

WEIL CLASSES ON ABELIAN SIXFOLDS OF WEIL TYPE FOR $\mathbb{Q}(i)$ WITH DISCRIMINANT -3

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ABSTRACT. Let $K = \mathbb{Q}(i)$. We prove that the Weil classes of every polarized complex abelian sixfold of Weil type for K whose Hermitian discriminant is the non-split class $[-3]$ are algebraic, and that every Hodge class on every power of such a sixfold is algebraic when its Hodge group is the full special unitary group. The proof starts from Galois covers of the projective line with monodromy group $W(E_8) \cdot \mu_4$, branched over twelve points: the multiplicity space of the natural representation in the cohomology of such a cover is the first cohomology of an abelian sixfold of Weil type, and a certified rank computation at one explicit cover shows that these sixfolds fill an open set of the nine-dimensional moduli space. Algebraic cycles reach these sixfolds through Milne's theorem on Lefschetz classes, from two sources: Schoen's cycles on cyclic covers, which a computer search shows do not suffice here, and products of determinant lines of characters on abelian covers of curves. For the second source we prove a general theorem (Theorem C): if such a product is of Hodge type at every Galois conjugate, and each of the cyclic covers involved has at least two branch points and order dividing 12 after removing the common factor of its branch exponents, then the product spans a space of algebraic classes. Its proof runs Schoen's symmetric-product construction on each factor, localises at a special fibre, transfers to a product of Fermat hypersurfaces, and applies Shioda's theorem on products of Fermat varieties.

1. INTRODUCTION

1.1. Weil classes. Let A be a complex abelian variety of dimension $2n$ with an action $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ of an imaginary quadratic field K , such that the two embeddings of K in $\mathbb{C}$ each occur with multiplicity n on the tangent space. Then the K -line

$$W_K(A) = \bigwedge_K^{2n} H^1(A, \mathbb{Q}) \subset H^{2n}(A, \mathbb{Q})$$

is a two-dimensional space of Hodge classes of type (n, n) , the *Weil classes* of A [27]. For a general such A they are not in the ring generated by divisor classes, so they are a natural test case for the Hodge conjecture [24,

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§5]. A polarization compatible with ι determines a K -Hermitian form H of signature (n, n) on $H_1(A, \mathbb{Q})$, and its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ is an invariant of the polarized rational Hodge structure. The polarized abelian $2n$ -folds of Weil type for K with a fixed discriminant form one family, parametrized by a quotient of the Hermitian symmetric domain of $SU(n, n)$, of dimension n^2 . The class $(-1)^n$ is called *split*.

For $K = \mathbb{Q}(i)$ the group $\mathbb{Q}^\times / \text{Nm } K^\times$ is detected by the sign and by the parities of the valuations at the primes $p \equiv 3 \pmod{4}$, which are inert in K . For sixfolds ($n = 3$) the split class is $[-1]$; the class $[-3]$ is not split, because 3 is inert in K and so is not a norm.

1.2. What was known. For abelian fourfolds of Weil type the Weil classes are algebraic for every K and every discriminant (Markman [12, 13]). For sixfolds the split case is known: Koike for $\mathbb{Q}(i)$ [8], Schoen for $\mathbb{Q}(\sqrt{-3})$ [19], and Markman for every K [13, Theorem 1.5.1]. Mostaedd describes the non-split sixfold cases as open [16, Introduction]. For $K = \mathbb{Q}(\sqrt{-3})$ the companion paper [21] proves algebraicity for sixfolds of every discriminant, through split eightfolds and Schoen's product argument; the class -2 was treated earlier in [20]. For $K = \mathbb{Q}(i)$ we know of no non-split discriminant for which the Weil classes of sixfolds were previously known to be algebraic. Voisin's Bourbaki exposé [25] surveys this subject.

This paper uses no eightfolds and no product argument. The sixfolds come directly from a family of Galois covers, dominance of the family is certified at one smooth point, and, because Schoen's cycles alone do not suffice for this family (Appendix E), a second source of cycles is added: Theorem C below.

1.3. Results.

Theorem A. *Let $K = \mathbb{Q}(i)$. The Weil classes of every polarized abelian sixfold of Weil type for K with Hermitian discriminant $[-3]$ are algebraic.*

Theorem B. *Let A be an abelian sixfold as in Theorem A whose Hodge group is SU_H , the special unitary group of its Hermitian form. Then every Hodge class on every power A^m is algebraic.*

The hypothesis of Theorem B is on the member A itself. It holds outside a countable union of proper closed subvarieties of the moduli space [24, Theorem 6.11]. We make no claim about the further Hodge classes on powers of members with smaller Hodge group.

The new ingredient is a theorem about abelian covers of curves. Let $D \rightarrow X$ be a connected Galois cover of a smooth projective curve X of genus g_X with finite abelian group A . A character ψ of A of order $e \geq 3$ defines a cyclic cover $D/\ker \psi \rightarrow X$ of degree e . At each of its branch points b the local monodromy is ψ of a generator of inertia, say $e^{2\pi i \alpha_b / e}$; let s_ψ be the number of branch points, $d_\psi = \gcd(e, \alpha_b : b \text{ a branch point})$,

and call e/d_ψ the *reduced order*. For characters $\psi_1, \dots, \psi_R$ of orders $e_k \geq 3$ (repetitions allowed), consider the line

$$\ell = \det H^1(D, \mathbb{C})_{\psi_1} \otimes \cdots \otimes \det H^1(D, \mathbb{C})_{\psi_R} \subset H^*(J(D)^R, \mathbb{C}),$$

an external product with one factor $J(D)$ for each k , and let $\ell^{(t)}$ be the same line with every ψ_k replaced by ψ_k^t .

Theorem C (Theorem 5.1). *Suppose $g_X \geq 1$, and that for every k the cyclic cover $D/\ker \psi_k \rightarrow X$ has at least two branch points and reduced order dividing 12. If every line $\ell^{(t)}$, t prime to the orders e_k , is of Hodge type, then the rational subspace whose complexification is the sum of the distinct lines $\ell^{(t)}$ is spanned by classes of algebraic cycles.*

In the application no single factor $\det H^1(D, \mathbb{C})_{\psi_k}$ is of Hodge type; only the product is. Section 5 states Theorem C on the product $\prod_k J(D/\ker \psi_k)$ (Theorem 5.1); the form above is Corollary 5.2.

1.4. The construction in one page.

$$\begin{aligned} G\text{-covers } C \rightarrow \mathbb{P}^1, G = W(E_8) \cdot \mu_4, & \implies \begin{array}{c} \sigma\text{-factor} \\ A_\sigma(C) \subset J(C), \dim 6 \end{array} \implies \\ \text{branched at 12 points} & \\ \implies \begin{array}{c} \text{every sixfold for } \mathbb{Q}(i) \\ \text{with discriminant } [-3] \end{array} & \end{aligned}$$

The sixfolds. Let $W = W(E_8) \subset GL_8(\mathbb{Q})$ and $G = W \cup iW \subset GL_8(\mathbb{Q}(i))$, of order 1 393 459 200, with σ its natural representation on K^8 , whose character field is $K = \mathbb{Q}(i)$. Take Galois covers $C \rightarrow \mathbb{P}^1$ with group G , branched over twelve points, with local monodromy ix_1 at one point ($x_1 \in W$, $x_1^2 = -1$), ix_2 at another ($x_2 \in W$ of order 12), and a reflection of W at each of the other ten. The representation σ occurs in $H^1(C)$ with multiplicity 6, and the multiplicity space is the first cohomology of an abelian sixfold $A_\sigma(C)$, the image in $J(C)$ of an endomorphism given by an element of $\mathbb{Q}[G]$ (Definition 3.2); the centre of $\mathbb{Q}[G]$ acting on σ is K , and it makes $A_\sigma(C)$ a sixfold of Weil type for K (Section 3). It is not of CM type (Remark 3.5).

Cycles. For a subgroup of G and a character of it, the corresponding quotients of C form a cyclic or abelian cover of a quotient curve, whose Jacobian is a factor of $J(C)$. Some of these factors carry known algebraic classes. The Weil classes of $A_\sigma(C)$ are produced from them by a correspondence, which is algebraic by Milne's theorem on Lefschetz classes [14] as soon as a certain class is invariant under the group of symplectic automorphisms of $H^1(C)$ commuting with G ; that invariance is an identity between integer vectors of length 45 (Section 4). Two kinds of auxiliary classes enter: Weil classes of Prym varieties of cyclic covers, algebraic by Schoen's theorem [18] (a computer search shows that these alone do not suffice; Appendix E), and products of determinant lines of several characters on abelian covers, algebraic by Theorem C. An explicit identity with 95 terms of the first kind and 3 of the second (Proposition 8.3) makes the Weil classes of $A_\sigma(C)$ algebraic for every cover C .

Theorem C. For one cyclic cover of a curve, Schoen’s construction [18, §2] produces a variety fibred over a Picard variety of the base curve whose relevant cohomology is one line. That line localises at a single special fibre, where the cover becomes, up to a vector bundle, a quotient of an open Fermat hypersurface; so the line is a Tate twist of a character line on a Fermat hypersurface. For several characters the Fermat lines multiply, and the Hodge condition on the product makes their product a Hodge class on a product of Fermat hypersurfaces of degrees dividing 12, algebraic by Shioda [23]. A cone over that cycle, and Schoen’s Abel–Prym correspondence, carry it back (Section 5).

Dominance and spreading. At one explicit cover C_0 the codifferential of the period map, a 9×9 matrix, has a certified nonzero determinant (Appendix B). So the family $A_\sigma(C)$ fills an open set of the nine-dimensional moduli space of sixfolds of its discriminant, which is $[-3]$ by an exact computation from the monodromy tuple (Proposition 8.2). The set of moduli points with algebraic Weil classes then contains an open set, and a countable-union argument makes it everything (Section 7).

What is new. Schoen’s cycles, Milne’s theorem, dominance followed by spreading [8], the passage from a single cyclic cover to a Fermat character, and Shioda’s theorem are known. New here are Theorem C, in which factors that are not of Hodge type are combined into a product that is, and its use as a second supply of cycles alongside Schoen’s.

Inputs. Published results: Schoen’s paper [18], with a repair of one step (Lemma 4.7); Shioda’s theorem on products of Fermat varieties [23, Theorem IV], which for degree 12 rests on a finite verification reported by Shioda and recomputed here (Subsection 5.6); Lieberman’s theorem on the Lefschetz operator [11, Theorem 3]; Milne’s theorem on Lefschetz classes [14, Theorem 3.2]; Landherr’s theorem and van Geemen’s description of the moduli [24]; the Chevalley–Weil formula and the codifferential of the period map [3]; standard sheaf theory and Hodge theory (proper base change, the Künneth formula, the Thom isomorphism, mixed Hodge structures). Levelt’s normal form for a regular singular point with semisimple local monodromy [10], used in Appendix B. Finite certificates, described in the appendices: that the tuple generates G (Appendix A); the character table data and the integer identity in $\mathbb{Z}^{45}$, with the branch data of the suppliers (Appendix C); the discriminant, by exact linear algebra over K (Proposition 8.2); and the rank of the codifferential at C_0 (Appendix B).

1.5. Relation to the companion paper. Sections 2, 3, 6, 7 and 9, the Schoen part of Section 4, and Lemma 4.7 adapt the corresponding parts of [21], which treats $\mathbb{Q}(\sqrt{-3})$. Apart from the published results listed above, the proof depends on the finite certificates. Their data are printed or summarised in the appendices, and their programs and outputs are in the supplementary files, the certificate archive [22]; a reader who wants to verify

them needs those files, not only this text. Every other step is proved here; in particular nothing is taken from [21] or [20] without proof.

