

EXPLICIT ONSETS, RATES AND THE TAIL OF THE COLLISION CONSTANT FOR HEAT-EVOLVED WEYL POLYNOMIALS

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ABSTRACT. Let h_k be the orthonormal Hermite polynomials for the standard Gaussian weight, and let $a(w) = \sum_{j \geq 0} a_j w^j$ be a real power series with radius of convergence greater than 1, $a(0) \neq 0$, no zeros on the unit circle and only simple zeros in the unit disk. For all large n the polynomial $\sum_{j=0}^n a_j h_{n-j}$ has exactly as many nonreal zeros as a has in the disk, but the known proof gives the onset by compactness. We make it explicit: if a has no zeros in $\{1-d \leq |w| \leq 1\}$, with d below an explicit threshold, the count is exact as soon as n exceeds a polynomial in $1/d$ and in the inverses of $|a(0)|$, $\min_{|w|=1} |a|$, $\min_{|w|=1-d} |a|$ and $\min_{[-1+d, 1-d]} \max(|a|, |a'|)$; no separation of the nonreal zeros of a is needed. By an exact identity this applies to the random Weyl polynomials G_n with independent standard Gaussian coefficients under the backward heat flow $G_{n,t} = \exp(-\frac{t}{2n} \partial_z^2) G_n$, whose real-rooting time converges in distribution to $T_* = R_*^{-2}$, where R_* is the smallest modulus of a nonreal zero of the Gaussian power series $q(w) = \sum_j \eta_j w^j$. We show that the convergence holds with the rate $n^{-\alpha}$ on $[1+\varepsilon, \infty)$ for every $\alpha < \frac{3}{50}$, an exponent that does not depend on ε ; the dependence on ε is removed by a super-exponential tail for the number of zeros of q in a disk. Under a coupling of the degrees that shares the top coefficients, the real-rooting time Λ_n satisfies $|\Lambda_n - T_*| \leq K/n$ for $n \geq n_0$, with K and n_0 depending on q . We bound K and n_0 by polynomials in condition parameters of q , with absolute degrees and, when $T_* \geq 1 + \varepsilon$, coefficients polynomial in $1/\varepsilon$. For the Gaussian power series this gives, on the event $\{T_* \geq 1 + \varepsilon\}$, a power tail with exponent $1/900$ for the largest scaled displacement D of the final collision of zeros, and without restriction the tail bound $\exp(-c(\log x / \log \log x)^{1/2})$; the inputs include closed forms of the two-point correlation functions of the zeros of q .

1. INTRODUCTION

1.1. Setting. This paper makes quantitative three results of the main paper [M], to which we refer for background and notation. Let $\xi_0, \xi_1, \dots$ be independent standard real Gaussian variables, let $G_n(z) = \sum_{k=0}^n \xi_k (\sqrt{n} z)^k / \sqrt{k!}$ be the real Weyl polynomial of degree n , and let $G_{n,t} = \exp(-\frac{t}{2n} \partial_z^2) G_n$, $t \geq 0$, be its evolution under the backward heat flow. The set of times at which $G_{n,t}$ has only real zeros is an interval $[\Lambda_n^{\text{std}}, \infty)$, and Λ_n^{std} is the *real-rooting time*. Let $\eta_0, \eta_1, \dots$ be independent standard real Gaussian variables, let $q(w) = \sum_{j \geq 0} \eta_j w^j$ be the *Gaussian power series*, let R_* be the smallest modulus of a nonreal zero of q in $\mathbb{D} = \{|w| < 1\}$, and $T_* = R_*^{-2}$. By [M, Theorem 1.1], $\Lambda_n^{\text{std}} \rightarrow T_*$ in distribution; moreover, for every $t > 1$, the number of nonreal zeros of $G_{n,t}$ converges in distribution to the number of nonreal zeros of q in $\{|w| < t^{-1/2}\}$, and $\mathbb{P}(G_{n,t} \text{ has only real zeros}) \rightarrow \mathbb{P}(T_* \leq t)$. (In [M] these statements are proved for more general coefficients; here the coefficients are standard Gaussian whenever they are random.) The proof goes through the reversed polynomials $P_{n,t}$ of [M, (1.5)], which are built from the coefficients of q and share them across all degrees; for each n , $(P_{n,t})_{t \geq 0}$ has the law of $(G_{n,t})_{t \geq 0}$, and the real-rooting time Λ_n of $P_{n,t}$ converges to T_* almost surely [M, Theorem 1.5].

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The deterministic input is an exact count of nonreal zeros [M, Theorem 1.4]. Let $h_k = \text{He}_k/\sqrt{k!}$ be the orthonormal polynomials for the standard Gaussian weight, and let $a(w) = \sum_{j \geq 0} a_j w^j$ be a real power series with radius of convergence greater than 1, $a(0) \neq 0$, no zeros on the unit circle and only simple zeros $w_1, \dots, w_k$ in $\mathbb{D}$. Then for every small $\varepsilon > 0$ there is n_0 such that for $n \geq n_0$ all zeros of $z \mapsto Q_n(\sqrt{n}z)$, where $Q_n = \sum_{j=0}^n a_j h_{n-j}$, are simple; exactly one of them lies within ε of each point $w_i + w_i^{-1}$, and it is real if and only if w_i is real; and the other $n - k$ are real and lie in $(-2 - \varepsilon, 2 + \varepsilon)$. In particular Q_n has exactly as many nonreal zeros as a has in $\mathbb{D}$. By the identity [M, (1.6)], $P_{n,t}(z)$ is, up to the factor $t^{n/2}$ and the dilation $z \mapsto z/\sqrt{t}$, the polynomial $Q_n(\sqrt{n} \cdot)$ for the symbol $a^{(t)}(w) = q(w/\sqrt{t})$, which has radius of convergence $\sqrt{t} > 1$ when $t > 1$.

Both n_0 in [M, Theorem 1.4] and the constants of the theorem on the final collision [M, Theorem 9.1] are obtained by compactness. We bound them explicitly, and we draw two consequences for the Gaussian case: a rate of convergence in [M, Theorem 1.1], and a tail bound for the constant of the final collision.

1.2. An explicit onset.

Theorem 1.1 (An explicit onset). *Let $a(w) = \sum_{j \geq 0} a_j w^j$ be a real power series with $|a_j| \leq C_0 R_0^{-j}$ for all $j \geq 0$, where $R_0 > 1$, such that $a(0) \neq 0$ and a has no zeros on the unit circle. Let $0 < d \leq \frac{1}{2}$ be such that a has no zero in $\{1 - d \leq |w| \leq 1\}$ and the minimum ν_d of $\max(|a(x)|, |a'(x)|)$ over $x \in [-1 + d, 1 - d]$ is positive. If d and n satisfy the explicit conditions (i)–(iv) of Theorem 2.6, which hold for all large n once d is small enough, then the number of nonreal zeros of $Q_n = \sum_{j=0}^n a_j h_{n-j}$ equals the number of nonreal zeros of a in $\mathbb{D}$, both counted with multiplicity, and all real zeros of Q_n are simple.*

This is Theorem 2.6. Condition (i) is $d \leq \kappa/\sqrt{2}$, where κ depends only on $C_0, R_0, \min_{|w|=1} |a|$ and the number of zeros of a in $\mathbb{D}$. By Jensen's formula (Lemma 2.2(b)), (ii)–(iv) hold as soon as n exceeds a polynomial in $1/d$ and in the inverses of $|a(0)|, \min_{|w|=1} |a|, \min_{|w|=1-d} |a|$ and ν_d , with exponents depending only on R_0 . No separation of the nonreal zeros of a is needed, and they need not be simple. Theorem 2.7 adds the localisation of [M, Theorem 1.4], with an onset that depends also on the minimum of $|a|$ away from its zeros (Remark 2.8). An explicit onset is needed when the symbol depends on n , as in [C], or is random, as in Theorems 1.2 and 1.3 below.

1.3. A rate of convergence.

Theorem 1.2 (A rate of convergence). *Let the coefficients be standard Gaussian. For every $\varepsilon \in (0, 1]$ and every $\alpha \in (0, \frac{3}{50})$ there is a constant $C_{\varepsilon, \alpha} < \infty$ such that for all $n \geq 2$*

$$\sup_{t \geq 1+\varepsilon} |\mathbb{P}(\Lambda_n^{\text{std}} \leq t) - \mathbb{P}(T_* \leq t)| \leq C_{\varepsilon, \alpha} n^{-\alpha}.$$

Theorem 1.2 is part of Theorem 3.6, which also gives a rate $n^{-\theta}$, $\theta < \frac{1}{10}$, at each fixed $t > 1$, for the probability that $G_{n,t}$ has only real zeros and for the law of the number of its nonreal zeros. It rests on Theorem 1.1 and on a bound for the number of zeros of q in a disk (Lemma 3.5), which makes the exponent independent of ε . The exponent is far from optimal (Remark 3.7), and we have no rate uniform in $t > 1$.

1.4. Explicit constants and the tail of the collision constant. Let $q(w) = \sum_j \eta_j w^j$ now be a deterministic real power series that satisfies the conditions (G1)–(G3) of [M, Section 9]: $q(0) \neq 0$, the nonreal zeros of q of minimal modulus R_* inside its disk of convergence are one simple conjugate pair $u_* = R_* e^{i\theta_*}, \bar{u}_*$, with $0 < \theta_* < \pi$, q has no real zero of modulus R_* , and

its real zeros of smaller modulus are simple. Let $T_* = R_*^{-2}$, and let $P_{n,t}$ and Λ_n be built from the coefficients of q as above. By [M, Theorem 9.1] there are $\delta > 0$, $K < \infty$ and n_0 such that for $n \geq n_0$: $|\Lambda_n - T_*| \leq K/n$; all zeros of P_{n,Λ_n} are real, and exactly one of them, z_n , is multiple, with $|z_n - 2R_*^{-1} \cos \theta_*| \leq K/n$; for $T_* - \delta \leq t < \Lambda_n$ the polynomial $P_{n,t}$ has exactly two nonreal zeros; and Λ_n is the only $t \geq T_* - \delta$ at which $P_{n,t}$ has a multiple zero. For the Gaussian power series (G1)–(G3) hold almost surely [M, Corollary 9.14], so that the largest scaled displacement of the collision,

$$D = \sup_{n \geq 1} n(|\Lambda_n - T_*| + |z_n - 2R_*^{-1} \cos \theta_*|),$$

where for $n < n_0$ the point z_n is chosen by the fixed rule of [M, Section 9.6], is finite almost surely. Through the coupling of the proof of [M, Corollary 9.16], the tail of D controls the tightness of $n(\Lambda_n^{\text{std}} - \tilde{T}_{*,n})$ for the Weyl polynomials, where $\tilde{T}_{*,n}$ has the law of T_* .

Theorem 1.3 (Explicit constants and the tail of D). (a) *Let $0 < \varepsilon \leq 1$, and let q satisfy (G1)–(G3), with radius of convergence at least 1, $T_* \geq 1 + \varepsilon$, $\sup_j |\eta_j|(1 + \varepsilon/2)^{-j/4} < \infty$, and no multiple zero in $[-r_c, r_c]$, $r_c = (1 + \varepsilon/2)^{-1/2}$. Then the conclusions of [M, Theorem 9.1] hold with δ , K and n_0 such that*

$$\delta^{-1} \leq 6\varepsilon^{-1} \Pi, \quad K \leq c_\varepsilon \Pi^{5/2}, \quad n_0 \leq c_\varepsilon \Pi^{10},$$

where $c_\varepsilon \leq c\varepsilon^{-c}$ with an absolute constant c , and Π is the product of nine factors, each at least 1, built from condition parameters of q : the maxima of 1 and the inverses of $|u_* q'(u_*)|$, of $\sin \theta_*$, of the distance g from R_* to the moduli of the other zeros of q , of the minimum of $|q|$ on an annulus about the circle $\{|u| = R_*\}$ away from u_* and $\bar{u}_*$, of the minimum of $\max(|q|, |q'|)$ on $[-r_c, r_c]$ and of $|q(0)|$; the maximum of 1 and the coefficient bound $\sup_j |\eta_j|(1 + \varepsilon/2)^{-j/4}$; T_* ; and a factor that is exponential in the number of zeros of q in $\{|u| \leq r_c\}$. The precise definition of Π is given in Section 4.1.

(b) *Let q be the Gaussian power series. There is an absolute constant c such that for all $0 < \varepsilon \leq 1$ and $x \geq 1$*

$$\mathbb{P}(D > x, T_* \geq 1 + \varepsilon) \leq c\varepsilon^{-c} x^{-\beta_\varepsilon} \quad \text{with} \quad \beta_\varepsilon \geq \frac{\varepsilon}{35640 \log(40/\varepsilon)},$$

and $\mathbb{P}(D > x, T_* \geq 1 + \varepsilon) \leq C_\varepsilon x^{-1/900}$, where $\log C_\varepsilon \leq c\varepsilon^{-c}$.

(c) *There are absolute constants $c > 0$ and C such that for the Gaussian power series and all $x \geq e^e$*

$$\mathbb{P}(D > x) \leq C \exp\left(-c \left(\frac{\log x}{\log \log x}\right)^{1/2}\right).$$

Part (a) is Theorem 4.1, part (b) is Theorem 4.12, and part (c) is Corollary 4.13. In (a) the exponents are absolute; the factor exponential in the number of zeros is kept as a parameter because for random q it can be bounded either by Jensen's formula, which gives the exponent β_ε of order $\varepsilon/\log(1/\varepsilon)$ in (b), or through the number of zeros of q in a disk, which gives the exponent $1/900$, independent of ε , but a constant C_ε that is not polynomial in $1/\varepsilon$. The proof of (b) uses closed forms of the two-point correlation functions of the zeros of q (Proposition 4.5), which give a small-ball bound for the gap g between R_* and the other moduli of zeros (Theorem 4.6). Part (c) combines the first bound of (b) with the bound for $\mathbb{P}(T_* < 1 + \varepsilon)$ that follows from [M, Theorem 7.13]. Whether $\mathbb{P}(D > x)$ decays like a power of x is open (Question 5.2); a small-ball lower bound for $\text{Im } u_*$, which we expect but have not proved, would exclude exponents above 1 (Remark 4.14(3)).

1.5. Ideas of the proofs. *The explicit onset (Section 2).* The proof of [M, Theorem 1.4] deforms a , through $a_s(w) = p_{\text{in}}(w)p_{\text{out}}(sw)$, $s \in [0, 1]$, into the polynomial p_{in} with the same zeros in $\mathbb{D}$, and excludes real double zeros along the way by a bulk estimate: in the bulk, $(Q_n, Q'_n/\sqrt{n})$ is a 2×2 matrix, built from $O(\log n)$ steps of the backward three-term recurrence, applied to (h_n, h_{n-1}) , plus a tail controlled by a Christoffel–Darboux bound. Each step of that estimate has explicit constants once the coefficient bound of a_s and the minimum of $|a_s|$ on the unit circle are controlled uniformly in s . Lemma 2.2 controls both: the coefficients of a_s are bounded by a constant times Υ^k , and $|a_s| \geq \mathfrak{d}(a)|a|$ on the closed unit disk, where the distortion $\mathfrak{d}(a)$ of the outer factor p_{out} is at least $\min_{|w|=1} |a|$ times Υ^{-k} divided by a coefficient bound; here Υ depends only on R_0 , and Jensen’s formula bounds the number k of zeros of a in $\mathbb{D}$. In the exterior we compare $Q_n(\sqrt{n}\cdot)/h_n(\sqrt{n}\cdot)$, as a function of w with $z = w + w^{-1}$, with a on the single circle $\{|w| = 1 - d\}$, and inside it we pass from a to this function by a linear homotopy, along which real double zeros are excluded by ν_d ; this is why no separation of the zeros of a enters.

The rate (Section 3). For the random symbol $a^{(t)} = q(\cdot/\sqrt{t})$, each parameter in the conditions of Theorem 1.1 has a polynomial small-ball bound, uniformly in $t \geq 1 + \varepsilon$: the coefficient bound, the minimum of $|q|$ on a circle, the near-double real zeros and the zeros near the circle (Lemmas 3.1–3.4). The factor exponential in the number of zeros of $a^{(t)}$ in $\mathbb{D}$ would cost, through Jensen’s formula, an exponent of order $\varepsilon/\log(1/\varepsilon)$; instead we use that the number of zeros of q in a disk has super-exponential tails (Lemma 3.5). Large times are handled by the tail of T_* .

Explicit constants and the tail of D (Section 4). Each compactness step of the proof of [M, Theorem 9.1] can be made explicit, and Theorem 2.6 replaces [M, Theorem 1.4] at the two times $T_* \pm \delta$. This bounds δ^{-1} , K and n_0 by powers of Π , times powers of $1/\varepsilon$. For the Gaussian power series each factor of Π has a power tail on $\{T_* \geq 1 + \varepsilon\}$. For the modulus gap g this is a first-moment bound over pairs of zeros: the two-point correlation functions, which we write in closed form through the Gaussian regression on the zeros, are at most the products of the one-point intensities. The factor exponential in the number of zeros is bounded through the super-exponential tail of that number, as in the proof of Theorem 1.2.

1.6. Related work. For the literature on zeros of linear combinations of orthogonal polynomials, on the heat flow of random polynomials and on the de Bruijn–Newman constant we refer to the introduction of [M]. Times $t = t_n$ that tend to 1 are the subject of [B], and symbols that depend on n , varying weights and the Laguerre flow that of [C]. The one-point intensities of the zeros of the Gaussian power series go back to Kac [5] for the real zeros, and the correlation functions are Pfaffians, by Matsumoto and Shirai [7] and, for the mixed correlations of real and nonreal zeros, by Forrester [3]. We need only the two-point functions, which we write in closed form (Proposition 4.5), and the Rice formulas of Azaïs and Wschebor [2].

1.7. Organisation. Section 2 proves Theorem 1.1 (Theorem 2.6) and its form with localisation (Theorem 2.7). Section 3 proves Theorem 1.2 (Theorem 3.6) and bounds the distortion of the outer factor for the random symbol (Proposition 3.9). Section 4 proves Theorem 1.3: the explicit constants in Section 4.1, the two-point functions and the modulus gap in Section 4.2, small-ball bounds at the collision in Section 4.3, and the tail of D in Section 4.4. Section 5 contains remarks and open questions.

2. AN EXPLICIT ONSET FOR THE EXACT COUNT

The proof of [M, Theorem 1.4] in [M, Section 6] obtains n_0 by compactness. To make it explicit, we control the deformation $a_s(w) = p_{\text{in}}(w)p_{\text{out}}(sw)$ of that proof through the zeros of a in $\mathbb{D}$ (Lemma 2.2), and we treat the exterior with one circle and one real segment instead of

the disks about the points $\zeta_i = w_i + w_i^{-1}$ used there, so that the count needs no separation of the zeros.

Definition 2.1 (Condition parameters). For $C_0 < \infty$ and $R_0 > 1$ let $\mathcal{S}(C_0, R_0)$ be the set of real power series $a(w) = \sum_{j \geq 0} a_j w^j$ with $|a_j| \leq C_0 R_0^{-j}$ for all $j \geq 0$, and put $Q_n[a] = \sum_{j=0}^n a_j h_{n-j}$. For $a \in \mathcal{S}(C_0, R_0)$ and $0 < d \leq \frac{1}{2}$ let

$$\alpha = |a(0)|, \quad \mu = \min_{|w|=1} |a(w)|, \quad \mu_d = \min_{|w|=1-d} |a(w)|, \quad \nu_d = \min_{x \in [-1+d, 1-d]} \max(|a(x)|, |a'(x)|),$$

and let $w_1, \dots, w_k$ be the zeros of a in $\mathbb{D}$, repeated according to multiplicity. Finally let

$$R_1 = R_0^{1/2}, \quad \bar{M} = \frac{C_0}{1 - R_0^{-1/2}}, \quad \Upsilon = \frac{R_1 + 1}{R_1 - 1}, \quad E = \frac{\log \Upsilon}{\log R_1}.$$

Thus $|a| \leq \bar{M}$ on $\{|w| = R_1\}$, and Υ and E depend only on R_0 .

