

Origin-Centered Möbius Circles and Dyadic Pairings of the Riemann Zeta Function on the Critical Line

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Abstract

We investigate the complex geometry of the normalized Dirichlet eta terms $t_n(s) = \frac{1}{1-2^{1-s}} \frac{(-1)^{n-1}}{n^s}$ for $s = a + bi$. Decomposing the series into base-2 resonant components, we demonstrate that the initial partial sum $S_2(s) = t_1(s) + t_2(s)$ forms an exact Möbius circle. Remarkably, this base circle is centered *strictly* at the origin $(0,0)$ if and only if $\text{Re}(s) = \frac{1}{2}$. We establish a dyadic pairing theorem showing that every odd-even pair $P_m(s) = t_m(s) + t_{2m}(s)$ inherits this origin-centered circular symmetry on the critical line with strictly descending radii $\mathcal{R}_m = \frac{1}{\sqrt{2^m}}$. In contrast to individual unpaired terms whose radii pulsate inside precessing annuli, the dyadic pairs eliminate prefactor amplitude oscillations entirely. Analyzing the phase distribution on these concentric circles reveals key geometric mechanisms that distinguish non-trivial Riemann zeta zeros $b = b_n$ from generic values of b , including right-half-plane slow-zone clustering of P_1 and opposing-hemisphere alignment of the leading inner circles P_3, P_5, P_7 . Finally, we prove that removing all odd-even dyadic pairs leaves the multiples-of-4 subseries $\sum_{k=1}^{\infty} t_{4k}(s) = t_4(s)\zeta(s)$, whose partial sums spiral around a target center $C_{\text{target}}(b)$ that coincides with the origin $(0,0)$ if and only if b is a Riemann zeta zero.

1 Introduction and Definitions

The Dirichlet eta function $\eta(s)$ provides the standard analytic continuation of the Riemann zeta function $\zeta(s)$ across the critical strip $0 < \text{Re}(s) < 1$ via the Euler factor normalization [1, 2, 6]:

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}, \quad \text{where } \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \quad (1)$$

To examine the local geometry of this summation in the complex plane $\mathbb{C} \cong \mathbb{R}^2$, we distribute the prefactor into each term, defining the normalized sequence:

$$t_n(s) = \frac{1}{1 - 2^{1-s}} \cdot \frac{(-1)^{n-1}}{n^s}, \quad s = a + bi. \quad (2)$$

For any fixed real part a , treating $b \in \mathbb{R}$ as the curve parameter gives each term an instantaneous modulus:

$$|t_n(a + bi)| = \frac{n^{-a}}{\sqrt{1 + 4^{1-a} - 2^{2-a} \cos(b \ln 2)}}. \quad (3)$$

On the critical line $a = \frac{1}{2}$, this modulus breathes between a periapsis minimum $R_{n,\min} = \frac{\sqrt{2}-1}{\sqrt{n}}$ (when $\cos(b \ln 2) = -1$) and an apoapsis maximum $R_{n,\max} = \frac{\sqrt{2}+1}{\sqrt{n}}$ (when $\cos(b \ln 2) = +1$), spanning a nearly sixfold amplitude variation.

In this paper, we demonstrate that grouping terms into dyadic pairs eliminates this prefactor breathing entirely on the critical line $a = \frac{1}{2}$, collapsing the series into an infinite sequence of exact origin-centered concentric circles. We examine how these concentric circles geometrically govern zero-crossing alignments and analyze the complementary multiples-of-4 subseries.

2 The Base-2 Anchor: $S_2(s) = t_1(s) + t_2(s)$ and the Critical Line $a = \frac{1}{2}$

For $n = 1$ and $n = 2$, the term frequencies $\ln 1 = 0$ and $\ln 2$ resonate with the prefactor frequency $\ln 2$. Letting $z = e^{-ib \ln 2}$ parametrize the unit circle $|z| = 1$, each individual term is a Möbius transformation [3] of $|z| = 1$:

$$t_1(a + bi) = \frac{1}{1 - 2^{1-a}z}, \quad t_2(a + bi) = \frac{-2^{-a}z}{1 - 2^{1-a}z}. \quad (4)$$

