

DENSEST PACKINGS OF TWO TRANSLATES OF A LATTICE IN FIVE DIMENSIONS

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ABSTRACT. We prove that no packing of unit balls in $\mathbb{R}^5$ whose centres form two translates of a lattice is denser than the D_5 lattice packing. Up to congruence exactly three such packings attain this density: $\sqrt{2} D_5$, a layered packing of Leech, and the packing with Szöllősi's kissing configuration.

1. INTRODUCTION

The root lattice $D_5 = \{x \in \mathbb{Z}^5 : x_1 + \cdots + x_5 \in 2\mathbb{Z}\}$ [8, Ch. 4, §7.1], scaled by $\sqrt{2}$ so that its minimal distance is 2, carries a packing of unit balls of density $\pi^2/(15\sqrt{2})$. Korkine and Zolotarev proved that no lattice packing of balls in $\mathbb{R}^5$ is denser [11]. Whether D_5 is densest among all packings in $\mathbb{R}^5$ is open, and non-lattice packings of the same density exist: Leech constructed three [12], Conway and Sloane placed them in a family of layered packings [7], and Cohn and Rajagopal [6] found further examples after Szöllősi's discovery of a new kissing configuration of 40 points [16].

A periodic packing is a finite union of translates of one lattice. After lattices, the next case is two translates, $P = \Lambda \cup (\Lambda + b)$. Andreanov and Kallus [1] extended Voronoi's algorithm for perfect forms [17] to such sets. In dimension 5 their enumeration found that the maximal density is that of D_5 , attained by five configurations, three of which represent D_5 . Parts of their computation used floating-point arithmetic, and they present these results as numerical rather than rigorous [1, §4]. Cohn and Rajagopal cite the bound as a theorem of [1] [6, §4.2]. We prove the bound together with the list of equality cases. The four-dimensional statement is proved in [3].

Throughout, Λ_m and b_m denote the lattices and vectors of Table 1.1, and $P_m = \Lambda_m \cup (\Lambda_m + b_m)$. Each Λ_m has covolume 4 and minimal norm 2. Two pairs (Λ, b) and (Λ', b') are *congruent* if an isometry of $\mathbb{R}^5$ maps Λ onto Λ' and $\Lambda + b$ onto $\Lambda' + b'$.

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^5$ be a lattice and $b \in \mathbb{R}^5 \setminus \Lambda$ such that $|x - y| \geq 2$ for all distinct $x, y \in P = \Lambda \cup (\Lambda + b)$. Then $\text{covol}(\Lambda) \geq 16\sqrt{2}$. Equality holds if and only if (Λ, b) is congruent to $(\sqrt{2}\Lambda_m, \sqrt{2}b_m)$ for some $m \in \{0, \dots, 4\}$, and these five pairs are pairwise not congruent. Equivalently, the packing of unit balls centred at P has density at most $\pi^2/(15\sqrt{2})$, and equality holds exactly when P is congruent to $\sqrt{2} D_5$, $\sqrt{2} P_1$ or $\sqrt{2} P_2$.*

The sets P_0, P_3, P_4 are D_5 split into two cosets of a sublattice of index two; the 31 such sublattices form three orbits under the symmetries of D_5 , and $\Lambda_0, \Lambda_3, \Lambda_4$ represent them (Remark 6.3). The set $\sqrt{2} P_1$ is the uniform packing with Szöllősi's kissing configuration Q_5 , found

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m	Λ_m	b_m	$ \text{Min } \Lambda_m $	P_m
0	$\{x \in \mathbb{Z}^5 : x_1 + x_2 + x_3, x_4 + x_5 \in 2\mathbb{Z}\} = D_3 \oplus D_2$	$e_1 + e_4$	16	D_5
1	$A_4 \oplus \mathbb{Z} \frac{4}{5} \mathbf{1}, \quad A_4 = \{x \in \mathbb{Z}^5 : \sum x_i = 0\}$	$e_1 + e_5$	20	not a lattice
2	$(D_4 \times 0) + \mathbb{Z}(h + 2e_5), \quad h = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	$e_1 + e_5$	24	not a lattice
3	$\{x \in \mathbb{Z}^5 : x_1 + x_2 + x_3 + x_4, x_5 \in 2\mathbb{Z}\} = D_4 \oplus 2\mathbb{Z}$	$e_1 + e_5$	24	D_5
4	$\{x \in \mathbb{Z}^5 : x_1 + \dots + x_5 \in 4\mathbb{Z}\}$	$e_1 + e_2$	20	D_5

TABLE 1.1. The five extremal configurations, at minimal distance $\sqrt{2}$. Here $\mathbf{1} = (1, 1, 1, 1, 1)$, $D_n = \{x \in \mathbb{Z}^n : \sum x_i \in 2\mathbb{Z}\}$, and $|\text{Min } \Lambda_m|$ is the number of vectors of norm 2 in Λ_m .

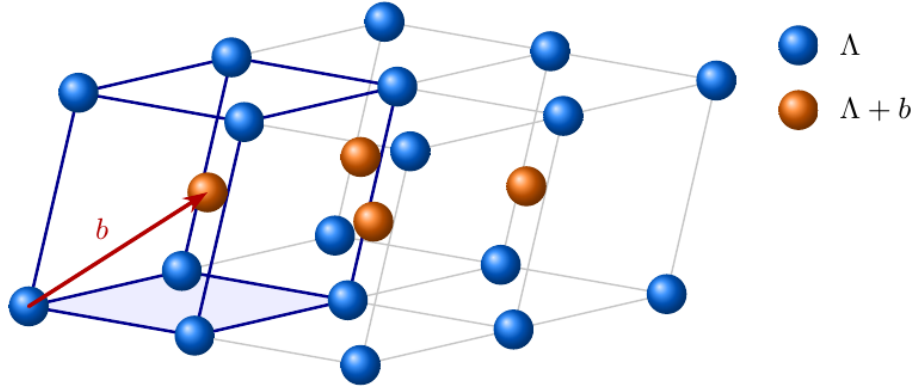


FIGURE 1.1. Two translates of a lattice, drawn in three dimensions: the points of Λ and of $\Lambda + b$, with a cell spanned by a basis of Λ . Such a set is described by the Gram matrix of the basis together with b . In five dimensions the densest packings of unit balls centred at such sets are $\sqrt{2} D_5$, $\sqrt{2} P_1$ and $\sqrt{2} P_2$ (Theorem 1.1).

by Andreanov and Kallus and described by Cohn and Rajagopal [6, Table 4.2]. The set $\sqrt{2} P_2$ is the layered packing of Leech whose kissing configuration is L_5 [12, 6]. The three sets are pairwise not congruent.

A lattice L is the union of two cosets of any sublattice Λ of index two, with $\text{covol}(L) = \frac{1}{2} \text{covol}(\Lambda)$. So Theorem 1.1 contains the five-dimensional case of the theorem of Korkine and Zolotarev: a lattice in $\mathbb{R}^5$ with minimal distance 2 has covolume at least $8\sqrt{2}$, with equality only for $\sqrt{2} D_5$. It also bounds the Voronoi cells.

Corollary 1.2. *Let $P \subset \mathbb{R}^5$ be a lattice, or a union of two translates of a lattice, such that distinct points of P are at distance at least 2. Every Voronoi cell of P has volume at least $8\sqrt{2}$, the volume of the Voronoi cell of $\sqrt{2} D_5$. Equality holds if and only if P is congruent to $\sqrt{2} D_5$, $\sqrt{2} P_1$ or $\sqrt{2} P_2$.*

Method. A pair (Λ, b) is encoded by the 6×6 Gram matrix J of a basis of Λ together with b . The packing condition becomes infinitely many linear inequalities $J[k] \geq 4$, which cut out a locally finite polyhedron $\mathcal{R}$ in the 21-dimensional space of symmetric matrices (Section 2). Packings are the points of $\mathcal{R}$ with $\det J = 0$, and the quantity to minimise is $\det Q$, where Q is

the upper left 5×5 block of J . Proposition 3.2 moves every packing, without increasing $\det Q$, to a vertex of $\mathcal{R}$ or to an interior point of an edge. The argument uses only the concavity of $\log \det Q$ and of the Schur complement of Q in J .

Up to the integral symmetries of the problem, $\mathcal{R}$ has 34 vertices and 480 edges at them (Proposition 4.3). This finite statement is computed in exact arithmetic and recorded in a certificate. Four independent implementations check the certificate, four independent enumerations recompute the extreme rays of the vertex cones, and a proof in Lean 4 checks in the kernel the minimal vectors and determinants of every vertex and the statements of the certificate on every listed edge (Appendix B states exactly what is formalized). The 25 unbounded edges are explicit (Table 4.2). In dimension four every vertex other than the optimum has $\det J > 0$; in dimension five two vertices have $\det J < 0$, and the bounded edges at them carry packings (Remark 4.4).

The equality case is proved by hand in Section 5. A deep hole of a lattice is surrounded by lattice points that span the space affinely (Lemma 5.1). Perturbing along the directions of $\mathcal{R}$ at an extremal configuration then shows that it is a vertex or an interior point of an edge of $\mathcal{R}$, so it appears in the finite list (Proposition 5.3). Section 6 identifies the five configurations.

2. TWO-PERIODIC SETS AND THE POLYHEDRON $\mathcal{R}$

2.1. Notation. Sym_n is the space of real symmetric $n \times n$ matrices, with any fixed norm $\|\cdot\|$. For $J \in \text{Sym}_6$ and $x \in \mathbb{R}^6$ we write $J[x] = x^\top J x$, and

$$J = \begin{pmatrix} Q & r \\ r^\top & s \end{pmatrix}, \quad Q = Q(J) \in \text{Sym}_5, \quad r \in \mathbb{R}^5, \quad s \in \mathbb{R}.$$

Let $\mathcal{M} = \mathbb{Z}^5 \times \{-1, 0, 1\} \subset \mathbb{Z}^6$ and $\mathcal{M}^* = \mathcal{M} \setminus \{0\}$. Elements of $\mathcal{M}$ are written $k = (n, l)$ with $n \in \mathbb{Z}^5$ and $l \in \{-1, 0, 1\}$. Define

$$\mathcal{R} = \{J \in \text{Sym}_6 : J[k] \geq 4 \text{ for all } k \in \mathcal{M}^*\}.$$

$\mathcal{R}$ is closed and convex. For $J \in \mathcal{R}$ let

$$\text{Min}(J) = \{k \in \mathcal{M}^* : J[k] = 4\}, \quad C_J = \{X \in \text{Sym}_6 : X[k] \geq 0 \text{ for all } k \in \text{Min}(J)\},$$

$$L_J = C_J \cap (-C_J) = \{X \in \text{Sym}_6 : X[k] = 0 \text{ for all } k \in \text{Min}(J)\},$$

and $d(J) = \dim L_J$. Since k and $-k$ give the same condition, we list one vector of each pair $\pm k$ when counting $\text{Min}(J)$.