Lemma 2.2 (Structure of a symbol). *Let $a \in \mathcal{S}(C_0, R_0)$ with $\alpha > 0$ and $\mu > 0$. Let $p_{\text{in}}(w) = a(0) \prod_{i=1}^k (1 - w/w_i)$, $p_{\text{out}} = a/p_{\text{in}}$ and $G = \alpha / \prod_i |w_i|$.*

- (a) $|w_i| \geq r_a := \alpha/(2C_0)$ and $1 - |w_i| \geq d_a := \mu R_0(1 - R_0^{-1})^2/C_0$ for every i .
- (b) $k \log R_1 \leq \log(\bar{M}/\alpha)$; hence $\Upsilon^k \leq (\bar{M}/\alpha)^E$.
- (c) p_{out} is analytic in $\{|w| < R_0\}$ and has no zeros in $\bar{\mathbb{D}}$, and

$$\mathfrak{d}(a) := \frac{\min_{\bar{\mathbb{D}}} |p_{\text{out}}|}{\max_{\bar{\mathbb{D}}} |p_{\text{out}}|} \geq \frac{\mu}{\bar{M}} \left(\frac{R_1 - 1}{2} \right)^k \geq \mathfrak{d}_0 := \frac{\mu}{\bar{M}} \Upsilon^{-k}.$$

- (d) For $s \in [0, 1]$ let $a_s(w) = p_{\text{in}}(w) p_{\text{out}}(sw)$. Then $a_s \in \mathcal{S}(\bar{M}\Upsilon^k, R_1)$, the coefficients of a_s are polynomials in s , $a_s(0) = a(0)$, $a_1 = a$, and for $s = 0$ the series a_s is the real polynomial p_{in} of degree k . The zeros of a_s in $\bar{\mathbb{D}}$ are $w_1, \dots, w_k$, and $|a_s(w)| \geq \mathfrak{d}(a) |a(w)|$ for $|w| \leq 1$.

Proof. (a) If $a(w_i) = 0$, then $\alpha \leq \sum_{j \geq 1} C_0 x^j = C_0 x/(1 - x)$ with $x = |w_i|/R_0$, so $|w_i| \geq R_0 \alpha/(C_0 + \alpha) \geq \alpha/(2C_0)$, as $\alpha \leq C_0$. With $u = w_i/|w_i|$ we have $\mu \leq |a(u) - a(w_i)| \leq (1 - |w_i|) \max_{\bar{\mathbb{D}}} |a'|$, and $\max_{\bar{\mathbb{D}}} |a'| \leq \sum_j j C_0 R_0^{-j} = C_0 R_0^{-1} (1 - R_0^{-1})^{-2}$.

(b) By Jensen's formula, $\sum \log(R_1/|z|) \leq \log(\bar{M}/\alpha)$, the sum over the zeros z of a in $\{|w| < R_1\}$, and each w_i contributes at least $\log R_1$. Since $\bar{M} \geq \alpha$, $\Upsilon^k \leq (\bar{M}/\alpha)^E$.

(c) The singularities of p_{out} at the w_i are removable. On $|w| = 1$, $|p_{\text{in}}(w)| = G \prod_i |w - w_i| \leq G 2^k$, so $|p_{\text{out}}| \geq \mu/(G 2^k)$ there, and on $\bar{\mathbb{D}}$ by the minimum modulus principle. On $|u| = R_1$, $|p_{\text{in}}(u)| \geq G(R_1 - 1)^k$ and $|a(u)| \leq \bar{M}$, so $\max_{\bar{\mathbb{D}}} |p_{\text{out}}| \leq \max_{|u| \leq R_1} |p_{\text{out}}| \leq \bar{M}/(G(R_1 - 1)^k)$ by the maximum principle. Dividing, G cancels; and $(R_1 - 1)/2 \geq (R_1 - 1)/(R_1 + 1) = \Upsilon^{-1}$.

(d) On $|w| = R_1$, $|p_{\text{in}}(w)| \leq G(1 + R_1)^k$ and $|p_{\text{out}}(sw)| \leq \bar{M}/(G(R_1 - 1)^k)$, so $|a_s(w)| \leq \bar{M}\Upsilon^k$; use Cauchy's estimates. For $|w| \leq 1$, $p_{\text{out}}(sw) \neq 0$ and $a_s(w) = a(w) p_{\text{out}}(sw)/p_{\text{out}}(w)$. $\square$

The next three lemmas are quantitative forms of [M, Proposition 4.3], of [M, Proposition 5.1] on a disk, and of [M, Lemma 2.6].

Lemma 2.3 (Explicit bulk estimate). *Let $R > 1$, $\mu_S > 0$, and let $(a_s)_{s \in S}$ be a family in $\mathcal{S}(C, R)$ such that $|a_s(w)| \geq \mu_S$ for all $s \in S$ and $|w| = 1$. Put $\vartheta = R^{-1/2}$ and*

$$\kappa = \min \left\{ 1, \frac{1}{2} \log R, \frac{\mu_S R (1 - \vartheta)^2}{6C} \right\}, \quad \Gamma = \frac{8CR^{1/2}}{(1 - \vartheta)^2},$$

$$M_n = \left\lceil \frac{2 \log n}{\log R} \right\rceil, \quad D_H = \frac{C(198/\kappa + 4M_n + 8)}{(1 - \vartheta)^4}.$$

If $n \geq 2$ satisfies

$$(2.1) \quad n \geq \frac{(M_n + 2)^2}{\kappa}, \quad n \geq D_H, \quad n \geq \frac{8(2\Gamma + 1)D_H}{\mu_S^2},$$

$$n^{3/2} \geq \frac{37(\Gamma + 1)C}{(1 - R^{-2})^{1/2}\mu_S^2},$$

then for all $s \in S$ and all real x with $|x| \leq 2\sqrt{n} \cosh \kappa$,

$$Q_n[a_s](x)^2 + \frac{Q_n[a_s]'(x)^2}{n} \geq c^2(h_n(x)^2 + h_{n-1}(x)^2), \quad c = \frac{\mu_S^2}{16(\Gamma + 1)}.$$

Proof. We follow the proof of [M, Proposition 4.3], with $L = 2 \cosh \kappa$ and $\kappa_0 = \frac{1}{2} \log R$, so that $e^{\kappa_0}/R = \vartheta$ and $\kappa \leq \kappa_0$.

The limit matrix. By [M, Lemma 4.2] with κ_0 , $\det H_{a_s}(z) \geq \mu_S^2$ for $|z| \leq 2$, and $\det H_{a_s}(z) = a_s(w)a_s(w^{-1})$ for $2 < |z| \leq L$, $z = w + w^{-1}$, w real, $e^{-\kappa} \leq |w| < 1$. Since $\max_{|u| \leq e^{\kappa_0}} |a_s'(u)| \leq C \sum_j j R^{-j} e^{(j-1)\kappa_0} = CR^{-1}(1 - \vartheta)^{-2}$, for real v with $e^{-\kappa} \leq |v| \leq e^{\kappa}$ and $\varsigma = \text{sign } v$,

$$|a_s(v) - a_s(\varsigma)| \leq (e^{\kappa} - 1)CR^{-1}(1 - \vartheta)^{-2} \leq e\kappa CR^{-1}(1 - \vartheta)^{-2} \leq \frac{e}{6}\mu_S < \frac{1}{2}\mu_S.$$

As $a_s(\varsigma)$ is real with $|a_s(\varsigma)| \geq \mu_S$, the real numbers $a_s(w)$ and $a_s(w^{-1})$ have the sign of $a_s(\varsigma)$ and modulus at least $\mu_S/2$. So $\det H_{a_s} \geq \mu_S^2/4$ on $[-L, L]$. By [M, Lemma 3.3], $\|G_j(z)\| \leq 4(j+2)e^{(j+1)\kappa}$, so $\|H_{a_s}(z)\| \leq 4Ce^{\kappa} \sum_j (j+2)\vartheta^j \leq \Gamma$ for $|z| \leq L$.

The head. The first condition gives $(M_n + 2)^2 \leq \kappa n$ with $\kappa \leq 1$. So for $j \leq M_n$ [M, Lemma 3.4], with $c_1 = 15/\kappa$, bounds the sum of the absolute values of the entries of the first row of $G_{n,j} - G_j$ by $c_1(j+1)^3 e^{j\kappa}/n$ (part (a)) and that of the second row by $(c_1 + 3)(j+1)^3 e^{j\kappa}/n$ (part (b)). Bounding the operator norm by the sum of the absolute values of the entries, $\|G_{n,j}(z) - G_j(z)\| \leq (2c_1 + 3)(j+1)^3 e^{j\kappa}/n \leq (33/\kappa)(j+1)^3 e^{j\kappa}/n$. For $j \geq 2$, [M, Lemma 2.5] gives $|A_j| \leq (j-1)e^{(j-2)\kappa}$, $|B_j|, |A_{j+1}| \leq j e^{(j-1)\kappa}$ and $|B_{j+1}| \leq (j+1)e^{j\kappa}$, so $\|G_j(z)\| \leq 4(j+1)e^{j\kappa}$. With $e^{\kappa}/R \leq \vartheta$, $\vartheta^{M_n} \leq 1/n$, $\sum_j (j+1)^3 \vartheta^j \leq 6(1 - \vartheta)^{-4}$ and $\sum_{j > M} (j+1)\vartheta^j \leq (M+2)\vartheta^{M+1}(1 - \vartheta)^{-2}$,

$$\|H_{n,M_n}[a_s](z) - H_{a_s}(z)\| \leq \frac{33C}{\kappa n} \sum_{j \geq 0} (j+1)^3 \vartheta^j + 4C \sum_{j > M_n} (j+1)\vartheta^j \leq \frac{D_H}{n} =: \delta_n$$

for $|z| \leq L$. The second and third conditions give $\delta_n \leq 1$ and $\delta_n(2\Gamma + 1) \leq \mu_S^2/8$. As in the proof of [M, Proposition 4.3], $\det H_{n,M_n}[a_s](z) \geq \mu_S^2/8$, $\|H_{n,M_n}[a_s](z)\| \leq \Gamma + 1$, and hence $s_{\min}(H_{n,M_n}[a_s](z)) \geq 2c$.

The tail. The tail estimate in the proof of [M, Proposition 4.3] holds with

$$\varepsilon_n = \sqrt{2}C(1 - R^{-2})^{-1/2}R^{-M_n}\sqrt{n(1 + L/2)} \leq 2.3C(1 - R^{-2})^{-1/2}n^{-3/2},$$

since $R^{-M_n} = \vartheta^{2M_n} \leq n^{-2}$ and $1 + L/2 \leq 1 + \cosh 1 < 2.55$. So $\varepsilon_n \leq c$ by the last condition, and by [M, (4.1)] the norm of the vector with entries $Q_n[a_s](x)$ and $Q_n[a_s]'(x)/\sqrt{n}$ is at least $(2c - \varepsilon_n)\|v_n(x)\| \geq c\|v_n(x)\|$. $\square$

Lemma 2.4 (Exterior error on a disk). *Let $a \in \mathcal{S}(C, R)$ with $R > 1$. For $0 < |w| < 1$ let $\varphi_n(w) = F_n(\mathcal{J}(w))$, where $\mathcal{J}(w) = w + w^{-1}$ and $F_n = Q_n[a](\sqrt{n}\cdot)/h_n(\sqrt{n}\cdot)$. Then φ_n extends analytically to $\mathbb{D}$, with $\varphi_n(0) = a(0)$ and $\varphi_n(\bar{w}) = \overline{\varphi_n(w)}$. For $0 < d < 1$ the function $X_n = \varphi_n - a$ satisfies*

$$\sup_{|w| \leq 1-d} |X_n(w)| \leq \frac{2\sqrt{2}C}{(1 - R^{-2})^{1/2}d^{7/2}n}, \quad \sup_{|x| \leq 1-d} |X_n'(x)| \leq \frac{64C}{(1 - R^{-2})^{1/2}d^{9/2}n},$$

the second supremum over real x .

Proof. The polynomials $Q_n[a](\sqrt{n} \cdot)$ and $h_n(\sqrt{n} \cdot)$ have degree n , their leading coefficients have ratio $a(0)$, and the zeros of the second lie in $(-2, 2)$ ([M, Lemma 2.4]). So φ_n is analytic on $\mathbb{D} \setminus \{0\}$ and tends to $a(0)$ at 0; real coefficients give the symmetry. Let $\rho = 1 - d$ and apply the last display of the proof of [M, Proposition 5.1] on the curve $\mathcal{J}(\{|w| = \rho\})$, which has distance at least $(1 - \rho)^2/\rho$ from $[-2, 2]$ by [M, (5.2)] and on which $|r| = \rho$. Since $\sum_{m \geq 1} m^2 \rho^{2m} = \rho^2(1 + \rho^2)(1 - \rho^2)^{-3} \leq 2\rho^2 d^{-3}$, this gives $|X_n| \leq 2C(1 - R^{-2})^{-1/2} \rho d^{-2} \cdot \sqrt{2} \rho d^{-3/2} n^{-1}$ on $|w| = \rho$, and the maximum principle extends the bound to $|w| \leq \rho$. For real $|x| \leq 1 - d$ the disk of radius $d/2$ about x lies in $\{|w| \leq 1 - d/2\}$, and Cauchy's estimate gives the second bound, as $2 \cdot 2\sqrt{2} \cdot 2^{7/2} = 64$. $\square$

Lemma 2.5 (Constancy in a symmetric domain). *Let Ω be a bounded Jordan domain, symmetric about $\mathbb{R}$, such that $\Omega \cap \mathbb{R}$ is an open interval I , and let U be a neighbourhood of $\bar{\Omega}$, symmetric about $\mathbb{R}$. For $s \in [0, 1]$ let f_s be analytic on U with $f_s(\bar{w}) = \overline{f_s(w)}$, such that $(s, w) \mapsto f_s(w)$ is continuous on $[0, 1] \times U$. If, for every s , f_s has no zero on $\partial\Omega$ and every real zero of f_s in I is simple, then the number of nonreal zeros of f_s in Ω , counted with multiplicity, does not depend on s .*

Proof. The number of zeros in Ω is constant by the argument principle. The number of real zeros in I is locally constant by the proof of [M, Lemma 2.6], with $[-\varrho, \varrho]$ replaced by $\bar{I}$, whose endpoints lie on $\partial\Omega$, where $f_{s_0} \neq 0$. $\square$

Theorem 2.6 (An explicit onset for the nonreal count). *Let $a \in \mathcal{S}(C_0, R_0)$ with $\alpha > 0$ and $\mu > 0$, and let $0 < d \leq \frac{1}{2}$ be such that a has no zero in $\{1 - d \leq |w| \leq 1\}$, $\mu_d > 0$ and $\nu_d > 0$. Put $C'_0 = \bar{M} \Upsilon^k$ and $\mu' = \mathfrak{d}_0 \mu$, and let κ, Γ, M_n and D_H be the constants of Lemma 2.3 for $C = C'_0, R = R_1$ and $\mu_S = \mu'$; thus $\kappa = \min\{1, \frac{1}{4} \log R_0, \mu' R_0^{1/2} (1 - \vartheta_0)^2 / (6C'_0)\}$ with $\vartheta_0 = R_0^{-1/4}$. Suppose that (i) $d \leq \kappa/\sqrt{2}$, and that $n \geq 2$ satisfies*

- (ii) $n \geq k$, and (2.1) holds for these constants;
- (iii) $n > 2\sqrt{2} C'_0 (1 - R_0^{-1})^{-1/2} d^{-7/2} / (\mathfrak{d}_0 \mu_d)$;
- (iv) $n > 64 C_0 (1 - R_0^{-2})^{-1/2} d^{-9/2} / \nu_d$.

Then the number of nonreal zeros of $Q_n[a]$ equals the number of nonreal zeros of a in $\mathbb{D}$, both counted with multiplicity, and all real zeros of $Q_n[a]$ are simple. Every zero z of $Q_n[a](\sqrt{n} \cdot)$ that is not of the form $\mathcal{J}(w)$ with $|w| < 1 - d$ is real and lies in $(-2 - 2d^2, 2 + 2d^2)$.

Size of the onset. By Lemma 2.2(b), $\Upsilon^k \leq (\bar{M}/\alpha)^E$, so all conditions are polynomial in $1/\mu, 1/\alpha, 1/\mu_d, 1/\nu_d$ and $1/d$, with exponents depending on R_0 only through E . For instance, normalising $C_0 = 1$ (the conclusion is invariant under $a \mapsto \lambda a$) and taking d of order $\mu^2 \Upsilon^{-2k}$, the largest value allowed by (i), conditions (iii) and (iv) read $n > C \Upsilon^{9k} \mu^{-8} \mu_d^{-1}$ and $n > C \Upsilon^{9k} \mu^{-9} \nu_d^{-1}$, with C depending only on R_0 . Neither the separation of the nonreal zeros of a nor $|a'|$ at its zeros enters. Real zeros enter through ν_d : if $x_1 < x_2$ are real zeros of a in $[-1 + d, 1 - d]$, then $\nu_d \leq \frac{1}{2}(x_2 - x_1)^2 \max_{[x_1, x_2]} |a''|$ by Rolle's theorem. For the symbols $a = q(\cdot/\sqrt{t})$ of [M, (1.6)] with independent standard Gaussian coefficients η_j , ν_d has a polynomial small-ball bound (Lemma 3.3).

Proof of Theorem 2.6. Let $\rho_0 = 1 - d$, let Ω_{in} be the interior of the ellipse $\mathcal{J}(\{|w| = \rho_0\})$, and $\Omega_{\text{out}} = \mathbb{C} \setminus \bar{\Omega}_{\text{in}} = \mathcal{J}(\{0 < |w| < \rho_0\})$; then $\Omega_{\text{in}} \cap \mathbb{R} = (-b, b)$ with $b = \rho_0 + \rho_0^{-1} \leq 2 + 2d^2$. Let a_s be as in Lemma 2.2(d), $\widehat{Q}_{n,s} = Q_n[a_s](\sqrt{n} \cdot)$, and $\varphi_{n,s}, X_{n,s} = \varphi_{n,s} - a_s$ as in Lemma 2.4 for a_s .

Step 1: the exterior count along the deformation. The zeros of a in $\mathbb{D}$ lie in $\{|w| < \rho_0\}$, so by Lemma 2.2(d) a_s has exactly k zeros there, and $|a_s| \geq \mathfrak{d}_0 \mu_d$ on $|w| = \rho_0$. Lemma 2.4 for $a_s \in \mathcal{S}(C'_0, R_1)$, where $1 - R_1^{-2} = 1 - R_0^{-1}$, and (iii) give $|X_{n,s}| < |a_s|$ on $|w| = \rho_0$. By Rouché's theorem $\varphi_{n,s}$ has exactly k zeros in $\{|w| < \rho_0\}$ and none on its boundary. As $\varphi_{n,s}(0) \neq 0$, $\mathcal{J}$

maps $\{0 < |w| < \rho_0\}$ conformally onto Ω_{out} and $h_n(\sqrt{n} \cdot)$ has no zeros off $[-2, 2]$, $\widehat{Q}_{n,s}$ has no zero on $\partial\Omega_{\text{in}}$, k zeros in Ω_{out} and $n - k$ in Ω_{in} .

Step 2: the bulk. By Lemma 2.2(d), $\min_{|w|=1} |a_s| \geq \mathfrak{d}_0 \mu = \mu'$. Lemma 2.3 with $C = C'_0$, $R = R_1$ and $\mu_S = \mu'$, whose conditions are (ii), shows that $\widehat{Q}_{n,s}$ has no real multiple zero in $[-L, L]$, $L = 2 \cosh \kappa$. By (i), $b - 2 \leq 2d^2 \leq \kappa^2 \leq L - 2$, so $(-b, b) \subset [-L, L]$.

Step 3: the interior. By Steps 1–2 and Lemma 2.5 with $\Omega = \Omega_{\text{in}}$, the number of nonreal zeros of $\widehat{Q}_{n,s}$ in Ω_{in} does not depend on s . For $s = 0$, $a_s = p_{\text{in}}$ has degree $k \leq n$ and $p_{\text{in}}(0) \neq 0$, so by [M, Lemma 2.7] $\widehat{Q}_{n,0}$ has at least $n - k$ distinct real zeros in $(-2, 2) \subset \Omega_{\text{in}}$; by Step 1 these are all its zeros in Ω_{in} . Hence all zeros of $\widehat{Q}_{n,1} = Q_n[a](\sqrt{n} \cdot)$ in Ω_{in} are real, and they are simple by Step 2.

