

THE MIRROR CLAUSE OF THE GEOMETRIC SELECTION CONJECTURE IS EMPTY

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ABSTRACT. The Geometric Selection Conjecture asks for a functional on the heterotic landscape, built from the Weil–Petersson metric and invariant under mirror symmetry, whose unique minimum is a phenomenologically viable vacuum. It was shown in our paper on the Dwork quintic that the functional carries no information beyond its minimum: the conjecture holds if and only if there is a viable vacuum fixed by the mirror involution wherever that involution is defined. We prove that the fixed-point clause is never satisfied. A vacuum fixed by the mirror involution has a threefold isomorphic to its own mirror, hence equal Hodge numbers $h^{1,1} = h^{2,1}$, vanishing Euler number and, on its component of the landscape, a bundle with $c_3 = 0$; its net number of families is therefore zero and it violates the three-family condition. Consequently no vacuum built on the tangent bundle of any Calabi–Yau threefold, or on any deformation of it, can be selected, and the conjecture is equivalent to the bare statement that one viable vacuum exists on a component of the landscape not of standard-embedding type, with no condition involving mirror symmetry. The same holds when invariance under any compact group of isometries containing the mirror involution is demanded. The existence statement itself remains open.

1. INTRODUCTION

The heterotic string on a product of four-dimensional space-time with a Calabi–Yau threefold X and a holomorphic bundle V on X gives a four-dimensional gauge theory whose charged chiral multiplets are counted by the cohomology of V [8, 11]. A conjecture of Bhattacharjee, Bhattacharya and Ghosh [3], the Geometric Selection Conjecture, proposes that the vacuum of the real world is singled out among all such pairs (X, V) by a functional constructed from the natural metric on their moduli, proper on every component, with a unique minimum, invariant under mirror symmetry, and attaining its minimum at a vacuum that meets nine phenomenological conditions (C1)–(C9). In [4] we proved that the functional is determined by the vacuum it selects: the conjecture holds if and only if there is a viable vacuum $\mathcal{V}^*$ with $\mu(\mathcal{V}^*) = \mathcal{V}^*$ whenever $\mathcal{V}^*$ lies in the domain $\mathcal{L}^\mu$ of the mirror involution μ (Theorem D there, proved as Theorem 22). The content of the conjecture is therefore an existence statement with one clause attached to it, the clause on mirror symmetry.

This note removes that clause. The argument is short. A vacuum fixed by μ is isomorphic to its mirror, so its threefold has $h^{1,1} = h^{2,1}$ and Euler number zero. The domain $\mathcal{L}^\mu$ is made of the components of the landscape that contain a vacuum of the standard embedding $V = T_X$, on which the Chern classes of V are constant, so $c_3(V) = c_3(T_X) = \chi(X) = 0$ there. The index theorem then gives zero net families, and condition (C2), three net families, fails. So no fixed point of μ is viable, every viable vacuum lies outside $\mathcal{L}^\mu$, and the clause is vacuous.

Theorem 1.1. *The Geometric Selection Conjecture holds if and only if the heterotic landscape contains a viable vacuum outside the domain $\mathcal{L}^\mu$ of the mirror involution, that is, on a component that contains no vacuum of the standard embedding. No vacuum of the form (X, V) with V a deformation of the tangent bundle of X , on any Calabi–Yau threefold X or any free quotient of one, is the minimum of a functional with the properties the conjecture demands, whether or not it is viable.*

Remark 24 of [4] already noted that for a viable vacuum off the standard embedding the clause on μ is empty. The new point is that the clause can never be met inside $\mathcal{L}^\mu$, so the vacua of standard type drop out

2020 *Mathematics Subject Classification.* Primary 81T30, 14J32; Secondary 14J33, 32G20.

Key words and phrases. heterotic string, Calabi–Yau threefold, mirror symmetry, generations, Geometric Selection Conjecture. Corresponding author; second e-mail address d.bhattacharjee@erl-forschung.de.

of the conjecture altogether and what remains is existence on the other components. Section 2 recalls the setting of [3, 4], Section 3 proves the theorem, and Section 4 states what is left of the conjecture.

