi^2$
	8	(6, 2, 0, 1, 0)	(6, 0, 2, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	8	(6, 2, 0, 1, 0)	(0, 4, 6, 0, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	8	(0, 6, 4, 0, 0)	(6, 0, 2, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	8	(0, 6, 4, 0, 0)	(0, 4, 6, 0, 0)	(2, 12, 12, 2, 0)	6	(0, 1, 1, 1, 3)	$250\psi^6$
	9	(6, 1, 1, 1, 0)	(6, 1, 1, 1, 0)	(14, 4, 4, 4, 0)	4	(1, 0, 0, 1, 2)	$50\psi^4$
	9	(6, 1, 1, 1, 0)	(0, 5, 5, 0, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	9	(0, 5, 5, 0, 0)	(6, 1, 1, 1, 0)	(8, 8, 8, 3, 0)	0	(0, 0, 0, 0, 0)	1
	9	(0, 5, 5, 0, 0)	(0, 5, 5, 0, 0)	(2, 12, 12, 2, 0)	6	(0, 1, 1, 1, 3)	$250\psi^6$

The octic. Here $b_i \leq 6$ and $b_1 + b_2 + b_3 + b_4 = 8$, and the shifts to consider are $s = 2$ and $s = 4$. Since $b_i + s \leq 10 < 16$, we have $a_i = 1$ exactly when $b_i \geq 8 - s$, and the excluded residue is $b_i = 7 - s$. So $\sum_i a_i = s$ asks for exactly $s/2$ entries with $b_i \geq 8 - s$.

At $s = 2$ exactly one entry equals six and none equals five, and the other three entries add up to two. Up to a permutation of the coordinates this leaves $(6, 2, 0, 0)$ and $(6, 1, 1, 0)$, with images $(0, 4, 2, 2)$ and $(0, 3, 3, 2)$; there are twelve labels of each kind, which gives the $12 + 12$ pairs of Proposition 8.4(b). At $s = 4$ exactly two entries are at least four and none equals three; two entries of at least four already add up to eight, so $\mathbf{b}$ is a permutation of $(4, 4, 0, 0)$, with image the complementary permutation $(0, 0, 4, 4)$, which gives three pairs.

The labels of the three lists are distinct: the sources at $s = 2$ have an entry six, their images have at most one entry larger than three, and the labels at $s = 4$ have two entries four and no entry six. So no label lies in two pairs, the 27 pairs are the classes, and the 149 labels carry $149 - 27 = 122$ characters.

The dectic. Here $b_i \leq 8$ for $i \leq 3$, $b_4 \leq 3$ and $b_1 + b_2 + b_3 + 2b_4 = 10$, and the shifts to consider are $s = 2$ and $s = 4$. For $i \leq 3$, $b_i + s \leq 12 < 20$, so $a_i = 1$ exactly when $b_i \geq 10 - s$, and the excluded residue is $b_i = 9 - s$. For x_4 we have $a_4 = \lfloor (b_4 + s)/5 \rfloor$ and the excluded residue is $b_4 + s \equiv 4 \pmod{5}$. The condition is

$$\#\{i \leq 3 : b_i \geq 10 - s\} + a_4 = s/2.$$

At $s = 2$ the value $b_4 = 2$ is excluded, $a_4 = 1$ exactly when $b_4 = 3$, and no b_i equals seven. Either one b_i equals eight and $b_4 \leq 1$, which leaves $(8, 0, 0, 1)$, $(8, 2, 0, 0)$ and $(8, 1, 1, 0)$ up to a permutation of the first three coordinates, with images $(0, 2, 2, 3)$, $(0, 4, 2, 2)$ and $(0, 3, 3, 2)$ under the same permutation; or no b_i is at least eight and $b_4 = 3$, which leaves the fifteen labels $(c_1, c_2, c_3, 3)$ with $c_1 + c_2 + c_3 = 4$, with images $(c_1 + 2, c_2 + 2, c_3 + 2, 0)$.