Step 4: the exterior. Let $X = X_{n,1}$. Lemma 2.4 for $a \in \mathcal{S}(C_0, R_0)$ and (iv) give $|X(x)| < \nu_d$ and $|X'(x)| < \nu_d$ for real $|x| \leq \rho_0$; and, as $C_0 \leq C'_0$, $\mathfrak{d}_0 \leq 1$ and $1 - R_0^{-2} \geq 1 - R_0^{-1}$, (iii) gives $|X| < \mu_d \leq |a|$ on $|w| = \rho_0$. Let $f_\lambda = a + \lambda X$, $\lambda \in [0, 1]$; it satisfies $f_\lambda(\bar{w}) = \overline{f_\lambda(w)}$ and has no zero on $|w| = \rho_0$. If $f_\lambda(x) = f'_\lambda(x) = 0$ for some real $|x| < \rho_0$, then $|a(x)| = \lambda |X(x)| < \nu_d$ and $|a'(x)| < \nu_d$, contrary to the definition of ν_d . So by Lemma 2.5 on $\{|w| < \rho_0\}$ the number of nonreal zeros of f_λ there does not depend on λ . For $\lambda = 0$ it is the number of nonreal zeros of a in $\mathbb{D}$. For $\lambda = 1$, $f_1 = \varphi_n$, and it is the number of nonreal zeros of $Q_n[a](\sqrt{n} \cdot)$ in Ω_{out} , because $\mathcal{J}$ maps the nonreal points of $\{0 < |w| < \rho_0\}$ onto those of Ω_{out} , as $\text{Im } \mathcal{J}(w) = \text{Im } w (1 - |w|^{-2})$. The real zeros of φ_n in $(-\rho_0, \rho_0)$ are simple, and hence so are the real zeros of $Q_n[a](\sqrt{n} \cdot)$ in Ω_{out} . With Steps 1 and 3 this proves the theorem. $\square$

Theorem 2.7 (The exact count with an explicit onset). *Let $a \in \mathcal{S}(C_0, R_0)$ be admissible, with the notation of Lemma 2.2 and Theorem 2.6. Let σ be the smallest distance between distinct zeros of a in $\mathbb{D}$ ($\sigma = 1$ if $k \leq 1$), and for $\rho > 0$ let $m(\rho)$ be the minimum of $|a(w)|$ over the w with $|w| \leq 1$ and $|w - w_i| \geq \rho$ for all i . Put $\delta'_0 = d_a \min(d_a, \sigma/2)$ and $\beta = 1 + 16C_0^2/\alpha^2$. Let $0 < \varepsilon \leq \delta'_0/3$, and let*

$$\rho = \min\left\{\frac{\sigma}{3}, \frac{d_a}{3}, \frac{r_a}{2}, \frac{\varepsilon}{\beta}\right\}, \quad d = \min\left\{\frac{1}{2}, \frac{d_a}{3}, \frac{\kappa}{\sqrt{2}}, \sqrt{\varepsilon/2}\right\}.$$

If n satisfies (ii) of Theorem 2.6 and

$$n > \frac{2\sqrt{2}C'_0}{(1 - R_0^{-1})^{1/2} d^{7/2} \mathfrak{d}_0 m(\rho)},$$

then all zeros of $Q_n(\sqrt{n} \cdot)$ are simple, and the conclusions (a) and (b) of [M, Theorem 1.4] hold for this ε .

Since $m(\rho) \geq \mu 2^{-k} \rho (\sigma/2)^{k-1}$ (Remark 2.8), the onset is also polynomial in $1/\varepsilon$.

Proof. By Lemma 2.2(a), $|w_i| \leq 1 - d_a < 1 - d$, and the points of $\{|w| = 1 - d\}$ have distance at least $d_a - d \geq \rho$ from the zeros. So $\mu_d \geq m(\rho)$, the displayed condition implies (iii), and Steps 1–3 of the proof of Theorem 2.6 apply: $Q_n(\sqrt{n} \cdot)$ has k zeros in Ω_{out} , and its other $n - k$ zeros are real, simple and lie in $(-b, b) \subset (-2 - \varepsilon, 2 + \varepsilon)$, as $2d^2 \leq \varepsilon$. The disks $B_i = \{|w - w_i| \leq \rho\}$ are disjoint, as $\rho \leq \sigma/3$; they lie in $\{|w| < 1 - d\}$, as $|w_i| + \rho \leq 1 - \frac{2}{3}d_a$; they do not meet $\mathbb{R}$ if $w_i \notin \mathbb{R}$, as $|\text{Im } w_i| \geq \sigma/2$, and are symmetric about $\mathbb{R}$ if $w_i \in \mathbb{R}$. Since $|a| \geq m(\rho) > |\varphi_n - a|$ on $\{|w| \leq 1 - d\} \setminus \bigcup_i \text{int } B_i$ (Lemma 2.4), Rouché's theorem gives exactly one zero of φ_n in each $\text{int } B_i$, real if and only if w_i is real, and no other zero in $\{|w| < 1 - d\}$. On B_i , $|w| \geq r_a/2$, so $|\mathcal{J}'(w)| = |1 - w^{-2}| \leq \beta$ and $\mathcal{J}(\text{int } B_i) \subset \{|z - \zeta_i| < \beta\rho \leq \varepsilon\}$. Finally, by [M, (5.2)], $\text{dist}(\zeta_i, [-2, 2]) \geq d_a^2$, and for $i \neq i'$ and nonreal ζ_i

$$|\zeta_i - \zeta_{i'}| = |w_i - w_{i'}| \frac{|1 - w_i w_{i'}|}{|w_i w_{i'}|} \geq \sigma d_a, \quad |\text{Im } \zeta_i| = |\text{Im } w_i| \frac{1 - |w_i|^2}{|w_i|^2} \geq \frac{\sigma d_a}{2}.$$

These are all at least $\delta'_0 \geq 3\varepsilon$, so the disks $\{|z - \zeta_i| < \varepsilon\}$ are disjoint from each other and from $(-2 - \varepsilon, 2 + \varepsilon)$, and from $\mathbb{R}$ when $\zeta_i \notin \mathbb{R}$. This gives (a) and (b) of [M, Theorem 1.4]. $\square$

Remark 2.8 (The depth $m(\rho)$). If $\rho \leq \sigma/2$, at most one w_i lies within $\sigma/2$ of a point w that enters $m(\rho)$, so $|p_{\text{in}}(w)| \geq G\rho(\sigma/2)^{k-1}$, and the proof of Lemma 2.2(c) gives $m(\rho) \geq \mu 2^{-k} \rho(\sigma/2)^{k-1}$. There is no lower bound polynomial in $\mu, \alpha, \sigma, \rho$: for $k \geq 2$ and $0 < r \leq \frac{1}{4}$, $a(w) = C_0(\frac{2}{3R_0})^k((w - \frac{1}{2})^k - r^k)$ lies in $\mathcal{S}(C_0, R_0)$, since $\binom{k}{j} 2^{j-k} \leq (\frac{3}{2})^k$; α and μ lie in $[\frac{1}{2}, 2] \cdot C_0(3R_0)^{-k}$ for every r ; and $\sigma = 2r \sin(\pi/k) \in [4r/k, 2\pi r/k]$. With $\rho = r/k$, the point $\frac{1}{2} + r + \rho$ gives $m(\rho) \leq 4ek\alpha\rho(k\sigma/2)^{k-1} = O(\sigma^k)$ as $r \downarrow 0$, for every fixed k . So an onset obtained by comparing $|\varphi_n - a|$ with $|a|$ near the zeros depends on $m(\rho)$, not only on μ, α, σ . Whether the true onset is that large is not known; it would need a lower bound for $|X_n|$ near the cluster.

3. A RATE OF CONVERGENCE

For Gaussian coefficients Theorem 2.6 gives a rate of convergence in [M, Theorem 1.1] for $t \geq 1 + \varepsilon$, once the condition parameters of the symbol $a^{(t)} = q(\cdot/\sqrt{t})$ of [M, (1.6)] have small-ball bounds uniform in t . As in [M], we work with the polynomials $P_{n,t}$ of [M, (1.5)], built from the coefficients of q , and with their real-rooting time Λ_n ([M, Theorem 1.5]). We use the notation of Section 2: the class $\mathcal{S}(C_0, R_0)$, the parameters $\alpha, \mu, \mu_d, \nu_d$ and the constants $R_1, \bar{M}, \Upsilon, E$ of Definition 2.1, the number k of zeros of the symbol in $\mathbb{D}$, the bound $\mathfrak{d}_0$ of Lemma 2.2, and the quantities $C'_0, \mu', \kappa, \vartheta_0, M_n$ and conditions (i)–(iv) of Theorem 2.6. In this section q is the Gaussian power series [M, (1.3)], $0 < \varepsilon \leq 1$, $r_1 = (1 + \varepsilon)^{-1/2}$ and $r_2 = \frac{1}{2}(1 + r_1)$. By concavity, $1 - r_1 \geq (1 - 2^{-1/2})\varepsilon \geq \varepsilon/4$, so $1 - r_2 \geq \varepsilon/8$. For $t \geq 1 + \varepsilon$ the zeros of $a^{(t)}$ in $\bar{\mathbb{D}}$ are $\sqrt{t}$ times the zeros of q in $\{|u| \leq t^{-1/2}\} \subset \{|u| \leq r_1\}$. Put $\ell_\delta = \log(1/\delta)$ and

$$Y_i = \sup_{|u| \leq r_2} |q^{(i)}(u)| \leq S_i := \sum_{j \geq i} \frac{j!}{(j-i)!} |\eta_j| r_2^{j-i} \quad (i = 1, 2).$$

If $S = \sum_j c_j |\eta_j|$ with $c_j \geq 0$, then $\mathbb{P}(S > \mathbb{E}S + y) \leq \exp(-y^2/(2\sum_j c_j^2))$ for $y \geq 0$: indeed $\mathbb{E}e^{\lambda|\eta_j|} = 2e^{\lambda^2/2}\Phi(\lambda) \leq e^{\lambda\mathbb{E}|\eta_j| + \lambda^2/2}$ for $\lambda \geq 0$, by the concavity of $\log \Phi$, where Φ is the standard normal distribution function. Here $\mathbb{E}S_1 = \sqrt{2/\pi}(1 - r_2)^{-2} \leq 52\varepsilon^{-2}$, $\sum_j j^2 r_2^{2j-2} = (1 + r_2^2)(1 - r_2^2)^{-3} \leq 2(1 - r_2)^{-3}$, $\mathbb{E}S_2 = 2\sqrt{2/\pi}(1 - r_2)^{-3} \leq 818\varepsilon^{-3}$ and $\sum_j j^2(j-1)^2 r_2^{2j-4} = 4(1 + 4r_2^2 + r_2^4)(1 - r_2^2)^{-5} \leq 24(1 - r_2)^{-5}$. Hence, for $0 < \delta \leq e^{-1}$,

$$(3.1) \quad \mathbb{P}(Y_i > y_i(\delta)) \leq \delta, \quad y_1(\delta) = \frac{52 + 46\ell_\delta^{1/2}}{\varepsilon^2}, \quad y_2(\delta) = \frac{818 + 1255\ell_\delta^{1/2}}{\varepsilon^3}.$$

Lemma 3.1 (Coefficients). *Let $\lambda_\varepsilon = (1 + \varepsilon)^{1/4}$ and $C_q = \sup_j |\eta_j| \lambda_\varepsilon^{-j}$. Then $a^{(t)} \in \mathcal{S}(C_q, \lambda_\varepsilon)$ for every $t \geq 1 + \varepsilon$, $\mathbb{P}(C_q > x) \leq (7/\varepsilon)e^{-x^2/2}$ for $x \geq 1$, and $\mathbb{P}(|\eta_0| < \delta) \leq \delta$ for $\delta > 0$.*

Proof. For $t \geq 1 + \varepsilon$, $|\eta_j| t^{-j/2} \leq |\eta_j| \lambda_\varepsilon^{-2j} \leq C_q \lambda_\varepsilon^{-j}$. For $x \geq 1$, $\mathbb{P}(|\eta_j| > x \lambda_\varepsilon^j) \leq e^{-x^2 \lambda_\varepsilon^{2j}/2} \leq e^{-x^2/2} \lambda_\varepsilon^{-j}$, since $\lambda_\varepsilon^{2j} - 1 \geq 2j \log \lambda_\varepsilon$. Summing over j gives $e^{-x^2/2}/(1 - \lambda_\varepsilon^{-1})$, and $1 - \lambda_\varepsilon^{-1} \geq (1 - 2^{-1/4})\varepsilon \geq \varepsilon/7$ by concavity in ε . $\square$

Lemma 3.2 (Small values on a circle). *For $0 < r \leq r_1$ and $0 < \delta \leq e^{-1}$,*

$$\mathbb{P}\left(\min_{|u|=r} |q(u)| < \delta\right) \leq \frac{7000}{\varepsilon^2 r} \delta \ell_\delta^{3/2}.$$

Proof. Let $y = y_1(\delta)$, $N = \lceil 2\pi y/\delta \rceil$ and $\theta_k = 2\pi k/N$. On $\{Y_1 \leq y\}$, $|\frac{d}{d\theta} q(re^{i\theta})| \leq y$, so if $|q| < \delta$ somewhere on the circle, then $|q(re^{i\theta_k})| < 2\delta$ for some k . Let $u = re^{i\theta}$. From $\mathbb{E}|q(u)|^2 = (1 - r^2)^{-1}$

and $\mathbb{E}q(u)^2 = (1-u^2)^{-1}$ we get $\text{Var Re } q(u) = \frac{1}{2}((1-r^2)^{-1} + \text{Re}(1-u^2)^{-1}) \geq \frac{1}{2}$, as $\text{Re}(1-u^2) > 0$, and

$$\det \text{Cov}(\text{Re } q(u), \text{Im } q(u)) = \frac{1}{4} \left(\frac{1}{(1-r^2)^2} - \frac{1}{|1-u^2|^2} \right) = \frac{r^2 \sin^2 \theta}{(1-r^2)^2 |1-u^2|^2}.$$

So $\mathbb{P}(|q(u)| < 2\delta)$ is at most $4\delta/\sqrt{\pi}$, and also at most $4\pi\delta^2(1-r^2)|1-u^2|/(2\pi r|\sin \theta|) \leq 4\delta^2/(r|\sin \theta|)$. We use the first bound at the at most $N\delta + 4$ points θ_k with $|\sin \theta_k| < \delta$, and the second at the others, for which $\sum_k |\sin \theta_k|^{-1} \leq 4/\delta + (2N/\pi) \log(2/\delta)$. With $N\delta \leq 2\pi y + 1$ and $r \leq 1$,

$$\mathbb{P}\left(\min_{|u|=r} |q| < \delta\right) \leq \delta + \frac{4\delta(2\pi y + 5)}{\sqrt{\pi}} + \frac{16\delta}{r} + \frac{8\delta(2\pi y + 1)}{\pi r} \log \frac{2}{\delta} \leq \delta \left[1 + \frac{(2\pi y + 5)(6 + 3 \log \frac{2}{\delta})}{r} \right],$$

and $2\pi y + 5 \leq 621 \ell_\delta^{1/2} \varepsilon^{-2}$ and $6 + 3 \log(2/\delta) \leq 11.1 \ell_\delta$ by (3.1) and $\ell_\delta \geq 1$. $\square$

Lemma 3.3 (Near-double real zeros). *Let $t \geq 1 + \varepsilon$ and $0 < d \leq \frac{1}{2}$, and let $\nu_d(t)$ be the parameter ν_d of Definition 2.1 for $a^{(t)}$. For $0 < \delta \leq e^{-1}$,*

$$\mathbb{P}(\nu_d(t) < \delta) \leq (4000 \varepsilon^{-3} \ell_\delta^{1/2} + 4t)\delta.$$

Proof. We may assume $\delta t < \frac{1}{4}$, since otherwise the bound exceeds 1. If $\nu_d(t) < \delta$, there is a real u with $|u| \leq r_1$, $|q(u)| < \delta$ and $|q'(u)| < \delta' := \delta\sqrt{t}$. Let $y = y_2(\delta)$ and take a grid in $[-r_1, r_1]$ of spacing δ'/y , with at most $2y/\delta' + 1$ points. On $\{Y_2 \leq y\}$ the grid point g nearest to u has $|q'(g)| < \frac{3}{2}\delta'$ and $|q(g)| < \delta + \delta'^2/y$. The real vector $(q(g), q'(g))$ has covariance determinant $(1 - g^2)^{-4} \geq 1$, so its density is at most $1/(2\pi)$. Hence

$$\mathbb{P}(\nu_d(t) < \delta) \leq \delta + \left(\frac{2y}{\delta'} + 1\right) \frac{2}{\pi} \cdot \frac{3\delta'}{2} \left(\delta + \frac{\delta'^2}{y}\right) = \delta + \frac{3}{\pi} (2y + \delta') \left(\delta + \frac{\delta'^2}{y}\right) \leq \delta \left(1 + \frac{6y}{\pi} + 2t\right),$$

where we used $\delta t < \frac{1}{4}$. Then (3.1) gives the claim. $\square$

Lemma 3.4 (Zeros near the unit circle). *For $t \geq 1 + \varepsilon$ and $0 < d \leq \frac{1}{2}$, the probability that $a^{(t)}$ has a zero in $\{1 - d \leq |w| \leq 1\}$ is at most $5d(1 + \varepsilon)^3 \varepsilon^{-3}$.*

Proof. This is the event that q has a zero in $\{(1-d)r \leq |u| \leq r\}$, where $r = t^{-1/2} \leq r_1$. By [7, Theorem 2.7], as in the proof of [M, Lemma 7.9], the nonreal zeros of q have the first intensity $2|\text{Im } u|/(\pi|1-u^2|(1-|u|^2)^2) \leq 2/(\pi(1-r_1^2)^3)$ on $\{|u| \leq r_1\}$. The real zeros have the intensity $1/(\pi(1-x^2))$ on $(-1, 1)$ [5]; this is the Rice formula [2, Theorem 6.2] in dimension one, since $q(x)$ has variance $(1-x^2)^{-1}$ and $q'(x)$ has conditional variance $(1-x^2)^{-3}$ given $q(x)$. So the expected number of zeros in the annulus is at most $\frac{2}{\pi(1-r_1^2)^3} \cdot 2\pi d + \frac{2d}{\pi(1-r_1^2)} \leq \frac{5d}{(1-r_1^2)^3}$, and $1 - r_1^2 = \varepsilon/(1 + \varepsilon)$. $\square$

The bound $\Upsilon^k \leq (\overline{M}/\alpha)^E$ of Lemma 2.2(b) for the factor Υ^k in Theorem 2.6 is attained only by symbols with many zeros forced by a small α . For the random symbol the number of zeros in a disk has super-exponential tails. For $0 < r < 1$ let $\mathcal{Z}_r$ be the number of zeros of q in $\{|u| \leq r\}$, counted with multiplicity.

Lemma 3.5 (Overcrowding). *Let $0 < r < 1$, $L_r = \log \frac{1+r}{1-r}$ and $A_r = 2r^3/(1-r^2)^3$, and for $m \geq 1$ put $a = \lceil m/2 \rceil$ and $b = \lceil m/4 \rceil$. Then*

$$\mathbb{P}(\mathcal{Z}_r \geq m) \leq \left(\frac{eL_r}{\sqrt{2\pi a}} \right)^a + \left(\frac{\sqrt{2}eA_r}{\sqrt{b}} \right)^b.$$

Consequently, for $\Upsilon > 1$ and $\beta > 0$, with $m_0 = 4 + \max(e^2 L_r^2 \Upsilon^{4\beta}/\pi, 8e^2 A_r^2 \Upsilon^{8\beta})$,

$$\mathbb{P}(\mathcal{Z}_r \geq m) \leq 2 \Upsilon^{-\beta m} \quad (m \geq m_0), \quad \mathbb{P}(\Upsilon^{\mathcal{Z}_r} > s) \leq C(r, \Upsilon, \beta) s^{-\beta} \quad (s \geq 1),$$

with $C(r, \Upsilon, \beta) = \max(2, \Upsilon^{\beta m_0})$.