Let $\Lambda \subset \mathbb{R}^5$ be a lattice with basis $a = (a_1, \dots, a_5)$ and let $b \in \mathbb{R}^5$. The Gram matrix of $(a_1, \dots, a_5, b)$ is denoted $J(a, b)$, so that

$$(2.1) \quad J(a, b)[(n, l)] = \left| \sum_{i=1}^5 n_i a_i + lb \right|^2.$$

Lemma 2.1. *Let $b \notin \Lambda$. The distinct points of $\Lambda \cup (\Lambda + b)$ are at distance at least 2 if and only if $J(a, b) \in \mathcal{R}$.*

Proof. Since Λ and $\Lambda + b$ are disjoint, the differences of distinct points are the vectors of $(\Lambda \setminus 0) \cup (\Lambda + b) \cup (\Lambda - b)$, that is, the vectors $\sum n_i a_i + lb$ with $(n, l) \in \mathcal{M}^*$. Apply (2.1). $\square$

Lemma 2.2. *For $J \in \mathcal{R}$ the block $Q(J)$ is positive definite.*

Proof. We have $Q[n] = J[(n, 0)] \geq 4$ for $n \in \mathbb{Z}^5 \setminus 0$. If $Q[x] < 0$ for some $x \in \mathbb{R}^5$, the open cone $\{Q < 0\}$ contains a rational point and hence an integral point $n \neq 0$ with $Q[n] < 0$, a contradiction. So Q is positive semidefinite. Suppose $Qv = 0$ with $\max_i |v_i| = 1$. By Dirichlet's theorem on simultaneous approximation, for every integer $N \geq 1$ there are an integer $1 \leq m \leq$

N^5 and $n \in \mathbb{Z}^5$ with $\max_i |mv_i - n_i| \leq 1/N$. Then $n \neq 0$ because $\max_i |mv_i| = m \geq 1 > 1/N$, and $Q[n] = Q[n - mv] \leq 25\|Q\|_\infty/N^2$, where $\|Q\|_\infty$ is the largest absolute value of an entry. For large N this is less than 4, a contradiction. Hence Q is positive definite. $\square$

Let $\text{Sym}_6^+ = \{J \in \text{Sym}_6 : Q(J) \text{ positive definite}\}$, an open convex set containing $\mathcal{R}$. For $J \in \text{Sym}_6^+$ put

$$q(J) = Q^{-1}r, \quad \sigma(J) = s - r^\top Q^{-1}r.$$

Lemma 2.3. *Let $J \in \text{Sym}_6^+$.*

- (1) $J[(x, l)] = Q[x + lq(J)] + l^2\sigma(J)$ for $x \in \mathbb{R}^5$ and $l \in \mathbb{R}$.
- (2) $\sigma(J) = \min_{x \in \mathbb{R}^5} J[(x, 1)]$, attained at $x = -q(J)$. In particular $\sigma(J') \leq J'[u]$ for every $J' \in \text{Sym}_6^+$ and every $u \in \mathbb{R}^5 \times \{1\}$, and σ is concave on Sym_6^+ .
- (3) $\det J = \det Q \cdot \sigma(J)$.

Proof. (1) is completing the square, (2) follows from (1) because Q is positive definite, and a minimum of affine functions of J is concave. (3) is the Schur complement formula. $\square$

Lemma 2.4. *Let $J \in \mathcal{R}$.*

- (1) $\sigma(J) = 0$ if and only if $J = J(a, b)$ for a basis a of a lattice $\Lambda \subset \mathbb{R}^5$ and a vector $b \notin \Lambda$ such that the distinct points of $\Lambda \cup (\Lambda + b)$ are at distance at least 2. In that case $\det Q(J) = \text{covol}(\Lambda)^2$.
- (2) If $\sigma(J) < 0$, then $J + |\sigma(J)|e_6e_6^\top \in \mathcal{R}$, its Q -block is $Q(J)$, and its σ is 0.

Proof. (1) If $\sigma(J) = 0$, then J is positive semidefinite of rank 5 by Lemma 2.3(1), (3) and Lemma 2.2, hence the Gram matrix of six vectors $a_1, \dots, a_5, b$ of $\mathbb{R}^5$, and $a_1, \dots, a_5$ are linearly independent because Q is positive definite. If b were $\sum n_i a_i$, then $J[(-n, 1)] = 0 < 4$. Lemma 2.1 gives the packing property. Conversely $J(a, b)$ is a Gram matrix of six vectors in $\mathbb{R}^5$, so $\det J(a, b) = 0$ and $\sigma = 0$ by Lemma 2.3(3). Finally $\det Q = \det(a_i \cdot a_j) = \text{covol}(\Lambda)^2$. (2) Adding $|\sigma|e_6e_6^\top$ changes $J[(n, l)]$ by $l^2|\sigma| \geq 0$, leaves Q and r unchanged and raises s , hence σ , by $|\sigma|$. $\square$

Lemma 2.5. *For $J \in \mathcal{R}$, $\det Q(J) \geq 64\pi^4/225$.*

Proof. Q is the Gram matrix of a lattice Λ with no nonzero vector of length less than 2. The open ball of radius 2 in $\mathbb{R}^5$ has volume $2^5 \cdot 8\pi^2/15$. By Minkowski's convex body theorem [5, Ch. III] it contains a nonzero point of Λ if its volume exceeds $2^5 \text{covol}(\Lambda)$. Hence $\text{covol}(\Lambda) \geq 8\pi^2/15$. $\square$

2.2. Local structure.

- Lemma 2.6.** (1) *For every compact set $K \subset \text{Sym}_6^+$ the set $\mathcal{M}_K = \{k \in \mathcal{M}^* : J[k] \leq 5 \text{ for some } J \in K\}$ is finite.*
- (2) *For every $J \in \mathcal{R}$ there is $\varepsilon > 0$ such that for all $X \in \text{Sym}_6$ with $\|X\| \leq \varepsilon$ we have $J + X \in \mathcal{R}$ if and only if $X \in C_J$.*
- (3) *For every $J \in \mathcal{R}$, $\text{Min}(J)$ is finite and $\mathcal{R} \subseteq J + C_J$.*

Proof. (1) On K the smallest eigenvalue of Q is at least some $\mu > 0$, and $|q(J)| \leq \rho$, $|\sigma(J)| \leq \rho$ for some ρ , by continuity. For $k = (n, l) \in \mathcal{M}^*$ and $J \in K$, Lemma 2.3(1) gives $J[k] \geq \mu|n + lq(J)|^2 - \rho$. If $|n| > \rho + \sqrt{(5 + \rho)/\mu}$, then $|n + lq(J)| \geq |n| - \rho$ and $J[k] > 5$. (2) Choose $\varepsilon_0 > 0$ such that the closed ball $K = \bar{B}(J, \varepsilon_0)$ lies in Sym_6^+ . The finitely many $k \in \mathcal{M}_K \setminus \text{Min}(J)$ have $J[k] > 4$, so there is $\varepsilon \leq \varepsilon_0$ with $(J + X)[k] > 4$ for all of them whenever $\|X\| \leq \varepsilon$. For such X , the vectors $k \notin \mathcal{M}_K$ satisfy $(J + X)[k] > 5$, so $J + X \in \mathcal{R}$ if and only if $(J + X)[k] = 4 + X[k] \geq 4$ for all $k \in \text{Min}(J)$. (3) $\text{Min}(J) \subseteq \mathcal{M}_{\{J\}}$ is finite. If

$J' \in \mathcal{R}$, then $J + \lambda(J' - J) \in \mathcal{R}$ for $\lambda \in [0, 1]$, and by (2) $\lambda(J' - J) \in C_J$ for small $\lambda > 0$; C_J is a cone. $\square$

Lemma 2.7. $\mathcal{R}$ contains no line.

Proof. If $J + \tau X \in \mathcal{R}$ for all $\tau \in \mathbb{R}$, then $X[k] = 0$ for every $k \in \mathcal{M}^*$. The matrices $kk^\top$ for $k = e_i$ ($1 \leq i \leq 6$) and $k = e_i + e_j$ ($1 \leq i < j \leq 6$), all in $\mathcal{M}^*$, span Sym_6 , so $X = 0$. $\square$

Definition 2.8. A *vertex* of $\mathcal{R}$ is a point $J \in \mathcal{R}$ with $d(J) = 0$. Then C_J is a pointed polyhedral cone, defined by the finitely many inequalities $X[k] \geq 0$, $k \in \text{Min}(J)$, whose linear forms span the 21-dimensional dual of Sym_6 . An *extreme ray* of C_J is a ray $\mathbb{R}_{\geq 0}Y$, $Y \neq 0$, with $Y \in C_J$ such that the forms $X \mapsto X[k]$ for the $k \in \text{Min}(J)$ with $Y[k] = 0$ span a 20-dimensional space. For an extreme ray Y put

$$t^*(J, Y) = \sup\{t \geq 0 : J + tY \in \mathcal{R}\} \in (0, \infty],$$

which is positive by Lemma 2.6(2). The set $\{J + tY : 0 \leq t \leq t^*\}$ (with $t < \infty$) is the *edge* of $\mathcal{R}$ at J in direction Y ; it is *bounded* if $t^* < \infty$.

We use two standard facts on polyhedral cones [18, Lecture 1]: a pointed polyhedral cone has finitely many extreme rays and is their conical hull, and a nonzero vector of the cone spans an extreme ray exactly when the inequalities it satisfies with equality have rank one less than the dimension.

Lemma 2.9. Let J be a vertex and Y an extreme ray of C_J , and let $Z = \{k \in \text{Min}(J) : Y[k] = 0\}$.

