

The Number of Terms in Greedy Fibonacci–Lucas Representations

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ARTICLE HISTORY

Compiled October 22, 2026

Article type: research

ABSTRACT

We prove Kimberling’s conjecture on greedy representations by Fibonacci and Lucas numbers: the ratio of the number of Zeckendorf summands of $1, \dots, n$ to the number of terms of their greedy representations tends to a limit in $(1.2, 1.4)$, and the limit is $(5 - \sqrt{5})/2$. We also prove the conjectures of Greathouse and Kurkov on the number of terms of the greedy representation.

KEYWORDS

Fibonacci numbers; Lucas numbers; Zeckendorf representation; greedy algorithm; summatory function; formal verification

1. Introduction

Let F_k and L_k be the Fibonacci and Lucas numbers, $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, let $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$, and let

$$S = \{1, 2, 3, 4, 5, 7, 8, 11, 13, 18, 21, 29, \dots\}$$

be the set of positive Fibonacci and Lucas numbers (A116470 in the OEIS [8]). For $n \geq 1$ let $g(n)$ be the largest element of S that is at most n . The greedy algorithm subtracts $g(n)$ from n and repeats; the number of terms it produces is

$$a(0) = 0, \quad a(n) = 1 + a(n - g(n)) \quad (n \geq 1),$$

which is A214973. Let $z(n)$ be the number of terms in the Zeckendorf representation of n (A007895), and let

$$W(n) = \sum_{i=1}^n a(i), \quad V(n) = \sum_{i=1}^n z(i)$$

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be A214981 and A179180. Representations by Lucas numbers go back to Carlitz, Scoville and Hoggatt [1]. Kimberling [3, Section 6] recorded the following conjecture, also stated in A214981.

Conjecture 1.1 (Kimberling). The limit of $V(n)/W(n)$ exists and lies in the interval $(1.2, 1.4)$.

Two more conjectures, from 2021, are recorded in A214973. Let $A049651(j) = (F_{3j+1} - 1)/2$.

Conjecture 1.2 (Greathouse). For every $j \geq 1$, $a(A049651(j)) = j$, and this is the first occurrence of j in A214973; that is, apart from its initial term, A049651 is the sequence of record positions of A214973.

The statement that $A049651(j)$ is the least n with $a(n) = j$ appeared earlier, without proof, in a 2012 comment by Kimberling in A049651.

Conjecture 1.3 (Kurkov). For every $n \geq 1$, $a(n) = A329320(A048679(n))$.

Here, if $n = F_{k_1} + \dots + F_{k_s}$ is the Zeckendorf representation of n , with $k_1 > \dots > k_s \geq 2$ and $k_i - k_{i+1} \geq 2$, then $A003714(n) = \sum_i 2^{k_i-2}$, whose binary word has no two adjacent ones. Preceded by a zero, such a word is a concatenation of the words 0 and 01 in exactly one way, and $A048679(n)$ is the number whose binary word is obtained by replacing each factor 01 by 1. Finally $A035263(m)$ is the parity of the 2-adic valuation $\nu_2(2m)$, and

$$A329320(y) = \sum_{k=0}^{\lfloor \log_2 y \rfloor} (1 - A035263(1 + \lfloor y/2^k \rfloor)), \quad A329320(0) = 0.$$

For $x \geq 1$ let $K(x)$ be the index $k \geq 2$ with $F_k \leq x < F_{k+1}$, and put $K(0) = 0$. We prove the three conjectures.

Theorem 1.4. Let $T(x) = W(x-1)$ and $Z(x) = V(x-1)$ for $x \geq 1$, and let $c = (5 - \sqrt{5})/2 = 2 + \psi$. For every $x \geq 1$,

$$|5T(x) - K(x)x| \leq 19x, \quad |5Z(x) - cK(x)x| \leq 13x. \quad (1.1)$$

Consequently, for every $n \geq 1$,

$$\left| W(n) - \frac{1}{5} n \log_{\varphi} n \right| \leq 10n, \quad \left| V(n) - \frac{c}{5} n \log_{\varphi} n \right| \leq 10n, \quad (1.2)$$

$$\left| \frac{V(n)}{W(n)} - c \right| \leq \frac{51}{K(n+1) - 19} \quad \text{whenever } K(n+1) > 19, \quad (1.3)$$

and $V(n)/W(n) \rightarrow (5 - \sqrt{5})/2 = 1.38196\dots$, which lies in $(1.2, 1.4)$. In particular Conjecture 1.1 holds.