2. THE LANDSCAPE, THE MIRROR INVOLUTION AND VIABILITY

We use the setting of [3] as recalled in [4, Section 9]. A heterotic vacuum is a pair (X, V) of a Calabi–Yau threefold X , that is a compact Kähler threefold with $K_X \cong \mathcal{O}_X$ and $h^{1,0} = h^{2,0} = 0$, and a slope-polystable holomorphic vector bundle V on X with $c_1(V) = 0$ for which $c_2(T_X) - c_2(V)$ is effective. The landscape $\mathcal{L}$ is the set of isomorphism classes of vacua, where $(X, V) \cong (X', V')$ when there is an isomorphism $X \rightarrow X'$ carrying V' to V . It is the disjoint union of its components $\mathcal{L}_\tau$ of fixed topological type τ ; the type records the diffeomorphism type of X together with the rank and the Chern classes of V . Each component carries the metric $g = G^{\text{WP}} \oplus G^V$ made of the Weil–Petersson metric on the complex structures of X and the L^2 metric on the bundle moduli. Two facts about a component are used below: the Euler number $\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X))$ is constant on it, since X only deforms, and the Chern classes of V are constant on it, since they are part of τ .

The *standard embedding* is the vacuum (X, T_X) . Its component is the component of *standard type* of X ; it contains the deformations of T_X as a polystable bundle, and on all of it $c_3(V) = c_3(T_X) = \chi(X)$.

The mirror involution μ is defined on the union $\mathcal{L}^\mu$ of the components of standard type and sends a vacuum to its mirror: the threefold of $\mu(X, V)$ is a mirror $\widehat{X}$ of X , so that

$$h^{1,1}(\widehat{X}) = h^{2,1}(X), \quad h^{2,1}(\widehat{X}) = h^{1,1}(X),$$

and μ is an isometry of g . This is the involution of [3], recorded there for the Weil–Petersson metric, and it is the mirror map of $(2, 2)$ vacua of [9, 7] extended along the components they lie on; nothing below depends on how the extension is made, only on the two displayed equalities for the underlying threefolds.

A vacuum is *viable* when it satisfies the nine conditions (C1)–(C9) of [3]. Only the second is used here:

(C2) the vacuum has three net families, that is, the index of the Dirac operator coupled to V on X , counted with the Wilson line if X is a free quotient, equals three in absolute value.

In every case of the heterotic dictionary the index in question is the holomorphic Euler characteristic $\chi(X, V) = \sum_q (-1)^q h^q(X, V)$, up to sign; for the standard embedding this is [4, Theorem 6] and in general it is the index formula of [10, Chapter 14]. A Wilson line on a free quotient $Y = X/H$ breaks the gauge group but does not change the index of V_Y on Y , which is $\chi(X, V)/|H|$; and (Y, V_Y) is itself a point of $\mathcal{L}$, so it is enough to treat an arbitrary vacuum and let X stand for Y .

The conjecture, in the form stated in [3] and recalled in [4], reads as follows.

Conjecture 2.1 (Geometric Selection Conjecture). There is a functional $\mathcal{S} : \mathcal{L} \rightarrow [0, \infty)$, constructed from the metric g , such that (G1) $\mathcal{S}$ is smooth and proper on each component $\mathcal{L}_\tau$; (G2) $\mathcal{S}$ has a unique global minimum $\mathcal{V}^* \in \mathcal{L}$; (G3) $\mathcal{V}^*$ is viable; (G4) $\mathcal{S} \circ \mu = \mathcal{S}$ on $\mathcal{L}^\mu$.