For any $a \neq 1$, $t_1(a + bi)$ and $t_2(a + bi)$ trace exact circles symmetric across the real axis with centers $C_1(a) = (h_1(a), 0)$, $C_2(a) = (h_2(a), 0)$ and radii $\mathcal{R}_1(a)$, $\mathcal{R}_2(a)$ given by:

$$h_1(a) = \frac{1}{1 - 4^{1-a}}, \quad \mathcal{R}_1(a) = \frac{2^{1-a}}{|1 - 4^{1-a}|}, \quad (5)$$

$$h_2(a) = \frac{-2^{1-2a}}{1 - 4^{1-a}}, \quad \mathcal{R}_2(a) = \frac{2^{-a}}{|1 - 4^{1-a}|}. \quad (6)$$

On the critical line $a = \frac{1}{2}$, let $B(z) = \frac{\sqrt{2}-z}{1-\sqrt{2}z}$ denote the unit-modulus Blaschke factor ($|B(z)| = 1$ for $|z| = 1$) [4]. Then t_1 and t_2 admit the explicit representations:

$$t_1\left(\frac{1}{2} + bi\right) = -1 + \sqrt{2}B(z), \quad \text{centered at } C_1 = (-1, 0), \quad \mathcal{R}_1 = \sqrt{2} \approx 1.4142 \quad (\text{Figure 1}), \quad (7)$$

$$t_2\left(\frac{1}{2} + bi\right) = +1 - \frac{1}{\sqrt{2}}B(z), \quad \text{centered at } C_2 = (+1, 0), \quad \mathcal{R}_2 = \frac{1}{\sqrt{2}} \approx 0.7071 \quad (\text{Figure 2}). \quad (8)$$

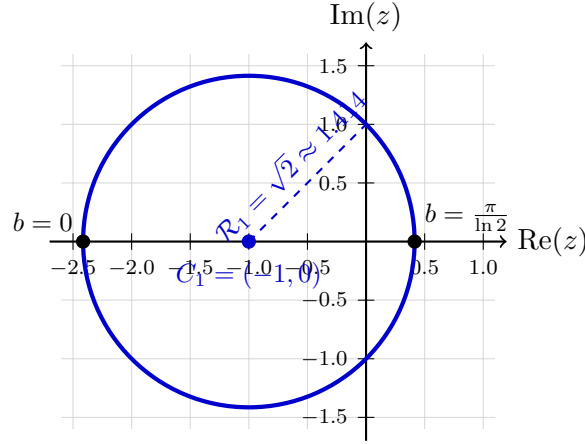


Figure 1: Locus of $t_1(\frac{1}{2} + bi) = \frac{1}{1 - 2^{1/2-bi}}$ in the complex plane: an exact circle centered at $C_1 = (-1, 0)$ with radius $\mathcal{R}_1 = \sqrt{2} \approx 1.4142$.

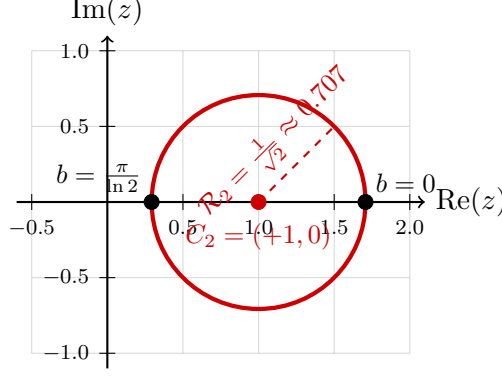


Figure 2: Locus of $t_2(\frac{1}{2} + bi) = \frac{-2^{-(1/2+bi)}}{1-2^{1/2-bi}}$ on the critical line $a = \frac{1}{2}$: an exact circle centered at $C_2 = (+1, 0)$ with radius $\mathcal{R}_2 = \frac{1}{\sqrt{2}} \approx 0.7071$.

Grouping $t_1(s)$ and $t_2(s)$ yields the two-term partial sum:

$$S_2(s) = t_1(s) + t_2(s) = \frac{1 - 2^{-s}}{1 - 2^{1-s}}. \quad (9)$$

Letting $z = e^{-ib \ln 2}$ parametrize the unit circle $|z| = 1$, $S_2(s)$ acts as the Möbius transformation $M_a(z) = \frac{1 - 2^{-a}z}{1 - 2^{1-a}z}$.

