

ON THE TIGHTNESS OF THE PARTHASARATHY AND SCHMIDT RANK BOUNDS FOR EXTREMAL TRACIAL COUPLINGS

DIPANKAR DEY, SOMNATH JANA, AND SOUMYASHANT NAYAK

ABSTRACT. Let $2 \leq a \leq b$, and let ρ_A, ρ_B be density operators on $\mathbb{C}^a$ and $\mathbb{C}^b$, respectively. Let $\mathcal{C}(\rho_A, \rho_B)$ denote the set of bipartite couplings on $\mathbb{C}^a \otimes \mathbb{C}^b$ with these marginals. Parthasarathy showed that every extreme coupling has rank at most $\text{KRP}(a, b) := \lfloor \sqrt{a^2 + b^2 - 1} \rfloor$. It is not known for which dimension pairs (a, b) this bound is tight.

We restrict attention to the tracial case $\rho_A = I_a/a$, $\rho_B = I_b/b$. For the Parthasarathy bound, we construct extreme couplings of rank r from pairs of length- r finite unit-norm tight frames (FUNTFs) in $\mathbb{C}^a$ and $\mathbb{C}^b$ satisfying a transversality condition. This yields an extreme coupling of rank b for every $2 \leq a < b$, and hence tightness of $\text{KRP}(a, b)$ whenever $a < b$ and $b \geq (a^2 - 1)/2$; in particular, along the whole qubit-qudit line $(2, b)$, $b \geq 3$. This recovers the two previously known tracial cases with $a < b$, namely $(2, 3)$ and $(3, 4)$.

The Schmidt decomposition and subadditivity of rank give the lower bound $\text{ES}(a, b) := \lfloor b/a \rfloor$ for the rank of every tracial coupling, extreme or not. For the Schmidt bound, we use sparse Kraus operators built from weighted partial permutation matrices. Setting $r := \text{ES}(a, b)$, we construct an extreme coupling of rank r whenever $b - (r - 1)a$ divides a , and show that for $a < b \leq 2a$, where this condition reads $b - a = \gcd(a, b)$, it is also necessary. For $b > a(a - 1)$, the divisibility condition is superfluous and the Schmidt bound is attained for every such b . Thus, for each fixed a , attainability of both bounds remains open for only finitely many $b \geq a$.

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1. INTRODUCTION

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the finite-dimensional Hilbert spaces of two quantum systems, A and B , of dimensions a and b , respectively, with $2 \leq a \leq b$. A *density operator* on a finite-dimensional Hilbert space $\mathcal{H}$ is a positive semidefinite operator with unit trace, and is used to represent quantum states. We denote the set of all density operators on $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$. We identify $\mathcal{H}_A$ with $\mathbb{C}^a$ and $\mathcal{H}_B$ with $\mathbb{C}^b$, each equipped with the standard inner product.

Let $\rho_A \in \mathcal{D}(\mathbb{C}^a)$ and $\rho_B \in \mathcal{D}(\mathbb{C}^b)$ be density operators. The corresponding *coupling set* with the prescribed marginal states ρ_A and ρ_B is defined as

$$\mathcal{C}(\rho_A, \rho_B) = \{ \rho \in \mathcal{D}(\mathbb{C}^a \otimes \mathbb{C}^b) : \text{Tr}_B(\rho) = \rho_A, \text{Tr}_A(\rho) = \rho_B \},$$

where Tr_A and Tr_B denote the partial traces over subsystems A, B , respectively.

The extremality criterion for couplings (for example, see Theorem 2.13) is rooted in Choi's characterization of extreme completely positive maps [Cho75], and later adapted to this setting by Rudolph [Rud04] and Price–Sakai [PS07]. In [Par05, Corollary, p. 261], Parthasarathy proved that the rank of every extremal coupling, ρ , is bounded above by,

$$(1) \quad \text{rank } \rho \leq \text{KRP}(a, b) := \left\lfloor \sqrt{a^2 + b^2 - 1} \right\rfloor,$$

and also established a slackness in the bound for the tracial two-qubit case, showing that every extremal tracial coupling in $\mathcal{C}(I_2/2, I_2/2)$ has rank $1 < \text{KRP}(2, 2) = 2$ (see [Par05, Theorem 3.1]). Rudolph subsequently raised the question of when this upper bound is attained [Rud04].

In this article, we focus exclusively on the case of *tracial couplings*,

$$\mathcal{C}(I_a/a, I_b/b),$$

where both marginal states are the maximally mixed quantum states (tracial states) in $\mathcal{D}(\mathbb{C}^a)$ and $\mathcal{D}(\mathbb{C}^b)$, respectively. We denote the maximum rank attained by an extreme point of $\mathcal{C}(I_a/a, I_b/b)$, by $R_{\max}(a, b)$. We say that the pair (a, b) is *KRP-tight* if

$$R_{\max}(a, b) = \text{KRP}(a, b),$$

meaning the Parthasarathy upper bound is achieved by an extremal tracial coupling.

Prior work on tightness of the Parthasarathy bound is of two kinds: constructions of extremal tracial couplings, and constructions that achieve the bound only with non-tracial marginals.

For tracial marginals, Ohno [Ohn10] computed $R_{\max}(3, 3) = 4$ and $R_{\max}(4, 4) = 5$ [Ohn10, Theorems 2.6 and 2.7] and established the diagonal lower bound $R_{\max}(a, a) \geq a$ for $a \geq 3$ [Ohn10, Theorem 2.8]. Kanmani and Satyanarayana [KS21] settled the qubit-qutrit pair in the tracial setting, showing $R_{\max}(2, 3) = \text{KRP}(2, 3) = 3$. Devendra, Dey, and Dey [DDD26] take tensor products of known extreme points and check whether the result remains extreme in the appropriate dimension, which allows them to build new examples from Ohno's extreme points. In this way they verified KRP tightness for the pairs $(5, 5)$, $(9, 9)$, and $(12, 12)$, for

the amplified families $(5k, 5k)$ with $3 \leq k \leq 14$, and for the rectangular pair $(3, 4)$. For the qubit-qudit family $(2, b)$ with $b \geq 4$, however, tightness required deforming the first marginal away from $I_2/2$, so that construction is non-tracial except at $b = 3$, where it coincides with the tracial case. Bala and Rana [BR26] verified KRP tightness for the infinite rectangular family $(a, a + m)$ with $a \geq 3$ and $m > (a^2 - 2a - 2)/2$, for the qubit case $(2, 2 + m)$ with $m \geq 1$, and for the single pair $(6, 6)$; as with the $(2, b)$ family, these constructions deform one marginal away from the maximally mixed state.

Via J_m (Theorem 2.10), a tracial coupling $\rho \in \mathcal{C}(\rho_A, \rho_B)$ corresponds to a CP map Φ satisfying the marginal constraints $\Phi^*(I_b) = I_a$, $\Phi(\rho_A^T) = \rho_B$, with extreme points of $\mathcal{C}(\rho_A, \rho_B)$ corresponding to extreme points of this set of CP maps, and $\text{rank}(\rho)$ equal to the Choi rank of Φ . Writing $K_1, \dots, K_r$ for the Kraus operators of the minimal Kraus representation of Φ , for most of the article we work directly with Kraus operators rather than couplings or Choi matrices. For tracial marginals, $J_m(\Phi) = \frac{1}{a}J(\Phi)$, and the marginal constraints become

$$(2) \quad \sum_{i=1}^r K_i^* K_i = I_a, \quad \sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b.$$

The simplest case is an equal-weight, rank-one collection

$$K_i = \sqrt{\frac{a}{r}} |v_i\rangle\langle u_i|,$$

with $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ and $\{v_i\}_{i=1}^r \subset \mathbb{C}^b$ unit vectors, for which (2) reduces to the defining identities for tight frames in $\mathbb{C}^a$ and $\mathbb{C}^b$.

Recall that a *finite unit-norm tight frame* (FUNTF) of length r in $\mathbb{C}^a$ is a collection of r unit vectors $\{u_i\}_{i=1}^r$ whose synthesis matrix U satisfies $UU^* = \frac{r}{a}I_a$. Such frames were introduced by Benedetto and Fickus [BF03]; their geometry has since been studied extensively (see [DS06, CMS17]) and they play an important role in signal processing.

Given a pair of FUNTFs $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ and $\{v_i\}_{i=1}^r \subset \mathbb{C}^b$ of the same length, define the bipartite state

$$(3) \quad \rho := \frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle\langle \bar{u}_i| \otimes |v_i\rangle\langle v_i|.$$

The tight-frame identities immediately give $\text{Tr}_B(\rho) = I_a/a$ and $\text{Tr}_A(\rho) = I_b/b$ so that ρ is a tracial coupling.

In Theorem 3.2, we note that the extremality of the state ρ is determined by the Gram matrices, G_U and G_V , of its constituent frames. More specifically, ρ is an extremal tracial coupling if and only if the linear map $\Theta_{U,V} : M_r(\mathbb{C}) \rightarrow M_b(\mathbb{C}) \oplus M_a(\mathbb{C})$, defined by

$$\Theta_{U,V}(X) := \left(V(X^T \circ G_U)V^*, U(X \circ G_V)U^* \right),$$

is injective. Part (ii) of that theorem tells us that injectivity of $\Theta_{U,V}$ guarantees $\text{rank } \rho = r$.

We say a pair of FUNTFs (U, V) is *transversal* if $\ker \Theta_{U,V} = \{(0, 0)\}$. Our strategy for attaining the KRP bound at a given dimension pair (a, b) is thus to construct a transversal pair of FUNTFs of length $r = \text{KRP}(a, b)$. The underlying bi-independence criterion is due to Choi and has been applied via explicit vector constructions in [Ohn10, Rud04]; our reduction to a single transversality map between two tight frames, however, appears to be new.

For every $2 \leq a < b$, we construct transversal FUNTF pairs (Theorem 4.1) in $\mathbb{C}^a, \mathbb{C}^b$ of length b which yields an extremal tracial coupling of rank b . This gives a lower bound

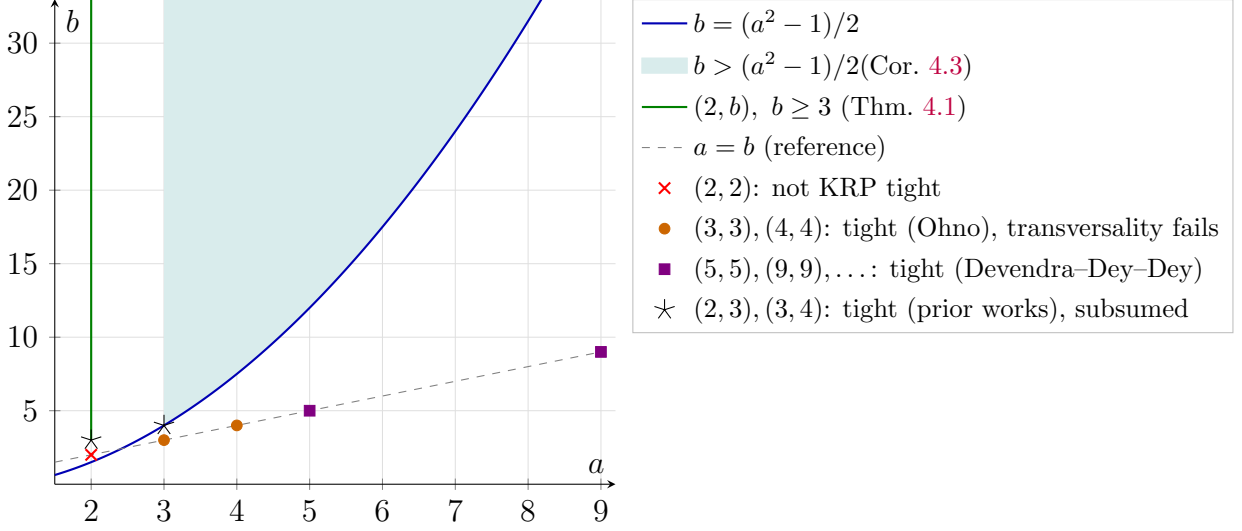


FIGURE 1. The (a, b) -plane for the tracial Parthasarathy rank bound problem. The curve $b = (a^2 - 1)/2$ is the boundary of the region (Corollary 4.3): every pair on or above it, with $a \geq 3$, is resolved by Theorem 4.1. The ray at $a = 2$ (a special case of Theorem 4.1) resolves every qubit-qudit pair $b \geq 3$. The point $(2, 2)$ is not KRP tight, while $(3, 3)$ and $(4, 4)$ are KRP tight but cannot be realized by the rank-one construction considered here. Square pairs resolved by Devendra–Dey–Dey are shown for context. The pairs $(2, 3)$ and $(3, 4)$, previously settled by Kanmani–Satyanarayana and Devendra–Dey–Dey respectively, lie on the boundaries of the regions resolved here.

$R_{\max}(a, b) \geq b$ (Corollary 4.2) supplementing Ohno’s result (see [Ohn10, Theorem 2.8]). It immediately follows that tightness of the Parthasarathy bound holds, that is, $R_{\max}(a, b) = \text{KRP}(a, b) = b$, along the qubit-qudit line $(2, b)$, $b \geq 3$, and throughout the region, $a \geq 3$, $b \geq \frac{a^2 - 1}{2}$ (see Corollary 4.3). This region naturally subsumes isolated known cases, such as the tracial $(3, 4)$ pair. Over the integers, our boundary condition $b \geq \frac{a^2 - 1}{2}$ is equivalent to the bound $m > (a^2 - 2a - 2)/2$ of Bala and Rana [BR26]. In their construction, one of the marginals is non-tracial and it does not appear to admit a tracial specialization, whereas ours achieves extremality directly within the tracial setting.

In low-dimensional square dimension pairs, however, our rank-one construction runs into obstructions. From Proposition 4.4 and Corollary 4.9, it follows that there are no transversal FUNTF pairs of length $\text{KRP}(a, a)$ for $(2, 2)$, $(3, 3)$, and $(4, 4)$. KRP tightness for $(3, 3)$ and $(4, 4)$ is nonetheless known, from Ohno’s results, leaving $(2, 2)$ as the only dimension pair known to fail KRP tightness (Theorem 2.7).

Figure ?? shows the KRP-tight region established here, the exceptional pairs $(2, 2)$, $(3, 3)$, $(4, 4)$, and the earlier results of Ohno, Kanmani–Satyanarayana, and Devendra–Dey–Dey that these results subsume or complement.

We conjecture that a transversal pair of FUNTFs of length $\text{KRP}(a, b)$ exists for every dimension pair (a, b) outside the set $\{(2, 2), (3, 3), (4, 4)\}$ (see Conjecture 6.1). In particular, echoing a similar question posed by Devendra, Dey, and Dey [DDD26], we conjecture that $(2, 2)$ is the only non-KRP-tight dimension pair (see Conjecture 6.2).

While the Parthasarathy bound caps the rank of an extremal coupling from above, in this article we introduce a companion lower bound, established in Theorem 2.3-(iii). A straightforward application of the Schmidt decomposition and the subadditivity of rank shows that a tracial coupling ρ , extremal or not, satisfies $\text{rank } \rho \geq \text{ES}(a, b) := \lceil b/a \rceil$, which we call the *Schmidt bound*. This raises the counterpart of Rudolph's question: for which dimension pairs is the Schmidt bound attained by an extremal tracial coupling?

We denote the minimum rank attained by an extreme point of $\mathcal{C}(I_a/a, I_b/b)$, by $R_{\min}(a, b)$. We say that the dimension pair (a, b) is *ES-tight* if

$$R_{\min}(a, b) = \text{ES}(a, b),$$

meaning the Schmidt lower bound is achieved by an extreme tracial coupling.