- (1) $\{t \geq 0 : J + tY \in \mathcal{R}\}$ is the interval $[0, t^*]$, or $[0, \infty)$ if $t^* = \infty$. If $t^* < \infty$, then $J + t^*Y$ is a vertex.
- (2) For $0 < t < t^*$ we have $\text{Min}(J + tY) = Z$ and $d(J + tY) = 1$.
- (3) Every $J' \in \mathcal{R}$ with $d(J') = 1$ lies on an edge: $J' = J_v + tY'$ for a vertex J_v , an extreme ray Y' of C_{J_v} , and $0 < t < t^*(J_v, Y')$.

Proof. (1) The set is convex and closed. Let $J_1 = J + t^*Y$ with $t^* < \infty$. Every $k \in Z$ has $J_1[k] = 4$, so $Z \subseteq \text{Min}(J_1)$. Since $J_1 + \eta Y \notin \mathcal{R}$ for $\eta > 0$, Lemma 2.6(2) gives $Y \notin C_{J_1}$, so some $k_1 \in \text{Min}(J_1)$ has $Y[k_1] < 0$. The forms of Z vanish on Y and span a 20-dimensional space, and the form of k_1 does not vanish on Y ; together they have rank 21, so $L_{J_1} = 0$. (2) For $k \in \mathcal{M}^*$ the function $\tau \mapsto (J + \tau Y)[k]$ is affine and at least 4 on $[0, t]$ and slightly beyond t . If it equals 4 at t , it is constant, so $J[k] = 4$ and $Y[k] = 0$. Hence $\text{Min}(J + tY) = Z$, and L_{J+tY} , cut out by the forms of Z of rank 20, has dimension 1. (3) Let $L_{J'} = \mathbb{R}X_0$. The set $I_0 = \{\tau : J' + \tau X_0 \in \mathcal{R}\}$ is a closed interval which contains a neighbourhood of 0 by Lemma 2.6(2) and is not $\mathbb{R}$ by Lemma 2.7. Let τ_e be a finite end point, $J_v = J' + \tau_e X_0$ and $Y' = -\text{sgn}(\tau_e)X_0$. Since $X_0[k] = 0$ for $k \in \text{Min}(J')$, we have $\text{Min}(J') \subseteq \text{Min}(J_v)$ and $L_{J_v} \subseteq L_{J'}$. If $X_0 \in L_{J_v}$, then $J_v \pm \eta X_0 \in \mathcal{R}$ for small η by Lemma 2.6(2), contradicting the choice of τ_e ; so $d(J_v) = 0$. The segment from J_v to J' lies in $\mathcal{R}$, so $Y' \in C_{J_v}$ by Lemma 2.6(2), and Y' vanishes on $\text{Min}(J')$, which has rank 20; so Y' spans an extreme ray of C_{J_v} , and $J' = J_v + |\tau_e|Y'$. Finally $|\tau_e| < t^*(J_v, Y')$, since otherwise J' would be a vertex by (1). $\square$

2.3. Symmetries. Let Γ be the group of matrices

$$T = \begin{pmatrix} U & v \\ 0 & \varepsilon \end{pmatrix}, \quad U \in \text{GL}_5(\mathbb{Z}), \quad v \in \mathbb{Z}^5, \quad \varepsilon \in \{1, -1\}.$$

Then $T(n, l) = (Un + lv, \varepsilon l)$, so T maps $\mathcal{M}$ bijectively onto itself, and $J \mapsto T^\top J T$ maps $\mathcal{R}$ bijectively onto itself with $(T^\top J T)[k] = J[Tk]$. Consequently $\text{Min}(T^\top J T) = T^{-1} \text{Min}(J)$, $X \mapsto$

$T^\top XT$ maps C_J onto $C_{T^\top JT}$, and vertices, extreme rays, edges and the values t^* correspond. Moreover $Q(T^\top JT) = U^\top QU$, so $\det Q$, $\det J$ and $\sigma = \det J / \det Q$ are Γ -invariant. We call J, J' *equivalent* if $J' = T^\top JT$ for some $T \in \Gamma$, and write Γ_J for the stabiliser of J .

Lemma 2.10. *Let a, a' be bases of lattices $\Lambda, \Lambda' \subset \mathbb{R}^5$ and $b, b' \in \mathbb{R}^5$. Then $J(a, b)$ and $J(a', b')$ are equivalent if and only if (Λ, b) and (Λ', b') are congruent.*

Proof. Suppose $J(a', b') = T^\top J(a, b)T$ with T as above. The vectors (a', b') and $(aU, \varepsilon b + \sum v_i a_i)$ have the same Gram matrix, and a' spans $\mathbb{R}^5$, so a linear isometry ψ maps a' to aU and b' to $\varepsilon b + \sum v_i a_i$. Then $\psi(\Lambda') = \Lambda$ and $\psi(\Lambda' + b') = \Lambda + \varepsilon b$. If $\varepsilon = -1$, replace ψ by $-\psi$, which maps Λ' onto Λ and $\Lambda' + b'$ onto $\Lambda + b$. Conversely, let an isometry f map Λ' onto Λ and $\Lambda' + b'$ onto $\Lambda + b$. Then $f(0) \in \Lambda$, and $\psi = f - f(0)$ is a linear isometry with $\psi(\Lambda') = \Lambda$ and $\psi(\Lambda' + b') = \Lambda + b$. So $\psi(a') = aU$ with $U \in \text{GL}_5(\mathbb{Z})$ and $\psi(b') = b + \sum v_i a_i$ with $v \in \mathbb{Z}^5$, and $J(a', b') = T^\top J(a, b)T$ with $\varepsilon = 1$. $\square$

2.4. Reaching all vertices.

Proposition 2.11. *Let v and w be vertices of $\mathcal{R}$. There are vertices $v = v_0, v_1, \dots, v_m = w$ such that $v_{s+1} = v_s + t^* Y_s$ for an extreme ray Y_s of C_{v_s} with $t^*(v_s, Y_s) < \infty$.*

Proof. Let $\ell(X) = \sum_{k \in \text{Min}(w)} X[k]$. Then $\ell \geq 0$ on C_w , and $\ell(X) = 0$ for $X \in C_w$ only if $X \in L_w = 0$. By compactness of $\{X \in C_w : \|X\| = 1\}$ there is $\delta > 0$ with $\ell(X) \geq \delta \|X\|$ on C_w . By Lemma 2.6(3), $\mathcal{R} \subseteq w + C_w$, so $\ell(J) > \ell(w)$ for $J \in \mathcal{R} \setminus \{w\}$, and for every c the set $S_c = \{J \in \mathcal{R} : \ell(J) \leq c\}$ is compact. A vertex J is the unique solution of the linear equations $J[k] = 4$, $k \in \text{Min}(J)$, and the vertices in S_c have $\text{Min}(J) \subseteq \mathcal{M}_{S_c}$, a finite set by Lemma 2.6(1). So S_c contains finitely many vertices.

Let u be a vertex with $u \neq w$. If $\ell(Y) \geq 0$ for every extreme ray of C_u , then $\ell \geq 0$ on C_u , and $\mathcal{R} \subseteq u + C_u$ gives $\ell(w) \geq \ell(u)$, which is false. So some extreme ray Y of C_u has $\ell(Y) < 0$. Along $u + tY$ the value of ℓ decreases, and $S_{\ell(u)}$ is compact, so $t^*(u, Y) < \infty$, and $u' = u + t^*Y$ is a vertex (Lemma 2.9) with $\ell(u') < \ell(u)$. Starting from $u = v$ and repeating, we obtain a sequence of vertices in $S_{\ell(v)}$ along which ℓ strictly decreases. It is finite, and it can only stop at w . $\square$

Corollary 2.12. *Let $J_0, \dots, J_N$ be vertices of $\mathcal{R}$ such that for every i and every extreme ray Y of C_{J_i} with $t^*(J_i, Y) < \infty$ the vertex $J_i + t^*Y$ is equivalent to some J_j . Then every vertex of $\mathcal{R}$ is equivalent to some J_j , and every edge of $\mathcal{R}$ is equivalent to an edge at some J_j .*

Proof. Let w be a vertex and take the path $J_0 = v_0, \dots, v_m = w$ of Proposition 2.11. If $v_s = T^\top J_i T$, then $Y' = T^{-\top} Y_s T^{-1}$ is an extreme ray of C_{J_i} with the same t^* , and $v_{s+1} = T^\top (J_i + t^* Y') T$, which is equivalent to some J_j by hypothesis. By induction w is equivalent to some J_j . An edge at a vertex $w = T^\top J_j T$ in direction Y is the image under T of the edge at J_j in direction $T^{-\top} Y T^{-1}$. $\square$

3. REDUCTION TO VERTICES AND EDGES

Lemma 3.1. *Let Q be positive definite and $H \in \text{Sym}_5$, $H \neq 0$. On the open interval of τ where $Q + \tau H$ is positive definite, $\psi(\tau) = \log \det(Q + \tau H)$ is strictly concave.*

Proof. With $Q_\tau = Q + \tau H$ we have $\psi'(\tau) = \text{tr}(Q_\tau^{-1} H)$ and $\psi''(\tau) = -\text{tr}((Q_\tau^{-1/2} H Q_\tau^{-1/2})^2) < 0$, because $Q_\tau^{-1/2} H Q_\tau^{-1/2}$ is a nonzero symmetric matrix. $\square$

Let

$$S = \{J \in \mathcal{R} : \sigma(J) \leq 0\}, \quad \mathcal{C} = \{J \in S : d(J) = 0\} \cup \{J \in \mathcal{R} : d(J) = 1, \sigma(J) = 0\}.$$

So $\mathcal{C}$ consists of the vertices with $\det J \leq 0$ and of the points with $\det J = 0$ in the relative interiors of edges (Lemma 2.9). By Lemma 2.4, the infimum of $\det Q$ over S equals the infimum of $\text{covol}(\Lambda)^2$ over the pairs of Theorem 1.1.

Proposition 3.2. *For every $J \in S$ there is $J' \in \mathcal{C}$ with $\det Q(J') \leq \det Q(J)$.*

Proof. Write $f(J) = \log \det Q(J)$ and $\pi(J) = 2d(J) + [\sigma(J) < 0]$, where $[\cdot]$ is 1 if the condition holds and 0 otherwise. We show: if $J \in S \setminus \mathcal{C}$, there is $J_1 \in S$ with $f(J_1) \leq f(J)$ and $\pi(J_1) < \pi(J)$. Induction on π then proves the proposition.