Theorem 2.2 ([4, Theorem 22]). *Conjecture 2.1 holds if and only if there is a viable vacuum $\mathcal{V}^*$ with $\mu(\mathcal{V}^*) = \mathcal{V}^*$ whenever $\mathcal{V}^* \in \mathcal{L}^\mu$.*

3. FIXED POINTS OF THE MIRROR INVOLUTION HAVE NO FAMILIES

Lemma 3.1. *Let X be a Calabi–Yau threefold and V a holomorphic vector bundle on X with $c_1(V) = 0$. Then $\chi(X, V) = \frac{1}{2} \int_X c_3(V)$.*

Proof. By Hirzebruch–Riemann–Roch, $\chi(X, V) = \int_X \text{ch}(V) \text{td}(X)$, and in degree six the integrand is $\text{ch}_3(V) + \text{ch}_1(V) \text{td}_2(X) + \text{rk}(V) \text{td}_3(X)$. Here $\text{ch}_1(V) = c_1(V) = 0$, $\text{td}_3(X) = \frac{1}{24} c_1(X) c_2(X) = 0$ since $c_1(X) = 0$, and $\text{ch}_3(V) = \frac{1}{6} (c_1^3 - 3c_1 c_2 + 3c_3)(V) = \frac{1}{2} c_3(V)$. $\square$

Lemma 3.2. *Let $(X, V) \in \mathcal{L}^\mu$ satisfy $\mu(X, V) = (X, V)$. Then $h^{1,1}(X) = h^{2,1}(X)$, $\chi(X) = 0$, $c_3(V) = 0$ and $\chi(X, V) = 0$. In particular (X, V) has no net families and is not viable.*

Proof. Since $\mu(X, V) = (X, V)$ in $\mathcal{L}$, the two vacua are isomorphic, so the threefold $\widehat{X}$ of $\mu(X, V)$ is isomorphic to X . Therefore $h^{1,1}(X) = h^{1,1}(\widehat{X}) = h^{2,1}(X)$, and $\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X)) = 0$. The vacuum lies in $\mathcal{L}^\mu$, so its component is of standard type and contains a vacuum (X_0, T_{X_0}) with X_0 a deformation of X . Chern classes are constant on the component, so $c_3(V) = c_3(T_{X_0}) = \chi(X_0)$, and Euler numbers are constant on it, so $\chi(X_0) = \chi(X) = 0$. Hence $c_3(V) = 0$, and $\chi(X, V) = 0$ by Lemma 3.1. The net number of families is $|\chi(X, V)| = 0 \neq 3$, so (C2) fails. $\square$

Proof of Theorem 1.1. By Theorem 2.2 the conjecture is equivalent to the existence of a viable vacuum $\mathcal{V}^*$ that is fixed by μ if it lies in $\mathcal{L}^\mu$. By Lemma 3.2 a vacuum in $\mathcal{L}^\mu$ fixed by μ is not viable. So a viable $\mathcal{V}^*$ satisfies the clause exactly when $\mathcal{V}^* \notin \mathcal{L}^\mu$, and the conjecture holds if and only if there is a viable vacuum outside $\mathcal{L}^\mu$. This is the first statement.

For the second, let $\mathcal{S}$ be any functional satisfying (G1)–(G4) and let $\mathcal{V}^*$ be its minimum. If $\mathcal{V}^* \in \mathcal{L}^\mu$ then $\mathcal{S}(\mu\mathcal{V}^*) = \mathcal{S}(\mathcal{V}^*)$ by (G4), so $\mu\mathcal{V}^*$ is also a global minimum and $\mu\mathcal{V}^* = \mathcal{V}^*$ by (G2); by Lemma 3.2 $\mathcal{V}^*$ then has no families, against (G3). So the minimum of every such functional lies outside $\mathcal{L}^\mu$, and no vacuum of standard type, which is to say no (X, V) with V a deformation of T_X on X or on a free quotient, is ever selected. This holds whether or not such a vacuum is viable: a vacuum of standard type has $|\chi(X)|/2$ net families by Lemmas 3.1 and 3.2, and the quotient of the Tian–Yau threefold by its free $\mathbb{Z}/3$ has $\chi = -6$, so three families of standard type do occur in the landscape; they are never the minimum. $\square$

Corollary 3.3. *If mirror pairs are identified in $\mathcal{L}$, each pair counted as a single point on the ground that both members define the same superconformal field theory, then μ is the identity on $\mathcal{L}^\mu$, condition (G4) is empty, and Conjecture 2.1 holds if and only if there is a viable vacuum of any type.*