The formula for $V(n)$ agrees with Lekkerkerker's theorem on the average number of Zeckendorf summands [5, 4], since $c/5 = 1/(\varphi^2 + 1)$. Pihko [7] proved the analogues of that theorem for the minimal and maximal Lucas representations and the maximal

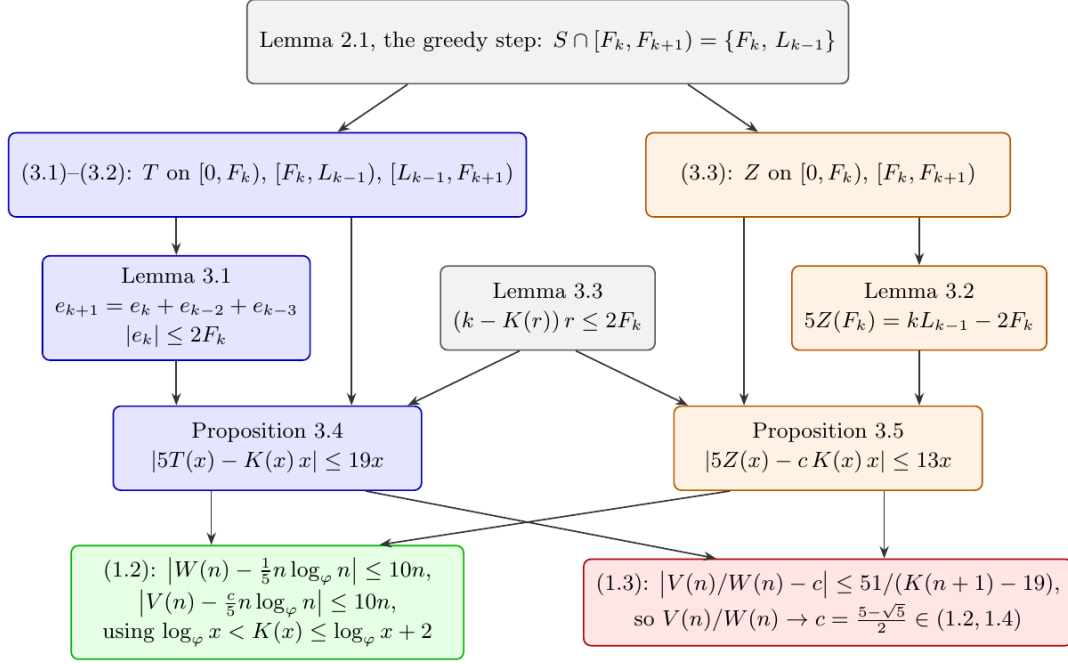


Figure 1. The structure of the proof of Theorem 1.4. The greedy step splits the summatory functions over the blocks of $[0, F_{k+1})$; the values at Fibonacci numbers and the gap lemma then give the bounds (1.1) by induction on x , and these give (1.2), (1.3) and the limit.

Fibonacci representation. Theorem 1.4 shows that the greedy algorithm over S uses on average $\frac{1}{5} \log_\varphi n$ terms.

Theorem 1.5. *For every $j \geq 1$, $a(\text{A049651}(j)) = j$ and $a(m) < j$ for $0 \leq m < \text{A049651}(j)$. Hence Conjecture 1.2 holds.*

Theorem 1.6. *For every $n \geq 0$, $a(n) = \text{A329320}(\text{A048679}(n))$. Hence Conjecture 1.3 holds.*

The proof of Theorem 1.6 gives $a(n)$ in terms of the Zeckendorf representation of n : if its indices form maximal chains $k, k-2, k-4, \dots$ of lengths $\ell_1, \dots, \ell_t$, then $a(n) = \sum_i \lceil \ell_i/2 \rceil$ (Corollary 5.1). The proofs are elementary, and every statement of Theorems 1.4–1.6 is also proved in the Lean 4 proof assistant (Section 6).

