

Largest Gaps between Eigenvalues of the Gaussian Orthogonal Ensemble

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AUTHOR'S NOTE

With the exception of this note, the entire document is machine-written. I have carefully supervised its preparation to ensure that the proofs are readable and that previous literature is appropriately cited. I welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We determine the asymptotic distribution of the largest gaps between consecutive eigenvalues of the Gaussian orthogonal ensemble (GOE) in the bulk of the spectrum. After an explicit centering and scaling, the largest gaps in a fixed interval converge to a Poisson point process with an exponential intensity, so the largest gap has a shifted Gumbel limit. The analogous result for the Gaussian unitary ensemble was proved by Feng and Wei. Our proof expresses the squared probability that the GOE has no eigenvalues in a union of intervals as the corresponding probability for the Gaussian unitary ensemble times a determinant of fixed size, for every matrix dimension. We control this determinant on intervals whose length in units of the local mean spacing grows with the dimension, using an uncertainty principle of Nazarov and exact identities for Hermite functions.

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Introduction

1.1 Background

The gaps between consecutive eigenvalues are among the central objects of random matrix theory. In the bulk of the spectrum, the limiting distribution of an individual gap is known for a wide range of models [17, 55]. The smallest and largest gaps within a fixed interval in the bulk are understood for far fewer models. Unlike an individual gap, they are not determined by the local eigenvalue statistics on any fixed scale. The motivation for their study is discussed in the introduction of [4], which includes a comparison with numerical data for the Riemann zeta function.

We first describe what is known about the largest gaps. For the Gaussian unitary ensemble (GUE) and the circular unitary ensemble (CUE) [14, 37], Ben Arous and Bourgade found the first-order asymptotics of the largest gaps in the bulk, which are of order $\sqrt{\log N}/N$, and left their fluctuations open [4]. Feng and Wei identified these fluctuations [21]. They proved that the largest gaps of the CUE and the GUE, after centering and rescaling, converge to a Poisson point process with an exponential intensity, so that the largest gap has a Gumbel limit. Charlier recently extended this result to unitary-invariant ensembles for a broad class of regular analytic potentials [6]. Bourgade proved universality of the smallest gaps in both symmetry classes, and of the largest gaps in the complex Hermitian class, for generalized Wigner matrices satisfying a smoothness assumption [5]. His proof combines new quantitative relaxation estimates for Dyson Brownian motion with reverse heat flow. Landon, Lopatto, and Marcinek proved comparison theorems for the largest gaps of real symmetric and complex Hermitian generalized Wigner matrices without a smoothness assumption [33]. In the complex Hermitian case, both works use the GUE result of Feng and Wei [21] to identify the limiting distribution. Both approaches also apply to the real symmetric largest-gap problem, but identifying its limiting distribution required the corresponding GOE result, which was not known.

The smallest gaps were studied earlier. Vinson's dissertation [52] was the first rigorous work on extremal gaps. It determined the limiting distribution of the smallest gap of the CUE and the GUE. Ben Arous and Bourgade found the joint limit of the smallest gaps of these two ensembles [4], building on Soshnikov's work on determinantal point processes [46]. The limiting distribution of the smallest gaps is now also known for the Gaussian orthogonal ensemble (GOE) [23], for the circular β -ensemble [27] with β a

positive integer [20], for the Gaussian symplectic ensemble [25], and for complex Wishart ensembles and one-cut unitary-invariant ensembles with real-analytic confining potentials [42]. Bourgade’s universality theorem [5] covers the smallest gaps of real symmetric and complex Hermitian generalized Wigner matrices with smooth entries, and Figalli and Guionnet proved limiting laws for the smallest gaps and first-order asymptotics for the largest gaps in perturbative several-matrix models with $\beta = 2$ [26]. For Wigner matrices and related models, lower bounds for the smallest gaps were proved in [40], and for sparse random matrices in [34]. Extremal gaps have also been studied for non-Hermitian matrices [35, 36, 38] and for the zeros of random functions [19, 22, 24].

The results above that identify limiting laws or first-order asymptotics for the largest eigenvalue gaps rest on a determinantal structure [45], either of the ensemble itself or of the Gaussian ensemble with which it is compared. The GOE does not have this structure, and the limiting distribution of its largest gaps has remained open; the problem was raised in [5, 33, 35]. For the circular orthogonal and symplectic ensembles (COE and CSE) [14, 37], Feng and Wei noted that even the order of the largest gaps is unknown [21]. Our main result, Theorem 1.1, determines the limiting distribution of the largest gaps in the bulk of the spectrum of the GOE.

The analysis of the largest gaps rests on precise asymptotics for the probability that a long interval contains no eigenvalue. For the sine process, which is the limit of the GUE eigenvalues after rescaling in the bulk [2], this probability is a Fredholm determinant of the sine kernel, and its asymptotics were predicted by des Cloizeaux and Mehta [12] and Dyson [13]. The leading term was proved by Widom [53], and the constant term, known as the Widom–Dyson constant, by Krasovsky [31], Ehrhardt [15], and Deift, Its, Krasovsky, and Zhou [8]. For $\beta = 1$ and $\beta = 4$, the corresponding asymptotics, including the constant terms, were proved by Ehrhardt [16]. For general $\beta > 0$, the bulk scaling limit of the Gaussian β -ensemble is the Sine_β process [50], and the probability that it has no point in a long interval was determined up to a constant factor by Valkó and Virág [51].

The eigenvalues of the GOE form a Pfaffian point process rather than a determinantal one: the k -point correlation function is the Pfaffian of a $2k \times 2k$ matrix built from a 2×2 matrix kernel, and one entry of this kernel contains the sign function $\text{sgn}(x - y)/2$ [37]. For even matrix dimensions, Tracy and Widom [48] expressed the square of the probability that the GOE has no eigenvalue in a union of intervals as the corresponding probability for the GUE, multiplied by the determinant of a matrix of fixed size; see also [1, 27, 28, 44, 49, 54] for the correlation kernels of orthogonal and symplectic ensembles, and [9–11] for their use in proving universality. Our proof builds on the identity of Tracy and Widom. However, we derive it differently, for even and odd matrix dimensions, and we analyze it on intervals whose length in units of the local mean spacing grows with N ; see Section 1.4.

1.2 Main results

Let H_N be an $N \times N$ matrix from the GOE, normalized so that its distribution has density proportional to $\exp(-N \operatorname{Tr} H^2/4)$ with respect to Lebesgue measure on real symmetric matrices. Equivalently, the entries H_{ij} with $i \leq j$ are independent centered Gaussian random variables, with variance $2/N$ if $i = j$ and $1/N$ if $i < j$. We let $\lambda_1 < \lambda_2 < \dots < \lambda_N$ denote the eigenvalues of H_N , and we let Z_N denote the normalizing constant in their joint density, which is

$$\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |\lambda_j - \lambda_i| \prod_{i=1}^N e^{-N\lambda_i^2/4} \quad (1.1)$$

on $\{\lambda_1 < \dots < \lambda_N\}$; see [2]. We define

$$\sigma(y) = \sqrt{4 - y^2}, \quad |y| \leq 2. \quad (1.2)$$

With this normalization, the empirical eigenvalue distribution converges to the semicircle law. This is the probability measure on $[-2, 2]$ with density $\sigma(y)/(2\pi)$. The mean spacing between consecutive eigenvalues near $y \in (-2, 2)$ is approximately $2\pi/(N\sigma(y))$.

We define the constant κ_1 and the function F by

$$\kappa_1 = 2^{13/24} \exp\left(\frac{3}{2}\zeta'(-1)\right), \quad F(s) = \frac{s^2}{64} + \frac{s}{8}, \quad s \geq 0, \quad (1.3)$$

where ζ denotes the Riemann zeta function. Both come from the asymptotics of the probability that the Sine₁ process has no point in a long interval, which we recall after Corollary 1.2. For $N \geq 2$, we let $b_N > 0$ denote the solution of $F(b_N) = \log N$, and we define $a_N = F'(b_N)$. Solving the quadratic equation $F(b_N) = \log N$, we find

$$b_N = 8\sqrt{\log N + \frac{1}{4}} - 4, \quad a_N = \frac{b_N + 4}{32}, \quad a_N b_N = 2 \log N - \frac{b_N}{8}. \quad (1.4)$$

We fix an interval $I = [y_-, y_+]$ with $-2 < y_- < y_+ < 2$, and we consider the gaps $\lambda_{i+1} - \lambda_i$ with both endpoints in I . We define

$$\sigma_I = \min_{y \in I} \sigma(y), \quad \partial_* I = \{\xi \in \{y_-, y_+\} : \sigma(\xi) = \sigma_I\}. \quad (1.5)$$

Observe that $\partial_* I$ is the set of endpoints of I at which the semicircle density is smallest. It consists of the endpoint with the larger absolute value if $y_- \neq -y_+$, and of the two endpoints if $y_- = -y_+$. In particular, $\xi \neq 0$ for all $\xi \in \partial_* I$. We define

$$\Lambda_I = \kappa_1 \sum_{\xi \in \partial_* I} \frac{\sigma_I^3}{|\xi|}. \quad (1.6)$$

We let $\mathcal{G}_N(I)$ denote the set of indices $i \in \{1, \dots, N-1\}$ such that $\lambda_i, \lambda_{i+1} \in I$, and for $i \in \mathcal{G}_N(I)$ we define the *rescaled gap*

$$\tau_{i,N} = a_N(N\sigma_I(\lambda_{i+1} - \lambda_i) - b_N) + \frac{9}{8} \log b_N. \quad (1.7)$$

Finally, we let $\mathcal{X}_N$ denote the point process on $\mathbb{R}$ whose points are the numbers $\tau_{i,N}$, $i \in \mathcal{G}_N(I)$. For a Borel set $A \subset \mathbb{R}$, we let $\mathcal{X}_N(A)$ denote the number of points of $\mathcal{X}_N$ in A , which is the number of indices $i \in \mathcal{G}_N(I)$ such that $\tau_{i,N} \in A$. Our main result is the following theorem.

Theorem 1.1. *Let $m \geq 1$, and let $A_1, \dots, A_m \subset \mathbb{R}$ be disjoint intervals, each of the form $(a, b]$ with $a < b$ or (a, ∞) . Then, as $N \rightarrow \infty$, the vector $(\mathcal{X}_N(A_1), \dots, \mathcal{X}_N(A_m))$ converges in distribution to a vector of independent Poisson random variables with means $\Lambda_I \int_{A_l} e^{-x} dx$, $1 \leq l \leq m$. In particular, $\mathcal{X}_N$ converges in distribution to the Poisson point process on $\mathbb{R}$ with intensity $\Lambda_I e^{-x} dx$, with respect to the vague topology.*

We let $\tau_{1,N}^\downarrow \geq \tau_{2,N}^\downarrow \geq \dots$ denote the values $\tau_{i,N}$, $i \in \mathcal{G}_N(I)$, arranged in decreasing order, and we set $\tau_{k,N}^\downarrow = -\infty$ if $\mathcal{G}_N(I)$ has fewer than k elements. We also let $m_{1,N}$ denote the largest gap $\lambda_{i+1} - \lambda_i$ with $i \in \mathcal{G}_N(I)$ (and we set $m_{1,N} = 0$ if $\mathcal{G}_N(I)$ is empty). For random variables Y_N and positive numbers θ_N , we write $Y_N = O_{\mathbb{P}}(\theta_N)$ if the sequence $(Y_N/\theta_N)_N$ is tight.

Corollary 1.2. *For every $k \geq 1$ and $x \in \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tau_{k,N}^\downarrow \leq x) = \exp(-\Lambda_I e^{-x}) \sum_{j=0}^{k-1} \frac{(\Lambda_I e^{-x})^j}{j!}. \quad (1.8)$$

In particular, $\tau_{1,N}^\downarrow - \log \Lambda_I$ converges in distribution to the standard Gumbel distribution, which has the distribution function $x \mapsto \exp(-e^{-x})$:

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tau_{1,N}^\downarrow - \log \Lambda_I \leq x) = \exp(-e^{-x}), \quad x \in \mathbb{R}. \quad (1.9)$$

Moreover,

$$N\sigma_I m_{1,N} = 8\sqrt{\log N} - 4 - \frac{9 \log \log N}{4\sqrt{\log N}} + O_{\mathbb{P}}((\log N)^{-1/2}). \quad (1.10)$$

We now explain where the constant κ_1 and the function F come from. For a Borel set $A \subset \mathbb{R}$, we define the *hole probability*

$$E_{1,N}(A) = \mathbb{P}(\lambda_i \notin A \text{ for all } i \in \{1, \dots, N\}). \quad (1.11)$$

The limit of the hole probability of a short interval is expressed through the sine kernel. We define the *sine kernel* S by

$$S(u) = \frac{\sin u}{\pi u}, \quad u \neq 0, \quad S(0) = \frac{1}{\pi}. \quad (1.12)$$

It is the kernel of the sine process, rescaled to have mean spacing π . For $t > 0$, we let $S_{+,t}$ and $S_{-,t}$ denote the integral operators on $L^2(0, t)$ with kernels $S(u - v) + S(u + v)$ and $S(u - v) - S(u + v)$, respectively, and we define the Fredholm determinants

$$D_+(t) = \det(\text{Id} - S_{+,t}), \quad D_-(t) = \det(\text{Id} - S_{-,t}), \quad q_1(s) = D_+(s/4), \quad (1.13)$$

for $t, s > 0$. We will show in Corollary 7.5 that $q_1(s)$ is the limit of $E_{1,N}((y, y + s/(N\sigma(y))))$ as $N \rightarrow \infty$, for all $y \in (-2, 2)$ and $s > 0$. The interval $(y, y + s/(N\sigma(y)))$ has length $s/(2\pi)$ in units of the mean spacing near y . So $q_1(s)$ is the probability that the Sine₁ process, normalized to have mean spacing 2π , has no point in a given interval of length s ; this expression for the gap probability is classical [37, 48]. The asymptotics of $D_\pm$ were determined by Ehrhardt [16, Theorem 4.8], confirming predictions of Dyson [13]. Ehrhardt proved that

$$\log D_\pm(t) = -\frac{t^2}{4} \mp \frac{t}{2} - \frac{\log t}{8} + \left(\frac{1}{24} \pm \frac{1}{4}\right) \log 2 + \frac{3}{2} \zeta'(-1) + o(1), \quad t \rightarrow \infty. \quad (1.14)$$

Setting $t = s/4$ in (1.14) (see Lemma 7.1), we have

$$q_1(s) = \kappa_1 s^{-1/8} e^{-F(s)} (1 + o(1)), \quad \frac{D_+(s/4)}{D_-(s/4)} = \sqrt{2} e^{-s/4} (1 + o(1)), \quad s \rightarrow \infty, \quad (1.15)$$

with κ_1 and F as in (1.3).

Remark 1.3. We comment on the constants in Theorem 1.1 and Corollary 1.2.

- (a) By (1.15), the function F is the exponent in the probability that the Sine₁ process has no point in a long interval. Its quadratic term, its linear term, and the power $s^{-1/8}$ in (1.15) agree with the asymptotics of Valkó and Virág for the Sine _{β} process at $\beta = 1$ [51], which are stated for intervals of length s in units in which the mean spacing is 2π . The constant κ_1 is the constant of Ehrhardt [16], after the change of variables $t = s/4$.
- (b) The constant -4 in (1.10) comes from the linear term $s/8$ in F . The coefficient $9/4$ of the third term comes from the power $s^{-1/8}$ in (1.15) and from the fact that the largest gaps are located within distance of order $(\log N)^{-1}$ of the points of $\partial_* I$; see Remark 9.2.
- (c) We consider the GUE with eigenvalue density proportional to $\prod_{i < j} |\lambda_j - \lambda_i|^2 \prod_i e^{-N\lambda_i^2/2}$. Its empirical eigenvalue distribution also converges to the semicircle law on $[-2, 2]$. We let $m_{1,N}^{\text{GUE}}$ denote the largest gap $\lambda_{i+1} - \lambda_i$ of the GUE with $\lambda_i, \lambda_{i+1} \in I$. It follows from [21, Theorem 2] that

$$N\sigma_I m_{1,N}^{\text{GUE}} = \sqrt{32 \log N} - \frac{5}{2\sqrt{2}} \frac{\log \log N}{\sqrt{\log N}} + O_{\mathbb{P}}((\log N)^{-1/2}). \quad (1.16)$$

The leading term for the GOE is larger by a factor $\sqrt{2}$. It agrees with the prediction $\sqrt{(64/\beta) \log N}$ at $\beta = 1$, which Feng and Wei formulated for the circular β -ensemble [21].

The following remark records a consequence of Corollary 1.2 for real symmetric generalized Wigner matrices, which was anticipated in [33] and [5]. We prove it in Section 10.

Remark 1.4. Let W_N be an $N \times N$ real symmetric generalized Wigner matrix satisfying [33, Definition 1.1]; in particular, the entries of $\sqrt{N}W_N$ have finite moments of all orders. For $k \geq 1$, let $\hat{T}_{k,N}$ denote the k -th largest of the gaps $\mu_{i+1} - \mu_i$ between consecutive eigenvalues $\mu_1 \leq \dots \leq \mu_N$ of W_N such that $\mu_i \in I$ (this is the convention of [33]), and let $\hat{T}_{k,N} = 0$ if there are fewer than k such gaps. We define

$$\hat{\tau}_{k,N} = a_N(N\sigma_I \hat{T}_{k,N} - b_N) + \frac{9}{8} \log b_N. \quad (1.17)$$

Then (1.8) holds with $\hat{\tau}_{k,N}$ in place of $\tau_{k,N}^\downarrow$. This follows from Corollary 1.2 and the comparison theorem [33, Corollary 1.6].

1.3 Notation

We use the following notation throughout the paper. For a statement P , we let $\mathbf{1}_P$ denote 1 if P holds and 0 otherwise, and for a set $A \subset \mathbb{R}$, we let $\mathbf{1}_A$ denote its indicator function. For a Borel set $A \subset \mathbb{R}$, we let $|A|$ denote its Lebesgue measure, and for $c > 0$, we write $cA = \{ca : a \in A\}$. We write $\|\cdot\|_2$ for the norm of $L^2(A)$, and $\|T\|$ for the operator norm of a bounded operator T on $L^2(A)$. We identify $L^2(A)$ with the space of functions in $L^2(\mathbb{R})$ that vanish outside A . For $t \in \mathbb{R}$, we write $(t)_+ = \max(t, 0)$. For $t, y \in \mathbb{R}$, we let $\int_t^y$ denote the oriented integral, so that $\int_t^y = -\int_y^t$ if $y < t$. In Section 1.4, C and c denote positive constants that do not depend on N , and whose values may change from line to line. They may depend on the fixed parameters δ, γ, r_0, k , and $x_1, \dots, x_k$ introduced there, and the implicit constants in $O(\cdot)$ have the same property. Limits are taken as $N \rightarrow \infty$.

1.4 Overview of the proof

We now describe the main steps of the proof. This overview can be read independently of the later sections; all objects in it are defined here, in Section 1.2, or in Section 1.3. The proof has two parts. The first part, which follows the strategy of Feng and Wei for the GUE [21], reduces Theorem 1.1 to three estimates for hole probabilities. The second part proves these estimates in four steps, and Steps 1–3 are new compared with the proof for the GUE. Where the GUE proof uses that the hole probability is a Fredholm determinant,

Step 1 uses an exact identity, valid for even and odd N , that writes the square of the GOE hole probability as the GUE hole probability times a determinant of fixed size. Step 2 replaces the sign function in this determinant, which does not decay, by functions that are concentrated near the endpoints of the intervals. Step 3 replaces the elementary bound on the inverse of the operator in the Fredholm determinant, which is a power of N in our setting, by a bound that is exponential in the expected number of eigenvalues in the intervals. Step 4 uses these results and the asymptotics of Hermite functions to compare the GOE hole probability with its limit.

Reduction to hole probabilities. For the Poisson point process with intensity $\Lambda_I e^{-x} dx$, the sum of $\prod_{j=1}^k (\tau_j - x_j)_+$ over all ordered k -tuples $(\tau_1, \dots, \tau_k)$ of distinct points has expectation

$$\prod_{j=1}^k \int_{\mathbb{R}} (t - x_j)_+ \Lambda_I e^{-t} dt = \prod_{j=1}^k \Lambda_I e^{-x_j}. \quad (1.18)$$

By a criterion of the type used in [21, Lemma 1], which we state in Lemma A.1, it therefore suffices to show that for every $k \geq 1$ and all $x_1, \dots, x_k \in \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \sum_{\substack{i_1, \dots, i_k \in \mathcal{G}_N(I) \\ \text{distinct}}} \prod_{j=1}^k (\tau_{i_j, N} - x_j)_+ = \prod_{j=1}^k \Lambda_I e^{-x_j}. \quad (1.19)$$

The positive parts $(\tau - x)_+$ are used instead of the indicators $\mathbf{1}_{\tau > x}$ because the left side of (1.19) can be expressed through hole probabilities, as follows. For $x \in \mathbb{R}$, we define

$$\ell_N(x) = \frac{1}{N\sigma_I} \left(b_N + \frac{x - \frac{9}{8} \log b_N}{a_N} \right), \quad \chi_N = a_N N \sigma_I. \quad (1.20)$$

By (1.7), we have $\tau_{i, N} - x = \chi_N (\lambda_{i+1} - \lambda_i - \ell_N(x))$ for all $i \in \mathcal{G}_N(I)$. Further, by Taylor's theorem and the definitions of a_N and b_N , we have

$$F(N\sigma_I \ell_N(x)) = \log N + x - \frac{9}{8} \log b_N + o(1). \quad (1.21)$$

We also call the interval $(\lambda_i, \lambda_{i+1})$ a gap. For a gap $(\lambda_i, \lambda_{i+1})$ and $\ell > 0$, the set of $y \in \mathbb{R}$ such that $[y, y + \ell] \subset (\lambda_i, \lambda_{i+1})$ is an interval of length $(\lambda_{i+1} - \lambda_i - \ell)_+$. For $y_1, \dots, y_k \in \mathbb{R}$, we let $\mathcal{W}_N(y_1, \dots, y_k)$ denote the event that the intervals $[y_j, y_j + \ell_N(x_j)]$, $1 \leq j \leq k$, are contained in distinct gaps $(\lambda_i, \lambda_{i+1})$ with $i \in \mathcal{G}_N(I)$. Integrating over the left endpoints $y_1, \dots, y_k$, we show in Lemma 10.1 that, for N large enough that $\ell_N(x_j) > 0$ for all j , the expectation in (1.19) equals

$$\chi_N^k \int_{y_-}^{y_+ - \ell_N(x_1)} \dots \int_{y_-}^{y_+ - \ell_N(x_k)} \mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k)) dy_k \dots dy_1. \quad (1.22)$$

If two of the intervals $[y_j, y_j + \ell_N(x_j)]$ intersect, then they cannot lie in distinct gaps, so $\mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k)) = 0$. Otherwise, the probability $\mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k))$ is bounded by the hole probability of the union of the intervals $(y_j, y_j + \ell_N(x_j))$. Conversely, suppose that these intervals are separated from each other, and from the endpoints of I , by distances at least $3N^{-1/3}$. Then $\mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k))$ is at least this hole probability minus the probability that some interval of length $N^{-1/3}$ in I contains no eigenvalue. The latter probability is $O(N^{-D})$ for every fixed $D > 0$. Indeed, every interval of length $N^{-1/3}$ in I contains one of $O(N^{1/3})$ fixed intervals of the form $(y, y + s/(N\sigma(y)))$ with $s = \gamma_0 \sqrt{\log N}$, where $\gamma_0 > 0$ is a constant with $\gamma_0^2/64 > D + 1/3$. By (1.23) below, applied with $\gamma = \gamma_0$, the hole probability of each of these intervals is at most $2q_1(\gamma_0 \sqrt{\log N}) \leq N^{-\gamma_0^2/64}$, and a union bound gives the claim.

It remains to estimate the hole probabilities of unions of at most k intervals in I with total length at most $C\sqrt{\log N}/N$. We need three estimates. The first is an asymptotic formula for one interval, which we prove in Corollary 7.5. For all $\delta \in (0, 1)$ and $\gamma > 0$, there exists $C > 0$ such that

$$E_{1,N}\left(\left(y, y + \frac{s}{N\sigma(y)}\right)\right) = q_1(s)(1 + O(e^{Cs}N^{-1/2})) \quad (1.23)$$

uniformly in $y \in [-2 + \delta, 2 - \delta]$ and $0 < s \leq \gamma\sqrt{\log N}$. The second and third estimates concern open intervals $A_1, \dots, A_r \subset [-2 + \delta, 2 - \delta]$ with disjoint closures and total length $\sum_j |A_j| \leq \gamma\sqrt{\log N}/N$, where the number r of intervals is at most a fixed integer r_0 . We prove in Corollary 8.5 that

$$E_{1,N}\left(\bigcup_{j=1}^r A_j\right) \leq e^{C\sqrt{\log N}} \prod_{j=1}^r E_{1,N}(A_j), \quad (1.24)$$

and that if, in addition, $\text{dist}(A_i, A_j) \geq N^{-1/2}$ for all $i \neq j$, then

$$E_{1,N}\left(\bigcup_{j=1}^r A_j\right) = (1 + O(e^{C\sqrt{\log N}}N^{-1/4})) \prod_{j=1}^r E_{1,N}(A_j). \quad (1.25)$$

We now explain how these estimates give (1.19). We fix $\delta > 0$ such that $I \subset [-2 + \delta, 2 - \delta]$, and we take $r_0 = k$. Since $N\ell_N(x) = (8 + o(1))\sqrt{\log N}/\sigma_I$ for every fixed x , we may also fix $\gamma > 0$ such that $N\sum_j \ell_N(x_j) \leq \gamma\sqrt{\log N}$ and $2N\ell_N(x_j) \leq \gamma\sqrt{\log N}$ for all j and all large N . We first consider the points $(y_1, \dots, y_k)$ in the domain of integration in (1.22) for which the intervals $[y_j, y_j + \ell_N(x_j)]$ are separated from each other, and from the endpoints of I , by distances at least $3N^{-1/3}$. For these points, we replace the probability $\mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k))$ by the hole probability of $\bigcup_j (y_j, y_j + \ell_N(x_j))$, with an error $O(N^{-D})$, where we choose $D > k + 1$ so that the error remains negligible after multiplication by $\chi_N^k \leq (N \log N)^k$. Next, we use (1.25) with $A_j = (y_j, y_j + \ell_N(x_j))$ and $r = k$ to replace

this hole probability by the product of the hole probabilities $E_{1,N}((y_j, y_j + \ell_N(x_j)))$. For the remaining points at which the intervals $[y_j, y_j + \ell_N(x_j)]$ are disjoint, we use (1.24) to bound $\mathbb{P}(\mathcal{W}_N(y_1, \dots, y_k))$ by $e^{C\sqrt{\log N}}$ times the same product. By (1.23), (1.15), and (1.21), and since q_1 is nonincreasing and $\sigma(y) \geq \sigma_I$ on I , each factor $\chi_N E_{1,N}((y_j, y_j + \ell_N(x_j)))$ is $O(\log N)$. Further, the set of these remaining points has k -dimensional Lebesgue measure $O(N^{-1/3})$. Their contribution is therefore $O(e^{C\sqrt{\log N}} N^{-1/3} (\log N)^k) = o(1)$, and (1.19) reduces to the statement

$$\lim_{N \rightarrow \infty} \chi_N \int_{y_-}^{y_+ - \ell_N(x)} E_{1,N}((y, y + \ell_N(x))) dy = \Lambda_I e^{-x}, \quad x \in \mathbb{R}, \quad (1.26)$$

which we prove in Section 9. To compute this integral, we apply (1.23) with $s = N\sigma(y)\ell_N(x)$. Note that

$$\begin{aligned} N\sigma(y)\ell_N(x) - N\sigma_I\ell_N(x) &= N\sigma_I\ell_N(x) \left(\frac{\sigma(y)}{\sigma_I} - 1 \right), \\ F'(b_N)b_N &= a_N b_N = 2 \log N (1 + o(1)), \end{aligned} \quad (1.27)$$

by (1.4). Set $s_N = N\sigma_I\ell_N(x)$ and $\Psi(y) = \sigma(y)/\sigma_I$. Using (1.23) and (1.15), we find that the integrand is

$$\chi_N q_1(s_N) \Psi(y)^{-1/8} \exp \left(-\frac{s_N^2}{64} (\Psi(y)^2 - 1) - \frac{s_N}{8} (\Psi(y) - 1) \right) (1 + o(1)), \quad (1.28)$$

uniformly in $y \in [y_-, y_+ - \ell_N(x)]$. Since $s_N^2/64 \sim \log N$, this shows that the integral is concentrated within distance of order $(\log N)^{-1}$ of the points of $\partial_* I$, where $\sigma(y) = \sigma_I$. Near $\xi \in \partial_* I$, we have

$$\frac{\sigma(y)}{\sigma_I} - 1 = \frac{|\xi|}{\sigma_I^2} |y - \xi| + O(|y - \xi|^2), \quad (1.29)$$

since $\sigma'(\xi) = -\xi/\sigma_I$. A Laplace-type computation then shows that the contribution of ξ to (1.26) is

$$\chi_N q_1(N\sigma_I\ell_N(x)) \frac{\sigma_I^2}{2|\xi| \log N} (1 + o(1)) = \frac{a_N b_N}{2 \log N} \kappa_1 \frac{\sigma_I^3}{|\xi|} e^{-x} (1 + o(1)), \quad (1.30)$$

where we used the asymptotics

$$q_1(N\sigma_I\ell_N(x)) = \kappa_1 e^{-x} b_N N^{-1} (1 + o(1)), \quad (1.31)$$

which follow from (1.15) and (1.21). In this computation, the factor $b_N^{9/8}$ coming from $e^{-F(s)}$ is compensated by the factor $s^{-1/8} = b_N^{-1/8} (1 + o(1))$ in (1.15) and by the

width $(\log N)^{-1}$ of the region near ξ , since $\chi_N/(N \log N) = 2\sigma_I b_N^{-1}(1 + o(1))$. Since $a_N b_N/(2 \log N) \rightarrow 1$, summing over $\xi \in \partial_* I$ gives the constant Λ_I .