1.6. Organization. Section 2 recalls abelian varieties of Weil type and their moduli. Section 3 constructs the σ -factor. Section 4 defines the two kinds of suppliers and proves that an integer identity among their characters gives algebraic Weil classes. Section 5 proves Theorem C. Section 6 computes the Hermitian discriminant from the monodromy tuple and states the dominance criterion. Section 7 spreads algebraicity from a dominant family to the moduli space. Section 8 applies all of this to $W(E_8) \cdot \mu_4$ and proves Theorem A, and Section 9 proves Theorem B. The appendices give the data: the monodromy tuple (Appendix A), the rank certificate (Appendix B), the supplier certificates (Appendix C), and the search showing that Schoen suppliers alone do not suffice (Appendix E).

2. ABELIAN VARIETIES OF WEIL TYPE

Throughout, K is an imaginary quadratic field and $\vartheta \in K$ is a nonzero element with $\bar{\vartheta} = -\vartheta$; for $K = \mathbb{Q}(i)$ we take $\vartheta = i$.

Definition 2.1. A *polarized abelian $2n$ -fold of Weil type for K* is a triple (A, ι, E) : a complex abelian variety A of dimension $2n$, an embedding $\iota: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$ such that both embeddings of K occur with multiplicity n on $T_0 A$, and a polarization E (an alternating form on $H_1(A, \mathbb{Q})$) with $E(\iota(a)x, y) = E(x, \iota(\bar{a})y)$.

Following van Geemen [24, Lemma 5.2], the form $H(x, y) = E(x, \iota(\vartheta)y) + \vartheta E(x, y)$ is K -Hermitian of signature (n, n) on the $2n$ -dimensional K -space $H_1(A, \mathbb{Q})$. Its discriminant $\text{disc } H \in \mathbb{Q}^\times / \text{Nm } K^\times$ does not depend on ϑ , and it is unchanged by a K -linear isogeny that multiplies the polarization by a positive rational number, since such a change multiplies $\det H$ by a $2n$ -th power of a rational number, which is a norm. By Landherr's theorem [24, §5.4], a nondegenerate K -Hermitian space is determined up to isometry by its dimension, signature and discriminant; the Hermitian spaces of signature (n, n) with an n -dimensional totally isotropic subspace are those of discriminant $(-1)^n$, called *split*. We write SU_H for the special unitary group of $(H_1(A, \mathbb{Q}), H)$, an algebraic group over $\mathbb{Q}$.

2.1. Moduli. Fix a K -Hermitian space (V, H) of dimension $2n$, signature (n, n) and discriminant $\mathfrak{d}$, and an $\mathcal{O}_K$ -stable lattice $\Lambda \subset V$ on which the alternating form $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$, scaled by a positive rational, is integral. The space $\mathcal{H}_n$ of n -dimensional complex subspaces of $V_{\mathbb{R}}$ (for the complex structure $\iota(\vartheta)/|\vartheta|$) on which H is positive is the Hermitian symmetric domain of $SU(n, n)$, connected of dimension n^2 . A point $V_+ \in \mathcal{H}_n$ determines a complex structure on $V_{\mathbb{R}}$ commuting with K , hence a Hodge structure of weight one on V of Weil type for K , polarized by $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$; every such polarized Hodge structure arises in this way [24, 5.10 and the proof of 6.11]. For $\ell \geq 3$ the congruence subgroup $\Gamma(\ell)$ of the unitary group of (Λ, H) is torsion

free, and $\mathcal{M} = \Gamma(\ell) \backslash \mathcal{H}_n$ is a smooth connected quasi-projective variety with a universal abelian scheme $f: \mathcal{A} \rightarrow \mathcal{M}$ with K -action [24, §§5.5–5.11].

Lemma 2.2 (Isogeny invariance). *If $\phi: A' \rightarrow A$ is a K -linear isogeny, then $W_K(A)$ is spanned by algebraic classes if and only if $W_K(A')$ is. Every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}$ is K -isogenous to a fibre of $\mathcal{A} \rightarrow \mathcal{M}$.*

Proof. Pullback and pushforward by ϕ are algebraic correspondences, and ϕ^* maps $W_K(A)$ isomorphically onto $W_K(A')$, since it is a K -linear isomorphism on $H^1(-, \mathbb{Q})$. For the second statement, Landherr's theorem gives a K -isometry from $(H_1(A, \mathbb{Q}), H)$ to (V, H) ; the image of $H_1(A, \mathbb{Z})$ is commensurable with Λ , so a multiple of it lies in Λ , and multiplication by that integer is a K -linear isogeny from A to the fibre of $\mathcal{A}$ at the point given by the complex structure of A . $\square$

3. ISOTYPIC FACTORS OF GALOIS COVERS

3.1. Hurwitz spaces. Fix a finite group G and conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ of G , $r \geq 4$ (the *class datum*). The *Nielsen class* is the set of tuples $(g_1, \dots, g_r)$ with $g_j \in \mathcal{C}_j$, $g_1 \cdots g_r = 1$, and $\langle g_1, \dots, g_r \rangle = G$. A connected Galois cover $C \rightarrow \mathbb{P}^1$ with group G , branched over r ordered points $p_1, \dots, p_r$, determines such a tuple up to simultaneous conjugation, once standard generators of $\pi_1(\mathbb{P}^1 - \{p_j\})$ are chosen: g_j is the monodromy along a loop around p_j . The Hurwitz space Hur of such covers is finite and étale over the configuration space of r distinct ordered points; its connected components are smooth, hence irreducible, of dimension $r - 3$ modulo PGL_2 , and correspond to orbits of the pure braid group on the Nielsen class modulo conjugation [4].

For an irreducible complex representation τ of G write $H^1(C, \mathbb{C}) = \bigoplus_{\tau} \tau \otimes M_{\tau}$ (Chevalley–Weil), and $m_{\tau} = \dim M_{\tau}$. The space M_{τ} carries a Hodge structure of weight one. For $g \in G$ write $c_{\tau}(g) = \text{rk}(\tau(g) - 1)$, and $\text{age } \tau(g) = \sum_k \alpha_k$ when the eigenvalues of $\tau(g)$ are $e^{2\pi i \alpha_k}$ with $0 \leq \alpha_k < 1$.

Proposition 3.1. *If τ is irreducible and nontrivial of dimension d , then*

$$m_{\tau} = -2d + \sum_{j=1}^r c_{\tau}(g_j), \quad \dim \text{Hom}_G(\tau, H^0(C, K_C)) = \sum_{j=1}^r \text{age } \tau(g_j^{-1}) - d.$$

Proof. Let $U_0 = \mathbb{P}^1 - \{p_j\}$ and $\mathbb{V}$ the local system on U_0 attached to τ . Then m_{τ} is the dimension of the parabolic cohomology $H^1(\mathbb{P}^1, j_* \mathbb{V})$, the image of $H_c^1(U_0, \mathbb{V}) \rightarrow H^1(U_0, \mathbb{V})$. Since τ is irreducible and nontrivial, $H^0(U_0, \mathbb{V}) = 0$, and $H^2(U_0, \mathbb{V}) = 0$ because U_0 is an affine curve; so $\dim H^1(U_0, \mathbb{V}) = (r - 2)d$. Likewise $H_c^0 = H_c^2 = 0$, the latter by duality with $H^0(U_0, \mathbb{V}^{\vee}) = 0$, so $\dim H_c^1(U_0, \mathbb{V}) = -\chi_c(U_0)d = (r - 2)d$. The long exact sequence of the pair formed by U_0 and small punctured discs Δ_j^* around the p_j begins

$$0 = H^0(U_0, \mathbb{V}) \rightarrow \bigoplus_j H^0(\Delta_j^*, \mathbb{V}) \rightarrow H_c^1(U_0, \mathbb{V}) \rightarrow H^1(U_0, \mathbb{V}) \rightarrow \cdots,$$

so the image of $H_c^1 \rightarrow H^1$ has dimension $(r-2)d - \sum_j \dim H^0(\Delta_j^*, \mathbb{V})$, and $\dim H^0(\Delta_j^*, \mathbb{V}) = d - c_\tau(g_j)$ is the dimension of the fixed space of $\tau(g_j)$. This gives the first formula. The second is the Chevalley–Weil formula [3, Theorem 2.10]. $\square$

3.2. The σ -factor. Let σ be an irreducible representation of G whose character field (the field generated by its values) is K and which is realizable over K , on $V_\sigma = K^d$. Since K is not real, the character takes non-real values and $\sigma \not\cong \bar{\sigma}$. The simple factor of $\mathbb{Q}[G]$ attached to σ is $\mathbb{Q}[G]_\varepsilon \cong \text{End}_K(V_\sigma) \cong M_d(K)$, with centre K . Averaging gives a G -invariant positive definite K -Hermitian form h_V on V_σ ; the Rosati involution $g \mapsto g^{-1}$ acts on $\mathbb{Q}[G]_\varepsilon$ as the h_V -adjoint and on its centre K as complex conjugation.

Definition 3.2. Choose $v \in V_\sigma$ with $h_V(v, v) \neq 0$, and let $e \in \mathbb{Q}[G]_\varepsilon$ be the h_V -orthogonal projection onto the line Kv ; it is a primitive idempotent of $\mathbb{Q}[G]$ fixed by the Rosati involution. Choose an integer $N_0 \geq 1$ with $N_0 e \in \mathbb{Z}[G]$; then $N_0 e$ is an endomorphism of $J(C)$ (not itself idempotent). For a cover C , the σ -factor is the abelian subvariety

$$A_\sigma(C) = \text{im}(N_0 e: J(C) \rightarrow J(C)),$$

with the restricted polarization E and the action ι of the centre K . It does not depend on N_0 , and different choices of v give K -isogenous factors.

The σ -isotypic part of $J(C)$ is isogenous to $A_\sigma(C)^d$. Such factors are often called Prym–Tyurin varieties [2, Ch. 12]; we do not use the exponent condition of that notion, and use the neutral name.

Lemma 3.3. *($A_\sigma(C), \iota, E$) is a polarized abelian variety of dimension m_σ , and both embeddings of K occur on its tangent space, with multiplicities the Hodge numbers (p_σ, q_σ) of M_σ . If $m_\sigma = 2n$ and $(p_\sigma, q_\sigma) = (n, n)$, it is a polarized abelian $2n$ -fold of Weil type for K .*

Proof. The σ -isotypic part of $H_1(J(C), \mathbb{Q})$ is $V_\sigma \otimes_K M$ with $M = \text{Hom}_{K[G]}(V_\sigma, H_1(J(C), \mathbb{Q})_\varepsilon)$, and $H_1(A_\sigma, \mathbb{Q}) = Kv \otimes_K M \cong M$. After extending scalars along the two embeddings of K , M becomes M_σ and $M_{\bar{\sigma}}$. The Rosati involution of E restricts on $\iota(K)$ to that of $\mathbb{Q}[G]$ on the centre, which is complex conjugation. $\square$

Lemma 3.4. *On a connected component S of Hur, the discriminant of the Hermitian form of $A_\sigma(C)$ does not depend on $C \in S$; we write $\mathfrak{d}(S)$ for it.*

Proof. The spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and E form a local system over S , since e and ι act by flat endomorphisms and E is flat; parallel transport is a K -isometry, and the discriminant is an isometry invariant. $\square$

Remark 3.5 ($A_\sigma(C)$ is not of CM type). In our application $(p_\sigma, q_\sigma) = (3, 3)$ (Lemma 8.1), and, as Section 8 shows by the certified rank computation of Appendix B, the family $A_\sigma(C)$ fills an open set of its moduli space. The Hodge group of a very general point of that moduli space is SU_H [24,

Theorem 6.11], so the same holds for $A_\sigma(C)$ with C outside a countable union of proper analytic subsets of the component; in particular $A_\sigma(C)$ is then not of CM type and not isogenous to a power of an elliptic curve. The situation is different when $p_\sigma q_\sigma = 0$, for instance when $m_\sigma = 1$: then the factor does not vary.