Let $J \in S \setminus \mathcal{C}$. Then either (a) $\sigma(J) < 0$ and $d(J) \geq 1$, or (b) $\sigma(J) = 0$ and $d(J) \geq 2$. In case (a) choose any $X \in L_J \setminus 0$. In case (b) let $u = (-q(J), 1) \in \mathbb{R}^6$, so that $J[u] = \sigma(J) = 0$ by Lemma 2.3(2), and choose $X \in L_J \setminus 0$ with $X[u] = 0$, which exists because $\dim L_J \geq 2$. Put $J(\tau) = J + \tau X$ and

$$I_0 = \{\tau \in \mathbb{R} : J(\tau) \in \mathcal{R}\}.$$

I_0 is a closed interval, it contains a neighbourhood of 0 by Lemma 2.6(2) because $\pm X \in C_J$, and $I_0 \neq \mathbb{R}$ by Lemma 2.7. Since $X[k] = 0$ for $k \in \text{Min}(J)$, we have $J(\tau)[k] = 4$ for these k , hence

$$(3.1) \quad \text{Min}(J) \subseteq \text{Min}(J(\tau)) \quad \text{and} \quad L_{J(\tau)} \subseteq L_J \quad (\tau \in I_0).$$

In case (b), Lemma 2.3(2) gives $\sigma(J(\tau)) \leq J(\tau)[u] = J[u] + \tau X[u] = 0$ for $\tau \in I_0$, and we put $I = I_0$. In case (a), $\tau \mapsto \sigma(J(\tau))$ is continuous on I_0 and negative at 0, and we let I be the connected component of $\{\tau \in I_0 : \sigma(J(\tau)) \leq 0\}$ containing 0. In both cases I is a closed interval, 0 is an interior point of I , $I \neq \mathbb{R}$, and $J(\tau) \in S$ for $\tau \in I$.

The function $\varphi(\tau) = f(J(\tau))$ is concave on I by Lemma 3.1 (it is constant if $Q(X) = 0$), and bounded below by Lemma 2.5. If $I = [\alpha, \beta]$ is bounded, concavity gives $\varphi(0) \geq \min(\varphi(\alpha), \varphi(\beta))$. If $I = [\alpha, \infty)$, then φ is nondecreasing, because a concave function that decreases somewhere on a half-line tends to $-\infty$; so $\varphi(\alpha) \leq \varphi(0)$, and similarly for $I = (-\infty, \beta]$. In every case there is a finite end point τ_0 of I with $\varphi(\tau_0) \leq \varphi(0)$. Put $J_1 = J(\tau_0)$. Then $J_1 \in S$ and $f(J_1) \leq f(J)$.

If τ_0 is an end point of I_0 , then $X \notin L_{J_1}$: otherwise $J_1 \pm \eta X \in \mathcal{R}$ for small $\eta > 0$ by Lemma 2.6(2). With (3.1), $d(J_1) \leq d(J) - 1$ and $\pi(J_1) \leq 2d(J) - 1 < \pi(J)$. If τ_0 is not an end point of I_0 , we are in case (a), $\sigma(J_1) = 0$ by the choice of I , and $\pi(J_1) = 2d(J_1) \leq 2d(J) < \pi(J)$. $\square$

4. THE VERTICES AND EDGES OF $\mathcal{R}$

Definition 4.1. For $u \in \mathbb{Z}^5$ and $a \in \mathbb{Z}$ let $p = (u, -a)$, $q = (u, -a - 1) \in \mathbb{Z}^6$ and $X_{u,a} = pq^\top + qp^\top$. Thus $Q(X_{u,a}) = 2uu^\top$ and

$$X_{u,a}[(n, l)] = 2(u \cdot n - al)(u \cdot n - (a + 1)l).$$

Lemma 4.2. $X_{u,a}[k] \geq 0$ for all $k \in \mathcal{M}$. Hence $J + tX_{u,a} \in \mathcal{R}$ for all $J \in \mathcal{R}$ and $t \geq 0$, and

$$\det Q(J + tX_{u,a}) = \det Q(J) + 2t \text{adj}(Q(J))[u].$$

Proof. For $k = (n, l)$ the two factors $u \cdot n - al$ and $u \cdot n - (a + 1)l$ are integers that differ by $l \in \{-1, 0, 1\}$; two such integers have a nonnegative product. The determinant formula is the matrix determinant lemma for the rank-one update $Q + 2t uu^\top$. $\square$

Proposition 4.3. *The matrices $J_0, \dots, J_{33}$ of Table A.1 are vertices of $\mathcal{R}$ with the invariants of Table 4.1. For each i let G_i be as in Table 4.1. Then:*

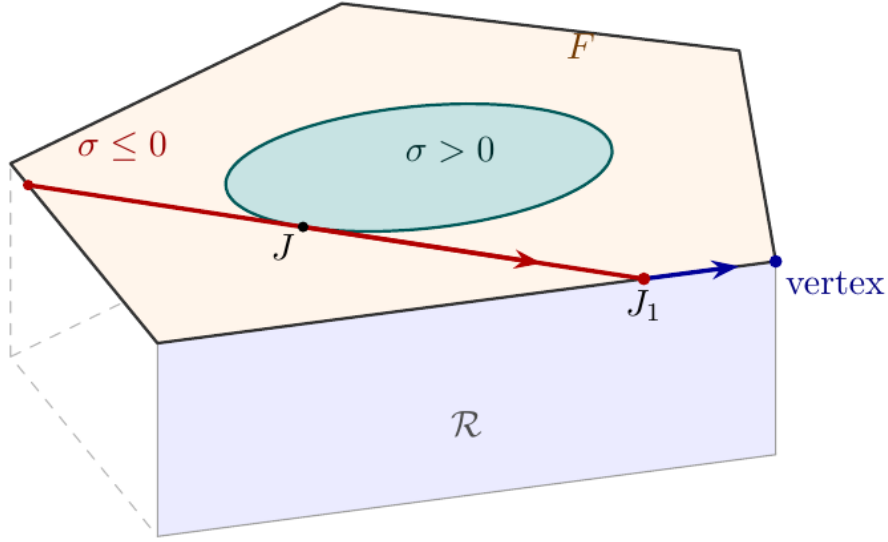


FIGURE 3.1. One step of the proof of Proposition 3.2 in a two-dimensional face F of $\mathcal{R}$, in case (b), $\sigma(J) = 0$. The set $F \cap \{\sigma > 0\}$ (shaded) is convex because σ is concave. On the line through J in the direction X , chosen with $X[u] = 0$, we have $\sigma \leq 0$, and $\log \det Q$ is concave on it, so one of the two ends of the chord, here J_1 , has $\det Q(J_1) \leq \det Q(J)$. The point J_1 lies on an edge of F . From there the same step moves along the edge to a vertex or to a point with $\sigma = 0$.

- (1) for every extreme ray Y of C_{J_i} with $t^*(J_i, Y) < \infty$, the vertex $J_i + t^*Y$ is equivalent to some J_j ; Table A.2 lists j for one ray of each G_i -orbit;
- (2) for every extreme ray Y of C_{J_i} and every $t \geq 0$ with $t \leq t^*(J_i, Y)$, either $\det(J_i + tY) > 0$ or $\det Q(J_i + tY) \geq 512$; and if $\det(J_i + tY) = 0$ and $\det Q(J_i + tY) = 512$, then $J_i + tY$ is equivalent to one of the matrices $E_0, \dots, E_4$ of Proposition 4.5;
- (3) every extreme ray Y of C_{J_i} with $t^*(J_i, Y) = \infty$ equals $X_{u,a}$ for some $u \in \mathbb{Z}^5$ and $a \in \{-2, -1, 0\}$, and is G_i -equivalent to one of the 25 rays of Table 4.2.

Proof. This is a finite computation in exact integer and rational arithmetic, described in Appendix B. The certificate lists, for each i , the minimal vectors of J_i , generators of G_i , all extreme rays of C_{J_i} , and one ray Y of each G_i -orbit; for a bounded edge it gives t^* , j and a matrix $T \in \Gamma$ with $T^T J_i T = J_i + t^*Y$, and for an unbounded edge the data (u, a) . For a ray gYg^T with $g \in G_i$, the edge at J_i in its direction is the image of the edge in direction Y under $J \mapsto g^T J g$ (Section 2.3), so the statements for the listed rays imply them for all rays. Statement (2) is decided on each listed edge from the polynomials $\det(J_i + tY)$, of degree at most 6, and $\det Q(J_i + tY)$, of degree at most 5, by Sturm sequences. $\square$

By Corollary 2.12 and Proposition 4.3(1), every vertex of $\mathcal{R}$ is equivalent to exactly one J_i , and every edge of $\mathcal{R}$ is equivalent to an edge at some J_i in the direction of one of the 480 listed rays.

Remark 4.4. The concavity of σ (Lemma 2.3(2)) locates the points of $\mathcal{C}$ on bounded edges. By Table 4.1, $\sigma(J_i)$ is negative for $i = 1, 2$, zero for $i = 0$ and positive otherwise. On a bounded edge whose end points have $\sigma \geq 0$, not both zero, σ is positive inside. So a point of $\mathcal{C}$ in the

i	$ \text{Min} $	$\det Q$	$\det J$	rays	orb.	U	i	$ \text{Min} $	$\det Q$	$\det J$	rays	orb.	U
0	32	512	0	1166	22	2	17	21	294	343	21	4	0
1	37	1024	-1024	373	7	1	18	21	288	351	21	8	1
2	30	768	-256	148	9	1	19	22	234	324	46	15	1
3	31	320	192	2621	38	2	20	21	252	351	21	10	0
4	24	512	256	56	6	1	21	21	384	448	21	3	0
5	36	128	192	38124	72	2	22	30	128	256	6336	20	1
6	25	324	243	261	11	1	23	27	162	243	621	11	1
7	29	256	256	2273	23	2	24	22	216	324	46	11	0
8	35	192	192	21196	56	2	25	22	180	324	46	11	0
9	24	432	216	56	8	1	26	22	162	324	46	6	0
10	21	378	324	21	11	1	27	21	180	351	21	8	0
11	30	192	256	6336	40	1	28	21	196	343	21	4	0
12	27	384	256	515	14	1	29	21	216	351	21	6	0
13	25	576	192	64	6	1	30	21	234	351	21	6	0
14	22	288	324	46	11	0	31	21	162	351	21	3	0
15	22	324	324	46	5	1	32	21	320	448	21	3	1
16	21	306	351	21	10	0	33	21	192	448	21	2	0

TABLE 4.1. The 34 vertices J_i of $\mathcal{R}$ up to equivalence; their entries are in Table A.1. $|\text{Min}|$ counts pairs $\pm k$; “rays” is the number of extreme rays of C_{J_i} , “orb.” the number of their orbits under the group $G_i \subseteq \Gamma_{J_i}$ generated by the automorphisms of J_i listed in the certificate, and U the number of these orbits that are unbounded edges. In all, 80,695 rays in 480 orbits, 25 of them unbounded. The pairs $(\det Q, \det J)$ are distinct, so no two J_i are equivalent.

interior of a bounded edge lies on an edge at J_1 or J_2 , or on an edge joining two vertices of class 0. There are two classes of the latter. On one, $\det J = 16t(t-4)(t-2)(t+2)$ with $t^* = 2$, which is positive inside. On the other, $\det J \equiv 0$ and $\det Q = 512(1+\tau-\tau^2)$ with $\tau = t/t^*$; this is the edge of [1, §4.3] contained in the set of packings, with $\det Q > 512$ inside. On ten classes of edges at J_1 or J_2 , $\det J$ has one root inside, where $\det Q$ equals 648, 673.59..., 697.00..., 711.11..., 714.19..., 715.45..., 734.56..., 737.28, 738.50... and 823.04... respectively; on the other edges at J_1 and J_2 , $\det J < 0$ inside. Proposition 4.3(2) records these facts; they are not used separately.