If $a \mid b$, then Proposition 5.2 gives an extremal tracial coupling of rank $\text{ES}(a, b) = \frac{b}{a}$, obtained by taking as Kraus operators a family of $\frac{b}{a}$ partial isometries from $\mathbb{C}^a$ into $\mathbb{C}^b$ with pairwise orthogonal ranges. For $a < b < 2a$, where $\text{ES}(a, b) = 2$, we give a complete characterization: the Schmidt bound, 2, is attained by an extreme point if and only if $b - a = \gcd(a, b)$ (Theorem 5.12). Thus, every consecutive pair $(a, a + 1)$ attains it while a pair like $(3, 5)$ does not. In general, with $r := \text{ES}(a, b)$ we construct a collection of weighted partial permutation Kraus operators that attains the Schmidt bound whenever $b - (r - 1)a$ divides a (Theorem 5.9), recovering the $r = 2$ characterization as a special case, and guaranteeing that for every $k \geq 2$, the window $(k - 1)a < b < ka$ contains a b such that (a, b) is a Schmidt-tight pair.

For $b > a(a - 1)$, the divisibility hypothesis becomes superfluous. The Schmidt bound is attained for every such b , by an explicit tight-frame construction (Theorem 5.14). Together, Theorems 5.9 and 5.14 leave only finitely many values for $b \geq a$ per a where Schmidt-bound attainment is not settled.

The constructions used to attain the Schmidt bound in §5 are quite different from those used for the Parthasarathy bound in §3: in place of rank-one Kraus operators and tight-frame geometry, they use weighted partial permutation matrices, and extremality is proved by a combinatorial argument (Lemma 5.6).

Organisation. The remainder of the paper is organized as follows. Section 2 reviews the convex-geometric formulation of the problem, the Parthasarathy/Schmidt rank bounds, and the extremality criterion for tracial couplings. Section 3 introduces the tight-frame construction and the linear map $\Theta_{U,V}$ whose injectivity characterizes extremality. Section 4 constructs a transversal FUNTF pair for every $a < b$ and gives obstructions to transversality for square pairs. Finally, Section 5 analyzes the companion lower bound, the Schmidt bound, detailing the complete characterization when $\text{ES}(a, b) = 2$ and a general construction for higher values of $\text{ES}(a, b)$.

1.1. Notations. To facilitate reading, we collect the conventions and notations used throughout the paper.

A note on conventions. Throughout, we use the Dirac convention for the inner product: $\langle x, y \rangle$ is conjugate-linear in the first argument and linear in the second. This is the standard convention in the quantum information literature. We also write a vector interchangeably in ket form, $|e_k\rangle$, or just, e_k : the ket is used inside Dirac expressions such as $|e_k\rangle\langle e_k|$, while the bare form appears when the vector is the argument of an operator applied in ordinary notation, such as $K_i e_k$; both denote the same vector.

Our basic reference for background material on completely positive maps, the Choi–Jamiołkowski isomorphism, and vectorization is Watrous’s *The Theory of Quantum Information* [Wat18], and we adopt its results freely. Purely for notational convenience, we deviate in one respect. In [Wat18], an operator $K \in L(X, Y)$ is vectorized into $Y \otimes X$ and the Choi representation is defined via $\Phi \otimes \mathcal{I}$, so that $J(\Phi) \in L(Y \otimes X)$. We instead vectorize a $b \times a$ matrix $K : \mathbb{C}^a \rightarrow \mathbb{C}^b$ into $\mathbb{C}^a \otimes \mathbb{C}^b$ (domain space first, codomain space second) and define the Choi matrix via $\mathcal{I} \otimes \Phi$ rather than $\Phi \otimes \mathcal{I}$, so that for $\Phi : M_a(\mathbb{C}) \rightarrow M_b(\mathbb{C})$ we consistently have $J(\Phi) \in M_a(\mathbb{C}) \otimes M_b(\mathbb{C})$, matching the order in which we write the coupling set $\mathcal{C}(\rho_A, \rho_B)$ and the subsystems $\mathbb{C}^a, \mathbb{C}^b$ throughout the paper. Any statement we cite from [Wat18] should be read with this relabeling of tensor factors understood; the mathematical content is unaffected.

Symbol	Definition
$M_a(\mathbb{C}), M_{a,b}(\mathbb{C})$	The space of $a \times a, a \times b$ complex matrices, respectively.
$\mathcal{C}(\rho_A, \rho_B)$	The coupling set; the space of all bipartite quantum states with marginals ρ_A and ρ_B .
$\text{KRP}(a, b)$	Parthasarathy’s KRP bound $\lfloor \sqrt{a^2 + b^2 - 1} \rfloor$, an upper bound on the rank of an extreme coupling.
$R_{\max}(a, b)$	The maximal rank of an extreme point of $\mathcal{C}(I_a/a, I_b/b)$.
$CP(M_a(\mathbb{C}), M_b(\mathbb{C}))$	Completely positive maps from $M_a(\mathbb{C})$ to $M_b(\mathbb{C})$.
U, V	Synthesis matrices of Finite Unit-Norm Tight Frames (FUNTFs) in $\mathbb{C}^a$ and $\mathbb{C}^b$, respectively.
G_U, G_V	The Gram matrices of the frames U and V , given by U^*U and V^*V .
P_U, P_V	The scaled Gram projections, $P_U = \frac{a}{r}G_U$ and $P_V = \frac{b}{r}G_V$.
$\Theta_{U,V}$	The linear map $\Theta_{U,V} : M_r \rightarrow M_b \oplus M_a$ whose injectivity characterizes extremality (Theorem 3.2).

TABLE 1. Notations used throughout the paper.

2. THE COUPLING SET AND ITS CONVEX GEOMETRY

In this section, we set up the convex-geometric language, prove the Parthasarathy and Schmidt rank bounds, and prove the extremality criteria (Lemma 2.2, Theorem 2.13) in the case of tracial couplings.

2.1. The Parthasarathy and Schmidt rank bounds. We give a linear algebraic formulation for the extreme points of $\mathcal{C}(\rho_A, \rho_B)$ which leads to the Parthasarathy bound. Let Herm_n be the real vector space of Hermitian $n \times n$ matrices, so $\dim_{\mathbb{R}} \text{Herm}_n = n^2$. Let $2 \leq a \leq b$ be the dimensions of the two subsystems. We denote the combined marginal map by,

$$\Lambda : \text{Herm}_{ab} \longrightarrow \text{Herm}_a \times \text{Herm}_b, \quad \Lambda(X) := (\text{Tr}_B X, \text{Tr}_A X).$$

In this language, $\mathcal{C}(\rho_A, \rho_B) = \{X \in \text{Herm}_{ab} : X \succeq 0, \Lambda(X) = (\rho_A, \rho_B)\}$ is the intersection of the closed convex cone of positive semidefinite matrices with an affine subspace, hence closed and convex; since every $X \succeq 0$ with $\text{Tr} X = 1$ has operator norm at most 1, it is also bounded. Thus $\mathcal{C}(\rho_A, \rho_B)$ is a compact convex set.

Proposition 2.1. *The image of Λ is equal to the hyperplane*

$$\{(P, Q) \in \text{Herm}_a \times \text{Herm}_b : \text{Tr } P = \text{Tr } Q\} \subset \text{Herm}_a \times \text{Herm}_b,$$

which has real dimension $a^2 + b^2 - 1$.

Proof. For any $H \in \text{Herm}_{ab}$, $\text{Tr}(\text{Tr}_B H) = \text{Tr } H = \text{Tr}(\text{Tr}_A H)$, so $\Lambda(H) = (\text{Tr}_B H, \text{Tr}_A H)$ satisfies $\text{Tr}(\text{Tr}_B H) = \text{Tr}(\text{Tr}_A H)$. Hence $\text{range } \Lambda$ lies in the hyperplane.

Recall the partial-trace identities $\text{Tr}_B(H_A \otimes H_B) = \text{Tr}(H_B) H_A$ and $\text{Tr}_A(H_A \otimes H_B) = \text{Tr}(H_A) H_B$. Let $(P, Q) \in \text{Herm}_a \times \text{Herm}_b$ with $\text{Tr } P = \text{Tr } Q$, and set

$$H_{AB} := P \otimes \frac{I_b}{b} + \frac{I_a}{a} \otimes Q - \text{Tr}(P) \frac{I_a}{a} \otimes \frac{I_b}{b}.$$

Using $\text{Tr}(Q) = \text{Tr}(P)$, we see that,

$$\text{Tr}_B(H_{AB}) = P + \text{Tr}(Q) \frac{I_a}{a} - \text{Tr}(P) \frac{I_a}{a} = P.$$

Similarly, $\text{Tr}_A(H_{AB}) = Q$. So $\Lambda(H_{AB}) = (P, Q)$, and $\text{range } \Lambda$ equals the hyperplane. This hyperplane is the kernel of the non-zero linear functional $(P, Q) \mapsto \text{Tr}(P) - \text{Tr}(Q)$ on $\text{Herm}_a \times \text{Herm}_b$, hence has real dimension $a^2 + b^2 - 1$. $\square$

For a complex subspace $W \subseteq \mathbb{C}^{ab}$, we define,

$$\text{Herm}(W) := \{H \in \text{Herm}_{ab} : \text{range } H \subseteq W\}.$$

Note that this is a real vector space of dimension $(\dim_{\mathbb{C}} W)^2$, since it is the space of self-adjoint maps from W to itself. With this notation, extremality admits the following characterization.

Lemma 2.2. *A coupling $X \in \mathcal{C}(\rho_A, \rho_B)$ is extreme if and only if*

$$\ker \Lambda \cap \text{Herm}(\text{range } X) = \{0\}.$$

Proof. Suppose that $\ker \Lambda \cap \text{Herm}(\text{range } X) = \{0\}$ and $X = \frac{1}{2}(Y_1 + Y_2)$ with $Y_1, Y_2 \in \mathcal{C}(\rho_A, \rho_B)$. Let $\xi \in \ker(X)$, so that $\langle X\xi, \xi \rangle = 0$. The positivity of Y_1, Y_2 forces $\langle Y_i \xi, \xi \rangle = 0$ as well, so $\ker(X) \subseteq \ker(Y_i)$ which implies $\text{range } Y_i \subseteq \text{range } X$, for $i = 1, 2$. Thus $Y_1, Y_2 \in \text{Herm}(\text{range } X)$. Hence $Y_1 - X \in \ker \Lambda \cap \text{Herm}(\text{range } X)$ which implies that $Y_1 = X = Y_2$. We conclude that X is an extreme point of $\mathcal{C}(\rho_A, \rho_B)$.

Conversely, let $H \neq 0$ be an element of $\ker \Lambda \cap \text{Herm}(\text{range } X)$. Since $\text{range } H \subseteq \text{range } X$, for all sufficiently small $\varepsilon > 0$ both $X + \varepsilon H$ and $X - \varepsilon H$ are positive semidefinite and satisfy $\Lambda(X \pm \varepsilon H) = \Lambda(X) = (\rho_A, \rho_B)$. In other words, $X \pm \varepsilon H \in \mathcal{C}(\rho_A, \rho_B)$. Thus, X being the midpoint of $X - \varepsilon H, X + \varepsilon H$, is not an extreme point of $\mathcal{C}(\rho_A, \rho_B)$. $\square$

We are now ready for a conceptual proof of the Parthasarathy upper bound for the ranks of extremal couplings, which is different from the original proof due to Parthasarathy.

Theorem 2.3. *Let $a \leq b$ be positive integers, and $\rho_A \in \mathcal{D}(\mathbb{C}^a), \rho_B \in \mathcal{D}(\mathbb{C}^b)$ be states.*

- (i) *For any $\rho \in \mathcal{C}(\rho_A, \rho_B)$, $\text{rank } \rho \leq (\text{rank } \rho_A)(\text{rank } \rho_B)$.*
- (ii) *(cf. [Par05, Corollary, p. 261]) If ρ is an extreme point of $\mathcal{C}(\rho_A, \rho_B)$, then*

$$\text{rank } \rho \leq \lfloor \sqrt{a^2 + b^2 - 1} \rfloor.$$

(iii) For every $\rho \in \mathcal{C}(\rho_A, \rho_B)$,

$$\text{rank } \rho \geq \left\lceil \frac{\text{rank } \rho_B}{a} \right\rceil.$$

In particular, if ρ is a tracial coupling, then

$$\text{rank } \rho \geq \left\lceil \frac{b}{a} \right\rceil.$$

Proof. (i) Let P_A, P_B be the orthogonal projections onto $\text{range}(\rho_A), \text{range}(\rho_B)$, and $P := P_A \otimes P_B$. Fix a unit vector $\xi \perp \text{range}(\rho_A)$. Since $\text{Tr}_B(\rho) = \rho_A$, $\langle \xi | \rho_A | \xi \rangle = 0$, so $\langle \xi \otimes \eta | \rho | \xi \otimes \eta \rangle = 0$ for every η ; as $\rho \succeq 0$, this forces $\rho(|\xi\rangle \otimes |\eta\rangle) = 0$ for every η . The same argument, using $\text{Tr}_A(\rho) = \rho_B$, applies to any $\eta \perp \text{range}(\rho_B)$. Together these give $\rho = P\rho P$, so

$$\text{rank } \rho \leq \dim \text{range}(P) = \dim (\text{range}(\rho_A) \otimes \text{range}(\rho_B)) = (\text{rank } \rho_A)(\text{rank } \rho_B).$$

(ii) Let $W := \text{range } \rho$. From Lemma 2.2 we see that $\Lambda|_{\text{Herm}(W)}$ is injective. Since its domain, $\text{Herm}(W)$, has real dimension $\dim_{\mathbb{C}}(W)^2 = (\text{rank } \rho)^2 = r^2$ and its codomain has real dimension $a^2 + b^2 - 1$, we obtain the bound,

$$r^2 \leq a^2 + b^2 - 1.$$

(iii) Let $\rho = \sum_{i=1}^r p_i |\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of ρ with $r = \text{rank } \rho$ and $p_i > 0$ for $1 \leq i \leq r$. Taking the partial trace over A gives

$$\rho_B = \sum_{i=1}^r p_i \text{Tr}_A(|\psi_i\rangle\langle\psi_i|).$$

For every $1 \leq i \leq r$, the reduced state $\text{Tr}_A(|\psi_i\rangle\langle\psi_i|)$ is the marginal of a pure state, whence its rank is bounded above by a (see [Wat18, Theorem 2.10]). Subadditivity of rank across the sum then gives

$$\text{rank } \rho_B \leq \sum_{i=1}^r \text{rank}(\text{Tr}_A(|\psi_i\rangle\langle\psi_i|)) \leq ra,$$

so $r \geq (\text{rank } \rho_B)/a$. Since r is an integer, we get the stated bound. Finally, when ρ is a tracial coupling, $\rho_B = I_b/b$ has rank b , and we get $\text{rank } \rho \geq \lceil b/a \rceil$. $\square$

In the rest of the article, we are only interested in the tracial case, that is, $\rho_A = \frac{1}{a}I_a, \rho_B = \frac{1}{b}I_b$.

Definition 2.4. Let $2 \leq a \leq b$ be positive integers. Motivated by the theorem above, we define

$$\text{KRP}(a, b) := \lfloor \sqrt{a^2 + b^2 - 1} \rfloor, \quad \text{ES}(a, b) := \left\lceil \frac{b}{a} \right\rceil,$$

named after the initials of K. R. Parthasarathy and E. Schmidt, respectively.