4. ALGEBRAIC WEIL CLASSES FROM SUPPLIERS

Keep the notation of Section 3, and assume $m_\sigma = 2n$ and $(p_\sigma, q_\sigma) = (n, n)$. We produce the Weil classes of $A_\sigma(C)$ by an algebraic correspondence from a product of auxiliary abelian varieties, each a factor of $J(C)$ carrying a known algebraic class. Milne's theorem on Lefschetz classes makes a class on a product of factors of $J(C)$ algebraic as soon as it is invariant under the centralizer U of G (Proposition 4.2); invariance under U is an identity between characters of the abelianization of U (Subsection 4.1). The auxiliary classes come from two sources, Schoen suppliers (Subsection 4.3) and product suppliers (Subsection 4.4), and Subsection 4.5 combines them.

Notation for this section

$U, T, X^*(T)$	centralizer of G in $Sp(H^1(C), E)$; its maximal abelian quotient; the characters of T
$V_\chi, a(\chi)$	the χ -eigenspace of $H' \leq G$ on $H^1(C, \mathbb{C})$; the character of U on $\det V_\chi$
e_σ	the character of U on the Weil line $\det M_\sigma$
$(H', \chi), P_\chi, h_\chi$	a Schoen supplier; the primitive part of $J(C/\ker \chi)$; $h_\chi = \dim V_\chi$
$(H', \mathbf{n}), a_{H'}(\mathbf{n})$	a product supplier, an integer vector on the characters of H' ; its character
$\psi_k, \ell_{\mathbf{n}}, W_{\mathbf{n}}$	the slot characters; the product of their determinant lines; its rational span
ς	the Hodge circle

4.1. The character group. Let U be the centralizer of G in the symplectic group of $(H^1(C, \mathbb{Q}), E)$, an algebraic group over $\mathbb{Q}$; we only need its complex points. Recall that an irreducible complex representation τ of a finite group is *orthogonal* or *quaternionic* if it is self-dual and carries a nonzero invariant symmetric, respectively alternating, bilinear form (Frobenius–Schur indicator $+1$ or -1), and that otherwise $\tau \not\cong \bar{\tau}$. In $H^1(C, \mathbb{C}) = \bigoplus_\tau \tau \otimes M_\tau$ the G -invariant alternating cup-product form pairs $\tau \otimes M_\tau$ with $\bar{\tau} \otimes M_{\bar{\tau}}$ and is zero between other summands. By Schur's lemma its restriction is the tensor product of the invariant form on τ , unique up to scalar, with a nondegenerate form on the multiplicity spaces: a duality between M_τ and $M_{\bar{\tau}}$ if $\tau \not\cong \bar{\tau}$; a symmetric form on M_τ if τ is quaternionic; an alternating one if τ is orthogonal. An automorphism commuting with G acts through $\prod_\tau GL(M_\tau)$, and preserving the cup-product form means preserving these forms. Hence $U(\mathbb{C})$ is the product of the groups $GL(M_\tau)$ over the pairs $\{\tau, \bar{\tau}\}$ of non-self-dual representations with $m_\tau > 0$ (acting on M_τ and contragrediently on $M_{\bar{\tau}}$), of $O(M_\tau)$ over the quaternionic τ with $m_\tau > 0$, and

of $Sp(M_\tau)$ over the orthogonal τ with $m_\tau > 0$. Let $T = U(\mathbb{C})/[U(\mathbb{C}), U(\mathbb{C})]$ be its maximal abelian quotient. Its character group is

$$X^*(T) = \mathbb{Z}^{\text{pairs}} \oplus (\mathbb{Z}/2)^{\text{quat}},$$

where “pairs” and “quat” run over the pairs and the quaternionic τ with $m_\tau > 0$: one free coordinate for each pair (the determinant on M_τ , for a chosen member τ of the pair; for the pair of σ we choose σ), and one coordinate of order two (the determinant of $O(M_\tau)$) for each quaternionic τ . For a subgroup $H' \leq G$ and a linear character χ of H' let

$$V_\chi = \bigoplus_\tau \text{Hom}_{H'}(\chi, \tau|_{H'}) \otimes M_\tau \subseteq H^1(C, \mathbb{C}),$$

the χ -eigenspace of H' , with Hodge numbers (p_χ, q_χ) . The line $\det V_\chi$ is a U -eigenline, with character

$$a(\chi) = (\langle \tau|_{H'}, \chi \rangle - \langle \bar{\tau}|_{H'}, \chi \rangle)_{\text{pairs}} \oplus (\langle \tau|_{H'}, \chi \rangle \bmod 2)_{\text{quat}}.$$

The line $\det M_\sigma \subset H^{2n}(A_\sigma(C), \mathbb{C})$, one of the two lines of $W_K(A_\sigma) \otimes \mathbb{C}$, has character e_σ : coordinate $+1$ at the pair of σ and 0 elsewhere.

Lemma 4.1. $a(\chi^{-1}) = -a(\chi)$ on the free coordinates, and $a(\chi^{-1}) = a(\chi)$ on the others. Moreover $p_{\chi^{-1}} = q_\chi$.

Proof. Complex conjugation of $H^1(C, \mathbb{C})$ maps V_χ to $V_{\chi^{-1}}$ and $H^{1,0}$ to $H^{0,1}$. It also gives $\langle \tau|_{H'}, \chi^{-1} \rangle = \langle \bar{\tau}|_{H'}, \chi \rangle$, which exchanges the two terms of each free coordinate; a quaternionic τ is self-conjugate. $\square$

The *Hodge circle* $\varsigma: S^1 \rightarrow GL(H^1(C, \mathbb{R}))$, acting by $z^p \bar{z}^q$ on $H^{p,q}$, commutes with G and preserves the polarization, so it lies in U ; it acts on $\det V_\chi$ through $z^{p_\chi} \bar{z}^{q_\chi}$. Restricting characters of T to it gives a homomorphism $\delta_T: X^*(T) \rightarrow \mathbb{Z}$ with $\delta_T(a(\chi)) = p_\chi - q_\chi$.

4.2. The transfer principle.

Proposition 4.2. Let $C \in \text{Hur}$, let $\mathfrak{X} = \prod_j X_j$ be a finite product of abelian varieties each isogenous to the image $\text{im}(N_j e_j) \subseteq J(C)$ of an endomorphism with e_j an idempotent of $\mathbb{Q}[G]$, and let $\Xi \in H^*(\mathfrak{X}, \mathbb{C})$ be fixed by $U(\mathbb{C})$, acting diagonally through the identifications $H^1(X_j) \subseteq H^1(C)$. Then Ξ is a complex linear combination of classes of algebraic cycles on $\mathfrak{X}$.

Proof. Choose one integer N_0 for all factors. The inclusions and the corestrictions of the $N_0 e_j$ give homomorphisms

$$\mathfrak{X} \xrightarrow{i_\mathfrak{X}} J(C)^{R'} \xrightarrow{p_\mathfrak{X}} \mathfrak{X}, \quad p_\mathfrak{X} \circ i_\mathfrak{X} = [N_0],$$

up to isogenies of the factors. They are built from elements of $\mathbb{Q}[G]$, so they commute with U , and $p_\mathfrak{X}^*$ is U -equivariant for the diagonal action of U on $H^1(J(C)^{R'}) = H^1(C)^{\oplus R'}$; hence $p_\mathfrak{X}^* \Xi$ is fixed by $U(\mathbb{C})$. Milne defines $S(A)$, for an abelian variety A , as the algebraic group whose points are the elements γ of the centralizer of $\text{End}^0(A)$ with $\gamma^\dagger \gamma = 1$ for the Rosati involution, “the largest algebraic subgroup of Sp whose elements commute

with the endomorphisms of A^r [14, §1]; it acts on $H^*(A^r) \cong \bigwedge(H^1(A)^{\oplus r})$ diagonally. Since $\mathbb{Q}[G] \subseteq \text{End}^0(J(C))$, the elements of $S(J(C))$ commute with G and preserve E , so $S(J(C)) \subseteq U$, compatibly with the diagonal actions, and $p_{\mathfrak{X}}^* \Xi$ is $S(J(C))$ -invariant. Milne's Theorem 3.2 [14] states that for any abelian variety A over an algebraically closed field and any $r \geq 0$, the $\mathbb{Q}$ -algebra $H^*(A^r)^{S(A)}$ is generated by divisor classes; by his Lemma 3.1 the invariants of the complexification are the complexification of the rational invariants. With $A = J(C)$ and $r = R'$, $p_{\mathfrak{X}}^* \Xi$ is a complex linear combination of products of divisor classes. Since $i_{\mathfrak{X}}^* p_{\mathfrak{X}}^* \Xi = [N_0]^* \Xi$ is a nonzero multiple of each homogeneous component of Ξ , the class Ξ is a complex combination of algebraic classes. $\square$

A class of the form (generator of an eigenline) $\boxtimes$ (generators of complementary eigenlines) is fixed by $U(\mathbb{C})$ exactly when its character in $X^*(T)$ is 0, since characters of $U(\mathbb{C})$ factor through T . This is how the proposition will be applied.

4.3. Schoen suppliers.

Definition 4.3. A *Schoen supplier* is a pair (H', χ) of a subgroup $H' \leq G$ and a character $\chi: H' \rightarrow \mathbb{C}^\times$ of order $m \geq 3$.

For $C \in \text{Hur}$, the curve $C' = C/\ker \chi$ is a connected cyclic cover of C/H' with group $H'/\ker \chi \cong \mathbb{Z}/m$. Its primitive part $P_\chi \subset J(C)$, the part on which $\mathbb{Z}/m$ acts through primitive characters, is cut out by a rational idempotent of $\mathbb{Q}[H']$, and $F_\chi = \mathbb{Q}(\mu_m)$ acts on it. For $t \in (\mathbb{Z}/m)^\times$ let $V_t \subset H^1(P_\chi, \mathbb{C})$ be the eigenspace where H' acts by χ^t ; then

$$V_t = \bigoplus_{\tau} \text{Hom}_{H'}(\chi^t, \tau|_{H'}) \otimes M_{\tau},$$

so $V_t = V_{\chi^t}$ and the character of $\det V_t$ is $a(\chi^t)$; all V_t have the same dimension h_χ , and the Weil classes $W_{F_\chi}(P_\chi) = \bigwedge_{F_\chi}^{h_\chi} H^1(P_\chi, \mathbb{Q})$, a one-dimensional F_χ -vector space, satisfy $W_{F_\chi}(P_\chi) \otimes \mathbb{C} = \bigoplus_t \det V_t$.

The branch data of $C' \rightarrow C/H'$ are read off from the tuple. The points of C/H' over p_j correspond to the double cosets $H' \backslash G / \langle g_j \rangle$; over the double coset of x the local monodromy is $x g_j^\ell x^{-1} \in H'$, with ℓ the least positive integer for which this lies in H' . Let q be the genus of C/H' , s the number of branch points of $C' \rightarrow C/H'$, and $\alpha \in (\mathbb{Z}/m - \{0\})^s$ their exponents with respect to a generator; then $h_\chi = 2q - 2 + s$ [18, Lemma 1.5]. Following Schoen [18, §1], α is *simple* if, after reordering, $\alpha_{2i} = -\alpha_{2i-1}$ for all i . A Schoen supplier is *Schoen-admissible* if h_χ is even and positive and α is simple (automatic when $s = 0$).

Example 4.4. Let $H' = \langle i \rangle N_W(P_3)$, of order 7776, for P_3 a Sylow 3-subgroup of $W = W(E_8)$, in the group $G = W(E_8) \cdot \mu_4$ of Section 8. It has a character χ of order 4 for which C/H' has genus $q = 384479$ and $C/\ker \chi \rightarrow C/H'$ has $s = 88$ branch points, all with exponent 2; so $h_\chi = 769044$, the exponents are simple ($2 \equiv -2$), and the supplier is

Schoen-admissible. It is one of the terms of the identity of Proposition 8.3 (Example 8.6).