2. The greedy step

We use $L_m = F_{m-1} + F_{m+1}$ for $m \geq 1$.

Lemma 2.1. *For $k \geq 4$,*

$$F_k < L_{k-1} < F_{k+1}, \quad L_{k-1} - F_k = F_{k-2}, \quad F_{k+1} - L_{k-1} = F_{k-3},$$

and $S \cap [F_k, F_{k+1}) = \{F_k, L_{k-1}\}$. Hence, for $F_k \leq n < F_{k+1}$,

$$a(n) = \begin{cases} 1 + a(n - F_k), & n < L_{k-1}, \\ 1 + a(n - L_{k-1}), & n \geq L_{k-1}. \end{cases} \quad (2.1)$$

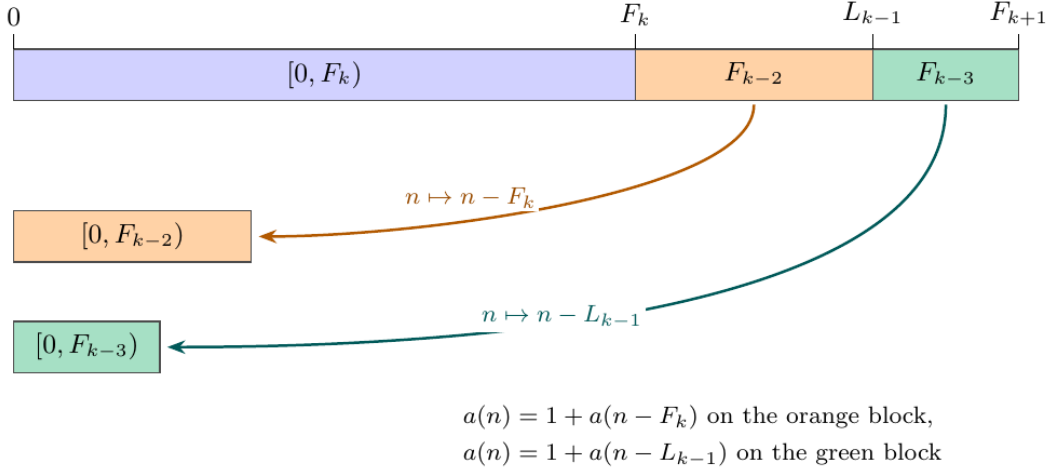


Figure 2. The greedy step on $[F_k, F_{k+1})$, drawn to scale for $k = 9$. By Lemma 2.1 the block $[F_k, L_{k-1})$, of length F_{k-2} , is moved onto $[0, F_{k-2})$ by subtracting F_k , and the block $[L_{k-1}, F_{k+1})$, of length F_{k-3} , onto $[0, F_{k-3})$ by subtracting L_{k-1} . The same three blocks give the decompositions (3.1) and (3.2) of the summatory function.

Proof. The identities follow from $L_{k-1} = F_{k-2} + F_k$ and $F_{k+1} - F_k - F_{k-2} = F_{k-1} - F_{k-2} = F_{k-3}$, and $F_{k-2}, F_{k-3} \geq 1$ gives the inequalities. The only Fibonacci number in $[F_k, F_{k+1})$ is F_k . For the Lucas numbers, $L_0 = 2 < F_k$, the sequence $(L_m)_{m \geq 1}$ is increasing, $L_{k-2} = F_{k-3} + F_{k-1} \leq F_{k-2} + F_{k-1} = F_k$, and $L_k = F_{k-1} + F_{k+1} > F_{k+1}$. So L_{k-1} is the only Lucas number in (F_k, F_{k+1}) . $\square$

For $1 \leq n \leq 2$ we have $a(n) = 1$. The Zeckendorf representation satisfies $z(n) = 1 + z(n - F_k)$ for $F_k \leq n < F_{k+1}$, $k \geq 2$, since its leading term is the largest Fibonacci number at most n .

