

A COUNTEREXAMPLE TO THE TABLE CONJECTURE ON THE SPHERE

PABLO SOBERÓN

ABSTRACT. We disprove Matschke's table conjecture, which asserts that every continuous real-valued function on S^2 takes a common value on the vertices of some rotated copy of every prescribed inscribed square. The counterexample is the cubic polynomial $H(x, y, z) = 14z^3 - 6z + y + xz$ and a square contained in a plane at distance $1/\sqrt{7}$ from the origin. We give an elementary proof.

DISCLAIMER

The results of this paper were found by an LLM, and the author only simplified and wrote them to be better understood. This is uploaded just as a research note on a personal academic website without the intention to publish or upload to arXiv. The purpose is just to show the extension of the methods involved, and to settle the table conjecture for those interested in the problem. The note was completely written by the author.

1. INTRODUCTION

In 1951, Dyson proved that every continuous function $f : S^2 \rightarrow \mathbb{R}$ takes the same value at the vertices of some square inscribed in a great circle [Dys51]. Later, Matschke conjectured that the condition of being on a great circle could be removed: for every square Q with vertices on S^2 , some rotation of Q should lie on a level set of f . This is known as the table conjecture on the sphere.

Conjecture 1 (Table conjecture, Matschke 2014 [Mat14]). *Let $f : S^2 \rightarrow \mathbb{R}$ be a continuous function, and $Q = \{q_1, q_2, q_3, q_4\} \subset S^2$ be a square. Then, there exists a rotation $R \in SO(3)$ such that $f(Rq_1) = f(Rq_2) = f(Rq_3) = f(Rq_4)$.*

It is known as the table conjecture since we can think of f as the height of a point on the sphere. Then, if we are given a table Q , we want to find a way to place the table on the sphere by placing its four legs at points of the same height. The purpose of this note is to disprove Conjecture 1.

Theorem 1. *Let $H(x, y, z) = 14z^3 - 6z + y + xz$ and $Q = \left\{ \frac{1}{\sqrt{7}}(\pm\sqrt{3}, \pm\sqrt{3}, 1) \right\} \subset S^2$. For every $R \in SO(3)$, the function H is not constant on RQ .*

There has been additional progress around Dyson's theorem and Conjecture 1. Livesay strengthened Dyson's result to any rectangle over a great circle [Liv54]. These results belong to a larger class of Knaster-type questions about placing finite configurations on the same level set of a function.

Crabb and Jaworowski reproved Dyson's theorem with arguments from cohomotopy [CJ09], and more recently Vassiliev presented an elegant short proof using characteristic classes [Vas26].

The research of P. Soberón is supported by NSF CAREER grant DMS-2237324 and a PSC-CUNY Track 1 award.

The table problem is related to planar inscription problems such as the square peg problem and Fenn's square table theorem [Fen70]. We recommend Matschke's survey on the square peg problem for the history and progress on that problem [Mat14].

There has been progress in variations of the table problem. If we replace the condition that Q is a square by simply being a cyclic planar quadrilateral, Karasev has shown that the corresponding conjecture fails [Kar16]. Hausel, Makai, and Szűcs showed that if the function f is even, then the table conjecture holds [HMS00]. Sadr showed a square table theorem for Riemannian surfaces of odd Euler characteristic, which also implies the even function case when passing to $\mathbb{RP}^2$ [Sad24].

The construction was motivated by the author's counterexample to Grünbaum's 1960 equipartition conjecture [Sob26, Grü60]. The common strategy is to begin with a symmetric candidate object (in this case, the function f) in which the solutions can be described explicitly. We then choose a perturbation that removes those solutions without creating others. Here the first candidate is $7z^3 - 3z$, and the perturbation is $y + xz$.

The methods to find counterexamples for the Grünbaum conjecture have been extended to other mass partition problems [MS26, Fri26]. The example constructed in this manuscript shows that the same ideas can help us find counterexample for conjectures beyond mass partition problems.

We present a simple proof of Theorem 1 in Section 2. We give an expanded explanation of how the search for the parameters was done in Section 3.