Definition 2.5. Let $2 \leq a \leq b$ be positive integers. We denote the maximum (minimum, respectively) of the ranks of extremal tracial couplings in $\mathcal{C}(I_a/a, I_b/b)$ by $R_{\max}(a, b), R_{\min}(a, b)$, respectively. We call the dimension pair (a, b) *KRP-tight* (*ES-tight*, respectively) if $R_{\max}(a, b) = \text{KRP}(a, b), (R_{\min}(a, b) = \text{ES}(a, b))$, respectively.

Remark 2.6 (The Birkhoff property for qubit channels). The fact used in Theorem 2.7 below, namely that every unital trace-preserving qubit channel is a convex combination of unitary conjugations, is a Birkhoff-type theorem for doubly stochastic maps on $M_2(\mathbb{C})$ with an interesting history. The result was proved by Bhat, Pati and Sunder in 1993 [BPS93],

and independently by Landau and Streater [LS93] in the same year. It appears to have been anticipated earlier by Tregub [Tre86]. The corresponding statement fails in dimensions $d \geq 3$. Kümmerer and Maassen [KM87] had exhibited extreme points of the set of unital channels on M_d , $d \geq 3$, that are not unitary conjugations. This failure makes the higher-dimensional analogue of Theorem 2.7 nontrivial. For completeness, we note that a more direct proof for the Theorem 2.7 was given by Parthasarathy [Par05].

Theorem 2.7. $R_{\max}(2, 2) = 1 < 2 = \text{KRP}(2, 2)$, that is, every extremal tracial coupling in the qubit-qubit case has rank 1.

Proof. Extreme points of $\mathcal{C}(I_2/2, I_2/2)$ correspond to extreme unital trace-preserving qubit channels. As noted in Remark 2.6, every extreme point of the set of unital trace-preserving qubit channels is a single unitary conjugation, whose Choi state is a maximally entangled pure state (rank 1). $\square$

Remark 2.8. The pair $(2, 2)$ is, to our knowledge, the only pair known to be non-KRP-tight.

2.2. The Choi–Jamiołkowski correspondence and the extremality criterion.

2.2.1. *Vectorization.* For a linear operator $K : \mathbb{C}^a \rightarrow \mathbb{C}^b$, define its *vectorization*

$$\text{vec}(K) := (I_a \otimes K)|\eta\rangle \in \mathbb{C}^a \otimes \mathbb{C}^b,$$

where $|\eta\rangle := \sum_{j=1}^a |e_j\rangle \otimes |e_j\rangle \in \mathbb{C}^a \otimes \mathbb{C}^a$. Specializing to the rank-one case, for $u \in \mathbb{C}^a, v \in \mathbb{C}^b$, we get,

$$(4) \quad \text{vec}(|v\rangle\langle u|) = |\bar{u}\rangle \otimes |v\rangle.$$

2.2.2. *The Choi matrix and the Choi–Kraus correspondence.* For a linear map $\Phi : M_a(\mathbb{C}) \rightarrow M_b(\mathbb{C})$, define its *Choi matrix*

$$J(\Phi) := (\mathcal{I} \otimes \Phi)(|\eta\rangle\langle\eta|) \in M_a(\mathbb{C}) \otimes M_b(\mathbb{C}),$$

where as above $|\eta\rangle = \sum_{j=1}^a |e_j\rangle \otimes |e_j\rangle$.

The following result of Choi [Cho75] (see [Wat18, Theorem 2.22]) relates complete positivity, positivity of the Choi matrix, and the Kraus representation, and defines the Choi rank.

Theorem 2.9 (Choi–Kraus correspondence). *Let $\Phi : M_a(\mathbb{C}) \rightarrow M_b(\mathbb{C})$ be linear. The following are equivalent:*

- (a) Φ is completely positive.
- (b) $J(\Phi) \succeq 0$.
- (c) (Kraus representation) $\Phi(X) = \sum_{i=1}^r K_i X K_i^*$ for some $r \geq 1$ and $K_i : \mathbb{C}^a \rightarrow \mathbb{C}^b$.

Moreover, $J : B(M_a(\mathbb{C}), M_b(\mathbb{C})) \rightarrow M_a(\mathbb{C}) \otimes M_b(\mathbb{C})$ is a linear bijection, restricting to a bijection between the completely positive maps $M_a(\mathbb{C}) \rightarrow M_b(\mathbb{C})$ and $\{Z \in M_a(\mathbb{C}) \otimes M_b(\mathbb{C}) : Z \succeq 0\}$. For a CP map Φ , the minimal number of terms r in a Kraus decomposition as in (c) equals $\text{rank } J(\Phi)$; this common value is called the Choi rank of Φ , denoted $CR(\Phi)$, and any minimal Kraus decomposition $\{K_i\}_{i=1}^{CR(\Phi)}$ has $\{|\text{vec}(K_i)\rangle\}$ a basis for $\text{range } J(\Phi)$ (and, a fortiori, linearly independent).

If Φ has Kraus form $\Phi(X) = \sum_{i=1}^r K_i X K_i^*$ with $K_i : \mathbb{C}^a \rightarrow \mathbb{C}^b$, then

$$(5) \quad J(\Phi) = \sum_{i=1}^r |\text{vec}(K_i)\rangle\langle\text{vec}(K_i)|,$$

so a Kraus decomposition of Φ corresponds to a decomposition of $J(\Phi)$ into rank-one positive semidefinite pieces $|\text{vec}(K_i)\rangle\langle\text{vec}(K_i)|$, each Kraus operator recovered (up to a global phase) as $K_i = \text{unvec}(v_i)$ for v_i a unit eigenvector of the i -th piece.

Theorem 2.10 ([DDD26, Proposition 2.7]). *Let $\rho_A \in \mathcal{D}(\mathbb{C}^a)$ be invertible and let $\rho_B \in \mathcal{D}(\mathbb{C}^b)$. Let $J : B(M_a(\mathbb{C}), M_b(\mathbb{C})) \rightarrow M_a(\mathbb{C}) \otimes M_b(\mathbb{C})$ denote the Choi–Jamiołkowski map, and define*

$$J_m(\Phi) := (\rho_A^{1/2} \otimes I_b) J(\Phi) (\rho_A^{1/2} \otimes I_b).$$

Then we have the following:

- (i) $J_m^{-1}(\mathcal{C}(\rho_A, \rho_B)) = \{\Phi \in CP(M_a(\mathbb{C}), M_b(\mathbb{C})) : \Phi^*(I_b) = I_a, \Phi(\rho_A^T) = \rho_B\}$, and this is a compact convex set.
- (ii) For $\rho \in \mathcal{C}(\rho_A, \rho_B)$, ρ is an extreme point of $\mathcal{C}(\rho_A, \rho_B)$ if and only if $J_m^{-1}(\rho)$ is an extreme point of $J_m^{-1}(\mathcal{C}(\rho_A, \rho_B))$.
- (iii) For every $\Phi \in CP(M_a(\mathbb{C}), M_b(\mathbb{C}))$, $\text{rank}(J_m(\Phi)) = \text{rank}(J(\Phi)) = CR(\Phi)$, the Choi rank of Φ .

Proof. (i) As ρ_A is invertible, $\rho_A^{1/2} \otimes I_b$ is invertible, so J_m , a composition of J with conjugation by an invertible operator, is also a linear bijection. For $\Phi \in B(M_a(\mathbb{C}), M_b(\mathbb{C}))$, note that,

$$J_m(\Phi) = \sum_{i,j=1}^a (\rho_A^{1/2} |e_i\rangle\langle e_j| \rho_A^{1/2}) \otimes \Phi(|e_i\rangle\langle e_j|)$$

We compute the marginals of $J_m(\Phi)$ below.

$$\begin{aligned} \text{Tr}_A(J_m(\Phi)) &= \sum_{i,j=1}^a \text{Tr}(\rho_A^{1/2} |e_i\rangle\langle e_j| \rho_A^{1/2}) \Phi(|e_i\rangle\langle e_j|) = \sum_{i,j=1}^a \langle e_j | \rho_A | e_i \rangle \Phi(|e_i\rangle\langle e_j|) \\ &= \sum_{i,j=1}^a (\rho_A^T)_{ij} \Phi(|e_i\rangle\langle e_j|) = \Phi(\rho_A^T), \\ \text{Tr}_B(J_m(\Phi)) &= \sum_{i,j=1}^a \rho_A^{1/2} |e_i\rangle\langle e_j| \rho_A^{1/2} \text{Tr}(\Phi(|e_i\rangle\langle e_j|)) \\ &= \rho_A^{1/2} \left(\sum_{i,j=1}^a \text{Tr}(\Phi^*(I_b) |e_i\rangle\langle e_j|) |e_i\rangle\langle e_j| \right) \rho_A^{1/2} \\ &= \rho_A^{1/2} (\Phi^*(I_b))^T \rho_A^{1/2}. \end{aligned}$$

So, $J_m(\{\Phi \in CP(M_a(\mathbb{C}), M_b(\mathbb{C})) : \Phi^*(I_b) = I_a, \Phi(\rho_A^T) = \rho_B\}) = \mathcal{C}(\rho_A, \rho_B)$

Since $\mathcal{C}(\rho_A, \rho_B)$ is compact and convex (§2.1) and J_m^{-1} is a linear bijection, $J_m^{-1}(\mathcal{C}(\rho_A, \rho_B))$ is compact and convex.

(ii). This follows from the general fact that an affine bijection carries extreme points to extreme points.

(iii). Since $\rho_A^{1/2} \otimes I$ is invertible, we have $\text{rank}(J_m(\Phi)) = \text{rank}(J(\Phi)) = CR(\Phi)$. $\square$

Remark 2.11. For tracial marginals, J_m identifies the coupling set with trace-preserving CP maps satisfying $\Phi(I_a/a) = I_b/b$. If one instead used the standard Choi matrix of a unital map (as in [Rud04, DDD26]), the rank-one construction $K_i = c |v_i\rangle\langle u_i|$ would lead to $c = \sqrt{b/r}$ and force $\sum |u_i\rangle\langle u_i| = (r/b)I_a$, which is *not* a tight frame condition in $\mathbb{C}^a$ when $a \neq b$. The

J_m formulation yields $c = \sqrt{a/r}$ and converts the two marginal constraints into the pair of FUNTF identities (2); this is why we work with J_m rather than J .

Remark 2.12. Specializing Theorem 2.10 to the tracial case, the marginal constraints in the Kraus representation, $\Phi(X) = \sum_{i=1}^r K_i X K_i^*$, take the form,

$$(6) \quad \sum_{i=1}^r K_i^* K_i = I_a, \quad \sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b.$$

Below we give a proof of the extremality criterion of extremal tracial couplings. The more general form of the theorem is proved in [Rud04] using a somewhat different approach.

Theorem 2.13 (cf. [Rud04, Theorem 1]). *Let Φ be a completely positive map lying in the compact convex set,*

$$\mathcal{A} := \{\Phi \in CP(M_a(\mathbb{C}) \rightarrow M_b(\mathbb{C})) : \Phi^*(I_b) = I_a, \Phi(I_a/a) = I_b/b\},$$

where Φ^* denotes the adjoint of Φ with respect to the Hilbert–Schmidt inner product $\langle A, B \rangle_{HS} = \text{Tr}(A^* B)$ (see [Wat18, Ch. 2]). Let $\Phi \in \mathcal{A}$ with minimal Kraus representation $\Phi(X) = \sum_{i=1}^r K_i X K_i^*$, where $K_i : \mathbb{C}^a \rightarrow \mathbb{C}^b$. Define the linear map $\Gamma : M_r(\mathbb{C}) \rightarrow M_a(\mathbb{C}) \oplus M_b(\mathbb{C})$ by

$$(7) \quad \Gamma(X) = \left(\sum_{i,j=1}^r x_{ij} K_j^* K_i, \sum_{i,j=1}^r x_{ij} K_i K_j^* \right).$$

Then Φ is an extreme point of $\mathcal{A}$ if and only if $\ker \Gamma = \{0\}$.

Proof. For Hermitian $Y = (y_{ij}) \in M_r(\mathbb{C})$, write $\Psi_Y(\cdot) := \sum_{i,j=1}^r y_{ij} K_i(\cdot) K_j^*$, so that $\Psi_{I_r} = \Phi$. A direct computation of the Hilbert–Schmidt adjoint gives $\Psi_Y^*(Z) = \sum_{i,j=1}^r y_{ij} K_j^* Z K_i$ (using $\bar{y}_{ij} = y_{ji}$), so

$$(8) \quad \Gamma(Y) = (\Psi_Y^*(I_b), \Psi_Y(I_a)).$$

In particular,

$$\Gamma(I_r) = (\Phi^*(I_b), \Phi(I_a)) = \left(I_a, \frac{a}{b} I_b \right),$$

encodes exactly the two conditions defining $\mathcal{A}$.

($\ker \Gamma \neq \{0\} \implies \text{Non-extreme}$). Let $0 \neq X \in \ker \Gamma$. Since $\Gamma(X^*) = \Gamma(X)^*$ by (7), $\ker \Gamma$ is closed under adjoints, so we may take X to be Hermitian, by replacing X with $X + X^*$ or $i(X - X^*)$ if needed. Fix $\varepsilon > 0$ small enough that $X_{\pm} := I_r \pm \varepsilon X \succ 0$, with square root $S_{\pm} := \sqrt{X_{\pm}}$. Writing $(X_{\pm})_{ij} = \sum_{l=1}^r (S_{\pm})_{il} \overline{(S_{\pm})_{jl}}$ and substituting into $\Psi_{X_{\pm}}$ gives the Kraus form

$$\Psi_{X_{\pm}}(Y) = \sum_{l=1}^r \tilde{K}_l^{\pm} Y (\tilde{K}_l^{\pm})^*, \quad \tilde{K}_l^{\pm} := \sum_{i=1}^r (S_{\pm})_{il} K_i,$$

so $\Psi_{\pm} := \Psi_{X_{\pm}}$ is CP. By linearity, $\Gamma(X_{\pm}) = \Gamma(I_r) \pm \varepsilon \Gamma(X) = \Gamma(I_r)$, and since $X_{\pm}$ is Hermitian, (8) gives $(\Psi_{\pm}^*(I_b), \Psi_{\pm}(I_a)) = (\Phi^*(I_b), \Phi(I_a)) = (I_a, \frac{a}{b} I_b)$; thus $\Psi_{\pm} \in \mathcal{A}$.

Since $X_+ + X_- = 2I_r$, $\Psi_+ + \Psi_- = 2\Psi_{I_r} = 2\Phi$, so $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$. The map $C \mapsto \sum_{i,j=1}^r c_{ij} |\text{vec}(K_i)\rangle \langle \text{vec}(K_j)|$ from $M_r(\mathbb{C})$ to $M_{ab}(\mathbb{C})$ is injective, since the r vectorizations $|\text{vec}(K_i)\rangle$ are linearly independent by minimality of the Kraus decomposition. As $X_+ \neq I_r$, this gives $J(\Psi_+) \neq J(\Phi)$, i.e. $\Psi_+ \neq \Phi$. Hence $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$ is a nontrivial convex combination in $\mathcal{A}$, and Φ is not extreme.

(*Non-extreme* $\implies \ker \Gamma \neq \{0\}$). Suppose $\Phi = \frac{1}{2}(\Psi_+ + \Psi_-)$, with $\Psi_{\pm} \in \mathcal{A}$ and $\Psi_+ \neq \Phi$.