Lemma 4.5. *For a subgroup $H' \leq G$ and a character χ of H' , the numbers q, s, h , the exponents α up to $(\mathbb{Z}/m)^\times$ and order, and the Hodge numbers of the eigenspaces V_t , depend only on G and the classes $\mathcal{C}_j$; in particular Schoen-admissibility does.*

Proof. Replacing g_j by a conjugate yg_jy^{-1} replaces the double coset of x by that of xy^{-1} , with the same local monodromy; Riemann–Hurwitz for $C \rightarrow C/H'$ gives q . The Hodge numbers of M_τ are given by Proposition 3.1 in terms of the classes. $\square$

Lemma 4.6 (Schoen). *If (H', χ) is Schoen-admissible, then $W_{F_\chi}(P_\chi)$ is spanned by classes of algebraic cycles, for every $C \in \text{Hur}$.*

Proof. We quote Schoen [18, §2]. Let C' be a smooth projective irreducible curve over $\mathbb{C}$ with a faithful action $\mathbb{Z}/m \hookrightarrow \text{Aut}(C')$, $m > 1$, whose invariant h is even, so that the quotient map has an even number $2r$ of branch points, with exponents $1 \leq \alpha_j \leq m-1$. *Theorem 2.0* (p. 11): if the $(\mathbb{Z}/m)^\times$ -orbit of $(\alpha_1, \dots, \alpha_{2r})$ is simple, then the relevant subspace U of $H^h(C'^h)$ is generated by fundamental classes of algebraic cycles. *Corollary 3.1* (p. 24), for $m > 2$: under the same hypotheses, the corresponding subspace of $H^h(B, \mathbb{Q})$ is generated by classes of algebraic cycles of codimension $h/2$, where $B \subseteq \text{Alb}(C')$ is the primitive part, the image of $\prod_{1 \neq Z' \leq \mathbb{Z}/m} \sum_{t \in Z'} (\text{Id} - t)$. Neither statement asks the exponents to be prime to m : Schoen’s §1 treats a branch point with $\gcd(\alpha_j, m) = d_j > 1$ through the local model $y^{m/d_j} = w^{\alpha_j/d_j}$. Simplicity is automatic for $r \leq 1$, and the unramified case $r = 0$ is included, with its own proof.

Apply this to $C' = C/\ker \chi \rightarrow C/H'$, which is smooth, projective and irreducible as a quotient of C , with $H'/\ker \chi \cong \mathbb{Z}/m$ acting faithfully. The subspace of Corollary 3.1 is W_{F_χ} of the primitive part, which is isogenous to P_χ compatibly with F_χ . Admissibility does not depend on C by Lemma 4.5. When every exponent is $m/2$ and $4 \mid m$, one step of Schoen’s proof is replaced by Lemma 4.7. $\square$

In Section 8 one Schoen supplier of the identity has every exponent equal to $m/2$ with $m = 4$ (Example 8.6). For such covers one step of Schoen’s proof needs a replacement, which we now give.

Recall Schoen’s construction [18, §2, pp. 11–19], for $C' \rightarrow X$ as in Lemma 4.6 with branch points ordered so that $\alpha_{2i} = -\alpha_{2i-1}$, $1 \leq i \leq r$. Let $W_0 = C'^h/G_0$, where $G_0 \subset (\mathbb{Z}/m)^h \rtimes S_h$ is the kernel of the sum map extended by S_h ; the residual group $\mathbb{Z}/m$ acts on W_0 , and $\beta_0: W_0 \rightarrow S^h X$ is a $\mathbb{Z}/m$ -cover branched along the divisors $S_j = b_j + S^{h-1}X$. Schoen chooses a linear subspace $P_0 \subset S^h X$ of dimension $h/2$ (his Lemma 2.3) and, for $i = 1, \dots, r$, blows up $(S^h X)_{i-1}$ along $K_{i-1} = S_{2i-1, i-1} \cap S_{2i, i-1}$, the intersection of the strict transforms of S_{2i-1} and S_{2i} ; the exceptional

divisor E_i is a $\mathbb{P}^1$ -bundle over K_{i-1} . Let W_i be the normalization of the pulled-back cover, $\beta_i: W_i \rightarrow (S^h X)_i$, and $F_i = \beta_i^{-1}(E_i)$. After r steps the strict transform of P_0 avoids the branch locus, its inverse image splits into m components, and these give the cycles of Theorem 2.0 (his Lemmas 2.4–2.6). To see that their classes on W_0 are nonzero, his Lemma 2.8 shows that no primitive character of $\mathbb{Z}/m$ occurs in $H^*(W_i, \mathbb{C})/H^*(W_{i-1}, \mathbb{C})$, and this reduces to the statement that no primitive character occurs in $H^*(F_i, \mathbb{C})$ [18, p. 18]. Schoen proves the latter (Sublemma 2.9, p. 19) by choosing a prime $p \mid m$ such that the $\mathbb{Z}/p$ -quotient of $C' \rightarrow X$ is ramified over b_{2i-1} . When every exponent equals $m/2$ and $4 \mid m$, no such prime exists.

Lemma 4.7 (Schoen’s Theorem 2.0 without the choice of a prime). *In the situation of Schoen’s Theorem 2.0, for every $i \leq r$ and every primitive character χ of $\mathbb{Z}/m$, the χ -part of $H^*(F_i, \mathbb{C})$ vanishes. Hence Theorem 2.0 and Corollary 3.1 of [18] hold under their stated hypotheses, without the choice of a prime in Sublemma 2.9; in particular they hold when every exponent is $m/2$ and $4 \mid m$.*

Proof. Since β_i is finite, $H^*(F_i, \mathbb{C})_\chi = H^*(E_i, (\beta_{i*}\mathbb{C})_\chi|_{E_i})$. The stalk of $\beta_{i*}\mathbb{C}$ at a point is the permutation module on the points above it, and its χ -part vanishes at every point whose stabilizer contains nontrivial inertia, because a primitive character is faithful. By Schoen’s Lemma 2.5, E_i is not a branch divisor of β_i : its multiplicity in the branch locus is $\alpha_{2i-1} + \alpha_{2i} \equiv 0 \pmod{m}$. So on E_i the sheaf $(\beta_{i*}\mathbb{C})_\chi$ is $j_!\mathcal{L}$, where $\mathcal{L}$ is a rank-one local system on the complement in E_i of the strict transforms $S_{j,i}$, with monodromy $\chi(\alpha_j) \neq 1$ around $S_{j,i}$. The strict transforms $S_{2i-1,i}$ and $S_{2i,i}$ meet E_i in two disjoint sections of $p: E_i \rightarrow K_{i-1}$, and every other $S_{j,i}$ meets E_i in a union of whole fibres. Such a fibre lies in the closed set $S_{j,i}$, so the restriction of $j_!\mathcal{L}$ to it is the zero sheaf. On any other fibre $\mathbb{P}^1$ it is the extension by zero of a rank-one local system on $\mathbb{C}^\times = \mathbb{P}^1 - \{0, \infty\}$ with monodromy $\chi(\alpha_{2i-1}) \neq 1$, whose compactly supported cohomology vanishes: $H_c^0 = 0$, H_c^2 is dual to the invariants of the dual system, which are 0, and the Euler characteristic is 0. Since p is proper and $j_!\mathcal{L}$ is constructible, proper base change identifies the stalk of $R^b p_*(j_!\mathcal{L})$ at $k \in K_{i-1}$ with $H^b(p^{-1}(k), j_!\mathcal{L}|_{p^{-1}(k)})$ [17, Theorem 11.3.6], which we have just shown to vanish. Hence $Rp_*(j_!\mathcal{L}) = 0$ and $H^*(E_i, j_!\mathcal{L}) = H^*(K_{i-1}, Rp_* j_!\mathcal{L}) = 0$. The rest of Schoen’s proof is unchanged. $\square$

4.4. Product suppliers. A single line $\det V_\chi$ is of Hodge type only when $p_\chi = q_\chi$. Several such lines, none of Hodge type, can have a product that is of Hodge type, and Section 5 shows that, under conditions on the branching, such a product spans algebraic classes.

Definition 4.8. A *product supplier* is a pair $(H', \mathbf{n})$ of a subgroup $H' \leq G$ and a finitely supported integer vector $\mathbf{n} = (n_\chi)$ on the linear characters χ of H' with $\dim V_\chi > 0$, such that every χ with $n_\chi \neq 0$ has order at least 3. Let $M_{\mathbf{n}}$ be the least common multiple of the orders of these χ . Since χ^t

depends only on t modulo the order of χ , the numbers

$$\delta_t(\mathbf{n}) = \sum_{\chi} n_{\chi} (p_{\chi^t} - q_{\chi^t}), \quad t \in (\mathbb{Z}/M_{\mathbf{n}})^{\times},$$

are well defined. The supplier is *balanced* if $\delta_t(\mathbf{n}) = 0$ for every $t \in (\mathbb{Z}/M_{\mathbf{n}})^{\times}$. Its character is $a_{H'}(\mathbf{n}) = \sum_{\chi} n_{\chi} a(\chi) \in X^*(T)$. An *integral lift* of a vector $v \in X^*(T)$ is an integer vector $\mathbf{n}$ with $a_{H'}(\mathbf{n}) = v$; different lifts of one vector give different products of lines.

By the last subsection, $\delta_1(\mathbf{n}) = \delta_T(a_{H'}(\mathbf{n}))$ depends only on the character of the supplier.

Fix a balanced product supplier and $C \in \text{Hur}$. Put $\kappa_{\mathbf{n}} = \bigcap_{n_{\chi} \neq 0} \ker \chi$, a normal subgroup of H' with abelian quotient $\mathbf{A} = H'/\kappa_{\mathbf{n}}$, and $D = C/\kappa_{\mathbf{n}}$, a connected Galois $\mathbf{A}$ -cover of $X = C/H'$; the characters χ with $n_{\chi} \neq 0$ factor through $\mathbf{A}$, and V_{χ} is identified with $H^1(D, \mathbb{C})_{\chi}$ by pullback. Let $R = \sum_{\chi} |n_{\chi}|$. A *slot* is one of the R occurrences: the character χ , used n_{χ} times, if $n_{\chi} > 0$, and the character χ^{-1} , used $|n_{\chi}|$ times, if $n_{\chi} < 0$. List the slot characters as $\psi_1, \dots, \psi_R$ and put

$$\ell_{\mathbf{n}} = \bigotimes_{k=1}^R \det H^1(D, \mathbb{C})_{\psi_k} \subset H^{d_{\mathbf{n}}}(J(D)^R, \mathbb{C}),$$

with $d_{\mathbf{n}} = \sum_k \dim H^1(D, \mathbb{C})_{\psi_k}$, an external product with one factor $J(D)$ for each slot. A negative coefficient thus contributes the determinant of the opposite character, in positive degree. For $t \in (\mathbb{Z}/M_{\mathbf{n}})^{\times}$ let $\ell_{\mathbf{n}}^{(t)}$ be the line obtained by replacing each ψ_k by ψ_k^t .

Lemma 4.9. *The distinct lines among the $\ell_{\mathbf{n}}^{(t)}$ are independent, and their sum is $W_{\mathbf{n}} \otimes \mathbb{C}$ for a $\mathbb{Q}$ -subspace $W_{\mathbf{n}} \subseteq H^{d_{\mathbf{n}}}(J(D)^R, \mathbb{Q})$ consisting of Hodge classes of type $(d_{\mathbf{n}}/2, d_{\mathbf{n}}/2)$. The line $\ell_{\mathbf{n}}$ is a U -eigenline with character $a_{H'}(\mathbf{n})$ on the free coordinates and $\sum_{\chi} |n_{\chi}| a(\chi)$ on the others.*

Proof. Two distinct lines $\ell^{(t)}$, $\ell^{(t')}$ differ in some slot, where they lie in the determinants of two different character eigenspaces of $\mathbf{A}$ on $H^1(D)$, each of positive dimension; so they lie in different $\mathbf{A}^R$ -isotypic summands of $H^*(J(D)^R, \mathbb{C})$, and the sum is direct. The eigenspaces are defined over $\mathbb{Q}(\mu_{M_{\mathbf{n}}})$, and the automorphism $\zeta \mapsto \zeta^t$ of that field maps $\ell_{\mathbf{n}}$ to $\ell_{\mathbf{n}}^{(t)}$, because $H^1(D)$ is defined over $\mathbb{Q}$; so the sum of the distinct lines is Galois stable and descends to $\mathbb{Q}$. The line $\ell^{(t)}$ has Hodge type $(P, d_{\mathbf{n}} - P)$ with $P - (d_{\mathbf{n}} - P) = \sum_k (p_{\psi_k^t} - q_{\psi_k^t}) = \delta_t(\mathbf{n})$, by Lemma 4.1 for the negative slots; balance makes it $(d_{\mathbf{n}}/2, d_{\mathbf{n}}/2)$. The character statement is Lemma 4.1. $\square$

Definition 4.10. A balanced product supplier $(H', \mathbf{n})$ is *product-admissible* if $X = C/H'$ has genus $g_X \geq 1$ and every slot character ψ_k satisfies hypotheses (H2) and (H3) of Theorem 5.1: the cyclic cover $D/\ker \psi_k \rightarrow X$ has at least two branch points, and e_k/d_k divides 12, with e_k the order of ψ_k and d_k the greatest common divisor of e_k and its branch exponents.