Proof. When mirror pairs are one point every point of $\mathcal{L}^\mu$ is fixed by μ , the clause of Theorem 2.2 is satisfied by every vacuum, and Theorem 2.2 gives the equivalence with existence. Lemma 3.2 does not apply, because a point that is a pair is a class of two vacua, not a vacuum isomorphic to its mirror. $\square$

Remark 3.4 (The compact-group reading). In [4, Remark 23] the words “constructed from the metric” were read as invariance under a compact group G of isometries of $(\mathcal{L}, g)$ containing μ , and the conjecture in that reading was shown equivalent to the existence of a viable vacuum fixed by G wherever G acts. A G -fixed vacuum in $\mathcal{L}^\mu$ is μ -fixed, so by Lemma 3.2 it is not viable, and the same argument gives: in the compact-group reading the conjecture holds if and only if there is a viable vacuum outside $\mathcal{L}^\mu$ fixed by G wherever G acts on the components outside $\mathcal{L}^\mu$. If G acts only on $\mathcal{L}^\mu$, as μ does, this is again the bare existence of a viable vacuum outside $\mathcal{L}^\mu$.

Remark 3.5 (Self-mirror threefolds). Lemma 3.2 does not say that a fixed point of μ is impossible, only that it is not viable. Threefolds isomorphic to their mirrors exist; the Schoen threefold with $h^{1,1} = h^{2,1} = 19$ and its free quotient by $\mathbb{Z}/3 \times \mathbb{Z}/3$ with $h^{1,1} = h^{2,1} = 3$ are examples, and on each of them the standard embedding has zero net families, as the lemma requires. The vacua of the standard embedding on the quintic pencil, on the three weighted pencils of Fermat type and on their free quotients, which were studied in [4], have $\chi \neq 0$ and are therefore never fixed by μ ; they fail (C2) for the independent reason computed there, namely that $|\chi|/2$ is $100/|H|$ on quintic quotients and never three on the weighted pencils.

4. WHAT REMAINS OF THE CONJECTURE

After Theorem 1.1 the Geometric Selection Conjecture is the sentence: *there is a heterotic vacuum, on a component that contains no standard embedding, satisfying (C1)–(C9)*. Nothing in it refers to a functional, to the Weil–Petersson metric or to mirror symmetry. The conditions divide into two groups.

Conditions (C1)–(C6), anomaly cancellation and stability, three net families, a Wilson line breaking to the Standard Model group, three families without chiral exotics, masses for the vectorlike exotics and doublet–triplet splitting, are satisfied by known vacua: (C1)–(C4) by the heterotic standard models of Braun, He, Ovrut and Pantev [6] and of Bouchard and Donagi [5], and (C1)–(C6) by line-bundle models of Anderson, Gray, Lukas and Palti [1, 2]. All of these have V of rank four or five with $c_3(V) \neq \chi(X)$, so they lie outside $\mathcal{L}^\mu$, as Theorem 1.1 requires of any candidate.

Conditions (C7)–(C9), stabilised moduli, an acceptable rank and hierarchy of the Yukawa masses, and consistent supersymmetry breaking, are stated in [3] by comparison with measurement. For none of the vacua above are all three established, and until the three conditions are given as inequalities with named constants there is no mathematical statement to prove or refute in them. The conjecture therefore stands exactly where Theorem 1.1 leaves it: equivalent to the existence of one vacuum satisfying (C1)–(C9), with the knowledge that the vacuum, if it exists, is not a deformation of the standard embedding, and with its remaining content entirely in the three conditions that are not yet mathematical.

DATA AVAILABILITY

No data were generated. The source of this note and a program that checks the lemma and the corollary on the Hodge numbers of self-mirror and three-family threefolds are in release 1.0.0 of the repository <https://github.com/creelie/geometric-selection-conjecture> (<https://github.com/creelie/geometric-selection-conjecture/releases/tag/v1.0.0>); the same check is the module `construct/gsc_mirror_fixed.py` of the Anaxia engine, <https://github.com/creelie/eBLLM>.

DECLARATIONS

Competing interests. The authors declare no competing interests. **Funding.** No funding was received. **Use of AI tools.** The authors used Claude Code (Anthropic) to draft the text of this note and the checking module from the authors’ argument; the authors verified every statement and take full responsibility for the content.

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