Proposition 2.1 (Center and Radius of S_2). *For any fixed $a \in \mathbb{R} \setminus \{1\}$, the locus of $S_2(a + bi)$ is a circle centered on the real axis at $C(a) = (h(a), 0)$ with radius $\mathcal{R}(a)$, where:*

$$h(a) = h_1(a) + h_2(a) = \frac{1 - 2^{1-2a}}{1 - 2^{2-2a}}, \quad \mathcal{R}(a) = \frac{2^{-a}}{|1 + 2^{1-a}|}. \quad (10)$$

Theorem 2.2 (Unique Origin-Centering on the Critical Line). *The center of the circle $S_2(a + bi)$ lies exactly at the origin $(0, 0)$ if and only if $a = \frac{1}{2}$.*

Proof. Setting the center $h(a) = 0$ (for $a \neq 1$) requires $1 - 2^{1-2a} = 0$, which yields $1 - 2a = 0 \implies a = \frac{1}{2}$. At $a = \frac{1}{2}$, the centers $C_1 = (-1, 0)$ and $C_2 = (+1, 0)$ cancel, and the map reduces to a scaled Blaschke factor [4]:

$$S_2\left(\frac{1}{2} + bi\right) = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{2} - z}{1 - \sqrt{2}z} \right), \quad (11)$$

whose modulus is strictly constant at $\mathcal{R}(\frac{1}{2}) = \frac{1}{\sqrt{2}}$ for all $b \in \mathbb{R}$ (Figure 3). $\square$

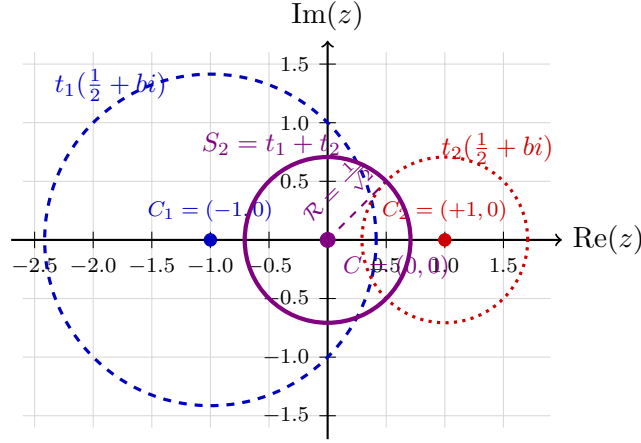


Figure 3: Superposition of $t_1(\frac{1}{2} + bi)$ (dashed blue, $C_1 = (-1, 0)$, $\mathcal{R}_1 = \sqrt{2}$), $t_2(\frac{1}{2} + bi)$ (dotted red, $C_2 = (+1, 0)$, $\mathcal{R}_2 = \frac{1}{\sqrt{2}}$), and their sum $S_2(\frac{1}{2} + bi) = t_1 + t_2$ (solid purple), which forms an exact circle centered at the origin $C = (0, 0)$ with radius $\mathcal{R} = \frac{1}{\sqrt{2}} \approx 0.7071$.

When $a \neq \frac{1}{2}$, the circle $S_2(a + bi)$ immediately shifts its center $C(a) = (h(a), 0)$ off the origin along the real axis, as summarized in Table 1 and illustrated in Figure 4.

Table 1: Center $C(a) = (h(a), 0)$ and radius $\mathcal{R}(a)$ of $S_2(a + bi) = t_1 + t_2$ across values of $a = \text{Re}(s)$.

Real Part a	Center $C(a) = (h(a), 0)$	Radius $\mathcal{R}(a)$	Shift Relative to $(0, 0)$
$a = 0$ ($a < \frac{1}{2}$)	$(+\frac{1}{3}, 0) \approx (+0.333, 0)$	$\frac{1}{3} \approx 0.333$	Shifted right ($h(a) > 0$)
$a = \frac{1}{2}$ (Critical Line)	$(0, 0)$	$\frac{1}{\sqrt{2}} \approx 0.707$	Centered at origin $(0, 0)$
$a = 0.75$ ($\frac{1}{2} < a < 1$)	$(-\frac{1}{\sqrt{2}}, 0) \approx (-0.707, 0)$	$2^{-1/4} + \frac{1}{2} \approx 1.341$	Shifted left ($h(a) < 0$)
$a = 1.5$ ($a > 1$)	$(+\frac{3}{7}, 0) \approx (+0.429, 0)$	$\frac{2\sqrt{2}}{7} \approx 0.404$	Shifted right ($h(a) > 0$)