Proof. Almost surely the zeros of q in $\mathbb{D} \setminus \{0\}$ are simple ([M, Lemma 7.4]), $q(0) \neq 0$, and q has no zero on $\{|u| = r\}$ ([M, Lemma 7.5](a)). So $\mathcal{Z}_r = N_{\mathbb{R}} + 2k_r$, where $N_{\mathbb{R}}$ is the number of real zeros in $[-r, r]$ and k_r the number of zeros in $\mathbb{D}_r^+$; and $\mathcal{Z}_r \geq m$ implies $N_{\mathbb{R}} \geq a$ or $k_r \geq b$. We use $\mathbb{P}(N \geq j) \leq \mathbb{E}\binom{N}{j}$ and $j! \geq (j/e)^j$.

Nonreal zeros. Step 1 of the proof of [M, Proposition 7.11] gives $\mathbb{E}\binom{k_r}{b} \leq (\pi r^2 C_r)^b (2b)^{b/2}/b!$ with $C_r = 2r/(\pi(1-r^2)^3)$, and $\pi r^2 C_r = A_r$. So $\mathbb{P}(k_r \geq b) \leq (\sqrt{2} e A_r / \sqrt{b})^b$.

Real zeros. By [7, Section 6.1, proof of Theorem 2.1], the a -point correlation function of the real zeros of q is

$$\rho_a(x_1, \dots, x_a) = (2\pi)^{-a/2} (\det \Sigma)^{1/2} \mathbb{E}|q(x_1) \cdots q(x_a)|, \quad \Sigma = \left(\frac{1}{1 - x_i x_j} \right)_{i,j \leq a},$$

and by Cauchy's determinant formula $\det \Sigma = \prod_i (1 - x_i^2)^{-1} \prod_{i < j} ((x_i - x_j)/(1 - x_i x_j))^2 \leq \prod_i (1 - x_i^2)^{-1}$. By Hölder's inequality, $\mathbb{E}|q(x_1) \cdots q(x_a)| \leq \prod_i (\mathbb{E}|q(x_i)|^a)^{1/a} = \mathbb{E}|\xi|^a \prod_i (1 - x_i^2)^{-1/2}$ with ξ standard Gaussian, and $\mathbb{E}|\xi|^a \leq ((2a-1)!!)^{1/2} \leq a^{a/2}$ by the inequality of arithmetic and geometric means. As $\int_{-r}^r (1 - x^2)^{-1} dx = L_r$,

$$\mathbb{E}\binom{N_{\mathbb{R}}}{a} = \frac{1}{a!} \int_{[-r,r]^a} \rho_a \leq \frac{a^{a/2} L_r^a}{a! (2\pi)^{a/2}} \leq \left(\frac{e L_r}{\sqrt{2\pi a}} \right)^a.$$

The consequences. Let $m \geq m_0$. Then $a \geq m/2 \geq e^2 L_r^2 \Upsilon^{4\beta}/(2\pi)$ and $b \geq m/4 \geq 2e^2 A_r^2 \Upsilon^{8\beta}$, so the two terms are at most $\Upsilon^{-2\beta a}$ and $\Upsilon^{-4\beta b}$, both at most $\Upsilon^{-\beta m}$. For $s \geq \Upsilon^{m_0}$ let $m = \lfloor \log s / \log \Upsilon \rfloor + 1 \geq m_0$; then $\mathbb{P}(\Upsilon^{\mathcal{Z}_r} > s) \leq \mathbb{P}(\mathcal{Z}_r \geq m) \leq 2\Upsilon^{-\beta m} \leq 2s^{-\beta}$. For $1 \leq s < \Upsilon^{m_0}$ the bound $C(r, \Upsilon, \beta)s^{-\beta} \geq 1$ is trivial. $\square$

So $\mathcal{Z}_r$, whose mean is of order $(1-r)^{-1}$, exceeds m with probability decaying like m^{-cm} , and $\Upsilon^{\mathcal{Z}_r}$ has finite moments of every order. The constants are crude: for $r = 0.8$ the first bound is below 1 only for m of order $3 \cdot 10^4$, while numerically $\mathbb{P}(\mathcal{Z}_{0.8} \geq m)$ falls from 0.32 at $m = 3$ to $5.5 \cdot 10^{-4}$ at $m = 8$.

Theorem 3.6 (A rate of convergence). *Let q be the Gaussian power series and $0 < \varepsilon \leq 1$. For every $\alpha \in (0, \frac{3}{50})$ there is $C_{\varepsilon, \alpha} < \infty$ such that for all $n \geq 2$*

$$\sup_{t \geq 1+\varepsilon} |\mathbb{P}(\Lambda_n^{\text{std}} \leq t) - \mathbb{P}(T_* \leq t)| \leq C_{\varepsilon, \alpha} n^{-\alpha}.$$

For every $\theta \in (0, \frac{1}{10})$ there is $C_{\varepsilon, \theta} < \infty$ such that for every $t \geq 1 + \varepsilon$ and $n \geq 2$ both $|\mathbb{P}(G_{n,t} \text{ has only real zeros}) - \mathbb{P}(T_* \leq t)|$ and the total variation distance between the laws of the number of nonreal zeros of $G_{n,t}$ and of the number of nonreal zeros of q in $\{|w| < t^{-1/2}\}$ are at most $C_{\varepsilon, \theta} t n^{-\theta}$.

Proof. Since Λ_n^{std} has the law of Λ_n and $G_{n,t}$ that of $P_{n,t}$ (proof of [M, Theorem 1.1]), we may work with $P_{n,t}$ and Λ_n . By Lemma 3.1, Theorem 2.6 applies to $a^{(t)}$ with $(C_0, R_0) = (C_q, \lambda_\varepsilon)$ for every $t \geq 1 + \varepsilon$; then $R_1 = R_\varepsilon := (1 + \varepsilon)^{1/8}$, $\Upsilon = \Upsilon_\varepsilon := (R_\varepsilon + 1)/(R_\varepsilon - 1)$ and $\vartheta_0 = \lambda_\varepsilon^{-1/4}$. Fix $\theta \in (0, \frac{1}{10})$ and put $\theta' = (1 - 10\theta)/20 > 0$. Let C and $n_{\varepsilon, \theta}$ denote constants depending only on ε and θ . For $n \geq 3$ put

$$\delta = n^{-\theta}, \quad x_n = \max(1, \sqrt{2\theta \log n}), \quad Z = \frac{x_n}{\delta(1 - \lambda_\varepsilon^{-1/2})},$$

$$W = Zn^{\theta'}, \quad c_1 = \frac{\lambda_\varepsilon^{1/2}(1 - \vartheta_0)^2}{6}, \quad \bar{d}_n = \frac{c_1 W^{-2}}{\sqrt{2}},$$

and $k_n = \lfloor \theta' \log n / \log \Upsilon_\varepsilon \rfloor$. Let $\mathcal{G}_n(t)$ be the event that $C_q \leq x_n$, $|\eta_0| \geq \delta$, the parameters of Definition 2.1 for $a^{(t)}$ satisfy $\mu, \mu_{\bar{d}_n}, \nu_{\bar{d}_n} \geq \delta$, $a^{(t)}$ has no zero in $\{1 - \bar{d}_n \leq |w| \leq 1\}$, and $\mathcal{Z}_{r_1} \leq k_n$.

On $\mathcal{G}_n(t)$ the count is exact. The number k of zeros of $a^{(t)}$ in $\mathbb{D}$ is the number of zeros of q in $\{|u| < t^{-1/2}\} \subset \{|u| \leq r_1\}$, so on $\mathcal{G}_n(t)$, $\Upsilon_\varepsilon^k \leq \Upsilon_\varepsilon^{k_n} \leq n^{\theta'}$. This replaces the bound of Lemma 2.2(b). Moreover $\bar{M} \leq x_n / (1 - \lambda_\varepsilon^{-1/2}) = \delta Z$ and $\mu \geq \delta$, so $\mathfrak{d}_0 = \mu / (\bar{M} \Upsilon_\varepsilon^k) \geq W^{-1}$, $\mu' \geq \delta W^{-1}$ and $C'_0 \leq \delta W$. So the third entry in the definition of κ in Theorem 2.6 is at least $c_1 W^{-2}$, and for $n \geq n_{\varepsilon, \theta}$ so are the other two; hence (i) holds with $d = \bar{d}_n$. In (ii), the constants of Lemma 2.3 for $C = C'_0$, $R = R_\varepsilon$ and $\mu_S = \mu'$ satisfy $\kappa \geq c_1 W^{-2}$, $M_n \leq C \log n$, $\Gamma \leq C \delta W$ and $D_H \leq C \delta W (W^2 + \log n)$; as $\delta W \geq 1$ and $\mu'^{-1} \leq W / \delta$, the four conditions (2.1) follow from $n \geq C W^2 \log^2 n$, $n \geq C W^3 \log n$, $n \geq C W^6 \log n$ and $n \geq C W^{8/3}$. Condition (iii) follows from $n \geq C W^9$, as $C'_0 / (\mathfrak{d}_0 \mu_{\bar{d}_n}) \leq W^2$ and $\bar{d}_n^{7/2} \leq C W^7$; condition (iv) from $n \geq C Z W^9$, as $C_q / \nu_{\bar{d}_n} \leq x_n / \delta \leq Z$ and $\bar{d}_n^{9/2} \leq C W^9$; and $n \geq k$ from $k \leq k_n \leq \log n$. Since $Z \leq C n^\theta (\log n)^{1/2}$, we have $Z W^9 \leq C n^{10\theta + 9\theta'} (\log n)^5 = C n^{1-11\theta'} (\log n)^5$ and $W^6 \leq C n^{6(\theta + \theta')} (\log n)^3$ with $6(\theta + \theta') < 1$; so all these hold for $n \geq n_{\varepsilon, \theta}$, uniformly in $t \geq 1 + \varepsilon$. Then Theorem 2.6 and [M, (7.1)] show that on $\mathcal{G}_n(t)$ the number $N_n(t)$ of nonreal zeros of $P_{n,t}$ equals the number N_t of nonreal zeros of q in $\{|w| < t^{-1/2}\}$. Since $\{\Lambda_n \leq t\} = \{N_n(t) = 0\}$ when $\eta_0 \neq 0$ ([M, Lemma 7.1](c)) and $\{N_t = 0\} = \{T_* \leq t\}$, the quantities $|\mathbb{P}(\Lambda_n \leq t) - \mathbb{P}(T_* \leq t)|$ and $\mathbb{P}(N_n(t) \neq N_t)$ are at most $\mathbb{P}(\mathcal{G}_n(t)^c)$ for $n \geq n_{\varepsilon, \theta}$.

The bad event. Let $1 + \varepsilon \leq t \leq T$ and $n \geq n_{\varepsilon, \theta}$, so that $\delta \leq e^{-1}$ and $\ell_\delta = \theta \log n$. As $e^{-x_n^2/2} \leq \delta$, Lemma 3.1 gives $\mathbb{P}(C_q > x_n) + \mathbb{P}(|\eta_0| < \delta) \leq (7/\varepsilon + 1)\delta$. The parameters μ and $\mu_{\bar{d}_n}$ are minima of $|q|$ on circles with radii in $[T^{-1/2}/2, r_1]$, so Lemma 3.2 bounds their contribution by $21000 \varepsilon^{-2} T^{1/2} \delta \ell_\delta^{3/2}$; Lemma 3.3 bounds that of $\nu_{\bar{d}_n}$ by $(4000 \varepsilon^{-3} \ell_\delta^{1/2} + 4T)\delta$, and Lemma 3.4 that of the annulus by $5(1 + \varepsilon)^3 \varepsilon^{-3} \bar{d}_n \leq C \delta^2$, as $\bar{d}_n \leq C W^{-2} \leq C Z^{-2} \leq C \delta^2$. Finally, Lemma 3.5 with $r = r_1$, $\Upsilon = \Upsilon_\varepsilon$ and $\beta = \theta/\theta'$ gives $\mathbb{P}(\mathcal{Z}_{r_1} \geq m) \leq 2 \Upsilon_\varepsilon^{-\beta m}$ for $m \geq m_0(r_1, \Upsilon_\varepsilon, \beta)$; with $m = k_n + 1 > \theta' \log n / \log \Upsilon_\varepsilon$ this gives $\mathbb{P}(\mathcal{Z}_{r_1} > k_n) \leq 2n^{-\theta}$ as soon as $k_n + 1 \geq m_0$, that is, for $\log n \geq m_0 \log \Upsilon_\varepsilon / \theta'$. Enlarging C to cover smaller n , we get for all $n \geq 2$

$$(3.2) \quad \sup_{1+\varepsilon \leq t \leq T} \left(|\mathbb{P}(\Lambda_n \leq t) - \mathbb{P}(T_* \leq t)| + \mathbb{P}(N_n(t) \neq N_t) \right) \leq C T n^{-\theta} (\log n)^{3/2}.$$

Large times. By the proof of [M, Lemma 7.9], with $r = T^{-1/2}$ and $\arctan \varrho \leq \varrho$, $\mathbb{P}(T_* > T) \leq 2T^{-3/2}$ for $T \geq 2$. For $t \geq T$, by monotonicity, $|\mathbb{P}(\Lambda_n \leq t) - \mathbb{P}(T_* \leq t)| \leq |\mathbb{P}(\Lambda_n \leq T) - \mathbb{P}(T_* \leq T)| + 2\mathbb{P}(T_* > T)$. With $T = \max(2, n^{2\theta/5})$ and (3.2) the supremum over $t \geq 1 + \varepsilon$ is at most $C n^{-3\theta/5} (\log n)^{3/2}$. Given $\alpha < \frac{3}{50}$, choose $\theta \in (\frac{5}{3}\alpha, \frac{1}{10})$; then $n^{-3\theta/5} (\log n)^{3/2} \leq C n^{-\alpha}$. The claims at a fixed t are (3.2) with $T = t$, applied with a slightly larger $\theta < \frac{1}{10}$ to absorb the logarithm, since $P_{n,t}$ has only real zeros exactly when $N_n(t) = 0$, and the total variation distance between the laws of $N_n(t)$ and N_t is at most $\mathbb{P}(N_n(t) \neq N_t)$. $\square$

Remark 3.7 (The exponent). The exponents do not depend on ε : the only place where the argument could lose a factor depending on ε is the factor Υ^k , and its bound $\Upsilon^k \leq (\bar{M}/\alpha)^E$ of Lemma 2.2(b), with $E \asymp \varepsilon^{-1} \log(1/\varepsilon)$, would give an exponent of order $\varepsilon / \log(1/\varepsilon)$; Lemma 3.5 replaces it. The constants are not explicit and not polynomial in $1/\varepsilon$: the proof needs $\log n \geq m_0 \log \Upsilon_\varepsilon / \theta'$, so $\log C_{\varepsilon, \alpha} = \varepsilon^{-O(1/(3/50 - \alpha))}$. The limits $\frac{1}{10}$ and $\frac{3}{50}$ come from the degree, about 9 in $1/\mu$, of the explicit onset (Theorem 2.6 and the paragraph after it). Since $\mathbb{P}(\mu < x)$ is of order x for the random symbol, the sufficient condition of that theorem fails with probability of order at least $n^{-1/9}$, so this argument cannot give a fixed- t exponent above $\frac{1}{9}$; a better exponent needs a sharper deterministic onset, in particular near the edges ± 2 of the bulk. The theorem gives no rate uniform in $t > 1$. In a Monte Carlo simulation (not reported here; $n \leq 1024$,

$t \in \{1.25, 1.5, 2, 3\}$, 24 000 to 48 000 samples for each pair (n, t) , the probability that exactly one of the events $\{P_{n,t} \text{ has only real zeros}\}$ and $\{T_* \leq t\}$ occurs decays with fitted exponents 0.68 to 0.82, and $\mathbb{P}(\Lambda_n \leq t) - \mathbb{P}(T_* \leq t)$ is positive at every computed point and decays with fitted exponents 0.4 to 0.7.

The distortion $\mathfrak{d}(a^{(t)})$ of the outer factor (Lemma 2.2(c)) also has a small-ball bound whose exponent does not depend on ε . We need the circle minimum uniformly in the radius.

Lemma 3.8 (Small values on a circle, uniformly in the radius). *For $t \geq 1 + \varepsilon$ and $0 < x \leq e^{-1}$,*

$$\mathbb{P}\left(\min_{|u|=t^{-1/2}} |q(u)| < x\right) \leq \frac{7000}{\varepsilon^2} \ell_x^{3/2} x^{1/2}.$$

Proof. Let $r = t^{-1/2} \leq r_1$. If $r \geq x^{1/2}$, Lemma 3.2 gives at most $7000\varepsilon^{-2}x\ell_x^{3/2}/r$, which is at most $7000\varepsilon^{-2}\ell_x^{3/2}x^{1/2}$. If $r < x^{1/2}$, then $|q(u) - \eta_0| \leq rY_1 \leq x^{1/2}Y_1$ on $|u| = r$, so the event implies $|\eta_0| < x + x^{1/2}Y_1$. By (3.1), $\mathbb{P}(Y_1 > y_1(x)) \leq x$, and $\mathbb{P}(|\eta_0| < v) \leq v$; so the probability is at most $2x + x^{1/2}(52 + 46\ell_x^{1/2})\varepsilon^{-2}$, which is at most $100\varepsilon^{-2}\ell_x^{1/2}x^{1/2}$. $\square$

Proposition 3.9 (The distortion). *Let $0 < \varepsilon \leq 1$, and let $\mathfrak{d}(a^{(t)})$ be the distortion of the symbol $a^{(t)}$, defined in Lemma 2.2(c) with $(C_0, R_0) = (C_q, \lambda_\varepsilon)$.*

- (a) *For $T \geq 1 + \varepsilon$ and $0 < \beta < 1$ there is $C_{\varepsilon, \beta}$ such that $\mathbb{P}(\mathfrak{d}(a^{(t)}) < \delta) \leq C_{\varepsilon, \beta} T^{1/2} \delta^\beta$ for all $t \in [1 + \varepsilon, T]$ and $\delta \in (0, 1]$.*
- (b) *For $0 < \beta < \frac{1}{2}$ there is $C_{\varepsilon, \beta}$ such that $\mathbb{P}(\mathfrak{d}(a^{(t)}) < \delta) \leq C_{\varepsilon, \beta} \delta^\beta$ for all $t \geq 1 + \varepsilon$ and $\delta \in (0, 1]$.*

Proof. By Lemma 2.2(c), $\mathfrak{d}(a^{(t)}) \geq (\mu/\overline{M})((R_\varepsilon - 1)/2)^k$, and $k \leq \mathcal{Z}_{r_1}$. Let $0 < \eta < 1$ and $\Upsilon' = 2/(R_\varepsilon - 1)$. On the complement of

$$\{\overline{M} > \delta^{-\eta/4}\} \cup \{\mu < \delta^{1-3\eta/4}\} \cup \{(\Upsilon')^{\mathcal{Z}_{r_1}} > \delta^{-\eta/2}\}$$

we have $\mathfrak{d} \geq \delta^{1-3\eta/4} \delta^{\eta/4} \delta^{\eta/2} = \delta$. The first event has probability at most $C_\varepsilon \exp(-c_\varepsilon \delta^{-\eta/2}) \leq C_{\varepsilon, \eta} \delta$ by Lemma 3.1, as $\overline{M} = C_q/(1 - \lambda_\varepsilon^{-1/2})$. The third has probability at most $C_{\varepsilon, \eta} \delta$ by Lemma 3.5 with $\Upsilon = \Upsilon'$ and exponent $2/\eta$. For the second, $\mu = \min_{|u|=t^{-1/2}} |q(u)|$: for (a), Lemma 3.2 with $r = t^{-1/2} \geq T^{-1/2}$ gives at most $C_\varepsilon T^{1/2} \delta^{1-3\eta/4} \log^{3/2}(1/\delta)$; for (b), Lemma 3.8 gives at most $C_\varepsilon \delta^{(1-3\eta/4)/2} \log^{3/2}(1/\delta)$. Take η small. $\square$

The exponent in (b) is limited by small radii, where $q(u) \approx \eta_0 + \eta_1 u$ on $|u| = r$ and the circle minimum is a small-ball problem for $|\eta_0| - r|\eta_1|$; we expect exponents up to 1 uniformly in t , but have not proved it.