2. PROOF OF THEOREM 1

For every rotated copy of Q we can find a positively oriented orthonormal basis (u, v, n) of $\mathbb{R}^3$ such that u, v are parallel to the sides of the rotated copy. More precisely, we can write the vertices as

$$p_{\sigma, \tau} = \frac{1}{\sqrt{7}}n + \frac{\sqrt{3}}{\sqrt{7}}(\sigma u + \tau v), \quad \sigma, \tau \in \{-1, 1\}. \quad (1)$$

We denote the coordinates $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $n = (n_1, n_2, n_3)$. To simplify the notation later in the proof, let $(u_3, v_3, n_3) = (r, s, t)$ the vector formed by the z -coordinates of u, v, n . Note that $r^2 + s^2 + t^2 = 1$.

The four values of H evaluated on the vertices $p_{\sigma, \tau}$ can be written uniquely as

$$\sqrt{\frac{7}{3}} \cdot H(p_{\sigma, \tau}) = \kappa_0 + \kappa_1 \sigma + \kappa_2 \tau + \kappa_{12} \sigma \tau.$$

The value of H on all four vertices is the same if and only if the three non-constant coefficients are zero. These are exactly the Walsch coefficients of the function $(\sigma, \tau) \mapsto \sqrt{7/3} \cdot H(p_{\sigma, \tau})$.

We can evaluate H on the vertices and use the fact that $r^2 + s^2 + t^2 = 1$ to obtain

$$\begin{aligned} \kappa_1 &= 12rs^2 + u_2 + \frac{tu_1 + rn_1}{\sqrt{7}}, \\ \kappa_2 &= 12r^2s + v_2 + \frac{tv_1 + sn_1}{\sqrt{7}}, \\ \kappa_{12} &= \sqrt{3} \left(12rst + \frac{su_1 + rv_1}{\sqrt{7}} \right). \end{aligned}$$

In Section 3 we expand the reduction to $\kappa_1, \kappa_2, \kappa_{12}$, as it is related to how the function was chosen. The equation $\kappa_1 = \kappa_2 = \kappa_{12} = 0$ is equivalent to

$$12rs \begin{bmatrix} r \\ s \\ t \end{bmatrix} + M = 0, \quad (2)$$

where

$$M = \begin{bmatrix} v_2 + \frac{tv_1 + sn_1}{\sqrt{7}} \\ u_2 + \frac{tu_1 + rn_1}{\sqrt{7}} \\ \frac{su_1 + rv_1}{\sqrt{7}} \end{bmatrix}.$$

We first want to show that Eq. (2) implies $|rs| < 1/8$. To do this, we bound $\|M\|$ first. The key is to first bound the magnitude of

$$M - \begin{bmatrix} v_2 \\ u_2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{7}} \begin{bmatrix} tv_1 + sn_1 \\ tu_1 + rn_1 \\ su_1 + rv_1 \end{bmatrix}.$$

We can use the equations $u_1^2 + v_1^2 + n_1^2 = 1$ and $ru_1 + sv_1 + tn_1 = 0$ (which follow from the fact that the matrix with columns u, v, n is orthogonal) to obtain

$$(tv_1 + sn_1)^2 + (tu_1 + rn_1)^2 + (su_1 + rv_1)^2 = 1 - 2(r^2u_1^2 + s^2v_1^2 + t^2n_1^2) \leq 1.$$

Therefore,

$$\|M\| \leq \sqrt{u_2^2 + v_2^2} + \frac{1}{\sqrt{7}} \leq 1 + \frac{1}{\sqrt{7}} < \frac{3}{2}.$$

Since (r, s, t) is a unit vector, Eq. (2) implies $12|rs| = \|M\| < 3/2$, which gives us $|rs| < 1/8$.

Since the basis u, v, n is positively oriented, we have $(u_2, v_2, n_2) = (r, s, t) \times (u_1, v_1, n_1)$, which implies $u_2 = sn_1 - tv_1$ and $v_2 = tu_1 - rn_1$.