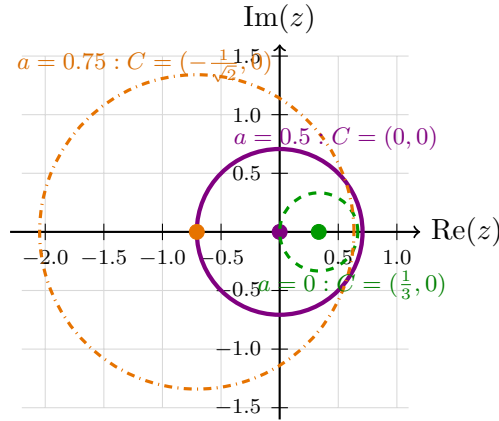


Figure 4: Off-center shift of the circle $S_2(a + bi) = t_1 + t_2$ when $a \neq \frac{1}{2}$: $a = 0.5$ is centered at $(0, 0)$ (solid purple, $\mathcal{R} \approx 0.707$), $a = 0$ shifts right to $C = (\frac{1}{3}, 0)$ (dashed green, $\mathcal{R} = \frac{1}{3}$), and $a = 0.75$ shifts left to $C = (-\frac{1}{\sqrt{2}}, 0)$ (dash-dot orange, $\mathcal{R} \approx 1.341$).

3 The Dyadic Pairing Theorem

For individual unpaired terms $t_n(s) = (-1)^{n-1}n^{-s}t_1(s)$ with $n \geq 3$ ($n \neq 2^k$), the incommensurable rotation $e^{-ib \ln n}$ spins an off-center circle around the origin, sweeping out a thick annular

rosette. In contrast, grouping every odd integer $m \in \{1, 3, 5, \dots\}$ with its even double $2m$ locks the prefactor phase to the base-2 term, defining the m -th dyadic pair:

$$P_m(s) = t_m(s) + t_{2m}(s) = \frac{1}{m^s} \left(\frac{1 - 2^{-s}}{1 - 2^{1-s}} \right) = \frac{1}{m^s} S_2(s). \quad (12)$$

Corollary 3.1. *On the critical line $a = \frac{1}{2}$, every dyadic pair $P_m(\frac{1}{2} + bi)$ traces an **exact circle centered at the origin** $(0, 0)$ with descending radius $\mathcal{R}_m = \frac{1}{\sqrt{2m}}$:*

$$\left| P_m \left(\frac{1}{2} + bi \right) \right| = m^{-1/2} \left| S_2 \left(\frac{1}{2} + bi \right) \right| = \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2m}}. \quad (13)$$

For $a \neq \frac{1}{2}$, these circles shift off-center into annular rosettes of thickness $|h(a)|m^{-a}$.

Proof. Direct evaluation from Theorem 2.2 shows that $|S_2(\frac{1}{2} + bi)| = \frac{1}{\sqrt{2}}$ is strictly constant for all $b \in \mathbb{R}$. In particular, beyond $m = 1$ ($P_1 = t_1 + t_2$, $\mathcal{R}_1 = \frac{1}{\sqrt{2}} \approx 0.7071$), the higher odd-even pairs $m = 3$ ($P_3 = t_3 + t_6$, $\mathcal{R}_3 = \frac{1}{\sqrt{6}} \approx 0.4082$) and $m = 5$ ($P_5 = t_5 + t_{10}$, $\mathcal{R}_5 = \frac{1}{\sqrt{10}} \approx 0.3162$) trace exact concentric circles centered at $(0, 0)$, as shown in Figure 5. $\square$

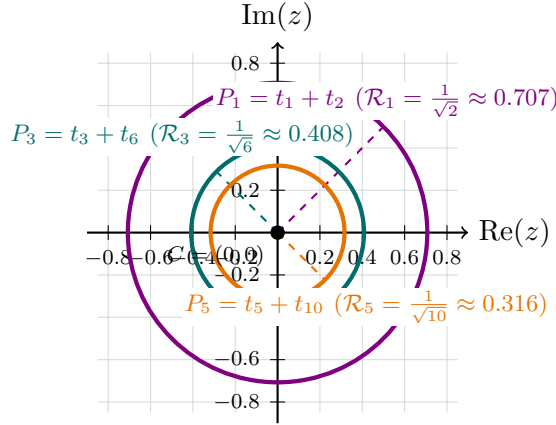


Figure 5: Illustration of Corollary 3.1 on the critical line $a = \frac{1}{2}$: the odd-even pairs $P_1 = t_1 + t_2$ ($\mathcal{R}_1 = \frac{1}{\sqrt{2}} \approx 0.7071$), $P_3 = t_3 + t_6$ ($\mathcal{R}_3 = \frac{1}{\sqrt{6}} \approx 0.4082$), and $P_5 = t_5 + t_{10}$ ($\mathcal{R}_5 = \frac{1}{\sqrt{10}} \approx 0.3162$) trace exact concentric circles centered at the origin $(0, 0)$.