The estimates (1.23), (1.24), and (1.25) are the main work of this paper. Two features make (1.23) delicate. First, at $s = N\sigma_I \ell_N(x)$, the probability $q_1(s)$ is of order $N^{-1} \sqrt{\log N}$ by (1.31). So we need a relative error bound for a polynomially small probability. Second, the interval contains of order $\sqrt{\log N}$ eigenvalues on average, so the convergence of local statistics on fixed scales does not suffice. For the GUE, the analogous estimates are proved in [21] using the determinantal structure. The asymptotics of the Fredholm determinant can then be derived from those of Toeplitz determinants [8], and the analogue of (1.24) holds without the factor $e^{C\sqrt{\log N}}$ by the negative association of determinantal point processes [4, Lemma 3.8]. Neither tool is available for the GOE. The GOE hole probability is the square root of the Fredholm determinant of an operator with a 2×2 matrix kernel [28, 44, 49], and one entry of this kernel contains the sign function $\text{sgn}(x - y)/2$, which does not decay as $|x - y| \rightarrow \infty$.

Hermite coordinates. For the analysis, it is convenient to rescale the eigenvalues by the factor $\sqrt{N/2}$. By (1.1), the vector $(\sqrt{N/2} \lambda_1, \dots, \sqrt{N/2} \lambda_N)$ has a density proportional to $\prod_{i < j} |x_j - x_i| \prod_i e^{-x_i^2/2}$ on $\{x_1 < \dots < x_N\}$, and its empirical distribution concentrates on $[-\sqrt{2N}, \sqrt{2N}]$. We call $x = \sqrt{N/2} \lambda$ the *Hermite coordinate*. In the rest of the overview, the variables x, y, z , and t denote points in Hermite coordinates, and J denotes a subset of $\mathbb{R}$ in Hermite coordinates. We let $\mathcal{P}_N(J)$ denote the probability that $\sqrt{N/2} \lambda_i \notin J$ for all i , so that $\mathcal{P}_N(\sqrt{N/2} A) = E_{1,N}(A)$ for every Borel set $A \subset \mathbb{R}$. For $\delta \in (0, 1/2]$, we define the *bulk*

$$\mathcal{B}_N(\delta) = [- (1 - \delta) \sqrt{2N}, (1 - \delta) \sqrt{2N}], \quad (1.32)$$

and we note that $\sqrt{N/2} [-2 + \delta, 2 - \delta] = \mathcal{B}_N(\delta/2)$. We say that an estimate holds in the bulk if, for every fixed $\delta \in (0, 1/2]$, it holds uniformly on $\mathcal{B}_N(\delta)$, with constants depending on δ . In Hermite coordinates, the mean spacing between consecutive eigenvalues near a point x in the bulk is approximately $\pi/\sqrt{2N - x^2}$.

The Hermite functions $h_0, h_1, \dots$ are obtained by applying the Gram–Schmidt procedure in $L^2(\mathbb{R})$ to the functions $x \mapsto x^n e^{-x^2/2}$, $n \geq 0$, and they form an orthonormal basis of $L^2(\mathbb{R})$. We define

$$K_N(x, y) = \sum_{n=0}^{N-1} h_n(x) h_n(y), \quad (K_N f)(y) = \int_{\mathbb{R}} K_N(y, x) f(x) dx, \quad (1.33)$$

for $x, y \in \mathbb{R}$ and for $f \in L^2(\mathbb{R})$ or f bounded. The restriction of K_N to $L^2(\mathbb{R})$ is the orthogonal projection onto the span of $h_0, \dots, h_{N-1}$. For a measurable set $J \subset \mathbb{R}$ of finite measure, we let K_J denote the integral operator on $L^2(J)$ with kernel $K_N(x, y)$, $x, y \in J$, and we define $\mathcal{D}_N(J) = \det(\text{Id} - K_J)$. This is the probability that the GUE of

Remark 1.3(c), with its eigenvalues multiplied by $\sqrt{N/2}$, has no eigenvalue in J [2]. The eigenvalues of K_J lie in $[0, 1)$, so $\text{Id} - K_J$ is invertible.

Step 1: An exact formula and the endpoint matrix. Let $J = \bigcup_{j=1}^r (a_j, b_j)$ with $a_1 < b_1 < a_2 < \dots < a_r < b_r$. For $t \in \mathbb{R}$, we define $\varepsilon_t(x) = \text{sgn}(x - t)/2$ and

$$g_t(y) = (K_N \varepsilon_t)(y) - \varepsilon_t(y), \quad y \in \mathbb{R}. \quad (1.34)$$

The function g_t measures how far the projection K_N is from preserving the step function ε_t . For $1 \leq i \leq r$, we define $G_i = (\text{Id} - K_J)^{-1}(\mathbf{1}_J g_{a_i}) \in L^2(J)$. The identity $G_i = \mathbf{1}_J g_{a_i} + K_N(\mathbf{1}_J G_i)$ holds almost everywhere on J , and we use its right side to define G_i at every point of J . For t in the closure of J , we write $G_i(t^+)$ and $G_i(t^-)$ for the one-sided limits of G_i at t , whenever they exist. We define the $r \times r$ endpoint matrix $\mathcal{A}_N(J)$ by

$$\mathcal{A}_N(J)_{ij} = \mathbf{1}_{i=j} - (G_i(b_j^-) - G_i(a_j^+)), \quad 1 \leq i, j \leq r. \quad (1.35)$$

We measure the size of J by its *microscopic length* $L(J) = \sqrt{2N}|J|$. For $J = \sqrt{N/2}A$, we have $L(J) = N|A|$, and $L(J)$ is comparable to the expected number of eigenvalues in J when J lies in the bulk. The first main step of the proof is the following estimate, which holds for even and odd N . For all $\delta \in (0, 1/2]$ and $r_0 \geq 1$, there exists $C > 0$ such that if $r \leq r_0$, $\bar{J} \subset \mathcal{B}_N(\delta)$, and $L(J) \leq N^{1/8}$, then

$$\mathcal{P}_N(J)^2 = \mathcal{D}_N(J) \left(\det \mathcal{A}_N(J) + O(e^{CL(J)} N^{-1/2}) \right), \quad |\det \mathcal{A}_N(J)| \leq C e^{CL(J)}. \quad (1.36)$$

The estimate (1.36) is the combination of an exact identity, Corollary 3.3, with Proposition 6.2. The exact identity expresses $\mathcal{P}_N(J)^2$, for every $N \geq 2$, as $\mathcal{D}_N(J)$ multiplied by the determinant of an explicit $(r+2) \times (r+2)$ matrix. This matrix contains $\mathcal{A}_N(J)$ as a principal submatrix, and we will not need its other entries in the overview (they are given in Definition 3.2). The square appears because $\mathcal{P}_N(J)$ is a Pfaffian, by de Bruijn's integration formula [7], and the square of a Pfaffian is a determinant. The Pfaffian involves the sign function on the complement of J . After an integration by parts (Lemma 3.5), the sign function appears only in r rank-one operators, one for each interval of J , and these produce the endpoint matrix. The two remaining rows of the $(r+2) \times (r+2)$ matrix involve the Hermite functions h_{N-1} and h_N of the highest degrees. Using the bound (1.46) below, we show that all entries of the matrix are $O(e^{CL(J)})$, and that these two rows are within $O(e^{CL(J)} N^{-1/2})$ of the corresponding rows of the identity matrix. Since the determinant is linear in each row, this gives (1.36).

For even N , an identity of this type was found by Tracy and Widom [48, (33)], who manipulated Fredholm determinants of operators with matrix kernels and justified these manipulations in detail for the sine kernel. We derive the exact identity differently, starting from de Bruijn's formula and using only finite-dimensional linear algebra. Our

derivation applies to even and odd N (this matters, since Theorem 1.1 concerns all N). For odd N , one of the two remaining rows involves a function whose smallness is not evident from its definition. To describe it, we define

$$(\varepsilon f)(x) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(x - z) f(z) \, dz, \quad p_N(y) = \int_{\mathbb{R}} K_N(x, y) \, dx, \quad x, y \in \mathbb{R}, \quad (1.37)$$

for all $N \geq 1$ and integrable functions f . Note that $\int_{\mathbb{R}} h_{N-1}(x) \, dx \neq 0$ when N is odd. For odd N , we define

$$c_N = \left(\frac{N}{2}\right)^{-1/4} \left(\int_{\mathbb{R}} h_{N-1}(x) \, dx\right)^{-1}, \quad \omega_N = \left(\frac{N}{2}\right)^{1/4} \varepsilon h_N + c_N. \quad (1.38)$$

In the bulk, the two terms of ω_N are of order one. However, we show in Lemma 2.6(b) that their sum is exactly a multiple of $1 - p_N$:

$$\omega_N = c_N(1 - p_N), \quad N \text{ odd}. \quad (1.39)$$

We also show that $p_N(y) = 1 + O(N^{-1/2})$ in the bulk, so (1.39) shows that $\omega_N = O(N^{-1/2})$ there. Landon and Sosoë observed a cancellation of this kind through an asymptotic analysis of integrals of Hermite functions [32, Lemma 8.14 and Appendix D].

Step 2: The sign function becomes local. Observe that the entries of $\mathcal{A}_N(J)$ involve the functions g_{a_i} only on J . The second step, which has no counterpart in the proof for the GUE, is the observation that g_t is concentrated near t , and that it can be computed from the kernel K_N in the bulk. We set $\varphi_N = (N/2)^{1/4} h_N$ and $\psi_N = (N/2)^{1/4} h_{N-1}$. We prove in Lemma 2.7 the exact identity

$$(K_N \varepsilon_t)(y) = (\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y) + \int_t^y K_N(x, y) \, dx, \quad t, y \in \mathbb{R}. \quad (1.40)$$

In the bulk, the product $(\varepsilon \varphi_N)(\varepsilon \psi_N)$ is $O(N^{-1/2})$. Indeed, one of its factors is a constant multiple of p_N , namely $\varepsilon \varphi_N = -c_N p_N$ for odd N by (1.39), and $\varepsilon \psi_N = -p_N / \int_{\mathbb{R}} \varphi_N$ for even N by Lemma 2.6(a). This factor is bounded in the bulk, since $p_N = 1 + O(N^{-1/2})$ there, and c_N and $(\int_{\mathbb{R}} \varphi_N)^{-1}$ are bounded. By parity, the other factor equals $\int_0^y \varphi_N$ or $\int_0^y \psi_N$, which is the integral of an oscillating function and is $O(N^{-1/2})$ in the bulk. The integral of the kernel between t and y is close to $\operatorname{sgn}(y - t)/2$ when $\sqrt{N}|y - t|$ is large. In Corollary 4.7, we deduce from (1.40) that

$$|g_t(y)| \leq \frac{C}{1 + \sqrt{N}|y - t|} + CN^{-1/4}, \quad t, y \in \mathcal{B}_N(\delta). \quad (1.41)$$

The term $CN^{-1/4}$ comes from the oscillation of the kernel far from the diagonal and is not optimal. Further, we define

$$g_{\text{sine}}(u) = -\operatorname{sgn}(u) \int_{|u|}^{\infty} S(v) \, dv, \quad u \in \mathbb{R}, \quad (1.42)$$

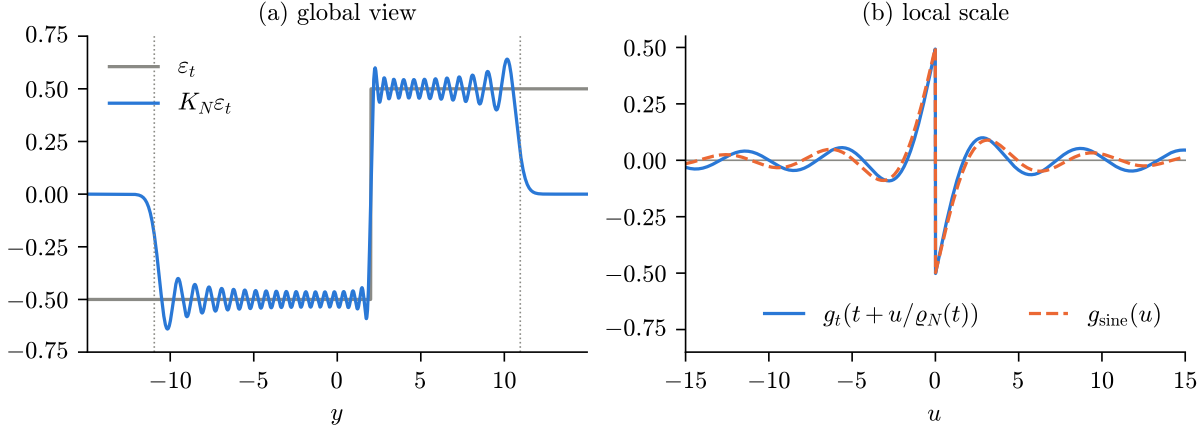


Figure 1.1. The function $g_t = K_N \varepsilon_t - \varepsilon_t$ for $N = 60$ and $t = 2$. Panel (a) shows $K_N \varepsilon_t$ and the step function ε_t ; the dotted lines are at $\pm\sqrt{2N}$. In the bulk, their difference is concentrated near t , up to small oscillations, while near and beyond the spectral edges it is of order one. Panel (b) shows $g_t(t + u/\varrho_N(t))$, where $\varrho_N(t) = \sqrt{2N - t^2}$, together with its limit $g_{\text{sine}}(u)$.

and we show that

$$\left| g_t\left(t + \frac{u}{\varrho_N(t)}\right) - g_{\text{sine}}(u) \right| \leq C(1 + |u|)N^{-1/2}, \quad t \in \mathcal{B}_N(\delta), \quad 0 < |u| \leq N^{1/4}, \quad (1.43)$$

where we write $\varrho_N(t) = \sqrt{2N - t^2}$; see Figure 1.1. The functions p_N , $\varepsilon\varphi_N$, and $\varepsilon\psi_N$ are defined by integrals over $\mathbb{R}$, but by parity each of them equals an explicit constant plus an integral over $[0, y]$ of Hermite functions, and the constants are given by exact formulas for $\int_{\mathbb{R}} h_n$. In particular, our arguments use the Hermite functions only in the bulk, and we will not need any estimates for them near the spectral edges $\pm\sqrt{2N}$.

The bound (1.41) is what makes the GOE hole probability factorize. We now assume, in addition to the assumptions of (1.36), that the distance between any two intervals of J is at least $1/\sqrt{2}$, which corresponds to the distance $N^{-1/2}$ in the original coordinates λ , and that $L(J) \leq \gamma\sqrt{\log N}$ for a fixed $\gamma > 0$. Then $g_{a_i}(y) = O(N^{-1/4})$ for y in the intervals (a_j, b_j) with $j \neq i$, by (1.41). Further, $|K_N(x, y)| \leq C$ for x and y in different intervals, while $K_N(x, x)$ is of order $\sqrt{N}$ in the bulk. Using these bounds and the bound (1.46) below, we show in Lemma 6.8 that the off-diagonal entries of $\mathcal{A}_N(J)$ are $O(e^{CL(J)}N^{-1/4})$, and that each diagonal entry $\mathcal{A}_N(J)_{jj}$ differs by $O(e^{CL(J)}N^{-1/2})$ from the endpoint matrix of the single interval $J_j = (a_j, b_j)$. The latter is a 1×1 matrix, which we identify with its entry. We write each interval J_j in the form of Step 4 below, with $x_0 = a_j$ and $s = 2\varrho_N(a_j)(b_j - a_j) \leq CL(J)$. The lower bound $\det \mathcal{A}_N(J_j) \geq ce^{-CL(J)}$ from Step 4 then shows that

$$\det \mathcal{A}_N(J) = (1 + O(e^{CL(J)}N^{-1/4})) \prod_{j=1}^r \mathcal{A}_N((a_j, b_j)). \quad (1.44)$$

Next, the operator K_J differs from the direct sum of the operators K_{J_j} by the operator whose kernel equals $K_N(x, y)$ for x and y in different intervals of J and vanishes otherwise. Its Hilbert–Schmidt norm is bounded by $C|J| \leq CL(J)N^{-1/2}$. Together with (1.46) below, a standard estimate for Fredholm determinants (Lemma 8.1) gives

$$\mathcal{D}_N(J) = (1 + O(e^{CL(J)}N^{-1})) \prod_{j=1}^r \mathcal{D}_N(J_j). \quad (1.45)$$

To prove (1.25), we set $J = \sqrt{N/2} \cup_j A_j$, which satisfies $\bar{J} \subset \mathcal{B}_N(\delta/2)$ and $L(J) = N \sum_j |A_j| \leq \gamma \sqrt{\log N} \leq N^{1/8}$. We apply (1.36) to J and to each J_j , and we insert (1.44) and (1.45). Using the lower bound on $\det \mathcal{A}_N(J_j)$ once more and taking square roots, we obtain (1.25).

Step 3: An inverse bound from the uncertainty principle. The estimates in Steps 1 and 2 involve the norm of $(\text{Id} - K_J)^{-1}$ on $L^2(J)$. Recall that the eigenvalues of K_J lie in $[0, 1)$. Therefore, $\|(\text{Id} - K_J)^{-1}\| \leq \mathcal{D}_N(J)^{-1}$. However, for an interval J , the right side grows like $e^{cL(J)^2}$, which is a power of N when $L(J)$ is of order $\sqrt{\log N}$, and this would overwhelm the polynomially small errors. The third main step is the much better bound

$$\|(\text{Id} - K_J)^{-1}\| \leq Ce^{CL(J)}, \quad |J| \leq 1, \quad (1.46)$$

which we prove in Proposition 5.1 using a quantitative uncertainty principle of Nazarov [29, 39]. To state it, we define the Fourier transform of $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}, \quad (1.47)$$

and we extend $\mathcal{F}$ to a unitary operator on $L^2(\mathbb{R})$ as usual. The principle states that for all measurable sets $A, \Sigma \subset \mathbb{R}$ of finite measure and all $f \in L^2(\mathbb{R})$,

$$\|f\|_2^2 \leq Ce^{C|A||\Sigma|} \left(\int_{\mathbb{R} \setminus A} |f(x)|^2 dx + \int_{\mathbb{R} \setminus \Sigma} |(\mathcal{F}f)(\xi)|^2 d\xi \right). \quad (1.48)$$

We let V_N denote the real span of $h_0, \dots, h_{N-1}$. Each h_n is an eigenfunction of $\mathcal{F}$, so $\mathcal{F}$ maps V_N to the complex span of $h_0, \dots, h_{N-1}$, and we show that for every f in this complex span, the integral of $|f|^2$ over the complement of $[-8\sqrt{N}, 8\sqrt{N}]$ is bounded by $e^{-N}\|f\|_2^2$. Applying (1.48) with $A = J$ and $\Sigma = [-8\sqrt{N}, 8\sqrt{N}]$ to $f \in V_N$, and using $|J||\Sigma| = 8\sqrt{2}L(J)$, we find that

$$\|f\|_2^2 \leq Ce^{CL(J)} (\|f\|_2^2 - \|\mathbf{1}_J f\|_2^2 + e^{-N}\|f\|_2^2), \quad f \in V_N. \quad (1.49)$$

Since $|J| \leq 1$, we have $L(J) \leq \sqrt{2N}$, so the term e^{-N} is negligible compared to $e^{-CL(J)}$, and we find $\|\mathbf{1}_J f\|_2^2 \leq (1 - ce^{-CL(J)})\|f\|_2^2$ for all $f \in V_N$. Since $\|K_J\|$ is the supremum of

$\|1_J f\|_2^2 / \|f\|_2^2$ over nonzero $f \in V_N$, this implies (1.46). For a bounded interval $J' \subset \mathbb{R}$, we let $S_{J'}$ denote the integral operator on $L^2(J')$ with kernel $S(u - v)$. Recall that $S(u - v)$ is the kernel of the orthogonal projection onto the space of functions in $L^2(\mathbb{R})$ whose Fourier transform vanishes outside $[-1, 1]$. The same argument, with V_N replaced by this space, $A = J'$, and $\Sigma = [-1, 1]$, so that $|A||\Sigma| = 2|J'|$ and there is no tail term, shows that $\|(\text{Id} - S_{J'})^{-1}\| \leq Ce^{C|J'|}$.

Step 4: Comparison with the sine kernel. We now explain the proof of the one-interval estimate (1.23). We fix a point $y_0 \in [-2 + \delta, 2 - \delta]$ in eigenvalue coordinates and a number $0 < s \leq \gamma\sqrt{\log N}$, and we define

$$x_0 = \sqrt{N/2} y_0, \quad \varrho = \varrho_N(x_0), \quad J = \left(x_0, x_0 + \frac{s}{2\varrho}\right). \quad (1.50)$$

Then $\varrho = \sqrt{N/2} \sigma(y_0)$, so $J = \sqrt{N/2} (y_0, y_0 + s/(N\sigma(y_0)))$, and $L(J) = s/\sigma(y_0) \leq Cs$. We apply (1.36) to J with $r = 1$ and $\delta/4$ in place of δ , so that the error term there is $O(e^{Cs} N^{-1/2})$. We show in Lemma 4.3, using the Plancherel–Rotach asymptotics of the Hermite functions [41, 47], that

$$\left| \frac{1}{\varrho} K_N\left(x_0 + \frac{u}{\varrho}, x_0 + \frac{v}{\varrho}\right) - S(u - v) \right| \leq CN^{-1/2}, \quad |u|, |v| \leq 2N^{1/4}. \quad (1.51)$$

This shows that after the change of variables $x = x_0 + u/\varrho$, the operator K_J is close to $S_{(0, s/2)}$. For $\ell > 0$, we let $\mathcal{A}_{\text{sine}}(\ell)$ denote the number given by the definitions of Step 1 with $r = 1$, after K_N , J , and $g_{a_1}(x_0 + \cdot/\varrho)$ are replaced by the integral operator with kernel $S(u - v)$ on $L^2(\mathbb{R})$, the interval $(0, \ell)$, and g_{sine} . So $\mathcal{A}_{\text{sine}}(s/2)$ is the sine-kernel analogue of $\det \mathcal{A}_N(J)$. Using (1.51), the convergence of $g_{x_0}(x_0 + u/\varrho)$ to $g_{\text{sine}}(u)$ from Step 2, and the bounds on $(\text{Id} - K_J)^{-1}$ and $(\text{Id} - S_{(0, s/2)})^{-1}$ from Step 3, we show that

$$\begin{aligned} \mathcal{D}_N(J) &= \det(\text{Id} - S_{(0, s/2)})(1 + O(e^{Cs} N^{-1/2})), \\ \det \mathcal{A}_N(J) &= \mathcal{A}_{\text{sine}}(s/2) + O(e^{Cs} N^{-1/2}). \end{aligned} \quad (1.52)$$

We translate $(0, s/2)$ to $(-s/4, s/4)$. On this symmetric interval, the operator with kernel $S(u - v)$ preserves the even and the odd functions. Its restrictions to these two subspaces are unitarily equivalent to $S_{+, s/4}$ and $S_{-, s/4}$, respectively. Therefore $\det(\text{Id} - S_{(0, s/2)}) = D_+(s/4)D_-(s/4)$. Further, by an identity of Tracy and Widom [48, (19), (25), (30)], we have $\mathcal{A}_{\text{sine}}(s/2) = D_+(s/4)/D_-(s/4)$. By (1.15) and the positivity and continuity of $D_{\pm}$, this ratio is bounded below by $ce^{-s/4}$ for all $s > 0$. Hence $\det \mathcal{A}_N(J) \geq ce^{-CL(J)}$, and the error term in (1.36) is small compared to $\det \mathcal{A}_N(J)$, where we used $s \leq \gamma\sqrt{\log N}$. Combining these estimates, we have

$$\begin{aligned} \mathcal{P}_N(J)^2 &= D_+(s/4)D_-(s/4) \cdot \frac{D_+(s/4)}{D_-(s/4)} (1 + O(e^{Cs} N^{-1/2})) \\ &= q_1(s)^2 (1 + O(e^{Cs} N^{-1/2})). \end{aligned} \quad (1.53)$$

By the description of J above, $\mathcal{P}_N(J)$ is the left side of (1.23) with $y = y_0$, so this is (1.23).

Finally, we prove the upper bound (1.24), which requires no separation assumption. Let $J_j = \sqrt{N/2} A_j$ for $1 \leq j \leq r$, and let $J = \bigcup_j J_j$. Then $\bar{J} \subset \mathcal{B}_N(\delta/2)$ and $L(J) \leq \gamma\sqrt{\log N} \leq N^{1/8}$. By (1.36) with $\delta/2$ in place of δ , we have $\mathcal{P}_N(J)^2 \leq Ce^{CL(J)} \mathcal{D}_N(J)$. Next, the operator $\text{Id} - K_J$ is positive definite, since $0 \leq K_J < \text{Id}$, so an inequality of Fischer type for Fredholm determinants of such operators gives $\mathcal{D}_N(J) \leq \prod_j \mathcal{D}_N(J_j)$. Finally, we write J_j in the form of the first part of this step, with y_0 the left endpoint of A_j and $s = N\sigma(y_0)|A_j| \leq 2\gamma\sqrt{\log N}$. Using (1.36) for the single interval J_j and the lower bound $\det \mathcal{A}_N(J_j) \geq ce^{-CL(J_j)}$, we find $\mathcal{D}_N(J_j) \leq Ce^{CL(J_j)} \mathcal{P}_N(J_j)^2$. Combining these bounds, taking square roots, and using $L(J) \leq \gamma\sqrt{\log N}$, we obtain (1.24).

1.5 Organization of the paper

Section 2 collects classical facts about Hermite functions and proves the exact identities, including (1.39) and (1.40). Section 3 proves the exact identity for $\mathcal{P}_N(J)^2$ described in Step 1. Section 4 contains the asymptotic estimates for Hermite functions in the bulk, and Section 5 proves the inverse bound (1.46). Section 6 reduces this identity to the endpoint matrix and shows that the endpoint matrix of separated intervals is nearly diagonal. Section 7 proves the one-interval estimate (1.23), and Section 8 proves the estimates (1.24) and (1.25). Section 9 computes the first moment (1.26), and Section 10 completes the proofs of Theorem 1.1, Corollary 1.2, and Remark 1.4. Appendix A contains the criterion for Poisson convergence. The main logical dependencies between the sections are shown in Figure 1.2.

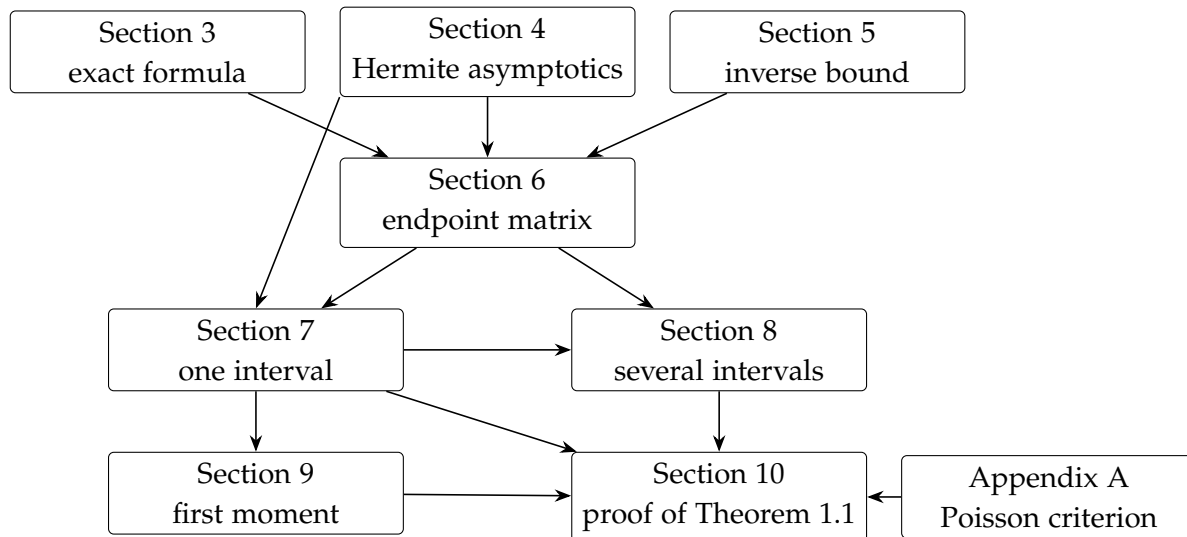


Figure 1.2. The main logical dependencies between the sections. An arrow from one section to another means that the second section uses results of the first. The classical facts and exact identities of Section 2 are used throughout.

Hermite functions and exact identities

In this section, we fix our notation and collect the facts about Hermite functions that will be used in the rest of the paper. The classical facts are recalled in Section 2.2. The identities in Section 2.4 are elementary, but our argument depends on them. They express every global quantity that we need, such as the integral of the Hermite kernel over a half-line, through quantities in the bulk of the spectrum. This allows us to avoid any asymptotic analysis of Hermite functions near the spectral edges.