At $s = 4$ the value $b_4 = 0$ is excluded, so $b_4 \geq 1$ and $a_4 = 1$; hence exactly one b_i is at least six, and none equals five. The other entries satisfy $b_j + b_k + 2b_4 = 10 - b_i \leq 4$ with $b_4 \geq 1$, which leaves $(6, 2, 0, 1)$, $(6, 1, 1, 1)$, $(6, 0, 0, 2)$, $(7, 1, 0, 1)$ and $(8, 0, 0, 1)$ up to a permutation of the first three coordinates, with images $(0, 6, 4, 0)$, $(0, 5, 5, 0)$, $(0, 4, 4, 1)$, $(1, 5, 4, 0)$ and $(2, 4, 4, 0)$.

Sorting the labels of these lists by b_4 shows which of them coincide. With $b_4 = 3$ there are the sources $(c_1, c_2, c_3, 3)$ and the image $(0, 2, 2, 3)$, which is one of them, with $\mathbf{c} = (0, 2, 2)$. With $b_4 = 2$ there are $(0, 4, 2, 2)$, $(0, 3, 3, 2)$ and $(6, 0, 0, 2)$, all different. With $b_4 = 1$ there are $(8, 0, 0, 1)$, which occurs at both shifts, and $(6, 2, 0, 1)$, $(6, 1, 1, 1)$, $(7, 1, 0, 1)$ and $(0, 4, 4, 1)$, all different. With $b_4 = 0$ there are $(8, 2, 0, 0)$ and $(8, 1, 1, 0)$, the images $(c_1 + 2, c_2 + 2, c_3 + 2, 0)$, all of whose entries are at least two, and $(0, 6, 4, 0)$, $(0, 5, 5, 0)$, $(1, 5, 4, 0)$ and $(2, 4, 4, 0)$, of which only the last has all entries at least two; it is the image of $\mathbf{c} = (0, 2, 2)$. So the only chains longer than one pair are

$$(8, 0, 0, 1) \xrightarrow{s=2} (0, 2, 2, 3) \xrightarrow{s=2} (2, 4, 4, 0) \xleftarrow{s=4} (8, 0, 0, 1)$$

and its permutations, the three triples of Proposition 8.4(c). The remaining pairs are the 12 with $\mathbf{c}$ not a permutation of $(2, 2, 0)$ and the $6 + 3$ others at $s = 2$, and the $6 + 3 + 3 + 6$ at $s = 4$, which are the pairs listed there: 39 pairs and 3 triples, 87 labels in 42 classes, and $145 - 87 + 42 = 100$ characters.

APPENDIX E. THE BLOCKS OF THE MASS MAP

Table 11 gives the data behind Table 4. For each row we take the representative with $(i, j, k, l) = (1, 2, 3, 4)$ and list, for $\mathbf{b} \in B_\chi$ and $\mathbf{c} \in B_{-\chi}$ in the order of Table 4, the vector $h + \mathbf{b} + \mathbf{c}$, its sector r , the vector k of (10) and the entry $\psi^r \prod_i w_i^{k_i}$ of Theorem 8.5. The other blocks of each row follow by permuting the coordinates of weight one. For the blocks of size 2×2 the number Δ of Lemma 8.7 is 6 for the self-dual sextic row, 0 and 8 for the second and third octic rows, and 0, 0, 10 and 10 for the sixth to ninth dectic rows; for instance in the eighth dectic row $\Delta = 4 + 6 - 0 - 0 = 10$, so the determinant is $50 \cdot 250 \psi^{10} - 1 = 12500 \psi^{10} - 1$.

APPENDIX F. EXACT CERTIFICATES FOR THE FIXED THREEFOLD

The following rational identities finish the smooth-seed proof. In each row $Aq + Bq' = 1$:

$$\begin{aligned}
q &= 2s^3 + 15s^2 + 6s + 1, & A &= (21s + 101)/37, \\
B &= -7s^2/37 - 307s/222 - 32/111; \\
q &= s^3 + 6s^2 + 15s + 2, & A &= -s/74 - 4/37, \\
B &= s^2/222 + 5s/111 + 3/37; \\
q &= s^4 + 8s^3 + 30s^2 + 8s + 1, & A &= s^2/6 + 47s/48 + 19/8, \\
B &= -s^3/24 - 21s^2/64 - 29s/24 - 11/64.
\end{aligned}$$

For $g = s^2 - 8s - 21$ and $h = s^4 - 16s^3 - 66s^2 - 16s + 89$ one has

$$\left(-\frac{3s^3}{22} + \frac{217s^2}{88} + \frac{85s}{22} - \frac{47}{8} \right) g + \left(\frac{3s}{22} - \frac{11}{8} \right) h = 1.$$

These are identities in $\mathbb{Q}[s]$ and can be verified by multiplication.