Proposition 4.5. *Let*

$E_0 = J_0$, $E_1 = J_3 + \frac{6}{5}X_{e_1,-1}$, $E_2 = J_5 + 3X_{e_5,0}$, $E_3 = J_7 + 2X_{e_5,0}$, $E_4 = J_8 + 2X_{e_1,-1}$.
Every $J \in \mathcal{C}$ satisfies $\det Q(J) \geq 512$, with equality if and only if J is equivalent to one of $E_0, \dots, E_4$.

Proof. Let $J \in \mathcal{C}$. If $d(J) = 0$, then J is equivalent to some J_i with $\sigma(J_i) \leq 0$, that is, $\det J_i \leq 0$. By Table 4.1, $i \in \{0, 1, 2\}$ and $\det Q(J) \in \{512, 1024, 768\}$, with 512 only for $J_0 = E_0$.

If $d(J) = 1$ and $\sigma(J) = 0$, Lemma 2.9(3) places J in the relative interior of an edge, and after applying an element of Γ we may assume $J = J_i + tY$ with Y an extreme ray of C_{J_i} and $0 < t < t^*(J_i, Y)$. Then $\det J = \det Q(J) \sigma(J) = 0$, and Proposition 4.3(2) gives $\det Q(J) \geq 512$, with equality only if J is equivalent to some E_m .

i	(u, a)	N	$\det Q(J_i + tX_{u,a})$	$\det(J_i + tX_{u,a})$	t_0	$\det Q$ at t_0
0	$(e_2, -1)$	4	$128(3t + 4)$	$-64t(3t + 4)$	—	—
0	$(e_4, 0)$	2	$256(t + 2)$	$-128t(t + 2)$	—	—
1	$(e_1, -1)$	5	$512(t + 2)$	$-256(t + 2)^2$	—	—
2	$(e_2, -1)$	3	$384(t + 2)$	$-64(t + 2)(3t + 2)$	—	—
3	$((1, -1, -1, 0, 0), -1)$	5	$64(4t + 5)$	$-64(2t^2 - 3)$	$\sqrt{6}/2$	$320 + 128\sqrt{6}$
3	$(e_1, -1)$	1	$160(t + 2)$	$-16(t + 2)(5t - 6)$	$6/5$	512
4	$(e_4, 0)$	2	$256(t + 2)$	$-128(t - 1)(t + 2)$	1	768
5	$(e_1, -1)$	16	$32(5t + 4)$	$-16(5t^2 - 12t - 12)$	$(6 + 4\sqrt{6})/5$	$320 + 128\sqrt{6}$
5	$(e_5, 0)$	5	$128(t + 1)$	$-64(t - 3)(t + 1)$	3	512
6	$(e_5, 0)$	4	$108(2t + 3)$	$-27(2t - 3)(2t + 3)$	$3/2$	648
7	$((1, 1, 0, -1, 0), -2)$	8	$256(t + 1)$	$-128(t - 2)(t + 1)$	2	768
7	$(e_5, 0)$	1	$128(t + 2)$	$-64(t - 2)(t + 2)$	2	512
8	$((0, 1, 1, 0, 1), 0)$	10	$96(3t + 2)$	$-48(t - 2)(3t + 2)$	2	768
8	$(e_1, -1)$	6	$32(5t + 6)$	$-16(t - 2)(5t + 6)$	2	512
9	$(e_5, 0)$	1	$216(t + 2)$	$-108(t - 1)(t + 2)$	1	648
10	$(e_5, 0)$	1	$54(4t + 7)$	$-108(t^2 - 3)$	$\sqrt{3}$	$378 + 216\sqrt{3}$
11	$(e_4, 0)$	6	$32(5t + 6)$	$-16(5t^2 - 8t - 16)$	$(4 + 4\sqrt{6})/5$	$320 + 128\sqrt{6}$
12	$(e_2, -1)$	4	$96(3t + 4)$	$-16(3t - 4)(3t + 4)$	$4/3$	768
13	$(e_2, -1)$	1	$288(t + 2)$	$-48(t + 2)(3t - 2)$	$2/3$	768
15	$(e_1, -1)$	1	$162(t + 2)$	$-81(t - 2)(t + 2)$	2	648
18	$(e_1, -1)$	1	$18(9t + 16)$	$-27(3t^2 - 2t - 13)$	$(1 + 2\sqrt{10})/3$	$342 + 108\sqrt{10}$
19	$(e_1, -1)$	1	$18(9t + 13)$	$-27(3t^2 - 4t - 12)$	$(2 + 2\sqrt{10})/3$	$342 + 108\sqrt{10}$
22	$(e_1, -1)$	16	$32(5t + 4)$	$-16(t - 4)(5t + 4)$	4	768
23	$((0, 1, 0, -1, -1), -2)$	10	$162(t + 1)$	$-81(t - 3)(t + 1)$	3	648
32	$(e_5, 0)$	1	$160(t + 2)$	$-16(t + 2)(5t - 14)$	$14/5$	768

TABLE 4.2. The unbounded edges $J_i + tX_{u,a}$, $t \geq 0$, one per G_i -orbit; N is the size of the orbit, and the N add up to 115. The last two columns give the positive root t_0 of $\det(J_i + tX_{u,a})$ and the value of $\det Q$ there. Each row is a direct computation from Table A.1 (the $\det Q$ column by Lemma 4.2); $320 + 128\sqrt{6} \approx 633.5$, $342 + 108\sqrt{10} \approx 683.5$, $378 + 216\sqrt{3} \approx 752.1$.

Conversely, $E_0 = J_0$ is a vertex with $\det J_0 = 0$ and $\det Q(J_0) = 512$. For $m \geq 1$, E_m is the point at t_0 on one of the four rows of Table 4.2 with $\det Q(t_0) = 512$; it lies in the relative interior of an unbounded edge, so $d(E_m) = 1$ by Lemma 2.9(2), and $\sigma(E_m) = \det E_m / \det Q(E_m) = 0$. $\square$

Corollary 4.6. $\det Q(J) \geq 512$ for every $J \in S$.

Proof. Propositions 3.2 and 4.5. $\square$

5. THE EQUALITY CASE

Lemma 5.1. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, $\rho = \max_{z \in \mathbb{R}^n} \text{dist}(z, \Lambda)$ its covering radius, and $c \in \mathbb{R}^n$ with $\text{dist}(c, \Lambda) = \rho$. Then the set $N(c) = \{v \in \Lambda : |v - c| = \rho\}$ is not contained in any affine hyperplane.

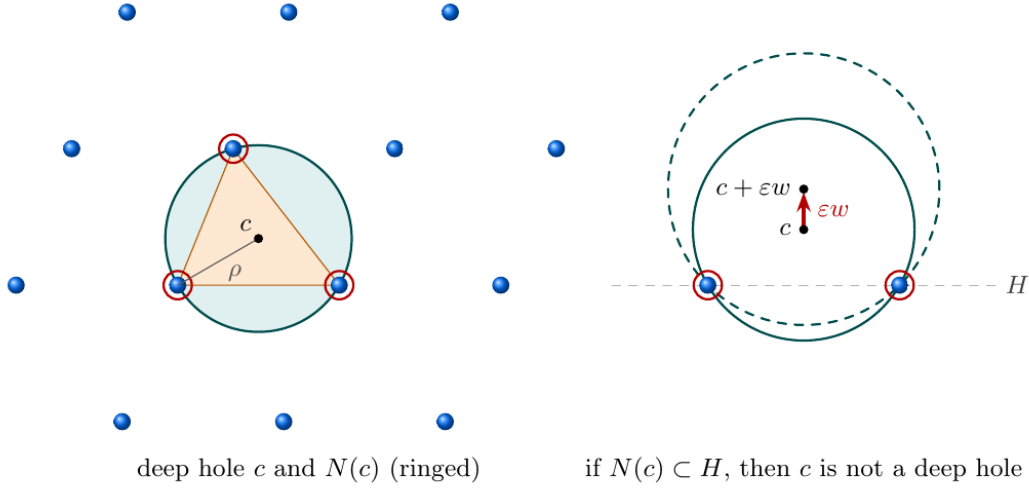


FIGURE 5.1. Lemma 5.1 in the plane. Left: a deep hole c of a lattice; the open disc of radius ρ about c contains no lattice point, and the points $N(c)$ on its boundary span the plane affinely. Right: if $N(c)$ lay on a line H , moving c away from H by a small εw would increase the distance to every point of $N(c)$, and to the whole lattice, beyond ρ .