2.1 Conventions

For real-valued measurable functions f and g on $\mathbb{R}$, we write

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x) dx \quad (2.1)$$

whenever the integral converges absolutely. All functions in this paper are real-valued, except for the Fourier transforms. For a measurable set $A \subset \mathbb{R}$, we let $\mathbf{1}_A$ denote its indicator function and $|A|$ its Lebesgue measure, and we set $A^c = \mathbb{R} \setminus A$. For a bounded measurable function Y and a function v in a space of functions on which $\langle Y, \cdot \rangle$ is defined, we let $v \otimes Y$ denote the rank-one operator

$$(v \otimes Y)f = \langle Y, f \rangle v. \quad (2.2)$$

We set $\text{sgn}(x) = x/|x|$ for $x \neq 0$ and $\text{sgn}(0) = 0$. For $t \in \mathbb{R}$, we define the step function

$$\varepsilon_t(x) = \frac{1}{2} \text{sgn}(x - t), \quad (2.3)$$

and for $f \in L^1(\mathbb{R})$ we define the function εf by

$$(\varepsilon f)(x) = \frac{1}{2} \int_{\mathbb{R}} \text{sgn}(x - z) f(z) dz. \quad (2.4)$$

We call ε the *sign operator*. The following properties are immediate from its definition. First, if $f \in L^1(\mathbb{R})$ is continuous, then εf is continuously differentiable with $(\varepsilon f)' = f$, and

$$\lim_{x \rightarrow \pm\infty} (\varepsilon f)(x) = \pm \frac{1}{2} \int_{\mathbb{R}} f(z) dz. \quad (2.5)$$

Further, if $f \in L^1(\mathbb{R})$ is continuously differentiable with $f' \in L^1(\mathbb{R})$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, then $\varepsilon(f') = f$. Finally, for all $f, g \in L^1(\mathbb{R})$, we have $\langle f, \varepsilon g \rangle = -\langle \varepsilon f, g \rangle$.

Throughout the paper, we let C and c denote large and small positive constants, respectively. They may depend on parameters that will be specified in each statement, but not on N or on the positions of the sets considered, and their values may change from line to line.

2.2 Classical facts

For every integer $n \geq 0$, we define the *Hermite function*

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}, \quad (2.6)$$

and we set $h_{-1} = 0$. The following facts are classical; see, for instance, [47]. The functions $(h_n)_{n \geq 0}$ form an orthonormal basis of $L^2(\mathbb{R})$, each h_n is a polynomial of degree n times $e^{-x^2/2}$, and $h_n(-x) = (-1)^n h_n(x)$. For all $n \geq 0$, we have the recurrence relations

$$\begin{aligned} x h_n(x) &= \sqrt{\frac{n+1}{2}} h_{n+1}(x) + \sqrt{\frac{n}{2}} h_{n-1}(x), \\ h'_n(x) &= \sqrt{\frac{n}{2}} h_{n-1}(x) - \sqrt{\frac{n+1}{2}} h_{n+1}(x). \end{aligned} \quad (2.7)$$

For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, let

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \quad (2.8)$$

denote its Fourier transform, extended as usual to a unitary operator on $L^2(\mathbb{R})$. The Hermite functions are eigenfunctions of this operator, with

$$\mathcal{F}h_n = (-i)^n h_n. \quad (2.9)$$

We also recall that $h_{2m+1}(0) = 0$ and

$$h_{2m}(0) = (-1)^m \pi^{-1/4} \frac{\sqrt{(2m)!}}{2^m m!} \quad (2.10)$$

for all $m \geq 0$. Evaluating (2.9) at $\xi = 0$, we have $\int_{\mathbb{R}} h_{2m+1} = 0$ and

$$\int_{\mathbb{R}} h_{2m}(x) dx = \sqrt{2\pi} (-1)^m h_{2m}(0) = \sqrt{2} \pi^{1/4} \frac{\sqrt{(2m)!}}{2^m m!}. \quad (2.11)$$

We set $V_0 = \{0\}$. For every integer $N \geq 1$, we let V_N denote the span of $h_0, \dots, h_{N-1}$, and we define

$$K_N(x, y) = \sum_{n=0}^{N-1} h_n(x) h_n(y). \quad (2.12)$$

We also let K_N denote the integral operator with kernel $K_N(x, y)$. This operator is the orthogonal projection of $L^2(\mathbb{R})$ onto V_N . For every bounded measurable f , we define $K_N f$ by the same formula. Then $K_N f \in V_N$. The kernel satisfies the Christoffel–Darboux formula

$$K_N(x, y) = \sqrt{\frac{N}{2}} \frac{h_N(x)h_{N-1}(y) - h_{N-1}(x)h_N(y)}{x - y}, \quad x \neq y, \quad (2.13)$$

which follows from the first identity in (2.7). Further, for $x \in \mathbb{R}$, we have $K_N(x, x) = \sum_{n < N} h_n(x)^2$, and the Cauchy–Schwarz inequality gives

$$|K_N(x, y)| \leq K_N(x, x)^{1/2} K_N(y, y)^{1/2}, \quad |f(x)|^2 \leq K_N(x, x) \|f\|_2^2 \quad \text{for } f \in V_N. \quad (2.14)$$

We will need a crude bound on Hermite functions far outside the spectrum.

Lemma 2.1. *For all integers $N \geq 1$ and $0 \leq n \leq N$, and all $x \in \mathbb{R}$ such that $|x| \geq 8\sqrt{N}$, we have $h_n(x)^2 \leq e^{-x^2/4}$.*

Proof. Let P_n denote the Hermite polynomial of degree n , normalized so that

$$h_n(x) = \frac{P_n(x)e^{-x^2/2}}{(2^n n! \sqrt{\pi})^{1/2}}. \quad (2.15)$$

These polynomials have the generating function (see [47])

$$\sum_{n \geq 0} P_n(x) z^n / n! = e^{2xz - z^2}. \quad (2.16)$$

By Cauchy's estimate on the circle $|z| = \sqrt{N}$, we have

$$|P_n(x)| \leq n! N^{-n/2} e^{2|x|\sqrt{N} + N}. \quad (2.17)$$

It follows that

$$h_n(x)^2 = \frac{P_n(x)^2 e^{-x^2}}{2^n n! \sqrt{\pi}} \leq \frac{n!}{N^n} e^{4|x|\sqrt{N} + 2N - x^2} \leq e^{4|x|\sqrt{N} + 2N - x^2}. \quad (2.18)$$

In the last inequality, we used

$$n! \leq n^n \leq N^n \quad (2.19)$$

for $1 \leq n \leq N$. The same bound holds for $n = 0$ by direct inspection. Finally, when $|x| \geq 8\sqrt{N}$, we have $4|x|\sqrt{N} \leq x^2/2$ and $2N \leq x^2/32$, so the exponent on the right side is bounded above by $-x^2/4$. $\square$

2.3 The GOE in Hermite coordinates

Recall that H_N denotes an $N \times N$ GOE matrix, normalized as in (1.1), with eigenvalues $\lambda_1 < \dots < \lambda_N$. We will work with the rescaled eigenvalues $\sqrt{N/2} \lambda_i$. By (1.1), the vector $(\sqrt{N/2} \lambda_1, \dots, \sqrt{N/2} \lambda_N)$ has the density

$$\frac{1}{\tilde{Z}_N} \prod_{1 \leq i < j \leq N} |x_j - x_i| \prod_{i=1}^N e^{-x_i^2/2} \quad (2.20)$$

on $\{x_1 < \dots < x_N\}$, where $\tilde{Z}_N$ is a normalizing constant. We call the variable

$$x = \sqrt{\frac{N}{2}} \lambda \quad (2.21)$$

the *Hermite coordinate*, and the variable λ the *eigenvalue coordinate*. In Hermite coordinates, the semicircle law is supported on $[-\sqrt{2N}, \sqrt{2N}]$, and the expected number of points per unit length near x_0 is approximately $\pi^{-1} \sqrt{2N - x_0^2}$.

Definition 2.2. For a measurable set $J \subset \mathbb{R}$ of finite measure, we let K_J denote the integral operator on $L^2(J)$ with kernel $K_N(x, y)$, $x, y \in J$, and we define

$$\mathcal{P}_N(J) = \mathbb{P}\left(\sqrt{\frac{N}{2}} \lambda_i \notin J \text{ for all } i \in \{1, \dots, N\}\right), \quad \mathcal{D}_N(J) = \det(\text{Id} - K_J). \quad (2.22)$$

We call $\mathcal{P}_N(J)$ the *GOE hole probability* of J .

The operator K_J is positive and trace class, with trace $\int_J K_N(x, x) dx < \infty$, since $x \mapsto K_N(x, x)$ is bounded. A nonzero element of V_N cannot vanish on the complement of a set of finite measure, which implies that $\|K_J\| < 1$. Hence $\mathcal{D}_N(J)$ is a well-defined Fredholm determinant in $(0, 1]$. It is the probability that the GUE, with eigenvalue density proportional to $\prod_{i < j} |x_j - x_i|^2 \prod_i e^{-x_i^2}$, has no eigenvalue in J (see, for instance, [2]), but we will not use this interpretation. Expanding K_N in the basis $(h_n)_{n < N}$, we also have

$$\mathcal{D}_N(J) = \det(\mathbf{1}_{m=n} - \langle h_m, \mathbf{1}_J h_n \rangle)_{m,n=0}^{N-1}. \quad (2.23)$$

Finally, for every measurable set $A \subset \mathbb{R}$ of finite measure, we have $E_{1,N}(A) = \mathcal{P}_N(\sqrt{N/2} A)$, where we recall that $E_{1,N}(A)$ is the hole probability defined in (1.11).

2.4 Top modes and exact identities

We now define the functions that measure how the GOE differs from the GUE. For every integer $N \geq 2$, we define the *top modes*

$$\varphi_N = \left(\frac{N}{2}\right)^{1/4} h_N, \quad \psi_N = \left(\frac{N}{2}\right)^{1/4} h_{N-1}. \quad (2.24)$$

With this normalization, $\varphi_N(x)\psi_N(y) = \sqrt{N/2}h_N(x)h_{N-1}(y)$, which is the factor appearing in (2.13). We also define the constant c_N and the function ω_N by

$$c_N = \begin{cases} \left(\int_{\mathbb{R}} \psi_N\right)^{-1} & \text{if } N \text{ is odd,} \\ 0 & \text{if } N \text{ is even,} \end{cases} \quad \omega_N = \varepsilon\varphi_N + c_N, \quad (2.25)$$

and the function

$$p_N(y) = \int_{\mathbb{R}} K_N(x, y) dx = (K_N \mathbf{1}_{\mathbb{R}})(y). \quad (2.26)$$

For odd N , the integer $N - 1$ is even, so $\int \psi_N \neq 0$ by (2.11), and c_N is well defined. Note that $p_N \in V_N$.

Lemma 2.3. *For all $N \geq 2$ and $x, y \in \mathbb{R}$, we have*

$$\partial_x K_N(x, y) + \partial_y K_N(x, y) = -\varphi_N(x)\psi_N(y) - \psi_N(x)\varphi_N(y). \quad (2.27)$$

Proof. By the second identity in (2.7), we have

$$\sum_{n=0}^{N-1} h'_n(x)h_n(y) = \sum_{n=1}^{N-1} \sqrt{\frac{n}{2}} h_{n-1}(x)h_n(y) - \sum_{n=1}^N \sqrt{\frac{n}{2}} h_n(x)h_{n-1}(y), \quad (2.28)$$

where we shifted the index in the second sum. We add the same identity with x and y interchanged. Then all terms with $n \leq N - 1$ cancel, and the remaining terms with $n = N$ give

$$-\sqrt{N/2} (h_N(x)h_{N-1}(y) + h_{N-1}(x)h_N(y)), \quad (2.29)$$

which is the right side of (2.27). $\square$

Lemma 2.4. *For all $N \geq 2$, we have*

$$p'_N = -\left(\int_{\mathbb{R}} \varphi_N\right)\psi_N - \left(\int_{\mathbb{R}} \psi_N\right)\varphi_N \quad (2.30)$$

and

$$p_N(0) = \sqrt{2} \sum_{0 \leq m \leq (N-1)/2} (-1)^m \binom{2m}{m} 4^{-m}, \quad |p_N(0) - 1| \leq 2N^{-1/2}. \quad (2.31)$$

Moreover, if N is even, then $\int \psi_N = 0$ and $1 \leq \int \varphi_N \leq 2$, and if N is odd, then $\int \varphi_N = 0$ and $1 \leq \int \psi_N \leq 2$. In particular, $c_N \in [1/2, 1]$ when N is odd.

Proof. Integrating (2.27) in x over $\mathbb{R}$, and using the fact that

$$\int_{\mathbb{R}} \partial_x K_N(x, y) dx = 0, \quad (2.32)$$

we obtain (2.30).

Next, since $h_n(0) = 0$ for odd n , we have

$$p_N(0) = \sum_{2m < N} h_{2m}(0) \int h_{2m}. \quad (2.33)$$

Inserting (2.10) and (2.11) into this identity, we obtain the formula in (2.31).

We now turn to the bound. Set

$$\omega_m = \binom{2m}{m} 4^{-m}. \quad (2.34)$$

An elementary induction using $\omega_{m+1} = \omega_m(2m+1)/(2m+2)$ shows that

$$\frac{1}{2\sqrt{m}} \leq \omega_m \leq \frac{1}{\sqrt{2m+1}}, \quad m \geq 1. \quad (2.35)$$

In particular, $(\omega_m)_{m \geq 0}$ decreases to 0, so the series $\sum_{m \geq 0} (-1)^m \omega_m$ converges. By Abel's theorem applied to the expansion

$$(1 - 4z)^{-1/2} = \sum_{m \geq 0} \omega_m (4z)^m \quad (2.36)$$

at $z = -1/4$, its value is $2^{-1/2}$. The alternating series bound then gives $|p_N(0) - 1| \leq \sqrt{2} \omega_{M+1}$ with $M = \lfloor (N-1)/2 \rfloor$, and (2.35) implies that

$$\sqrt{2} \omega_{M+1} \leq \sqrt{2} (2M+3)^{-1/2} \leq 2N^{-1/2}. \quad (2.37)$$

Finally, we have

$$\int h_n = 0 \quad (2.38)$$

for odd n . When $N = 2m$ is even, (2.11) gives

$$\int \varphi_N = m^{1/4} \sqrt{2} \pi^{1/4} \omega_m^{1/2}, \quad (2.39)$$

and (2.35) shows that this number belongs to $[\pi^{1/4}, 2^{1/4} \pi^{1/4}] \subset [1, 2]$. The same computation with $(N/2)^{1/4} = (m+1/2)^{1/4}$ shows that $\int \psi_N \in [1, 2]$ when $N = 2m+1$ is odd. $\square$

The next lemma records which antiderivatives of Hermite functions lie in V_N .

Lemma 2.5. *Let $n \geq 0$. Then the derivative maps V_n onto $\{f \in V_{n+1} : \int_{\mathbb{R}} f = 0\}$. Consequently, the following statements hold.*

- (a) *If n is odd, then $\varepsilon h_n \in V_n$.*
- (b) *If N is even, then $K_N(\varepsilon h_N) = 0$.*

(c) If N is odd, then $\varepsilon h_N \in V_N$ and $K_N(\varepsilon h_{N-1}) = 0$.

Proof. By (2.7), the derivative maps V_n into V_{n+1} . The image consists of functions with zero integral, and the derivative is injective on V_n , so the image has dimension n . The space $\{f \in V_{n+1} : \int f = 0\}$ also has dimension n , since $\int h_0 \neq 0$. Hence the image is all of this space, which is the first claim.

To prove (a), suppose that n is odd. Then $h_n \in V_{n+1}$ and $\int h_n = 0$, so the first claim gives $h_n = g'$ for some $g \in V_n$. The functions g and εh_n vanish at infinity and have derivative h_n , so they are equal, and (a) follows.

We now prove (b) and (c), using the identity $\langle h_m, \varepsilon f \rangle = -\langle \varepsilon h_m, f \rangle$. Let $m < N$. For odd m , (a) implies that $\varepsilon h_m \in V_m$, so εh_m is orthogonal to h_N and h_{N-1} when $m \leq N-2$. For even m , the function εh_m is odd.

Suppose first that N is even. When $m < N$ is odd, we have $m \leq N-1$, so $\langle \varepsilon h_m, h_N \rangle = 0$. When m is even, the function $\varepsilon h_m h_N$ is odd, so again $\langle \varepsilon h_m, h_N \rangle = 0$. We conclude that $K_N(\varepsilon h_N) = 0$, which is (b).

Finally, suppose that N is odd. Then $\varepsilon h_N \in V_N$ by (a). The same parity argument applied to h_{N-1} , which is even, gives $\langle \varepsilon h_m, h_{N-1} \rangle = 0$ for all $m < N$. This proves (c). $\square$

The following lemma contains the cancellation between the two terms in ω_N for odd N , which is the main reason for the definition (2.25).

Lemma 2.6. *Let $N \geq 2$.*

- (a) *If N is even, then $\varepsilon \psi_N = -p_N / \int \varphi_N$ and $\omega_N = \varepsilon \varphi_N$.*
- (b) *If N is odd, then $\varepsilon \varphi_N = -c_N p_N$, and therefore $\omega_N = c_N(1 - p_N)$.*
- (c) *In both cases, $K_N \omega_N = 0$, $\langle \psi_N, \omega_N \rangle = 0$, and $\langle \psi_N, \varepsilon \varphi_N \rangle = -c_N \int \psi_N$.*

Proof. Suppose first that N is even. By Lemma 2.4, we have

$$p'_N = -(\int \varphi_N) \psi_N, \quad (2.40)$$

so the derivative of $\varepsilon \psi_N + p_N / \int \varphi_N$ vanishes. As $y \rightarrow \infty$, we have $p_N(y) \rightarrow 0$, since $p_N \in V_N$, and (2.5) gives

$$(\varepsilon \psi_N)(y) \rightarrow (1/2) \int \psi_N = 0. \quad (2.41)$$

Therefore,

$$\varepsilon \psi_N + p_N / \int \varphi_N = 0, \quad (2.42)$$

and (a) follows, since $c_N = 0$.

Next, suppose that N is odd. Then Lemma 2.4 gives $p'_N = -c_N^{-1} \varphi_N$, so the derivative of $\varepsilon \varphi_N + c_N p_N$ vanishes. As $y \rightarrow \infty$, we have $p_N(y) \rightarrow 0$ and

$$(\varepsilon \varphi_N)(y) \rightarrow (1/2) \int \varphi_N = 0, \quad (2.43)$$

which gives (b).

We now prove (c). When N is even, Lemma 2.5(b) implies that

$$K_N \omega_N = K_N \varepsilon \varphi_N = 0. \quad (2.44)$$

For odd N , we have

$$p_N = K_N \mathbf{1}_{\mathbb{R}} \in V_N, \quad (2.45)$$

so $K_N p_N = p_N$, and therefore

$$K_N \omega_N = c_N (p_N - K_N p_N) = 0. \quad (2.46)$$

Since $\psi_N \in V_N$ and ω_N is bounded, we find

$$\langle \psi_N, \omega_N \rangle = \langle K_N \psi_N, \omega_N \rangle = \langle \psi_N, K_N \omega_N \rangle = 0. \quad (2.47)$$

Finally, we have

$$\langle \psi_N, \varepsilon \varphi_N \rangle = \langle \psi_N, \omega_N \rangle - c_N \int \psi_N = -c_N \int \psi_N. \quad (2.48)$$

□

Observe that for odd N , Lemma 2.6(b) and Lemma 2.4 show that $\varepsilon \varphi_N$ is close to the constant $-c_N$ wherever p_N is close to 1. We will see in Section 4 that this holds in the bulk of the spectrum, so that ω_N is small there, although the two terms in its definition are of order one.

The last identity of this section concerns the sign function. It shows that, up to the product $(\varepsilon \varphi_N)(\varepsilon \psi_N)$, the projection K_N applied to a step function only involves the kernel between the jump and the point of evaluation.

Lemma 2.7. *For all $N \geq 2$ and $y \in \mathbb{R}$, we have*

$$\frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(x - y) K_N(x, y) \, dx = (\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y). \quad (2.49)$$

Consequently, for all $t, y \in \mathbb{R}$, we have

$$(K_N \varepsilon_t)(y) = (\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y) + \int_t^y K_N(x, y) \, dx, \quad (2.50)$$

where the integral is oriented, so that $\int_t^y = -\int_y^t$ when $y < t$.

Proof. Let $G(y)$ denote the left side of (2.49). Then

$$2G(y) = \int_y^\infty K_N(x, y) \, dx - \int_{-\infty}^y K_N(x, y) \, dx. \quad (2.51)$$

Differentiating this identity, we have

$$G'(y) = -K_N(y, y) + \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(x - y) \partial_y K_N(x, y) dx. \quad (2.52)$$

We now insert (2.27) into the integral. Integrating over (y, ∞) and $(-\infty, y)$ separately, we see that the term with $-\partial_x K_N(x, y)$ gives $K_N(y, y)$. The remaining terms give

$$-\frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(x - y) (\varphi_N(x) \psi_N(y) + \psi_N(x) \varphi_N(y)) dx = \psi_N(y) (\varepsilon \varphi_N)(y) + \varphi_N(y) (\varepsilon \psi_N)(y), \quad (2.53)$$

where we used

$$\int \operatorname{sgn}(x - y) f(x) dx = -2(\varepsilon f)(y). \quad (2.54)$$

Therefore, $G'(y) = ((\varepsilon \varphi_N)(\varepsilon \psi_N))'(y)$. As $y \rightarrow \infty$, we have $G(y) \rightarrow 0$ and

$$(\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y) \rightarrow (1/4) \int \varphi_N \int \psi_N = 0, \quad (2.55)$$

since one of the two integrals vanishes by Lemma 2.4. This completes the proof of (2.49).

We now prove (2.50), whose two sides are continuously differentiable in t , with derivative $-K_N(t, y)$ by the symmetry of K_N . For $t = y$, the left side reduces to $G(y)$ and the right side to $(\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y)$, so (2.50) follows from (2.49). $\square$

An exact formula for the hole probability

In this section, we express $\mathcal{P}_N(J)^2$ as $\mathcal{D}_N(J)$ times the determinant of a matrix whose size depends only on the number of intervals in J . The identity holds for every $N \geq 2$ and every finite union of bounded intervals J . For even N , an identity of this type was obtained by Tracy and Widom in [48, (33)], who manipulated Fredholm determinants of operators with matrix kernels. They gave a detailed justification of these manipulations for the sine kernel in [48, Section VIII]. Our derivation is different. It starts from de Bruijn's integration formula and uses only finite-dimensional linear algebra, so no regularization of the sign kernel is required. It treats N of either parity and pairs the two endpoints of each interval from the start, which produces a matrix whose size is the number of intervals plus two, instead of twice the number of intervals plus one.

3.1 Statement

Throughout this section, we fix $N \geq 2$, an integer $r \geq 1$, and real numbers $a_1 < b_1 < a_2 < b_2 < \dots < a_r < b_r$. We set

$$J_j = (a_j, b_j), \quad J = \bigcup_{j=1}^r J_j. \quad (3.1)$$

For $1 \leq j \leq r$, we define the functions

$$k_j = K_N(\cdot, b_j) - K_N(\cdot, a_j), \quad w_j = \mathbf{1}_{J^c} \varepsilon_{a_j}. \quad (3.2)$$

Observe that $k_j \in V_N$ and $\langle k_j, f \rangle = f(b_j) - f(a_j)$ for all $f \in V_N$. Further, since $\operatorname{sgn}(x - a_j) = \operatorname{sgn}(x - b_j)$ for $x \notin [a_j, b_j]$, we have $w_j = \mathbf{1}_{J^c} \varepsilon_{b_j}$ almost everywhere. The function w_j vanishes on J and takes the value $-1/2$ on $J^c \cap (-\infty, a_j)$ and the value $1/2$ on $J^c \cap (b_j, \infty)$. We also define the function

$$Y_N(J) = \mathbf{1}_{J^c} (\omega_N - \varepsilon(\mathbf{1}_J \varphi_N)), \quad (3.3)$$

where we recall that ω_N was defined in (2.25), and the operator $T_N(J)$ on V_N by

$$T_N(J) = \operatorname{Id} - K_N \mathbf{1}_J - \sum_{j=1}^r k_j \otimes w_j + \psi_N \otimes Y_N(J), \quad (3.4)$$

where $K_N \mathbf{1}_J$ denotes the operator $f \mapsto K_N(\mathbf{1}_J f)$, and the rank-one operators were defined in (2.2). Finally, we define

$$\begin{aligned} X_N^\sharp(J) &= -\frac{1}{2} \sum_{j=1}^r k_j - \frac{1}{2} \left(\int_J \varphi_N \right) \psi_N, & d_N(J) &= 1 - c_N \int_J \psi_N, \\ Y_N^\sharp(J) &= -2c_N (\mathbf{1}_J \varepsilon \psi_N + \mathbf{1}_{J^c} \varepsilon (\mathbf{1}_J \psi_N)). \end{aligned} \quad (3.5)$$

Note that $X_N^\sharp(J) \in V_N$, and that $Y_N^\sharp(J)$ is bounded. For even N , we have $c_N = 0$, so $Y_N^\sharp(J) = 0$ and $d_N(J) = 1$.

Proposition 3.1. *Let $\mathbb{T}_N(J)$ denote the operator on $V_N \oplus \mathbb{R}$ defined by*

$$\mathbb{T}_N(J)(f, a) = (T_N(J)f + a X_N^\sharp(J), \langle Y_N^\sharp(J), f \rangle + a d_N(J)). \quad (3.6)$$

Then

$$\mathcal{P}_N(J)^2 = \det \mathbb{T}_N(J). \quad (3.7)$$

In particular, if N is even, then $\mathcal{P}_N(J)^2 = \det T_N(J)$.

To state the consequence of Proposition 3.1 that we will use, we write

$$Q_N(J) = \text{Id} - K_N \mathbf{1}_J, \quad (3.8)$$

viewed as an operator on V_N . In the basis $(h_n)_{n < N}$, the matrix of this operator is the matrix in (2.23), so $\det Q_N(J) = \mathcal{D}_N(J) > 0$. In particular, $Q_N(J)$ is invertible. We define $X_0, \dots, X_r \in V_N$ and bounded functions $Y_0, \dots, Y_r$ by

$$X_0 = -\psi_N, \quad X_j = k_j, \quad Y_0 = Y_N(J), \quad Y_j = w_j, \quad 1 \leq j \leq r, \quad (3.9)$$

so that $T_N(J) = Q_N(J) - \sum_{\alpha=0}^r X_\alpha \otimes Y_\alpha$. We index the rows and columns of the following matrices by $\{0, 1, \dots, r\}$ and $\{0, 1, \dots, r+1\}$, respectively.

Definition 3.2. We define the $(r+1) \times (r+1)$ matrix $\mathbf{M}_N(J)$ and the $(r+2) \times (r+2)$ matrix $\mathbf{M}_N^\sharp(J)$ by

$$\mathbf{M}_N(J)_{\alpha\beta} = \mathbf{1}_{\alpha=\beta} - \langle Y_\alpha, Q_N(J)^{-1} X_\beta \rangle, \quad 0 \leq \alpha, \beta \leq r, \quad (3.10)$$

and

$$\mathbf{M}_N^\sharp(J) = \begin{pmatrix} \mathbf{M}_N(J) & v_N(J) \\ z_N(J)^\top & d_N(J) \end{pmatrix}, \quad (3.11)$$

where $v_N(J), z_N(J) \in \mathbb{R}^{r+1}$ have the entries

$$v_N(J)_0 = \frac{1}{2} \int_J \varphi_N, \quad v_N(J)_j = -\frac{1}{2}, \quad z_N(J)_\beta = \langle Y_N^\sharp(J), Q_N(J)^{-1} X_\beta \rangle, \quad (3.12)$$

for $1 \leq j \leq r$ and $0 \leq \beta \leq r$.

Corollary 3.3. *For all $N \geq 2$ and J as in (3.1), we have*

$$\mathcal{P}_N(J)^2 = \mathcal{D}_N(J) \det \mathbf{M}_N^\sharp(J). \quad (3.13)$$

If N is even, then $z_N(J) = 0$ and $d_N(J) = 1$, so $\det \mathbf{M}_N^\sharp(J) = \det \mathbf{M}_N(J)$.