Remark 3.10 (Another deformation). Let $a \in \mathcal{S}(C_0, R_0)$ with $\alpha, \mu > 0$, let $p_{\text{in}}, p_{\text{out}}$ be as in Lemma 2.2, let ρ_{out} be the smallest modulus of a zero of a outside $\overline{\mathbb{D}}$ (or R_0 if it is smaller), and let $a_s^h = p_{\text{in}} p_{\text{out}}^s = p_{\text{in}} e^{s \log p_{\text{out}}}$, with the branch of $\log p_{\text{out}}$ that vanishes at 0; it is analytic in $\{|w| < \rho_{\text{out}}\}$ and has real coefficients. On $|w| = 1$, $|a_s^h| = |p_{\text{in}}|^{1-s} |a|^s$, so $\min_{s \in [0, 1], |w|=1} |a_s^h| = \min(\mu, m_{\text{in}})$ with $m_{\text{in}} = \min_{|w|=1} |p_{\text{in}}|$. Any continuous family from p_{in} to a has circle minimum at most $\min(\mu, m_{\text{in}})$ (look at its end points), so this family is optimal in that respect; and $m_{\text{in}} \geq \mathfrak{d}(a)\mu$, since $|p_{\text{out}}(0)| = 1$. But m_{in} is a product over the zeros of a in $\mathbb{D}$, and its lower bounds cost a factor exponential in k ; and if a zero w' of a with $|w'| = \rho_{\text{out}} < R_0$ has a multiplicity j with $js \notin \mathbb{Z}$ (for instance if the zeros on $|w| = \rho_{\text{out}}$ are simple and $0 < s < 1$), then a_s^h is not analytic at w' , so its coefficients $a_l(s)$ satisfy $\limsup_l |a_l(s)|^{1/l} = 1/\rho_{\text{out}}$, and $\rho_{\text{out}} - 1$ has no lower bound depending only on ε . So this deformation does not remove the dependence on k either; in Theorem 3.6 it is removed by Lemma 3.5 instead.

4. EXPLICIT CONSTANTS AND THE TAIL OF D

In this section we bound the constants of [M, Theorem 9.1] explicitly (Section 4.1) and, for the Gaussian power series, bound the tail of the largest scaled displacement D of the collision, the variable of the proof of [M, Corollary 9.16] (Sections 4.2–4.4). The notation is that of [M, Section 9]: $q(w) = \sum_j \eta_j w^j$ is a real power series, the polynomials $P_{n,t}$ and their real-rooting time Λ_n are built from its coefficients, and $u_* = R_* e^{i\theta_*}$, $0 < \theta_* < \pi$, is its nonreal zero of minimal modulus.

4.1. Explicit constants. The proof of [M, Theorem 9.1] obtains δ , K and n_0 by compactness. We now bound them explicitly, by polynomials in condition parameters of q whose degrees are absolute. Throughout this subsection $0 < \varepsilon \leq 1$, and q satisfies the conditions (G1)–(G3) of [M, Section 9], with radius of convergence $\rho_q \geq 1$ and $T_* \geq 1 + \varepsilon$. Put

$$\lambda_{\varepsilon/2} = (1 + \varepsilon/2)^{1/4}, \quad r_c = \lambda_{\varepsilon/2}^{-2}, \quad C_q = \sup_{j \geq 0} |\eta_j| \lambda_{\varepsilon/2}^{-j},$$

as in Lemma 3.1 with $\varepsilon/2$ in place of ε , and assume $C_q < \infty$. Let $\overline{M}_c = C_q/(1 - \lambda_{\varepsilon/2}^{-1/2})$, $\Upsilon_c = (\lambda_{\varepsilon/2}^{1/2} + 1)/(\lambda_{\varepsilon/2}^{1/2} - 1)$ and $E_c = \log \Upsilon_c / \log \lambda_{\varepsilon/2}^{1/2}$; these are $\overline{M}$, Υ and E of Definition 2.1 for $(C_0, R_0) = (C_q, \lambda_{\varepsilon/2})$. Let $s_* = \sin \theta_*$ and $m_* = \min(\theta_*, \pi - \theta_*)$, so that $m_* \geq s_*$, and recall $\gamma_* = u_* q'(u_*)$ from [M, Section 9.3]. The *modulus gap* $g \in (0, 1]$ is the minimum of 1 and the distance from R_* to the moduli of the zeros of q in $\{|u| < \rho_q\}$ other than u_* , $\bar{u}_*$, real zeros included; and

$$\nu^{\text{glob}} = \min\{\max(|q(x)|, |q'(x)|) : x \in \mathbb{R}, |x| \leq r_c\}.$$

We assume also $\nu^{\text{glob}} > 0$, that is, q has no multiple zero in $[-r_c, r_c]$; (G3) gives this only in $(-R_*, R_*)$. Let $\delta_1 = \min\{\varepsilon/4, gT_*^{3/2}/6\}$, let $\mathfrak{A}$ be the closed annulus $\{(T_* + 2\delta_1)^{-1/2} \leq |u| \leq (T_* - 2\delta_1)^{-1/2}\}$, as in [M, Section 9.3] with this δ_1 , and for $\rho > 0$ let $m_{\mathfrak{A}}(\rho)$ be the minimum of $|q|$ over the points of $\mathfrak{A}$ at distance at least ρ from u_* and $\bar{u}_*$. Put

$$c_F = \frac{|\gamma_*|}{8T_*}, \quad \overline{B}_2 = \frac{4000 \max(1, C_q)}{\varepsilon^3}, \quad r_1 = \min\left\{\frac{c_F}{\overline{B}_2}, \delta_1, \frac{m_*}{2}\right\}, \quad \rho_V = \frac{r_1 s_*}{6T_*^{3/2}}.$$

Finally let k_- be the number of zeros of q in $\{|u| < (T_* - \delta_1)^{-1/2}\}$, $\mathcal{U} = \Upsilon_c^{k_-}$, and

$$\begin{aligned} \Pi = & \max(1, |\gamma_*|^{-1}) \cdot s_*^{-1} \cdot g^{-1} \cdot \max(1, m_{\mathfrak{A}}(\rho_V)^{-1}) \\ & \cdot \max(1, (\nu^{\text{glob}})^{-1}) \cdot \max(1, |\eta_0|^{-1}) \cdot \max(1, C_q) \cdot T_* \cdot \mathcal{U}. \end{aligned}$$

Each of the nine factors of Π is at least 1; we call them $P_\gamma, P_s, P_g, P_{\mathfrak{A}}, P_\nu, P_\alpha, P_C, T_*, \mathcal{U}$, in this order. Below, c denotes absolute constants and c_ε numbers with $c_\varepsilon \leq c\varepsilon^{-c}$, not the same at each occurrence.

Theorem 4.1 (Explicit constants). *In this setting the conclusions (a)–(d) of [M, Theorem 9.1] hold with $\delta = \delta_1$ and with K, n_0 such that*

$$\delta^{-1} \leq 6\varepsilon^{-1} \Pi, \quad K \leq c_\varepsilon \Pi^{5/2}, \quad n_0 \leq c_\varepsilon \Pi^{10}.$$

Moreover $\mathcal{U} \leq (\overline{M}_c/|\eta_0|)^{E_c}$, and $\mathcal{U} \leq \Upsilon_c^{\mathcal{Z}}$, where $\mathcal{Z}$ is the number of zeros of q in $\{|u| \leq r_c\}$, counted with multiplicity.

The factor $\mathcal{U} = \Upsilon^k$ of Theorem 2.6 is kept as a parameter, because for random q it can be bounded either by Jensen's formula or through the number of zeros in a disk (Theorem 4.12).

Lemma 4.2. (a) *If $X \geq 1$ and $n \geq 16X(1 + \log X)^2$, then $n \geq X(1 + \log n)^2$. In particular this holds for $n \geq 64X^2$.*

(b) If $P_1, \dots, P_k$ are distinct factors of Π and $a_1, \dots, a_k \geq 0$, then $\prod_i P_i^{a_i} \leq \Pi^{\max_i a_i}$.

Proof. (a) The function $n \mapsto n/(1+\log n)^2$ is increasing for $n \geq e$. At $n_1 = 16X(1+\log X)^2 \geq 16$, using $\log(1+y) \leq y$, we have $1+\log n_1 \leq 1+\log 16+3\log X \leq 4(1+\log X)$, so $n_1/(1+\log n_1)^2 \geq X$. Moreover $(1+\log X)^2 \leq 4X$ for $X \geq 1$. (b) Each $P_i \geq 1$, so $P_i^{a_i} \leq P_i^{\max_i a_i}$, and a product of distinct factors of Π is at most Π . $\square$

By Lemma 4.2(b), a bound by c_ε times a monomial in the factors of Π is a bound by $c_\varepsilon \Pi^e$, where e is the largest exponent of a single factor.

Proof of Theorem 4.1. We go through the proof of [M, Theorem 9.1], which runs from [M, Section 9.2] to [M, Section 9.5], and bound each constant. The elementary numerical inequalities used below were also checked on grids by computer.

Step 1: δ_1 and the annulus. As $g \leq 1 \leq T_*$ and $\varepsilon \leq 1$, $\delta_1 \geq \min(\varepsilon/4, g/6) \geq \varepsilon g/6$, so $\delta_1^{-1} \leq 6\varepsilon^{-1}P_g$. Also $\delta_1 \leq \varepsilon/4 < T_*/4$ and $T_* - 2\delta_1 \geq 1 + \varepsilon/2$. By the mean value theorem,

$$(T_* - 2\delta_1)^{-1/2} - R_* \leq \delta_1 (T_*/2)^{-3/2} = 2^{3/2} \delta_1 T_*^{-3/2} \leq 0.48g, \quad R_* - (T_* + 2\delta_1)^{-1/2} \leq \delta_1 T_*^{-3/2} \leq \frac{g}{6}.$$

So every point of $\mathfrak{A}$ has modulus within $g/2$ of R_* , and by the definition of g the only zeros of q in $\mathfrak{A}$ are u_* and $\bar{u}_*$; this is condition (C2) of [M, Section 9.3]. Condition (C1) there holds because $(T_* - 2\delta_1)^{-1/2} \leq r_c < 1 \leq \rho_q$. For $t \in I_1 = [T_* - \delta_1, T_* + \delta_1] \subset [1 + 3\varepsilon/4, \infty)$ we have $|\eta_j| t^{-j/2} \leq C_q \lambda_{\varepsilon/2}^j \lambda_{\varepsilon/2}^{-2j} = C_q \lambda_{\varepsilon/2}^{-j}$, which is [M, (9.2)] with $(C_0, R_0) = (C_q, \lambda_{\varepsilon/2})$.

Step 2: κ . Take $\kappa = \kappa_c := \delta_1/(5T_*) \leq \varepsilon/20$. Since $\log(1+x) \geq x/(1+x)$, we have $\log \frac{T_*+2\delta_1}{T_*+\delta_1} \geq \frac{\delta_1}{T_*+2\delta_1} \geq 2\kappa_c$ and $\log \frac{T_*-\delta_1}{T_*-2\delta_1} \geq \frac{\delta_1}{T_*-\delta_1} \geq 2\kappa_c$, which is [M, (9.3)]. As $\log \lambda_{\varepsilon/2} \geq \varepsilon/10$, the number $\vartheta = e^{\kappa_c}/\lambda_{\varepsilon/2}$ satisfies $\log(1/\vartheta) \geq \varepsilon/20$, hence $1 - \vartheta \geq \varepsilon/21$; and $\ell = 40/\varepsilon$ satisfies $\ell \log(1/\vartheta) \geq 2$. Moreover $\kappa_c^{-1} \leq 30\varepsilon^{-1}T_*P_g$.

Step 3: [M, Proposition 9.7]. Let $X_3 = (\ell + 3)^2/\kappa_c$ and $n_3 = \lceil 64X_3^2 \rceil$. For $n \geq n_3$, $M_n + 2 \leq \ell \log n + 3 \leq (\ell + 3)(1 + \log n)$, and Lemma 4.2(a) gives $(M_n + 2)^2 \leq \kappa_c n$; so n_3 may serve as the n_3 of [M, Proposition 9.7], and $n_3 \leq c_\varepsilon T_*^2 P_g^2 \leq c_\varepsilon \Pi^2$. For $n \geq n_3$ we have $(M_n + 2)/n \leq \kappa_c/(M_n + 2) \leq 1$, so the paragraph after the proof of [M, Proposition 9.7] gives $C_H \leq C_q(1 - \vartheta)^{-4}(198/\kappa_c + 4) \leq 202(21/\varepsilon)^4 C_q \kappa_c^{-1}$. In the proof of [M, Proposition 9.7](b), $R_0^{-M_n} \leq n^{-2}$, so $\varepsilon_n \leq C'/n$ with $C' = \sqrt{2}(1 + \cosh \kappa_c)^{1/2} C_q(1 - \lambda_{\varepsilon/2}^{-2})^{-1/2} \leq 5.7\varepsilon^{-1/2} C_q$, using $1 - \lambda_{\varepsilon/2}^{-2} \geq \varepsilon/6$. Hence $C_W = C_H + C' \leq c_\varepsilon P_C T_* P_g \leq c_\varepsilon \Pi$.

Step 4: [M, Lemma 9.8](b). *The local part.* The c_F of the proof of [M, Lemma 9.8] satisfies $2c_F = \frac{1}{2}|\gamma_*| \min(1, \frac{1}{2T_*}) = |\gamma_*|/(4T_*)$, which is the c_F defined above. With $u = t^{-1/2}e^{i\theta}$, the function $F(t, \theta) = q(u)$ has

$$F_{\theta\theta} = -(uq' + u^2q''), \quad F_{t\theta} = -\frac{i(uq' + u^2q'')}{2t}, \quad F_{tt} = \frac{uq' + u^2q''}{4t^2} + \frac{uq'}{2t^2},$$

with q', q'' evaluated at u . For $t \in I_1$ we have $t \geq 1$ and $|u| < r_c$, so each second derivative is at most

$$\sup_{|u| \leq r_c} (|q'(u)| + |q''(u)|) \leq C_q r_c^{-2} \sum_{j \geq 1} j^2 (\lambda_{\varepsilon/2} r_c)^j \leq \frac{3C_q}{(1 - \lambda_{\varepsilon/2} r_c)^3} \leq \bar{B}_2,$$

by $\sum_j j^2 x^j \leq 2(1-x)^{-3}$, $r_c^{-2} \leq \frac{3}{2}$ and $1 - \lambda_{\varepsilon/2} r_c \geq \varepsilon/11$. Taylor's formula with integral remainder gives $|F(T_* + \alpha, \theta_* + \beta) - F_t \alpha - F_\theta \beta| \leq \frac{1}{2} \bar{B}_2 (|\alpha| + |\beta|)^2$, the first derivatives being taken at (T_*, θ_*) . Hence $|F| \geq c_F (|\alpha| + |\beta|)$ for $|\alpha|, |\beta| \leq r_1$, as $2r_1 \leq 2c_F/\bar{B}_2$. So r_1 may serve as the r_1 of the proof of [M, Lemma 9.8]: it is at most δ_1 and at most $\frac{1}{2} \min(\theta_*, \pi - \theta_*)$, and $s_0 = \sin(m_*/2) \geq s_*/\pi$. The computation of that proof gives

$$s_{\min}(H_{a(t)}(2 \cos \theta)) \geq \frac{1}{16} s_0 c_F (|t - T_*| + |y - y_*|) \quad (|t - T_*| \leq r_1, |\theta - \theta_*| \leq r_1).$$

The neighbourhood V . Let $r_2 = 0.6 s_* r_1$ and $V = \{(t, y) : |t - T_*| < r_1, |y - y_*| < r_2\}$. If $\theta \in [0, \pi]$ and $|\theta - \theta_*| \geq r_1$, then, since $\cos$ is decreasing on $[0, \pi]$ and $\theta_* \pm r_1 \in [0, \pi]$, and $\sin x \geq \frac{2}{\pi} \min(x, \pi - x)$ on $[0, \pi]$,

$$|\cos \theta - \cos \theta_*| \geq 2 \sin \frac{r_1}{2} \min_{\pm} \sin \left(\theta_* \pm \frac{r_1}{2} \right) \geq 2 \cdot \frac{r_1}{\pi} \cdot \frac{2}{\pi} \cdot \frac{3m_*}{4} \geq 0.3 s_* r_1.$$

So $y = 2 \cos \theta \in [-2, 2]$ and $|y - y_*| < r_2$ force $|\theta - \theta_*| < r_1$. Moreover $r_2 \leq 0.6 m_* \cdot m_*/2 < 4m_*^2/\pi^2 \leq 2 - |y_*|$, so $V \subset I_1 \times (-2, 2)$, and the local bound holds on V .

The global part. By [M, Lemma 3.3], $\|G_j(y)\| \leq 4(j+2)e^{(j+1)\kappa_c}$ for $|y| \leq L = 2 \cosh \kappa_c$, so for $t \in I_1$

$$\|H_{a(t)}(y)\| \leq 4C_q e^{\kappa_c} \sum_{j \geq 0} (j+2) \vartheta^j \leq \Gamma_c := \frac{8C_q e^{\kappa_c}}{(1-\vartheta)^2} \leq \frac{3720 C_q}{\varepsilon^2}.$$

Let $(t, y) \in (I_1 \times [-L, L]) \setminus V$ and $m = m_{\mathfrak{A}}(\rho_V)$. If $|y| \leq 2$, write $y = 2 \cos \theta$ with $\theta \in [0, \pi]$ and $u = t^{-1/2} e^{i\theta} \in \mathfrak{A}$. If $|t - T_*| \geq r_1$, then $||u| - R_*| \geq \frac{1}{2} r_1 (T_* + \delta_1)^{-3/2} \geq 0.35 r_1 T_*^{-3/2}$. Otherwise $|y - y_*| \geq r_2$, so $|\theta - \theta_*| \geq r_2/2$, and as $|u| \geq (1.25 T_*)^{-1/2}$,

$$|u - u_*| \geq 2(|u|R_*)^{1/2} \sin \frac{|\theta - \theta_*|}{2} \geq \frac{2}{\pi} \cdot 0.94 R_* \cdot 0.3 s_* r_1 \geq 0.17 R_* s_* r_1.$$

Since also $|u - \bar{u}_*| \geq \operatorname{Im} u + \operatorname{Im} u_* \geq R_* s_*$ and $R_* \geq T_*^{-3/2}$, in both cases u has distance at least ρ_V from u_* and $\bar{u}_*$, and $\det H_{a(t)}(y) = |q(u)|^2 \geq m^2$. If $2 < |y| \leq L$, then $\det H_{a(t)}(y) = q(w/\sqrt{t}) q(1/(w\sqrt{t}))$ with $w = r(y)$ real; both points are real, lie in $\mathfrak{A}$ by [M, (9.3)], and have distance at least $\operatorname{Im} u_* = R_* s_* \geq \rho_V$ from $u_*, \bar{u}_*$, and $\det H_{a(t)}(y) \geq 0$ by [M, Lemma 9.8](a); so again $\det H_{a(t)}(y) \geq m^2$. In both cases $s_{\min}(H_{a(t)}(y)) = \det H / \|H\| \geq c_3 := m^2 / \Gamma_c$. As $\delta_1 + 2L \leq 4.3$, the constant $c_* = \min\{s_0 c_F / 16, c_3 / 4.3\}$ satisfies [M, Lemma 9.8](b), and $c_*^{-1} \leq c_\varepsilon \max(P_s T_* P_\gamma, P_C P_{\mathfrak{A}}^2) \leq c_\varepsilon \Pi^2$.