Example 4.11. In the group G of Section 8, let $w_1 \in W$ have order 3 with centralizer of order 311 040, and $H' = \langle i \rangle C_W(w_1)$. Then $X = C/H'$ has

genus 3259, and there are characters χ_2, χ_7 of H' such that $\mathbf{n} = -\chi_2 - \chi_7$ is a balanced product supplier with two slots $\psi_1 = \bar{\chi}_2$ (order 4, 6732 branch points) and $\psi_2 = \bar{\chi}_7$ (order 12, 20 branch points), both with $d_k = 1$. Neither $\det H^1(D)_{\psi_k}$ is of Hodge type, while their product is, at every t . Example 8.5 follows it through the proof of Theorem C.

By Lemma 4.5 product-admissibility depends only on the class datum. Balance is hypothesis (H4) of Theorem 5.1, since $2p_{\psi_k^t} - h_k = p_{\psi_k^t} - q_{\psi_k^t}$. Hence:

Corollary 4.12. *If $(H', \mathbf{n})$ is a product-admissible supplier, then for every $C \in \text{Hur}$ the space $W_{\mathbf{n}}$ is spanned by classes of algebraic cycles on $J(D)^R$.*

Proof. Theorem 5.1 and Corollary 5.2, applied to $D = C/\kappa_{\mathbf{n}} \rightarrow X = C/H'$ and the slot characters. $\square$

4.5. The contraction.

Definition 4.13. Let $L(G; \mathcal{C}_1, \dots, \mathcal{C}_r) \subseteq X^*(T)$ be the subgroup generated by the $a(\chi)$ of the Schoen-admissible suppliers and the $a_{H'}(\mathbf{n})$ of the product-admissible suppliers. *Condition (L)* is that $e_{\sigma} \in L$.

Proposition 4.14. *Let $C \in \text{Hur}$ and let $X' = \prod_j X'_j$ be a finite product of abelian subvarieties $X'_j = \text{im}(N_j e_j) \subseteq J(C)$, with e_j idempotents of $\mathbb{Q}[G]$. Let $\ell \subset H^p(X', \mathbb{C})$ be an eigenline of U whose character ξ lies in L . Then ℓ lies in the complex span of classes of algebraic cycles on X' .*

We first record the complements of the supplier lines.

Lemma 4.15 (Complements). (a) *Let (H', χ) be a Schoen supplier with eigenspaces V_t , $t \in (\mathbb{Z}/m)^\times$, and $t_0 = \pm 1$. Choose generators z_t of the lines $\det V_t$ and put $Z = \prod_{t \neq t_0} z_t$. Then Z is a U -eigenline with character $-t_0 a(\chi)$ on the free coordinates and $a(\chi)$ on the others, and for $v \in \det V_t$, $v \cup Z = 0$ unless $t = t_0$, while $\cup Z$ is nonzero on $\det V_{t_0}$. (b) Let $(H', \mathbf{n})$ be a balanced product supplier. On the copy of $J(D)$ belonging to a slot with character ψ , choose a generator of the top exterior power of the sum of all A -eigenspaces of $H^1(D, \mathbb{C})$ other than $H^1(D, \mathbb{C})_{\psi}$, and let Z be the external product of these over the slots. Then Z has the character opposite to that of $\ell_{\mathbf{n}}$, $\cup Z$ is nonzero on $\ell_{\mathbf{n}}$, and $v_L \cup Z = 0$ for every line $L \neq \ell_{\mathbf{n}}$ among the $\ell_{\mathbf{n}}^{(t)}$.*

Proof. (a) The product $\prod_t z_t$ spans the top cohomology of P_{χ} , on which U acts trivially, so Z has the character opposite to $a(\chi^{t_0})$, which Lemma 4.1 computes. For $t \neq t_0$ the product $v \cup Z$ contains $\det V_t$ twice. (b) The product of Z with $\ell_{\mathbf{n}}$ spans the top cohomology of $J(D)^R$, on which U acts trivially. A line $L \neq \ell_{\mathbf{n}}$ differs from $\ell_{\mathbf{n}}$ in some slot, where its factor already occurs in the complement. $\square$

Proof of Proposition 4.14. Write $\xi = \sum_i n_i a(\chi_i) + \sum_l a_{H'_l}(\mathbf{n}_l)$ with Schoen-admissible suppliers (H'_i, χ_i) , $n_i \neq 0$, and product-admissible suppliers

$(H'_l, \mathbf{n}_l)$; a product term with a coefficient c is absorbed by replacing $\mathbf{n}_l$ by $c\mathbf{n}_l$. This is again product-admissible: $\delta_t(c\mathbf{n}_l) = c\delta_t(\mathbf{n}_l) = 0$; its slots are those of $\mathbf{n}_l$ repeated $|c|$ times, inverted when $c < 0$; and ψ^{-1} has the same branch points, order and gcd d as ψ (its exponents are the negatives), so (H2) and (H3) still hold. Its character is $a_{H'_l}(c\mathbf{n}_l) = c a_{H'_l}(\mathbf{n}_l)$. Put $t_i = \text{sign } n_i$, $P_i = P_{\chi_i}$, $F_i = F_{\chi_i}$. By Lemma 4.6 choose a nonzero rational algebraic class $y_i \in W_{F_i}(P_i)$, and by Corollary 4.12 a nonzero rational algebraic class $y_l \in W_{\mathbf{n}_l}$ on $J(D_l)^{R_l}$, $D_l = C/\kappa_{\mathbf{n}_l}$. Put $\mathfrak{B} = \prod_i P_i^{|n_i|} \times \prod_l J(D_l)^{R_l}$ and $\mathbf{y} = \boxtimes_i y_i^{\boxtimes |n_i|} \boxtimes \boxtimes_l y_l \in H^*(\mathfrak{B}, \mathbb{Q})$, an algebraic class.

The correspondence. With the complements Z_i (for $t_0 = t_i$) and Z_l of Lemma 4.15 and a generator x of ℓ , put

$$\Xi = \left(\boxtimes_i Z_i^{\boxtimes |n_i|} \boxtimes \boxtimes_l Z_l \right) \boxtimes x \in H^*(\mathfrak{B} \times X', \mathbb{C}).$$

By Lemmas 4.15 and 4.9 its character is $\xi - \sum_i n_i a(\chi_i) - \sum_l a_{H'_l}(\mathbf{n}_l) = 0$ on the free coordinates. On the coordinates of order two, Z_i has character $a(\chi_i)$ whatever the sign of n_i , and $|n_i|$ copies give $|n_i|a(\chi_i) \equiv n_i a(\chi_i) \pmod{2}$. Likewise Z_l has the character of $\ell_{\mathbf{n}_l}$ there, $\sum_{\chi} |n_{l,\chi}| a(\chi) \equiv a_{H'_l}(\mathbf{n}_l) \pmod{2}$, because $a(\chi^{-1}) = a(\chi)$ on these coordinates (Lemma 4.1). So the character of Ξ is $\xi - \sum_i n_i a(\chi_i) - \sum_l a_{H'_l}(\mathbf{n}_l) = 0$ there too. (For the group of Section 8 there are no coordinates of order two.) Every factor of $\mathfrak{B} \times X'$ is isogenous to the image of an endomorphism $N_0 e'$ of $J(C)$ with e' an idempotent of $\mathbb{Q}[G]$: for P_i and X'_j by construction, and for $J(D_l)$ with e' the averaging idempotent of $\kappa_{\mathbf{n}_l}$, since pullback identifies $H^1(D_l)$ with $H^1(C)^{\kappa_{\mathbf{n}_l}}$. So Proposition 4.2 makes Ξ a complex combination of algebraic classes.

The contraction is nonzero. Write $y_i = \sum_t c_{i,t} v_{i,t}$ with $v_{i,t} \in \det V_{i,t}$ normalized by $\int_{P_i} v_{i,t_i} \cup Z_i = 1$. Each $c_{i,t}$ is nonzero, because $W_{F_i}(P_i)$ is a one-dimensional F_i -vector space and a nonzero vector has a nonzero component under every embedding of F_i . Similarly write $y_l = \sum_{L} c_{l,L} v_L$ over the distinct lines L of Lemma 4.9, with $v_{\ell_{\mathbf{n}_l}}$ normalized against Z_l ; the coefficient $c_{l,\ell_{\mathbf{n}_l}}$ is nonzero because y_l is rational and nonzero and the Galois action on coefficients permutes the distinct lines transitively. By Lemma 4.15,

$$\Xi_*(\mathbf{y}) = \text{pr}_{2*}(\text{pr}_1^* \mathbf{y} \cup \Xi) = \pm \prod_i c_{i,t_i}^{|n_i|} \prod_l c_{l,\ell_{\mathbf{n}_l}} x \neq 0,$$

and x lies in the complex span of algebraic classes. $\square$

Corollary 4.16. *If condition (L) holds, then $W_K(A_\sigma(C))$ is spanned by classes of algebraic cycles for every $C \in \text{Hur}$, on every component.*

Proof. $W_K(A_\sigma) \otimes \mathbb{C} = \det M_\sigma \oplus \det M_{\bar{\sigma}}$, two eigenlines of U with characters $\pm e_\sigma$ on the free coordinates and 0 on the others, so both characters lie in L . Proposition 4.14 puts both in the complex span of algebraic classes. If $\mathcal{Z}$ is the rational span of algebraic classes and $W \otimes \mathbb{C} \subseteq \mathcal{Z} \otimes \mathbb{C}$ for a rational subspace W , then $W \subseteq \mathcal{Z}$. $\square$

Remark 4.17. The identity $e_\sigma \in L$ must hold with integer coefficients. An identity for a multiple ke_σ would give a k -fold tensor power of the target line, and no k -th root of it. Large coefficients only require finitely many copies of each factor.

5. PRODUCTS OF DETERMINANT LINES ON ABELIAN COVERS: THEOREM C

This section is independent of the rest of the paper: it uses only the definitions of Subsection 5.1, and Corollary 5.2 connects it with Section 4. The proof runs as follows. For a single cyclic cover $D' \rightarrow X$ with a character ψ , Schoen constructs a variety W_0 fibred over $\text{Pic}^h(X)$ whose ψ -cohomology is the single line $\det H^1(D')_\psi$. That line is supported over a single point L_0 of $\text{Pic}^h(X)$, and over L_0 the cover is, up to a vector bundle, an open piece of a Fermat hypersurface divided by a finite group. So the line is, up to a Tate twist, a character line on a Fermat hypersurface. For a product of factors only the total Hodge type matters, so a product of lines that is of Hodge type corresponds to a Hodge class on a product of Fermat hypersurfaces. That class is algebraic by Shioda's theorem, and a cone over its cycle, followed by Schoen's Abel–Prym correspondence, gives algebraic classes in the product line. For a single factor, Schoen observed that the Hodge condition is exactly the condition for the corresponding Fermat character to be of Hodge type [18, Corollary 1.9 and Remark 1.10, p. 10]; the construction below turns this correspondence of Hodge structures into one of cycles.