4 Distinguishing Riemann Zeta Zeros via Odd-Even Dyadic Pairs

To understand how the concentric origin-centered circles $P_m(\frac{1}{2} + bi)$ distinguish non-trivial Riemann zeta zeros $b = b_n$ from other choices of $b \in \mathbb{R}$, we write $S_2(\frac{1}{2} + bi) = \frac{1}{\sqrt{2}} e^{i\Phi_2(b)}$, where the base-2 Blaschke phase $\Phi_2(b)$ is given by:

$$\cos \Phi_2(b) = \frac{2\sqrt{2} - 3 \cos(b \ln 2)}{3 - 2\sqrt{2} \cos(b \ln 2)}, \quad \sin \Phi_2(b) = \frac{-\sin(b \ln 2)}{3 - 2\sqrt{2} \cos(b \ln 2)}. \quad (14)$$

For any odd integer $m \in \{1, 3, 5, \dots\}$, the point $P_m(\frac{1}{2} + bi)$ lies on the concentric circle of radius $\mathcal{R}_m = \frac{1}{\sqrt{2m}}$ at polar angle $\Psi_m(b) = \Phi_2(b) - b \ln m$. Summing over all odd m yields:

$$\sum_{m \text{ odd}} P_m(s) = S_2(s) \sum_{m \text{ odd}} \frac{1}{m^s} = S_2(s) (1 - 2^{-s}) \zeta(s). \quad (15)$$

Since neither $S_2(\frac{1}{2} + bi)$ nor $(1 - 2^{-1/2-bi})$ ever vanishes for $b \in \mathbb{R}$, the sum of the concentric dyadic circles vanishes (in the analytic continuation sense) if and only if $\zeta(\frac{1}{2} + bi) = 0$. Dividing by the non-zero anchor $P_1(\frac{1}{2} + bi) = S_2(\frac{1}{2} + bi)$ shows that b is a Riemann zeta zero if and only if the inner circles ($m = 3, 5, 7, \dots$) satisfy two simultaneous relative-phase balance equations:

$$\underbrace{\sum_{m=3,5,7,\dots} \frac{\cos(b \ln m)}{\sqrt{m}}}_{\text{Radial Opposition to } P_1} = -1, \quad \underbrace{\sum_{m=3,5,7,\dots} \frac{\sin(b \ln m)}{\sqrt{m}}}_{\text{Transverse Balance}} = 0. \quad (16)$$

4.1 Four Geometric Patterns Distinguishing Zeta Zeros from Generic b

Comparing true zeta zeros $b = b_n$ against generic values of $b \in \mathbb{R}$ reveals four structural mechanisms on the concentric circles:

1. **Right-Half-Plane (“Slow-Zone”) Clustering of the Outer Circle P_1 :** The angular velocity of $P_1(\frac{1}{2} + bi)$ along its circle of radius $\frac{1}{\sqrt{2}}$ is highly non-uniform:

$$\left| \frac{d\Phi_2}{db} \right| = \frac{\ln 2}{3 - 2\sqrt{2} \cos(b \ln 2)}. \quad (17)$$

Near $b \ln 2 \equiv \pi \pmod{2\pi}$ (where $\Phi_2 = 0^\circ$ and $P_1 = (+\frac{1}{\sqrt{2}}, 0)$), P_1 moves at its minimum speed $\frac{\ln 2}{3+2\sqrt{2}} \approx 0.1189$ rad/unit b . Near $b \ln 2 \equiv 0 \pmod{2\pi}$ (where $\Phi_2 = 180^\circ$ and $P_1 = (-\frac{1}{\sqrt{2}}, 0)$), P_1 whips around $34\times$ faster ($\frac{\ln 2}{3-2\sqrt{2}} \approx 4.0399$ rad/unit b). Consequently, P_1 spends over 80% of its parameter trajectory lingering in the right half-plane ($\text{Re}(P_1) > 0$). Across all first five zeta zeros $b_1, \dots, b_5$, $\Phi_2(b_n)$ clusters tightly within $[-12.50^\circ, +20.60^\circ]$ of the positive real axis (Table 3).