At the level of Choi matrices, $J(\Psi_-) \succeq 0$ implies that $J(\Psi_+) = 2J(\Phi) - J(\Psi_-) \preceq 2J(\Phi)$, hence $\text{range } J(\Psi_+) \subseteq \text{range } J(\Phi)$. Since the minimal Kraus operators of Φ form a basis for $\text{range } J(\Phi) = \text{span}\{|\text{vec}(K_i)\rangle : 1 \leq i \leq r\}$, $J(\Psi_+)$ is supported within this subspace and, being Hermitian, may be expanded in this basis with a Hermitian coefficient matrix $X \in M_r(\mathbb{C})$:

$$J(\Psi_+) = \sum_{i,j=1}^r x_{ij} |\text{vec}(K_i)\rangle \langle \text{vec}(K_j)| \implies \Psi_+ = \Psi_X.$$

Since X is Hermitian, (8) applies at $Y = X$ and at $Y = I_r$, giving

$$\Gamma(X) = (\Psi_+^*(I_b), \Psi_+(I_a)), \quad \Gamma(I_r) = (\Phi^*(I_b), \Phi(I_a)).$$

Since $\Psi_+, \Phi \in \mathcal{A}$, we note that $\Gamma(X) = \Gamma(I_r) = (I_a, \frac{a}{b}I_b)$, i.e. $\Gamma(X - I_r) = 0$. Since $\Psi_+ \neq \Phi$, we have $X \neq I_r$, so $X - I_r$ is a non-zero element of $\ker \Gamma$. $\square$

3. CONSTRUCTION OF EXTREMAL TRACIAL COUPLINGS USING TIGHT FRAMES

In §2, we noted that Theorem 2.13 reduces extremality of tracial couplings to a linear-algebraic condition on the associated Kraus operators K_i . In this section, we pursue the simplest possible strategy of building extremal tracial couplings using linearly independent rank-one Kraus operators of the form $K_i = |v_i\rangle \langle u_i|$, with $\{u_i\}$ in $\mathbb{C}^a$ and $\{v_i\}$ in $\mathbb{C}^b$ being tight frames of the same length. The marginal constraints hold automatically because of tightness, so the $\ker \Gamma = \{(0, 0)\}$ condition of Theorem 2.13 reduces extremality to injectivity of an explicit linear map $\Theta_{U,V}$ built from the two frames alone.

3.1. Frames. A sequence $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ is a *frame* if there exist $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^r |\langle u_i, x \rangle|^2 \leq B\|x\|^2 \quad (x \in \mathbb{C}^a).$$

The frame is *tight* if $A = B$, and a *finite unit-norm tight frame* (FUNTF) if, in addition, $\|u_i\| = 1$ for all i . For a FUNTF, taking the trace of $\sum_{i=1}^r |u_i\rangle \langle u_i| = AI_a$ gives $A = r/a$, and one obtains the reconstruction formula

$$x = \frac{a}{r} \sum_{i=1}^r \langle u_i, x \rangle u_i \quad \text{for all } x \in \mathbb{C}^a.$$

FUNTFs exist for every $r \geq a$. The unit-norm and tightness conditions ensure that the construction below produces a state with maximally mixed marginals (Theorem 3.1(i)).

It is convenient to express frame operations via matrices. Given a frame $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ with *synthesis matrix* $U \in \mathbb{C}^{a \times r}$ (the frame vectors as columns, so $Uc = \sum_{i=1}^r c_i u_i$), we record four objects used in the paper:

- **Analysis operator** $U^* : \mathbb{C}^a \rightarrow \mathbb{C}^r$, the adjoint of U , mapping x to its coefficient sequence $(\langle u_i, x \rangle)_{i=1}^r$.
- **Frame operator** $UU^* \in \mathbb{C}^{a \times a}$: tightness reads $UU^* = \frac{r}{a}I_a$.
- **Gram matrix** $G_U = U^*U \in \mathbb{C}^{r \times r}$, recording the pairwise inner products $(G_U)_{ij} = \langle u_i, u_j \rangle$. For a FUNTF, its diagonal entries are all 1.
- **Spectrum of G_U** : since G_U shares its non-zero eigenvalues with UU^* , a tight FUNTF has $\text{sp}(G_U) = \{r/a \text{ (multiplicity } a), 0 \text{ (multiplicity } r - a)\}$.

A second FUNTF $\{v_i\}_{i=1}^r \subset \mathbb{C}^b$ of the same length r is recorded identically, with synthesis matrix $V \in \mathbb{C}^{b \times r}$ and Gram matrix $G_V = V^*V$.

3.2. The rank-one tuple and its extremality criterion. Associated to a pair of FUNTFs U, V of length r is the equal-weight, rank-one Kraus tuple

$$(9) \quad K_i = \sqrt{\frac{a}{r}} |v_i\rangle\langle u_i|, \quad i = 1, \dots, r.$$

The specific coefficient $\sqrt{a/r}$ is dictated by the trace-preserving constraint $\sum K_i^* K_i = I_a$ under the J_m correspondence (see Remark 2.11).

Note that $|\text{vec}(K_i)\rangle = \sqrt{a/r} |\bar{u}_i\rangle \otimes |v_i\rangle$, and

$$|\text{vec}(K_i)\rangle\langle\text{vec}(K_i)| = \frac{a}{r} (|\bar{u}_i\rangle \otimes |v_i\rangle)(\langle\bar{u}_i| \otimes \langle v_i|) = \frac{a}{r} |\bar{u}_i\rangle\langle\bar{u}_i| \otimes |v_i\rangle\langle v_i|.$$

Consider the quantum channel given by $\Phi(X) = \sum_{i=1}^r K_i X K_i^*$. We may directly verify that Φ satisfies the tracial marginal constraints (2). Tightness gives $\sum_{i=1}^r |u_i\rangle\langle u_i| = \frac{r}{a} I_a$ and $\sum_{i=1}^r |v_i\rangle\langle v_i| = \frac{r}{b} I_b$. Since $K_i^* = \sqrt{a/r} |u_i\rangle\langle v_i|$ and $\|v_i\| = \|u_i\| = 1$, we have

$$\begin{aligned} K_i^* K_i &= \frac{a}{r} \|v_i\|^2 |u_i\rangle\langle u_i| = \frac{a}{r} |u_i\rangle\langle u_i|, & \sum_{i=1}^r K_i^* K_i &= \frac{a}{r} \cdot \frac{r}{a} I_a = I_a, \\ K_i K_i^* &= \frac{a}{r} \|u_i\|^2 |v_i\rangle\langle v_i| = \frac{a}{r} |v_i\rangle\langle v_i|, & \sum_{i=1}^r K_i K_i^* &= \frac{a}{r} \cdot \frac{r}{b} I_b = \frac{a}{b} I_b. \end{aligned}$$

Using (5), we observe that the (normalized) Choi state of the quantum channel is

$$(10) \quad \rho := J_m(\Phi) = \frac{1}{a} J(\Phi) = \frac{1}{a} \sum_{i=1}^r |\text{vec}(K_i)\rangle\langle\text{vec}(K_i)| = \frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle\langle\bar{u}_i| \otimes |v_i\rangle\langle v_i|.$$

Theorem 3.1. *Let $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ and $\{v_i\}_{i=1}^r \subset \mathbb{C}^b$ be finite unit-norm tight frames (FUNTFs) of the same length r , and let ρ be the state defined in (10),*

$$\frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle\langle\bar{u}_i| \otimes |v_i\rangle\langle v_i|.$$

Then we have the following:

- (i) $\rho \in \mathcal{C}(I_a/a, I_b/b)$.
- (ii) $\text{rank } \rho = \text{rank}(\overline{G_U} \circ G_V)$, where G_U and G_V are the Gram matrices of the frames.
- (iii) $\text{rank } \rho = r$ if and only if the product vectors $\bar{u}_i \otimes v_i$ are linearly independent.

Proof. (i) Tracing out the second subsystem,

$$\text{Tr}_B \rho = \frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle\langle\bar{u}_i| \|v_i\|^2 = \frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle\langle\bar{u}_i| = \frac{I_a}{a},$$

using $\|v_i\| = 1$ and the tightness of $\{u_i\}$, which gives $\sum_{i=1}^r |u_i\rangle\langle u_i| = \frac{r}{a} I_a$. Tracing out the first subsystem,

$$\text{Tr}_A \rho = \frac{1}{r} \sum_{i=1}^r \|\bar{u}_i\|^2 |v_i\rangle\langle v_i| = \frac{1}{r} \sum_{i=1}^r |v_i\rangle\langle v_i| = \frac{I_b}{b},$$

using $\|\bar{u}_i\| = \|u_i\| = 1$ and the tightness of $\{v_i\}$, $\sum_{i=1}^r |v_i\rangle\langle v_i| = \frac{r}{b} I_b$. Hence $\rho \in \mathcal{C}(I_a/a, I_b/b)$.

(ii) Let $|w_i\rangle = |\bar{u}_i\rangle \otimes |v_i\rangle$ and let W be the matrix with columns $|w_1\rangle, \dots, |w_r\rangle$, so that $\rho = \frac{1}{r}WW^*$ and $\text{rank } \rho = \text{rank}(W^*W)$. The matrix W^*W has entries

$$(W^*W)_{ij} = \langle w_i, w_j \rangle = \langle \bar{u}_i, \bar{u}_j \rangle \langle v_i, v_j \rangle = \overline{(G_U)_{ij}} (G_V)_{ij},$$

i.e. $W^*W = \overline{G_U} \circ G_V$, giving $\text{rank } \rho = \text{rank}(\overline{G_U} \circ G_V)$.

(iii) By (ii), $\text{rank } \rho = \text{rank } W$ equals the dimension of the span of the r vectors $|w_i\rangle$, which is r precisely when they are linearly independent. $\square$

Whether ρ is extreme depends on the inner products within the two frames. We write $\circ$ for the Hadamard (entrywise) product on $M_r(\mathbb{C})$.

Theorem 3.2 (Transversality criterion). *Let $\{u_i\}_{i=1}^r \subset \mathbb{C}^a$ and $\{v_i\}_{i=1}^r \subset \mathbb{C}^b$ be FUNTFs of the same length r , with synthesis matrices U, V , and let*

$$\rho := \frac{1}{r} \sum_{i=1}^r |\bar{u}_i\rangle \langle \bar{u}_i| \otimes |v_i\rangle \langle v_i|$$

be the corresponding state, as in Theorem 3.1. Let $\Theta_{U,V} : M_r(\mathbb{C}) \rightarrow M_b(\mathbb{C}) \oplus M_a(\mathbb{C})$ be the linear map given by

$$\Theta_{U,V}(X) := \left(V(X^T \circ G_U)V^*, U(X \circ G_V)U^* \right).$$

Then:

- (i) ρ is an extreme point of the coupling set $\mathcal{C}(I_a/a, I_b/b)$ if and only if $\Theta_{U,V}$ is injective, that is, $\ker \Theta_{U,V} = \{(0, 0)\}$.
- (ii) If $\Theta_{U,V}$ is injective, the product vectors $\bar{u}_i \otimes v_i$ are linearly independent, and consequently $\text{rank } \rho = r$.

Proof. From Theorem 2.13 and Theorem 3.1, we see that ρ is an extreme point if and only if the simultaneous equations

$$\sum_{i,j=1}^r x_{ij} K_i^* K_j = 0 \quad \text{and} \quad \sum_{i,j=1}^r x_{ij} K_j K_i^* = 0$$

imply $X = (x_{ij}) = 0$.

(i) Since $\frac{r}{a} K_i^* K_j = \langle v_i, v_j \rangle |u_i\rangle \langle u_j| = (G_V)_{ij} |u_i\rangle \langle u_j|$ and the (i, j) -th entry of $X \circ G_V$ is $x_{ij}(G_V)_{ij}$, summing against x_{ij} yields

$$\frac{r}{a} \sum_{i,j=1}^r x_{ij} K_i^* K_j = \sum_{i,j=1}^r x_{ij} (G_V)_{ij} |u_i\rangle \langle u_j| = U(X \circ G_V)U^*.$$

Similarly, since $\frac{r}{a} K_j K_i^* = (G_U)_{ji} |v_j\rangle \langle v_i|$, and $x_{ij}(G_U)_{ji}$ is the (j, i) entry of $X^T \circ G_U$, we have,

$$\frac{r}{a} \sum_{i,j=1}^r x_{ij} K_j K_i^* = V(X^T \circ G_U)V^*.$$

The two sums therefore vanish simultaneously if and only if $\Theta_{U,V}(X) = (0, 0)$. Thus, the extremality criterion of Theorem 2.13 reduces to $\ker \Theta_{U,V} = \{0\}$.

(ii) Suppose $\sum_{i=1}^r c_i K_i = 0$ for scalars $c_1, \dots, c_r \in \mathbb{C}$, and write $c := (c_1, \dots, c_r) \in \mathbb{C}^r$. Taking the adjoint gives $\sum_{i=1}^r \bar{c}_i K_i^* = 0$. Let $\{e_k\}_{k=1}^r$ be the standard basis of $\mathbb{C}^r$. Fix $1 \leq k \leq r$ and define $X = |\bar{c}\rangle\langle e_k|$, meaning $x_{ij} = \bar{c}_i \delta_{jk}$. Then

$$\sum_{i,j=1}^r x_{ij} K_i^* K_j = \left(\sum_{i=1}^r \bar{c}_i K_i^* \right) K_k = 0, \quad \sum_{i,j=1}^r x_{ij} K_j K_i^* = K_k \left(\sum_{i=1}^r \bar{c}_i K_i^* \right) = 0.$$

This demonstrates that $X \in \ker \Theta_{U,V}$. Injectivity forces $X = 0$, hence $c = 0$, i.e. $c_i = 0$ for every i , proving the Kraus operators K_i are linearly independent. Because vectorization is a linear isomorphism mapping K_i to $\sqrt{a/r} \bar{u}_i \otimes v_i$, the product vectors $\bar{u}_i \otimes v_i$ are linearly independent. This shows that $\text{rank } \rho = r$. $\square$

Definition 3.3 (Transversal pair). We call a pair of FUNTFs U, V of the same length *transversal* when $\ker \Theta_{U,V} = \{0\}$.

By Theorem 3.2, existence of a single transversal pair of length r guarantees the existence of a rank- r extreme point in $\mathcal{C}(I_a/a, I_b/b)$. Thus for a given dimension pair (a, b) , the tightness of the KRP bound can be established by finding one transversal pair of length $r = \text{KRP}(a, b)$.

Remark 3.4. The rank-one, equal-weight form of the K_i is a restriction we impose; it is not forced by extremality, and we do not expect it to produce every KRP-tight state. Ohno's extreme points at $(3, 3)$ and $(4, 4)$ require Kraus operators of rank greater than one and fall outside it. Its advantage is that extremality becomes the injectivity of the explicit map $\Theta_{U,V}$.

3.3. The projection formulation. The kernel of $\Theta_{U,V}$ admits a description that depends on two orthogonal projections, the scaled Gram matrices:

$$(11) \quad P_U := \frac{a}{r} G_U, \quad P_V := \frac{b}{r} G_V.$$

The projections P_U and P_V project onto the row spaces of U and V , respectively.

Lemma 3.5. *The matrices P_U and P_V are orthogonal projections satisfying:*

$$\text{rank } P_U = a, \quad \text{rank } P_V = b, \quad (P_U)_{ii} = \frac{a}{r}, \quad (P_V)_{ii} = \frac{b}{r} \quad \text{for all } i.$$

Proof. By the tightness of the frame U , its frame operator is $UU^* = \frac{r}{a} I_a$, so

$$P_U^2 = \left(\frac{a}{r} \right)^2 U^* (UU^*) U = \left(\frac{a}{r} \right)^2 \frac{r}{a} U^* U = P_U.$$

Since G_U is Hermitian, $P_U = P_U^*$, so P_U is an orthogonal projection. Its rank is a because U has full row rank, and its diagonal entries are $\frac{a}{r}$ because the columns of U have unit norm. The same argument applies to P_V . $\square$

Lemma 3.6. *For $X \in M_r(\mathbb{C})$, $\Theta_{U,V}(X) = 0$ if and only if*

$$(12) \quad P_V(X^T \circ P_U)P_V = 0 \quad \text{and} \quad P_U(X \circ P_V)P_U = 0.$$

Consequently, transversality depends on the two scaled Gram projections P_U and P_V relative to the standard coordinate basis used for the Hadamard product.