Proof. The maximum exists because $z \mapsto \text{dist}(z, \Lambda)$ is continuous and Λ -periodic. $N(c)$ is finite and nonempty, and since Λ is discrete there is $\delta > 0$ with $|v - c| \geq \rho + \delta$ for $v \in \Lambda \setminus N(c)$. Suppose $\langle v, z \rangle = \alpha$ for all $v \in N(c)$, with $|z| = 1$. Put $\beta = \alpha - \langle c, z \rangle$, so that $\langle v - c, z \rangle = \beta$ on $N(c)$, and let $w = -z$ if $\beta > 0$ and $w = z$ otherwise. For $0 < \varepsilon < \delta$ and $v \in N(c)$,

$$|v - c - \varepsilon w|^2 = \rho^2 - 2\varepsilon \langle v - c, w \rangle + \varepsilon^2 = \rho^2 + 2\varepsilon |\beta| + \varepsilon^2 > \rho^2,$$

and for $v \in \Lambda \setminus N(c)$, $|v - c - \varepsilon w| \geq \rho + \delta - \varepsilon > \rho$. So $\text{dist}(c + \varepsilon w, \Lambda) > \rho$, contradicting the definition of ρ . $\square$

A point c with $\text{dist}(c, \Lambda) = \rho$ is a *deep hole* of Λ .

Lemma 5.2. Let $J \in \text{Sym}_6^+$, let A be a real 5×5 matrix with $A^\top A = Q(J)$, and put $\Lambda = A\mathbb{Z}^5$ and $c = -Aq(J)$. Then $J[(n, 0)] = |An|^2$ and $J[(n, 1)] = |An - c|^2 + \sigma(J)$ for $n \in \mathbb{Z}^5$. In particular $J \in \mathcal{R}$ if and only if $|v| \geq 2$ for $v \in \Lambda \setminus 0$ and $\text{dist}(c, \Lambda)^2 \geq 4 - \sigma(J)$; and for $J \in \mathcal{R}$ we have $(n, 1) \in \text{Min}(J)$ if and only if $|An - c|^2 = 4 - \sigma(J)$.

Proof. By Lemma 2.3(1), $J[(n, l)] = |A(n + lq(J))|^2 + l^2 \sigma(J) = |An - lc|^2 + l^2 \sigma(J)$. The vectors $(n, -1)$ give the same values as $(-n, 1)$. $\square$

Proposition 5.3. Let $J \in \mathcal{S}$ with $\det Q(J) = 512$. Then $\sigma(J) = 0$ and $d(J) \leq 1$. Hence $J \in \mathcal{C}$, and J is equivalent to one of $E_0, \dots, E_4$.

Proof. We first record two consequences of Corollary 4.6 for any $J' \in \mathcal{S}$ with $\det Q(J') = 512$ and any $X \in L_{J'}$ that satisfies $\sigma(J') < 0$ or $X[u'] = 0$, where $u' = (-q(J'), 1)$:

- (a) $Q(X) = 0$;
- (b) writing $X = \begin{pmatrix} 0 & x \\ x^\top & y \end{pmatrix}$, we have $2x \cdot n + y = 0$ for every $(n, 1) \in \text{Min}(J')$.

For (a): $J' + \tau X \in \mathcal{R}$ for small $|\tau|$ by Lemma 2.6(2), since $\pm X \in C_{J'}$. If $\sigma(J') < 0$, then $\sigma(J' + \tau X) < 0$ for small $|\tau|$ by continuity. If $X[u'] = 0$, then $\sigma(J' + \tau X) \leq (J' + \tau X)[u'] =$

m	a_1, a_2, a_3, a_4, a_5	b
0	$(01\bar{1}00)(10100)(1\bar{1}000)(0001\bar{1})(00011)$	$(010\bar{1}0)$
1	$\frac{1}{5}(1\bar{9}\bar{4}\bar{4}\bar{4})(\bar{1}0001)(\bar{1}1000)(0\bar{1}010)(0\bar{1}100)$	$\frac{1}{5}(1\bar{4}\bar{4}\bar{4}1)$
2	$(10100)(01010)(00\bar{1}10)(00\bar{1}\bar{1}0) - \frac{1}{2}(11114)$	(11211)
3	$(010\bar{1}0)(10\bar{1}00)(1\bar{1}000)(00110)(00002)$	$(01\bar{1}\bar{1}\bar{1})$
4	$(\bar{1}\bar{1}\bar{1}0\bar{1})(0100\bar{1})(010\bar{1}0)(1\bar{1}000)(001\bar{1}0)$	$(\bar{1}0\bar{1}1\bar{1})$

TABLE 6.1. Bases of Λ_m and vectors $b \equiv b_m \pmod{\Lambda_m}$; the Gram matrix of $\sqrt{2}(a_1, \dots, a_5, b)$ is E_m . Coordinates are written without commas, and $k = -k$.

$\sigma(J') \leq 0$ by Lemma 2.3(2). In both cases $J' + \tau X \in S$ on an interval around 0, where $\varphi(\tau) = \log \det Q(J' + \tau X)$ is concave, at least $\log 512$ by Corollary 4.6, and equal to $\log 512$ at $\tau = 0$. A concave function with an interior minimum on an interval is constant there, so $Q(X) = 0$ by Lemma 3.1. For (b): with $Q(X) = 0$ we have $X[(n, 1)] = 2x \cdot n + y$, which vanishes on $\text{Min}(J')$ because $X \in L_{J'}$.

Now let J be as in the proposition, and let A, Λ, c be as in Lemma 5.2. Let ρ be the covering radius of Λ . Since $J \in \mathcal{R}$ and $\sigma(J) \leq 0$, Lemma 5.2 gives $\rho^2 \geq \text{dist}(c, \Lambda)^2 \geq 4 - \sigma(J) \geq 4$.

Suppose $\rho > 2$. Choose a deep hole c' of Λ , put $q' = -A^{-1}c'$ and $\sigma' = 4 - \rho^2 < 0$, and let

$$J' = \begin{pmatrix} Q & Qq' \\ q'^T Q & \sigma' + Q[q'] \end{pmatrix}, \quad Q = Q(J).$$

Then $Q(J') = Q$, $q(J') = q'$ and $\sigma(J') = \sigma'$. By Lemma 5.2, $J' \in \mathcal{R}$, because Λ has minimal distance at least 2 and $\text{dist}(c', \Lambda)^2 = \rho^2 = 4 - \sigma'$; moreover $(n, 1) \in \text{Min}(J')$ whenever $An \in N(c')$. So $J' \in S$ and $\det Q(J') = 512$. Let $X \in L_{J'}$. Since $\sigma(J') < 0$, (a) and (b) apply: $Q(X) = 0$, and the affine function $v \mapsto 2x \cdot A^{-1}v + y$ vanishes on $N(c')$. By Lemma 5.1, $N(c')$ is not contained in an affine hyperplane, so $x = 0$ and $y = 0$, that is, $X = 0$. Hence $d(J') = 0$: J' is a vertex with $\det J' = 512\sigma' < 0$. By Proposition 4.3 it is equivalent to J_1 or J_2 , the only J_i with $\det J_i < 0$; but $\det Q(J_1) = 1024$ and $\det Q(J_2) = 768$. This contradiction shows $\rho = 2$.

Then $4 \geq \text{dist}(c, \Lambda)^2 \geq 4 - \sigma(J)$ gives $\sigma(J) = 0$ and $\text{dist}(c, \Lambda) = 2 = \rho$, so c is a deep hole of Λ , and $(n, 1) \in \text{Min}(J)$ if and only if $An \in N(c)$. Let $u = (-q(J), 1)$ and let $X \in L_J$ with $X[u] = 0$. By (a) and (b), $Q(X) = 0$ and $v \mapsto 2x \cdot A^{-1}v + y$ vanishes on $N(c)$, so $X = 0$ by Lemma 5.1. Thus the linear map $L_J \rightarrow \mathbb{R}$, $X \mapsto X[u]$, is injective, and $d(J) \leq 1$. So $J \in \mathcal{C}$, and Proposition 4.5 completes the proof. $\square$

6. THE FIVE EXTREMAL CONFIGURATIONS

Lemma 6.1. *For each m , the vectors $a_1, \dots, a_5$ of Table 6.1 form a basis of Λ_m , the vector b satisfies $b - b_m \in \Lambda_m$, and the Gram matrix of $\sqrt{2}a_1, \dots, \sqrt{2}a_5, \sqrt{2}b$ is E_m . Each Λ_m has covolume 4, minimal norm 2 and $|\text{Min } \Lambda_m|$ as in Table 1.1.*

Proof. Λ_0, Λ_3 and Λ_4 have index 4 in $\mathbb{Z}^5$. Λ_1 is the orthogonal sum of A_4 , of determinant 5, and $\mathbb{Z}_5^4 \mathbf{1}$, of norm $\frac{16}{5}$; Λ_2 consists of the layers $x_5 = 2j$, each a translate of D_4 , of covolume 2, at spacing 2. So all five have covolume 4. The vectors a_i lie in Λ_m and $|\det(a_1, \dots, a_5)| = 4$, so they form a basis; $b - b_m \in \Lambda_m$ and the Gram matrices are direct computations. The norm-2 vectors are: the $4 \cdot 3 + 4 \cdot 1 = 16$ vectors $\pm e_i \pm e_j$ of $D_3 \oplus D_2$; the 20 roots of A_4 , since Λ_1 has no other vector of norm less than $\frac{16}{5}$; the 24 roots of $D_4 \times 0$, since a vector of Λ_2 with $x_5 = 2j \neq 0$ has norm at least $1 + 4 = 5$; the 24 roots of $D_4 \times 0$ in Λ_3 ; and the 20 vectors $e_i - e_j$ in Λ_4 . $\square$

- Lemma 6.2.** (1) $P_0 = P_3 = P_4 = D_5$.
 (2) For $m = 1, 2$, $2b_m \notin \Lambda_m$; the set P_m is not a lattice, and its group of translations $\{y : P_m + y = P_m\}$ is Λ_m .
 (3) The five pairs (Λ_m, b_m) are pairwise not congruent, and P_1, P_2, D_5 are pairwise not congruent.