Proof. We write $Q = Q_N(J)$ and $\tilde{X}_\alpha = Q^{-1}X_\alpha$. By (3.5) and (3.12), we have

$$X_N^\sharp(J) = \sum_{\alpha} v_N(J)_\alpha X_\alpha. \quad (3.14)$$

Therefore, $\mathbb{T}_N(J) = (Q \oplus 1) \mathbb{T}'$, where

$$\mathbb{T}'(f, a) = \left(f - \sum_{\alpha=0}^r \tilde{X}_\alpha (\langle Y_\alpha, f \rangle - a v_N(J)_\alpha), a + \langle Y_N^\sharp(J), f \rangle + a(d_N(J) - 1) \right). \quad (3.15)$$

We now write $\mathbb{T}'$ as the identity minus a sum of $r + 2$ rank-one operators. Set $U_\beta = (\tilde{X}_\beta, 0)$ for $\beta \leq r$ and $U_{r+1} = (0, 1)$. We define

$$\Phi_\alpha(f, a) = \langle Y_\alpha, f \rangle - a v_N(J)_\alpha \quad (3.16)$$

for $\alpha \leq r$ and

$$\Phi_{r+1}(f, a) = -\langle Y_N^\sharp(J), f \rangle - a(d_N(J) - 1), \quad (3.17)$$

and we let $U \otimes \Phi$ denote the operator $u \mapsto \Phi(u)U$. Then

$$\mathbb{T}' = \text{Id} - \sum_{\alpha=0}^r (\tilde{X}_\alpha, 0) \otimes \Phi_\alpha - (0, 1) \otimes \Phi_{r+1}, \quad (3.18)$$

so Sylvester's determinant identity implies that $\det \mathbb{T}'$ is the determinant of the $(r + 2) \times (r + 2)$ matrix with entries $\mathbf{1}_{\alpha=\beta} - \Phi_\alpha(U_\beta)$. Comparing entries, we see that this matrix coincides with $\mathbf{M}_N^\sharp(J)$. Since $\det(Q \oplus 1) = \mathcal{D}_N(J)$, Proposition 3.1 now gives (3.13). The last claim follows from $c_N = 0$ for even N . $\square$

3.2 De Bruijn's formula

Let $A \subset \mathbb{R}$ be a measurable set. We define the skew-symmetric bilinear form

$$\Omega_A(f, g) = \langle \mathbf{1}_A f, \varepsilon(\mathbf{1}_A g) \rangle, \quad f, g \in V_{N+1}, \quad (3.19)$$

and, for odd N , the skew-symmetric bilinear form on $V_N \oplus \mathbb{R}$ given by

$$\hat{\Omega}_A((f, a), (g, b)) = \Omega_A(f, g) - \frac{b}{2} \langle \mathbf{1}_A, f \rangle + \frac{a}{2} \langle \mathbf{1}_A, g \rangle. \quad (3.20)$$

With a slight abuse of notation, we also let Ω_A denote the matrix of Ω_A in the basis $(h_n)_{n < N}$ of V_N , and we let $\hat{\Omega}_A$ denote the matrix of $\hat{\Omega}_A$ in the basis of $V_N \oplus \mathbb{R}$ given by the vectors $(h_n, 0)$, $n < N$, and $(0, 1)$.

Lemma 3.4. *If N is even, then $\det \Omega_{\mathbb{R}} \neq 0$ and $\mathcal{P}_N(J)^2 = \det \Omega_{J^c} / \det \Omega_{\mathbb{R}}$. If N is odd, then $\det \hat{\Omega}_{\mathbb{R}} \neq 0$ and $\mathcal{P}_N(J)^2 = \det \hat{\Omega}_{J^c} / \det \hat{\Omega}_{\mathbb{R}}$.*

Proof. Since h_n is a polynomial of degree n times $e^{-x^2/2}$, there exists a constant $\varsigma_N \neq 0$ such that

$$\det(h_m(x_i))_{m=0, i=1}^{N-1, N} = \varsigma_N \prod_{i < j} (x_j - x_i) \prod_i e^{-x_i^2/2}. \quad (3.21)$$

The product $\prod_{i < j} (x_j - x_i)$ is positive on $\{x_1 < \dots < x_N\}$. Comparing with (2.20), we find that for every measurable A ,

$$\mathbb{P}\left(\sqrt{\frac{N}{2}} \lambda_i \in A \text{ for all } i\right) = \frac{1}{\varsigma_N \tilde{Z}_N} \int_{x_1 < \dots < x_N, x_i \in A} \det(h_m(x_i)) dx_1 \cdots dx_N. \quad (3.22)$$

For even N , de Bruijn's formula [7] states that the integral on the right is the Pfaffian of the matrix with entries $\int_{A \times A} \text{sgn}(y - x) h_m(x) h_n(y) dx dy$. Observe that this entry equals $-2\Omega_A(h_m, h_n)$. For odd N , the corresponding formula (see [3, Theorem 6.1] or [44, (2.2)]) states that the integral is the Pfaffian of the same matrix bordered by the column $(\int_A h_m)_{m < N}$, its negative transpose, and a zero. This bordered matrix coincides with $-2\hat{\Omega}_A$.

Taking $A = \mathbb{R}$ shows that the Pfaffian is nonzero in either case. Taking $A = J^c$, dividing by the case $A = \mathbb{R}$, and squaring, we obtain the claim, since the square of a Pfaffian is the determinant and the factors of -2 cancel. $\square$

3.3 The derivative trick

To compute the determinants in Lemma 3.4, we replace the first argument of Ω_A by derivatives. Here and below, we write $\partial f = f'$. The idea is that ε inverts the derivative, so that $\Omega_{\mathbb{R}}(\partial f, g) = -\langle f, g \rangle$. However, ∂f does not lie in V_N in general, and its projection $K_N \partial f$ differs from ∂f by a multiple of the top mode h_N . The following lemma carries out this computation for $A = \mathbb{R}$ and $A = J^c$. Recall from (3.3) that $Y_N(J) = \mathbf{1}_{J^c}(\omega_N - \varepsilon(\mathbf{1}_J \varphi_N))$.

Lemma 3.5. *For all $f, g \in V_N$, we have*

$$\Omega_{\mathbb{R}}(K_N \partial f, g) = -\langle f, g \rangle - \langle \psi_N, f \rangle \langle \varepsilon \varphi_N, g \rangle, \quad (3.23)$$

$$\Omega_{J^c}(K_N \partial f, g) = -\langle f, T_N(J)g \rangle + c_N \langle \psi_N, f \rangle \langle \mathbf{1}_{J^c}, g \rangle. \quad (3.24)$$

Proof. By (2.7), we have $\partial f \in V_{N+1}$ and

$$\langle h_N, \partial f \rangle = -\langle \partial h_N, f \rangle = -\sqrt{N/2} \langle h_{N-1}, f \rangle. \quad (3.25)$$

Therefore,

$$K_N \partial f = \partial f + \sqrt{\frac{N}{2}} \langle h_{N-1}, f \rangle h_N = \partial f + \langle \psi_N, f \rangle \varphi_N. \quad (3.26)$$

We first consider $A = \mathbb{R}$. Integrating by parts and using $\partial \varepsilon g = g$, we find that

$$\Omega_{\mathbb{R}}(\partial f, g) = \langle \partial f, \varepsilon g \rangle = -\langle f, g \rangle. \quad (3.27)$$

Also, $\Omega_{\mathbb{R}}(\varphi_N, g) = -\langle \varepsilon \varphi_N, g \rangle$. Applying $\Omega_{\mathbb{R}}(\cdot, g)$ to (3.26) and inserting these identities, we obtain (3.23).

We now consider $A = J^c$. We set

$$\Psi_g = \varepsilon g - \mathbf{1}_{J^c} \varepsilon(\mathbf{1}_{J^c} g) = \mathbf{1}_J \varepsilon g + \mathbf{1}_{J^c} \varepsilon(\mathbf{1}_J g), \quad (3.28)$$

so that

$$\Omega_{J^c}(\partial f, g) = \langle \partial f, \varepsilon g \rangle - \langle \partial f, \Psi_g \rangle = -\langle f, g \rangle - \langle \partial f, \Psi_g \rangle. \quad (3.29)$$

The restriction of Ψ_g to each interval J_j extends continuously to the closure $\bar{J}_j$, and it has derivative g on J_j . It is constant on each connected component of J^c , since $\varepsilon(\mathbf{1}_J g)$ has derivative $\mathbf{1}_J g$, so its jumps occur only at the points a_j and b_j . We have

$$\Psi_g(a_j^-) - \Psi_g(a_j^+) = (\varepsilon(\mathbf{1}_J g))(a_j) - (\varepsilon g)(a_j) = -(\varepsilon(\mathbf{1}_{J^c} g))(a_j) = \langle w_j, g \rangle, \quad (3.30)$$

where the last equality follows from $(\varepsilon h)(t) = -\langle \varepsilon_t, h \rangle$ and $\mathbf{1}_{J^c} \varepsilon_{a_j} = w_j$. In the same way, we find that

$$\Psi_g(b_j^-) - \Psi_g(b_j^+) = (\varepsilon(\mathbf{1}_{J^c} g))(b_j) = -\langle w_j, g \rangle, \quad (3.31)$$

where we used $\mathbf{1}_{J^c} \varepsilon_{b_j} = w_j$ almost everywhere.

Integrating by parts on each interval where Ψ_g is continuous, we find

$$\begin{aligned} \langle \partial f, \Psi_g \rangle &= \sum_{j=1}^r (f(a_j) - f(b_j)) \langle w_j, g \rangle - \langle \mathbf{1}_J f, g \rangle \\ &= -\sum_{j=1}^r \langle f, k_j \rangle \langle w_j, g \rangle - \langle f, K_N(\mathbf{1}_J g) \rangle, \end{aligned} \quad (3.32)$$

where in the last step we used $f \in V_N$. Further, the skew-symmetry of ε gives

$$\Omega_{J^c}(\varphi_N, g) = -\langle \mathbf{1}_{J^c} \varepsilon(\mathbf{1}_{J^c} \varphi_N), g \rangle = -\langle Y_N(J), g \rangle + c_N \langle \mathbf{1}_{J^c}, g \rangle, \quad (3.33)$$

where we used

$$\varepsilon(\mathbf{1}_{J^c} \varphi_N) = \varepsilon \varphi_N - \varepsilon(\mathbf{1}_J \varphi_N) = \omega_N - c_N - \varepsilon(\mathbf{1}_J \varphi_N). \quad (3.34)$$

Applying $\Omega_{J^c}(\cdot, g)$ to (3.26) and combining the identities above, we have

$$\begin{aligned} \Omega_{J^c}(K_N \partial f, g) &= -\langle f, g \rangle + \langle f, K_N(\mathbf{1}_J g) \rangle + \sum_{j=1}^r \langle f, k_j \rangle \langle w_j, g \rangle \\ &\quad - \langle \psi_N, f \rangle (\langle Y_N(J), g \rangle - c_N \langle \mathbf{1}_{J^c}, g \rangle), \end{aligned} \quad (3.35)$$

which is (3.24) by the definition (3.4). $\square$

3.4 Proof of Proposition 3.1

Let B denote the $N \times N$ matrix with entries $B_{\ell m} = \langle h_\ell, \partial h_m \rangle$. Then

$$K_N \partial h_m = \sum_{\ell} B_{\ell m} h_\ell, \quad (3.36)$$

so for every measurable A , the matrix with entries $\Omega_A(K_N \partial h_m, h_n)$ is given by $B^\top \Omega_A$.

Suppose first that N is even. Then $c_N = 0$, and Lemma 3.5 shows that the matrices $-B^\top \Omega_{\mathbb{R}}$ and $-B^\top \Omega_{J^c}$ are the matrices of the operators $\text{Id} + \psi_N \otimes \varepsilon \varphi_N$ and $T_N(J)$ in the basis $(h_n)_{n < N}$, respectively. The first operator has determinant $1 + \langle \varepsilon \varphi_N, \psi_N \rangle = 1$, by Lemma 2.6(c) with $c_N = 0$. In particular, $\det B \neq 0$, and Lemma 3.4 gives

$$\mathcal{P}_N(J)^2 = \frac{\det \Omega_{J^c}}{\det \Omega_{\mathbb{R}}} = \frac{\det(-B^\top \Omega_{J^c})}{\det(-B^\top \Omega_{\mathbb{R}})} = \det T_N(J). \quad (3.37)$$

Suppose now that N is odd, and set

$$\nu = 2c_N(N/2)^{1/4}. \quad (3.38)$$

We define the vectors

$$u_m = (K_N \partial h_m, -\nu \mathbf{1}_{m=N-1}), \quad 0 \leq m < N, \quad u_N = (\nu h_{N-1}, 0), \quad (3.39)$$

in $V_N \oplus \mathbb{R}$, and we set $e_n = (h_n, 0)$ for $n < N$ and $e_N = (0, 1)$. Let Γ denote the matrix with

$$u_\alpha = \sum_{\beta} \Gamma_{\alpha\beta} e_\beta. \quad (3.40)$$

Then the matrix with entries $-\hat{\Omega}_A(u_\alpha, e_\beta)$ is given by $-\Gamma \hat{\Omega}_A$. We now compute these entries for $A \in \{\mathbb{R}, J^c\}$, using Lemma 3.5 and the identity $\nu \mathbf{1}_{m=N-1}/2 = c_N \langle \psi_N, h_m \rangle$.

We begin with the case $m, n < N$. Then (3.24) gives

$$-\hat{\Omega}_{J^c}(u_m, e_n) = -\Omega_{J^c}(K_N \partial h_m, h_n) + c_N \langle \psi_N, h_m \rangle \langle \mathbf{1}_{J^c}, h_n \rangle = \langle h_m, T_N(J) h_n \rangle, \quad (3.41)$$

and in the same way, (3.23) gives $-\hat{\Omega}_{\mathbb{R}}(u_m, e_n) = \langle h_m, (\text{Id} + \psi_N \otimes \omega_N) h_n \rangle$, where we used $\omega_N = \varepsilon \varphi_N + c_N$.

Next, let $m < N$ and $n = N$. Then the definition (3.20) gives $-\hat{\Omega}_A(u_m, e_N) = \langle \mathbf{1}_A, K_N \partial h_m \rangle / 2$. When $A = \mathbb{R}$, this quantity equals

$$\langle p_N, \partial h_m \rangle / 2 = -\langle p'_N, h_m \rangle / 2 = 0, \quad (3.42)$$

where we used (2.30), $\int \varphi_N = 0$, and the orthogonality of φ_N to V_N . When $A = J^c$, we combine the case $A = \mathbb{R}$ with (3.26) to obtain

$$\begin{aligned} \frac{1}{2} \langle \mathbf{1}_{J^c}, K_N \partial h_m \rangle &= -\frac{1}{2} \langle \mathbf{1}_J, \partial h_m + \langle \psi_N, h_m \rangle \varphi_N \rangle \\ &= -\frac{1}{2} \sum_{j=1}^r \langle k_j, h_m \rangle - \frac{1}{2} \langle \psi_N, h_m \rangle \int_J \varphi_N = \langle h_m, X_N^\#(J) \rangle. \end{aligned} \quad (3.43)$$

It remains to compute the entries with $\alpha = N$. For $n < N$, we have $-\widehat{\Omega}_A(u_N, e_n) = -\nu \Omega_A(h_{N-1}, h_n)$. When $A = \mathbb{R}$, Lemma 2.5(c) implies that

$$\Omega_{\mathbb{R}}(h_{N-1}, h_n) = -\langle \varepsilon h_{N-1}, h_n \rangle = 0, \quad (3.44)$$

so this entry vanishes. To treat $A = J^c$, we expand $\mathbf{1}_{J^c} = 1 - \mathbf{1}_J$ and use the skew-symmetry of ε . This gives, for all $f, g \in V_N$,

$$\Omega_{J^c}(f, g) - \Omega_{\mathbb{R}}(f, g) = \langle \mathbf{1}_J \varepsilon f + \mathbf{1}_{J^c} \varepsilon(\mathbf{1}_J f), g \rangle. \quad (3.45)$$

Taking $f = h_{N-1}$ and using $\nu h_{N-1} = 2c_N \psi_N$, we find

$$-\widehat{\Omega}_{J^c}(u_N, e_n) = \langle Y_N^\sharp(J), h_n \rangle. \quad (3.46)$$

Finally, we have

$$-\widehat{\Omega}_A(u_N, e_N) = (\nu/2) \langle \mathbf{1}_A, h_{N-1} \rangle = c_N \langle \mathbf{1}_A, \psi_N \rangle, \quad (3.47)$$

which takes the value 1 for $A = \mathbb{R}$ and the value $d_N(J)$ for $A = J^c$.

Combining these computations, we see that $-\Gamma \widehat{\Omega}_{J^c}$ is the matrix of $\mathbb{T}_N(J)$ in the basis (e_β) , and that $-\Gamma \widehat{\Omega}_{\mathbb{R}}$ is the matrix of $(\text{Id} + \psi_N \otimes \omega_N) \oplus 1$, which has determinant $1 + \langle \omega_N, \psi_N \rangle = 1$ by Lemma 2.6(c). In particular, $\det \Gamma \neq 0$, and Lemma 3.4 gives

$$\mathcal{P}_N(J)^2 = \frac{\det \widehat{\Omega}_{J^c}}{\det \widehat{\Omega}_{\mathbb{R}}} = \frac{\det(-\Gamma \widehat{\Omega}_{J^c})}{\det(-\Gamma \widehat{\Omega}_{\mathbb{R}})} = \det \mathbb{T}_N(J). \quad (3.48)$$

This completes the proof. □

Estimates in the bulk

In this section, we prove the asymptotic estimates for Hermite functions that we need. They all concern points in the bulk of the spectrum, and they follow from the Plancherel–Rotach asymptotics together with the identities of Section 2.4. The main results are Lemma 4.3, which compares the Hermite kernel with the sine kernel on growing windows, and Lemma 4.6 and Corollary 4.7, which control integrals of the kernel against step functions.

For $\delta \in (0, 1/2]$, we define the *bulk*

$$\mathcal{B}_N(\delta) = [- (1 - \delta)\sqrt{2N}, (1 - \delta)\sqrt{2N}], \quad (4.1)$$

and for $x_0 \in \mathcal{B}_N(\delta)$ we define the *local scale*

$$\varrho_N(x_0) = \sqrt{2N - x_0^2}. \quad (4.2)$$

Note that $\sqrt{2\delta N} \leq \varrho_N(x_0) \leq \sqrt{2N}$ for $x_0 \in \mathcal{B}_N(\delta)$. We also define the *sine kernel*

$$S(t) = \frac{\sin t}{\pi t}, \quad t \neq 0, \quad S(0) = \frac{1}{\pi}. \quad (4.3)$$

Throughout this section, we fix $\delta \in (0, 1/2]$. The constants C , c , and N_0 depend only on δ , and all statements hold for $N \geq N_0$.

4.1 Plancherel–Rotach asymptotics

For $n \geq 1$ and $|x| < \sqrt{2n+1}$, we define $\theta_n(x) \in (0, \pi)$ by $x = \sqrt{2n+1} \cos \theta_n(x)$, and we set

$$\begin{aligned} \alpha_n(x) &= \frac{2^{1/4}}{\sqrt{\pi} n^{1/4}} (\sin \theta_n(x))^{-1/2}, \\ \Theta_n(x) &= \left(\frac{n}{2} + \frac{1}{4}\right) (\sin 2\theta_n(x) - 2\theta_n(x)) + \frac{3\pi}{4}. \end{aligned} \quad (4.4)$$

Lemma 4.1. *There exists $C > 0$ such that for all integers n with $|n - N| \leq 2$ and all $x \in \mathcal{B}_N(\delta)$, we have*

$$h_n(x) = \alpha_n(x) (\sin \Theta_n(x) + \rho_n(x)), \quad |\rho_n(x)| \leq \frac{C}{N}. \quad (4.5)$$

Moreover, the following estimates hold for such n and x .

- (i) We have $\Theta'_n(x) = \sqrt{2n+1-x^2}$, $|\Theta''_n(x)| \leq C$, and $|\theta'_n(x)| \leq CN^{-1/2}$.
- (ii) We have $C^{-1}N^{-1/4} \leq \alpha_n(x) \leq CN^{-1/4}$ and $|\alpha'_n(x)| \leq CN^{-3/4}$.
- (iii) The functions defined by $\mathfrak{a} = \alpha_{N-1}/\alpha_N$ and $\vartheta = \Theta_{N-1} - \Theta_N$ satisfy $|\mathfrak{a}(x) - 1| \leq C/N$, $|\mathfrak{a}'(x)| \leq CN^{-3/2}$, $|\vartheta(x) - \theta_N(x)| \leq C/N$, and $|\vartheta'(x)| \leq CN^{-1/2}$.

Proof. When $N \geq N_0$, we have $|x| \leq (1 - \delta/2)\sqrt{2n+1}$ for $|n - N| \leq 2$ and $x \in \mathcal{B}_N(\delta)$, so $\theta_n(x)$ belongs to a compact subset of $(0, \pi)$ depending only on δ . The expansion (4.5) is then the Plancherel–Rotach asymptotic formula [41] in the form of [47, Theorem 8.22.9], multiplied by the normalizing factor $(2^n n! \sqrt{\pi})^{-1/2}$ in (2.6).

For (i), we have

$$\theta'_n(x) = -(2n+1-x^2)^{-1/2} \quad (4.6)$$

and

$$\partial_\theta((2n+1)/4)(\sin 2\theta - 2\theta) = -(2n+1) \sin^2 \theta. \quad (4.7)$$

Therefore,

$$\Theta'_n(x) = \sqrt{2n+1} \sin \theta_n(x) = \sqrt{2n+1-x^2}, \quad (4.8)$$

and

$$\Theta''_n(x) = -x(2n+1-x^2)^{-1/2}. \quad (4.9)$$

These are bounded as claimed, since $2n+1-x^2 \geq cN$.

The bounds on α_n in (ii) follow from $\sin \theta_n(x) \in [c, 1]$. Further, we have

$$(\log \alpha_n)'(x) = x/(2(2n+1-x^2)), \quad (4.10)$$

which is bounded by $CN^{-1/2}$ in absolute value.

For (iii), the identity for $(\log \alpha_n)'$ gives

$$(\log \mathfrak{a})' = x/(2(2N-1-x^2)) - x/(2(2N+1-x^2)), \quad (4.11)$$

which is bounded by $CN^{-3/2}$ in absolute value. The bound on $\mathfrak{a} - 1$ follows from $(N/(N-1))^{1/4} = 1 + O(N^{-1})$ and $\sin \theta_{N-1}(x)/\sin \theta_N(x) = 1 + O(N^{-1})$. Next, we have

$$\vartheta' = \sqrt{2N-1-x^2} - \sqrt{2N+1-x^2}, \quad (4.12)$$

which is bounded by $CN^{-1/2}$ in absolute value.

Finally, we regard n as a continuous variable. Differentiating (4.4) in n at fixed x , and using

$$\partial_n \theta_n(x) = \cos \theta_n(x) / ((2n+1) \sin \theta_n(x)), \quad (4.13)$$

we have

$$\partial_n \Theta_n(x) = \frac{1}{2} (\sin 2\theta_n - 2\theta_n) - (2n+1) \sin^2 \theta_n \cdot \frac{\cos \theta_n}{(2n+1) \sin \theta_n} = -\theta_n(x). \quad (4.14)$$

Integrating this identity, and using the bound $|\partial_n \theta_n(x)| \leq C/N$ for $n \in [N-1, N]$, we conclude that

$$\vartheta(x) = \int_{N-1}^N \theta_t(x) dt = \theta_N(x) + O(N^{-1}). \quad (4.15)$$

□

Lemma 4.2. *For all $x, y \in \mathcal{B}_N(\delta)$, the following estimates hold.*

- (a) *We have $|h_n(x)| \leq CN^{-1/4}$ for $|n - N| \leq 2$. In particular, $|\varphi_N(x)| + |\psi_N(x)| \leq C$.*
- (b) *We have $K_N(x, x) \leq C\sqrt{N}$ and $K_{N+1}(x, x) \leq C\sqrt{N}$.*
- (c) *We have $|K_N(x, y)| \leq C\sqrt{N}(1 + \sqrt{N}|x - y|)^{-1}$.*
- (d) *We have $|\partial_x K_N(x, y)| \leq CN$.*

Proof. The bounds in (a) follow from Lemma 4.1. For (b), letting $y \rightarrow x$ in (2.13) and using the second identity in (2.7) for h'_N and h'_{N-1} , we find

$$\begin{aligned} K_N(x, x) &= \frac{N}{2} (h_{N-1}(x)^2 + h_N(x)^2) - \frac{\sqrt{N(N+1)}}{2} h_{N+1}(x) h_{N-1}(x) \\ &\quad - \frac{\sqrt{N(N-1)}}{2} h_{N-2}(x) h_N(x). \end{aligned} \quad (4.16)$$

By (a), each term on the right side is bounded by $CN \cdot N^{-1/2}$. The same argument with $N+1$ in place of N bounds $K_{N+1}(x, x)$, since it only involves h_n with $|n - N| \leq 2$, which are covered by (a).

To prove (c), we first suppose that $|x - y| \geq N^{-1/2}$. Then (2.13) and (a) give

$$|K_N(x, y)| \leq CN^{1/2} \cdot N^{-1/2} \cdot |x - y|^{-1}, \quad (4.17)$$

which implies the claim. In the remaining case $|x - y| < N^{-1/2}$, the claim follows from (2.14) and (b).

For (d), we write

$$\partial_x K_N(x, y) = \sum_{n < N} h'_n(x) h_n(y), \quad (4.18)$$

and the Cauchy–Schwarz inequality gives

$$|\partial_x K_N(x, y)| \leq \left(\sum_{n=0}^{N-1} h'_n(x)^2 \right)^{1/2} K_N(y, y)^{1/2}. \quad (4.19)$$

To bound the sum on the right side, we use (2.7), which gives

$$h'_n(x)^2 \leq n h_{n-1}(x)^2 + (n+1) h_{n+1}(x)^2, \quad (4.20)$$

so the sum is bounded by

$$2NK_{N+1}(x, x) \leq CN^{3/2}, \quad (4.21)$$

where we used (b) in the last step. Using (b) again, we have

$$|\partial_x K_N(x, y)| \leq CN^{3/4}N^{1/4}. \quad (4.22)$$

□

4.2 The local sine kernel

Lemma 4.3. *For all $x_0 \in \mathcal{B}_N(\delta)$ and all $u, v \in \mathbb{R}$ such that $|u|, |v| \leq 2N^{1/4}$, we have, setting $\varrho = \varrho_N(x_0)$,*

$$\left| \frac{1}{\varrho} K_N\left(x_0 + \frac{u}{\varrho}, x_0 + \frac{v}{\varrho}\right) - S(u - v) \right| \leq CN^{-1/2}. \quad (4.23)$$

Proof. Set $x = x_0 + u/\varrho$ and $y = x_0 + v/\varrho$, and define

$$\mathcal{K}(u, v) = \varrho^{-1} K_N(x, y). \quad (4.24)$$

Observe that

$$|x - x_0| + |y - x_0| \leq CN^{-1/4}. \quad (4.25)$$

Hence $x, y \in \mathcal{B}_N(\delta/2)$ when $N \geq N_0$, and we may apply Lemmas 4.1 and 4.2 with $\delta/2$ in place of δ .

We first suppose that $|u - v| \geq N^{-1/2}$. Using (2.13) and $x - y = (u - v)/\varrho$, we find

$$\mathcal{K}(u, v) = \sqrt{\frac{N}{2}} \frac{h_N(x)h_{N-1}(y) - h_{N-1}(x)h_N(y)}{u - v}. \quad (4.26)$$

We insert (4.5). The terms involving ρ_N or ρ_{N-1} contribute to $\mathcal{K}(u, v)$ at most

$$CN^{1/2} \cdot N^{-1/2}N^{-1}|u - v|^{-1} \leq CN^{-1/2}. \quad (4.27)$$

We set $z_1 = \Theta_N(x)$ and $z_2 = \Theta_N(y)$. In the notation of Lemma 4.1(iii), the remaining terms in the numerator equal

$$\alpha_N(x)\alpha_N(y) \left(\mathfrak{a}(y) \sin z_1 \sin(z_2 + \vartheta(y)) - \mathfrak{a}(x) \sin(z_1 + \vartheta(x)) \sin z_2 \right). \quad (4.28)$$

Lemma 4.1(iii) implies that

$$|\mathfrak{a}(y) - \mathfrak{a}(x)| \leq CN^{-3/2}|x - y| \quad (4.29)$$

and

$$|\vartheta(y) - \vartheta(x)| \leq CN^{-1/2}|x - y|. \quad (4.30)$$

Replacing $\mathfrak{a}(y)$ by $\mathfrak{a}(x)$ and $\vartheta(x)$ by $\vartheta(y)$ therefore changes the numerator by at most $CN^{-1}|x - y|$, and changes $\mathcal{K}(u, v)$ by at most

$$CN^{1/2}N^{-1}|x - y|/|u - v| = CN^{-1/2}\varrho^{-1} \leq CN^{-1}. \quad (4.31)$$

After these replacements, we use the identity

$$\sin z_1 \sin(z_2 + w) - \sin(z_1 + w) \sin z_2 = \sin w \sin(z_1 - z_2), \quad z_1, z_2, w \in \mathbb{R}, \quad (4.32)$$

with $w = \vartheta(y)$, and we find that

$$\mathcal{K}(u, v) = \sqrt{\frac{N}{2}} \alpha_N(x) \alpha_N(y) \mathfrak{a}(x) \sin \vartheta(y) \frac{\sin(z_1 - z_2)}{u - v} + O(N^{-1/2}). \quad (4.33)$$

We now evaluate the prefactor. It follows from Lemma 4.1(iii) and (i) that

$$\sin \vartheta(y) = \sin \theta_N(y) + O(N^{-1}) \quad (4.34)$$

and

$$\sin \theta_N(x) / \sin \theta_N(y) = 1 + O(N^{-3/4}), \quad (4.35)$$

since $|x - y| \leq CN^{-1/4}$. Using (4.4), we have

$$\begin{aligned} \sqrt{\frac{N}{2}} \alpha_N(x) \alpha_N(y) \mathfrak{a}(x) \sin \vartheta(y) &= \frac{1}{\pi} \left(\frac{\sin \theta_N(y)}{\sin \theta_N(x)} \right)^{1/2} (1 + O(N^{-1})) \\ &= \frac{1}{\pi} (1 + O(N^{-3/4})). \end{aligned} \quad (4.36)$$

Next, Lemma 4.1(i) and Taylor's theorem give

$$\begin{aligned} z_1 - z_2 &= \Theta'_N(x_0)(x - y) + O(|x - y|(|x - x_0| + |y - x_0|)) \\ &= (u - v)(1 + O(N^{-3/4})), \end{aligned} \quad (4.37)$$

where we used the bound $|x - x_0| + |y - x_0| \leq CN^{1/4}/\varrho$ and

$$\Theta'_N(x_0) = \sqrt{\varrho^2 + 1} = \varrho(1 + O(N^{-1})). \quad (4.38)$$

Using the inequality

$$|\sin(w(1 + w')) - \sin w| \leq |ww'|, \quad (4.39)$$

which holds for all $w, w' \in \mathbb{R}$, we deduce that

$$\sin(z_1 - z_2)/(u - v) = \sin(u - v)/(u - v) + O(N^{-3/4}). \quad (4.40)$$

Inserting this estimate and the evaluation of the prefactor into (4.33), we obtain (4.23) when $|u - v| \geq N^{-1/2}$.