Notation for this section

X, g_X	the base curve and its genus
$A, D \rightarrow X$	finite abelian group, connected Galois A -cover
$\psi_k, k = 1, \dots, R$	the slots: characters of A , of orders $e_k \geq 3$
D_k, Γ_k	the cyclic cover $D/\ker \psi_k \rightarrow X$ and its group
s_k, h_k, d_k, e'_k	branch points, $h_k = 2g_X - 2 + s_k$, gcd of e_k and the exponents, $e'_k = e_k/d_k$
M_0	$\text{lcm}_k e_k$
$h_{\text{tot}}, q_{\text{tot}}, \nu$	$\sum_k h_k, Rg_X, \frac{1}{2} \sum_k (s_k - 2)$
L_0, Π_0	the special point $K_X + \sum_j b_j$ of $\text{Pic}^h(X)$; the common base locus $\sum_j b_j + K_X $ of the hyperplanes H_j in $ L_0 $
Ω	the complement of s general hyperplanes in $\mathbb{P}^{s-2}$
ψ	the character $\boxtimes_k \psi_k$ of $\Gamma_{\text{tot}} = \prod_k \Gamma_k$

5.1. Setting and statement. Let X be a smooth projective curve of genus g_X , A a finite abelian group, and $D \rightarrow X$ a connected Galois cover with group A . Fix characters $\psi_1, \dots, \psi_R$ of A (repetitions allowed), the *slots*, of orders $e_k \geq 3$. For each slot:

- $D_k = D/\ker \psi_k \rightarrow X$ is a connected cyclic cover with group $\Gamma_k = A/\ker \psi_k$, cyclic of order e_k , on which ψ_k is faithful;

- its branch points are the points $b \in X$ whose inertia group in $\mathbf{A}$ is not contained in $\ker \psi_k$; there are s_k of them, and $h_k = 2g_X - 2 + s_k$;
- the *exponent* $\alpha_{k,b} \in \mathbb{Z}/e_k - \{0\}$ at b is defined by $\psi_k(g_b) = e^{2\pi i \alpha_{k,b}/e_k}$, for the generator g_b of inertia given by a small positive loop around b ; and $d_k = \gcd(e_k, \alpha_{k,b} : b \text{ a branch point})$, $e'_k = e_k/d_k$, the *reduced order*.

By Chevalley–Weil, $H^1(D_k, \mathbb{C})_{\psi_k}$ has dimension h_k [18, Lemma 1.5], and pullback identifies it with $H^1(D, \mathbb{C})_{\psi_k}$. For t prime to M_0 put $p_{k,t} = \dim H^{1,0}(D_k)_{\psi_k^t}$. Define the lines

$$\ell^{(t)} = \bigotimes_k \det H^1(D_k, \mathbb{C})_{\psi_k^t} \subset H^{h_{\text{tot}}}(\prod_k J(D_k), \mathbb{C}), \quad h_{\text{tot}} = \sum_k h_k.$$

The distinct lines among them are independent, their sum is stable under the Galois action on coefficients, and so it is $W \otimes \mathbb{C}$ for a rational subspace $W \subseteq H^{h_{\text{tot}}}(\prod_k J(D_k), \mathbb{Q})$; the proof is that of Lemma 4.9.

Theorem 5.1 (Theorem C). *Assume*

- (H1) $g_X \geq 1$;
- (H2) $s_k \geq 2$ for every k ;
- (H3) e'_k divides 12 for every k ;
- (H4) $\sum_k (2p_{k,t} - h_k) = 0$ for every $t \in (\mathbb{Z}/M_0)^\times$.

Then W is spanned by classes of algebraic cycles on $\prod_k J(D_k)$.

Corollary 5.2. *Under the same hypotheses, the space $W_{\mathbf{n}}$ of Lemma 4.9, formed on $J(D)^R$ with one copy of $J(D)$ for each slot, is spanned by classes of algebraic cycles.*

Proof. For each slot, the norm homomorphism $J(D) \rightarrow J(D_k)$ induces on H^1 the pullback $H^1(D_k) \rightarrow H^1(D)$, which identifies $H^1(D_k, \mathbb{C})_{\psi_k^t}$ with $H^1(D, \mathbb{C})_{\psi_k^t}$ for every t . The product of these homomorphisms $J(D)^R \rightarrow \prod_k J(D_k)$ pulls $\ell^{(t)}$ back to $\ell_{\mathbf{n}}^{(t)}$, so it maps W onto $W_{\mathbf{n}}$, and pullback by a homomorphism of abelian varieties preserves algebraic classes. $\square$

(H4) says that every $\ell^{(t)}$ is of Hodge type: $\ell^{(t)}$ has type $(P_t, h_{\text{tot}} - P_t)$ with $P_t = \sum_k p_{k,t}$. Under (H1) and (H2) we have $h_k \geq 2g_X \geq 2$, and h_{tot} is even by (H4). Hypothesis (H1) is used in Lemma 5.6, where the special fibre is a single point of $\text{Pic}^h(X)$ with base locus $\Pi_0 \cong \mathbb{P}^{g_X-1}$; when $g_X = 0$, $\text{Pic}^h(X)$ is a point and no localisation is needed, but we do not treat that case. In the application $g_X \geq 3259$. Hypothesis (H3) is used only in Lemma 5.11, to map Fermat varieties of the single degree 12 onto the Fermat models of all slots; the same proof works whenever all e'_k divide one integer m for which Shioda's theorem gives the Hodge conjecture for products of Fermat varieties of degree m (m prime or $m \leq 20$, Theorem 5.10). We state the case $m = 12$, which the application needs.

Example 5.3. For the supplier of Example 4.11, $g_X = 3259$, and the two slots have $(e_k, s_k, d_k) = (4, 6732, 1)$ and $(12, 20, 1)$, so $h_1 = 13\,248$, $h_2 =$

6536, $e'_1 = 4$ and $e'_2 = 12$. The Fermat models of Subsection 5.4 are the Fermat hypersurfaces of degree 4 and dimension 6730 and of degree 12 and dimension 18. Example 8.5 computes their Hodge types and the cone.

Conventions. Fix one slot and drop the index k until Subsection 5.5. Write $D' = D_k$, $\Gamma = \mathbb{Z}/e$, ψ faithful, branch points $b_1, \dots, b_s$ with exponents $\alpha_1, \dots, \alpha_s \neq 0$ as above, and $h = 2g_X - 2 + s$. Loops around points of X , and meridians of divisors below, are oriented positively for the complex orientation. With these conventions, the Chevalley–Weil formula reads $p = \dim H^{1,0}(D')_\psi = g_X - 1 + \sum_j \{-\alpha_j/e\}$, with $\{\cdot\}$ the fractional part; the opposite convention replaces ψ by $\bar{\psi}$ throughout, which does not affect the statements.

Three standard facts. (F1) $\sum_j \alpha_j \equiv 0 \pmod{e}$: in π_1 of X minus the branch points, the product of the loops around the branch points is a product of commutators, whose image in the abelian group Γ is trivial. (F2) For $L \in \text{Pic}^h(X)$ the divisor S_j meets the fibre $|L|$ of π transversally, in the hyperplane $H_j = |L - b_j|$: the restriction of $\mathcal{O}(S_j)$ to $|L|$ is $\mathcal{O}(1)$. So a meridian of H_j in $|L|$ is a meridian of S_j in $S^h X$. (F3) The ψ -parts below are defined over $\mathbb{Q}(\mu_e)$, not over $\mathbb{Q}$; “mixed Hodge structure” on a ψ -part means complex mixed Hodge structure.

5.2. Schoen’s construction. Following [18, §2], let $\hat{G} = \Gamma^h \rtimes S_h$ act on D^h , let $\mathbf{N} \subset \Gamma^h$ be the kernel of the sum map $\Gamma^h \rightarrow \Gamma$, and $G_0 = \mathbf{N} \rtimes S_h$. Put $W_0 = D^h/G_0$. The quotient $\hat{G}/G_0 \cong \Gamma$ acts on W_0 with quotient $S^h X$, giving a Γ -cover $\beta: W_0 \rightarrow S^h X$, and the Abel–Jacobi map $\pi: S^h X \rightarrow \text{Pic}^h(X)$ is a projective bundle with fibres $|L| \cong \mathbb{P}^{h-g_X}$, since $h \geq 2g_X - 1$. Let $S_j = b_j + S^{h-1}X \subset S^h X$, the divisors of the branch points.

Lemma 5.4 ([18, Lemmas 1.1–1.2, pp. 5–6]). *The ψ -isotypic part of $H^*(W_0, \mathbb{C})$ for the action of Γ is one-dimensional, concentrated in degree h , and its pullback to D^h is the line $U_\psi = \bigwedge^h H^1(D', \mathbb{C})_\psi$, embedded in $H^h(D^h, \mathbb{C})$ by the Künneth formula.*

Proof. $H^*(W_0, \mathbb{C}) = H^*(D^h, \mathbb{C})^{G_0}$. Decompose $H^*(D^h) = H^*(D')^{\otimes h}$ into the tensor products $\bigotimes_i H^*(D')_{\chi_i}$ of eigenspaces of Γ . An element $(\gamma_i) \in \mathbf{N}$ acts on such a product by $\prod_i \chi_i(\gamma_i)$, and this is 1 for all (γ_i) with $\sum_i \gamma_i = 0$ exactly when all χ_i are equal; the residual Γ then acts by the common character. For $\chi = \psi$ faithful, $H^0(D')$ and $H^2(D')$ have no ψ -part, so the ψ -part is $H^1(D')_\psi^{\otimes h}$, and invariance under S_h , with the Koszul sign, gives its top exterior power $\bigwedge^h H^1(D')_\psi$, which is one-dimensional because $\dim H^1(D')_\psi = h$. $\square$

By Schoen’s Remark 1.4 [18, pp. 6–7], U_ψ is the pullback, under the sum map $D^h \rightarrow J(D')$, of the line $\det H^1(D', \mathbb{C})_\psi \subset H^h(J(D'), \mathbb{C})$.

5.3. Localisation.

Lemma 5.5. *Let j be the inclusion of $S^h X - \bigcup_j S_j$. The ψ -isotypic part of $\beta_* \mathbb{C}$ is $j_! \mathcal{L}$ for a rank-one local system $\mathcal{L}$ whose monodromy around S_j is $\psi(g_{b_j}) \neq 1$.*

Proof. The stalk of $\beta_* \mathbb{C}$ at a point is $\mathbb{C}[\Gamma/\Gamma_y]$, with Γ_y the stabilizer of a point y above it, and its ψ -part is nonzero exactly when ψ is trivial on Γ_y . At a divisor containing some b_j , with multiplicity $\mu \geq 1$, a point of D^h above it has μ coordinates over b_j , each fixed by the inertia group $\langle g_{b_j} \rangle$; the element of $\hat{G}$ that applies g_{b_j} to one such coordinate fixes the point, and its image in Γ is g_{b_j} . So $\Gamma_y \ni g_{b_j}$, on which the faithful ψ is nontrivial, and the ψ -part vanishes there. Off $\bigcup_j S_j$, coincident points are permuted by $S_h \subset G_0$, so β is étale for the residual action, and $(\beta_* \mathbb{C})_\psi$ is a rank-one local system $\mathcal{L}$. Near a general point of S_j , write the divisor as $x_1 + x_2 + \dots + x_h$ with x_1 near b_j and the other points distinct and unbranched. Moving x_1 once around b_j lifts, in D^h , to a path from a point $(y_1, \dots, y_h)$ to $(g_{b_j} y_1, y_2, \dots, y_h)$, which differs from the starting point by the element $(g_{b_j}, 0, \dots, 0)$ of Γ^h , whose image in $\hat{G}/G_0 = \Gamma$ is g_{b_j} . So the monodromy of $\mathcal{L}$ around S_j is $\psi(g_{b_j})$. $\square$

For $L \in \text{Pic}^h(X)$ the divisors in $|L|$ that contain b_j form a hyperplane $H_j = |L - b_j|$: since $\deg(L - b_j) = h - 1 \geq 2g_X - 1$, the point b_j is not a base point. By Riemann–Roch, $h^0(L - \sum_j b_j)$, of degree $2g_X - 2$, equals $g_X - 1$ unless $L = L_0 := K_X + \sum_j b_j$, when it is g_X ; and any $s - 1$ of the hyperplanes are independent, because L minus $s - 1$ of the points has degree $2g_X - 1$.