Lemma 2.2. Let $n = F_{k_1} + \dots + F_{k_s}$ be the Zeckendorf representation of $n \geq 1$. If $s \geq 2$ and $k_2 = k_1 - 2$, then $a(n) = 1 + a(F_{k_3} + \dots + F_{k_s})$; otherwise $a(n) = 1 + a(F_{k_2} + \dots + F_{k_s})$.

Proof. If $k_1 \leq 3$ then $n \in \{1, 2\}$, $s = 1$ and $a(n) = 1$. Let $k = k_1 \geq 4$, so $F_k \leq n < F_{k+1}$, and let $r = n - F_k$. If $k_2 = k - 2$, then $n \geq F_k + F_{k-2} = L_{k-1}$ and $n - L_{k-1} = F_{k_3} + \dots + F_{k_s}$. Otherwise $r = 0$ or $r < F_{k_2+1} \leq F_{k-2}$, so $n < L_{k-1}$. In both cases (2.1) gives the claim. $\square$

So the greedy algorithm over S is the Zeckendorf algorithm in which a leading summand F_k followed by F_{k-2} is merged into L_{k-1} .

3. The summatory functions

Write $T(x) = \sum_{0 \leq i < x} a(i)$ and $Z(x) = \sum_{0 \leq i < x} z(i)$, so that $W(n) = T(n+1)$ and $V(n) = Z(n+1)$. If $f(i) = 1 + f(i-b)$ for $b \leq i < d$, then $\sum_{i < x} f(i) = \sum_{i < b} f(i) + (x-b) + \sum_{i < x-b} f(i)$ for $b \leq x \leq d$. With Lemma 2.1 this gives, for $k \geq 4$,

$$T(x) = T(F_k) + (x - F_k) + T(x - F_k), \quad F_k \leq x \leq L_{k-1}, \quad (3.1)$$

$$T(x) = T(L_{k-1}) + (x - L_{k-1}) + T(x - L_{k-1}), \quad L_{k-1} \leq x \leq F_{k+1}, \quad (3.2)$$

and, for $k \geq 2$,

$$Z(x) = Z(F_k) + (x - F_k) + Z(x - F_k), \quad F_k \leq x \leq F_{k+1}. \quad (3.3)$$

Lemma 3.1. *Let $e_k = 5T(F_k) - kF_k$. Then $e_{k+1} = e_k + e_{k-2} + e_{k-3}$ for $k \geq 4$, and $|e_k| \leq 2F_k$ for $k \geq 1$.*

Proof. Taking $x = L_{k-1}$ in (3.1) and $x = F_{k+1}$ in (3.2),

$$T(F_{k+1}) = T(F_k) + T(F_{k-2}) + T(F_{k-3}) + F_{k-1} \quad (k \geq 4).$$

Multiplying by 5 and subtracting $(k+1)F_{k+1}$, the recursion for e_k follows from

$$kF_k + (k-2)F_{k-2} + (k-3)F_{k-3} + 5F_{k-1} = (k+1)F_{k+1},$$

which holds because $F_{k-2} + F_{k-3} = F_{k-1}$ reduces it to $5F_{k-1} - 2F_{k-2} - 3F_{k-3} = F_{k+1}$, and both sides equal $2F_{k-1} + F_{k-2}$. Since $T(1) = 0$, $T(2) = 1$ and $T(3) = 2$, we get $(e_1, e_2, e_3, e_4) = (-1, -2, -1, -2)$, so $|e_k| \leq 2F_k$ for $k \leq 4$, and by induction $|e_{k+1}| \leq 2(F_k + F_{k-2} + F_{k-3}) = 2F_{k+1}$. $\square$