Proof. (1) $\Lambda_m \subset D_5$ for $m = 0, 3, 4$, with index 2 since $\text{covol}(D_5) = 2$, and $b_m \in D_5 \setminus \Lambda_m$. So P_m is the union of the two cosets of Λ_m in D_5 . (2) $2b_1 = 2e_1 + 2e_5$ has coordinate sum 4, so its component along $\mathbf{1}$ is $\frac{4}{5}\mathbf{1}$, and $2b_1 - \frac{4}{5}\mathbf{1} = \frac{2}{5}(3, -2, -2, -2, 3) \notin \mathbb{Z}^5$; so $2b_1 \notin \Lambda_1$. $2b_2 = 2e_1 + 2e_5$ has $x_5 = 2$, while the vectors of Λ_2 with $x_5 = 2$ lie in $(D_4 + h) \times \{2\}$, which does not contain $2e_1$. Let y be a translation of $P = \Lambda \cup (\Lambda + b)$, where $b \notin \Lambda$ and $2b \notin \Lambda$. Then $y = 0 + y \in P$. If $y \in \Lambda + b$, then $y + b \in \Lambda + 2b$, which meets neither Λ nor $\Lambda + b$. So $y \in \Lambda$, and conversely every $y \in \Lambda$ is a translation. As $P \neq \Lambda$, P is not a lattice. (3) A congruence of pairs maps Λ onto Λ' , so it preserves $|\text{Min } \Lambda|$, and it preserves whether $2b \in \Lambda$. The pairs $(|\text{Min } \Lambda_m|, [2b_m \in \Lambda_m])$ are $(16, 1)$, $(20, 0)$, $(24, 0)$, $(24, 1)$, $(20, 1)$, all distinct. A congruence of point sets carries the group of translations of one onto that of the other. D_5 is a lattice and P_1, P_2 are not, and a congruence $P_1 \rightarrow P_2$ would map Λ_1 onto Λ_2 , which have 20 and 24 vectors of minimal norm. $\square$

Remark 6.3. (1) Every splitting of D_5 into the two cosets of a sublattice of index two satisfies the packing condition, so the theorem lists each such splitting up to congruence. D_5 has 31 sublattices of index two, the kernels of the 31 nonzero homomorphisms $D_5 \rightarrow \mathbb{Z}/2$: $x \mapsto x_i$, $x \mapsto x_i + x_j$ and $x \mapsto \frac{1}{2} \sum_i \varepsilon_i x_i$ modulo 2, with $\varepsilon \in \{\pm 1\}^5$ up to sign; there are 5, 10 and 16 of them. Signed permutations of the coordinates map them to Λ_3, Λ_0 and Λ_4 respectively.

(2) Translations by Λ and the point reflection $x \mapsto b - x$ preserve $P = \Lambda \cup (\Lambda + b)$, and together they act transitively on P , so every packing of the theorem is uniform. In P_1 and P_2 every point has 40 neighbours at distance $\sqrt{2}$. The isometry $x \mapsto -(x_4, x_2, x_3, x_1, x_5)$ maps Λ_1 onto itself and b_1 onto $-e_4 - e_5$, so it maps P_1 onto the packing given by the lattice basis $e_1 - e_2, \dots, e_4 - e_5, \frac{4}{5}\mathbf{1}$ and the translation vectors $0, (0, 0, 0, -1, -1)$ in [6, Table 4.2]: the uniform packing with kissing configuration Q_5 , which is $J_{5,5}$ of [1] [6, §1].

(3) $P_2 = \bigcup_{i \in \mathbb{Z}} (D_4 + c_i) \times \{i\}$ with $c_i = 0, e_1, h, e_1 + h$ for $i \equiv 0, 1, 2, 3 \pmod{4}$. The neighbours of 0 are the 24 roots of D_4 in the layer $i = 0$, the 8 points $(\pm e_k, 1)$, $1 \leq k \leq 4$, and the 8 points $(\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1), -1)$ with an odd number of minus signs. This is Leech's configuration L_5 [12], as described in [6, §2], reflected in the hyperplane $x_5 = 0$. By Lemma 6.2(2), P_2 is 2-periodic in the sense of [6], so by [6, Theorem 4.3] and [7, §5], $\sqrt{2}P_2$ is the 2-periodic one of Leech's three uniform packings.

7. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

Proof of Theorem 1.1. Let Λ and b be as in the theorem, a a basis of Λ and $J = J(a, b)$. By Lemmas 2.1 and 2.4, $J \in \mathcal{R}$ and $\sigma(J) = 0$, so $J \in S$, and $\det Q(J) = \text{covol}(\Lambda)^2$. Corollary 4.6 gives $\text{covol}(\Lambda) \geq \sqrt{512} = 16\sqrt{2}$.

If $\text{covol}(\Lambda) = 16\sqrt{2}$, then $\det Q(J) = 512$, and Proposition 5.3 shows that J is equivalent to some E_m . By Lemma 6.1, $E_m = J(\sqrt{2}a^{(m)}, \sqrt{2}b^{(m)})$, where $a^{(m)}, b^{(m)}$ are the vectors of Table 6.1, $a^{(m)}$ is a basis of Λ_m and $\Lambda_m + b^{(m)} = \Lambda_m + b_m$. By Lemma 2.10, (Λ, b) is congruent to $(\sqrt{2}\Lambda_m, \sqrt{2}b_m)$.

Conversely, $E_m \in \mathcal{C} \subset \mathcal{R}$ and $\sigma(E_m) = 0$ (Proposition 4.5), so by Lemmas 2.4 and 6.1 the pair $(\sqrt{2}\Lambda_m, \sqrt{2}b_m)$ satisfies the hypotheses: $b_m \notin \Lambda_m$, and $\text{covol}(\sqrt{2}\Lambda_m)^2 = \det Q(E_m) = 512$. The five pairs are pairwise not congruent by Lemma 6.2(3).

Finally, P has two points per fundamental domain of Λ , so the density of the packing of unit balls centred at P is $2 \cdot \frac{8\pi^2}{15} / \text{covol}(\Lambda) \leq \pi^2 / (15\sqrt{2})$. Equality holds exactly when (Λ, b) is congruent to some $(\sqrt{2}\Lambda_m, \sqrt{2}b_m)$, that is, when P is congruent to some $\sqrt{2}P_m$; by Lemma 6.2, these sets are $\sqrt{2}D_5$, $\sqrt{2}P_1$ and $\sqrt{2}P_2$. $\square$

Proof of Corollary 1.2. If $P = L$ is a lattice with basis $l_1, \dots, l_5$, let Λ be the sublattice of index two spanned by $2l_1, l_2, \dots, l_5$ and $b = l_1$; then $P = \Lambda \cup (\Lambda + b)$. So in both cases $P = \Lambda \cup (\Lambda + b)$ with $b \notin \Lambda$. The map $x \mapsto b - x$ is an isometry of P exchanging Λ and $\Lambda + b$; it maps the Voronoi cell V_0 of 0 onto the cell V_b of b . Translations by Λ permute the cells, and $V_0 \cup V_b$ is a fundamental domain of Λ up to a null set. Hence every cell has volume $\frac{1}{2} \text{covol}(\Lambda) \geq 8\sqrt{2}$ by Theorem 1.1, with equality exactly for the sets listed there. The Voronoi cell of $\sqrt{2}D_5$ has volume $\text{covol}(\sqrt{2}D_5) = 2 \cdot 2^{5/2} = 8\sqrt{2}$. $\square$

APPENDIX A. TABLES

APPENDIX B. COMPUTATION AND VERIFICATION

The computation behind Proposition 4.3 follows [1] and Voronoi's algorithm [17, 15], in exact arithmetic throughout. The programs, the data, the certificate and the Lean formalization are archived at Zenodo [4].

Enumeration. Starting from J_0 , for each new vertex J we compute $\text{Min}(J)$ by the method of Fincke and Pohst [9], applied to the decomposition of Lemma 2.3(1); we compute the extreme rays of C_J by the double description method [10] in exact rational arithmetic; we compute automorphisms of J in Γ and the orbits of the rays under the group G they generate; for one ray Y per orbit we decide whether $t^* = \infty$ (Lemma 4.2) and otherwise find t^* exactly; and we test Γ -equivalence of $J + t^*Y$ with the known vertices by a search over images of minimal vectors. The search closes after 34 classes, with 80,695 rays in 480 orbits. The certificate records, for each vertex, its entries, its minimal vectors, generators of G_i and all extreme rays of C_{J_i} as primitive integer vectors; for each orbit, a representative ray with t^* , j and a matrix $T \in \Gamma$ with $T^\top J_j T = J_i + t^*Y$ (bounded) or its data (u, a) (unbounded); and the five points E_m .

Independent checks of the certificate. Four independent implementations, in C and C++ with rational arithmetic on multiprecision integers, in Julia and in Python with exact rationals, read the certificate and check, without using the enumeration: that each J_i lies in $\mathcal{R}$ and has exactly the listed minimal vectors, by an enumeration of all $k \in \mathcal{M}^*$ with $J_i[k] \leq 4$ from an exact $LDL^\top$ decomposition; that these have rank 21; that the listed generators lie in Γ and fix J_i ; that the listed rays are distinct, primitive, lie in C_{J_i} and are extreme (rank 20); that the generators permute them and that the representatives meet each orbit once, with the stated orbit sizes; for each bounded representative, that $J_i + t^*Y \in \mathcal{R}$, that some minimal vector k of $J_i + t^*Y$ has $Y[k] < 0$, and that $T^\top J_j T = J_i + t^*Y$ with $T \in \Gamma$; for each unbounded representative, that $Y = X_{u,a}$ and $Y[k] \geq 0$ on $\mathcal{M}$; and, on every representative edge, statement (2) of Proposition 4.3, by computing $\det(J_i + tY)$ and $\det Q(J_i + tY)$ by interpolation and counting the roots of $\det(J_i + tY)$ in $(0, t^*)$ with $\det Q < 512$, $= 512$ and > 512 by Sturm sequences. They find the five points $E_0, \dots, E_4$ and no other point with $\det J = 0$ and $\det Q \leq 512$.