Now suppose that $|u - v| < N^{-1/2}$. Let $u' = v - \operatorname{sgn}(v)N^{-1/2}$ if $v \neq 0$, and $u' = N^{-1/2}$ if $v = 0$. Then $|u' - v| = N^{-1/2}$, $|u'| \leq 2N^{1/4}$, and $|u - u'| \leq 2N^{-1/2}$. By Lemma 4.2(d), we have

$$|\partial_u \mathcal{K}(u, v)| = \varrho^{-2} |\partial_x K_N(x, y)| \leq C, \quad (4.41)$$

and $|S'| \leq 1$. Therefore, by the case already treated,

$$|\mathcal{K}(u, v) - S(u - v)| \leq |\mathcal{K}(u', v) - S(u' - v)| + (C + 1)|u - u'| \leq CN^{-1/2}. \quad (4.42)$$

This completes the proof. $\square$

4.3 Oscillatory integrals and global functions

Lemma 4.4. *For $n \in \{N-1, N\}$, all $a < b$ in $\mathcal{B}_N(\delta)$, and every continuously differentiable function f on $[a, b]$, we have*

$$\left| \int_a^b h_n(x) f(x) dx \right| \leq CN^{-3/4} \left(\sup_{[a,b]} |f| + \int_a^b |f'(x)| dx \right) + CN^{-5/4} \int_a^b |f(x)| dx. \quad (4.43)$$

In particular, $|\int_a^b \varphi_N| + |\int_a^b \psi_N| \leq CN^{-1/2}$.

Proof. By (4.5), the contribution of ρ_n to the integral is at most

$$CN^{-1/4} N^{-1} \int_a^b |f|. \quad (4.44)$$

To bound the main term, we integrate by parts using $\sin \Theta_n = -(\cos \Theta_n)' / \Theta_n'$, and we find

$$\int_a^b \alpha_n f \sin \Theta_n dx = \left[-\frac{\alpha_n f}{\Theta_n'} \cos \Theta_n \right]_a^b + \int_a^b \cos \Theta_n \left(\frac{\alpha_n}{\Theta_n'} \right)' f dx + \int_a^b \cos \Theta_n \frac{\alpha_n}{\Theta_n'} f' dx. \quad (4.45)$$

By Lemma 4.1, we have $|\alpha_n / \Theta_n'| \leq CN^{-3/4}$ and $|(\alpha_n / \Theta_n')'| \leq CN^{-5/4}$ on $\mathcal{B}_N(\delta)$. Together with the last display, these bounds imply (4.43). The last claim follows by taking $f = 1$ and using $b - a \leq C\sqrt{N}$ and $(N/2)^{1/4} \leq N^{1/4}$. $\square$

We now show that the global functions introduced in Section 2.4 are close to their limiting values in the bulk. Recall that p_N , ω_N , and c_N were defined in (2.25) and (2.26).

Lemma 4.5. *For all $y \in \mathcal{B}_N(\delta)$, the following estimates hold.*

- (a) *We have $|p_N(y) - 1| \leq CN^{-1/2}$.*
- (b) *We have $|(\varepsilon \varphi_N)(y)| + |(\varepsilon \psi_N)(y)| \leq C$ and $|(\varepsilon \varphi_N)(y) (\varepsilon \psi_N)(y)| \leq CN^{-1/2}$. If N is odd, then $|(\varepsilon \psi_N)(y)| \leq CN^{-1/2}$.*
- (c) *We have $|\omega_N(y)| \leq CN^{-1/2}$.*

Proof. To prove (a), we use (2.30), which gives

$$p_N(y) - p_N(0) = - \left(\int_{\mathbb{R}} \varphi_N \right) \int_0^y \psi_N(x) dx - \left(\int_{\mathbb{R}} \psi_N \right) \int_0^y \varphi_N(x) dx. \quad (4.46)$$

Lemmas 2.4 and 4.4 show that the right side is bounded by $CN^{-1/2}$ in absolute value, since $[0, y] \subset \mathcal{B}_N(\delta)$. Recalling that $|p_N(0) - 1| \leq 2N^{-1/2}$, we obtain (a).

For (b), observe that $(\varepsilon f)(0) = 0$ for every even integrable function f , so that

$$(\varepsilon f)(y) = \int_0^y f. \quad (4.47)$$

Suppose first that N is even. Then φ_N is even, and Lemma 4.4 implies that $|(\varepsilon\varphi_N)(y)| \leq CN^{-1/2}$. Further, Lemma 2.6(a), part (a), and $\int \varphi_N \geq 1$ give

$$|(\varepsilon\psi_N)(y)| = p_N(y) / \int \varphi_N \leq 2. \quad (4.48)$$

Now suppose that N is odd. Then ψ_N is even, so $|(\varepsilon\psi_N)(y)| \leq CN^{-1/2}$, while Lemma 2.6(b) gives

$$|(\varepsilon\varphi_N)(y)| = c_N p_N(y) \leq 2. \quad (4.49)$$

This proves (b).

It remains to prove (c). If N is even, then $\omega_N = \varepsilon\varphi_N$, which was bounded in the proof of (b). If N is odd, then Lemma 2.6(b) gives $\omega_N = c_N(1 - p_N)$, and the claim follows from (a) and $c_N \leq 1$. $\square$

4.4 Integrals of the kernel

Lemma 4.6. *For all $a, y \in \mathcal{B}_N(\delta)$ with $a < y$, the following estimates hold.*

(a) *We have*

$$\left| \int_a^y K_N(x, y) dx - \frac{1}{2} \right| \leq \frac{C}{1 + \sqrt{N}(y - a)} + CN^{-1/4}. \quad (4.50)$$

(b) *Set $\varrho = \varrho_N(y)$ and $T = \varrho(y - a)$. If $T \leq 2N^{1/4}$, then*

$$\left| \int_a^y K_N(x, y) dx - \int_0^T S(t) dt \right| \leq C(1 + T)N^{-1/2}. \quad (4.51)$$

The same estimates hold for $\int_y^a K_N(x, y) dx$ when $a > y$, with $y - a$ replaced by $a - y$.

Proof. The last sentence of the lemma follows from the first two claims and the symmetry $K_N(-x, -y) = K_N(x, y)$, so we may suppose that $a < y$. For (b), the change of variables $x = y - t/\varrho$ gives

$$\int_a^y K_N(x, y) dx = \int_0^T \varrho^{-1} K_N(y - t/\varrho, y) dt, \quad (4.52)$$

and Lemma 4.3 with $x_0 = y$, $u = -t$, and $v = 0$ shows that the integrand is $S(t) + O(N^{-1/2})$, since S is even. Integrating, we obtain (b).

For (a), suppose first that $T \leq N^{1/4}$. We have

$$\left| \frac{1}{2} - \int_0^T S(t) dt \right| = \left| \int_T^\infty \frac{\sin t}{\pi t} dt \right| \leq \min\left(\frac{1}{2}, \frac{2}{\pi T}\right), \quad (4.53)$$

where we integrated by parts once for the second bound. Combining (4.53) with (b), and using $T \geq c\sqrt{N}(y - a)$ and $(1 + T)N^{-1/2} \leq 2N^{-1/4}$, we obtain (4.50).

Now suppose that $T > N^{1/4}$, and set

$$a' = y - N^{1/4}/\varrho \in (a, y). \quad (4.54)$$

The case already treated gives

$$\int_{a'}^y K_N(x, y) dx = 1/2 + O(N^{-1/4}), \quad (4.55)$$

and (2.13) yields

$$\int_a^{a'} K_N(x, y) dx = \sqrt{\frac{N}{2}} \left(h_{N-1}(y) \int_a^{a'} \frac{h_N(x)}{x-y} dx - h_N(y) \int_a^{a'} \frac{h_{N-1}(x)}{x-y} dx \right). \quad (4.56)$$

We apply Lemma 4.4 with $f(x) = (x-y)^{-1}$ on $[a, a']$. Here $\sup |f| = (y-a')^{-1}$,

$$\int |f'| \leq (y-a')^{-1}, \quad (4.57)$$

and

$$\int |f| \leq \log((y-a)/(y-a')) \leq C \log N. \quad (4.58)$$

We also have

$$(y-a')^{-1} = \varrho N^{-1/4} \leq CN^{1/4}, \quad (4.59)$$

so these bounds give

$$\left| \int_a^{a'} h_n(x)(x-y)^{-1} dx \right| \leq CN^{-1/2} \quad (4.60)$$

for $n \in \{N-1, N\}$. Together with Lemma 4.2(a), this implies that

$$\left| \int_a^{a'} K_N(x, y) dx \right| \leq CN^{1/2} N^{-1/4} N^{-1/2} = CN^{-1/4}. \quad (4.61)$$

In this case, we have $\sqrt{N}(y-a) \geq cN^{1/4}$, and (4.50) follows. $\square$

We now apply these estimates to the projection of a step function. For $t \in \mathbb{R}$, we define

$$g_t = K_N \varepsilon_t - \varepsilon_t, \quad (4.62)$$

where ε_t was defined in (2.3), and we define the bounded function

$$g_{\text{sine}}(u) = -\text{sgn}(u) \int_{|u|}^{\infty} S(v) dv, \quad u \in \mathbb{R}. \quad (4.63)$$

The function g_t measures how far the projection K_N is from preserving the step function ε_t . The next corollary shows that it is concentrated near t , where it is approximated by g_{sine} on the local scale.

Corollary 4.7. *For all $t, y \in \mathcal{B}_N(\delta)$ with $y \neq t$, the following statements hold.*

(a) *We have*

$$g_t(y) = (\varepsilon\varphi_N)(y) (\varepsilon\psi_N)(y) + \operatorname{sgn}(y - t) \left(\int_{\min(t,y)}^{\max(t,y)} K_N(x, y) \, dx - \frac{1}{2} \right). \quad (4.64)$$

(b) *We have $|g_t(y)| \leq C(1 + \sqrt{N}|y - t|)^{-1} + CN^{-1/4}$.*

(c) *Set $\varrho = \varrho_N(t)$, and write $y = t + u/\varrho$. If $0 < |u| \leq N^{1/4}$, then $|g_t(y) - g_{\sin}(u)| \leq C(1 + |u|)N^{-1/2}$.*

Proof. The identity (a) is a restatement of (2.50), since $\varepsilon_t(y) = \operatorname{sgn}(y - t)/2$ and the oriented integral $\int_t^y$ equals $\operatorname{sgn}(y - t) \int_{\min(t,y)}^{\max(t,y)}$. The bound (b) follows from (a), Lemma 4.5(b), and Lemma 4.6(a).

To prove (c), we apply Lemma 4.6(b) with the local scale $\varrho_N(y)$ in place of ϱ . Set

$$T = \varrho_N(y)|y - t| = |u| \varrho_N(y) / \varrho. \quad (4.65)$$

Since $\varrho \geq c\sqrt{N}$ and

$$|\varrho_N(y) - \varrho| \leq C|y - t| \leq C|u|/\varrho, \quad (4.66)$$

we have

$$|T - |u|| \leq C|u|^2/N \leq CN^{-1/2}, \quad (4.67)$$

and in particular $T \leq 2N^{1/4}$. By Lemma 4.6(b) and $|S| \leq 1/\pi$, we find

$$\int_{\min(t,y)}^{\max(t,y)} K_N(x, y) \, dx = \int_0^{|u|} S(v) \, dv + O((1 + |u|)N^{-1/2}). \quad (4.68)$$

Inserting this into (4.64), and using Lemma 4.5(b) and the identity

$$\int_0^{|u|} S - 1/2 = - \int_{|u|}^\infty S, \quad (4.69)$$

we obtain (c). □

An inverse bound from the uncertainty principle

The entries of the matrix $\mathbf{M}_N^\sharp(J)$ in Corollary 3.3 involve the inverse of $\text{Id} - K_J$. The goal of this section is to show that its norm grows at most exponentially in the length of J measured on the microscopic scale. Given a bounded measurable set $J \subset \mathbb{R}$, we define its *microscopic length*

$$L(J) = \sqrt{2N} |J|. \quad (5.1)$$

In eigenvalue coordinates, we have $L(J) = N|A|$ for $J = \sqrt{N/2}A$, so $L(J)$ is comparable to the expected number of eigenvalues in J when J is in the bulk.

Since every eigenvalue of K_J belongs to $[0, 1)$, we have $\|(\text{Id} - K_J)^{-1}\| \leq \det(\text{Id} - K_J)^{-1}$. Since $\det(\text{Id} - K_J)$ decays like $e^{-cL(J)^2}$ for an interval, this only gives a bound of the form $\|(\text{Id} - K_J)^{-1}\| \leq e^{cL(J)^2}$. However, this is not enough for our purposes. We will take $L(J)$ of order $\sqrt{\log N}$, where $e^{cL(J)^2}$ is a power of N and would overwhelm the polynomially small errors in Section 4. In this regime, the bound $e^{cL(J)}$ proved below is much smaller than any power of N . The main input to its proof is a quantitative uncertainty principle of Nazarov [39], in the form given by Jaming [29].

Proposition 5.1. *There exist absolute constants $C, N_0 > 0$ such that the following holds. Let $N \geq N_0$, and let $J \subset \mathbb{R}$ be measurable with $|J| \leq 1$. Then*

$$\|(\text{Id} - K_J)^{-1}\|_{L^2(J) \rightarrow L^2(J)} \leq Ce^{cL(J)}. \quad (5.2)$$

Further, for every measurable set $J \subset \mathbb{R}$ of finite measure, the integral operator S_J on $L^2(J)$ with kernel $S(u - v)$, $u, v \in J$, satisfies $\|(\text{Id} - S_J)^{-1}\| \leq Ce^{c|J|}$.

We will use the following one-dimensional form of Nazarov's uncertainty principle [39], as stated in [29, Theorem]. Recall that $\mathcal{F}$ denotes the unitary Fourier transform from (2.8).

Lemma 5.2. *There exists an absolute constant $C_0 > 0$ such that for all measurable sets $A, \Sigma \subset \mathbb{R}$ of finite measure and all $f \in L^2(\mathbb{R})$,*

$$\|f\|_2^2 \leq C_0 e^{C_0|A||\Sigma|} \left(\int_{A^c} |f(x)|^2 dx + \int_{\Sigma^c} |(\mathcal{F}f)(\xi)|^2 d\xi \right). \quad (5.3)$$

Proof. We define

$$\widehat{f}(\eta) = \int_{\mathbb{R}} f(x) e^{2\pi i x \eta} dx. \quad (5.4)$$

In [29], the inequality is stated with a Fourier transform of this form, normalized so that it is unitary on $L^2(\mathbb{R})$; the sign in the exponent is irrelevant, since replacing Σ by $-\Sigma$ does not change $|\Sigma|$. In dimension one, the factor $e^{C \min(\cdot)}$ there is bounded by $e^{C|A||\Sigma|}$, and the result is due to Nazarov [39]. We set

$$\Sigma' = -\Sigma/(2\pi). \quad (5.5)$$

Using

$$(\mathcal{F}f)(\xi) = (2\pi)^{-1/2} \widehat{f}(-\xi/(2\pi)) \quad (5.6)$$

and changing variables, we have

$$\int_{\Sigma^c} |\mathcal{F}f|^2 = \int_{\Sigma'^c} |\widehat{f}|^2. \quad (5.7)$$

Together with $|\Sigma'| = |\Sigma|/(2\pi)$ and the Plancherel identity $\|\widehat{f}\|_2 = \|f\|_2$, this gives (5.3). $\square$

Proof of Proposition 5.1. Set

$$\Sigma_N = [-8\sqrt{N}, 8\sqrt{N}]. \quad (5.8)$$

We first show that every $f \in V_N$ satisfies

$$\int_{\Sigma_N^c} |f(x)|^2 dx \leq e^{-N} \|f\|_2^2. \quad (5.9)$$

Indeed, (2.14) and Lemma 2.1 imply that

$$|f(x)|^2 \leq K_N(x, x) \|f\|_2^2 \leq Ne^{-x^2/4} \|f\|_2^2 \quad (5.10)$$

for $|x| \geq 8\sqrt{N}$, and

$$\int_{|x| \geq 8\sqrt{N}} Ne^{-x^2/4} dx \leq 2N \int_{8\sqrt{N}}^{\infty} e^{-2\sqrt{N}x} dx \leq e^{-N}, \quad (5.11)$$

where we used $x^2/4 \geq 2\sqrt{N}x$ for $x \geq 8\sqrt{N}$ in the first inequality, and $N \geq N_0$ in the second.

Now let $f \in V_N$. It follows from (2.9) that the function $\mathcal{F}f$ belongs to the complex span of $h_0, \dots, h_{N-1}$, and $\|\mathcal{F}f\|_2 = \|f\|_2$. The proof of (5.9) applies verbatim to this complex span. Applying Lemma 5.2 with $A = J$ and $\Sigma = \Sigma_N$, and then (5.9) to $\mathcal{F}f$, we find

$$\|f\|_2^2 \leq C_0 e^{16C_0\sqrt{N}|J|} \left(\int_{J^c} |f|^2 + e^{-N} \|f\|_2^2 \right). \quad (5.12)$$

Since $16\sqrt{N}|J| \leq 12L(J)$ and $L(J) \leq \sqrt{2N}$, we have

$$C_0 e^{16C_0\sqrt{N}|J| - N} \leq 1/2 \quad (5.13)$$

for $N \geq N_0$. Therefore,

$$\|\mathbf{1}_J f\|_2^2 \leq (1 - c e^{-CL(J)}) \|f\|_2^2, \quad f \in V_N, \quad (5.14)$$

with $c = (2C_0)^{-1}$ and $C = 12C_0$. This means that the operator $K_N \mathbf{1}_J K_N$ on $L^2(\mathbb{R})$ has norm at most $1 - c e^{-CL(J)}$. The operator K_J on $L^2(J)$ equals $(\mathbf{1}_J K_N)(\mathbf{1}_J K_N)^*$, while $K_N \mathbf{1}_J K_N = (\mathbf{1}_J K_N)^*(\mathbf{1}_J K_N)$, so the two operators have the same norm. Since K_J is self-adjoint and nonnegative, we conclude that

$$\|(\text{Id} - K_J)^{-1}\| \leq c^{-1} e^{CL(J)}. \quad (5.15)$$

Finally, we consider the sine kernel. Let P denote the orthogonal projection of $L^2(\mathbb{R})$ onto the functions f with $\mathcal{F}f$ supported in $[-1, 1]$. Then $S_J = \mathbf{1}_J P \mathbf{1}_J$. Applying Lemma 5.2 with $A = J$ and $\Sigma = [-1, 1]$ to such f , we find

$$\|\mathbf{1}_J f\|_2^2 \leq (1 - C_0^{-1} e^{-2C_0|J|}) \|f\|_2^2, \quad (5.16)$$

and the rest of the argument is the same as above. $\square$

We will use Proposition 5.1 through the following consequence for values at single points. For a bounded measurable set J and a bounded function f on J , the function $(\text{Id} - K_J)^{-1}f$ is a priori only an element of $L^2(J)$. However, we have

$$(\text{Id} - K_J)^{-1}f = f + K_J(\text{Id} - K_J)^{-1}f, \quad (5.17)$$

and the second term on the right is the restriction to J of a function in V_N . We therefore use the right side of (5.17) to define $(\text{Id} - K_J)^{-1}f$ at every point of the closure of J at which f is defined. In particular, if J is a finite union of open intervals and f is continuous on the closure of each of them, then $(\text{Id} - K_J)^{-1}f$ has one-sided limits at all endpoints.

Lemma 5.3. *Let $\delta \in (0, 1/2]$. There exist $C, N_0 > 0$ depending only on δ such that the following holds for $N \geq N_0$. Let $J \subset \mathcal{B}_N(\delta)$ be measurable with $|J| \leq 1$, let f be a bounded function on J , and set $G = (\text{Id} - K_J)^{-1}f$. Then*

$$\left| \int_J K_N(t, z) G(z) dz \right| \leq C e^{CL(J)} \sup_J |f|, \quad t \in \mathcal{B}_N(\delta). \quad (5.18)$$

Consequently, $|G(t)| \leq C e^{CL(J)} \sup_J |f|$ for every $t \in J$, with $G(t)$ defined through (5.17). The same bound holds for boundary values obtained as limits from within J .

Proof. Using the Cauchy–Schwarz inequality, Lemma 4.2(b), and Proposition 5.1, we find that the left side of (5.18) is bounded by

$$K_N(t, t)^{1/2} \|G\|_{L^2(J)} \leq C N^{1/4} e^{CL(J)} |J|^{1/2} \sup_J |f| \leq C e^{CL(J)} L(J)^{1/2} \sup_J |f|. \quad (5.19)$$

Since $L(J)^{1/2} \leq e^{L(J)}$, the last display implies (5.18), and the bound for G on J follows from (5.17). Passing to limits gives the boundary assertion. $\square$

The endpoint matrix

In this section, we show that, up to a small error, the determinant of $\mathbf{M}_N^\sharp(J)$ in Corollary 3.3 is the determinant of an $r \times r$ matrix $\mathcal{A}_N(J)$ whose entries are differences of values of $(\text{Id} - K_J)^{-1}g_{a_i}$ at the endpoints of J . Recall that g_{a_i} was defined in (4.62). We then show that $\mathcal{A}_N(J)$ is close to a diagonal matrix when the intervals of J are far apart.

6.1 Setting

Throughout this section, we fix $\delta \in (0, 1/2]$ and an integer $r_0 \geq 1$. The constants C , c , and N_0 depend only on δ and r_0 , and all statements hold for $N \geq N_0$. We consider sets $J = \bigcup_{j=1}^r (a_j, b_j)$ as in (3.1), with $r \leq r_0$, and we assume that

$$\bar{J} \subset \mathcal{B}_N(\delta), \quad L(J) \leq N^{1/8}, \quad (6.1)$$

where $L(J)$ is the microscopic length from (5.1). In particular, $|J| \leq 1$, so that Proposition 5.1 and Lemma 5.3 apply. We also define the *microscopic separation* of the intervals of J by

$$\text{sep}(J) = \sqrt{2N} \min_{i \neq j} \text{dist}(J_i, J_j) \quad (6.2)$$

when $r \geq 2$, and $\text{sep}(J) = \infty$ when $r = 1$. Given a function G on J that has one-sided limits at the endpoints of J , we set

$$\Delta_j G = G(b_j^-) - G(a_j^+), \quad 1 \leq j \leq r. \quad (6.3)$$

Definition 6.1. We define the *endpoint matrix* of J to be the $r \times r$ matrix $\mathcal{A}_N(J)$ with entries

$$\mathcal{A}_N(J)_{ij} = \mathbf{1}_{i=j} - \Delta_j \left[(\text{Id} - K_J)^{-1} (\mathbf{1}_J g_{a_i}) \right], \quad 1 \leq i, j \leq r. \quad (6.4)$$

Note that the restriction of g_{a_i} to each interval J_k is bounded and extends continuously to the closure $\bar{J}_k$, where at the point a_i we use the value $\lim_{y \downarrow a_i} g_{a_i}(y)$. In particular, the one-sided limits in (6.4) are defined through (5.17).

The main result of this section is the following proposition.

Proposition 6.2. *Under the assumptions (6.1), we have*

$$|\det \mathbf{M}_N^\sharp(J) - \det \mathcal{A}_N(J)| \leq C e^{CL(J)} N^{-1/2}, \quad |\det \mathcal{A}_N(J)| \leq C e^{CL(J)}. \quad (6.5)$$

Consequently, $\mathcal{P}_N(J)^2 \leq C e^{CL(J)} \mathcal{D}_N(J)$.

6.2 Reduction to local quantities

The entries of $\mathbf{M}_N^\sharp(J)$ are pairings of bounded functions Y on $\mathbb{R}$ with $Q_N(J)^{-1}X$, where $X \in V_N$. We first show that these pairings depend on Y only through $\langle Y, X \rangle$ and the restriction of $K_N Y$ to J , and for $X = k_j$ only through the latter.

Lemma 6.3. *Let Y be a bounded measurable function on $\mathbb{R}$, and let $X \in V_N$. Then*

$$\langle Y, Q_N(J)^{-1}X \rangle = \langle Y, X \rangle + \langle \mathbf{1}_J K_N Y, (\text{Id} - K_J)^{-1}(\mathbf{1}_J X) \rangle. \quad (6.6)$$

Moreover, for $1 \leq j \leq r$, we have $\langle Y, Q_N(J)^{-1}k_j \rangle = \Delta_j[(\text{Id} - K_J)^{-1}(\mathbf{1}_J K_N Y)]$.

Proof. Let $f \in V_N$, and set

$$G = (\text{Id} - K_J)^{-1}(\mathbf{1}_J f) \in L^2(J). \quad (6.7)$$

Then $f + K_N(\mathbf{1}_J G) \in V_N$, and

$$\begin{aligned} Q_N(J)(f + K_N(\mathbf{1}_J G)) &= f + K_N(\mathbf{1}_J G) - K_N(\mathbf{1}_J f + \mathbf{1}_J K_N(\mathbf{1}_J G)) \\ &= f - K_N(\mathbf{1}_J f) + K_N(\mathbf{1}_J(\text{Id} - K_J)G) = f, \end{aligned} \quad (6.8)$$

where we used $\mathbf{1}_J K_N(\mathbf{1}_J G) = K_J G$ on J . Therefore $Q_N(J)^{-1}f = f + K_N(\mathbf{1}_J G)$. Pairing with Y and using the symmetry of K_N , we obtain (6.6) with $f = X$.

To prove the second claim, we take $X = k_j$. Since $K_N Y \in V_N$, we have $\langle Y, k_j \rangle = \Delta_j(K_N Y)$. Also, by the symmetry of $(\text{Id} - K_J)^{-1}$,

$$\begin{aligned} \langle \mathbf{1}_J K_N Y, (\text{Id} - K_J)^{-1}(\mathbf{1}_J k_j) \rangle &= \langle (\text{Id} - K_J)^{-1}(\mathbf{1}_J K_N Y), \mathbf{1}_J k_j \rangle \\ &= \Delta_j[K_N(\mathbf{1}_J(\text{Id} - K_J)^{-1}(\mathbf{1}_J K_N Y))]. \end{aligned} \quad (6.9)$$

Adding the two terms and using (5.17), we obtain the claim. $\square$

We now compute $\mathbf{1}_J K_N w_i$, where $w_i = \mathbf{1}_{J^c \varepsilon_{a_i}}$ was defined in (3.2). This is the step where the sign function becomes local.

Lemma 6.4. *For $1 \leq i \leq r$, we have*

$$\mathbf{1}_J K_N w_i = (\text{Id} - K_J)(\mathbf{1}_J \varepsilon_{a_i}) + \mathbf{1}_J g_{a_i}. \quad (6.10)$$

Consequently, $\mathbf{M}_N(J)_{ij} = \mathcal{A}_N(J)_{ij}$ for all $1 \leq i, j \leq r$.

Proof. We have $K_N w_i = K_N \varepsilon_{a_i} - K_N(\mathbf{1}_J \varepsilon_{a_i})$, and $\mathbf{1}_J K_N(\mathbf{1}_J \varepsilon_{a_i}) = K_J(\mathbf{1}_J \varepsilon_{a_i})$. The definition (4.62) also gives $\mathbf{1}_J K_N \varepsilon_{a_i} = \mathbf{1}_J \varepsilon_{a_i} + \mathbf{1}_J g_{a_i}$. Combining these identities, we obtain (6.10).

Applying Lemma 6.3 with $Y = w_i$ and $X = k_j$, and using (6.10), we have

$$\langle w_i, Q_N(J)^{-1}k_j \rangle = \Delta_j[\mathbf{1}_J \varepsilon_{a_i}] + \Delta_j[(\text{Id} - K_J)^{-1}(\mathbf{1}_J g_{a_i})]. \quad (6.11)$$

The function ε_{a_i} is constant on each interval J_j , with the value $1/2$ on J_i and on the intervals to the right of J_i , and the value $-1/2$ on the intervals to the left of J_i . Hence $\Delta_j[\mathbf{1}_J \varepsilon_{a_i}] = 0$ for all j . Recall that $Y_i = w_i$ and $X_j = k_j$ in (3.9). The claim now follows from (3.10) and (6.4). $\square$

Remark 6.5. Lemma 6.4 is where the nondecaying sign function in the GOE kernel is removed. The function w_i does not decay, and a direct estimate of the pairing $\langle w_i, Q_N(J)^{-1} k_j \rangle$ would require control of the projection K_N on the whole real line. However, w_i is locally constant away from the endpoints of J , and (6.10) shows that after discarding the term $(\text{Id} - K_J)(\mathbf{1}_J \varepsilon_{a_i})$, which does not contribute to the endpoint differences, only the function g_{a_i} remains. By Corollary 4.7(b), we have

$$|g_{a_i}(y)| \leq C(1 + \sqrt{N}|y - a_i|)^{-1} + CN^{-1/4}, \quad (6.12)$$

so this function is small away from a_i on the microscopic scale and can be computed in terms of the kernel in the bulk.