Lemma 3.2. *For $k \geq 1$, $5Z(F_k) = k(2F_k - F_{k-1}) - 2F_k$, and $|5Z(F_k) - ckF_k| \leq 4F_k$.*

Proof. The formula holds for $k = 1, 2, 3$, where $Z(F_k) = 0, 0, 1$. Taking $x = F_{k+1}$ in (3.3) gives $Z(F_{k+1}) = Z(F_k) + F_{k-1} + Z(F_{k-1})$ for $k \geq 2$, and the formula for $k+1$ follows from the formulas for k and $k-1$ and the identity

$$k(2F_k - F_{k-1}) + (k-1)(2F_{k-1} - F_{k-2}) + 3F_{k-1} - 2F_k = (k+1)(2F_{k+1} - F_k) - 2F_{k+1},$$

in which, after $F_{k+1} = F_k + F_{k-1}$, the coefficients of k agree because $F_{k-1} + F_{k-2} = F_k$, and the remaining terms are $-(2F_{k-1} - F_{k-2}) + 3F_{k-1} - 2F_k = -F_k$ on the left and $(2F_{k+1} - F_k) - 2F_{k+1} = -F_k$ on the right. For the bound, $\psi F_k + F_{k-1} = \psi^k$ by induction on k (using $\psi^2 = \psi + 1$), so

$$5Z(F_k) - ckF_k = -k(F_{k-1} + \psi F_k) - 2F_k = -k\psi^k - 2F_k,$$

and $|\psi| < 1$ and $k \leq 2F_k$ give $|5Z(F_k) - ckF_k| \leq k + 2F_k \leq 4F_k$. $\square$

Lemma 3.3. *If $r \geq 0$ and $k \geq K(r)$, then $0 \leq (k - K(r))r \leq 2F_k$.*

Proof. The claim is clear if $r = 0$ or $k = K(r)$. Otherwise let $j = K(r) < k$ and $d = k - j - 1 \geq 0$. Then $r < F_{j+1}$, and $F_k = F_j F_d + F_{j+1} F_{d+1} \geq F_{j+1} F_{d+1}$ together with $d+1 \leq 2F_{d+1}$ gives $(k-j)r \leq (d+1)F_{j+1} \leq 2F_k$. $\square$

Proposition 3.4. *For every $x \geq 0$, $|5T(x) - K(x)x| \leq 19x$.*

Proof. By induction on x . For $x = 0, 1, 2$ the left side is 0, 2 and 1. Let $x \geq 3$ and $k = K(x) \geq 4$.

If $x \leq L_{k-1}$, let $r = x - F_k$. Then $r \leq F_{k-2}$, so $K(r) \leq k$, and by (3.1)

$$5T(x) - kx = e_k + 5r + (5T(r) - K(r)r) - (k - K(r))r.$$

By Lemmas 3.1 and 3.3 and the induction hypothesis, this lies between $-4F_k - 14r$ and $2F_k + 24r$. Since $r \leq F_k$ and $x = F_k + r$, both bounds are at most $19x$ in absolute value.

If $x > L_{k-1}$, let $r = x - L_{k-1} < F_{k-3}$. By (3.2) and (3.1) at $x = L_{k-1}$, and since $x = F_k + F_{k-2} + r$,

$$5T(x) - kx = e_k + e_{k-2} + 3F_{k-2} + 5r + (5T(r) - K(r)r) - (k - K(r))r,$$

which lies between $-4F_k + F_{k-2} - 14r$ and $2F_k + 5F_{k-2} + 24r$; both are at most $19x$ in absolute value. $\square$

Proposition 3.5. *For every $x \geq 0$, $|5Z(x) - cK(x)x| \leq 13x$.*

Proof. By induction on x ; the case $x = 0$ is trivial. Let $x \geq 1$, $k = K(x) \geq 2$ and $r = x - F_k < F_{k-1}$, so $K(r) \leq k$ and $r \leq F_k$. By (3.3),

$$5Z(x) - ckx = (5Z(F_k) - ckF_k) + 5r + (5Z(r) - cK(r)r) - c(k - K(r))r.$$

Since $0 < c \leq 2$, Lemmas 3.2 and 3.3 and the induction hypothesis place this between $-8F_k - 8r$ and $4F_k + 18r$, and both are at most $13x = 13F_k + 13r$ in absolute value. $\square$