Lemma 5.6. *Assume (H1) and (H2).*

- (a) *For $L \neq L_0$, $|L| - \bigcup H_j \cong (\mathbb{C}^\times)^{s-1} \times \mathbb{A}^{g_X-1}$, and $H_c^*(|L| - \bigcup H_j, \mathcal{L}) = 0$.*
- (b) *If $s \geq 3$, all H_j in $|L_0|$ contain $\Pi_0 = \sum_j b_j + |K_X| \cong \mathbb{P}^{g_X-1}$, and projection from Π_0 makes $|L_0| - \bigcup H_j$ the total space of a vector bundle of rank g_X over $\Omega = \mathbb{P}^{s-2}$ minus s hyperplanes in general position.*
- (c) *If $s = 2$, then $H_1 = H_2$ in $|L_0|$, and the complement $|L_0| - H_1$ is $\mathbb{A}^{g_X}$, on which $\mathcal{L}$ is trivial.*

Proof. (a) Off L_0 the s linear forms defining the H_j are linearly independent on $H^0(L)$, of dimension $h - g_X + 1 \geq s$. Choose homogeneous coordinates $x_0, \dots, x_{h-g_X}$ on $|L|$ with $H_j = \{x_{j-1} = 0\}$ for $j = 1, \dots, s$. Then $|L| - \bigcup H_j$ is the set where $x_0, \dots, x_{s-1}$ are nonzero, and in the chart $x_0 = 1$ it is $(\mathbb{C}^\times)^{s-1} \times \mathbb{A}^{h-g_X-s+1}$, with $h - g_X - s + 1 = g_X - 1$. A small loop in the j -th factor $\mathbb{C}^\times$, $j = 1, \dots, s - 1$, is a meridian of H_{j+1} , around which $\mathcal{L}$ has monodromy $\psi(g_{b_{j+1}}) \neq 1$ (Lemma 5.5 and (F2)). The compactly supported cohomology of $\mathbb{C}^\times$ with a nontrivial rank-one local system vanishes, so the Künneth formula for compact supports [17, Corollary 11.3.11] gives the vanishing. (b) At L_0 the intersection of all H_j is $|L_0 - \sum b_j| = \sum b_j + |K_X|$, of dimension $g_X - 1$. Hyperplanes containing the centre of projection descend

to hyperplanes of $\mathbb{P}^{h-2g_X} = \mathbb{P}^{s-2}$, and they are in general position because any $s-1$ of the H_j are independent. (c) For $s=2$, $L_0 - b_1 = K_X + b_2$ has b_2 as a base point, so $H_1 = b_1 + b_2 + |K_X| = H_2$, a hyperplane of $|L_0| \cong \mathbb{P}^{g_X}$; its complement $\mathbb{A}^{g_X}$ is simply connected. $\square$

Proposition 5.7 (Localisation). *Assume (H1) and (H2). Put $W^\circ = \beta^{-1}(|L_0| - \bigcup_j H_j)$. Restriction gives isomorphisms*

$$U_\psi \cong H^h(W_0, \mathbb{C})_\psi \xrightarrow{\sim} H^h(\beta^{-1}|L_0|, \mathbb{C})_\psi \cong H_c^h(W^\circ, \mathbb{C})_\psi$$

of (complex) mixed Hodge structures on the ψ -parts, and both sides are one-dimensional. More generally, in every degree b , restriction $H^b(W_0, \mathbb{C})_\psi \rightarrow H^b(\beta^{-1}|L_0|, \mathbb{C})_\psi$ and extension by zero $H_c^b(W^\circ, \mathbb{C})_\psi \rightarrow H^b(\beta^{-1}|L_0|, \mathbb{C})_\psi$ are isomorphisms, and these spaces vanish for $b \neq h$.

Proof. By Lemma 5.5, $H^*(W_0, \mathbb{C})_\psi = H^*(S^h X, j_! \mathcal{L}) = H^*(\text{Pic}^h(X), R\pi_* j_! \mathcal{L})$, hypercohomology of the derived direct image. Since π is proper and $j_! \mathcal{L}$ is constructible, proper base change [17, Theorem 11.3.6 and (11.3.3)] identifies the stalk of $R^b \pi_* j_! \mathcal{L}$ at L with $H^b(|L|, j_! \mathcal{L}) = H_c^b(|L| - \bigcup H_j, \mathcal{L})$, which vanishes for $L \neq L_0$ by Lemma 5.6(a). So every cohomology sheaf of $R\pi_* j_! \mathcal{L}$ is supported at the point L_0 , the complex is a skyscraper at L_0 , and its hypercohomology in degree b is its stalk $H^b(|L_0|, j_! \mathcal{L}) = H^b(\beta^{-1}|L_0|, \mathbb{C})_\psi$, which is $H_c^b(|L_0| - \bigcup H_j, \mathcal{L})$ through extension by zero; the restriction map to $\beta^{-1}|L_0|$ is this identification, in every degree. The same lemma identifies the stalk with $H_c^b(W^\circ, \mathbb{C})_\psi$, compactly supported cohomology of the open part. Lemma 5.4 gives the dimension and the degree $b = h$. The maps are restrictions and the identification of a $j_!$ -extension with compactly supported cohomology of the open part, so they are morphisms of mixed Hodge structures. $\square$

5.4. The special fibre and the Fermat hypersurface. Assume first $s \geq 3$, and put $r = s - 2$, $d = \gcd(e, \alpha_1, \dots, \alpha_s)$, $e' = e/d$. Choose coordinates in which $\Omega = \{x \in \mathbb{P}^{s-1} : \sum_j x_j = 0, x_j \neq 0 \text{ for all } j\}$ (any s hyperplanes in general position in $\mathbb{P}^{s-2}$ can be put in this form). Let

$$F = F_{e'}^r = \left\{ Y \in \mathbb{P}^{s-1} : \sum_j Y_j^{e'} = 0 \right\},$$

the Fermat hypersurface of degree e' and dimension r , and $F^\circ \subset F$ the open set where all $Y_j \neq 0$. The map $Y \mapsto (Y_j^{e'})_j$ is a Galois cover $F^\circ \rightarrow \Omega$ with group $\Gamma_F = \mu_{e'}^s / \mu_{e'}$ (diagonal). A vector $a \in (\mathbb{Z}/e')^s$ with $\sum_j a_j = 0$ defines a character $\zeta \mapsto \prod_j \zeta_j^{a_j}$ of Γ_F . For such a character b let $V(b) = H^r(F, \mathbb{C})_b$.

Lemma 5.8 (The special fibre). *Assume (H1) and $s \geq 3$, and let $a = \varepsilon(\alpha_j/d)_j$, where $\varepsilon \in \{\pm 1\}$ is the sign relating the deck action on $F^\circ \rightarrow \Omega$ to monodromy; it depends only on the conventions, not on the slot or on t . By (F1), $\sum_j a_j = 0$, so a is a character of Γ_F . Let H'_F be the kernel of the character a of Γ_F and $S'^\circ = F^\circ / H'_F$.*

- (a) W° has d connected components, permuted transitively by Γ . Each is isomorphic to the pullback of the cyclic cover $S'^\circ \rightarrow \Omega$ to the rank- g_X vector bundle of Lemma 5.6(b).
- (b) There are isomorphisms of complex mixed Hodge structures

$$H_c^h(W^\circ, \mathbb{C})_\psi \cong H_c^r(S'^\circ, \mathbb{C})_a(-g_X) \cong H_c^r(F^\circ, \mathbb{C})_a(-g_X) \cong V(a)(-g_X),$$

and $V(a)$ is one-dimensional.

- (c) $V(a)$ has Hodge type $(p - g_X, h - p - g_X)$, where $p = \dim H^{1,0}(D')_\psi$.

Proof. (a) Over the vector bundle, W° is pulled back from a Γ -torsor over Ω (the branch divisors have been removed) whose monodromy around the j -th hyperplane is g_{b_j} , of exponent α_j (Lemma 5.5). Torsors under the abelian group Γ over the connected Ω are classified by homomorphisms $H_1(\Omega, \mathbb{Z}) \rightarrow \Gamma$, and $H_1(\Omega, \mathbb{Z})$ is generated by the meridians of the hyperplanes: a loop in Ω bounds a disc in the simply connected $\mathbb{P}^{s-2}$, which can be moved transverse to the hyperplanes away from their intersections, and removing small discs at the intersection points writes the loop as a product of conjugates of meridians. The image of the homomorphism is $d\mathbb{Z}/e$, so the torsor is induced from a connected $d\mathbb{Z}/e$ -torsor and has d components, permuted transitively by Γ ; each component is a connected cyclic cover of Ω of degree e' whose meridian monodromies, in $d\mathbb{Z}/e \cong \mathbb{Z}/e'$, have exponents α_j/d . The quotient $S'^\circ \rightarrow \Omega$ of $F^\circ \rightarrow \Omega$ is a connected cyclic cover with group $\Gamma_F/H'_F \cong \mu_{e'}$ whose meridian monodromies are given by the character a ; so, after identifying $\mu_{e'}$ with $d\mathbb{Z}/e$ (which fixes ε), each component is S'° pulled back to the bundle.

(b) $H_c^*(W^\circ)$ is induced from the stabilizer $d\mathbb{Z}/e$ of a component, so by Frobenius reciprocity its ψ -part is the $\psi|_{d\mathbb{Z}/e}$ -part of one component; the restriction of a faithful character of $\mathbb{Z}/e$ to $d\mathbb{Z}/e \cong \mathbb{Z}/e'$ is faithful, and under the identification of (a) it is the character a . For a rank- g_X algebraic vector bundle the Thom isomorphism shifts compactly supported cohomology by $2g_X$ and twists the mixed Hodge structure by $(-g_X)$. The a -part of $H_c^r(S'^\circ) = H_c^r(F^\circ)^{H'_F}$ is the a -part of $H_c^r(F^\circ)$, since a is trivial on H'_F . Finally, $F - F^\circ$ is the union of the coordinate strata $F_J = F \cap \{Y_j = 0 \text{ for } j \in J\}$, $J \neq \emptyset$. Filter it by the closed sets $F_{\geq c} = \bigcup_{|J| \geq c} F_J$; the difference $F_{\geq c} - F_{\geq c+1}$ is the disjoint union of the locally closed strata F_J° , $|J| = c$, where exactly the coordinates in J vanish. Each F_J° is fixed pointwise by the factor $\mu_{e'}$ of Γ_F in a coordinate $j \in J$, which therefore acts trivially on $H_c^*(F_J^\circ)$, while it acts on the a -part through the nontrivial character $\zeta \mapsto \zeta^{a_j}$ ($a_j \neq 0$). So the a -part of $H_c^*(F_J^\circ)$ vanishes, and the long exact sequences of compactly supported cohomology for the decompositions $F_{\geq c} = (F_{\geq c} - F_{\geq c+1}) \sqcup F_{\geq c+1}$ give $H_c^*(F - F^\circ)_a = 0$; then the sequence for $F = F^\circ \sqcup (F - F^\circ)$ gives $H_c^r(F^\circ)_a \cong H^r(F)_a$. By Shioda [23, Theorem I, p. 176], $V(a)$ is one-dimensional for a with all entries nonzero and sum zero.