6.3 Bounds on the remaining entries

We now bound the entries of $\mathbf{M}_N^\sharp(J)$, including those that involve the top modes. We will use repeatedly that

$$\int_J |K_N(y, z)| dz \leq C \log(2 + L(J)), \quad y \in \mathcal{B}_N(\delta). \quad (6.13)$$

To see this, note that the bound $C\sqrt{N}(1 + \sqrt{N}|y - z|)^{-1}$ from Lemma 4.2(c) is a decreasing function of $|y - z|$. So the integral over J is bounded by the integral of this bound over the interval $\{z : |z - y| \leq |J|/2\}$, which is equal to $2C \log(1 + \sqrt{N}|J|/2)$. Since $\sqrt{N}|J| \leq L(J)$, the claim follows.

Lemma 6.6. *For $1 \leq i \leq r$, we have $\sup_J |K_N w_i| \leq C \log(2 + L(J))$. Further, with $Y_0 = Y_N(J)$ and $Y_N^\sharp(J)$ as in (3.3) and (3.5),*

$$\sup_J |K_N Y_0| + \sup_J |K_N Y_N^\sharp(J)| \leq C(1 + L(J))^2 N^{-1/2}. \quad (6.14)$$

Proof. The first claim follows from (6.10), (6.13), and Corollary 4.7(b).

Next, we observe that for all integrable f ,

$$\mathbf{1}_{J^c} \varepsilon(\mathbf{1}_J f) = \sum_{k=1}^r \left(\int_{J_k} f \right) w_k \quad (6.15)$$

almost everywhere. Indeed, $\text{sgn}(x - z) = \text{sgn}(x - a_k)$ for $x \notin \overline{J_k}$ and $z \in J_k$. Therefore

$$Y_0 = \mathbf{1}_{J^c} \omega_N - \sum_{k=1}^r \left(\int_{J_k} \varphi_N \right) w_k, \quad Y_N^\#(J) = -2c_N \left(\mathbf{1}_J \varepsilon \psi_N + \sum_{k=1}^r \left(\int_{J_k} \psi_N \right) w_k \right). \quad (6.16)$$

To bound $\sup_J |K_N Y_0|$, we recall from Lemma 2.6(c) that $K_N \omega_N = 0$, so that $K_N(\mathbf{1}_{J^c} \omega_N) = -K_N(\mathbf{1}_J \omega_N)$. Then Lemma 4.5(c) and (6.13) imply that

$$\sup_J |K_N(\mathbf{1}_J \omega_N)| \leq C N^{-1/2} \log(2 + L(J)). \quad (6.17)$$

Further, Lemma 4.2(a) gives

$$\left| \int_{J_k} \varphi_N \right| \leq C |J| \leq C L(J) N^{-1/2}. \quad (6.18)$$

Combining these bounds with the first claim, we obtain the desired bound.

For $Y_N^\#(J)$, we may assume that N is odd, since otherwise $Y_N^\#(J) = 0$. Then Lemma 4.5(b) gives $|\varepsilon \psi_N| \leq C N^{-1/2}$ on J , and the same argument applies, using $c_N \leq 1$. $\square$

Lemma 6.7. *The entries of $\mathbf{M}_N^\#(J)$ satisfy the following bounds, for $1 \leq i, j \leq r$ and $0 \leq \beta \leq r$:*

$$\begin{aligned} |\mathbf{M}_N(J)_{i0}| + |\mathbf{M}_N(J)_{ij}| &\leq C e^{CL(J)}, \\ |\mathbf{M}_N(J)_{00} - 1| + |\mathbf{M}_N(J)_{0j}| + |z_N(J)_\beta| + |d_N(J) - 1| + |v_N(J)_0| &\leq C e^{CL(J)} N^{-1/2}. \end{aligned} \quad (6.19)$$

Proof. We first consider the entries in the columns $j \geq 1$. By Lemma 6.3, for all bounded Y , we have

$$\langle Y, Q_N(J)^{-1} k_j \rangle = \Delta_j [(\text{Id} - K_J)^{-1} (\mathbf{1}_J K_N Y)], \quad (6.20)$$

and Lemma 5.3 bounds this by $C e^{CL(J)} \sup_J |K_N Y|$. Taking $Y = w_i$, $Y = Y_0$, and $Y = Y_N^\#(J)$, and using Lemma 6.6, we obtain the bounds on $\mathbf{M}_N(J)_{ij}$, $\mathbf{M}_N(J)_{0j}$, and $z_N(J)_j$.

Next, we consider the column $\beta = 0$, where $X_0 = -\psi_N$. Lemma 6.3 also gives, for all bounded Y ,

$$\langle Y, Q_N(J)^{-1} \psi_N \rangle = \langle Y, \psi_N \rangle + \langle \mathbf{1}_J K_N Y, (\text{Id} - K_J)^{-1} (\mathbf{1}_J \psi_N) \rangle. \quad (6.21)$$

Using Proposition 5.1 and Lemma 4.2(a), we bound the second term in (6.21) by

$$|J|^{1/2} \sup_J |K_N Y| \cdot C e^{CL(J)} \|\mathbf{1}_J \psi_N\|_2 \leq C e^{CL(J)} \sup_J |K_N Y|. \quad (6.22)$$

It remains to bound $\langle Y, \psi_N \rangle$ for the three choices of Y . For $Y = w_i$, we have

$$\langle w_i, \psi_N \rangle = \langle \varepsilon_{a_i}, \psi_N \rangle - \langle \mathbf{1}_J \varepsilon_{a_i}, \psi_N \rangle = -(\varepsilon \psi_N)(a_i) - \langle \mathbf{1}_J \varepsilon_{a_i}, \psi_N \rangle, \quad (6.23)$$

which Lemma 4.5(b) and Lemma 4.2(a) bound by C in absolute value. When $Y = Y_0$, we use $\langle \omega_N, \psi_N \rangle = 0$ from Lemma 2.6(c) and the formula for Y_0 in the proof of Lemma 6.6 to write

$$\langle Y_0, \psi_N \rangle = -\langle \mathbf{1}_J \omega_N, \psi_N \rangle - \sum_{k=1}^r \left(\int_{J_k} \varphi_N \right) \langle w_k, \psi_N \rangle, \quad (6.24)$$

which is bounded by $CN^{-1/2}|J| + CL(J)N^{-1/2}$ in absolute value. The same computation for $Y = Y_N^\sharp(J)$ with N odd gives

$$|\langle Y_N^\sharp(J), \psi_N \rangle| \leq CN^{-1/2}|J| + C|J| \max_k |\langle w_k, \psi_N \rangle|, \quad (6.25)$$

and the right side can be bounded by $CL(J)N^{-1/2}$. Combining these bounds with Lemma 6.6, and recalling that $\mathbf{M}_N(J)_{\alpha 0} = \mathbf{1}_{\alpha=0} + \langle Y_\alpha, Q_N(J)^{-1} \psi_N \rangle$ and $z_N(J)_0 = -\langle Y_N^\sharp(J), Q_N(J)^{-1} \psi_N \rangle$, we obtain the bounds for the column $\beta = 0$.

Finally, Lemma 4.2(a) gives

$$|d_N(J) - 1| = c_N \left| \int_J \psi_N \right| \leq C|J| \quad (6.26)$$

and $|v_N(J)_0| \leq C|J|$. This completes the proof, since $|J| \leq CL(J)N^{-1/2}$ and $L(J) \leq e^{L(J)}$. $\square$

Proof of Proposition 6.2. We order the rows and columns of $\mathbf{M}_N^\sharp(J)$ as $0, 1, \dots, r, r+1$, where the index $r+1$ corresponds to the last row and column in (3.11). Let $\mathbf{E}$ denote the matrix obtained from $\mathbf{M}_N^\sharp(J)$ by replacing the row with index 0 by the unit vector e_0 and the row with index $r+1$ by the unit vector e_{r+1} . Lemma 6.7 implies that every entry of $\mathbf{M}_N^\sharp(J)$ is bounded by $Ce^{CL(J)}$ in absolute value, and that the rows with index 0 and $r+1$ differ from e_0 and e_{r+1} by vectors with entries bounded by $Ce^{CL(J)}N^{-1/2}$. Since the determinant is linear in each row and $r \leq r_0$, we find

$$|\det \mathbf{M}_N^\sharp(J) - \det \mathbf{E}| \leq Ce^{CL(J)}N^{-1/2}. \quad (6.27)$$

Expanding $\det \mathbf{E}$ along the two unit rows, we find that $\det \mathbf{E}$ is the determinant of the submatrix of $\mathbf{M}_N^\sharp(J)$ with rows and columns indexed by $1, \dots, r$. Lemma 6.4 shows that this submatrix coincides with $\mathcal{A}_N(J)$, which gives the first bound in (6.5). The second bound follows from Lemma 6.7 and $r \leq r_0$, and the last claim follows from Corollary 3.3. $\square$

6.4 Separated intervals

We now show that the endpoint matrix of a union of well-separated intervals is close to the diagonal matrix formed by the endpoint matrices of the individual intervals.

Lemma 6.8. *Under the assumptions (6.1), suppose that $r \geq 2$ and $\text{sep}(J) \geq 1$. Then for all $1 \leq i, j \leq r$ with $i \neq j$,*

$$\begin{aligned} |\mathcal{A}_N(J)_{ij}| &\leq Ce^{CL(J)} (\text{sep}(J)^{-1} + N^{-1/4}), \\ |\mathcal{A}_N(J)_{jj} - \mathcal{A}_N(J_j)| &\leq Ce^{CL(J)} \text{sep}(J)^{-1}, \end{aligned} \quad (6.28)$$

where $\mathcal{A}_N(J_j)$ is the endpoint matrix of the single interval J_j , which is a real number.

Proof. Throughout the proof, we write $L = L(J)$ and $d = \text{sep}(J)$. Let K_0 denote the operator on $L^2(J) = \bigoplus_k L^2(J_k)$ that acts as K_{J_k} on each $L^2(J_k)$, and set $R = K_J - K_0$. The kernel of this operator is equal to $K_N(x, y)$ for x and y in different intervals of J , and to 0 otherwise. Using Lemma 4.2(c), we see that $|K_N(x, y)| \leq C\sqrt{N}d^{-1}$ when x and y belong to different intervals, so the Hilbert–Schmidt norm of R satisfies

$$\|R\|_2^2 \leq \sum_{k \neq l} |J_k| |J_l| C N d^{-2} \leq CL^2 d^{-2}. \quad (6.29)$$

By the resolvent identity, we have

$$(\text{Id} - K_J)^{-1} = (\text{Id} - K_0)^{-1} + (\text{Id} - K_J)^{-1} R (\text{Id} - K_0)^{-1}. \quad (6.30)$$

Fix i . We define

$$G = (\text{Id} - K_J)^{-1} (\mathbf{1}_J g_{a_i}) \quad (6.31)$$

and

$$G_0 = (\text{Id} - K_0)^{-1} (\mathbf{1}_J g_{a_i}). \quad (6.32)$$

By Corollary 4.7(b), we have $|g_{a_i}| \leq C$ on J , and $|g_{a_i}(y)| \leq C(d^{-1} + N^{-1/4})$ for $y \in J_k$ with $k \neq i$, since then $\sqrt{N}|y - a_i| \geq d/\sqrt{2}$. Applying Proposition 5.1 to each J_k separately, we have

$$\|\mathbf{1}_{J_k} G_0\|_2 \leq Ce^{CL} |J|^{1/2} (d^{-1} + N^{-1/4}), \quad k \neq i, \quad \|G_0\|_2 \leq Ce^{CL} |J|^{1/2}. \quad (6.33)$$

Combining (6.30) and (6.29) with Proposition 5.1, we also have

$$\|G - G_0\|_2 \leq Ce^{CL} \|R\|_2 \|G_0\|_2 \leq Ce^{CL} d^{-1} |J|^{1/2}, \quad (6.34)$$

where in the last inequality we absorbed the factor L from (6.29) into e^{CL} .

We now evaluate G at an endpoint $t \in \{a_j, b_j\}$ with $j \neq i$. By (5.17), we have

$$G(t) = g_{a_i}(t) + \sum_{k=1}^r \int_{J_k} K_N(t, z) G(z) dz. \quad (6.35)$$

The first term is bounded by $C(d^{-1} + N^{-1/4})$, since $\sqrt{N}|t - a_i| \geq d/\sqrt{2}$. When $k \neq j$, we have $|K_N(t, z)| \leq C\sqrt{N}d^{-1}$ for $z \in J_k$. Then the Cauchy–Schwarz inequality, together with (6.33) and (6.34), shows that the term with index k is bounded by

$$C\sqrt{N}d^{-1} |J|^{1/2} \|G\|_2 \leq Ce^{CL} L d^{-1}. \quad (6.36)$$

For $k = j$, the Cauchy–Schwarz inequality and Lemma 4.2(b) give

$$\left| \int_{J_j} K_N(t, z) G(z) \, dz \right| \leq CN^{1/4} (\|\mathbf{1}_{J_j} G_0\|_2 + \|G - G_0\|_2) \leq Ce^{CL} (d^{-1} + N^{-1/4}), \quad (6.37)$$

where we used $N^{1/4}|J|^{1/2} \leq L^{1/2}$ and $j \neq i$. Taking the difference between $t = b_j$ and $t = a_j$, we obtain the first bound in (6.28).

To prove the second bound, we define

$$G^{(i)} = (\text{Id} - K_{J_i})^{-1}(\mathbf{1}_{J_i} g_{a_i}). \quad (6.38)$$

By the definition of the endpoint matrix, $\mathcal{A}_N(J_i) = 1 - \Delta_i G^{(i)}$. We also have $\mathbf{1}_{J_i} G_0 = G^{(i)}$. For $t \in \{a_i, b_i\}$, we can then write

$$G(t) - G^{(i)}(t) = \sum_{k \neq i} \int_{J_k} K_N(t, z) G(z) \, dz + \int_{J_i} K_N(t, z) (G - G_0)(z) \, dz. \quad (6.39)$$

As above, the first sum is bounded by $Ce^{CL} L d^{-1}$. Using (6.34), we bound the last term by $CN^{1/4} \|G - G_0\|_2 \leq Ce^{CL} d^{-1}$. This gives the second bound in (6.28), after renaming i as j . $\square$

The hole probability of one interval

In this section, we prove that the GOE hole probability of an interval $(x_0, x_0 + s/(2\varrho_N(x_0)))$ in the bulk, with $s \leq \gamma\sqrt{\log N}$, is asymptotic to its limit q_1 , with a relative error that is polynomially small in N . We first recall the limiting objects.

7.1 Sine-kernel determinants

Recall from (1.13) that $D_{\pm}(t) = \det(\text{Id} - S_{\pm,t})$ for $t > 0$, where $S_{\pm,t}$ is the integral operator on $L^2(0, t)$ with kernel $S(u - v) \pm S(u + v)$, and S is the sine kernel (4.3). We set $D_{\pm}(0) = 1$. The operators $S_{\pm,t}$ are the restrictions of the integral operator with kernel $S(u - v)$ on $L^2(-t, t)$ to even and odd functions, respectively, so their eigenvalues lie in $[0, 1)$, and

$$D_+(t)D_-(t) = \det(\text{Id} - S_{(-t,t)}) = \det(\text{Id} - S_{(0,2t)}), \quad (7.1)$$

where S_A denotes the operator on $L^2(A)$ with kernel $S(u - v)$, and the second equality follows from translation invariance; see [48, Section IV] or [16, (4)]. The functions $D_{\pm}$ are continuous and positive on $[0, \infty)$, and they are nonincreasing. To see the monotonicity, note that $S_{\pm,t}$ is a compression of $S_{\pm,t'}$ for $t < t'$, so its eigenvalues increase with t by the Courant–Fischer min-max principle. By definition, $q_1(s) = D_+(s/4)$. We define

$$\mu(s) = \frac{D_+(s/4)}{D_-(s/4)}, \quad s \geq 0, \quad (7.2)$$

and we set $q_1(0) = 1$. (The interpretation of q_1 as a gap probability, recalled in Section 1.2, is not used in our proofs.) Recall from (1.14) the asymptotics of $D_{\pm}(t)$ as $t \rightarrow \infty$, which were proved by Ehrhardt [16, Theorem 4.8].

Lemma 7.1. *Recall the function $F(s) = s^2/64 + s/8$ and the constant κ_1 from (1.3). Then the following statements hold.*

- (a) *We have $q_1(s) = \kappa_1 s^{-1/8} e^{-F(s)}(1 + o(1))$ as $s \rightarrow \infty$, and q_1 is nonincreasing.*
- (b) *We have $\mu(s) = \sqrt{2} e^{-s/4}(1 + o(1))$ as $s \rightarrow \infty$, and there exists $c > 0$ such that $\mu(s) \geq c e^{-s/4}$ for all $s \geq 0$.*

Proof. Setting $t = s/4$ in (1.14) with the upper sign, we find

$$\log q_1(s) = -\frac{s^2}{64} - \frac{s}{8} - \frac{1}{8} \log s + \frac{1}{4} \log 2 + \frac{7}{24} \log 2 + \frac{3}{2} \zeta'(-1) + o(1), \quad (7.3)$$

which is the asymptotic formula in (a), since $1/4 + 7/24 = 13/24$. The monotonicity in (a) was shown above. Subtracting the two cases of (1.14), we find

$$\log \mu(s) = -s/4 + (\log 2)/2 + o(1). \quad (7.4)$$

Since μ is continuous and positive on $[0, \infty)$, this implies the lower bound in (b). $\square$

Next, we identify the limit of the endpoint matrix of one interval. For $\ell > 0$, we define

$$\mathcal{A}_{\text{sine}}(\ell) = 1 - \left(((\text{Id} - S_{(0,\ell)})^{-1} g_{\text{sine}})(\ell^-) - ((\text{Id} - S_{(0,\ell)})^{-1} g_{\text{sine}})(0^+) \right), \quad (7.5)$$

where g_{sine} was defined in (4.63), and the values at the endpoints are defined through the analogue of (5.17) for S , with $g_{\text{sine}}(0^+) = -1/2$. The following identity is due to Tracy and Widom.

Lemma 7.2. *For all $\ell > 0$, we have $\mathcal{A}_{\text{sine}}(\ell) = D_+(\ell/2)/D_-(\ell/2)$. In particular, $\mathcal{A}_{\text{sine}}(s/2) = \mu(s)$ for all $s > 0$.*

Proof. Set $t = \ell/2$ and $J = (-t, t)$, and define

$$\Delta G = G(t^-) - G((-t)^+). \quad (7.6)$$

Since the sine kernel is translation invariant and g_{sine} is evaluated relative to the left endpoint, we have

$$\mathcal{A}_{\text{sine}}(\ell) = 1 - \Delta[(\text{Id} - S_J)^{-1}(\mathbf{1}_J g_{\text{sine}}(\cdot + t))]. \quad (7.7)$$

Let $R(x, y)$, $x \in \mathbb{R}$, $y \in \bar{J}$, denote the kernel of $(\text{Id} - S\mathbf{1}_J)^{-1}S\mathbf{1}_J$. Then

$$R(x, y) = S(x - y) + \int_J S(x - z)R(z, y) dz, \quad (7.8)$$

and R is symmetric on $J \times J$.

Set $w = \mathbf{1}_{J^c} \varepsilon_{-t}$. Tracy and Widom showed [48, (19), (25), (30)] that

$$1 - \int_{J^c} w(x)(R(x, t) - R(x, -t)) dx = \frac{D_+(t)}{D_-(t)}, \quad (7.9)$$

where the integral over J^c is the limit as $M \rightarrow \infty$ of the integral over $J^c \cap [-M, M]$ (we show below that this limit exists). To deduce this identity from the equations of Tracy and Widom, set

$$a_{\pm} = \int_{J^c} w(x)R(x, \pm t) dx. \quad (7.10)$$

Since $\mathbf{1}_{J^c} \varepsilon_{-t} = \mathbf{1}_{J^c} \varepsilon_t$ almost everywhere, the determinant in [48, (19)] with $m = 1$ is

$$\det \begin{pmatrix} 1 + a_- & -a_+ \\ a_- & 1 - a_+ \end{pmatrix} = 1 - (a_+ - a_-), \quad (7.11)$$

which is the left side of (7.9). This determinant is computed in [48, (25)] and is equal to $D_+(t)/D_-(t)$ by [48, (30)].

It remains to identify the left side of (7.9) with $\mathcal{A}_{\text{sine}}(\ell)$. To that end, for $z \in \bar{J}$, we set

$$\Phi(z) = \lim_{M \rightarrow \infty} \int_{J^c \cap [-M, M]} w(x) S(x - z) dx. \quad (7.12)$$

A direct computation using $\int_0^\infty S = 1/2$ gives

$$\lim_{M \rightarrow \infty} \int_{-M}^M \varepsilon_{-t}(x) S(x - z) dx = \int_0^{z+t} S(v) dv = \varepsilon_{-t}(z) + g_{\text{sine}}(z + t), \quad (7.13)$$

and the convergence is uniform in $z \in \bar{J}$, since $|\int_M^\infty S| \leq 2/M$. Therefore

$$\Phi = (\text{Id} - S_J)(\mathbf{1}_J \varepsilon_{-t}) + \mathbf{1}_J g_{\text{sine}}(\cdot + t). \quad (7.14)$$

Inserting the equation for R into the left side of (7.9) and exchanging the order of integration over $[-M, M]$ and J , we find

$$\int_{J^c} w(x) R(x, y) dx = \Phi(y) + \int_J \Phi(z) R(z, y) dz = ((\text{Id} - S_J)^{-1} \Phi)(y), \quad y = \pm t. \quad (7.15)$$

Since ε_{-t} is equal to $1/2$ on J , we have

$$\Delta[(\text{Id} - S_J)^{-1} \Phi] = \Delta[(\text{Id} - S_J)^{-1} (\mathbf{1}_J g_{\text{sine}}(\cdot + t))], \quad (7.16)$$

which completes the proof. $\square$

7.2 The comparison

Proposition 7.3. *Let $\delta \in (0, 1/2]$ and $\gamma > 0$. There exist $C, N_0 > 0$ depending only on δ and γ such that the following holds for $N \geq N_0$. Let $x_0 \in \mathcal{B}_N(\delta)$, set $q = q_N(x_0)$, and let $0 < s \leq \gamma \sqrt{\log N}$. Then the interval defined by $J = (x_0, x_0 + s/(2q))$ satisfies*

$$\left| \frac{\mathcal{P}_N(J)}{q_1(s)} - 1 \right| \leq C e^{Cs} N^{-1/2}, \quad \left| \frac{\mathcal{D}_N(J)}{D_+(s/4)D_-(s/4)} - 1 \right| \leq C e^{Cs} N^{-1/2}. \quad (7.17)$$

Before giving the proof, we record a standard bound for determinants. We let $\|Z\|_1$ and $\|Z\|_2$ denote the trace norm and the Hilbert–Schmidt norm of a trace class operator Z .

Lemma 7.4. *Let Z be a trace class operator on a Hilbert space with $\|Z\|_2 \leq 1/2$, and let (y_k) denote its eigenvalues, repeated according to algebraic multiplicity. We define $\log \det(\text{Id} - Z) = \sum_k \log(1 - y_k)$, where $\log$ denotes the principal branch of the logarithm. Then*

$$|\log \det(\text{Id} - Z) + \text{Tr } Z| \leq \|Z\|_2^2. \quad (7.18)$$

Proof. Recall that, by Lidskii's theorem,

$$\mathrm{Tr} Z = \sum_k y_k \quad (7.19)$$

and

$$\det(\mathrm{Id} - Z) = \prod_k (1 - y_k), \quad (7.20)$$

and that, by Weyl's inequality (see [43]),

$$\sum_k |y_k|^2 \leq \|Z\|_2^2. \quad (7.21)$$

In particular, $|y_k| \leq 1/2$ for all k . The claim now follows from the elementary inequality $|\log(1 - y) + y| \leq |y|^2$ for complex y with $|y| \leq 1/2$. $\square$

Proof of Proposition 7.3. Set $\ell = s/2$. Then $J = (x_0, x_0 + \ell/\varrho)$ and

$$L(J) = \sqrt{2N\ell}/\varrho \leq Cs. \quad (7.22)$$

Since $s \leq \gamma\sqrt{\log N}$, we may assume that $\bar{J} \subset \mathcal{B}_N(\delta/2)$, $L(J) \leq N^{1/8}$, and $Ce^{Cs}N^{-1/2} \leq N^{-1/4}$ for every fixed C , by taking N_0 large. We use the results of Sections 4 to 6 with $\delta/2$ in place of δ .

We define a map from $L^2(J)$ to $L^2(0, \ell)$ by

$$(Uf)(u) = \varrho^{-1/2}f(x_0 + u/\varrho). \quad (7.23)$$

This map is unitary. Let $\mathcal{K}_\ell$ denote the integral operator on $L^2(0, \ell)$ with kernel

$$\mathcal{K}(u, v) = \varrho^{-1}K_N(x_0 + u/\varrho, x_0 + v/\varrho). \quad (7.24)$$

Note that UK_JU^{-1} coincides with this operator. We abbreviate $S_\ell = S_{(0, \ell)}$, and we set $\Xi = \mathcal{K}_\ell - S_\ell$. By Lemma 4.3, we have $|\Xi(u, v)| \leq CN^{-1/2}$ for $u, v \in [0, \ell]$. Further, Proposition 5.1 implies that $\|(\mathrm{Id} - \mathcal{K}_\ell)^{-1}\|$ and $\|(\mathrm{Id} - S_\ell)^{-1}\|$ are bounded by Ce^{Cs} .

We first prove the second bound in (7.17). By (7.1), we have $D_+(s/4)D_-(s/4) = \det(\mathrm{Id} - S_\ell)$, and $\mathcal{D}_N(J) = \det(\mathrm{Id} - \mathcal{K}_\ell)$. Set $Z = (\mathrm{Id} - S_\ell)^{-1}\Xi$. Then

$$\det(\mathrm{Id} - \mathcal{K}_\ell) = \det(\mathrm{Id} - S_\ell) \det(\mathrm{Id} - Z). \quad (7.25)$$

We have $\|\Xi\|_2 \leq C\ell N^{-1/2}$ and $\|Z\|_2 \leq Ce^{Cs}N^{-1/2}$. The operators $\mathcal{K}_\ell$ and S_ℓ are positive and have continuous kernels, so their traces are the integrals of their kernels over the diagonal. Therefore, $|\mathrm{Tr} \Xi| \leq C\ell N^{-1/2}$. Writing $(\mathrm{Id} - S_\ell)^{-1} = \mathrm{Id} + S_\ell(\mathrm{Id} - S_\ell)^{-1}$, we also have

$$\begin{aligned} |\mathrm{Tr} Z| &\leq |\mathrm{Tr} \Xi| + |\mathrm{Tr}(S_\ell(\mathrm{Id} - S_\ell)^{-1}\Xi)| \\ &\leq |\mathrm{Tr} \Xi| + \|S_\ell\|_2 \|(\mathrm{Id} - S_\ell)^{-1}\| \|\Xi\|_2 \leq Ce^{Cs}N^{-1/2}, \end{aligned} \quad (7.26)$$

where we used $\|S_\ell\|_2 \leq \ell/\pi$. Then Lemma 7.4 implies that $|\log \det(\text{Id} - Z)| \leq Ce^{Cs}N^{-1/2}$, which gives the second bound in (7.17).

Next, we compare the endpoint matrices. We define

$$G = (\text{Id} - K_J)^{-1}(\mathbf{1}_J g_{x_0}). \quad (7.27)$$

The endpoint matrix of J is then the number $\mathcal{A}_N(J) = 1 - \Delta_1 G$. Set

$$\tilde{g}(u) = g_{x_0}(x_0 + u/\varrho) \quad (7.28)$$

and

$$\tilde{G}(u) = G(x_0 + u/\varrho) \quad (7.29)$$

for $u \in (0, \ell)$. Using (5.17) and the change of variables $z = x_0 + v/\varrho$, we find that

$$\tilde{G} = \tilde{g} + \mathcal{K}_\ell \tilde{G}, \quad (7.30)$$

and this identity extends to $u \in \{0, \ell\}$.

Let $G_{\text{sine}} = (\text{Id} - S_\ell)^{-1}g_{\text{sine}}$. By Corollary 4.7(c), we have $|\tilde{g}(u) - g_{\text{sine}}(u)| \leq C(1 + \ell)N^{-1/2}$ for $u \in (0, \ell)$. Note that

$$\tilde{G} - G_{\text{sine}} = (\text{Id} - \mathcal{K}_\ell)^{-1}((\tilde{g} - g_{\text{sine}}) + \Xi G_{\text{sine}}) \quad (7.31)$$

and that $\|G_{\text{sine}}\|_2 \leq Ce^{Cs}\ell^{1/2}$. Therefore,

$$\|\tilde{G} - G_{\text{sine}}\|_2 \leq Ce^{Cs}N^{-1/2}. \quad (7.32)$$

For $u \in \{0, \ell\}$, we then have

$$\begin{aligned} |\tilde{G}(u) - G_{\text{sine}}(u)| &\leq |\tilde{g}(u) - g_{\text{sine}}(u)| + \int_0^\ell |\Xi(u, v)| |\tilde{G}(v)| \, dv \\ &\quad + \int_0^\ell |S(u - v)| |\tilde{G}(v) - G_{\text{sine}}(v)| \, dv \\ &\leq Ce^{Cs}N^{-1/2}, \end{aligned} \quad (7.33)$$

by the Cauchy–Schwarz inequality. Therefore, by Lemma 7.2,

$$|\mathcal{A}_N(J) - \mu(s)| = |\mathcal{A}_N(J) - \mathcal{A}_{\text{sine}}(\ell)| \leq Ce^{Cs}N^{-1/2}. \quad (7.34)$$

We now conclude. By Corollary 3.3, Proposition 6.2, (7.34), and Lemma 7.1(b), we have

$$\frac{\mathcal{P}_N(J)^2}{\mathcal{D}_N(J)} = \det \mathbf{M}_N^\#(J) = \mu(s) + O(e^{Cs}N^{-1/2}) = \mu(s)(1 + O(e^{Cs}N^{-1/2})). \quad (7.35)$$

Combining this with the second bound in (7.17) and $D_+(s/4)D_-(s/4)\mu(s) = q_1(s)^2$, we find that $\mathcal{P}_N(J)^2 = q_1(s)^2(1 + O(e^{Cs}N^{-1/2}))$. Taking square roots, we obtain the first bound in (7.17). $\square$

We now translate Proposition 7.3 to eigenvalue coordinates. Recall from the introduction that $E_{1,N}(A)$ denotes the probability that H_N has no eigenvalue in A , and that $\sigma(y) = \sqrt{4 - y^2}$.