Proof of Theorem 1.4. The bounds (1.1) are Propositions 3.4 and 3.5. Let $x = n + 1$ and $K = K(x)$. Then $W(n) = T(x)$ and $V(n) = Z(x)$, and

$$5V(n) - 5cW(n) = (5Z(x) - cKx) - c(5T(x) - Kx)$$

has absolute value at most $13x + 2 \cdot 19x = 51x$, while $5W(n) \geq (K - 19)x$. This gives (1.3), and $K(n + 1) \rightarrow \infty$ gives the limit. Finally $6/5 < (5 - \sqrt{5})/2 < 7/5$ because $121 < 125 < 169$.

For (1.2), write $\lambda(t) = \log_\varphi t$. Since $F_{k+1} \leq \varphi^k$ and $\varphi^{k-2} \leq F_k$ for $k \geq 2$, and $F_K \leq x < F_{K+1}$, we have $\lambda(x) < K \leq \lambda(x) + 2$. Also $K \leq x + 1$, because $F_k \geq k - 1$ for every $k \geq 1$. Next,

$$x\lambda(x) - n\lambda(n) = \lambda(x) + n \frac{\log(1 + 1/n)}{\log \varphi},$$

and $n \log(1 + 1/n) \leq 1 < 3 \log \varphi$, because $\varphi^3 = 2\varphi + 1 > e$. Hence $0 \leq x\lambda(x) - n\lambda(n) \leq K + 3 \leq n + 5$. In the identity

$$5W(n) - n\lambda(n) = (5T(x) - Kx) + (K - \lambda(x))x + (x\lambda(x) - n\lambda(n))$$

the three terms lie in $[-19x, 19x]$, $[0, 2x]$ and $[0, n + 5]$, so $-19(n + 1) \leq 5W(n) - n\lambda(n) \leq 21(n + 1) + n + 5 \leq 48n$ for $n \geq 1$. In the same way, since $0 < c \leq 2$,

$$5V(n) - cn\lambda(n) = (5Z(x) - cKx) + c(K - \lambda(x))x + c(x\lambda(x) - n\lambda(n))$$

lies between $-13(n + 1)$ and $13(n + 1) + 4(n + 1) + 2(n + 5) \leq 46n$. Dividing by 5 gives (1.2). $\square$

4. The records

Let $R_j = F_2 + F_5 + \dots + F_{3j-1} = \sum_{i < j} F_{3i+2}$. Since $F_{3j+1} + 2F_{3j+2} = F_{3j+3} + F_{3j+2} = F_{3j+4}$, induction gives $2R_j + 1 = F_{3j+1}$, so $R_j = A049651(j)$. Moreover $R_j < F_{3j}$ for $j \geq 1$, since $2R_j < F_{3j+1} \leq 2F_{3j}$.

Proof of Theorem 1.5. First, $a(R_j) = j$. This holds for $j = 0, 1$. For $j \geq 2$ let $k = 3j - 1 \geq 5$. Then $R_j = F_k + R_{j-1}$ and $R_{j-1} < F_{k-2}$, so $F_k \leq R_j < L_{k-1}$, and (2.1) gives $a(R_j) = 1 + a(R_{j-1}) = j$.

Second, $a(m) \geq j$ implies $m \geq R_j$. We use induction on m . The cases $j \leq 1$ and $m \leq 2$ are clear, so let $j = t + 2$ with $t \geq 0$, and let $m \geq 3$ and $k = K(m) \geq 4$. By (2.1) there is r with $a(r) = a(m) - 1 \geq t + 1$, where either $r = m - F_k < F_{k-2}$ or $r = m - L_{k-1} < F_{k-3}$. By induction $r \geq R_{t+1} \geq F_{3t+2}$, so $F_{3t+2} < F_{k-2}$, hence $3t + 2 < k - 2$ and $F_{3t+5} \leq F_k$. In both cases $m \geq F_k + r \geq F_{3t+5} + R_{t+1} = R_{t+2}$.