Proof. Since the global scalar factors $\frac{a}{r}$ and $\frac{b}{r}$ do not alter the kernel, it suffices to show that for any matrix $A \in M_r(\mathbb{C})$, $VAV^* = 0$ is equivalent to $P_V A P_V = 0$, and that $UAU^* = 0$ is equivalent to $P_U A P_U = 0$. Applying the first with $A = X^T \circ G_U$ and the second with $A = X \circ G_V$ then gives (12) from the definition of $\Theta_{U,V}$.

Since

$$VV^* = \frac{r}{b}I_b \quad \text{and} \quad UU^* = \frac{r}{a}I_a,$$

we have

$$VAV^* = 0 \implies V^*VAVV^* = 0 \implies P_VAP_V = 0$$

and

$$P_VAP_V = 0 \implies V^*VAV^*V = 0 \implies V(V^*VAV^*V)V^* = 0 \implies VAV^* = 0,$$

proving the first equivalence. The identical argument, with U, P_U in place of V, P_V , proves $UAU^* = 0 \iff P_UAP_U = 0$. $\square$

4. TIGHTNESS OF THE PARTHASARATHY RANK BOUND

In this section, we construct a rank- b extremal tracial coupling for every $a < b$, establishing KRP tightness in the region $b \geq (a^2 - 1)/2$ and along the qubit-qudit line $(2, b)$, $b \geq 3$. For frames of length $a + 1$ in $\mathbb{C}^a$, however, we show that the transversality operator $\Theta_{U,V}$ always has non-trivial kernel, which obstructs the construction at $(3, 3)$ and $(4, 4)$.

4.1. Extremal tracial couplings of rank b for $a < b$. When the lengths of the FUNTF pair equal b , the second FUNTF is square, and a square tight frame is unitary. Since conjugation by a unitary is injective, taking $V = I_b$ reduces injectivity of $\Theta_{U,V}$ to injectivity of its first component alone: the map $X \mapsto X^T \circ G_U$ on $M_r(\mathbb{C})$, which is injective exactly when G_U has no zero entries. We construct such a U in the following way for every $2 \leq a < b$: one fixed vector, together with $b - 1$ vectors at an equal angle to it, obtained from one another by rotating within its orthogonal complement via the $(b - 1)$ -th roots of unity.

Theorem 4.1. *For every dimension pair (a, b) satisfying $2 \leq a < b$, there exists a transversal pair of FUNTFs of length b in $\mathbb{C}^a$ and $\mathbb{C}^b$. Consequently, there is an extreme point of $\mathcal{C}(I_a/a, I_b/b)$ of rank b .*

Proof. Take $V = I_b$, a square unitary FUNTF, so $G_V = I_b$ and $\Theta_{U,V}(X) = (X^T \circ G_U, U \operatorname{diag}(X)U^*)$. Transversality of (U, I_b) is then equivalent to injectivity of $X \mapsto X^T \circ G_U$ on $M_b(\mathbb{C})$, which holds exactly when $G_U = U^*U$ has no zero entries. So it suffices to construct a FUNTF $U \in \mathbb{C}^{a \times b}$ whose Gram matrix has all entries non-zero.

Let $\{|e_k\rangle\}_{k=0}^{a-1}$ be the standard basis of $\mathbb{C}^a$. Define weights

$$\alpha = \frac{b-a}{a(b-1)}, \quad \beta = \frac{b(a-1)}{a(b-1)},$$

so that $\alpha + \beta = 1$ and $\alpha, \beta \in (0, 1)$. Let $\zeta = e^{2\pi i/(b-1)}$, a primitive $(b-1)$ -th root of unity, and for $j = 0, 1, \dots, b-2$ define the unit vector $|w_j\rangle \in \operatorname{span}\{|e_1\rangle, \dots, |e_{a-1}\rangle\}$ by

$$|w_j\rangle = \frac{1}{\sqrt{a-1}} \sum_{k=1}^{a-1} \zeta^{kj} |e_k\rangle.$$

The b columns of U are

$$|u_0\rangle = |e_0\rangle, \quad |u_j\rangle = \sqrt{\alpha} |e_0\rangle + \sqrt{\beta} |w_{j-1}\rangle, \quad j = 1, \dots, b-1.$$

Since $\langle w_{j-1} | w_{j-1} \rangle = 1$ and $\alpha + \beta = 1$, every $|u_j\rangle$ is a unit vector, and so is $|u_0\rangle$.

We compute the matrix elements of $UU^* = |u_0\rangle\langle u_0| + \sum_{j=1}^{b-1} |u_j\rangle\langle u_j|$ in the standard basis. For $k, k' \in \{1, \dots, a-1\}$,

$$\langle e_k | UU^* | e_{k'} \rangle = \beta \sum_{j=1}^{b-1} \langle e_k | w_{j-1} \rangle \langle w_{j-1} | e_{k'} \rangle = \frac{\beta}{a-1} \sum_{m=0}^{b-2} \zeta^{(k-k')m} = \frac{\beta(b-1)}{a-1} \delta_{k,k'} = \frac{b}{a} \delta_{k,k'},$$

where we re-indexed $m = j-1$ and used that $\zeta^{k-k'} \neq 1$ for $0 < |k-k'| \leq a-2 < b-1$. For the off-diagonal elements involving $|e_0\rangle$ and $|e_k\rangle$ with $k \geq 1$,

$$\langle e_0 | UU^* | e_k \rangle = \sqrt{\alpha\beta} \sum_{j=1}^{b-1} \langle w_{j-1} | e_k \rangle = \frac{\sqrt{\alpha\beta}}{\sqrt{a-1}} \sum_{m=0}^{b-2} \zeta^{-km} = 0,$$

since $1 \leq k \leq a-1 < b-1$ ensures $\zeta^{-k} \neq 1$. Finally,

$$\langle e_0 | UU^* | e_0 \rangle = 1 + (b-1)\alpha = \frac{b}{a}.$$

Thus $UU^* = \frac{b}{a}I_a$, and U is a FUNTF of b vectors in $\mathbb{C}^a$.

Note that,

$$\langle u_0 | u_j \rangle = \sqrt{\alpha} \neq 0 \quad \text{for every } j \geq 1$$

For $1 \leq j < j' \leq b-1$, set $d = j' - j$ with $1 \leq d \leq b-2$, and let $S(d) = \sum_{k=1}^{a-1} \zeta^{kd}$. Then

$$\langle u_j | u_{j'} \rangle = \alpha + \beta \langle w_{j-1} | w_{j'-1} \rangle = \alpha + \beta \frac{S(d)}{a-1}.$$

Since $S(d)$ is a sum of roots of unity, it is an algebraic integer. If the inner product were zero, we would have $S(d) = -(a-1)\alpha/\beta = -(b-a)/b$, which is rational but not an integer (as $0 < b-a < b$). A rational algebraic integer is an integer ([DF03, Chapter 15, Prop.28]), so equality is impossible. Hence $\langle u_j | u_{j'} \rangle \neq 0$ for all $j \neq j'$. Therefore G_U has no zero entries, $\ker \Theta_{U, I_b} = \{(0, 0)\}$, and (U, I_b) is a transversal pair of length b . $\square$

Corollary 4.2. *For $a \leq b$, we have $R_{\max}(a, b) \geq b$. This supplements Ohno's result that $R_{\max}(a, a) \geq a$.*

Corollary 4.3. *If $2 \leq a < b$ and $\frac{a^2-1}{2} \leq b$, then $R_{\max}(a, b) = \text{KRP}(a, b) = b$.*

Proof. Since $b \geq \frac{1}{2}(a^2 - 1)$, we have $b^2 \leq a^2 + b^2 - 1 < (b+1)^2$. Hence $\text{KRP}(a, b) = \lfloor \sqrt{a^2 + b^2 - 1} \rfloor = b$, and Theorem 4.1 produces an extreme coupling of rank b , so $R_{\max}(a, b) = b$. The KRP bound is tight in particular for

$$(2, b), \quad b \geq 3; \quad (3, b), \quad b \geq 4; \quad (4, b), \quad b \geq 8; \quad (5, b), \quad b \geq 12. \quad \square$$

4.2. Some obstructions to transversality.

Proposition 4.4. *For every pair of FUNTFs U, V of length a in $\mathbb{C}^a$, $\ker \Theta_{U, V}$ consists of the matrices with zeroes on the diagonal. Thus,*

$$\dim_{\mathbb{C}}(\ker \Theta_{U, V}) = a^2 - a.$$

Proof. Because U and V are FUNTFs of length $r = a$, their square synthesis matrices satisfy $UU^* = I_a$ and $VV^* = I_a$. Thus U and V are unitary matrices, whence $G_U = G_V = I_a$.

For any $X \in M_a(\mathbb{C})$, $X \circ I_a = X^T \circ I_a = \text{diag}(X)$, so the transversality map reduces to:

$$\Theta_{U, V}(X) = \left(V (X^T \circ G_U) V^*, U (X \circ G_V) U^* \right) = \left(V \text{diag}(X) V^*, U \text{diag}(X) U^* \right).$$

Because U and V are invertible, $\Theta_{U,V}(X) = 0$ if and only if $\text{diag}(X) = 0$.

Therefore, the kernel of $\Theta_{U,V}$ is the subspace of matrices with zeros on their main diagonal, which has complex dimension $a^2 - a$. $\square$

The failure at $(2, 2)$ may be explained by the fact that there are no rank-2 extreme points in $\mathcal{C}(I_2/2, I_2/2)$ at all, let alone one obtained from a transversal pair of FUNTFs. The cases $(3, 3)$ and $(4, 4)$ are different, and rest on the following elementary facts about rank-one projections.

Lemma 4.5. *Let Q be a rank-one orthogonal projection on $\mathbb{C}^r$ with constant diagonal entries. Then each diagonal entry equals $1/r$, and $Q = \frac{1}{r}|p\rangle\langle p|$ for some $p \in \mathbb{C}^r$ all of whose entries are of unit-modulus, that is, $|p_i| = 1$ for every $1 \leq i \leq r$.*

Proof. Since Q has rank one, $Q = |w\rangle\langle w|$ for some unit vector $w \in \mathbb{C}^r$. Its diagonal entries $|w_i|^2$ sum to $\langle w|w\rangle = 1$; being all equal, each equals $1/r$. Writing $p := \sqrt{r}w$ gives $Q = \frac{1}{r}|p\rangle\langle p|$, and $|p_i|^2 = r|w_i|^2 = 1$ for every i . $\square$

For a FUNTF of length $r = a + 1$, the correlation projections P_U, P_V (Lemma 3.5) have rank $a = r - 1$ and constant diagonal a/r , so $I_r - P_U, I_r - P_V$ are rank-one orthogonal projections with constant diagonal. By Lemma 4.5,

$$I_r - P_U = \frac{1}{r}|p\rangle\langle p|, \quad I_r - P_V = \frac{1}{r}|q\rangle\langle q|$$

for unit-modulus $p, q \in \mathbb{C}^r$ (the Naimark complements of U, V). To rule out a transversal pair it suffices to exhibit one non-zero element of $\ker \Theta_{U,V}$, which we now do.

By Lemma 3.6, with $P_U = I_r - \frac{1}{r}|p\rangle\langle p|$ and $P_V = I_r - \frac{1}{r}|q\rangle\langle q|$ as above, we seek a non-zero $X \in M_r(\mathbb{C})$ with

$$P_V(X^T \circ P_U)P_V = 0 \quad \text{and} \quad P_U(X \circ P_V)P_U = 0.$$

The next lemma describes the solutions for X in either condition.

Lemma 4.6. *Let $P = I - \frac{1}{r}|q\rangle\langle q|$ be the orthogonal projection onto $(\text{span}\{|q\rangle\})^\perp$ for a vector $q \in \mathbb{C}^r$ with $\langle q|q\rangle = r$. For any matrix $A \in M_r(\mathbb{C})$,*

$$PAP = 0 \quad \Longleftrightarrow \quad A = |q\rangle\langle x| + |y\rangle\langle q| \text{ for some } x, y \in \mathbb{C}^r.$$

Proof. If $A = |q\rangle\langle x| + |y\rangle\langle q|$, then $PAP = (P|q\rangle)\langle x|P + P|y\rangle(\langle q|P) = 0$ since $P|q\rangle = 0$ and $\langle q|P = 0$. Conversely, if $PAP = 0$, the range of AP lies entirely in $\ker P = \text{span}\{|q\rangle\}$. Thus, $AP = |q\rangle\langle \tilde{x}|$ for some vector $\tilde{x} \in \mathbb{C}^r$. Decomposing A via the projection yields $A = A(P + I - P) = AP + \frac{1}{r}A|q\rangle\langle q|$. Substituting $AP = |q\rangle\langle \tilde{x}|$ gives $A = |q\rangle\langle \tilde{x}| + \left(\frac{A|q\rangle}{r}\right)\langle q|$. Setting $x = \tilde{x}$ and $y = \frac{A|q\rangle}{r}$ recovers the desired form. $\square$

Proposition 4.7. *For every pair of FUNTFs U, V of length $a + 1$ in $\mathbb{C}^a$, the map $\Theta_{U,V}$ has non-trivial kernel.*

Proof. Set $r = a + 1$. With $P_U = I_r - \frac{1}{r}|p\rangle\langle p|, P_V = I_r - \frac{1}{r}|q\rangle\langle q|$ for $p, q \in \mathbb{C}^r$ whose entries have unit modulus, by Lemma 4.6,

$$P_V(X^T \circ P_U)P_V = 0 \quad \Longleftrightarrow \quad X^T \circ P_U = |q\rangle\langle x| + |y\rangle\langle q|$$

for some $x, y \in \mathbb{C}^r$. Every entry of P_U is non-zero with $(P_U)_{ii} = \frac{r-1}{r}, (P_U)_{ij} = -\frac{1}{r}p_i\bar{p}_j$ for $i \neq j$ since p has entries of unit modulus. Taking the simplest nontrivial choice $x = 0, y = q$, we

get $X^T \circ P_U = |q\rangle\langle q|$. Dividing entrywise by $(P_U)_{ij} = \delta_{ij} - \frac{1}{r}p_i\bar{p}_j$, gives $X^T = \frac{r^2}{r-1}I_r - r|s\rangle\langle s|$, where $s = \bar{p} \circ q$. Equivalently,

$$X := \frac{r^2}{r-1}I_r - r|\bar{s}\rangle\langle \bar{s}|.$$

Since p and q have unit-modulus entries, $s = \bar{p} \circ q$ does as well, and

$$(X \circ P_V)_{ii} = 1 = p_i\bar{p}_i \quad \text{and} \quad (X \circ P_V)_{ij} = (-r\bar{s}_i s_j) \left(\frac{1}{r}q_i\bar{q}_j \right) = p_i\bar{p}_j, \quad \text{for } i \neq j.$$

Hence, $X \circ P_V = |p\rangle\langle p|$. Since $P_V|q\rangle = 0$ and $P_U|p\rangle = 0$, both conditions of Lemma 3.6 hold, so $X \in \ker \Theta_{U,V}$. Finally,

$$\text{Tr } X = \text{Tr} \left(\frac{r^2}{r-1}I_r - r|\bar{s}\rangle\langle \bar{s}| \right) = \frac{r^3}{r-1} - r^2 = \frac{r^2}{r-1} > 0,$$

so $X \neq 0$. □

Remark 4.8. Let $a \geq 2$ be a positive integer. The frame length relevant to KRP tightness at (a, a) is $r = \text{KRP}(a, a)$, and Proposition 4.7 applies precisely when $r = a + 1$, i.e.

$$a + 1 \leq \sqrt{2a^2 - 1} < a + 2.$$

For $a \geq 2$, we have

$$a^2 + 2a + 1 \leq 2a^2 - 1 < a^2 + 4a + 4,$$

equivalently $a^2 - 2a - 2 \geq 0$ and $a^2 - 4a - 5 < 0$. The first holds for integers $a \geq \lceil 1 + \sqrt{3} \rceil = 3$, and the second holds for integers $a < 5$, so that $a \in \{3, 4\}$. Thus, among square dimension pairs, $\text{KRP}(a, a) = a + 1$ occurs only at $(3, 3)$ and $(4, 4)$.