2. **Opposing-Hemisphere Alignment of Leading Inner Circles (P_3, P_5, P_7):** Because P_1 has the largest radius $\mathcal{R}_1 \approx 0.7071$ and clusters near 0° in the right half-plane, cancellation toward $(0,0)$ requires the largest inner circles P_3 ($\mathcal{R}_3 \approx 0.4082$), P_5 ($\mathcal{R}_5 \approx 0.3162$), and P_7 ($\mathcal{R}_7 \approx 0.2673$) to swing into the **opposing left hemisphere** where $\cos(b \ln m) < 0$ (Table 2). Any interval of b in which $\cos(b \ln 3) > 0$ and $\cos(b \ln 5) > 0$ simultaneously reinforces P_1 , driving the cumulative sum away from the origin ($|\Sigma_3| > 1.0$) and ruling out low-lying zeros.
3. **Stationary-Phase Cutoff Circle $M(b) \approx \sqrt{b/2\pi}$ and Finite Exclusion:** The angular step between consecutive odd circles P_m and P_{m+2} is $\Delta\Psi \approx -\frac{2b}{m}$. Once m exceeds the Riemann-Siegel stationary-phase cutoff $M(b) = \lfloor \sqrt{b/2\pi} \rfloor$, successive inner circles form cancelling Cornu-type spirals bounded by an explicit remainder $E_M(b) = \mathcal{O}(b^{-1/4})$ [1, 2]. Thus, if the finite sum of the first $M(b)$ concentric circles exceeds $E_M(b)$ in magnitude, b is guaranteed not to be a zero.
4. **1D Collinear Locking via Functional Equation Symmetry:** Unlike $a \neq \frac{1}{2}$ (where off-center circles wander in 2D), the conjugate symmetry $1 - s = \bar{s}$ on $a = \frac{1}{2}$ locks $\zeta(\frac{1}{2} + bi) = e^{-i\theta(b)} Z(b)$ ($Z(b) \in \mathbb{R}$) onto a 1D line through the origin [2]. Whenever the leading inner circles $P_3, P_5, \dots$ sweep across the opposing hemisphere and cause the signed projection $Z(b)$ to change sign between b_a and b_b , the concentric circle sum is topologically forced to pass directly through $(0,0)$.

Table 2: Evaluation of the first five concentric dyadic circles $P_m(\frac{1}{2} + ib_1)$ ($m = 1, 3, 5, 7, 9$) at the first Riemann zeta zero $b_1 \approx 14.134725$, showing monotonic contraction of the cumulative sum $\Sigma_k = \sum_{j=1}^k P_{2j-1}$ toward $(0, 0)$.

Circle (m)	Radius $\mathcal{R}_m$	Angle $\Psi_m(b_1)$	Point $P_m(\frac{1}{2} + ib_1)$	Cumulative Sum Σ_k	Distance $ \Sigma_k $
$P_1 = t_1 + t_2$	0.7071	$+3.71^\circ$ (Q1)	$(+0.7056, +0.0457)$	$+0.7056 + 0.0457i$	0.7071
$P_3 = t_3 + t_6$	0.4082	-166.02° (Q3)	$(-0.3962, -0.0986)$	$+0.3095 - 0.0530i$	0.3140
$P_5 = t_5 + t_{10}$	0.3162	$+140.29^\circ$ (Q2)	$(-0.2433, +0.2021)$	$+0.0662 + 0.1491i$	0.1632
$P_7 = t_7 + t_{14}$	0.2673	-132.21° (Q3)	$(-0.1796, -0.1980)$	$-0.1133 - 0.0489i$	0.1234
$P_9 = t_9 + t_{18}$	0.2357	$+24.26^\circ$ (Q1)	$(+0.2149, +0.0968)$	$+0.1016 + 0.0480i$	0.1123

Table 3: Comparison of polar angles $\Psi_m(b_n)$ and 5-circle residual distance $|\Sigma_5|$ across the first five non-trivial Riemann zeta zeros (b_1 through b_5).