So $a(m) < j$ for $m < R_j$, and $a(R_j) = j$. Thus R_j is the first occurrence of j , and since $R_1 < R_2 < \dots$, the records of A214973 occur exactly at $R_1, R_2, \dots$. $\square$

5. Kurkov's formula

Let $n = F_{k_1} + \dots + F_{k_s}$ be the Zeckendorf representation of $n \geq 1$. A *chain* is a maximal run $k_i, k_{i+1}, \dots, k_{i+\ell-1}$ of consecutive indices with $k_p - k_{p+1} = 2$ for $i \leq p < i + \ell - 1$; the indices split into chains of lengths $\ell_1, \dots, \ell_t$, and consecutive chains are separated by gaps $k_p - k_{p+1} \geq 3$.

Corollary 5.1. $a(n) = \sum_{i=1}^t \lceil \ell_i / 2 \rceil$.

Proof. Apply Lemma 2.2 repeatedly. Along a chain it merges the first and second terms, the third and fourth, and so on, giving $\lceil \ell / 2 \rceil$ terms; it never merges the last term of a chain with the first term of the next, since they differ by at least 3. $\square$

Lemma 5.2. $A048679(n) = \sum_{i=1}^s 2^{h_i}$ with $h_i = k_i - 2 - (s - i)$. The exponents satisfy $h_1 > \dots > h_s \geq 0$, and $h_i - h_{i+1} = 1$ if and only if $k_i - k_{i+1} = 2$.

Proof. The binary word of $A003714(n)$, preceded by a zero, consists of the factors 01 occupying positions $k_i - 1, k_i - 2$ ($1 \leq i \leq s$; position 0 is the units digit), separated by zeros. These factors are disjoint because $k_{i+1} - 1 \leq k_i - 3$. Replacing each 01 by 1 moves the digit at position $k_i - 2$ down by one place for each of the $s - i$ factors below it, so it lands at position h_i . Finally $h_i - h_{i+1} = k_i - k_{i+1} - 1 \geq 1$ and $h_s = k_s - 2 \geq 0$. $\square$

Lemma 5.3. For $y \geq 0$, $A329320(y) = \sum \lceil \ell / 2 \rceil$, the sum over the maximal blocks of consecutive ones in the binary word of y , of lengths ℓ .

Proof. Since $\nu_2(2m) = 1 + \nu_2(m)$, we have $1 - A035263(1 + u) = \nu_2(u + 1) \pmod{2}$, and $\nu_2(\lfloor y / 2^k \rfloor + 1)$ is the number of consecutive ones of y starting at position k and going up. This number is 0 if the digit at position k is 0, and, along a block of ℓ ones, it takes the values $\ell, \ell - 1, \dots, 1$, of which $\lceil \ell / 2 \rceil$ are odd. The sum defining $A329320(y)$ runs over the positions $0, \dots, \lfloor \log_2 y \rfloor$, which contain every block. $\square$

Proof of Theorem 1.6. For $n = 0$ both sides are 0. For $n \geq 1$, Lemma 5.2 shows that the blocks of ones of $A048679(n)$ correspond to the chains of the Zeckendorf representation of n , with the same lengths. Lemma 5.3 and Corollary 5.1