Step 5: [M, Proposition 9.9]. Part (a) holds for $n \geq n_3$ with $K_1 = C_W / c_* \leq c_\varepsilon \Pi^2$, the monomials being $P_C T_*^2 P_g P_s P_\gamma$ and $P_C^2 T_* P_g P_{\mathfrak{A}}^2$. In part (b), $\varrho_0 = 2 + |\eta_0|^{-1} C_q (1 - \lambda_{\varepsilon/2}^{-2})^{-1/2} \leq c_\varepsilon P_\alpha P_C$. We replace the compactness argument of (b) by an explicit one. Let $t \in I_1$, $y \in J = [-\varrho_0, -L] \cup [L, \varrho_0]$ and $w = r(y)$. Then w is real, $\varrho_0^{-1} < |w| \leq e^{-\kappa_c} \leq 1 - \kappa_c/2$, and $|r'(y)| = w^2 / (1 - w^2) \leq (1 - e^{-2\kappa_c})^{-1} \leq \kappa_c^{-1}$. In the notation of that proof, $\Phi_n(t, y) - \Phi(t, y) = X_n(w)$ and $\partial_y(\Phi_n - \Phi)(t, y) = X'_n(w) r'(y)$, where X_n is the function of Lemma 2.4 for $a^{(t)} \in \mathcal{S}(C_q, \lambda_{\varepsilon/2})$. That lemma with $d = \kappa_c/2$ gives

$$|\Phi_n - \Phi| + |\partial_y \Phi_n - \partial_y \Phi| \leq \frac{E'}{n}, \quad E' = \frac{1481 C_q}{(1 - \lambda_{\varepsilon/2}^{-2})^{1/2} \kappa_c^{11/2}},$$

as $2\sqrt{2} \cdot 2^{7/2} + 64 \cdot 2^{9/2} \leq 1481$. On the other hand, with $u = w/\sqrt{t}$, which is real with $|u| < r_c$, and $|r'(y)| \geq w^2$,

$$\begin{aligned} |\Phi(t, y)| + |\partial_y \Phi(t, y)| &= |q(u)| + t^{-1/2} |q'(u)| |r'(y)| \\ &\geq w^2 t^{-1/2} (|q(u)| + |q'(u)|) \geq m_1 := \frac{\nu^{\text{glob}}}{\varrho_0^2 (T_* + \delta_1)^{1/2}}. \end{aligned}$$

So for $n \geq n_b := \lfloor E' / m_1 \rfloor + 1$ the functions $\Phi_n(t, \cdot)$ and $\partial_y \Phi_n(t, \cdot)$ have no common zero on J , and the rest of the proof of (b) applies. Here $n_b \leq c_\varepsilon P_C (T_* P_g)^{11/2} (P_\alpha P_C)^2 T_*^{1/2} P_\nu \leq c_\varepsilon \Pi^6$. Part (c) holds for $n \geq n_5 := \max(n_3, n_b)$.

Step 6: the anchor times $t_\pm = T_* \pm \delta_1$. We apply Theorem 2.6 to $a^{(t_\pm)} \in \mathcal{S}(C_q, \lambda_{\varepsilon/2})$, whose zeros in $\mathbb{D}$ are $\sqrt{t_\pm}$ times the zeros of q in $\{|u| < t_\pm^{-1/2}\}$; their number $k_\pm$ is at most k_- , so $\Upsilon_c^{k_\pm} \leq \mathcal{U}$. Let $\kappa_\pm$ be the κ of that theorem for $a^{(t_\pm)}$ and $d_\pm = \min\{\kappa_\pm / \sqrt{2}, \delta_1 / (8T_*)\}$. For $\varrho \in [1 - d_\pm, 1]$

the circle $\{|u| = \varrho t_{\pm}^{-1/2}\}$ lies in $\mathfrak{A}$, and its radius differs from R_* by at least $0.35 \delta_1 T_*^{-3/2} \geq \rho_V$. Indeed, for t_+ , $T_*^{-1/2} - (T_* + \delta_1)^{-1/2} \geq \frac{1}{2} \delta_1 (1.25 T_*)^{-3/2}$ and $(1 - d_+)(T_* + \delta_1)^{-1/2} \geq (T_* + 2\delta_1)^{-1/2}$; for t_- , $(T_* - \delta_1)^{-1/2} - T_*^{-1/2} \geq \frac{1}{2} \delta_1 T_*^{-3/2}$, while $d_-(T_* - \delta_1)^{-1/2} \leq 1.16 d_- T_*^{-1/2} \leq 0.15 \delta_1 T_*^{-3/2}$. Hence $a^{(t_{\pm})}$ has no zero in $\{1 - d_{\pm} \leq |w| \leq 1\}$, by (C2), and its parameters of Definition 2.1 satisfy $\mu, \mu_{d_{\pm}} \geq m$ and $\nu_{d_{\pm}} \geq (T_* + \delta_1)^{-1/2} \nu^{\text{glob}}$; also $\alpha = |\eta_0|$. Since $m \leq \mu \leq \overline{M}_c$, the quantities of Lemma 2.2 and Theorem 2.6 satisfy

$$\mathfrak{d}_0 \geq \frac{m}{\overline{M}_c \mathcal{U}}, \quad C'_0 \leq \overline{M}_c \mathcal{U}, \quad \mu' \geq \frac{m^2}{\overline{M}_c \mathcal{U}}, \quad \kappa_{\pm} \geq c'_1 \frac{m^2}{\overline{M}_c \mathcal{U}^2},$$

with $c'_1 = \min\{\frac{1}{4} \log \lambda_{\varepsilon/2}, \lambda_{\varepsilon/2}^{1/2} (1 - \lambda_{\varepsilon/2}^{-1/4})^2 / 6\} \geq \varepsilon^2 / 9600$. As $\overline{M}_c \leq 21\varepsilon^{-1} C_q$, we get $\kappa_{\pm}^{-1} \leq c_{\varepsilon} P_C^2 \mathcal{U}^2 P_{\mathfrak{A}}^2$ and $d_{\pm}^{-1} \leq c_{\varepsilon} \max(P_C^2 \mathcal{U}^2 P_{\mathfrak{A}}^2, T_* P_g) \leq c_{\varepsilon} \Pi^2$. So (i) of Theorem 2.6 holds. In (ii), the constants of Lemma 2.3 for $C = C'_0$, $R = \lambda_{\varepsilon/2}^{1/2}$, $\mu_S = \mu'$ satisfy $M_n + 2 \leq (40/\varepsilon + 3)(1 + \log n)$, $D_H \leq c_{\varepsilon} C'_0 (\kappa_{\pm}^{-1} + 1 + \log n)$ and $\Gamma \leq c_{\varepsilon} C'_0$. With Lemma 4.2, the conditions of (ii)–(iv) hold as soon as n exceeds each of the following bounds, at both anchors:

- $n \geq k_{\pm}$: for $n \geq \mathcal{U}$, as $k_{\pm} \leq \log \mathcal{U} / \log \Upsilon_c$ and $\log \Upsilon_c \geq 1$;
- $n \geq (M_n + 2)^2 / \kappa_{\pm}$: for $n \geq c_{\varepsilon} \kappa_{\pm}^{-2}$, at most $c_{\varepsilon} \Pi^4$;
- $n \geq D_H$: for $n \geq c_{\varepsilon} C'_0 \kappa_{\pm}^{-1}$ and $n \geq c_{\varepsilon} \max(1, C'_0)^2$, at most $c_{\varepsilon} \Pi^3$;
- $n \geq 8(2\Gamma + 1)D_H / \mu'^2$: the right side is at most $c_{\varepsilon} (P_C^6 \mathcal{U}^6 P_{\mathfrak{A}}^6 + P_C^4 \mathcal{U}^4 P_{\mathfrak{A}}^4 (1 + \log n))$, so $n \geq c_{\varepsilon} \Pi^8$ suffices;
- $n^{3/2} \geq 37(\Gamma + 1)C'_0 (1 - \lambda_{\varepsilon/2}^{-1})^{-1/2} \mu'^{-2}$: the right side is at most $c_{\varepsilon} P_C^4 \mathcal{U}^4 P_{\mathfrak{A}}^4$, so $n \geq c_{\varepsilon} \Pi^{8/3}$ suffices;
- (iii): we have $C'_0 / (\mathfrak{d}_0 \mu_{d_{\pm}}) \leq c_{\varepsilon} P_C^2 \mathcal{U}^2 P_{\mathfrak{A}}^2$, so with the factor $d_{\pm}^{-7/2}$ the bound is at most $c_{\varepsilon} \max(P_C^9 \mathcal{U}^9 P_{\mathfrak{A}}^9, P_C^2 \mathcal{U}^2 P_{\mathfrak{A}}^2 T_*^{7/2} P_g^{7/2}) \leq c_{\varepsilon} \Pi^9$;
- (iv): $64C_q (1 - \lambda_{\varepsilon/2}^{-2})^{-1/2} d_{\pm}^{-9/2} / \nu_{d_{\pm}} \leq c_{\varepsilon} \max(P_C^{10} \mathcal{U}^9 P_{\mathfrak{A}}^9 T_*^{1/2} P_{\nu}, P_C T_*^5 P_g^{9/2} P_{\nu}) \leq c_{\varepsilon} \Pi^{10}$.

Let n_7 be the least integer such that every $n \geq n_7$ satisfies all of them; then $n_7 \leq c_{\varepsilon} \Pi^{10}$. For $n \geq n_7$, Theorem 2.6 and [M, (9.1)] give [M, (9.4)]: the nonreal zeros of $a^{(t_{-})}$ in $\mathbb{D}$ are $\sqrt{t_{-}} u_*$ and $\sqrt{t_{-}} \bar{u}_*$, as in [M, Section 9.5], and $a^{(t_{+})}$ has none; so $N_n(t_{-}) = 2$ and $N_n(t_{+}) = 0$, the real zeros of $P_{n,t_{-}}$ are simple, and its two nonreal zeros are simple, being a conjugate pair.

Step 7: conclusion. With $n_0 = \max(n_5, n_7) \leq c_{\varepsilon} \Pi^{10}$, the proof in [M, Section 9.5] applies without change and gives (a)–(d) with $K = K_1 (1 + \sqrt{T_* + \delta_1} + (T_* - \delta_1)^{-1/2}) \leq 3.2 K_1 T_*^{1/2} \leq c_{\varepsilon} \Pi^{5/2}$.

The bounds for $\mathcal{U}$. Lemma 2.2(b) for $a^{(t_{-})}$, which has $\alpha = |\eta_0| > 0$ and no zero on the unit circle, gives $\mathcal{U} = \Upsilon_c^{k_{-}} \leq (\overline{M}_c / |\eta_0|)^{E_c}$. And $(T_* - \delta_1)^{-1/2} < r_c$, so $k_{-} \leq \mathcal{Z}$. $\square$

Remark 4.3. (1) The exponent 10 comes from condition (iv) at the anchor times, through $d_{\pm} \asymp \kappa_{\pm} \asymp (\overline{M}_c \mathcal{U} / m)^{-2}$, as in the paragraph on the size of the onset after Theorem 2.6. The exponents are crude: Lemma 4.2(b) does not distinguish the factors of Π . (2) Theorem 4.1 needs no lower bound for $|q|$ near clustered zeros other than $m_{\mathfrak{A}}$, no separation of the zeros inside the disk of radius R_* , and no lower bound for $|q'|$ at the real zeros: real zeros enter only through ν^{glob} . (3) Without the factor $\mathcal{U}$, that is, with $\mathcal{U} \leq (\overline{M}_c / |\eta_0|)^{E_c}$, the bounds are polynomial with degrees of order $E_c \asymp \varepsilon^{-1} \log(1/\varepsilon)$.

4.2. Pairs of zeros of the Gaussian power series. In this subsection and the next two, q is the Gaussian power series [M, (1.3)]. We write $\mathbb{D}_r = \{|w| < r\}$, $\mathbb{D}^+ = \mathbb{D} \cap \{\text{Im } w > 0\}$ and

$\mathbb{D}_r^+ = \mathbb{D}_r \cap \mathbb{D}^+$. The zeros of q have the one-point intensities

$$\rho_r(x) = \frac{1}{\pi(1-x^2)} \quad (x \in (-1, 1)), \quad \rho_c(z) = \frac{2 \operatorname{Im} z}{\pi |1-z^2| (1-|z|^2)^2} = \frac{b_z}{\pi(1-|z|^2)} \quad (z \in \mathbb{D}^+),$$

for the real zeros and for the zeros in $\mathbb{D}^+$, with respect to Lebesgue measure [7, Theorems 2.1 and 2.7], [M, Lemma 7.9], where $b_z = 2 \operatorname{Im} z / ((1-|z|^2)|1-z^2|)$. Let $\psi(a, b) = |a-b|/|1-a\bar{b}|$ be the pseudo-hyperbolic distance on $\mathbb{D}$. For $a \in \mathbb{D}$ let $B_a(w) = (w-a)/(1-\bar{a}w)$ if a is real, and $B_a(w) = (w-a)(w-\bar{a})/((1-\bar{a}w)(1-aw))$ if a is not real. These are Blaschke products with real Taylor coefficients, and $|B_x(y)| = \psi(x, y)$, $|B_z(x)| = \psi(x, z)^2$, $|B_z(w)| = \psi(z, w)\psi(z, \bar{w})$ and $|B'_z(z)| = b_z$. Rice formulas are used in the form of [2, Theorem 6.2], for Gaussian fields $Z : U \rightarrow \mathbb{R}^d$ on open sets $U \subset \mathbb{R}^d$ with analytic paths, a nondegenerate law of $Z(t)$ for each $t \in U$, and almost surely no zero t with $\det Z'(t) = 0$; the last condition holds below because the zeros of q in $\mathbb{D} \setminus \{0\}$ are simple [M, Lemma 7.4].

Lemma 4.4 (Regression). *Let $a_1, \dots, a_m \in \mathbb{D}$ be distinct points whose set is closed under complex conjugation, and let $\mathfrak{B}$ be the product of the factors B_x over the real points x and B_z over the pairs $\{z, \bar{z}\}$. Then the conditional law of q given $q(a_1) = \dots = q(a_m) = 0$ (the Gaussian regression) is the law of $\mathfrak{B}\tilde{q}$, where $\tilde{q}$ has the law of q . In particular, conditionally on these values, $(q'(a_i))_{i \leq m}$ has the law of $(\mathfrak{B}'(a_i)\tilde{q}'(a_i))_{i \leq m}$. The density at 0 of the real vector formed by the values $q(a_i)$ (real and imaginary parts at the nonreal points, one point of each pair) is the product, over the points taken in any order, of $(2\pi)^{-1/2}(1-x^2)^{1/2}/|\mathfrak{B}_<(x)|$ for a real point x and $1/(\pi b_z |\mathfrak{B}_<(z)|^2)$ for a pair z , where $\mathfrak{B}_<$ is the product of the factors of the points taken before.*

Proof. For one real point x this is [7, Lemma 4.2]: the covariance $1/(1-uv)$ of q minus its regression on $q(x)$ is $B_x(u)B_x(v)/(1-uv)$. For one conjugate pair it is [7, Lemma 7.2], with the same identity for the covariance and the pseudo-covariance. For several points we condition on one point or pair at a time: conditioning a Gaussian vector on a block of coordinates and then on the next block is conditioning on both. Once q has been replaced by $\mathfrak{B}_A\hat{q}$, the remaining conditions $q(a) = 0$ are equivalent to $\hat{q}(a) = 0$, because $\mathfrak{B}_A(a) \neq 0$; this proves the first claim by induction, also when real points and pairs are mixed. The derivative identity follows by differentiating $\mathfrak{B}\tilde{q}$ at a_i , where $\mathfrak{B}$ vanishes. For the densities, the density at 0 of a Gaussian vector is the product of the conditional densities at 0 of its blocks, and by the first claim the conditional law of $q(a)$ given the earlier values is that of $\mathfrak{B}_<(a)\tilde{q}(a)$. Finally $\operatorname{Var} q(x) = (1-x^2)^{-1}$, and $(\det \operatorname{Cov}(\operatorname{Re} q(z), \operatorname{Im} q(z)))^{1/2} = b_z/2$ (proof of Lemma 3.2). $\square$

Proposition 4.5 (Two-point functions). *Let ρ_{rr} , ρ_{rc} and ρ_{cc} be the two-point correlation functions, with respect to Lebesgue measure, of pairs of real zeros, of a real zero and a zero in $\mathbb{D}^+$, and of pairs of zeros in $\mathbb{D}^+$. For distinct $x, y \in (-1, 1)$ and $z, w \in \mathbb{D}^+$,*

$$\begin{aligned} \rho_{rr}(x, y) &= \psi(\psi + \sqrt{1-\psi^2} \arcsin \sqrt{1-\psi^2}) \rho_r(x) \rho_r(y), & \psi &= \psi(x, y), \\ (4.1) \quad \rho_{rc}(x, z) &= \psi^2(2-\psi^2) \rho_r(x) \rho_c(z), & \psi &= \psi(x, z), \\ \rho_{cc}(z, w) &= ab(3-a-b) \rho_c(z) \rho_c(w), & a &= \psi(z, w)^2, \quad b = \psi(z, \bar{w})^2. \end{aligned}$$

In particular $\rho_{rr} \leq \rho_r \rho_r$, $\rho_{rc} \leq \rho_r \rho_c$ and $\rho_{cc} \leq \rho_c \rho_c$.

Proof. The mixed pair. The field $Z(x, z) = (q(x), \operatorname{Re} q(z), \operatorname{Im} q(z))$ on $U = (-1, 1) \times \mathbb{D}^+$ has nondegenerate values, since by Lemma 4.4 the density of $Z(x, z)$ at 0 is finite, and its zeros in $I \times B$ are the pairs formed by a real zero in I and a zero in B . Its Jacobian determinant is $q'(x)|q'(z)|^2$, by the Cauchy–Riemann equations. So [2, Theorem 6.2] gives $\mathbb{E}[N_{\mathbb{R}}(I)N_+(B)] =$

$\int_I \int_B \rho_{rc}$ with

$$\rho_{rc}(x, z) = p_{Z(x, z)}(0) \mathbb{E}[|q'(x)| |q'(z)|^2 \mid q(x) = 0, q(z) = 0].$$

By Lemma 4.4 with $\mathfrak{B} = B_x B_z$: $p_{Z(x, z)}(0) = (2\pi)^{-1/2} (1 - x^2)^{1/2} (\pi b_z \psi^2)^{-1}$, since $|B_x(z)|^2 = \psi^2$; $q'(x) = B'_x(x) B_z(x) \check{q}(x)$ with $|B'_x(x)| = (1 - x^2)^{-1}$ and $|B_z(x)| = \psi^2$; and $|q'(z)| = \psi b_z |\check{q}(z)|$. Hence $\rho_{rc} = \psi^2 (1 - x^2)^{-1/2} b_z (2\pi)^{-1/2} \pi^{-1} M$ with $M = \mathbb{E}[|X| |Y|^2]$, $X = \check{q}(x)$ and $Y = \check{q}(z)$, viewed in $\mathbb{R}^2$. Write $Y = \mathbf{a}X + W$ with W independent of X ; the cross term vanishes because $\mathbb{E}[|X|X] = 0$, and $\mathbb{E}|X|^3 = 2\sqrt{2/\pi} \sigma_X^3$. So

$$M = \mathbb{E}|X| \left(\mathbb{E}|Y|^2 + \frac{|\mathbb{E}XY|^2}{\mathbb{E}X^2} \right) = \sqrt{\frac{2}{\pi}} \frac{1 + (1 - \psi^2)}{(1 - x^2)^{1/2} (1 - |z|^2)},$$

since $\mathbb{E}X^2 = (1 - x^2)^{-1}$, $\mathbb{E}|Y|^2 = (1 - |z|^2)^{-1}$, $\mathbb{E}XY = (1 - xz)^{-1}$, $|1 - xz| = |1 - x\bar{z}|$ and $(1 - x^2)(1 - |z|^2)/|1 - x\bar{z}|^2 = 1 - \psi^2$. With $\rho_r(x)\rho_c(z) = b_z/(\pi^2(1 - x^2)(1 - |z|^2))$ this is the second line of (4.1).

Two nonreal zeros. Use the field $(\operatorname{Re} q(z), \operatorname{Im} q(z), \operatorname{Re} q(w), \operatorname{Im} q(w))$ on $\{(z, w) \in (\mathbb{D}^+)^2 : z \neq w\}$ and $\mathfrak{B} = B_z B_w$, with $|B_z(w)| = \psi(z, w)\psi(z, \bar{w}) = \sqrt{ab}$. The density at 0 is $(\pi b_z)^{-1}(\pi b_w ab)^{-1}$, and $|q'(z)|^2 |q'(w)|^2 = b_z^2 b_w^2 a^2 b^2 |\check{q}(z)|^2 |\check{q}(w)|^2$. By Isserlis' formula $\mathbb{E}|X|^2 |Y|^2 = \mathbb{E}|X|^2 \mathbb{E}|Y|^2 + |\mathbb{E}XY|^2 + |\mathbb{E}X\bar{Y}|^2$ for complex Gaussian X, Y with real covariance structure, and $(1 - |z|^2)(1 - |w|^2)/|1 - z\bar{w}|^2 = 1 - a$, $(1 - |z|^2)(1 - |w|^2)/|1 - zw|^2 = 1 - b$; so the ratio of the moment to $\mathbb{E}|X|^2 \mathbb{E}|Y|^2$ is $3 - a - b$, which gives the third line.