	11	12	13	14	15	16	22	23	24	25	26	33	34	35	36	44	45	46	55	56	66
J_0	4	-2	-2	0	0	2	4	2	0	0	0	4	0	0	-2	4	0	-2	4	-2	4
J_1	4	0	0	0	0	2	4	0	0	0	2	4	0	0	-2	4	0	-2	4	-2	4
J_2	4	0	-2	0	0	4	4	0	0	0	2	4	0	0	-4	4	0	-2	4	-2	8
J_3	8	-2	-4	2	2	4	4	2	0	0	0	4	-2	-2	-2	4	2	0	4	0	4
J_4	4	0	-2	0	0	4	4	2	0	0	2	4	0	0	-2	4	0	-2	4	-2	8
J_5	4	0	-2	-2	-2	6	4	2	-2	-2	4	4	0	0	-2	4	2	-6	4	-6	16
J_6	4	-1	-2	1	1	1	4	2	-1	-1	2	4	-2	-2	1	4	1	-2	4	-2	4
J_7	4	0	-2	-2	0	4	4	2	-2	0	2	4	0	0	-2	4	0	-4	4	-2	8
J_8	4	0	-2	0	-2	4	4	2	-2	0	2	4	-2	2	-2	4	0	-2	4	-4	8
J_9	4	-1	-2	1	0	2	4	2	-1	0	1	4	-2	0	-1	4	0	-1	4	-2	4
J_{10}	4	-1	-2	1	0	2	4	2	-1	1	1	4	-2	1	-1	4	-1	-1	4	-1	4
J_{11}	4	0	-2	0	0	4	4	2	-2	2	2	4	-2	2	-2	4	-2	-2	4	0	8
J_{12}	4	-2	-2	0	0	2	4	2	0	0	0	4	0	0	-2	4	2	-4	4	-4	8
J_{13}	4	0	-2	0	0	4	4	0	0	0	2	4	0	0	-4	4	2	-4	4	-4	12
J_{14}	8	-3	-4	2	3	3	4	2	1	0	-1	4	-2	-2	-2	6	3	-1	4	0	4
J_{15}	8	-1	-4	1	2	5	4	1	2	1	0	4	-1	-2	-3	4	2	0	4	0	6
J_{16}	6	0	-3	3	3	3	4	1	1	2	1	4	-2	-2	-2	4	3	1	6	1	4
J_{17}	6	-2	-3	2	2	2	4	2	0	1	0	4	-1	-1	-2	4	2	-1	4	0	4
J_{18}	6	-1	-3	2	1	3	4	1	1	2	0	4	-2	-1	-2	4	2	0	4	0	4
J_{19}	6	0	-3	3	2	3	4	1	1	2	1	4	-2	-1	-2	4	2	1	4	1	4
J_{20}	10	-2	-5	4	2	5	4	2	0	1	0	4	-2	-1	-3	4	2	1	4	0	6
J_{21}	4	0	-2	0	0	4	4	2	0	0	2	4	0	0	-2	4	2	-4	4	-4	12
J_{22}	4	-2	-4	0	0	2	8	6	-4	-4	8	8	-2	-2	2	4	2	-6	4	-6	16
J_{23}	4	-1	-3	-1	-1	4	4	3	-2	-2	2	6	0	0	-3	4	1	-4	4	-4	10
J_{24}	6	0	-3	-3	-3	9	6	2	-4	-2	7	4	0	1	-4	6	3	-10	4	-8	26
J_{25}	8	0	-3	-4	-4	12	4	2	-3	-1	4	4	0	1	-4	6	3	-10	4	-8	26
J_{26}	4	-1	-3	-1	-1	4	4	3	-2	-2	3	6	0	0	-2	4	1	-4	4	-5	12
J_{27}	8	0	-4	-4	-5	11	4	3	-3	-2	4	6	0	1	-3	6	4	-9	6	-10	22
J_{28}	6	1	-2	-4	-4	9	4	3	-3	-3	6	6	-1	-1	0	6	4	-9	6	-10	22
J_{29}	6	1	-1	-4	-4	9	6	3	-5	-4	8	4	-2	-1	1	8	5	-11	6	-11	24
J_{30}	8	1	-4	-5	-5	12	4	2	-3	-2	6	6	1	1	-4	6	4	-11	6	-10	26
J_{31}	6	0	-3	-3	-3	8	4	1	-2	-1	5	4	1	2	-4	4	2	-7	4	-6	20
J_{32}	4	2	-2	-2	0	6	8	2	-4	0	8	4	0	0	-2	4	0	-6	4	-2	16
J_{33}	4	0	-2	0	0	4	4	2	-2	2	2	4	-2	2	-2	4	-2	-2	4	-2	12
E_1	$\frac{52}{5}$	-2	-4	2	2	$\frac{26}{5}$	4	2	0	0	0	4	-2	-2	-2	4	2	0	4	0	4
E_2	4	0	-2	-2	-2	6	4	2	-2	-2	4	4	0	0	-2	4	2	-6	10	-9	16
E_3	4	0	-2	-2	0	4	4	2	-2	0	2	4	0	0	-2	4	0	-4	8	-4	8
E_4	8	0	-2	0	-2	6	4	2	-2	0	2	4	-2	2	-2	4	0	-2	4	-4	8

TABLE A.1. The entries J_{ab} , $a \leq b$, of the vertices $J_0, \dots, J_{33}$ and of the matrices $E_1, \dots, E_4$ of Proposition 4.5; $E_0 = J_0$.

Completeness of the extreme rays. The certificate lists all extreme rays of each cone C_{J_i} , and the checks above confirm that every listed ray is extreme. That the lists are complete is confirmed four times, by four enumerations of the extreme rays of all 34 cones whose outputs agree with the certificate: an implementation of the double description method with the adjacency test of Fukuda and Prodon [10], an implementation of the reverse search method of Avis and Fukuda [2], and two further implementations of the double description method written independently, in C and in Julia, which start from a simplicial cone, add the inequalities one at a time and keep exactly the extreme rays by the combinatorial adjacency test.

The Lean formalization. A proof in Lean 4 with Mathlib [14, 13], in which every step is checked by the kernel with no evaluation by compiled code and which uses only the axioms

i	classes j of the end points $J_i + t^*Y$ of the listed edges at J_i
0	$0^2 1 2 3^3 4 5^3 6 7 8^3 9 10 11 12 \infty^2$
1	$0 2 3 6 8 11 \infty$
2	$0 1 3 6 8 11 12 13 \infty$
3	$0^3 1 2 3^5 4 5^3 6 7^2 8^4 9 10^2 11^3 12 13 14 15 16 17 18 19 20 \infty^2$
4	$0 3 5 11 21 \infty$
5	$0^3 3^3 4 5^9 6 7^4 8^4 10^2 11^6 12^2 14^2 15 16^2 17 18 19^3 20^2 22^4 23 24^2 25^4 26^2 27^3 28^3 29^2$ $30 31 \infty^2$
6	$0 1 2 3 5 8 9 10 11 14 \infty$
7	$0 3^2 5^4 7 8^2 9 11 15 18 19 20 22 23 24 30 32 \infty^2$
8	$0^3 1 2 3^4 5^4 6 7^2 8^6 9 10 11^5 12^2 13 14^2 16^2 17^2 18 19^2 20^2 22^3 23 24^3 25 28 29 30 \infty^2$
9	$0 3 6 7 8 10 14 \infty$
10	$0 3^2 5^2 6 8 9 11 16 \infty$
11	$0 1 2 3^3 4 5^6 6 7 8^5 10 11^2 12 13 14 16 18 19 20 21 22 23 24 25 27 29 32 33 \infty$
12	$0 2 3 5^2 8^2 11 12 13 14 16 21 \infty$
13	$2 3 8 11 12 \infty$
14	$3 5^2 6 8^2 9 11 12 16 20$
15	$3 5 7 18 \infty$
16	$3 5^2 8^2 10 11 12 14 19$
17	$3 5 8^2$
18	$3 5 7 8 11 15 19 \infty$
19	$3 5^3 7 8^2 11 16 18 20 23 29 30 \infty$
20	$3 5^2 7 8^2 11 14 19 24$
21	$4 11 12$
22	$5^4 7 8^3 11 22 23^2 24 25 26 27 30 31 33 \infty$
23	$5 7 8 11 19 22^2 24 25 26 \infty$
24	$5^2 7 8^3 11 20 22 23 30$
25	$5^4 8 11 22 23 27^2 29$
26	$5^2 22 23 27 31$
27	$5^3 11 22 25^2 26$
28	$5^3 8$
29	$5^2 8 11 19 25$
30	$5 7 8 19 22 24$
31	$5 22 26$
32	$7 11 \infty$
33	$11 22$

TABLE A.2. The edges at J_i , one per G_i -orbit of extreme rays. An entry j^e means that e of the listed edges end at a vertex equivalent to J_j ; ∞^e counts the unbounded edges.

of propositional extensionality, choice and quotient soundness, proves the following for the 34 matrices J_i and the 480 representative edges of the certificate. Each J_i lies in $\mathcal{R}$; $\text{Min}(J_i)$ is the listed set up to sign; $\det J_i$ and $\det Q(J_i)$ have the values of Table 4.1. Along every representative edge, $\det(J_i + tY) > 0$ or $\det Q(J_i + tY) \geq 512$, with $\det = 0$ and $\det Q = 512$ only at the listed points. For a bounded edge, $t^* > 0$, $J_i + t^*Y = T^\top J_j T$ with $T \in \Gamma$, and

some $k \in \mathcal{M}^*$ has $(J_i + tY)[k] < 4$ for all $t > t^*$; for an unbounded edge, $Y[k] \geq 0$ for all $k \in \mathcal{M}^*$. Here $\mathcal{R}$, $\mathcal{M}^*$, Γ and the determinants are the Mathlib notions for real matrices. Three ingredients of Proposition 4.3 are not formalized: the rank conditions that make the J_i vertices and the listed rays extreme, the completeness of the lists of extreme rays, and the orbit decompositions. They rest on the exact computations above. A formal certificate of completeness would have to account for every set of 20 independent minimal vectors of a vertex with up to 36 minimal vectors, or for the faces of the many degenerate rays, and we have not produced one.

Equality configurations. A separate exact computation recomputes the points of $\mathcal{C}$ with $\det Q = 512$ from the enumeration and classifies, independently of Section 5, all pairs (Q, c) with Q the Q -block of such a point and c a deep hole of the lattice of Q , using the Voronoi cell of the lattice; it finds the same five classes. Further exact computations check that the three lattice cases are $\sqrt{2}D_5$, and check Lemma 6.1, Lemma 6.2 and the statements of Remark 6.3 that concern P_1 , P_2 and the sublattices of D_5 .

Floating-point numbers are used only to choose enumeration boxes, which are then enlarged; every decision is made in exact arithmetic. The values 512, $320 + 128\sqrt{6}$, 648 and 768 of $\det Q$ in Table 4.2 correspond to the centre densities $1/(8\sqrt{2})$, $(\sqrt{3} - \sqrt{2})/4$, $1/(9\sqrt{2})$ and $1/(8\sqrt{3})$ found in [1, §4.3], and the 34 vertices, the two with $\det J < 0$, and the edge on which $\det J \equiv 0$ agree with [1, §4.3].

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STATEMENTS AND DECLARATIONS

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