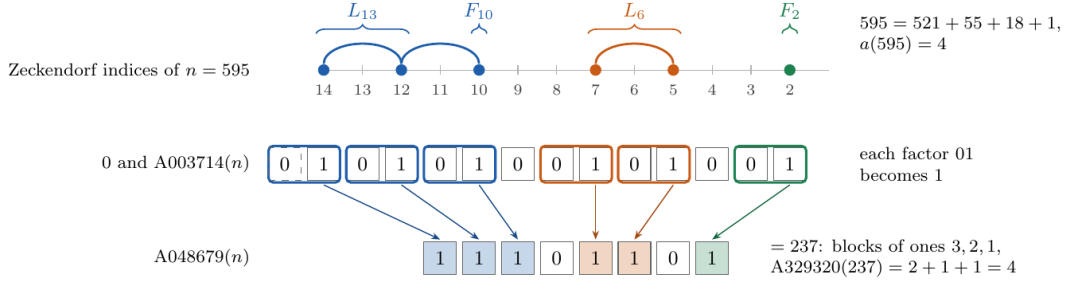


Figure 3. Kurkov's formula for $n = 595 = F_{14} + F_{12} + F_{10} + F_7 + F_5 + F_2$. The Zeckendorf indices form the chains 14, 12, 10, then 7, 5, then 2, and the greedy algorithm uses $L_{13} = F_{14} + F_{12}$, F_{10} , $L_6 = F_7 + F_5$ and F_2 , so $a(595) = 4$ (Corollary 5.1). The binary word of $A003714(595)$, preceded by a zero, splits into the factors 01 and 0; replacing each 01 by 1 gives $A048679(595) = 237$, whose blocks of ones have the lengths of the chains (Lemma 5.2), and $A329320(237) = 4$ (Lemma 5.3).

give $A329320(A048679(n)) = \sum_i \lceil \ell_i/2 \rceil = a(n)$. Figure 3 follows the argument for $n = 595$. $\square$

6. Formal verification

Theorems 1.4, 1.5 and 1.6 are formalized in Lean 4 [2] with Mathlib [6], version 4.34.1, and the formal proofs are checked by the Lean kernel. The formalization defines the sequences directly from their entries in the OEIS: a by the greedy recursion over the positive Fibonacci and Lucas numbers, z as the length of the Zeckendorf representation, W and V as sums over $1 \leq i \leq n$, $A048679$ by the rewriting of binary digits described in Section 1, and $A329320$ by the displayed sum. It proves the bounds (1.1), the explicit formulas (1.2), the limit $V(n)/W(n) \rightarrow (5 - \sqrt{5})/2$ together with $6/5 < (5 - \sqrt{5})/2 < 7/5$, Theorem 1.5 for every j , and Theorem 1.6 for every $n \geq 0$. In the formal proof of Theorem 1.6 the induction runs on the Zeckendorf representation through the recursion $A329320(2^{Q+1} + u) = A329320(2^{Q+1} + 2^Q + u) = 1 + A329320(u)$ for $u < 2^Q$, in place of Lemma 5.3. The proofs depend only on the axioms of Lean's logic, namely propositional extensionality, choice and the soundness of quotients; they have no unproved step and add no axiom.

Independent computations in exact integer arithmetic, from the definitions of the sequences, confirm (1.1) for $x \leq 10^8 + 1$, Lemma 3.1 and the formula of Lemma 3.2 for $F_k \leq 10^8 + 1$, Theorems 1.5 and 1.6 and Lemma 2.2 for $n \leq 10^8$, and Lemmas 2.1, 3.1 and 3.2 for $k \leq 400$; they agree with the terms of the sequences recorded in the OEIS. The ratio $V(n)/W(n)$ equals 1.3421 at $n = 10^6$ and 1.3588 at $n = 10^8$.

Acknowledgements

The authors used the generative AI tool Claude (Anthropic), through Claude Code, to assist with the proofs, the Lean formalization, the computations and the preparation of this article. The authors reviewed and edited the content and take full responsibility for it.

Disclosure statement

Competing interests: none.

Funding

None.

Data availability

Deep Bhattacharjee computed the Lean formalization and the Julia, C, C++, shell and Python computations with the help of AI assistance (Claude, Anthropic). The Lean formalization of the proofs in this paper is archived at Zenodo, doi:10.5281/zenodo.23385265. The paper is available at <https://github.com/creelie/musical-rotary-phone/blob/main/preprintKimberlingConjecture/KimberlingConjecture.pdf>.

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MSC2020: Primary 11B39; Secondary 11A67, 68V15