Two real zeros. Use the field $(q(x), q(y))$ on $\{x \neq y\}$, or [7, Lemma 6.1]. The density at 0 is $(2\pi)^{-1} (1 - x^2)^{1/2} (1 - y^2)^{1/2} / \psi$, the Jacobian factor is $\psi^2 (1 - x^2)^{-1} (1 - y^2)^{-1} |\check{q}(x)\check{q}(y)|$, and $\mathbb{E}|XY| = \frac{2}{\pi} \sigma_X \sigma_Y (\sqrt{1 - c^2} + c \arcsin c)$ for the correlation $c = \sqrt{1 - \psi^2}$ of $X = \check{q}(x)$, $Y = \check{q}(y)$.

The bounds. $t(2 - t) \leq 1$ for all real t . On $[0, 1]^2$, $\partial_a[ab(3 - a - b)] = b(3 - 2a - b) \geq 0$, and symmetrically in b , so $ab(3 - a - b) \leq 1$. With $\psi = \cos \varphi$, $\varphi \in [0, \frac{\pi}{2}]$, the first factor is $\cos^2 \varphi + \varphi \sin \varphi \cos \varphi \leq 1$, because $\varphi \leq \tan \varphi$. $\square$

The bound $\rho_{rr} \leq \rho_r \rho_r$ is [7, Corollary 2.2], and the closed form for ρ_{rr} is equivalent to [7, (2.6)]; the closed form for ρ_{cc} is the case $n = 2$ of [7, Theorem 2.7], and the bound $\rho_{cc} \leq \rho_c \rho_c$ is stated after it. Mixed correlations are not treated in [7]; Forrester [3, (30)] gives all mixed correlations as a Pfaffian, and ρ_{rc} can also be read off its entries [3, (49)–(52)]. All three closed forms agree with a direct numerical evaluation of the Kac–Rice formula. Only the three bounds are used below.

Theorem 4.6 (The modulus gap). *For $0 < \varepsilon \leq 1$ and $s > 0$,*

$$\mathbb{P}(T_* \geq 1 + \varepsilon, g < s) \leq 107 \varepsilon^{-4} s.$$

Proof. Let $r_\varepsilon = (1 + \varepsilon)^{-1/2}$ and $r'_\varepsilon = \frac{1}{2}(1 + r_\varepsilon)$. By concavity $1 - r_\varepsilon \geq \varepsilon/4$, so $1 - r'_\varepsilon \geq \varepsilon/8$, and as $r_\varepsilon \geq 2^{-1/2}$, $1 - r'_\varepsilon \geq 1.853 \cdot \varepsilon/8 \geq 0.2316 \varepsilon$. If $s > (1 - r_\varepsilon)/2$, the right side exceeds $107 \varepsilon^{-3}/8 > 1$; so let $s \leq (1 - r_\varepsilon)/2 \leq 0.15$.

On the event, $u_* \in \mathbb{D}^+$ is a zero with $|u_*| = R_* \leq r_\varepsilon$, and some zero $u' \notin \{u_*, \bar{u}_*\}$ of q in $\mathbb{D}$ has $||u'| - R_*| < s$, hence $|u'| < r'_\varepsilon$. If u' is not real, then u' or $\bar{u}'$ is a zero in $\mathbb{D}^+$ other than u_* , of the same modulus. So the event implies the existence of an ordered pair (u, v) of distinct zeros, with $u \in \mathbb{D}_{r_\varepsilon}^+$, $v \in \mathbb{D}^+$ or v real, and $||v| - |u|| < s$. By Markov's inequality, the Rice formula for second factorial moments and Proposition 4.5, its probability is at most

$$\int_{\mathbb{D}_{r_\varepsilon}^+} \rho_c(u) \left[\int_{A(|u|) \cap \mathbb{D}^+} \rho_c(v) dA(v) + \int_{\{x \in \mathbb{R} : ||x| - |u|| < s\}} \rho_r(x) dx \right] dA(u),$$

where $A(\varrho) = \{\varrho - s < |v| < \varrho + s\}$. On $\{|v| \leq r'_\varepsilon\}$ we have $\rho_c(v) \leq 2 \operatorname{Im} v / (\pi(1 - r_\varepsilon'^2)^3)$, as $|1 - v^2| \geq 1 - |v|^2$, and in polar coordinates $\int_{A(\varrho) \cap \mathbb{D}^+} \operatorname{Im} v dA(v) \leq \frac{2}{3}((\varrho + s)^3 - (\varrho - s)^3) = 4\varrho^2 s + \frac{4}{3}s^3 \leq 4.03s$, also when $\varrho < s$. With $(1 - r_\varepsilon'^2)^{-3} \leq 80.5\varepsilon^{-3}$ the first inner integral is at most $2 \cdot 80.5 \cdot 4.03s / (\pi\varepsilon^3) \leq 207s\varepsilon^{-3}$. The second is at most $4s / (\pi(1 - r_\varepsilon'^2)) \leq 5.5s\varepsilon^{-1}$. The outer integral is $\varkappa(r_\varepsilon)$ of [M, Lemma 7.9], and since $\arctan \varrho \leq \pi/4$, $\varkappa(r) \leq \int_0^r \varrho(1 - \varrho^2)^{-2} d\varrho = r^2 / (2(1 - r^2))$, so $\varkappa(r_\varepsilon) \leq 1/(2\varepsilon)$. Altogether the probability is at most $\frac{1}{2\varepsilon}(207\varepsilon^{-3} + 5.5\varepsilon^{-1})s \leq 107\varepsilon^{-4}s$. $\square$

The same integral bounds $\varkappa$ for small r . By the proof of [M, Lemma 7.9], and since $\arctan \varrho \leq \varrho$,

$$\varkappa(r) = \frac{4}{\pi} \int_0^r \frac{\varrho \arctan \varrho}{(1 - \varrho^2)^2} d\varrho \leq \frac{4}{\pi} \int_0^r \frac{\varrho^2 d\varrho}{(1 - \varrho^2)^2} \leq \frac{4r^3}{3\pi(1 - r^2)^2},$$

which is at most $2r^3$ for $r \leq 0.7$, as $(1 - r^2)^2 \geq 0.26 > 2/(3\pi)$ there. For $0.7 < r < 0.8$, $\varkappa(r) \leq r^2 / (2(1 - r^2)) \leq 2r^3$, as $4r(1 - r^2) \geq 1$ there; and $2r^3 > 1$ for $r \geq 0.8$. As $\{R_* < \varrho\} = \{T_* > \varrho^{-2}\}$, [M, Lemma 7.9] gives

$$(4.2) \quad \mathbb{P}(R_* < \varrho) \leq 2\varrho^3 \quad (\varrho > 0), \quad \mathbb{P}(T_* > s) \leq 2s^{-3/2} \quad (s > 1).$$

[M, Lemma 9.12] is the qualitative form of Theorem 4.6. Numerically, $\mathbb{P}(T_* \geq 1.5, g < s)$ lies between $2.1s$ and $2.4s$ for $10^{-3} \leq s \leq 10^{-2}$, so the exponent 1 is the true one and the constant $107\varepsilon^{-4}$ is crude.

4.3. Small balls at the collision. Let $r_\varepsilon = (1 + \varepsilon)^{-1/2}$ and $r_c = (1 + \varepsilon/2)^{-1/2}$ as above, and $r_3 = \frac{1}{2}(1 + r_c)$, so that $1 - r_3 \geq \varepsilon/12$. Let $Y'_2 = \sup_{|u| \leq r_3} |q''(u)|$. As in (3.1), with r_3 in place of r_2 , $\mathbb{E} \sum_j j(j-1) |\eta_j| r_3^{j-2} = 2\sqrt{2/\pi} (1 - r_3)^{-3} \leq 2758\varepsilon^{-3}$ and $\sum_j j^2(j-1)^2 r_3^{2j-4} \leq 24(1 - r_3)^{-5}$, so

$$(4.3) \quad \mathbb{P}(Y'_2 > y_3(\ell)) \leq e^{-\ell} \quad (\ell \geq 0), \quad y_3(\ell) = \frac{2758 + 3456\ell^{1/2}}{\varepsilon^3}.$$

Lemma 4.7 (Small balls for $q'(u_*)$, γ_* and $\sin \theta_*$). *For $0 < r < 1$, $\tau > 0$ and $0 < s \leq 1$:*

- (a) $\mathbb{E} \#\{u \in \mathbb{D}_r^+ : q(u) = 0, |q'(u)| \leq \tau\} \leq 6\pi^{-4/3}(1 - r^2)^{-1}\tau^2$;
- (b) $\mathbb{E} \#\{u \in \mathbb{D}_r^+ : q(u) = 0, |uq'(u)| \leq \tau\} \leq 2.862(1 - r^2)^{-3/2}\tau^{3/2}$;
- (c) $\mathbb{E} \#\{u \in \mathbb{D}_r^+ : q(u) = 0, \operatorname{Im} u < s|u|\} \leq \frac{4}{3\pi}r^3(1 - r^2)^{-3}s^2$.

Consequently

$$\mathbb{P}(T_* \geq 1 + \varepsilon, |q'(u_*)| \leq \tau) \leq \frac{2.61\tau^2}{\varepsilon}, \quad \mathbb{P}(T_* \geq 1 + \varepsilon, |\gamma_*| \leq \tau) \leq \frac{8.1\tau^{3/2}}{\varepsilon^{3/2}},$$

$$\mathbb{P}(T_* \geq 1 + \varepsilon, s_* < s) \leq \frac{1.21s^2}{\varepsilon^3}.$$

Proof. (c) On $\mathbb{D}_r^+$, $\rho_c(u) \leq 2 \operatorname{Im} u / (\pi(1 - r^2)^3)$. In polar coordinates $u = \varrho e^{i\varphi}$ the set $\{\sin \varphi < s\}$ contributes at most $\int_0^r \varrho^2 d\varrho \cdot 2 \int_0^{\arcsin s} \sin \varphi d\varphi \cdot \frac{2}{\pi(1 - r^2)^3}$, and $2(1 - \sqrt{1 - s^2}) \leq 2s^2$.

(a) We use the weighted Rice formula [2, Theorem 6.4], in the form restated in [6, Appendix A.1, Theorem 1]: for a Gaussian field $Z : U \rightarrow \mathbb{R}^d$ with C^2 paths and nondegenerate values on an open $U \subset \mathbb{R}^d$, a compact $\Lambda \subset U$, random vectors Y^t continuous in t with (Z, Y^t) jointly Gaussian, and a bounded continuous g ,

$$\mathbb{E} \sum_{t \in \Lambda, Z(t)=0} g(Y^t) = \int_{\Lambda} \mathbb{E}[|\det Z'(t)| g(Y^t) \mid Z(t) = 0] p_{Z(t)}(0) dt.$$

Apply it to $Z = (\operatorname{Re} q, \operatorname{Im} q)$ on $U = \mathbb{D}^+$, with $Y^u = q'(u)$, a compact $\Lambda \subset \mathbb{D}_r^+$, and $g_k(v) = \min(1, \max(0, 1 - k(|v| - \tau)))$, which is bounded, continuous and decreases to $\mathbf{1}_{\{|v| \leq \tau\}}$ as

$k \rightarrow \infty$. Let $k \rightarrow \infty$: on the left by dominated convergence, the sum being at most the number of zeros of q in $\mathbb{D}_r^+$, whose mean $\varkappa(r)$ is finite; on the right by monotone convergence. Then let $\Lambda \uparrow \mathbb{D}_r^+$. So the left side of (a) is $\int_{\mathbb{D}_r^+} I(u) dA(u)$ with

$$I(u) = p_{q(u)}(0) \mathbb{E}[|q'(u)|^2 \mathbf{1}\{|q'(u)| \leq \tau\} \mid q(u) = 0].$$

By Lemma 4.4, given $q(u) = 0$ we have $q'(u) = B'_u(u)\xi$, with $|B'_u(u)| = b := b_u$ and ξ distributed as $q(u)$. The density of ξ is at most $p_{q(u)}(0) = 1/(\pi b)$, and $\text{Var Re } \xi \geq \frac{1}{2}$ (proof of Lemma 3.2), so $\mathbb{P}(|\xi| \leq \varrho) \leq 2\varrho/\sqrt{\pi}$. Hence, with $y = \text{Im } u \leq b$ (as $(1 - |u|^2)|1 - u^2| \leq 2$),

$$I(u) \leq (\pi b)^{-1} \tau^2 \mathbb{P}(|\xi| \leq \tau/b) \leq \frac{2\tau^3}{\pi^{3/2} b^2} \leq \frac{2\tau^3}{\pi^{3/2} y^2}.$$

Also $I(u) \leq \rho_c(u) \leq Ay$ with $A = 2/(\pi(1 - r^2)^3)$. Integrating $\min(Ay, B\tau^3|u|^{-3}y^{-2})$, $B = 2\pi^{-3/2}$, over $y \in (0, \infty)$ gives $\frac{3}{2}A^{1/3}B^{2/3}\tau^2$, and the range of $\text{Re } u$ has length at most 2; and $3A^{1/3}B^{2/3} = 6\pi^{-4/3}(1 - r^2)^{-1}$.

(b) The same, with τ replaced by $\tau/|u|$: $I(u) \leq \min(Ay, B\tau^3|u|^{-3}y^{-2})$. In polar coordinates $u = \varrho e^{i\varphi}$ we have $y = \varrho \sin \varphi$, and $2\varphi/\pi \leq \sin \varphi \leq \varphi$ on $[0, \frac{\pi}{2}]$; by the symmetry $\varphi \mapsto \pi - \varphi$ the φ -integral of I is at most twice the integral over $(0, \infty)$ of $\min(A\varrho\varphi, \frac{\pi^2}{4}B\tau^3\varrho^{-5}\varphi^{-2})$, which is $\beta\tau^2\varrho^{-3}$ with $\beta = 3A^{1/3}(\pi^2B/4)^{2/3} = 3 \cdot 2^{-1/3}(1 - r^2)^{-1}$, and it is also at most $2 \int_0^{\pi/2} A\varrho\varphi d\varphi = a\varrho$ with $a = A\pi^2/4 = \pi/(2(1 - r^2)^3)$. Integrating $\varrho \min(a\varrho, \beta\tau^2\varrho^{-3})$ over $\varrho \in (0, \infty)$ gives $\frac{4}{3}a^{1/4}\beta^{3/4}\tau^{3/2}$, and $\frac{4}{3}\pi^{1/4}3^{3/4}2^{-1/2} < 2.862$.

For the consequences, $u_* \in \mathbb{D}_{r_\varepsilon}^+$ on $\{T_* \geq 1 + \varepsilon\}$ (almost surely $R_* \neq r_\varepsilon$, by [M, Lemma 7.5(a)]), $1 - r_\varepsilon^2 = \varepsilon/(1 + \varepsilon) \geq \varepsilon/2$ and $r_\varepsilon^3(1 - r_\varepsilon^2)^{-3} \leq 2^{3/2}\varepsilon^{-3}$; and the probability that some zero has a property is at most the expected number of such zeros. $\square$

The exponents 2, 3/2 and 2 are the natural ones (heuristically: a nonreal zero of modulus about ϱ near 0 forces $\eta_0 \approx \varrho^2$ and $\eta_1 \approx \varrho$, so $\gamma_* \approx \varrho^2$, while $\mathbb{P}(R_* < \varrho) \asymp \varrho^3$). Parts (a) and (b) are the only places where the weighted Rice formula is used. The next lemma, which needs no Rice formula, gives exponent 2 for $|q'(u_*)|$ up to a logarithm, hence exponent 6/5 for $|\gamma_*|$ up to a logarithm (through $\mathbb{P}(R_* < \varrho) \leq 2\varrho^3$, (4.2), with $\varrho = \tau^{2/5}$); this suffices for Theorem 4.12.

Lemma 4.8 (Near-critical values). *Let $y \geq 2758\varepsilon^{-3}$, $0 < \tau \leq 1$ and $0 \leq \delta \leq \tau^2/(4y)$. Then*

$$\mathbb{P}\left(Y'_2 \leq y \text{ and } \exists u \in \overline{\mathbb{D}}_{r_c} : |q(u)| \leq \delta, |q'(u)| < \tau\right) \leq \tau^2(19 + 5.4 \log(2.04y)).$$

Proof. Put $h = \tau/(2y) \leq \varepsilon^3/5516 \leq \min(\frac{1}{3}, r_3 - r_c)$. Since $q(\bar{u}) = \overline{q(u)}$ we may assume $\text{Im } u \geq 0$. Work on $\{Y'_2 \leq y\}$; all Taylor bounds below use segments in $\overline{\mathbb{D}}_{r_3}$.

Points with $\text{Im } u \geq 2h$. Let g be the point of the grid $h(\mathbb{Z} + i(\mathbb{Z} + \frac{1}{2}))$ nearest to u , so $|g - u| \leq h/\sqrt{2}$ and $\text{Im } g \geq \frac{3}{2}h$. Then $|q'(g)| < \tau + yh/\sqrt{2} \leq 1.354\tau =: \tau'$ and $|q(g)| \leq \delta + \tau h/\sqrt{2} + yh^2/4 \leq 0.667\tau^2/y =: \delta'$. For a fixed g , let $b = b_g \geq \text{Im } g$. The density of $q(g)$ is at most $1/(\pi b)$, so $\mathbb{P}(|q(g)| \leq \delta') \leq \delta'^2/b$. Given $q(g) = v$, the law of $q'(g)$ is Gaussian with a mean linear in v and a covariance that does not depend on v ; by Lemma 4.4 its centred part has the law of $B'_g(g)\xi$ with ξ distributed as $q(g)$. By Anderson's inequality [1] the disk $\{|\cdot| < \tau'\}$ has conditional probability at most $\mathbb{P}(|B'_g(g)\xi| < \tau') \leq \min(1, 2\tau'/(\sqrt{\pi}b))$. So

$$\mathbb{P}(|q(g)| \leq \delta', |q'(g)| < \tau') \leq f(\text{Im } g), \quad f(y') = \frac{\delta'^2}{y'} \min\left(1, \frac{2\tau'}{\sqrt{\pi}y'}\right),$$

and f is decreasing. The grid points with $|\text{Re } g| \leq 1$ lie on the rows $\text{Im } g = (j + \frac{1}{2})h$, $j \geq 1$, with at most $3/h$ points each. Comparing the sum with an integral, $\sum_{j \geq 1} f((j + \frac{1}{2})h) \leq$

$f(\frac{3}{2}h) + h^{-1} \int_{3h/2}^{\infty} f \leq \delta'^2 \frac{4y}{3\tau} + h^{-1} \delta'^2 (1 + \log(2.04y))$, so these points contribute at most $\frac{3\delta'^2}{h^2} (\frac{5}{3} + \log(2.04y)) \leq \tau^2 (8.9 + 5.4 \log(2.04y))$.