Corollary 4.9. *Every pair of FUNTFs U, V of length 4 in $\mathbb{C}^3$, and every pair of FUNTFs U, V of length 5 in $\mathbb{C}^4$, has $\ker \Theta_{U,V} \neq \{0\}$. Thus, there are no transversal pairs to be found in the $(3, 3)$ or $(4, 4)$ case.*

Proof. Both cases correspond to FUNTFs of length $n + 1$ in $\mathbb{C}^n$, with $n = 3$ and $n = 4$, respectively. The assertion follows immediately from Proposition 4.7. □

Remark 4.10. Corollary 4.9 shows why the transversal-pair construction cannot handle the dimension pairs $(3, 3)$ and $(4, 4)$. Both pairs are nevertheless KRP tight: Ohno's constructions give extreme points of rank 4 and 5 [Ohn10], respectively, using Kraus operators of rank greater than one.

5. TIGHTNESS OF THE SCHMIDT RANK BOUND

Theorem 2.3 bounds the rank of an extremal tracial coupling between the Schmidt bound $\text{ES}(a, b) = \lceil b/a \rceil$ and the Parthasarathy bound $\text{KRP}(a, b) = \lfloor \sqrt{a^2 + b^2 - 1} \rfloor$. Section 4 concerned dimension pairs at which the Parthasarathy upper bound is attained. Here, we explore the analogous question for the Schmidt bound: *For which values of (a, b) is this lower bound tight?*

5.1. Extremal tracial couplings with rank $\text{ES}(a, b)$. Let $2 \leq a \leq b$ be positive integers. We start with the case where $a \mid b$.

Proposition 5.1. *Let $2 \leq a$ be a positive integer. Then the dimension pair (a, a) is ES-tight, that is, $R_{\min}(a, a) = \text{ES}(a, a) = 1$.*

Proof. Let $\psi := \frac{1}{\sqrt{a}} \sum_{i=1}^a |e_i\rangle \otimes |e_i\rangle$, and set $\rho := |\psi\rangle\langle\psi|$. A direct computation shows $\text{Tr}_A(\rho) = \text{Tr}_B(\rho) = I_a/a$, so $\rho \in \mathcal{C}(I_a/a, I_a/a)$. Since ρ is a pure state, it is an extremal tracial coupling of rank 1, establishing $R_{\min}(a, a) = \text{ES}(a, a) = 1$. $\square$

Proposition 5.2. *Let $a \leq b$ be positive integers such that a divides b , and set $r = \frac{b}{a}$. Then there exists an extreme point of the tracial coupling set $\mathcal{C}(I_a/a, I_b/b)$ of rank r .*

Proof. Because $b = ra$, we can orthogonally decompose the target space as $\mathbb{C}^b = \bigoplus_{i=1}^r B_i$, where $\dim B_i = a$. For each i , let $V_i : \mathbb{C}^a \rightarrow \mathbb{C}^b$ be an isometry with range B_i . Note that $V_i^* V_j = \delta_{ij} I_a$ and $\sum_{i=1}^r V_i V_i^* = I_b$. Defining the Kraus operators as $K_i = \frac{1}{\sqrt{r}} V_i$, we obtain the marginal constraints:

$$\sum_{i=1}^r K_i^* K_i = \frac{1}{r} \sum_{i=1}^r I_a = I_a, \quad \sum_{i=1}^r K_i K_i^* = \frac{1}{r} \sum_{i=1}^r V_i V_i^* = \frac{a}{b} I_b.$$

Thus, the Choi state ρ associated with $\{K_i\}_{i=1}^r$ is a tracial coupling of rank r . To establish extremality, suppose $X = (x_{ij}) \in M_r(\mathbb{C})$ satisfies $\sum_{i,j=1}^r x_{ij} K_i K_j^* = 0$. Left-multiplying by V_m^* and right-multiplying by V_n isolates a single coefficient:

$$V_m^* \left(\sum_{i,j=1}^r x_{ij} \frac{1}{r} V_i V_j^* \right) V_n = \frac{1}{r} x_{mn} I_a = 0,$$

which forces $x_{mn} = 0$ for all m, n . Thus, from Theorem 2.13, it follows that ρ is an extreme point of rank r . $\square$

We will need the following lemma.

Lemma 5.3. *Let $a \leq b$ and $r := \lceil \frac{b}{a} \rceil$. Let $K_1, \dots, K_r : \mathbb{C}^a \rightarrow \mathbb{C}^b$ satisfy the constraints $\sum_{i=1}^r K_i^* K_i = I_a$ and $\sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b$. Then $\text{rank } K_i = a$ for every $1 \leq i \leq r$.*

Proof. Set $c := a/b$. Since $K_i K_i^* \geq 0$ and $\sum_{i=1}^r K_i K_i^* = c I_b$, each $K_i K_i^* \leq c I_b$, so $\text{sp}(K_i K_i^*) \subseteq [0, c]$. As $K_i K_i^*$ and $K_i^* K_i$ have the same non-zero eigenvalues, this gives $\text{sp}(K_i^* K_i) \subseteq [0, c]$ as well, i.e., $K_i^* K_i \leq c I_a$, for every $1 \leq i \leq r$.

Fix $1 \leq j \leq r$. Summing the bound over the remaining indices and using $\sum_{i=1}^r K_i^* K_i = I_a$ gives

$$K_j^* K_j = I_a - \sum_{i \neq j} K_i^* K_i \geq I_a - (r-1)c I_a = (1 - (r-1)c) I_a.$$

By definition of r , $r-1 < \frac{b}{a} \leq r$, so $1 - (r-1)c > 0$. Thus $K_j^* K_j$ is invertible, so $\ker K_j = \{0\}$, and by rank-nullity, $\text{rank } K_j = a$. $\square$

Definition 5.4. Let $A \in M_{a,b}(\mathbb{C})$. We say that A is a *partial permutation matrix* if it has entries from $\{0, 1\}$ with at most one 1 in each row and each column. More generally, we say that A is a *weighted partial permutation matrix* if it has at most one non-zero entry in each row and each column.

Observation 5.5. Let $a \leq b$ be positive integers. To an injective mapping $\pi : \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ and a mapping $w : \{1, \dots, a\} \rightarrow (0, \infty)$, we may associate a $b \times a$ weighted partial permutation matrix as follows. We view $\{1, \dots, a\}, \{1, \dots, b\}$, respectively, as the indexing set for the set of columns, rows, respectively, of a $b \times a$ matrix. In this context, $\pi(k)$ is the row of the unique non-zero entry in column k and $w(k)$ its value. We define the entries of the weighted partial permutation matrix $A : \mathbb{C}^a \rightarrow \mathbb{C}^b$ associated with (π, w) by:

$$A_{m,k} = \begin{cases} w(k) & \text{if } m = \pi(k), \\ 0 & \text{otherwise.} \end{cases}$$

Consider injective mappings $\pi_1, \pi_2 : \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ and mappings $w_1, w_2 : \{1, \dots, a\} \rightarrow (0, \infty)$. Let A_1, A_2 , respectively, be the weighted partial permutation matrices associated with $(\pi_1, w_1), (\pi_2, w_2)$, respectively. The product $A_1 A_2^*$ has entries:

$$(A_1 A_2^*)_{m,n} = \begin{cases} w_1(k)w_2(k) & \text{if } \exists k \text{ such that } m = \pi_1(k) \text{ and } n = \pi_2(k), \\ 0 & \text{otherwise.} \end{cases}$$

Because π_1 and π_2 are injective, this k is unique when it exists. Moreover, the product $A_1^* A_2$ has entries:

$$(A_1^* A_2)_{k,l} = \begin{cases} w_1(k)w_2(l) & \text{if } \pi_1(k) = \pi_2(l), \\ 0 & \text{otherwise.} \end{cases}$$

Because the $K_i K_j^*$ have small and mostly disjoint supports, a single entry of $\sum_{i,j} x_{ij} K_i K_j^*$ usually involves only one x_{ij} ; the next lemma exploits this.

Lemma 5.6. For $1 \leq i \leq r$, consider injective mappings $\pi_i : \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ and mappings $w_i : \{1, \dots, a\} \rightarrow (0, \infty)$. Let $K_i : \mathbb{C}^a \rightarrow \mathbb{C}^b$ be the weighted partial permutation matrix associated with (π_i, w_i) as in Observation 5.5. Assume that $\{K_i\}_{i=1}^r$ satisfies the tracial marginal constraints $\sum_{i=1}^r K_i^* K_i = I_a$ and $\sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b$ and that the following holds:

- (a) For each i , there exists a row index m in the range of π_i that is not in the range of any other π_j ($j \neq i$).
- (b) For each distinct pair $i \neq j$, there exists a column index $k \in \{1, \dots, a\}$ such that the coordinate pair $(\pi_i(k), \pi_j(k))$ is not equal to $(\pi_{i'}(k'), \pi_{j'}(k'))$ for $(i', j') \neq (i, j)$ and any column index k' .

Then the Choi state of $\{K_i\}$ is an extreme point of $\mathcal{C}(I_a/a, I_b/b)$.

Proof. By Theorem 2.13, it suffices to show that $\sum_{i,j} x_{ij} K_i K_j^* = 0$ implies $X = (x_{ij}) = 0$. By Observation 5.5, the non-zero entries of $K_i K_j^*$ occur at the coordinates $(\pi_i(k), \pi_j(k))$.

For the diagonal entries of X , fix i and let $m = \pi_i(k)$ be the exclusive row provided by condition (a). Evaluating the matrix sum at entry (m, m) yields only the (i, i) term, forcing $x_{ii} w_i(k)^2 = 0$. Since $w_i(k) > 0$, we have $x_{ii} = 0$.

For the off-diagonal entries, fix distinct $i \neq j$ and let k be the column provided by condition (b), mapping to $(m, n) = (\pi_i(k), \pi_j(k))$. Evaluating the matrix sum at entry (m, n) yields only the (i, j) term, forcing $x_{ij} (w_i(k) w_j(k)) = 0$. Hence, $x_{ij} = 0$. $\square$

We illustrate Lemma 5.6 in the smallest nontrivial case, $(a, b) = (2, 3)$.

Example 5.7. Consider the dimension pair $(a, b) = (2, 3)$. We first define the injective mappings $\pi_1, \pi_2 : \{1, 2\} \rightarrow \{1, 2, 3\}$ and the weight functions $w_1, w_2 : \{1, 2\} \rightarrow (0, \infty)$ as follows:

$$\begin{aligned} \pi_1(1) &= 1, & w_1(1) &= \sqrt{1/3}, \\ \pi_1(2) &= 2, & w_1(2) &= \sqrt{2/3}, \\ \pi_2(1) &= 3, & w_2(1) &= \sqrt{2/3}, \\ \pi_2(2) &= 1, & w_2(2) &= \sqrt{1/3}. \end{aligned}$$

As given in Observation 5.5, to these pairs (π_i, w_i) , we associate the weighted partial permutation matrices $K_1, K_2 : \mathbb{C}^2 \rightarrow \mathbb{C}^3$:

$$K_1 = \begin{pmatrix} \sqrt{1/3} & 0 \\ 0 & \sqrt{2/3} \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{1/3} \\ 0 & 0 \\ \sqrt{2/3} & 0 \end{pmatrix}.$$

Direct computation verifies the tracial marginals:

$$K_1^* K_1 + K_2^* K_2 = I_2, \quad K_1 K_1^* + K_2 K_2^* = \frac{2}{3} I_3.$$

We verify the two conditions given in Lemma 5.6.

- (a) The range of π_1 is $\{1, 2\}$ and the range of π_2 is $\{1, 3\}$. Row index $m = 2$ is exclusive to range of π_1 , and row index $m = 3$ is exclusive to the range of π_2 .
- (b) There is only one pair (i, j) with $i \neq j$, namely $(1, 2)$. Choosing the column $k = 1$ yields the coordinate pair $(\pi_1(1), \pi_2(1)) = (1, 3)$. No other pair of operators (i', j') at any column k' maps to $(1, 3)$.

The resulting extreme point is the Choi state $\rho = \frac{1}{2} \sum_{i=1}^2 |\text{vec}(K_i)\rangle \langle \text{vec}(K_i)|$. Vectorizing the operators, with $\{|e_j\rangle\}_{j=1}^2, \{|f_m\rangle\}_{m=1}^3$ the standard bases of $\mathbb{C}^2, \mathbb{C}^3$, yields:

$$\begin{aligned} |\text{vec}(K_1)\rangle &= \frac{1}{\sqrt{3}} |e_1\rangle \otimes |f_1\rangle + \sqrt{\frac{2}{3}} |e_2\rangle \otimes |f_2\rangle, \\ |\text{vec}(K_2)\rangle &= \sqrt{\frac{2}{3}} |e_1\rangle \otimes |f_3\rangle + \frac{1}{\sqrt{3}} |e_2\rangle \otimes |f_1\rangle. \end{aligned}$$

In the lexicographic basis $|e_1\rangle \otimes |f_1\rangle, |e_1\rangle \otimes |f_2\rangle, |e_1\rangle \otimes |f_3\rangle, |e_2\rangle \otimes |f_1\rangle, |e_2\rangle \otimes |f_2\rangle, |e_2\rangle \otimes |f_3\rangle$, this produces the 6×6 bipartite density matrix:

$$\rho = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 1 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 5.8. *Let $2 \leq a \leq b$ be positive integers such that $b \equiv 1 \pmod{a}$. Then $\mathcal{C}(I_a/a, I_b/b)$ contains an extreme point of rank $\lceil \frac{b}{a} \rceil$.*

Proof. Let $r := \lceil \frac{b}{a} \rceil$. Since $b \equiv 1 \pmod{a}$, $b = (r-1)a + 1$. For $1 \leq i \leq r-1$, define $\pi_i : \{1, \dots, a\} \rightarrow \{1, \dots, b\}$ as translation by $(i-1)a$:

$$\pi_i(k) = k + (i-1)a, \quad 1 \leq k \leq a.$$

Define π_r as the cyclic translation by $(r-1)a = b-1$:

$$\pi_r(1) = b, \quad \pi_r(k) = k-1 \quad (2 \leq k \leq a).$$

$$\begin{array}{ccccccc} & \text{image of } \pi_r & & & & & \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ b & 1 & 2 & \cdots & a-1 & a & \\ & \underbrace{\hspace{1.5cm}} & & & & & \\ & \text{image of } \pi_1 & & \text{image of } \pi_2 & & \text{image of } \pi_{r-1} & \end{array}$$

Note that each π_i is injective. Define strictly positive weights

$$w_i(k) := \begin{cases} \sqrt{k/b}, & i = 1, \\ \sqrt{a/b}, & 2 \leq i \leq r-1, \\ \sqrt{(a-k+1)/b}, & i = r; \end{cases}$$

on $\{1, \dots, a\}$.