Corollary 7.5. *Let $\delta \in (0, 1)$ and $\gamma > 0$. There exist $C, N_0 > 0$ depending only on δ and γ such that for $N \geq N_0$, all $y \in [-2 + \delta, 2 - \delta]$, and all $0 < s \leq \gamma\sqrt{\log N}$,*

$$\left| E_{1,N}\left(y, y + \frac{s}{N\sigma(y)}\right) - q_1(s) \right| \leq Ce^{Cs} N^{-1/2} q_1(s). \quad (7.36)$$

Proof. Set $x_0 = \sqrt{N/2}y$. Then $x_0 \in \mathcal{B}_N(\delta/2)$, $\varrho_N(x_0) = \sqrt{N/2}\sigma(y)$, and the image of the interval $(y, y + s/(N\sigma(y)))$ under $\lambda \mapsto \sqrt{N/2}\lambda$ is $(x_0, x_0 + s/(2\varrho_N(x_0)))$. The claim therefore follows from Proposition 7.3. $\square$

Several intervals

In this section, we prove two estimates for unions of intervals, which are used in the proof of Theorem 1.1. The first states that the hole probability of a union of well-separated intervals factorizes. The second states that the hole probability of an arbitrary union of disjoint intervals is bounded by e^{CL} times the product of the individual hole probabilities. For the GUE, the second estimate holds without the factor e^{CL} , by the negative association of determinantal point processes [4, Lemma 3.8]. For the GOE, the factor e^{CL} comes from the endpoint matrix. This factor is harmless, since L is of order $\sqrt{\log N}$ in our application.

Throughout this section, we fix $\delta \in (0, 1/2]$, an integer $r_0 \geq 1$, and $\gamma > 0$, and we consider sets $J = \bigcup_{j=1}^{r_0} J_j$ as in (3.1) with

$$r \leq r_0, \quad \bar{J} \subset \mathcal{B}_N(\delta), \quad L(J) \leq \gamma \sqrt{\log N}. \quad (8.1)$$

The constants C , c , and N_0 depend only on δ , r_0 , and γ , and all statements hold for $N \geq N_0$.

8.1 The determinantal part

Lemma 8.1. *Let $J', J'' \subset \mathbb{R}$ be disjoint bounded measurable sets. Then*

$$\mathcal{D}_N(J' \cup J'') \leq \mathcal{D}_N(J') \mathcal{D}_N(J''). \quad (8.2)$$

Moreover, if J' and J'' satisfy the assumptions of Proposition 5.1 and $\|K_{J'J''}\|_2 \leq ce^{-C(L(J')+L(J''))}$, where $K_{J'J''}$ denotes the operator from $L^2(J'')$ to $L^2(J')$ with kernel $K_N(x, y)$, then

$$\left| \log \frac{\mathcal{D}_N(J' \cup J'')}{\mathcal{D}_N(J') \mathcal{D}_N(J'')} \right| \leq Ce^{C(L(J')+L(J''))} \|K_{J'J''}\|_2^2. \quad (8.3)$$

Proof. With respect to the decomposition $L^2(J' \cup J'') = L^2(J') \oplus L^2(J'')$, we write

$$\text{Id} - K_{J' \cup J''} = \begin{pmatrix} \text{Id} - K_{J'} & -K_{J'J''} \\ -K_{J''J'} & \text{Id} - K_{J''} \end{pmatrix}. \quad (8.4)$$

Factoring this operator into block triangular operators whose diagonal blocks are $\text{Id} - K_{J'}$ and the Schur complement, and using the multiplicativity of Fredholm determinants, we have

$$\frac{\mathcal{D}_N(J' \cup J'')}{\mathcal{D}_N(J') \mathcal{D}_N(J'')} = \det(\text{Id} - Z), \quad (8.5)$$

where we define

$$Z = (\text{Id} - K_{J''})^{-1/2} K_{J''J'} (\text{Id} - K_{J'})^{-1} K_{J'J''} (\text{Id} - K_{J''})^{-1/2}. \quad (8.6)$$

Note that the operator Z is trace class and nonnegative. Since $\text{Id} - K_{J' \cup J''}$ is nonnegative, so is its Schur complement $\text{Id} - K_{J''} - K_{J''J'} (\text{Id} - K_{J'})^{-1} K_{J'J''}$, and therefore $Z \leq \text{Id}$. Hence the eigenvalues of Z lie in $[0, 1]$, and $\det(\text{Id} - Z) \leq 1$, which proves (8.2).

We now prove the second claim. Under the additional assumptions, Proposition 5.1 gives

$$\|Z\|_1 \leq C e^{C(L(J') + L(J''))} \|K_{J'J''}\|_2^2 \leq 1/2. \quad (8.7)$$

Therefore, (8.3) follows from

$$|\log \det(\text{Id} - Z)| \leq \sum_k |\log(1 - z_k)| \leq 2\|Z\|_1, \quad (8.8)$$

where we let $z_k \in [0, 1/2]$ denote the eigenvalues of Z . $\square$

8.2 Factorization and the quasi-product bound

We first record a lower bound for the endpoint matrix of a single interval.

Lemma 8.2. *Let J' be an interval satisfying (8.1). Then $\det \mathbf{M}_N^\sharp(J') \geq c e^{-CL(J')}$ and $\mathcal{A}_N(J') \geq c e^{-CL(J')}$.*

Proof. We write

$$J' = (x_0, x_0 + s/(2\varrho)) \quad (8.9)$$

with $\varrho = \varrho_N(x_0)$, so that $s \leq 2L(J')$. By Corollary 3.3 and Proposition 7.3, we have

$$\det \mathbf{M}_N^\sharp(J') = \mathcal{P}_N(J')^2 / \mathcal{D}_N(J') = \mu(s)(1 + O(e^{Cs} N^{-1/2})) \quad (8.10)$$

and Lemma 7.1(b) gives $\mu(s) \geq c e^{-s/4}$. This yields the first claim for $N \geq N_0$, since $e^{Cs} N^{-1/2} \leq N^{-1/4}$. The second claim follows from the first and Proposition 6.2. $\square$

Proposition 8.3. *Assume (8.1), and suppose that $\text{sep}(J) \geq \sqrt{N}$. Then*

$$\left| \frac{\mathcal{P}_N(J)}{\prod_{j=1}^r \mathcal{P}_N(J_j)} - 1 \right| \leq C e^{CL(J)} N^{-1/4}. \quad (8.11)$$

Proof. We write $L = L(J)$ and $d = \text{sep}(J)$. We first consider the determinantal part. Set

$$J'_k = J_1 \cup \cdots \cup J_{k-1} \quad (8.12)$$

for $2 \leq k \leq r$. By Lemma 4.2(c), we have $|K_N(x, y)| \leq C\sqrt{N}d^{-1}$ for $x \in J'_k$ and $y \in J_k$, which implies that

$$\|K_{J'_k J_k}\|_2^2 \leq CNd^{-2}|J|^2 \leq CL^2d^{-2} \leq CL^2N^{-1}. \quad (8.13)$$

Applying (8.3) for $k = r, r-1, \dots, 2$, we find

$$\mathcal{D}_N(J) = \left(\prod_{j=1}^r \mathcal{D}_N(J_j) \right) (1 + O(e^{CL}N^{-1})). \quad (8.14)$$

Next, we consider the endpoint matrices. Let $\mathcal{A}_0$ denote the diagonal matrix with entries $\mathcal{A}_N(J_j)$, and write $\mathcal{A}_N(J) = \mathcal{A}_0 + E$. It follows from Lemma 6.8 and $d \geq \sqrt{N}$ that the entries of E are bounded by $Ce^{CL}N^{-1/4}$ in absolute value. By Lemma 8.2, the diagonal entries of $\mathcal{A}_0$ are at least ce^{-CL} . Therefore, we have

$$\det \mathcal{A}_N(J) = \det \mathcal{A}_0 \det(\text{Id} + \mathcal{A}_0^{-1}E) = \left(\prod_{j=1}^r \mathcal{A}_N(J_j) \right) (1 + O(e^{CL}N^{-1/4})). \quad (8.15)$$

In particular, $\det \mathcal{A}_N(J) \geq ce^{-CL}$.

We now apply Proposition 6.2 to J and to each J_j . Together with Lemma 8.2, this gives

$$\det \mathbf{M}_N^\sharp(J) = \det \mathcal{A}_N(J) (1 + O(e^{CL}N^{-1/2})) \quad (8.16)$$

and

$$\det \mathbf{M}_N^\sharp(J_j) = \mathcal{A}_N(J_j) (1 + O(e^{CL}N^{-1/2})). \quad (8.17)$$

Hence

$$\det \mathbf{M}_N^\sharp(J) = \left(\prod_{j=1}^r \det \mathbf{M}_N^\sharp(J_j) \right) (1 + O(e^{CL}N^{-1/4})). \quad (8.18)$$

Combining (8.14) and (8.18) with Corollary 3.3, we find

$$\mathcal{P}_N(J)^2 = \prod_j \mathcal{P}_N(J_j)^2 (1 + O(e^{CL}N^{-1/4})). \quad (8.19)$$

Taking square roots completes the proof. $\square$

Proposition 8.4. *Assume (8.1). Then*

$$\mathcal{P}_N(J) \leq Ce^{CL(J)} \prod_{j=1}^r \mathcal{P}_N(J_j). \quad (8.20)$$

Proof. Combining Proposition 6.2, Lemma 8.1, Corollary 3.3, and Lemma 8.2, we have

$$\begin{aligned} \mathcal{P}_N(J)^2 &\leq Ce^{CL(J)} \mathcal{D}_N(J) \leq Ce^{CL(J)} \prod_{j=1}^r \mathcal{D}_N(J_j) \\ &= Ce^{CL(J)} \prod_{j=1}^r \frac{\mathcal{P}_N(J_j)^2}{\det \mathbf{M}_N^\sharp(J_j)} \leq Ce^{CL(J)} \prod_{j=1}^r \mathcal{P}_N(J_j)^2. \end{aligned} \quad (8.21)$$

Taking square roots, we obtain (8.20). $\square$

We now translate these results to eigenvalue coordinates.

Corollary 8.5. *Let $\delta \in (0, 1)$, $r_0 \geq 1$, and $\gamma > 0$. There exist $C, N_0 > 0$ depending only on δ , r_0 , and γ such that the following holds for $N \geq N_0$. Let $A_1, \dots, A_r \subset [-2 + \delta, 2 - \delta]$ be open intervals with disjoint closures, where $r \leq r_0$, and suppose that $L \leq \gamma\sqrt{\log N}$, where we set $L = N \sum_j |A_j|$. Then*

$$E_{1,N}\left(\bigcup_{j=1}^r A_j\right) \leq Ce^{CL} \prod_{j=1}^r E_{1,N}(A_j). \quad (8.22)$$

If, in addition, $\text{dist}(A_i, A_j) \geq N^{-1/2}$ for all $i \neq j$, then

$$\left| E_{1,N}\left(\bigcup_{j=1}^r A_j\right) - \prod_{j=1}^r E_{1,N}(A_j) \right| \leq Ce^{CL} N^{-1/4} \prod_{j=1}^r E_{1,N}(A_j). \quad (8.23)$$

Proof. Let J_j denote the image of A_j under $\lambda \mapsto \sqrt{N/2} \lambda$, and set $J = \bigcup_j J_j$. These images are intervals, and their union satisfies (8.1) with $\delta/2$ in place of δ . Moreover, we have $L(J) = L$ and

$$\text{sep}(J) = N \min_{i \neq j} \text{dist}(A_i, A_j). \quad (8.24)$$

So (8.22) follows from Proposition 8.4. Under the additional assumption that $\text{dist}(A_i, A_j) \geq N^{-1/2}$ for all $i \neq j$, we have $\text{sep}(J) \geq \sqrt{N}$, and (8.23) follows from Proposition 8.3. $\square$

The first moment

In this section, we compute the integral of the one-interval hole probability over the starting point of the interval, at the threshold relevant for the largest gaps. This calculation determines the centering, the scaling, and the constant Λ_I in Theorem 1.1. We work in eigenvalue coordinates, with the notation of Section 1.2. Recall the quantities $\ell_N(x)$ and χ_N from (1.20). For $x \in \mathbb{R}$, we also define $s_N(x) = N\sigma_I \ell_N(x)$, so that

$$s_N(x) = b_N + \frac{x - \frac{9}{8} \log b_N}{a_N}, \quad \ell_N(x) = \frac{s_N(x)}{N\sigma_I}, \quad \chi_N = a_N N\sigma_I. \quad (9.1)$$

These definitions are chosen so that, for $i \in \mathcal{G}_N(I)$,

$$\tau_{i,N} - x = \chi_N (\lambda_{i+1} - \lambda_i - \ell_N(x)). \quad (9.2)$$

Note that $s_N(x) = b_N + O_x(\log \log N / \sqrt{\log N})$, where O_x indicates that the implicit constant depends on x .

Proposition 9.1. *For every $x \in \mathbb{R}$, we have*

$$\lim_{N \rightarrow \infty} \chi_N \int_{y_-}^{y_+ - \ell_N(x)} E_{1,N}((y, y + \ell_N(x))) \, dy = \Lambda_I e^{-x}. \quad (9.3)$$

Moreover, there exist $C_x, N_x > 0$ such that for all $N \geq N_x$,

$$\sup_{y \in [y_-, y_+ - \ell_N(x)]} \chi_N E_{1,N}((y, y + \ell_N(x))) \leq C_x \log N. \quad (9.4)$$

Proof. Throughout the proof, x is fixed, and we abbreviate $s_N = s_N(x)$ and $\ell_N = \ell_N(x)$. We begin with the value of q_1 at s_N . Recall that $F(b_N) = \log N$, $F'(b_N) = a_N$, and $F'' = 1/32$. Taylor's theorem then gives

$$F(s_N) = \log N + x - \frac{9}{8} \log b_N + \frac{1}{64} \left(\frac{x - \frac{9}{8} \log b_N}{a_N} \right)^2 = \log N + x - \frac{9}{8} \log b_N + o(1), \quad (9.5)$$

where we used $a_N \geq c\sqrt{\log N}$. Using Lemma 7.1(a), (9.5), and $s_N/b_N \rightarrow 1$, we have

$$q_1(s_N) = \kappa_1 s_N^{-1/8} e^{-F(s_N)} (1 + o(1)) = \frac{\kappa_1 b_N e^{-x}}{N} (1 + o(1)). \quad (9.6)$$

For $y \in I$, we set

$$\Psi(y) = \sigma(y)/\sigma_I \geq 1, \quad (9.7)$$

and for $y \in [y_-, y_+ - \ell_N]$, we set

$$s(y) = N\sigma(y)\ell_N = s_N\Psi(y), \quad (9.8)$$

so that

$$(y, y + \ell_N) = (y, y + s(y)/(N\sigma(y))). \quad (9.9)$$

Since $I \subset (-2, 2)$ is compact and $s(y) \leq C\sqrt{\log N}$, Corollary 7.5 gives

$$E_{1,N}((y, y + \ell_N)) = q_1(s_N\Psi(y))(1 + o(1)) \quad (9.10)$$

uniformly in $y \in [y_-, y_+ - \ell_N]$.

Next, (9.10) and (9.6) imply that

$$\chi_N E_{1,N}((y, y + \ell_N)) \leq C\chi_N q_1(s_N) \leq C_x a_N b_N, \quad (9.11)$$

where we used that q_1 is nonincreasing and $\Psi \geq 1$. This implies (9.4), since $a_N b_N \leq 2\log N$ by (1.4).

We now prove (9.3). Let $d_0 \in (0, |I|/2)$ be a small constant, which will be chosen below depending only on I , and let $\mathcal{U}$ denote the set of $y \in I$ with $\text{dist}(y, \partial_* I) < d_0$. Since Ψ is continuous on I and attains the value 1 only on $\partial_* I$, there exists $c_0 > 0$ such that $\Psi \geq 1 + c_0$ on $I \setminus \mathcal{U}$. It follows from Lemma 7.1(a) and (9.5) that

$$q_1((1 + c_0)s_N) \leq C e^{-(1+c_0)^2 s_N^2 / 64} \leq N^{-(1+c_0)^2 + o(1)}, \quad (9.12)$$

so (9.10) and the monotonicity of q_1 imply that the contribution of $I \setminus \mathcal{U}$ to the left side of (9.3) is bounded by $C\chi_N N^{-(1+c_0)^2 + o(1)} \rightarrow 0$.

Fix $\xi \in \partial_* I$. Recall that

$$|\xi| = \max(|y_-|, |y_+|) > 0, \quad (9.13)$$

since I is nondegenerate. We parametrize the points of $\mathcal{U}$ near ξ as $y = \xi - \text{sgn}(\xi)t$ with $t \in [0, d_0]$, so that t is the distance from y to ξ , and y moves into the interior of I as t increases. Note that $\sigma'(\xi) = -\xi/\sigma_I$, which gives

$$\Psi(\xi - \text{sgn}(\xi)t) = 1 + \frac{|\xi|}{\sigma_I^2} t + O(t^2). \quad (9.14)$$

We choose d_0 so small that the function $t \mapsto \Psi(\xi - \text{sgn}(\xi)t)$ is increasing on $[0, d_0]$, with derivative between $|\xi|/(2\sigma_I^2)$ and $2|\xi|/\sigma_I^2$.

We now change variables by setting

$$z = (\log N)(\Psi(y) - 1), \quad (9.15)$$

so that $s(y) = s_N(1 + z/\log N)$, and $dt = \sigma_I^2(|\xi| \log N)^{-1}(1 + O(t)) dz$. In the integral, t ranges over $[0, d_0)$ if $\xi = y_-$, and over $[\ell_N, d_0)$ if $\xi = y_+$. In both cases, we let $[\underline{z}_N, \bar{z}_N)$ denote the corresponding range of z . We have

$$0 \leq \underline{z}_N \leq C\ell_N \log N \rightarrow 0 \quad (9.16)$$

and $\bar{z}_N \geq c \log N$.

Setting $w = z/\log N \geq 0$, we have

$$F(s_N(1 + w)) - F(s_N) = \frac{s_N^2}{64}(2w + w^2) + \frac{s_N}{8}w. \quad (9.17)$$

By (9.5), we have $s_N^2/64 = \log N + O(\sqrt{\log N})$. Therefore, the right side of (9.17) equals $2z + o(1)$ uniformly for z in compact sets. Further, it is at least z for all $z \geq 0$ when N is large. Combining this with Lemma 7.1(a), which shows that the ratio $q_1(s)/(\kappa_1 s^{-1/8} e^{-F(s)})$ takes values in $[1/2, 2]$ for $s \geq s_N$ and N large, we find

$$\lim_{N \rightarrow \infty} \frac{q_1(s_N(1 + z/\log N))}{q_1(s_N)} = e^{-2z}, \quad \frac{q_1(s_N(1 + z/\log N))}{q_1(s_N)} \leq 4e^{-z}, \quad (9.18)$$

where the limit is locally uniform in $z \geq 0$, and the bound holds for all $z \geq 0$ and N large.

It follows from (9.10) that the contribution of the part of $\mathcal{U}$ near ξ to the left side of (9.3) is

$$(1 + o(1)) \chi_N q_1(s_N) \frac{\sigma_I^2}{|\xi| \log N} \int_{\underline{z}_N}^{\bar{z}_N} \frac{q_1(s_N(1 + z/\log N))}{q_1(s_N)} (1 + O(t)) dz. \quad (9.19)$$

By (9.18) and dominated convergence, the integral converges to

$$\int_0^\infty e^{-2z} dz = 1/2. \quad (9.20)$$

Using (9.6) and $\chi_N = a_N N \sigma_I$, we find that the contribution converges to

$$\lim_{N \rightarrow \infty} \frac{a_N b_N}{2 \log N} \cdot \kappa_1 \frac{\sigma_I^3}{|\xi|} e^{-x} = \kappa_1 \frac{\sigma_I^3}{|\xi|} e^{-x}, \quad (9.21)$$

where we used that $a_N b_N / (2 \log N) \rightarrow 1$, which follows from (1.4). Summing over $\xi \in \partial_* I$, we obtain (9.3). $\square$

Remark 9.2. The proof of Proposition 9.1 explains the form of the centering in Theorem 1.1. Heuristically, the integral of $E_{1,N}((y, y + \ell))$ over y is the expected value of $\sum_i (\lambda_{i+1} - \lambda_i - \ell)_+$, so the expected number of gaps longer than ℓ is minus its derivative in ℓ . With $\ell = s/(N \sigma_I)$, the proof shows that the integral is approximately $q_1(s) \sigma_I^2 / (2|\xi| \log N)$ for each $\xi \in \partial_* I$, and differentiating $q_1(s) \approx \kappa_1 s^{-1/8} e^{-F(s)}$ produces a factor $F'(s) \approx s/32$. So

the expected number of gaps longer than $s/(N\sigma_I)$ is of order $Ns q_1(s)/\log N$, which is of order $Nq_1(s)/s$ since $s^2 \approx 64 \log N$. This number is of order one when $Nq_1(s) \approx s \approx b_N$, or equivalently when $F(s) \approx \log N - (9/8) \log b_N$. The linear term $s/8$ in F , which comes from the term $\mp t/2$ in (1.14), produces the constant -4 in (1.10). By contrast, for the GUE the corresponding exponent $s^2/32$ has no linear term.

Factorial moments and the proof of Theorem 1.1

In this section, we complete the proofs of Theorem 1.1 and Corollary 1.2. We follow the approach of Feng and Wei [21], who reduced the Poisson convergence to the convergence of the moments in (10.1) below; we state and prove this reduction as Lemma A.1 in Appendix A. The new input is the use of the estimates for several intervals from Section 8 in place of the negative association of determinantal processes used in [21].

10.1 The positive-part moments

Fix $k \geq 1$ and $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$, and set $\ell_j = \ell_N(x_j)$, with $\ell_N(x)$ as in (9.1). We work with N sufficiently large that $0 < \ell_j < |I|$ for every j . We define

$$M_{N,k}(\mathbf{x}) = \mathbb{E} \sum_{\substack{i_1, \dots, i_k \in \mathcal{G}_N(I) \\ \text{distinct}}} \prod_{j=1}^k (\tau_{i_j, N} - x_j)_+, \quad (10.1)$$

where we write $(t)_+ = \max(t, 0)$. For a realization of the eigenvalues, we let $\mathcal{Y}_N$ denote the set of $\mathbf{y} = (y_1, \dots, y_k) \in \prod_{j=1}^k [y_-, y_+ - \ell_j]$ for which there exist distinct $i_1, \dots, i_k \in \mathcal{G}_N(I)$ such that

$$[y_j, y_j + \ell_j] \subset (\lambda_{i_j}, \lambda_{i_j+1}), \quad 1 \leq j \leq k. \quad (10.2)$$

Lemma 10.1. *We have*

$$M_{N,k}(\mathbf{x}) = \chi_N^k \int_{\prod_j [y_-, y_+ - \ell_j]} \mathbb{P}(\mathbf{y} \in \mathcal{Y}_N) \, d\mathbf{y}. \quad (10.3)$$

Proof. Consider a gap $(\lambda_i, \lambda_{i+1})$ and $\ell > 0$. The set of y such that $[y, y + \ell] \subset (\lambda_i, \lambda_{i+1})$ is an interval of length $(\lambda_{i+1} - \lambda_i - \ell)_+$. When $i \in \mathcal{G}_N(I)$, this interval is a subset of $[y_-, y_+ - \ell]$. Given $\mathbf{y} \in \mathcal{Y}_N$, the indices $i_1, \dots, i_k$ in (10.2) are determined by $\mathbf{y}$, since the gaps are disjoint. Therefore, $\mathcal{Y}_N$ is the disjoint union, over ordered k -tuples of distinct indices in $\mathcal{G}_N(I)$, of products of such intervals, and

$$|\mathcal{Y}_N| = \sum_{\substack{i_1, \dots, i_k \in \mathcal{G}_N(I) \\ \text{distinct}}} \prod_{j=1}^k (\lambda_{i_j+1} - \lambda_{i_j} - \ell_j)_+. \quad (10.4)$$

By (9.2), the right side equals χ_N^{-k} times the sum in (10.1). Taking expectations and using Fubini's theorem, we obtain (10.3). $\square$

Note that the event $\{\mathbf{y} \in \mathcal{Y}_N\}$ implies that the set $\bigcup_j (y_j, y_j + \ell_j)$ contains no eigenvalue. The converse holds when the intervals are separated, provided that every interval of moderate length in I contains an eigenvalue. To make this precise, we set

$$\eta_N = N^{-1/3}, \quad (10.5)$$

and we let $\mathcal{R}_N$ denote the event that every closed interval $B \subset I$ with $|B| \geq \eta_N$ contains an eigenvalue of H_N .

Lemma 10.2. *For every $D > 0$, there exists $N_0 > 0$ such that $\mathbb{P}(\mathcal{R}_N^c) \leq N^{-D}$ for $N \geq N_0$.*

Proof. Let $\gamma > 0$ be a parameter, which will be chosen at the end of the proof. Let $\delta > 0$ be such that $I \subset [-2 + \delta, 2 - \delta]$, and set $s = \gamma \sqrt{\log N}$. We define

$$B_m = [y_- + m\eta_N/2, y_- + (m+1)\eta_N/2] \quad (10.6)$$

for the integers $m \geq 0$ with $B_m \subset I$; there are at most $2|I|N^{1/3}$ of them. Every closed interval $B \subset I$ with $|B| \geq \eta_N$ contains one of the B_m .

Let y denote the left endpoint of B_m . Since $s/(N\sigma(y)) \leq \eta_N/2$ for $N \geq N_0$, the interval $(y, y + s/(N\sigma(y)))$ contains no eigenvalue whenever B_m contains none. Using Corollary 7.5 and Lemma 7.1(a), we find that the hole probability of this interval is bounded by

$$2q_1(s) \leq Ce^{-F(s)} \leq CN^{-\gamma^2/64}. \quad (10.7)$$

A union bound over m gives $\mathbb{P}(\mathcal{R}_N^c) \leq CN^{1/3-\gamma^2/64}$, and we choose γ so that $\gamma^2/64 > D + 1$. $\square$

Remark 10.3. Lemma 10.2 also follows from the rigidity of eigenvalues proved in [18, Theorem 2.2], after a rescaling of the GOE by the factor $(1 + 1/N)^{-1/2}$ that makes the row sums of the variance matrix equal to one. However, the argument above requires only the one-interval estimate.

Proposition 10.4. *For every $k \geq 1$ and $\mathbf{x} \in \mathbb{R}^k$, we have*

$$\lim_{N \rightarrow \infty} M_{N,k}(\mathbf{x}) = \prod_{j=1}^k \Lambda_I e^{-x_j}. \quad (10.8)$$

Proof. We define

$$\mathcal{H}_j(y) = E_{1,N}((y, y + \ell_j)), \quad \mathcal{H}(\mathbf{y}) = E_{1,N}\left(\bigcup_{j=1}^k (y_j, y_j + \ell_j)\right). \quad (10.9)$$

By Proposition 9.1, we have

$$\lim_{N \rightarrow \infty} \chi_N \int_{y_-}^{y_+ - \ell_j} \mathcal{H}_j(y) dy = \Lambda_I e^{-x_j}, \quad \sup_{y \in [y_-, y_+ - \ell_j]} \chi_N \mathcal{H}_j(y) \leq C \log N. \quad (10.10)$$

Here, and for the rest of this proof, the constants C may depend on k , $\mathbf{x}$, and I . Let $\mathcal{S}_N$ denote the set of $\mathbf{y} \in \prod_j [y_-, y_+ - \ell_j]$ such that

$$|y_i - y_j| \geq 4\eta_N \quad (i \neq j), \quad y_j - y_- \geq 4\eta_N, \quad y_+ - (y_j + \ell_j) \geq 4\eta_N, \quad (10.11)$$

and let $\mathcal{S}_N^c$ denote its complement in $\prod_j [y_-, y_+ - \ell_j]$. We may assume that $\ell_j \leq \eta_N$ for all j , since $\ell_j \leq C\sqrt{\log N}/N$.