Points with $0 \leq \operatorname{Im} u < 2h$. Let $x = \operatorname{Re} u$. Then $|q'(x)| \leq \tau + 2hy = 2\tau$ and $|q(x)| \leq \delta + 2h\tau + 2h^2y \leq 1.75\tau^2/y$. Let x_g be the point nearest to x of the grid of spacing τ/y in $[-r_c, r_c]$, which has at most $2y/\tau + 1$ points. Then $|q'(x_g)| \leq 2.5\tau$ and $|q(x_g)| \leq 2.875\tau^2/y$. The density of $(q(x_g), q'(x_g))$ is at most $1/(2\pi)$, because its covariance determinant is $(1 - x_g^2)^{-4} \geq 1$. So each grid point contributes at most $(2\pi)^{-1} (5.75\tau^2/y)(5\tau) \leq 4.58\tau^3/y$, and these points contribute at most $9.16\tau^2 + 4.58\tau^3/y \leq 10.1\tau^2$. $\square$

For $\rho > 0$ let

$$\mathbf{m}(\rho) = \inf \{ |q(u)| : |u| \leq r_c, q \text{ has no zero in } \{|v - u| < \rho\} \},$$

which is nondecreasing in ρ . In the setting of Section 4.1, every point of $\mathfrak{A}$ has modulus within $g/2$ of R_* (Step 1 of the proof of Theorem 4.1), every zero of q other than $u_*, \bar{u}_*$ has modulus at distance at least g from R_* , and $\mathfrak{A} \subset \mathbb{D}_{r_c}$. Hence

$$(4.4) \quad m_{\mathfrak{A}}(\rho) \geq \mathbf{m}(\min(\rho, g/2)) \quad (\rho > 0).$$

Lemma 4.9 (Depth away from the zeros). *For $0 < \rho \leq \varepsilon/12$ and $0 < \delta \leq e^{-1}$, with $y = y_3(\log(1/\delta))$,*

$$\mathbb{P}(\mathbf{m}(\rho) < \delta) \leq \delta + \left(\frac{16\delta^2}{\rho^2} + 4y\delta \right) (19 + 5.4 \log(2.04y)).$$

Proof. Let $\tau = \max(4\delta/\rho, 2\sqrt{y\delta})$. If $\tau > 1$, the right side exceeds 1. Otherwise work on $\{Y'_2 \leq y\}$, whose complement has probability at most δ by (4.3). Let $|u| \leq r_c$, $|q(u)| < \delta$ and $|q'(u)| \geq \tau$, and put $s = 2\delta/|q'(u)| \leq \rho/2$. On $|v - u| = s$, $|q(v) - q'(u)(v - u)| \leq \delta + ys^2/2 \leq \frac{3}{2}\delta < 2\delta = |q'(u)(v - u)|$, because $ys^2 \leq 4y\delta^2/\tau^2 \leq \delta$. By Rouché's theorem q has a zero in $\{|v - u| < s\} \subset \{|v - u| < \rho\}$. So $\mathbf{m}(\rho) < \delta$ implies that some $u \in \mathbb{D}_{r_c}$ has $|q(u)| < \delta$ and $|q'(u)| < \tau$. As $\tau^2 \geq 4y\delta$, Lemma 4.8 applies. $\square$

Lemma 4.10 (Scale-free depth). *Let*

$$V_{\mathbf{m}} = \sup_{0 < \rho \leq \varepsilon/12} \frac{\rho^{3/2}}{\mathbf{m}(\rho)}.$$

Then $\mathbb{P}(V_{\mathbf{m}} > s) \leq c\varepsilon^{-c}s^{-1}(1 + \log s)$ for $s \geq 1$, with an absolute constant c .

Proof. Let $\rho_k = 2^{-k-1}\varepsilon/12$ and $\delta_k = (2\rho_k)^{3/2}/s \leq e^{-1}$ for $k \geq 0$. If $V_{\mathbf{m}} > s$, then some $\rho \in (\rho_k, 2\rho_k]$ has $\mathbf{m}(\rho) < \rho^{3/2}/s$, and $\mathbf{m}(\rho_k) \leq \mathbf{m}(\rho) < \delta_k$. By Lemma 4.9, with $y_k = y_3(\log(1/\delta_k))$ and $\delta_k^2/\rho_k^2 = 8\rho_k/s^2$,

$$\mathbb{P}(\mathbf{m}(\rho_k) < \delta_k) \leq \delta_k + (128\rho_k s^{-2} + 4y_k \delta_k) (19 + 5.4 \log(2.04y_k)).$$

Here $\delta_k \leq 2^{-3k/2}/s$, $\sum_k \rho_k \leq \varepsilon/12$, and $\log(1/\delta_k) \leq \log s + 2(k+1) + \frac{3}{2} \log(12/\varepsilon)$, so $y_k \leq c\varepsilon^{-3}(1 + \log s + k + \log(1/\varepsilon))^{1/2}$. Summing over k gives the claim. $\square$

4.4. The tail of D . Let

$$D = \sup_{m \geq 1} m(|\Lambda_m - T_*| + |z_m - 2R_*^{-1} \cos \theta_*|),$$

with $z_m = F_m(\eta_0, \dots, \eta_m)$, as in the proof of [M, Corollary 9.16]. By [M, Corollary 9.14], $D < \infty$ almost surely. For the first degrees we use two crude bounds.

Lemma 4.11 (Small degrees). *Let $\Sigma_3 = \sum_{j \geq 3} (j+1)2^{3-j}|\eta_j|$ and $\Lambda^{\sup} = \max\{30, (1.42|\eta_2| + 45.5\Sigma_3)/|\eta_0|\}$.*

- (a) For every $m \geq 2$, $\Lambda_m \leq \Lambda^{\sup}$, and $\mathbb{P}(\Lambda^{\sup} > x) \leq 365/x$ for $x \geq 30$.
- (b) For $m \geq 2$ and $t \geq 0$, every zero z of $P_{m,t}$ satisfies $|z| \leq 4(1+mt)^{1/2}H_m$, with $H_m = \max(1, \max_{j \leq m} |\eta_j|/|\eta_0|)$.
- (c) $Y' = \sup_{j \geq 0} |\eta_j|/(2 \log(j+2))^{1/2}$ satisfies $\mathbb{P}(Y' > y) \leq 2 \cdot 2^{-y^2}$ for $y \geq 2$.

Proof. (a) Let $t \geq 30$ and $a_j = \eta_j t^{-j/2}$. In [M, Lemma 8.2(c)] with $\kappa = 1$ we have $c_\kappa = \coth 1 + \frac{1}{2} < 1.82$, so for $j \leq \sqrt{m}$ (when $j \leq m/2$) $\beta_{m,j} \leq 2e^{1.82}(j+1)e^j \leq 12.4(j+1)e^j$, and for $j > \sqrt{m}$, $\beta_{m,j} \leq \sqrt{m} < j$; so $\beta_{m,j} \leq 12.4(j+1)e^j$ for all $j \geq 3$. As $t \geq 30 > 4e^2$, $e/\sqrt{t} \leq \frac{1}{2}$, and the left side of [M, (8.3)] is at most

$$\frac{\sqrt{2}|\eta_2|}{t} + 12.4 \frac{e^3}{t^{3/2}} \sum_{j \geq 3} (j+1)2^{3-j}|\eta_j| \leq \frac{1.42|\eta_2| + 45.5\Sigma_3}{t},$$

which is less than $|\eta_0|$ for $t > \Lambda^{\sup}$. So $Q_m^{(t)}$, and hence $P_{m,t}$, has only real zeros for every $t > \Lambda^{\sup}$, and $\Lambda_m \leq \Lambda^{\sup}$. Conditionally on $(\eta_j)_{j \geq 1}$, $\mathbb{P}(|\eta_0| < v/x) \leq v/x$; and $\mathbb{E}[1.42|\eta_2| + 45.5\Sigma_3] = \sqrt{2/\pi}(1.42 + 455) \leq 365$, as $\sum_{j \geq 3} (j+1)2^{3-j} = 10$.

- (b) Write $P_{m,t}(z) = \sum_k c_k z^k$. From $\exp(-\frac{t}{2m}\partial^2)z^l = \sum_i \frac{(-t/2m)^i}{i!} \frac{l!}{(l-2i)!} z^{l-2i}$,

$$c_k = \sum_{i \leq (m-k)/2} \frac{(-t/2m)^i}{i!} \frac{(k+2i)!}{k!} \eta_{m-k-2i} \frac{m^{(k+2i)/2}}{\sqrt{(k+2i)!}}, \quad c_m = \frac{\eta_0 m^{m/2}}{\sqrt{m!}}.$$

Since $m^{(l-m)/2} \sqrt{m!/l!} \leq 1$ for $l \leq m$ and $(k+2i)!/k! \leq m^{2i}$, the i -th term of $|c_k/c_m|$ is at most $H_m(mt/2)^i/i! \leq H_m \max(1, mt/2)^{p/2}$, where $p = m - k$. So $|c_k/c_m|^{1/p} \leq H_m(p/2 + 1)^{1/p} \max(1, mt/2)^{1/2} \leq 2H_m(1+mt)^{1/2}$, and Fujiwara's bound $|z| \leq 2 \max_{k < m} |c_k/c_m|^{1/(m-k)}$ [4] gives the claim.

- (c) $\mathbb{P}(|\eta_j| > y(2 \log(j+2))^{1/2}) \leq (j+2)^{-y^2}$, and $\sum_{k \geq 2} k^{-y^2} \leq 2 \cdot 2^{-y^2}$ for $y \geq 2$. $\square$

Theorem 4.12 (The tail of D). *Let q be the Gaussian power series, $0 < \varepsilon \leq 1$, and $E_c = \log \Upsilon_c / \log \lambda_{\varepsilon/2}^{1/2}$ as in Section 4.1. There is an absolute constant c such that for all $x \geq 1$:*

- (a) $\mathbb{P}(D > x, T_* \geq 1 + \varepsilon) \leq c \varepsilon^{-c} x^{-\beta_\varepsilon}$, where $\beta_\varepsilon = 1/(1782E_c) \geq \varepsilon/(35640 \log(40/\varepsilon))$;
- (b) $\mathbb{P}(D > x, T_* \geq 1 + \varepsilon) \leq C_\varepsilon x^{-1/900}$, where $\log C_\varepsilon \leq c \varepsilon^{-c}$.

Proof. Work on $\Omega = \{T_* \geq 1 + \varepsilon\}$, intersected with the almost sure event on which (G1)–(G3) hold with $\rho_q = 1$, $C_q < \infty$, $\nu^{\text{glob}} > 0$ (as $\eta_0 \neq 0$ and the zeros of q in $\mathbb{D} \setminus \{0\}$ are simple, [M, Lemma 7.4]), and the conclusions of [M, Corollary 9.14] hold. Theorem 4.1 applies.

A deterministic bound. For $m \geq n_0$ the m -th term of D is at most $2K$. For $m = 1$ it is $T_* + 2T_*^{1/2}$, as $\Lambda_1 = 0$ and $z_1 = 0$. For $2 \leq m < n_0$, z_m is 0 or a zero of P_{m,Λ_m} , so by Lemma 4.11(a), (b) the m -th term is at most $n_0(\Lambda^{\sup} + 3T_* + 4(1 + n_0\Lambda^{\sup})^{1/2}H_{n_0})$. With $Y'' = \max(1, Y')$ we have $H_{n_0} \leq P_\alpha Y''(2 \log(n_0 + 2))^{1/2} \leq 2P_\alpha Y'' n_0^{1/10}$. Hence $D \leq c(K + n_0^{1.6}\Lambda^{\sup}T_*P_\alpha Y'')$, and Theorem 4.1 with Lemma 4.2(b) gives $D \leq c_\varepsilon(\Pi \Lambda^{\sup} Y'')^{17}$. Next, by (4.4) and the monotonicity of $\mathfrak{m}$, with $\rho' = \min(\rho_V, g/2, \varepsilon/12)$,

$$P_{\mathfrak{A}} = \max(1, m_{\mathfrak{A}}(\rho_V)^{-1}) \leq \max(1, V_{\mathfrak{m}}) \rho'^{-3/2},$$

and $\rho'^{-1} \leq \max(6T_*^{3/2}/(r_1 s_*), 2P_g, 12/\varepsilon) \leq c_\varepsilon \Pi_1^{5/2}$ with $\Pi_1 = P_\gamma P_s P_g P_C T_*$, since $r_1^{-1} \leq c_\varepsilon \max(T_* P_\gamma P_C, P_g, P_s)$. Let Π_3 be the product of the eleven factors $P_\gamma, P_s, P_g, P_C, T_*, \max(1, V_{\mathfrak{m}}), P_\nu, P_\alpha, \mathcal{U}, \Lambda^{\sup}$ and Y'' , each at least 1. Then $\Pi \leq c_\varepsilon \max(1, V_{\mathfrak{m}}) \Pi_1^{19/4} P_\nu P_\alpha \mathcal{U}$, and

$$(4.5) \quad D \leq c_\varepsilon \Pi_3^{81}.$$

Tails. Each factor V of Π_3 satisfies $\mathbb{P}(V > s, \Omega) \leq c\varepsilon^{-c}s^{-b}(1 + \log s)^2$ for $s \geq 1$, with the following b :

- P_γ : $b = 6/5$, by Lemma 4.8 with $\delta = 0$ and $y = y_3(2 \log(1/\tau))$, and $\mathbb{P}(R_* < \varrho) \leq 2\varrho^3$ (by (4.2)) with $\varrho = \tau^{2/5}$ (or $b = 3/2$ by Lemma 4.7(b));
- P_s : $b = 2$ (Lemma 4.7(c)); P_g : $b = 1$ (Theorem 4.6); P_α : $b = 1$; T_* : $b = 3/2$, by (4.2);
- P_C and Y'' : every b , by Lemma 3.1 with $\varepsilon/2$ in place of ε , and Lemma 4.11(c); $\Lambda^{\sup}$: $b = 1$ (Lemma 4.11(a));
- $\max(1, V_m)$: $b = 1$ (Lemma 4.10); P_ν : $b = 1$, because the proof of Lemma 3.3, with $\varepsilon/2$ in place of ε and $\delta' = \delta$, gives $\mathbb{P}(\nu^{\text{glob}} < \delta) \leq (32000\varepsilon^{-3} \log^{1/2}(1/\delta) + 4)\delta$;
- $\mathcal{U}$: for (a), $\mathcal{U} \leq (\overline{M}_c/|\eta_0|)^{E_c}$ gives $\mathbb{P}(\mathcal{U} > s) \leq \mathbb{P}(\overline{M}_c > s^{1/(2E_c)}) + \mathbb{P}(|\eta_0| < s^{-1/(2E_c)}) \leq c\varepsilon^{-c}s^{-1/(2E_c)}$; for (b), $\mathcal{U} \leq \Upsilon_c^Z$ and Lemma 3.5 with $r = r_c$, $\Upsilon = \Upsilon_c$ and exponent 1 give $\mathbb{P}(\mathcal{U} > s) \leq C(r_c, \Upsilon_c, 1)s^{-1}$, where $\log C(r_c, \Upsilon_c, 1) \leq c\varepsilon^{-15}$ because $\log((1 + r_c)/(1 - r_c)) \leq \log(12/\varepsilon)$, $2r_c^3/(1 - r_c^2)^3 \leq 54\varepsilon^{-3}$ and $\Upsilon_c \leq 40/\varepsilon$.

By (4.5), $D > x$ forces some factor of Π_3 to exceed $(x/c_\varepsilon)^{1/891}$. Summing the eleven bounds gives (a), the smallest exponent being $1/(2E_c)$ (note $E_c \geq 72$) and $E_c \leq 20\varepsilon^{-1} \log(40/\varepsilon)$, and (b), all exponents being at least 1 and the logarithms being absorbed by the margin $1/891 - 1/900$. $\square$

Corollary 4.13. *There are absolute constants $c > 0$ and C such that for $x \geq e^e$*

$$\mathbb{P}(D > x) \leq C \exp\left(-c\left(\frac{\log x}{\log \log x}\right)^{1/2}\right).$$

Proof. By [M, Theorem 7.13] there is an absolute C_0 with $\mathbb{P}(T_* < 1 + \varepsilon) \leq C_0 e^{-0.8/\varepsilon}$ for $0 < \varepsilon \leq 1$. With Theorem 4.12(a),

$$\mathbb{P}(D > x) \leq C_0 e^{-0.8/\varepsilon} + c\varepsilon^{-c} \exp\left(-\frac{\varepsilon \log x}{35640 \log(40/\varepsilon)}\right).$$

Take $\varepsilon = (\log \log x / \log x)^{1/2}$. For large x , $\log(40/\varepsilon) \leq \log \log x$, so the second term is at most $\exp(c \log \log x - (\log x / \log \log x)^{1/2} / 35640)$, and the first is $C_0 \exp(-0.8(\log x / \log \log x)^{1/2})$. Enlarge C for the remaining x . $\square$

Remark 4.14. (1) The tail of D on $\{T_* \geq 1 + \varepsilon\}$ rests on no unproved hypothesis: the small-ball bound for the modulus gap is Theorem 4.6, and the weighted Rice formula is not needed. (2) In (a) the constant is polynomial in $1/\varepsilon$ and the exponent is of order $\varepsilon / \log(1/\varepsilon)$; it comes from Jensen's bound for $\mathcal{U}$. In (b) the exponent does not depend on ε , because the number of zeros of q in a disk of radius r_c has super-exponential tails (Lemma 3.5), but $\log C_\varepsilon$ is of order $\varepsilon^{-O(1)}$. (3) Whether $\mathbb{P}(D > x)$ has a power bound without the restriction $T_* \geq 1 + \varepsilon$ is Question 5.2. By [M, Theorem 1.7] the limit points of $n(\Lambda_n - T_*)$ fill the interval $[\tau_c - W, \tau_c + W]$, so $D \geq W = (2(\text{Im } u_*)^2)^{-1}$; a lower bound $\mathbb{P}(\text{Im } u_* < \varrho) \geq c\varrho^2$, which we expect but have not proved, would give $\mathbb{P}(D > x) \geq c'x^{-1}$, so that no exponent above 1 is possible. (4) The exponents $1/900$ and $1/(1782E_c)$ come from the exponent 10 of Theorem 4.1 and a union bound over eleven factors, and are not optimised.

5. REMARKS AND QUESTIONS

5.1. Rates. The exponents in Theorem 3.6, any $\alpha < \frac{3}{50}$ and any $\theta < \frac{1}{10}$ at fixed t , do not depend on ε , but the constants are not polynomial in $1/\varepsilon$ (Remark 3.7). By Remark 3.7 this argument cannot give a fixed- t exponent above $\frac{1}{9}$; the limit is the degree of the explicit onset of Theorem 2.6, through the vertex loss of Lemma 2.4 near the edges ± 2 and the bulk constant. The distortion of the outer factor has an ε -independent small-ball exponent, uniformly in $t \geq 1 + \varepsilon$ (Proposition 3.9(b)), and the natural alternative deformation is no better (Remark 3.10).

Question 5.1. (i) For the Gaussian power series, is there a rate $n^{-\alpha}$ in Theorem 3.6 with $\alpha > \frac{1}{10}$, or with constants polynomial in $1/\varepsilon$? The Monte Carlo exponents of Remark 3.7 are 0.4 to 0.82. (ii) Does Proposition 3.9(b) hold for every $\beta < 1$, uniformly in $t \geq 1 + \varepsilon$? (iii) Is there a deformation of $a^{(t)}$ into a polynomial with the same zeros in $\mathbb{D}$ along which the minimum of $|a_s|$ on the unit circle and the coefficient bound of a_s are controlled by powers of μ , $|\eta_0|$ and C_q whose exponents do not depend on ε ? By Remark 3.10, any such deformation has circle minimum at most $\min_{|w|=1} |p_{\text{in}}(w)|$, a product over the zeros of $a^{(t)}$ in $\mathbb{D}$.

5.2. The constant in the final collision. [M, Theorem 9.1] gives $|\Lambda_n - T_*| = O(1/n)$ with a constant that depends on q . By Theorem 4.1 this constant is bounded, when $T_* \geq 1 + \varepsilon$, by a polynomial in condition parameters of q with absolute degrees and coefficients polynomial in $1/\varepsilon$, and for Gaussian coefficients $\mathbb{P}(D > x, T_* \geq 1 + \varepsilon) \leq C_\varepsilon x^{-1/900}$ (Theorem 4.12). The constant C_ε is not polynomial in $1/\varepsilon$, and with a constant polynomial in $1/\varepsilon$ we have only an exponent of order $\varepsilon/\log(1/\varepsilon)$; so for D itself we obtain only $\mathbb{P}(D > x) \leq C \exp(-c(\log x/\log \log x)^{1/2})$ (Corollary 4.13). By Remark 4.14(3), no bound better than x^{-1} can be expected.

Question 5.2. Does $\mathbb{P}(D > x)$ decay like a power of x for the Gaussian power series, without the restriction $T_* \geq 1 + \varepsilon$? By [M, Theorem 7.13] this would follow from Theorem 4.12(b) with $\log C_\varepsilon = o(1/\varepsilon)$, or from Theorem 4.12(a) with an exponent bounded below. What is the true exponent of $\mathbb{P}(D > x, T_* \geq 1 + \varepsilon)$?

STATEMENT ON THE USE OF AI

The proofs, computations and text of this paper were produced with AI systems from Anthropic, OpenAI and Google, directed by the author. The proofs were checked by independent AI referee rounds; no human expert has yet read them.

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