Let $\{|e_k\rangle\}_{k=1}^a$, $\{|f_m\rangle\}_{m=1}^b$ be the standard bases of $\mathbb{C}^a$, $\mathbb{C}^b$, and define

$$K_i := \sum_{k=1}^a w_i(k) |f_{\pi_i(k)}\rangle \langle e_k|.$$

Claim 1. For every $k \in \{1, \dots, a\}$, the values $\pi_1(k), \dots, \pi_r(k)$ are pairwise distinct.

Proof of Claim. For $i \leq r-1$, $\pi_i(k) = (i-1)a + k$ occupies the k -th position within the i -th of the $r-1$ disjoint length- a blocks pictured above. Since the blocks are disjoint, $\pi_1(k), \dots, \pi_{r-1}(k)$ lie in different blocks and so are automatically pairwise distinct, one per block. It remains to compare each with $\pi_r(k)$. At $k=1$, $\pi_r(1) = b$ lies one step beyond the last block, which ends at $(r-1)a = b-1$, so it exceeds every $\pi_i(1)$. At $k \geq 2$, $\pi_r(k) = k-1$ is strictly less than $\pi_1(k) = k$, and hence less than every $\pi_i(k)$ for $i \leq r-1$ as well, since $\pi_i(k) \geq \pi_1(k)$. Either way, $\pi_r(k)$ differs from all the others.

Claim 2. $\sum_{i=1}^r K_i^* K_i = I_a$ and $\sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b$. Moreover, for each i there is a column k_i such that the value $m_i := \pi_i(k_i)$ is attained by π_i but by no other π_j , that is, $m_i \in \text{range}(\pi_i) \setminus \bigcup_{j \neq i} \text{range}(\pi_j)$. (This is the hypothesis (a) of Lemma 5.6.) Explicitly,

$$m_1 = a \text{ at } k_1 = a, \quad m_i = ia \text{ at } k_i = a \quad (2 \leq i \leq r-1), \quad m_r = b \text{ at } k_r = 1.$$

Proof of Claim. By injectivity of π_i , $\langle f_{\pi_i(k)} | f_{\pi_i(k')} \rangle = \delta_{kk'}$, so $K_i^* K_i = \sum_{k=1}^a w_i(k)^2 |e_k\rangle \langle e_k|$ is diagonal. Fixing k ,

$$\sum_{i=1}^r w_i(k)^2 = w_1(k)^2 + \sum_{i=2}^{r-1} w_i(k)^2 + w_r(k)^2 = \frac{k}{b} + (r-2) \frac{a}{b} + \frac{a-k+1}{b} = \frac{(r-1)a+1}{b} = 1,$$

so $\sum_{i=1}^r K_i^* K_i = I_a$.

For $K_i K_i^* = \sum_{k=1}^a w_i(k)^2 |f_{\pi_i(k)}\rangle \langle f_{\pi_i(k)}|$: grouping the ra terms across all i by target row m ,

$$\sum_{i=1}^r K_i K_i^* = \sum_{m=1}^b \left(\sum_{(i,k): \pi_i(k)=m} w_i(k)^2 \right) |f_m\rangle \langle f_m|.$$

The inner coefficient is a/b for every m , checked case by case:

- $1 \leq m \leq a-1$: exactly two pairs land on m , namely $(1, m)$ and $(r, m+1)$, contributing $w_1(m)^2 + w_r(m+1)^2 = \frac{m}{b} + \frac{a-m}{b} = \frac{a}{b}$;
- $m = a$: only $(1, a)$ lands, contributing $w_1(a)^2 = a/b$;
- $a+1 \leq m \leq (r-1)a$: writing $m = k + (i-1)a$ uniquely for $2 \leq i \leq r-1$, only (i, k) lands, contributing $w_i(k)^2 = a/b$;
- $m = b$: only $(r, 1)$ lands, contributing $w_r(1)^2 = a/b$.

Hence $\sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b$.

The rows $m_1, \dots, m_r$ listed in the claim are exactly the singly-covered rows identified in this case analysis ($m = a$, $m = ia$ for $2 \leq i \leq r-1$, and $m = b$), each received by exactly the operator named.

Claim 3. $K_1, \dots, K_r$ are linearly independent, so the Choi state has rank r .

Proof of Claim. Suppose $\sum_l c_l K_l = 0$. Pairing with $\langle f_{m_i}|$ on the left and $|e_{k_i}\rangle$ on the right,

$$0 = \left\langle f_{m_i} \left| \sum_l c_l K_l \right| e_{k_i} \right\rangle = c_i w_i(k_i),$$

since m_i is exclusive to i , so every term but $l = i$ vanishes. As $w_i(k_i) > 0$, this forces $c_i = 0$; since i was arbitrary, $c = 0$.

Claim 4. $\{K_i K_j^*\}_{1 \leq i, j \leq r}$ are linearly independent, so the Choi state is extreme.

Proof of Claim. Suppose $\sum_{m,n} x_{mn} K_m K_n^* = 0$. Fix $i, j \in \{1, \dots, r\}$, and pair the relation with $\langle f_{\pi_i(k_j)}|$ on the left and $|f_{m_j}\rangle$ on the right:

$$0 = \sum_{m,n} x_{mn} \langle f_{\pi_i(k_j)} | K_m K_n^* | f_{m_j} \rangle.$$

Since m_j is exclusive to j , $K_n^* |f_{m_j}\rangle = 0$ for every $n \neq j$, while $K_j^* |f_{m_j}\rangle = w_j(k_j) |e_{k_j}\rangle$; so only the $n = j$ term survives:

$$0 = w_j(k_j) \sum_m x_{mj} \langle f_{\pi_i(k_j)} | K_m | e_{k_j} \rangle.$$

By the column-wise distinctness claim,

$$\langle f_{\pi_i(k_j)} | K_m | e_{k_j} \rangle = w_m(k_j) \langle f_{\pi_i(k_j)} | f_{\pi_m(k_j)} \rangle = \begin{cases} w_i(k_j), & m = i, \\ 0, & m \neq i, \end{cases}$$

so the sum collapses to its single $m = i$ term, $w_j(k_j) w_i(k_j) x_{ij} = 0$. Since both weights are positive, $x_{ij} = 0$; as i, j were arbitrary, $X = 0$.

By Theorem 2.13, the Choi state corresponding to $\{K_i\}$ has rank r and is an extremal tracial coupling. $\square$

Theorem 5.9. *Let $a \leq b$ be positive integers, and $r := \lceil \frac{b}{a} \rceil$. If $b - (r - 1)a$ divides a , then $\mathcal{C}(I_a/a, I_b/b)$ contains an extreme point of rank r .*

Proof. If $a = b$, then $r = 1$ and Proposition 5.1 applies. Assume $b > a$ and set $p := b - (r - 1)a > 0$. By hypothesis $p \mid a$; write $a = pt$ with $t \geq 1$ and $m := (r - 1)t + 1$, so $(a, b) = p(t, m)$.

We first construct a rank- r extreme point for the base pair (t, m) . If $t \geq 2$, then $m \equiv 1 \pmod{t}$ and Lemma 5.8 applies directly, yielding Kraus operators $K'_1, \dots, K'_r : \mathbb{C}^t \rightarrow \mathbb{C}^m$. If $t = 1$ (equivalently $a \mid b$ and $m = r$), set $K'_i := \frac{1}{\sqrt{r}}|f_i\rangle\langle e_1|$ where $\{|f_i\rangle\}_{i=1}^r$ and $|e_1\rangle$ are standard bases of $\mathbb{C}^r$ and $\mathbb{C}^1$. One checks $\sum_i K'_i{}^* K'_i = I_1$, $\sum_i K'_i K'_i{}^* = \frac{1}{r}I_r = \frac{t}{m}I_m$, and the operators $K'_i K'_j{}^*$ are linearly independent.

Now identify $\mathbb{C}^a \cong \mathbb{C}^p \otimes \mathbb{C}^t$ and $\mathbb{C}^b \cong \mathbb{C}^p \otimes \mathbb{C}^m$, and define

$$K_i := I_p \otimes K'_i, \quad i = 1, \dots, r.$$

Since $K_i{}^* K_j = \delta_{ij} I_p \otimes K'_i{}^* K'_j$ and $K_i K_j{}^* = I_p \otimes K'_i K'_j{}^*$, summing gives

$$\sum_{i=1}^r K_i{}^* K_i = I_p \otimes I_t = I_a, \quad \sum_{i=1}^r K_i K_i{}^* = I_p \otimes \frac{t}{m} I_m = \frac{a}{b} I_b.$$

Suppose $\sum_{i,j} x_{ij} K_i K_j{}^* = 0$. This reads $I_p \otimes Y = 0$ with $Y = \sum_{i,j} x_{ij} K'_i K'_j{}^*$, which forces $Y = 0$, and linear independence of $\{K'_i K'_j{}^*\}$ gives $X = 0$. $\square$

5.2. A complete characterization for $a < b \leq 2a$. For $\text{ES}(a, b) = 2$, which corresponds to $a < b \leq 2a$, the divisibility condition of Theorem 5.9 is also necessary. The proof compares the spectra of $K_1{}^* K_1, K_2{}^* K_2$ with those of $K_1 K_1{}^*, K_2 K_2{}^*$. The comparison produces a bijection between index sets with constant step $(b - a)/b$, and Lemma 5.10 shows that such a bijection exists only if $b - a$ divides a .

Lemma 5.10. *Let p, q be positive integers, and let $n = p + q$. Let $S, T \subset \{1, \dots, n\}$ with $|S| = |T| = q$ and $S \cup T = \{1, \dots, n\}$. Let $\nu_1, \dots, \nu_n, \alpha, \beta$ be real numbers such that $\nu_i = \alpha$ for $i \notin S$ and $\nu_j = \beta$ for $j \notin T$, with $\alpha \neq \beta$. If there exist a bijection $\phi : T \rightarrow S$ and a non-zero constant δ such that $\nu_i = \nu_{\phi(i)} + \delta$ for every $i \in T$, then $p \mid q$.*

Proof. Let us encode ϕ as a directed graph $G = (V, E)$ on the vertex set $V = \{1, \dots, n\}$, with a directed edge $i \rightarrow \phi(i)$ for every $i \in T$. Since $\phi : T \rightarrow S$ is a bijection and $T \cup S = V$, we observe that:

- Every vertex in T has out-degree 1, while every vertex outside T ($i \notin T$) is a sink and has out-degree 0.
- Every vertex in S has in-degree 1, while every vertex outside S ($i \notin S$) is a source and has in-degree 0 in G .

At each step $i \rightarrow \phi(i)$ of a directed path in G , the relation $\nu_i = \nu_{\phi(i)} + \delta$ implies that ν strictly decreases by δ . Thus G has no directed cycles.

Since $T \cup S = V$, there are no isolated vertices, and the graph G decomposes into a disjoint union of simple directed paths covering all of V . The number of sources in G is equal to $|V \setminus S| = n - |S| = (p + q) - q = p$ and similarly, the number of sinks in G is equal to p . It follows that there are p directed paths in the above decomposition of G .

Along each directed path in the decomposition, the value of ν drops by δ at each step. By definition, every source $i \notin S$ satisfies $\nu_i = \alpha$, and every sink $j \notin T$ satisfies $\nu_j = \beta$. Thus, every path in the decomposition has the same number of edges, $L = (\alpha - \beta)/\delta$.

Because each path in the decomposition consists of L edges and $L + 1$ vertices, and the p disjoint paths partition all n vertices, we have:

$$p + q = n = p(L + 1),$$

which implies that $q = pL$. Thus $p \mid q$. $\square$

Proposition 5.11. *Let $a < b$ be positive integers. If a rank-2 tracial coupling of $\mathcal{C}(I_a/a, I_b/b)$ exists, then $b - a \mid a$.*

Proof. Let $K_1, K_2 : \mathbb{C}^a \rightarrow \mathbb{C}^b$ be the Kraus operators of the coupling, satisfying

$$K_1 K_1^* + K_2 K_2^* = \frac{a}{b} I_b, \quad K_1^* K_1 + K_2^* K_2 = I_a.$$

Because they sum to a multiple of the identity, $K_1 K_1^*$ and $K_2 K_2^*$ commute and are simultaneously diagonalizable in $M_b(\mathbb{C})$. On every joint eigenspace, their eigenvalues sum to a/b . Let $p := b - a$. Since $a < b$ by hypothesis, note that $p > 0$. If $p = a$, then $b - a \mid a$ holds trivially, so we may assume $p \neq a$.

Since the coupling has rank 2, Theorem 2.3(iii) gives $\lceil \frac{b}{a} \rceil \leq 2$. As $a < b$, we also have $\lceil \frac{b}{a} \rceil \geq 2$. So $\lceil \frac{b}{a} \rceil = 2$, and Lemma 5.3 applies to K_1, K_2 , giving $\text{rank } K_1 = \text{rank } K_2 = a$. Thus, 0 is an eigenvalue of $K_1 K_1^*$ with multiplicity p . Let V_1 denote the p -dimensional 0-eigenspace for $K_1 K_1^*$. Then V_1 is the same as the $\frac{a}{b}$ -eigenspace of $K_2 K_2^*$. Similarly, if V_2 denotes the 0-eigenspace for $K_2 K_2^*$, the $\frac{a}{b}$ -eigenspace of $K_1 K_1^*$ is the same as V_2 . Since $0 \neq a/b$, V_1 and V_2 are eigenspaces of the Hermitian operator $K_1 K_1^*$ for two distinct eigenvalues, and are therefore mutually orthogonal. The space $\mathbb{C}^b \ominus (V_1 \oplus V_2)$ has dimension $q := b - 2p = a - p$ and the restrictions of both $K_1 K_1^*$ and $K_2 K_2^*$ to this space are invertible positive operators, thus possessing strictly positive eigenvalues.

The matrices $K_1^* K_1$ and $K_2^* K_2$ sum to I_a , so they are simultaneously diagonalizable in $\mathbb{C}^a$. Let their joint eigenvalues be ν_i and $1 - \nu_i$, respectively, for $i = 1, \dots, a$. The multiset of strictly positive eigenvalues of $K_m^* K_m$ is equal to that of $K_m K_m^*$ for both $m \in \{1, 2\}$. Thus, the multiset $\{\nu_i\}_{i=1}^a$ contains p copies of a/b alongside $a - p$ values strictly less than a/b , defining an index subset $S_1 \subset \{1, \dots, a\}$ of size p where $\nu_i = a/b$. Symmetrically, the multiset $\{1 - \nu_i\}_{i=1}^a$ for $K_2^* K_2$ also contains p copies of a/b , defining an index subset $S_2 \subset \{1, \dots, a\}$ of size p where $1 - \nu_i = a/b$, which gives us $\nu_i = 1 - a/b = p/b$.