For completeness, a reproducible description fixes every entry of the matrix in (13). Order all bases lexicographically. For $L_1 L_2$ the source is $H^2(A, \mathcal{O}(-4, -5, 0, 1))$ and the target is $H^2(A, \mathcal{O}(-2, -3, 2, 3))$. Represent each tensor monomial by its vector v of y exponents, using the Čech convention of the proof of Theorem 14.2. The source vectors range over

$$\{-3, -2, -1\} \times \{-4, -3, -2, -1\} \times \{0\} \times \{0, 1\},$$

and the target vectors range over

$$\{-1\} \times \{-2, -1\} \times \{0, 1, 2\} \times \{0, 1, 2, 3\}.$$

Expand F_1 as $\sum_e A_e \prod_i x_i^{2-e_i} y_i^{e_i}$, using (11). The integer matrix is

$$M(1)_{w,v} = A_{w-v},$$

where an absent coefficient is zero. Terms outside the target ranges represent zero Čech classes. This specifies a 24 by 24 integer matrix without a numerical approximation to π . Its four six-dimensional character blocks and their exact determinants are recorded below. The algorithm for their bases takes the first member of each b -orbit and replaces the pair (v, bv) by $v + \varepsilon_b(-1)^q bv$ in ambient degree q ; fixed vectors are kept only for their actual eigenvalue. The generator a is already diagonal.

$$M_{++} = \begin{pmatrix} 5 & 8 & 9 & 2 & 3 & 4 \\ 2 & 2 & 5 & 5 & 0 & 1 \\ 0 & 5 & 5 & 0 & 2 & 2 \\ 11 & 16 & 16 & 11 & 2 & 15 \\ 2 & 9 & 12 & 5 & 3 & 4 \\ 1 & 5 & 2 & 2 & 0 & 1 \end{pmatrix}, \quad \det M_{++} = 792.$$

$$M_{+-} = \begin{pmatrix} 5 & 6 & -5 & -2 & 3 & -2 \\ 2 & -2 & 5 & -5 & 0 & 1 \\ 0 & 5 & -5 & 0 & 2 & -2 \\ 11 & -6 & 6 & -11 & 2 & 11 \\ 2 & 5 & -2 & -5 & 3 & -2 \\ 1 & -5 & 2 & -2 & 0 & 1 \end{pmatrix}, \quad \det M_{+-} = -792.$$

$$M_{-+} = \begin{pmatrix} 5 & 1 & 2 & 2 & 0 & 1 \\ 2 & 9 & 12 & 5 & 3 & 4 \\ 11 & 16 & 16 & 11 & 2 & 15 \\ 0 & 5 & 5 & 0 & 2 & 2 \\ 2 & 2 & 5 & 5 & 0 & 1 \\ 1 & 12 & 9 & 2 & 3 & 4 \end{pmatrix}, \quad \det M_{-+} = 792.$$

$$M_{--} = \begin{pmatrix} 5 & -1 & 2 & -2 & 0 & 1 \\ 2 & 5 & -2 & -5 & 3 & -2 \\ 11 & -6 & 6 & -11 & 2 & 11 \\ 0 & 5 & -5 & 0 & 2 & -2 \\ 2 & -2 & 5 & -5 & 0 & 1 \\ 1 & 2 & -5 & -2 & 3 & -2 \end{pmatrix}, \quad \det M_{--} = -792.$$

The matrix code uses exact rational elimination and checks equivariance by reconstructing every image from the target eigenbasis. The nonvanishing assertion is a finite integer certificate, combined with the analytic transcendence argument for the actual parameter. It is not a finite search over moduli and is not a claimed Lean formalization.

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