

A 256-STEP CERTIFICATE FOR TURÁN'S PURE POWER-SUM CONSTANT

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ABSTRACT. We give an exact-input certificate for the upper bound $C_{42} \leq 0.6888863712$ in the pure power-sum problem. The construction refines the step profiles of Griego, Röhrig and Plata to 256 middle blocks, without changing their coefficient-cancellation mechanism. It improves the strongest public proposal located in our 22 October 2026 audit, 0.688970, by 0.0000836288. An unmodified public interval checker and a separately written Arb checker verify the strict finite inequalities. The asymptotic implication is proved below; no explicit finite threshold, optimality, human peer review or formal verification is claimed. All material remains local and unsubmitted.

1. STATEMENT AND PRIOR CONSTRUCTIONS

Define

$$R_n = \min_{\substack{z_1, \dots, z_n \in \mathbb{C} \\ \max_j |z_j| = 1}} \max_{1 \leq k \leq n} \left| \sum_{j=1}^n z_j^k \right|, \quad C_{42} = \limsup_{n \rightarrow \infty} R_n.$$

This is constant 42a in Tao and contributors' table [1]. The condition is maximum modulus one; the other roots may lie inside the unit disk.

Theorem 1. *The pure power-sum constant satisfies*

$$C_{42} \leq \frac{215276991}{312500000} = 0.6888863712.$$

The table lists Harcos's 0.69368 and Griego's proposed 0.6906538 [2, 3]. Open PR 184 proposes Röhrig's eight-step bound 0.688983 [4]. A separate public repository by Kerrigan Plata proposes 0.688970 using six nonuniform middle levels [5]. We reproduced both latter certificates before optimizing. Plata's value is the strongest public upper proposal located, and is the comparison used here; both proposals explicitly lack expert peer review. The local audit is bounded, not an exhaustive priority claim.

We retain the generating function, free final block and Newton-identity argument of these works. The changes are a finer middle profile and new rational inputs. No new power-sum reduction theorem is needed.

2. A FINITE-STEP CRITERION

Put $\tau = 1/3$ and $B = 2/3$. Let

$$\tau = t_0 < t_1 < \dots < t_M = B, \quad \alpha = a + ib, \quad a > 0, \quad s = 1 - \alpha,$$

and choose middle values $\eta_0, \dots, \eta_{M-1}$. Set $\delta_0 = \eta_0 - s$ and $\delta_j = \eta_j - \eta_{j-1}$ for $j > 0$. Define

$$I_\alpha(x) = \int_0^x \frac{v^{\alpha-1}}{1-v} dv, \quad K = I_\alpha(\tau), \quad D = I_a(\tau) > 0.$$

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For $t + r < 1$ put

$$J_\alpha(t, r) = \int_0^{1-t-r} \frac{v^{\alpha-1}}{1-v} \log \frac{(1-t-v)(1-r-v)}{tr} dv,$$

and put $J_\alpha(t, r) = 0$ otherwise. Positive real powers use real logarithms. This also equals the integral of $(1-u-v)^{\alpha-1}/(uv)$ over $u \geq t, v \geq r, u+v \leq 1$. Finally set

$$(1) \quad Y = 1 + sK - \sum_j \delta_j (I_\alpha(1-t_j) - K) + \frac{1}{2} \sum_{i,j} \delta_i \delta_j J_\alpha(t_i, t_j).$$

Proposition 2. *If $|s| < C$, $|\eta_j| < C$ for all j , and $|Y| < CD$, then $R_n \leq C$ for every sufficiently large integer n . Consequently $C_{42} \leq C$.*

Proof. Put $A = \lfloor n/3 \rfloor$. Split the indices into $E = \{1, \dots, A\}$, $M_n = \{A+1, \dots, n-A-1\}$ and $F = \{n-A, \dots, n\}$. On E set $S_m = s$. On M_n put

$$w_m = \sum_j \delta_j \mathbf{1}_{m > \lfloor nt_j \rfloor}, \quad S_m = s + w_m.$$

Every such value is one of the η_j . Write $\beta_l = (\alpha)_l/l!$ and $P_m = n\beta_{n-m}/m$ on F . These numbers are nonzero because $a > 0$. Set

$$Y_n = n\beta_n - n \sum_{m \in M_n} \frac{w_m \beta_{n-m}}{m} + \frac{n}{2} \sum_{\substack{u,v \in M_n \\ u+v \leq n}} \frac{w_u w_v \beta_{n-u-v}}{uv} + s \sum_{m \in F} P_m,$$

$$S_m = \frac{Y_n}{\sum_{r \in F} |P_r|} \frac{\overline{P_m}}{|P_m|} \quad (m \in F).$$

Consider coefficients b_l defined by

$$\mathcal{B}(z) = \frac{1}{1-z} \exp \left(- \sum_{m=1}^n \frac{S_m z^m}{m} \right) = \sum_{l \geq 0} b_l z^l.$$

Relative to $(1-z)^{-\alpha}$ every correction index exceeds A . Since $3(A+1) > n$, at most two corrections contribute to b_n . A final correction plus any other correction has index at least $n+1$. Expanding the exponential therefore gives exactly $nb_n = Y_n - \sum_{m \in F} S_m P_m = 0$.

Let $y_2, \dots, y_n$ be the roots of $Z^{n-1} + b_1 Z^{n-2} + \dots + b_{n-1}$, with multiplicity, and let $y_1 = 1$. The logarithmic derivative of $\mathcal{B}$ and Newton's identities give the same triangular recurrence for S_k and $Q_k = \sum_j y_j^k$ through $k = n$; at the last step use $b_n = 0$. Hence $Q_k = S_k$ for $1 \leq k \leq n$. Dividing all roots by $\Lambda = \max_j |y_j| \geq 1$ cannot increase these power-sum moduli.

It remains to bound the final powers. The gamma-ratio asymptotic is

$$\beta_l = \frac{l^{\alpha-1}}{\Gamma(\alpha)} (1 + O(l^{-1})), \quad \sum_{l \leq L} |\beta_l| = O((L+1)^\alpha).$$

Away from $l = 0$, ordinary Riemann sums give

$$\Gamma(\alpha) n^{-\alpha} Y_n \longrightarrow Y, \quad |\Gamma(\alpha)| n^{-\alpha} \sum_{m \in F} |P_m| \longrightarrow D.$$

For completeness, discard first $l < \varepsilon n$ in each sum. In a single sum $m \asymp n$. In a double sum group by $l = n - u - v$: there are at most n pairs and $uv \gg n^2$. After the relevant normalization the discarded contribution is $O(\varepsilon^a + n^{-a})$ in both cases. This permits $\varepsilon \downarrow 0$, including the singular endpoint. The finite number of step discontinuities and the integer floors do not affect the limit. Thus $|Y_n|/\sum_F |P_m| \rightarrow |Y|/D < C$. All final powers have modulus below C eventually, which proves the assertion for all sufficiently large n , not merely a subsequence. $\square$

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