Substituting these expressions into Eq. (2), and including the orthogonality equation $ru_1 + sv_1 + tn_1 = 0$ gives us a system of four homogeneous linear equations that we can write as follows:

$$\begin{pmatrix} t & \frac{s}{\sqrt{7}} & \frac{s}{\sqrt{7}} - r & r \\ \frac{t}{\sqrt{7}} & -t & s + \frac{r}{\sqrt{7}} & s \\ \frac{s}{\sqrt{7}} & r & 0 & t \\ \frac{r}{\sqrt{7}} & \frac{r}{\sqrt{7}} & 0 & t \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ n_1 \\ 12rs \end{pmatrix} = 0. \quad (3)$$

The vector in Eq. (3) is nonzero, since

$$u_1^2 + v_1^2 + n_1^2 = 1.$$

Therefore the determinant D of the matrix in Eq. (3) must vanish.

Expanding this determinant yields

$$D = \frac{(r^2 - s^2)^2 + 5t^2(r^2 + s^2) + 8t^4}{7} + \frac{2rs(r^2 - s^2)}{\sqrt{7}}. \quad (4)$$

Using $r^2 + s^2 = 1 - t^2$, this becomes

$$D = t^2 + \frac{(1 - 2t^2)^2}{7} + \frac{2rs(r^2 - s^2)}{\sqrt{7}} - \frac{4r^2s^2}{7}. \quad (5)$$

The first two terms have a uniform positive lower bound:

$$t^2 + \frac{(1 - 2t^2)^2}{7} = \frac{1}{7} + \frac{3}{7}t^2 + \frac{4}{7}t^4 \geq \frac{1}{7}.$$

Also, $|r^2 - s^2| \leq 1$, so

$$D \geq \frac{1}{7} - \frac{2|rs|}{\sqrt{7}} - \frac{4r^2s^2}{7}.$$

Using $|rs| < 1/8$ and $2/\sqrt{7} < 1$,

$$D > \frac{1}{7} - \frac{1}{8} - \frac{1}{112} = \frac{1}{112} > 0.$$

This contradicts the necessary condition $D = 0$. Hence the four values of H cannot all be equal.

3. INTUITION BEHIND THE CONSTRUCTION OF THE COUNTEREXAMPLE.

3.1. Structure of the example. In this section we explain how the example function was chosen. In the introduction, we mentioned that H could be interpreted as a nice function $(7z^3 - 3z)$, with a perturbation $y + xz$. We first explain why the function $7z^3 - 3z$ allows us to characterize the set of copies of Q in a level set, and why the function is natural.

Let us analyze the values constructed above $\kappa_1, \kappa_2, \kappa_{12}$ for the function $7z^3 - 3z$.

Given a function $W = A + B\sigma + C\tau$ with $\sigma, \tau \in \{+1, -1\}$, we can expand the Walsh coefficients of W^3 as

$$\begin{aligned} [\sigma]W^3 &= B(3A^2 + B^2 + 3C^2) \\ [\tau]W^3 &= C(3A^2 + 3B^2 + C^2) \\ [\sigma\tau]W^3 &= 6ABC. \end{aligned}$$

Now, when $W = \sqrt{7} \cdot z(p_{\sigma,\tau})$, then $A = t$, $B = \sqrt{3}r$, and $C = \sqrt{3}s$. So, for example, the coefficient of σ in W^3 is

$$\sqrt{3} \cdot r(3t^2 + 3r^2 + 9s^2) = \sqrt{3} \cdot r(3 + 6s^2)$$

Notice that the $W^3 - 3W = \sqrt{7}(7z^3 - 3z)$ over the vertices $p_{\sigma,\tau}$. The equation above shows why the linear term is useful, as the coefficient for σ in $W^3 - 3W$ is now $6\sqrt{3} \cdot rs^2$. Similarly, the coefficient for τ in $W^3 - 3W$ is $6\sqrt{3}r^2s$. Finally, the linear part does not contribute to the coefficient of $\sigma\tau$, so it is only $18rst$. Overall, up to normalizations, for the Walsh coefficients to be zero, we need the vector $rs(r, s, t)$ to be zero. Since (r, s, t) is a unit vector, this implies $rs = 0$.

Therefore, we can characterize the set of good tables for the function $7z^3 - 3z$.