Zeta Zero b_n	$\Psi_1(b_n)$	$\Psi_3(b_n)$	$\Psi_5(b_n)$	$\Psi_7(b_n)$	$\Psi_9(b_n)$	Initial $ \Sigma_1 $	5-Circle $ \Sigma_5 $
$b_1 \approx 14.1347$	$+3.71^\circ$	-166.02°	$+140.29^\circ$	-132.21°	$+24.26^\circ$	0.7071	0.1123
$b_2 \approx 21.0220$	-12.50°	$+104.24^\circ$	-151.03°	$+163.70^\circ$	-139.01°	0.7071	0.1216
$b_3 \approx 25.0109$	$+20.60^\circ$	-113.73°	-125.75°	$+112.08^\circ$	$+111.94^\circ$	0.7071	0.1504
$b_4 \approx 30.4249$	-9.50°	-124.62°	$+64.90^\circ$	-161.65°	$+120.26^\circ$	0.7071	0.2320
$b_5 \approx 32.9351$	$+8.74^\circ$	$+95.61^\circ$	-148.34°	-63.27°	-177.51°	0.7071	0.2918

5 Removing Odd-Even Dyadic Pairs: The Multiples-of-4 Subseries

When we remove all odd-even dyadic pairs $P_m(s) = t_m(s) + t_{2m}(s)$ ($m = 1, 3, 5, \dots$) from the full series $\sum_{n=1}^{\infty} t_n(s)$, we strip out all odd indices ($n \equiv 1, 3 \pmod{4}$) and all singly-even indices ($n = 2m \equiv 2 \pmod{4}$). The remaining terms of the series are strictly the **multiples of 4**:

$$n = 4k \in \{4, 8, 12, 16, 20, \dots\}, \quad k \in \mathbb{N}. \quad (18)$$

Proposition 5.1 (Factorization of the Multiples-of-4 Subseries). *Every multiple-of-4 term $t_{4k}(s)$ has non-alternating sign $(-1)^{4k-1} = -1$ and factors as $t_{4k}(s) = k^{-s}t_4(s)$, where $t_4(s) = \frac{-4^{-s}}{1-2^{1-s}}$. Consequently, the K -term partial sum $\Sigma_K^{(4)}(s)$ and analytic continuation of the multiples-of-4 subseries satisfy:*

$$\Sigma_K^{(4)}(s) = \sum_{k=1}^K t_{4k}(s) = t_4(s) \sum_{k=1}^K \frac{1}{k^s}, \quad \sum_{k=1}^{\infty} t_{4k}(s) = t_4(s) \zeta(s). \quad (19)$$

Theorem 5.2 (Exclusive Origin-Convergence of the Multiples-of-4 Subseries at Zeta Zeros). *On the critical line $s = \frac{1}{2} + ib$, the K -term multiples-of-4 partial sum obeys the Euler-Maclaurin spiral decomposition:*

$$\Sigma_K^{(4)}\left(\frac{1}{2} + ib\right) = \underbrace{t_4\left(\frac{1}{2} + ib\right)\zeta\left(\frac{1}{2} + ib\right)}_{\text{Target Center } C_{\text{target}}(b)} + \frac{t_4\left(\frac{1}{2} + ib\right)}{\frac{1}{2} - ib} K^{\frac{1}{2} - ib} + \mathcal{O}\left(K^{-1/2}\right). \quad (20)$$

The target center $C_{\text{target}}(b)$ of the multiples-of-4 subseries lies at the origin $(0, 0)$ **if and only if** $b = b_n$ is a zero of $\zeta(\frac{1}{2} + ib)$. For all non-zero choices of b , the subseries spirals around a non-zero point $C_{\text{target}}(b) \neq (0, 0)$.

Proof. On the critical line $a = \frac{1}{2}$, the conic radius of the anchor term $t_4(\frac{1}{2} + ib)$ is:

$$R_4(b) = \left| t_4 \left(\frac{1}{2} + ib \right) \right| = \frac{1}{2\sqrt{3 - 2\sqrt{2}\cos(b\ln 2)}} \in \left[\frac{\sqrt{2}-1}{2} \approx 0.2071, \frac{\sqrt{2}+1}{2} \approx 1.2071 \right]. \quad (21)$$