We first bound the integral of the product of the $\mathcal{H}_j$ over $\mathcal{S}_N^c$. The set $\mathcal{S}_N^c$ is covered by the sets where $|y_i - y_j| < 4\eta_N$ for some $i \neq j$, and the sets where y_j lies within $4\eta_N$ of y_- or of $y_+ - \ell_j$ for some j . To bound the integral over the set where $|y_i - y_j| < 4\eta_N$, we integrate first in y_i , bound $\mathcal{H}_i$ by its supremum, and integrate the remaining factors over their full ranges. Using (10.10), we see that the contribution of this set to $\chi_N^k \int_{\mathcal{S}_N^c} \prod_j \mathcal{H}_j(y_j) d\mathbf{y}$ is bounded by $C\eta_N \log N$. The boundary sets are treated in the same way. Summing these contributions, we have

$$\chi_N^k \int_{\mathcal{S}_N^c} \prod_{j=1}^k \mathcal{H}_j(y_j) d\mathbf{y} \leq C\eta_N \log N. \quad (10.12)$$

We now prove the upper bound in (10.8). Suppose first that two of the intervals $[y_j, y_j + \ell_j]$ intersect. Then they cannot be contained in two disjoint gaps, so $\mathbb{P}(\mathbf{y} \in \mathcal{Y}_N) = 0$. Otherwise, we have

$$\mathbb{P}(\mathbf{y} \in \mathcal{Y}_N) \leq \mathcal{H}(\mathbf{y}), \quad (10.13)$$

and the intervals $(y_j, y_j + \ell_j)$ have disjoint closures and lie in I . Set

$$L = N \sum_j \ell_j \leq C\sqrt{\log N}. \quad (10.14)$$

By (8.22), we have

$$\mathcal{H}(\mathbf{y}) \leq Ce^{C\sqrt{\log N}} \prod_j \mathcal{H}_j(y_j). \quad (10.15)$$

When moreover $\mathbf{y} \in \mathcal{S}_N$, the intervals are separated by at least $3\eta_N \geq N^{-1/2}$, and (8.23) gives

$$\mathcal{H}(\mathbf{y}) = (1 + O(e^{C\sqrt{\log N}} N^{-1/4})) \prod_j \mathcal{H}_j(y_j). \quad (10.16)$$

Inserting these bounds into (10.3) and using (10.12), we find

$$M_{N,k}(\mathbf{x}) \leq (1 + O(e^{C\sqrt{\log N}} N^{-1/4})) \prod_{j=1}^k \left(\chi_N \int_{y_-}^{y_+ - \ell_j} \mathcal{H}_j \right) + Ce^{C\sqrt{\log N}} \eta_N \log N. \quad (10.17)$$

By (10.10), the right side converges to $\prod_j \Lambda_I e^{-x_j}$.

To prove the lower bound, suppose that $\mathbf{y} \in \mathcal{S}_N$, that the set $\bigcup_j (y_j, y_j + \ell_j)$ contains no eigenvalue, and that $\mathcal{R}_N$ holds. We order the intervals from left to right. The interval between y_- and the first prescribed interval, the intervals between consecutive prescribed intervals, and the interval between the last prescribed interval and y_+ all have length at least $3\eta_N$ and lie in I , so each contains an eigenvalue. Further, the points y_j and $y_j + \ell_j$ are almost surely not eigenvalues. Hence each prescribed interval $[y_j, y_j + \ell_j]$ is contained in a gap $(\lambda_i, \lambda_{i+1})$ with $\lambda_i, \lambda_{i+1} \in I$, and different prescribed intervals belong to different gaps. This shows that $\mathbf{y} \in \mathcal{Y}_N$, and therefore

$$\mathbb{P}(\mathbf{y} \in \mathcal{Y}_N) \geq \mathcal{H}(\mathbf{y}) - \mathbb{P}(\mathcal{R}_N^c), \quad \mathbf{y} \in \mathcal{S}_N. \quad (10.18)$$

Using (10.3), (8.23), Lemma 10.2 with $D = k + 1$, and $\chi_N \leq CN\sqrt{\log N}$, we have

$$M_{N,k}(\mathbf{x}) \geq (1 - O(e^{C\sqrt{\log N}} N^{-1/4})) \chi_N^k \int_{\mathcal{S}_N} \prod_{j=1}^k \mathcal{H}_j(y_j) d\mathbf{y} - C(N\sqrt{\log N})^k N^{-k-1}. \quad (10.19)$$

Finally, (10.12) implies that χ_N^k times the difference between the integrals of $\prod_j \mathcal{H}_j(y_j)$ over $\mathcal{S}_N$ and over $\prod_j [y_-, y_+ - \ell_j]$ is $o(1)$. By (10.10), the latter integral, multiplied by χ_N^k , converges to $\prod_j \Lambda_I e^{-x_j}$. This completes the proof. $\square$

10.2 Proofs of the main results

Proof of Theorem 1.1. Proposition 10.4 shows that the point processes $\mathcal{X}_N$ satisfy the hypothesis of Lemma A.1 with $\Lambda = \Lambda_I$. Hence Theorem 1.1 follows from Lemma A.1. $\square$

Proof of Corollary 1.2. For $x \in \mathbb{R}$, let $\mathcal{X}_N((x, \infty))$ denote the number of $i \in \mathcal{G}_N(I)$ with $\tau_{i,N} > x$. Then

$$\{\tau_{k,N}^\downarrow \leq x\} = \{\mathcal{X}_N((x, \infty)) \leq k - 1\}, \quad (10.20)$$

with the convention that $\tau_{k,N}^\downarrow = -\infty$ if $|\mathcal{G}_N(I)| < k$. By Theorem 1.1, $\mathcal{X}_N((x, \infty))$ converges in distribution to a Poisson random variable with mean $\Lambda_I e^{-x}$, which gives (1.8). Taking $k = 1$, we obtain (1.9).

We now prove (1.10). Let $\mathcal{E}_N = \{\mathcal{G}_N(I) \neq \emptyset\}$. By (1.8) with $k = 1$, we have $\limsup_{N \rightarrow \infty} \mathbb{P}(\mathcal{E}_N^c) \leq \exp(-\Lambda_I e^{-x})$ for every $x \in \mathbb{R}$. Letting $x \rightarrow -\infty$ shows that $\mathbb{P}(\mathcal{E}_N) \rightarrow 1$. Let $\tilde{\tau}_{1,N}$ equal $\tau_{1,N}^\downarrow$ on $\mathcal{E}_N$ and 0 on its complement. Then (1.9) implies that $\tilde{\tau}_{1,N}$ is tight. On $\mathcal{E}_N$, the definition (1.7) gives

$$N\sigma_I m_{1,N} = b_N + a_N^{-1}(\tilde{\tau}_{1,N} - (9/8) \log b_N). \quad (10.21)$$

From (1.4), we have

$$\begin{aligned} b_N &= 8\sqrt{\log N} - 4 + O((\log N)^{-1/2}), \\ \frac{1}{a_N} &= \frac{4}{\sqrt{\log N + 1/4}} = \frac{4}{\sqrt{\log N}} + O((\log N)^{-3/2}), \end{aligned} \quad (10.22)$$

and $\log b_N = (1/2) \log \log N + \log 8 + o(1)$. Inserting these expansions on $\mathcal{E}_N$ and using $\mathbb{P}(\mathcal{E}_N) \rightarrow 1$, we obtain (1.10). $\square$

Proof of the claims in Remark 1.4. Given a real symmetric $N \times N$ matrix M with eigenvalues $\mu_1 \leq \dots \leq \mu_N$, a set $I' \subset \mathbb{R}$, and $k \geq 1$, we let $T_k(M; I')$ denote the k -th largest of the gaps $\mu_{i+1} - \mu_i$ with $\mu_i \in I'$, with $T_k(M; I') = 0$ if there are fewer than k such gaps. We also set

$$\widehat{\tau}_k(M; I') = a_N(N\sigma_I T_k(M; I') - b_N) + \frac{9}{8} \log b_N, \quad (10.23)$$

so that the quantity $\widehat{\tau}_{k,N}$ in Remark 1.4 is $\widehat{\tau}_k(W_N; I)$. Throughout the proof, we fix $k \geq 1$, and we let $F_k(x)$ denote the right side of (1.8).

We first compare W_N with a Gaussian matrix. Set

$$\theta_N = (1 + 1/N)^{-1/2} \quad (10.24)$$

and $G_N = \theta_N H_N$. The entries $(G_N)_{ij}$, $i \leq j$, are independent centered Gaussian random variables with variance $(1 + \mathbf{1}_{i=j})/(N+1)$. Hence every row of the variance matrix of G_N sums to 1, and G_N is the GOE matrix in the normalization of [33, Definition 1.1].

Let $\Phi \in C_c^\infty(\mathbb{R})$, and set $\nu = N/\sqrt{\log N}$ and

$$S_N(t) = \Phi\left(a_N\sigma_I\sqrt{\log N}t - a_Nb_N + \frac{9}{8} \log b_N\right), \quad t \in \mathbb{R}, \quad (10.25)$$

so that

$$\Phi(\widehat{\tau}_k(M; I)) = S_N(\nu T_k(M; I)). \quad (10.26)$$

The derivatives of S_N of order at most five are bounded by $C(\log N)^5$, since $a_N\sigma_I\sqrt{\log N} \leq C \log N$. Using [33, Corollary 1.6], whose error term involves derivatives of the test function of order at most five, we find

$$|\mathbb{E}\Phi(\widehat{\tau}_k(W_N; I)) - \mathbb{E}\Phi(\widehat{\tau}_k(G_N; I))| \leq CN^{-c}(\log N)^5. \quad (10.27)$$

Next, we show that $\mathbb{P}(\widehat{\tau}_k(G_N; I) \leq x) \rightarrow F_k(x)$ for all $x \in \mathbb{R}$. Set $I_N = \theta_N^{-1}I$. The matrix G_N has eigenvalues $\theta_N\lambda_i$, so $T_k(G_N; I) = \theta_N T_k(H_N; I_N)$. Fix $x \in \mathbb{R}$ and $\eta > 0$. By (10.23) and (9.1), the events $\{\widehat{\tau}_k(G_N; I) > x\}$ and $\{T_k(H_N; I_N) > \ell_N(x)/\theta_N\}$ coincide for N large. Note that $\ell_N(x + \eta) - \ell_N(x) = \eta/\chi_N$ is much larger than $\ell_N(x)(\theta_N^{-1} - 1) \leq C\sqrt{\log N}/N^2$. Hence, for N large, we have

$$\ell_N(x) \leq \ell_N(x)/\theta_N \leq \ell_N(x + \eta), \quad (10.28)$$

and therefore

$$\mathbb{P}(\widehat{\tau}_k(H_N; I_N) > x + \eta) \leq \mathbb{P}(\widehat{\tau}_k(G_N; I) > x) \leq \mathbb{P}(\widehat{\tau}_k(H_N; I_N) > x). \quad (10.29)$$

It remains to compare $\widehat{\tau}_k(H_N; I_N)$ with $\tau_{k,N}^\downarrow$. Let $x' \in \mathbb{R}$. The events $\{\widehat{\tau}_k(H_N; I_N) > x'\}$ and $\{\tau_{k,N}^\downarrow > x'\}$ can differ only if there exists a gap $(\lambda_i, \lambda_{i+1})$ of length at least $\ell_N(x')$ such that exactly one of the conditions $\lambda_i \in I_N$ and $i \in \mathcal{G}_N(I)$ holds. The closure of such a gap contains a point within distance $2/N$ of y_- or of y_+ , since $|\theta_N^{-1}y_\pm - y_\pm| \leq 2/N$. A gap of length at least $\ell_N(x')$ whose closure contains a point within distance $2/N$ of $y_\pm$ contains one of the intervals

$$\left(y_\pm - \frac{\ell_N(x')}{2} + \frac{2}{N}, y_\pm - \frac{2}{N}\right), \quad \left(y_\pm + \frac{2}{N}, y_\pm + \frac{\ell_N(x')}{2} - \frac{2}{N}\right). \quad (10.30)$$

These four intervals lie in $[-2 + \delta, 2 - \delta]$ for some $\delta > 0$ depending on I . Further, $\sigma(y) \geq \sigma_I - C\ell_N(x')$ on them, so each has the form $(y, y + s/(N\sigma(y)))$ with $s \geq s_N(x')/2 - C$, where we recall that

$$s_N(x') = N\sigma_I\ell_N(x'). \quad (10.31)$$

Using Corollary 7.5, Lemma 7.1(a), and the monotonicity of q_1 , we see that the probability that one of them contains no eigenvalue is bounded by $Cq_1(s_N(x')/2 - C) \leq N^{-1/4+o(1)}$. Therefore,

$$|\mathbb{P}(\widehat{\tau}_k(H_N; I_N) > x') - \mathbb{P}(\tau_{k,N}^\downarrow > x')| \leq N^{-1/4+o(1)} \quad (10.32)$$

for every fixed x' . Inserting this with $x' = x$ and $x' = x + \eta$ into (10.29) and using (1.8), we find that every limit point of $\mathbb{P}(\widehat{\tau}_k(G_N; I) \leq x)$ belongs to $[F_k(x), F_k(x + \eta)]$. Letting $\eta \downarrow 0$ and using the continuity of F_k , we have

$$\mathbb{P}(\widehat{\tau}_k(G_N; I) \leq x) \rightarrow F_k(x). \quad (10.33)$$

Since F_k is the distribution function of a probability measure on $\mathbb{R}$, it follows that

$$\mathbb{E}\Phi(\widehat{\tau}_k(G_N; I)) \rightarrow \int \Phi dF_k \quad (10.34)$$

for all $\Phi \in C_c^\infty(\mathbb{R})$. Then (10.27) implies that the same holds for W_N in place of G_N . Convergence of $\mathbb{E}\Phi$ for all $\Phi \in C_c^\infty(\mathbb{R})$ to the integral against a probability measure implies convergence in distribution, so

$$\mathbb{P}(\widehat{\tau}_k(W_N; I) \leq x) \rightarrow F_k(x) \quad (10.35)$$

for all $x \in \mathbb{R}$. □

A criterion for Poisson convergence

In this appendix, we prove the criterion used in the proof of Theorem 1.1. It is a variant of [21, Lemma 1], where the conclusion is stated for half-lines.

Lemma A.1. *Let $\Lambda > 0$, and for each N , let $\mathcal{X}_N = \sum_{i \in \mathcal{I}_N} \delta_{X_{i,N}}$ be a point process on $\mathbb{R}$ with an almost surely finite index set $\mathcal{I}_N$. Suppose that for every $k \geq 1$ and $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \sum_{\substack{i_1, \dots, i_k \in \mathcal{I}_N \\ \text{distinct}}} \prod_{j=1}^k (X_{i_j, N} - x_j)_+ = \prod_{j=1}^k \Lambda e^{-x_j}. \quad (\text{A.1})$$

Let $A_1, \dots, A_m \subset \mathbb{R}$ be disjoint intervals, each of the form $(a, b]$ with $a < b$ or (a, ∞) . Then $(\mathcal{X}_N(A_1), \dots, \mathcal{X}_N(A_m))$ converges in distribution to a vector of independent Poisson random variables with means $\Lambda \int_{A_l} e^{-x} dx$, $1 \leq l \leq m$. Consequently, $\mathcal{X}_N$ converges in distribution to $\text{PPP}(\Lambda e^{-x} dx)$ with respect to the vague topology.

Proof. For $\mathbf{x} \in \mathbb{R}^k$, we define

$$\mathcal{I}_N(\mathbf{x}) = \mathbb{E} \sum_{\substack{i_1, \dots, i_k \in \mathcal{I}_N \\ \text{distinct}}} \prod_{j=1}^k \mathbf{1}\{X_{i_j, N} > x_j\}, \quad f(x) = \Lambda e^{-x}. \quad (\text{A.2})$$

We first show that $\mathcal{I}_N(\mathbf{x}) \rightarrow \prod_j f(x_j)$. Given $h > 0$ and $x \in \mathbb{R}$, we set

$$u_{x,h}(t) = h^{-1}((t - x)_+ - (t - x - h)_+), \quad (\text{A.3})$$

so that

$$\mathbf{1}\{t > x + h\} \leq u_{x,h}(t) \leq \mathbf{1}\{t > x\}. \quad (\text{A.4})$$

Expanding the product $\prod_j u_{x_j, h}(X_{i_j, N})$ into 2^k terms and applying (A.1) to each of them, we find

$$\lim_{N \rightarrow \infty} \mathbb{E} \sum_{\text{distinct}} \prod_{j=1}^k u_{x_j, h}(X_{i_j, N}) = \prod_{j=1}^k \frac{f(x_j) - f(x_j + h)}{h}. \quad (\text{A.5})$$

Combining this with (A.4), we find

$$\limsup_{N \rightarrow \infty} \mathcal{I}_N(\mathbf{x}) \leq \prod_{j=1}^k \frac{f(x_j - h) - f(x_j)}{h}, \quad \liminf_{N \rightarrow \infty} \mathcal{I}_N(\mathbf{x}) \geq \prod_{j=1}^k \frac{f(x_j) - f(x_j + h)}{h}, \quad (\text{A.6})$$

where in the first bound we used (A.4) with $x_j - h$ in place of x_j . Letting $h \downarrow 0$, we conclude that

$$\mathcal{I}_N(\mathbf{x}) \rightarrow \prod_j (-f'(x_j)) = \prod_j f(x_j). \quad (\text{A.7})$$

Next, let $k_1, \dots, k_m \geq 0$. We define the falling factorial by

$$(n)_k = n(n-1) \cdots (n-k+1). \quad (\text{A.8})$$

Since the intervals A_l are disjoint, $\mathbb{E} \prod_l (\mathcal{X}_N(A_l))_{k_l}$ is the expected number of families of distinct indices $(i_{l,p})_{1 \leq l \leq m, 1 \leq p \leq k_l}$ with $X_{i_{l,p},N} \in A_l$ for all l and p . To compute this expectation, we write

$$\mathbf{1}\{t \in (a, b]\} = \mathbf{1}\{t > a\} - \mathbf{1}\{t > b\} \quad (\text{A.9})$$

and

$$\mathbf{1}\{t \in (a, \infty)\} = \mathbf{1}\{t > a\}, \quad (\text{A.10})$$

and we expand the product of these indicators. This expresses $\mathbb{E} \prod_l (\mathcal{X}_N(A_l))_{k_l}$ as a finite linear combination of the quantities $\mathcal{I}_N(\mathbf{x})$, with $\mathbf{x}$ having coordinates among the endpoints of the A_l . Using the first part of the proof, we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \prod_{l=1}^m (\mathcal{X}_N(A_l))_{k_l} = \prod_{l=1}^m \left(\Lambda \int_{A_l} e^{-x} dx \right)^{k_l}, \quad (\text{A.11})$$

which is the corresponding factorial moment of the claimed Poisson vector. The method of moments then gives the convergence in distribution of $(\mathcal{X}_N(A_1), \dots, \mathcal{X}_N(A_m))$, since a vector of independent Poisson random variables is determined by its moments, and moments are finite linear combinations of factorial moments.

Finally, let $\mathcal{X}$ be a Poisson point process on $\mathbb{R}$ with intensity $\Lambda e^{-x} dx$. It is simple and has a nonatomic intensity measure. The convergence of count vectors proved above implies that $\mathcal{X}_N(B)$ converges in distribution to $\mathcal{X}(B)$ for every finite union B of disjoint bounded intervals of the form $(a, b]$. This implies local tightness of the counts, convergence of the void probabilities, and convergence of the probabilities of at least two points. The one-dimensional version of Kallenberg's criterion [30], as stated in [4, Theorem 2.2] for point processes on $\mathbb{R}^2$, therefore gives the convergence in distribution of $\mathcal{X}_N$ to $\mathcal{X}$ with respect to the vague topology. Intervals of other types are handled by approximation using the expected-count limits above, since the limiting intensity has no atoms; see also [4, Proposition 2.1] for the same deduction. This completes the proof. $\square$

References

- [1] M. Adler, P. J. Forrester, T. Nagao, and P. van Moerbeke. Classical skew orthogonal polynomials and random matrices. *J. Stat. Phys.*, 99(1-2):141–170, 2000. doi: 10.1023/A:1018644606835.
- [2] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. doi: 10.1017/CBO9780511801334.
- [3] Jinho Baik and Eric M. Rains. Algebraic aspects of increasing subsequences. *Duke Math. J.*, 109(1):1–65, 2001. doi: 10.1215/S0012-7094-01-10911-3.
- [4] Gérard Ben Arous and Paul Bourgade. Extreme gaps between eigenvalues of random matrices. *Ann. Probab.*, 41(4):2648–2681, 2013. doi: 10.1214/11-AOP710.
- [5] Paul Bourgade. Extreme gaps between eigenvalues of Wigner matrices. *J. Eur. Math. Soc. (JEMS)*, 24(8):2823–2873, 2022. doi: 10.4171/JEMS/1141.
- [6] Christophe Charlier. Largest gaps between bulk eigenvalues of unitary-invariant random Hermitian matrices. Preprint, arXiv:2602.07524, 2026.
- [7] N. G. de Bruijn. On some multiple integrals involving determinants. *J. Indian Math. Soc. (N.S.)*, 19:133–151, 1955.
- [8] P. Deift, A. Its, I. Krasovsky, and X. Zhou. The Widom–Dyson constant for the gap probability in random matrix theory. *J. Comput. Appl. Math.*, 202(1):26–47, 2007. doi: 10.1016/j.cam.2005.12.040.
- [9] Percy Deift and Dimitri Gioev. Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices. *Comm. Pure Appl. Math.*, 60(6):867–910, 2007. doi: 10.1002/cpa.20164.
- [10] Percy Deift and Dimitri Gioev. Universality in random matrix theory for orthogonal and symplectic ensembles. *Int. Math. Res. Pap. IMRP*, 2007:Art. ID rpm004, 116 pp., 2007. doi: 10.1093/imrp/rpm004.
- [11] Percy Deift and Dimitri Gioev. *Random matrix theory: invariant ensembles and universality*, volume 18 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2009.
- [12] J. des Cloizeaux and M. L. Mehta. Asymptotic behavior of spacing distributions for the eigenvalues of random matrices. *J. Math. Phys.*, 14:1648–1650, 1973. doi:

- 10.1063/1.1666239.
- [13] F. J. Dyson. Fredholm determinants and inverse scattering problems. *Comm. Math. Phys.*, 47:171–183, 1976. doi: 10.1007/BF01608375.
 - [14] Freeman J. Dyson. Statistical theory of the energy levels of complex systems. I. *J. Math. Phys.*, 3:140–156, 1962. doi: 10.1063/1.1703773.
 - [15] Torsten Ehrhardt. Dyson’s constant in the asymptotics of the Fredholm determinant of the sine kernel. *Comm. Math. Phys.*, 262(2):317–341, 2006. doi: 10.1007/s00220-005-1493-4.
 - [16] Torsten Ehrhardt. Dyson’s constants in the asymptotics of the determinants of Wiener–Hopf–Hankel operators with the sine kernel. *Comm. Math. Phys.*, 272(3): 683–698, 2007. doi: 10.1007/s00220-007-0239-x.
 - [17] László Erdős and Horng-Tzer Yau. Gap universality of generalized Wigner and β -ensembles. *J. Eur. Math. Soc. (JEMS)*, 17(8):1927–2036, 2015. doi: 10.4171/JEMS/548.
 - [18] László Erdős, Horng-Tzer Yau, and Jun Yin. Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.*, 229(3):1435–1515, 2012. doi: 10.1016/j.aim.2011.12.010.
 - [19] Renjie Feng and Stephen Muirhead. Poisson approximation of the largest gaps between zeros of a stationary Gaussian process. Preprint, arXiv:2605.22146, 2026.
 - [20] Renjie Feng and Dongyi Wei. Small gaps of circular β -ensemble. *Ann. Probab.*, 49(2): 997–1032, 2021. doi: 10.1214/20-AOP1468.
 - [21] Renjie Feng and Dongyi Wei. Large gaps of CUE and GUE. *Ann. Probab.*, 53(5): 1668–1702, 2025. doi: 10.1214/24-AOP1747.
 - [22] Renjie Feng and Dong Yao. Smallest distances between zeros of Gaussian analytic functions. *Adv. Math.*, 496:Paper No. 110999, 2026. doi: 10.1016/j.aim.2026.110999.
 - [23] Renjie Feng, Gang Tian, and Dongyi Wei. Small gaps of GOE. *Geom. Funct. Anal.*, 29 (6):1794–1827, 2019. doi: 10.1007/s00039-019-00520-5.
 - [24] Renjie Feng, Friedrich Götze, and Dong Yao. Smallest gaps between zeros of stationary Gaussian processes. *J. Funct. Anal.*, 287(4):Paper No. 110493, 2024. doi: 10.1016/j.jfa.2024.110493.
 - [25] Renjie Feng, Jiaming Li, and Dong Yao. Small gaps of GSE. Preprint, arXiv:2409.03324, 2024.
 - [26] Alessio Figalli and Alice Guionnet. Universality in several-matrix models via approximate transport maps. *Acta Math.*, 217(1):81–176, 2016. doi: 10.1007/s11511-016-0142-4.
 - [27] Peter J. Forrester. *Log-gases and random matrices*, volume 34 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2010.

- [28] Peter J. Forrester and Anthony Mays. A method to calculate correlation functions for $\beta = 1$ random matrices of odd size. *J. Stat. Phys.*, 134(3):443–462, 2009. doi: 10.1007/s10955-009-9684-6.
- [29] Philippe Jaming. Nazarov’s uncertainty principles in higher dimension. *J. Approx. Theory*, 149(1):30–41, 2007. doi: 10.1016/j.jat.2007.04.005.
- [30] Olav Kallenberg. *Random measures*. Akademie-Verlag, Berlin, third edition, 1983.
- [31] I. V. Krasovsky. Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. *Int. Math. Res. Not.*, 2004(25): 1249–1272, 2004. doi: 10.1155/S1073792804140221.
- [32] Benjamin Landon and Philippe Sosoe. Almost-optimal bulk regularity conditions in the CLT for Wigner matrices. Preprint, arXiv:2204.03419, 2022.
- [33] Benjamin Landon, Patrick Lopatto, and Jake Marcinek. Comparison theorem for some extremal eigenvalue statistics. *Ann. Probab.*, 48(6):2894–2919, 2020. doi: 10.1214/20-AOP1439.
- [34] Patrick Lopatto and Kyle Luh. Tail bounds for gaps between eigenvalues of sparse random matrices. *Electron. J. Probab.*, 26:Paper No. 130, 2021. doi: 10.1214/21-EJP669.
- [35] Patrick Lopatto and Matthew Meeker. Smallest gaps between eigenvalues of real Gaussian matrices. *Bernoulli*, 32(1):325–349, 2026. doi: 10.3150/25-BEJ1860.
- [36] Patrick Lopatto and Moritz Otto. Maximum gap in complex Ginibre matrices. Preprint, arXiv:2501.04611, 2025.
- [37] M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.
- [38] Philippe Moreillon. Largest bulk gap of the complex Ginibre ensemble. Preprint, arXiv:2608.20101, 2026.
- [39] F. L. Nazarov. Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. *Algebra i Analiz*, 5(4):3–66, 1993. In Russian; English translation in *St. Petersburg Math. J.* 5 (1994), no. 4, 663–717.
- [40] Hoi Nguyen, Terence Tao, and Van Vu. Random matrices: tail bounds for gaps between eigenvalues. *Probab. Theory Related Fields*, 167(3-4):777–816, 2017. doi: 10.1007/s00440-016-0693-5.
- [41] M. Plancherel and W. Rotach. Sur les valeurs asymptotiques des polynomes d’Hermite. *Comment. Math. Helv.*, 1(1):227–254, 1929. doi: 10.1007/BF01208365.
- [42] Dai Shi and Yunjiang Jiang. Smallest gaps between eigenvalues of random matrices with complex Ginibre, Wishart and universal unitary ensembles. Preprint, arXiv:1207.4240, 2012.

- [43] Barry Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005.
- [44] Christopher D. Sinclair. Correlation functions for $\beta = 1$ ensembles of matrices of odd size. *J. Stat. Phys.*, 136(1):17–33, 2009. doi: 10.1007/s10955-009-9771-8.
- [45] Alexander Soshnikov. Determinantal random point fields. *Russian Math. Surveys*, 55(5):923–975, 2000. doi: 10.1070/rm2000v055n05ABEH000321.
- [46] Alexander Soshnikov. Statistics of extreme spacing in determinantal random point processes. *Mosc. Math. J.*, 5(3):705–719, 2005.
- [47] G. Szegő. *Orthogonal polynomials*, volume XXIII of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, fourth edition, 1975.
- [48] Craig A. Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177(3):727–754, 1996. doi: 10.1007/BF02099545.
- [49] Craig A. Tracy and Harold Widom. Correlation functions, cluster functions and spacing distributions for random matrices. *J. Stat. Phys.*, 92(5-6):809–835, 1998. doi: 10.1023/A:1023084324803.
- [50] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the Brownian carousel. *Invent. Math.*, 177(3):463–508, 2009. doi: 10.1007/s00222-009-0180-z.
- [51] Benedek Valkó and Bálint Virág. Large gaps between random eigenvalues. *Ann. Probab.*, 38(3):1263–1279, 2010. doi: 10.1214/09-AOP508.
- [52] Jade P. Vinson. *Closest spacing of eigenvalues*. PhD thesis, Princeton University, 2001. Posted 2011 as arXiv:1111.2743.
- [53] H. Widom. The asymptotics of a continuous analogue of orthogonal polynomials. *J. Approx. Theory*, 77(1):51–64, 1994. doi: 10.1006/jath.1994.1033.
- [54] Harold Widom. On the relation between orthogonal, symplectic and unitary matrix ensembles. *J. Stat. Phys.*, 94(3-4):347–363, 1999. doi: 10.1023/A:1004536018336.
- [55] Albert Zhang. Quantitative gap universality for Wigner matrices. *Electron. J. Probab.*, 31:Paper No. 150, 2026. doi: 10.1214/26-EJP1601.