Therefore, choosing normalizations carefully, the function $H_\varepsilon = 7z^3 - 3z + \varepsilon(y + xz)$ the key equation (Eq. (2)) becomes

$$6rs \begin{bmatrix} r \\ s \\ t \end{bmatrix} + \varepsilon \cdot M = 0,$$

where $\varepsilon > 0$ is a (potentially large) real number and M corresponds to the Walsh coefficients from $y + xz$. Since (r, s, t) is a unit vector, by choosing ε sufficiently small, we can guarantee that rs will be bounded, as we would have $6|rs| = \varepsilon\|M\| \leq \varepsilon(1 + 1/\sqrt{7})$.

The perturbation $y + xz$ is chosen so that Eq. (5) is positive if rs is sufficiently small. When $rs = 0$, the determinant becomes $D = t^2 + \frac{1}{7}(1 - 2t^2)^2 \geq \frac{1}{7}$. The fact that $\varepsilon = 1/2$ works is fortunate, but this explains why the method example can be described as a simple equation with a suitable perturbation.

3.2. How the search was performed. The initial search for the example was much less insightful. Given the success of cubic and quartic polynomials in the Grünbaum equipartition problem, an LLM was tasked with finding similar counterexamples for the table conjecture. Given that the space of possible tables can be parametrized with $SO(3)$, there was some possibility that an example could be found this way.

The analysis of zeros is done much more crudely, by sampling methods. For every sensible candidate function, a reparametrization of the problem using $\kappa_1, \kappa_2, \kappa_{12}$ can reduce the problem to showing that some polynomials do not share a common zero in four copies of $[0, 1]^3$. This can be done with a Bernstein subdivision argument. This search was successful, but had the inconvenient feature that the final verification required 5,104 cases.

The computational example motivated the search for a cubic whose solutions could be described completely. Once the initial function $7z^3 - 3z$ was found, an LLM found the correct perturbation to finish the proof.

REFERENCES

- [CJ09] Michael C. Crabb and Jan Jaworowski, *Theorems of Kakutani and Dyson revisited*, J. Fixed Point Theory Appl. **5** (2009), no. 2, 227–236. MR2529498
- [Dys51] Freeman J. Dyson, *Continuous functions defined on spheres*, Ann. of Math. (2) **54** (1951), 534–536. MR44620
- [Fen70] Roger Fenn, *The table theorem*, Bull. London Math. Soc. **2** (1970), 73–76. MR271940
- [Fri26] Florian Frick, *Counterexamples to the ramos conjecture for two hyperplanes*, arXiv preprint arXiv:2609.26723 (2026).
- [Grü60] Branko Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261. MR124818
- [HMS00] Tamás Hausel, Endre Makai Jr., and András Szűcs, *Inscribing cubes and covering by rhombic dodecahedra via equivariant topology*, Mathematika **47** (2000), no. 1-2, 371–397. MR1924512
- [Kar16] Roman N. Karasev, *A note on Makeev’s conjectures*, J. Math. Sci. (N.Y.) **212** (2016), no. 5, 521–526. MR3473007
- [Liv54] George R. Livesay, *On a theorem of F. J. Dyson*, Ann. of Math. (2) **59** (1954), 227–229. MR59546
- [Mat14] Benjamin Matschke, *A survey on the square peg problem*, Notices Amer. Math. Soc. **61** (2014), no. 4, 346–352. MR3184501
- [MS26] Leonardo Martínez-Sandoval, *Counterexamples and symmetry for uneven orthogonal mass partitions in the plane*, arXiv preprint arXiv:2609.16757 (2026).
- [Sad24] Ali Naseri Sadr, *A table theorem for surfaces with odd euler characteristic*, arXiv preprint arXiv:2412.01977 (2024).
- [Sob26] Pablo Soberón, *Four hyperplanes do not always equipartition a mass in $\mathbb{R}^4$* , arXiv preprint arXiv:2608.23312 (2026).
- [Vas26] Victor A. Vassiliev, *Inscribed squares of level sets of functions on the sphere*, arXiv preprint arXiv:2607.13052 (2026).

BARUCH COLLEGE & THE GRADUATE CENTER, CITY UNIVERSITY OF NEW YORK, ONE BERNARD
BARUCH WAY, NEW YORK, NY 10010, UNITED STATES
Email address: psoberon@gc.cuny.edu