On the q -dimensional space $\mathbb{C}^b \ominus (V_1 \oplus V_2)$, the eigenvalues of $K_1 K_1^*$ are $\{\nu_i \mid i \notin S_1\}$, and the eigenvalues of $K_2 K_2^*$ are $\{a/b - \nu_i \mid i \notin S_1\}$; these are the non- a/b eigenvalues of $K_2 K_2^*$, since its a/b -eigenspace is V_1 . On the other side, the non- a/b eigenvalues of $K_2^* K_2$ are exactly $\{1 - \nu_j \mid j \notin S_2\}$. Since the full non-zero spectra of $K_2^* K_2$ and $K_2 K_2^*$ agree as multisets, and each contains exactly p copies of a/b (indexed by S_2 , respectively supported on V_1), removing these leaves the two remaining size- q multisets equal:

$$\{1 - \nu_j \mid j \notin S_2\} = \{a/b - \nu_i \mid i \notin S_1\}.$$

Thus there is a bijection $\phi : \{1, \dots, a\} \setminus S_2 \rightarrow \{1, \dots, a\} \setminus S_1$ such that:

$$1 - \nu_j = \frac{a}{b} - \nu_{\phi(j)} \quad \text{for every } j \notin S_2.$$

Rearranging this equation yields:

$$\nu_j = \nu_{\phi(j)} + 1 - \frac{a}{b} = \nu_{\phi(j)} + \frac{p}{b}.$$

We are now ready to apply Lemma 5.10. Set $n = a$, and let

$$T = \{1, \dots, a\} \setminus S_2, \quad S = \{1, \dots, a\} \setminus S_1.$$

Then $|S| = |T| = q = a - p$. The corresponding constant values are

$$\alpha = \frac{a}{b} \quad \text{for } i \notin S, \quad \beta = \frac{p}{b} \quad \text{for } j \notin T.$$

Since $p \neq a$, we have $\alpha \neq \beta$ as well as $S_1 \cap S_2 = \emptyset$. Thus,

$$S \cup T = \{1, \dots, a\}.$$

Also, $p > 0$, so the step size $\delta = p/b$ is non-zero. Thus all the hypotheses of Lemma 5.10 are satisfied, and we see that

$$p \mid q = a - p$$

and hence $p \mid a$. Recalling that $p = b - a$, we obtain $b - a \mid a$, as required. $\square$

Theorem 5.12. *Let $a < b \leq 2a$. There is an extreme point of $\mathcal{C}(I_a/a, I_b/b)$ of rank 2 if and only if $b - a = \gcd(a, b)$.*

Proof. Since $a < b \leq 2a$, we have $r = \lceil \frac{b}{a} \rceil = 2$. By Theorem 5.9 and Proposition 5.11, a rank-2 extreme point exists if and only if $b - a$ divides a .

If $b - a = \gcd(a, b)$, then $b - a$ divides a and sufficiency follows. Conversely, suppose $b - a$ divides a . Then it also divides $b = a + (b - a)$, so $b - a$ is a common divisor of a and b ; hence $b - a \leq \gcd(a, b)$. But $\gcd(a, b)$ divides $b - a$ as well, giving $\gcd(a, b) \leq b - a$. Thus $b - a = \gcd(a, b)$. $\square$

5.3. An extremal tracial coupling of rank $\text{ES}(a, b)$ for $b \geq a(a - 1)$. The divisibility hypothesis of Theorem 5.9 is not needed once b is large enough.

Lemma 5.13 (Harmonic Frame). *For every $a \geq 2$ and every $1 \leq p \leq a$, there exists a unit-norm tight frame $\{v_1, \dots, v_a\} \subset \mathbb{C}^p$ with frame operator $\frac{a}{p}I_p$, that is, $\sum_{i=1}^a |v_i\rangle\langle v_i| = \frac{a}{p}I_p$.*

Proof. Let $\omega = e^{2\pi i/a}$ and fix any subset $S \subset \{0, \dots, a - 1\}$ with $|S| = p$; let $\{|e_s\rangle\}_{s \in S}$ be the standard basis of $\mathbb{C}^p$. For $i = 1, \dots, a$, define

$$|v_i\rangle = \frac{1}{\sqrt{p}} \sum_{s \in S} \omega^{s(i-1)} |e_s\rangle.$$

Each $|v_i\rangle$ is a unit vector because $|\omega^{s(i-1)}| = 1$. The frame operator has matrix entries

$$\langle e_s | \left(\sum_{i=1}^a |v_i\rangle\langle v_i| \right) | e_{s'} \rangle = \frac{1}{p} \sum_{i=1}^a \omega^{(s-s')(i-1)} = \delta_{s,s'} \frac{a}{p},$$

since $0 < |s - s'| < a$ for distinct $s, s' \in S$. Hence $\sum_{i=1}^a |v_i\rangle\langle v_i| = \frac{a}{p}I_p$. $\square$

Any linear map $K : \mathbb{C}^a \rightarrow \mathbb{C}^b$ can be written column by column as

$$K = \sum_{k=1}^a \lambda_k |u_k\rangle\langle e_k|,$$

with unit vectors $u_k \in \mathbb{C}^b$ and $\lambda_k \geq 0$. Writing each Kraus operator this way, $K_i = \sum_{k=1}^a \lambda_{i,k} |u_{i,k}\rangle\langle e_k|$, gives

$$K_i K_j^* = \sum_{k=1}^a \lambda_{i,k} \lambda_{j,k} |u_{i,k}\rangle\langle u_{j,k}|,$$

so whether $\{K_i K_j^*\}_{i,j=1}^r$ is linearly independent (which suffices for extremality) depends only on the ra vectors $u_{i,k}$. We choose them as follows.

Theorem 5.14. *Let $a \geq 2$ and let $b > a(a-1)$ be positive integers. Then there is an extreme point of $\mathcal{C}(I_a/a, I_b/b)$ of rank $\lceil \frac{b}{a} \rceil$.*

Proof. We first set up the notation used to construct the extreme point, working with its corresponding CP map and a minimal Kraus representation thereof, under the correspondence of Theorem 2.10.

Let $r := \lceil \frac{b}{a} \rceil$, $p := b - (r-1)a$ and $w_a := \frac{a}{b}$, $w_p := \frac{p}{b}$. Note that $0 < p \leq a$ and $(r-1)w_a + w_p = \frac{(r-1)a+p}{b} = 1$. Let $f_1, \dots, f_b$ denote the standard basis vectors of $\mathbb{C}^b$, and define,

$$\Omega := \{(i, k) \in \{1, \dots, r\} \times \{1, \dots, a\} : k \neq i\}.$$

Since $|\Omega| = ra - a = a(r-1) < b$, we may choose an injective map $\sigma : \Omega \rightarrow \{1, \dots, b\}$. Set

$$V := \text{span}\{f_{\sigma(i,k)} : (i, k) \in \Omega\}, \quad W := V^\perp,$$

so $\dim V = a(r-1)$ and $\dim W = p$. Using Lemma 5.13, fix a tight frame $\{v_1, \dots, v_a\} \subset W$ with $\sum_{i=1}^a |v_i\rangle\langle v_i| = \frac{a}{p}I_W$, $\|v_i\| = 1$.

Let $e_1, \dots, e_a$ denote the standard basis vectors of $\mathbb{C}^a$. For $1 \leq i \leq r$, $1 \leq k \leq a$, set

$$u_{i,k} := \begin{cases} f_{\sigma(i,k)}, & k \neq i, \\ v_i, & k = i, \end{cases} \quad \lambda_{i,k} := \begin{cases} \sqrt{w_a}, & k \neq i, \\ \sqrt{w_p}, & k = i, \end{cases}.$$

Note that $(i, k) \notin \Omega$ if and only if $k = i$, and thus for every $1 \leq k \leq a$, we have

$$(13) \quad \sum_{i=1}^a \lambda_{i,k}^2 = w_p + (r-1)w_a = 1.$$

Define $K_1, \dots, K_r : \mathbb{C}^a \rightarrow \mathbb{C}^b$ by

$$K_i := \sum_{k=1}^a \lambda_{i,k} |u_{i,k}\rangle\langle e_k|.$$

Claim 5. If $(i, k) \in \Omega$, then $u_{i,k} \perp u_{i',k'}$ for every $(i', k') \neq (i, k)$. In particular, $u_{i,k} \perp u_{i,k'}$ for $k \neq k'$ and $1 \leq i \leq r$, and $u_{i,k} \perp u_{i',k}$ for $i \neq i'$ and $1 \leq k \leq a$.

Proof of Claim. If also $(i', k') \in \Omega$, then $\sigma(i', k') \neq \sigma(i, k)$ since σ is injective and $(i', k') \neq (i, k)$, so $u_{i,k} = f_{\sigma(i,k)}$ and $u_{i',k'} = f_{\sigma(i',k')}$ are distinct standard basis vectors. If instead $(i', k') \notin \Omega$, then $u_{i',k'} = v_{i'} \in W$, while $u_{i,k} = f_{\sigma(i,k)} \in V$, and $V \perp W$.

For $(i, k), (i', k') \notin \Omega$ the vectors $u_{i,k} = v_i$ and $u_{i',k'} = v_{i'}$ need not be orthogonal; there we use the tight-frame identity instead.

Claim 6. $\sum_{i=1}^r K_i^* K_i = I_a$ and $\sum_{i=1}^r K_i K_i^* = \frac{a}{b} I_b$.

Proof of Claim. From $K_i = \sum_{k=1}^a \lambda_{i,k} |u_{i,k}\rangle\langle e_k|$, we get

$$(14) \quad K_i^* K_i = \sum_{1 \leq k, k' \leq a} \lambda_{i,k} \lambda_{i,k'} |u_{i,k}\rangle\langle u_{i,k'}| |e_k\rangle\langle e_{k'}|,$$

$$(15) \quad K_i K_i^* = \sum_{1 \leq k, k' \leq a} \lambda_{i,k} \lambda_{i,k'} \langle e_k | e_{k'} \rangle |u_{i,k}\rangle\langle u_{i,k'}|.$$

In the first sum, the previous claim gives $\langle u_{i,k}, u_{i,k'} \rangle = 0$ for $k \neq k'$. Since $\langle u_{i,k}, u_{i,k} \rangle = 1$, for $1 \leq i \leq r$ we have,

$$K_i^* K_i = \sum_{k=1}^a \lambda_{i,k}^2 |e_k\rangle \langle e_k|.$$

By interchanging the order of summation, we observe that

$$\sum_{i=1}^r K_i^* K_i = \sum_{i=1}^r \sum_{k=1}^a \lambda_{i,k}^2 |e_k\rangle \langle e_k| = \sum_{k=1}^a \left(\sum_{i=1}^r \lambda_{i,k}^2 \right) |e_k\rangle \langle e_k| = \sum_{k=1}^a |e_k\rangle \langle e_k| = I_a.$$

In the second sum, using $\langle e_k, e_{k'} \rangle = \delta_{kk'}$, we have

$$K_i K_i^* = \sum_{k=1}^a \lambda_{i,k}^2 |u_{i,k}\rangle \langle u_{i,k}|.$$

As before, by interchanging the order of summation, we observe that,

$$\begin{aligned} \sum_{i=1}^r K_i K_i^* &= w_a \sum_{(i,k) \in \Omega} |f_{\sigma(i,k)}\rangle \langle f_{\sigma(i,k)}| + w_p \sum_{i=1}^a |v_i\rangle \langle v_i| \\ &= w_a E_V + w_p \cdot \frac{a}{p} E_W = \frac{a}{b} E_V + \frac{a}{b} E_W = \frac{a}{b} I_b. \end{aligned}$$

Claim 7. For each $k \in \{1, \dots, a\}$, the vectors $K_1 e_k, \dots, K_r e_k$ are linearly independent. In particular, $K_1, \dots, K_r$ are linearly independent, and the corresponding Choi state has rank r .

Proof of Claim. Let $1 \leq k \leq a$. By Claim 5, $u_{1,k}, \dots, u_{r,k}$ are mutually orthogonal. Since $K_i e_k = \lambda_{i,k} u_{i,k}$ with $\lambda_{i,k} > 0$, the vectors $K_1 e_k, \dots, K_r e_k$ are non-zero and mutually orthogonal, hence linearly independent.

If $K_1, \dots, K_r$ satisfied a non-trivial linear relation $\sum_{j=1}^r c_j K_j = 0$ for scalars c_j , evaluating at any fixed e_k would give a non-trivial linear relation among $K_1 e_k, \dots, K_r e_k$, leading us to a contradiction.

Claim 8. The operators $\{K_i K_j^*\}_{1 \leq i, j \leq r}$ are linearly independent.

Proof of Claim. Suppose $\sum_{i,j=1}^r x_{ij} K_i K_j^* = 0$. Let $\ell \in \{1, \dots, r\}$. Since $a \geq 2$, choose $k' \neq \ell$ with $1 \leq k' \leq a$, so $(\ell, k') \in \Omega$. By Claim 5, $u_{\ell, k'}$ is orthogonal to every $u_{j,k}$ with $(j, k) \neq (\ell, k')$. From $K_j^* = \sum_{k=1}^a \lambda_{j,k} |e_k\rangle \langle u_{j,k}|$, we have

$$K_j^* u_{\ell, k'} = \sum_{k=1}^a \lambda_{j,k} \langle u_{j,k} | u_{\ell, k'} \rangle e_k = \begin{cases} 0, & j \neq \ell, \\ \lambda_{\ell, k'} e_{k'}, & j = \ell. \end{cases}$$

Applying the relation to $u_{\ell, k'}$, we get,

$$0 = \sum_{1 \leq i, j \leq r} x_{ij} K_i K_j^* u_{\ell, k'} = \lambda_{\ell, k'} \sum_{i=1}^r x_{i\ell} K_i e_{k'}.$$

Since $\ell \neq k'$, $\lambda_{\ell, k'} = \sqrt{w_a} \neq 0$. This shows $\sum_{i=1}^r x_{i\ell} K_i e_{k'} = 0$; by the previous claim, $K_1 e_{k'}, \dots, K_r e_{k'}$ are linearly independent, so $x_{i\ell} = 0$ for every i . As ℓ was arbitrary, we conclude that $x_{ij} = 0$ for $1 \leq i, j \leq r$.

By Theorem 2.13, this claim establishes that the Choi state is extreme. $\square$

6. CONCLUDING REMARKS

Our results settle KRP tightness throughout the region, $b \geq \frac{1}{2}(a^2 - 1)$, and settle ES-tightness at $\text{ES}(a, b) = 2$, that is, $a < b \leq 2a$, and throughout the region $b \geq a(a - 1)$ (Theorems 4.1, 5.12, 5.9, and 5.14). We close with two conjectures and two open problems.

Conjecture 6.1. *For every (a, b) with $a, b \geq 2$ and $(a, b) \notin \{(2, 2), (3, 3), (4, 4)\}$, there exists a transversal pair of FUNTFs of length $r = \text{KRP}(a, b)$ in $\mathbb{C}^a$ and $\mathbb{C}^b$.*

If the conjecture holds, the problem of KRP tightness would be settled for every pair except $(2, 2)$, which motivates the following conjecture (see also Devendra–Dey–Dey [DDD26]):

Conjecture 6.2. *$R_{\max}(a, b) = \text{KRP}(a, b)$ for every $(a, b) \neq (2, 2)$ with $a, b \geq 2$.*

For $\text{ES}(a, b) = 2$, that is, $a < b < 2a$, Theorem 5.12 gives a complete characterization: the Schmidt bound is attained if and only if $b - a = \gcd(a, b)$. Two problems remain open.

First, for $r := \text{ES}(a, b) \geq 3$, Theorem 5.9 shows that the Schmidt bound is attained whenever $b - (r - 1)a$ divides a , and Theorem 5.14 shows that this condition is not needed once $b > a(a - 1)$. Whether the condition is necessary in the intermediate range $2a < b \leq a(a - 1)$, as it is when $r = 2$, is open.

Second, Ohno’s constructions show that $(3, 3)$ and $(4, 4)$ are KRP tight, while Corollary 4.9 shows that no transversal pair of FUNTFs reaches them. It is open whether a modification of the transversality approach, allowing unequal weights or Kraus operators of rank greater than one, produces maximal-rank extremal tracial couplings at these two pairs, and more generally wherever equal-weight rank-one pairs fail.

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STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE, MYSORE ROAD,
RVCE POST, BENGALURU, KARNATAKA - 560 059, INDIA
Email address: dip101525@gmail.com

STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE, MYSORE ROAD,
RVCE POST, BENGALURU, KARNATAKA - 560 059, INDIA
Email address: janasonmath364@gmail.com

STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE, MYSORE ROAD,
RVCE POST, BENGALURU, KARNATAKA - 560 059, INDIA
Email address: soumyashant@isibang.ac.in