Because $R_4(b) \geq 0.2071 > 0$ for all $b \in \mathbb{R}$, the factor $t_4(\frac{1}{2} + ib)$ never vanishes. Applying the Euler-Maclaurin summation formula [2, 6] to $\sum_{k=1}^K k^{-s}$ and multiplying by $t_4(s)$ yields (20). Thus, $C_{\text{target}}(b) = t_4(\frac{1}{2} + ib)\zeta(\frac{1}{2} + ib) = (0, 0)$ if and only if $\zeta(\frac{1}{2} + ib) = 0$. $\square$

Table 4 illustrates Theorem 5.2: at true zeta zeros $(b_1, b_2, b_3, b_4, b_5)$, the anchor conic $t_4(b_n)$ is compressed near its periapsis minimum (0.2071) and the first five terms $t_4, t_8, t_{12}, t_{16}, t_{20}$ contract strongly toward $C_{\text{target}}(b_n) = (0, 0)$. Conversely, at non-zero values of b (such as $b = 0$, $b = \frac{\pi}{\ln 2} \approx 4.5324$, $b = 10.0$, and the Gram point $b = 17.8456$), adding $t_8, \dots, t_{20}$ reinforces t_4 toward the off-origin center $C_{\text{target}}(b) \neq (0, 0)$. Finally, applying the dyadic pairing theorem recursively to the multiples of 4 by pairing $(t_{4m} + t_{8m}), (t_{16m} + t_{32m}), \dots$ restores alternating signs and sums the complete binary tower of each odd m to $\mathcal{T}_m(s) = \sum_{r=0}^{\infty} t_{2^r m}(s) = (1 - 2^{-s})^{-1} m^{-s}$.

Table 4: Comparison of the multiples-of-4 subseries $\Sigma_5^{(4)}(b) = \sum_{k=1}^5 t_{4k}(\frac{1}{2} + ib)$ at non-zero values of b versus the first five Riemann zeta zeros (b_1 through b_5).

Input b	Category	Radius $R_4(b)$	Target $ C_{\text{target}}(b) $	5-Term $ \Sigma_5^{(4)}(b) $	Ratio $\frac{ \Sigma_5^{(4)} }{R_4(b)}$
$b = 0.0000$	Non-Zero (Real Axis)	1.2071	$1.7620 \neq 0$	3.9010	3.232 (Diver)
$b = \frac{\pi}{\ln 2} \approx 4.5324$	Non-Zero (Half-Period)	0.2071	$0.1398 \neq 0$	0.2477	1.196 (Off-O
$b = 10.0000$	Non-Zero	0.5791	$0.8972 \neq 0$	0.7469	1.290 (Off-O
$b = 17.8456$	Non-Zero (Gram Point g_1)	1.0514	$2.4606 \neq 0$	2.2878	2.176 (Off-O
$b_1 \approx 14.1347$	1st Zeta Zero	0.2107	0.0000	0.0467	0.222 (77.8%
$b_2 \approx 21.0220$	2nd Zeta Zero	0.2443	0.0000	0.0536	0.219 (78.1%
$b_3 \approx 25.0109$	3rd Zeta Zero	0.2968	0.0000	0.0752	0.253 (74.7%
$b_4 \approx 30.4249$	4th Zeta Zero	0.2293	0.0000	0.0887	0.387 (61.3%
$b_5 \approx 32.9351$	5th Zeta Zero	0.2260	0.0000	0.0996	0.441 (55.9%

6 Conclusion

By unifying the Dirichlet prefactor with individual terms, we have shown that the critical line $a = \frac{1}{2}$ is distinguished not only algebraically by functional equation symmetry, but geometrically: it is the **unique real axis slice where the base-2 epicyclic anchor $S_2(s)$ and all its dyadic descendents $P_m(s)$ are perfectly centered at the origin $(0, 0)$** [5]. Furthermore, the phase distribution across the concentric circles $P_m(\frac{1}{2} + ib)$ and the exclusive origin-convergence of the complementary multiples-of-4 subseries $\sum_{k=1}^{\infty} t_{4k}(s) = t_4(s)\zeta(s)$ provide complementary geometric criteria that separate Riemann zeta zeros from all other points on the critical line.

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References

- [1] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford University Press, 1986.
- [2] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.
- [3] T. Needham, *Visual Complex Analysis*, Oxford University Press, 1997.
- [4] J. B. Garnett, *Bounded Analytic Functions*, Revised 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, 2007.
- [5] H. Bohr and R. Courant, “Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion,” *Journal für die reine und angewandte Mathematik*, vol. 144, pp. 249–274, 1914.
- [6] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, 1976.