(c) By (b) and Lemma 5.4, $V(a)(-g_X) \cong U_\psi$, which has type $(p, h-p)$; the twist $(-g_X)$ raises both indices by g_X . As a check, Shioda's Theorem I(ii)

[23, p. 176] gives $V(a)$ the type $(r + 1 - |a|, |a| - 1)$ with $|a| = \sum_j \langle a_j \rangle / e'$, $\langle a_j \rangle \in \{1, \dots, e' - 1\}$; for $a = (\alpha_j/d)_j$ and the convention for p above, $p = g_X - 1 + \sum_j (1 - \langle a_j \rangle / e') = g_X + r + 1 - |a|$, in agreement. $\square$

Slots with $s = 2$. By Lemma 5.6(c), W° is then a Γ -torsor over the simply connected $\mathbb{A}^{g_X}$, hence trivial: $W^\circ \cong \Gamma \times \mathbb{A}^{g_X}$, with Γ acting by translation on the first factor. We put $r = 0$, $F = F^\circ = \Gamma$ (a set of e points), $\Gamma_F = \Gamma$ acting by translation, $a = \psi$, $H'_F = 1$ and $S'^\circ = F$, so that W° is the trivial rank- g_X bundle over S'° and $V(a) = H^0(F, \mathbb{C})_\psi$ is one-dimensional. Lemma 5.8 then holds with these conventions: (a) and (b) by the Thom isomorphism for the trivial bundle, and (c) because $p = g_X - 1 + \{-\alpha_1/e\} + \{\alpha_1/e\} = g_X$ and $h = 2g_X$, so both sides have type $(0, 0)$ after the twist. With these conventions all statements below hold for slots with $s_k = 2$, including products in which some slots have $s_k = 2$ and others $s_k \geq 3$. In the application of Section 8 every slot has $s \geq 20$.

5.5. Hodge types on the product. Now restore the index k . Put $Y = \prod_k W_{0,k}$, an ordinary product over $\mathbb{C}$, with the action of $\Gamma_{\text{tot}} = \prod_k \Gamma_k$ and the character $\psi = \boxtimes_k \psi_k$, and $F_{\text{tot}} = \prod_k F_k$, of dimension $2\nu := \sum_k r_k$. Put $q_{\text{tot}} = Rg_X$, so that $h_{\text{tot}} = 2\nu + 2q_{\text{tot}}$.

Lemma 5.9 (Hodge type). *Under (H1)–(H2), the Künneth formula and Proposition 5.7 give*

$$H^{h_{\text{tot}}}(Y, \mathbb{C})_\psi = \bigotimes_k H^{h_k}(W_{0,k}, \mathbb{C})_{\psi_k} \cong \left(\bigotimes_k V(a_k) \right) (-q_{\text{tot}}),$$

a line, and the same for every simultaneous conjugate ψ^t , with a_k replaced by ta_k . The line $\bigotimes_k V(ta_k) \subset H^{2\nu}(F_{\text{tot}}, \mathbb{C})$ has type (ν, ν) if and only if $\sum_k (2p_{k,t} - h_k) = 0$. So (H4) holds if and only if every simultaneous conjugate is of type (ν, ν) .

Proof. The first statement is Künneth for a finite product over a field, together with the slot isomorphisms; the ψ -part of a product of Γ_{tot} -modules is the tensor product of the ψ_k -parts. By Lemma 5.8(c), $\bigotimes_k V(ta_k)$ has type $(\sum_k p_{k,t} - q_{\text{tot}}, h_{\text{tot}} - \sum_k p_{k,t} - q_{\text{tot}})$, which is (ν, ν) exactly when $\sum_k p_{k,t} = h_{\text{tot}}/2$. $\square$

In particular, (H4) forces $\sum_k r_k$ to be even, though no single r_k need be. Let $\mathcal{V} \subseteq H^{2\nu}(F_{\text{tot}}, \mathbb{Q})$ be the rational subspace whose complexification is the sum of the distinct lines $\bigotimes_k V(ta_k)$; under (H4) it consists of Hodge classes.

5.6. The Fermat input. For an integer $m \geq 1$ write $X_m^r \subset \mathbb{P}^{r+1}$ for the Fermat variety $x_0^m + \dots + x_{r+1}^m = 0$ of degree m and dimension r . Shioda proves:

Theorem 5.10 (Shioda [23, Theorem IV and the Corollary preceding it, p. 182]). Fix $m \geq 1$. Then the Hodge Conjecture for the product $X_m^{r_1} \times \dots \times X_m^{r_k}$ is true provided that it is true for X_m^r for all even $r \leq r_1 + \dots +$

$r_k + 2(k - 1)$. In particular, if m is a prime or $m \leq 20$, then the Hodge Conjecture for arbitrary product $X_m^{r_1} \times \cdots \times X_m^{r_k}$ is true.

Here “the Hodge Conjecture” means that every rational Hodge class is algebraic, and the dimensions $r_1, \dots, r_k$ are arbitrary; the parity condition concerns only the auxiliary single Fermat varieties in Shioda’s proof. The case $m \leq 20$ rests on Shioda’s Theorem III and a finite verification. Let M_m be the semigroup of nonnegative integer solutions $(x_1, \dots, x_{m-1}; y)$, $y > 0$, of the equations $\sum_{v=1}^{m-1} \langle tv \rangle x_v = my$ for all $t \in (\mathbb{Z}/m)^\times$, where $\langle tv \rangle \in \{0, \dots, m-1\}$ is the representative, and $M_m(y)$ its elements of length y . An element is *decomposable* if it is a sum of two elements of M_m , and *quasi-decomposable* if adding some element of $M_m(1)$ makes it decomposable in a different way [23, p. 180]. Shioda’s condition (P_m^n) is: *every indecomposable element of $M_m(y)$ with $3 \leq y \leq n/2 + 1$ is quasi-decomposable*; by his Theorem III [23, p. 180] it implies the Hodge conjecture for X_m^n . Shioda reports that he verified (P_m^n) for $12 \leq m \leq 20$ and all n , with a table of indecomposable elements computed with Maruyama [23, p. 182]. Theorem C needs it for $m = 12$, and we have recomputed this case exactly. The indecomposable elements of M_{12} are its Hilbert basis, computed with Normaliz [15]: 56 elements, of lengths $y = 1, 2, 3, 4, 5$ in numbers 6, 16, 22, 8, 4. Each of the 34 with $y \geq 3$ is quasi-decomposable, with an explicit witness; for example

$$\begin{aligned} & (0, 0, 0, 4, 0, 0, 0, 0, 0, 2, 0; 3) + (0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0; 1) \\ &= 2(0, 0, 0, 2, 0, 1, 0, 0, 0, 1, 0; 2). \end{aligned}$$

So (P_{12}^n) holds for every n , since the condition concerns only the finite Hilbert basis. The same program reproduces Shioda’s table on p. 181 for $m = 6, 8, 9, 10$, and finds that the condition fails for $m = 25$. The program and its output, with all 34 witnesses, are in the supplementary files. In the application, slots of reduced order 12 occur in each of the three product suppliers, so degree 12 cannot be avoided with the identity used here.

Lemma 5.11 (Fermat input). *Assume (H3) and (H4). Then every class in $\mathcal{V}$ is algebraic.*

Proof. For each k with $r_k \geq 1$, the coordinate map $\phi_k: X_{12}^{r_k} \rightarrow F_k, [Y_j] \mapsto [Y_j^{12/e'_k}]$, is defined because $e'_k \mid 12$; it has no base points and is finite and surjective (it is the quotient by $(\mu_{12/e'_k})^{s_k}$ modulo the diagonal). Slots with $r_k = 0$ are finite sets of points and are kept as they are. Let Φ be the product map. For rational cohomology $\Phi_*\Phi^* = (\deg \Phi) \text{id}$, so Φ^* is injective and carries Hodge classes to Hodge classes. A product of Fermat varieties of degree 12 with a finite set is a disjoint union of copies of a product of Fermat varieties of degree 12, so by Theorem 5.10 with $m = 12$ every Hodge class upstairs is algebraic. For $v \in \mathcal{V}$ choose an algebraic cycle $\tilde{z}$ with class Φ^*v ; then $v = (\deg \Phi)^{-1}\Phi_*[\tilde{z}]$ is algebraic. $\square$

The class in $\mathcal{V}$ is not a product of classes on the factors, which are not Hodge classes on their own; this is why Shioda's theorem for products is needed.

5.7. The cone cycle. Notation. In this subsection all (co)homology has complex coefficients unless stated otherwise; for a finite abelian group acting, $(\)_\chi$ denotes the χ -isotypic part, an exact functor. Put $W_{\text{tot}}^\circ = \prod_k W_k^\circ$, an open subset of the compact space $Y_{L_0} = \prod_k \beta_k^{-1}|L_{0,k}|$, which is closed in Y ; let $j: W_{\text{tot}}^\circ \hookrightarrow Y_{L_0}$ and $i: Y_{L_0} \hookrightarrow Y$ be the inclusions, and $B = Y_{L_0} - W_{\text{tot}}^\circ$. Similarly put $F_{\text{tot}}^\circ = \prod_k F_k^\circ$ with inclusion $j': F_{\text{tot}}^\circ \hookrightarrow F_{\text{tot}}$, and let $p: F_{\text{tot}} \rightarrow S'_{\text{tot}} = \prod_k F_k/H'_{F,k}$ and $p^\circ: F_{\text{tot}}^\circ \rightarrow S'_{\text{tot}}^\circ$ be the quotients by $H'_{F,\text{tot}} = \prod_k H'_{F,k}$; since Γ_{F_k} acts freely on F_k° , p° is finite étale. Write $\bar{\psi} = \boxtimes_k \bar{\psi}_k$, and $-a = (-a_k)_k$ for the corresponding character of $\Gamma_{F,\text{tot}} = \prod_k \Gamma_{F_k}$.

We use three standard facts about a locally compact space T that is a complex algebraic variety. (i) A closed subvariety $V \subseteq T$ of pure dimension m has a fundamental class $[V] \in H_{2m}^{BM}(T)$ in Borel–Moore homology, natural under restriction to open subsets and under pushforward along closed embeddings; for compact T , Borel–Moore homology is singular homology. (ii) For an open inclusion $j_T: T^\circ \hookrightarrow T$, a class $\alpha \in H_n^{BM}(T)$ and $\beta \in H_c^n(T^\circ)$, one has $\langle \alpha, j_{T!}\beta \rangle_T = \langle \alpha|_{T^\circ}, \beta \rangle_{T^\circ}$, where $j_{T!}$ is extension by zero and $\langle \ , \ \rangle$ is the evaluation pairing (cap product followed by the degree map): extension by zero is adjoint to restriction. (iii) For a complex vector bundle $\pi_E: E \rightarrow S$ of rank c with Thom class τ_E , the map $w \mapsto \pi_E^* w \cup \tau_E$ is an isomorphism $H_c^b(S) \rightarrow H_c^{b+2c}(E)$ (Thom isomorphism), and $\langle \pi_E^{-1}[V], \pi_E^* w \cup \tau_E \rangle_E = \pm \langle [V], w \rangle_S$ for a closed subvariety $V \subseteq S$, by the projection formula and $\int_{\text{fibre}} \tau_E = 1$.

Lemma 5.12. *Assume (H1) and (H2). For every b , $H^b(B)_{\bar{\psi}} = 0$, and extension by zero $j_!: H_c^b(W_{\text{tot}}^\circ)_{\bar{\psi}} \rightarrow H^b(Y_{L_0})_{\bar{\psi}}$ is an isomorphism; restriction $H^b(Y)_{\bar{\psi}} \rightarrow H^b(Y_{L_0})_{\bar{\psi}}$ is an isomorphism. Likewise $H_c^b(F_{\text{tot}} - F_{\text{tot}}^\circ)_{-a} = 0$, and $j'_!: H_c^b(F_{\text{tot}}^\circ)_{-a} \rightarrow H^b(F_{\text{tot}})_{-a}$ is an isomorphism.*

Proof. For one slot, Proposition 5.7, applied to the faithful character $\bar{\psi}_k$, which satisfies the same hypotheses, gives the two isomorphisms in every degree. The Künneth formula for the compact products Y and Y_{L_0} and for compactly supported cohomology of W_{tot}° [17, Corollary 11.3.11], together with the fact that the $\bar{\psi}$ -part of a tensor product of Γ_{tot} -modules is the tensor product of the $\bar{\psi}_k$ -parts, gives them for the products. The long exact sequence $\cdots \rightarrow H_c^b(W_{\text{tot}}^\circ) \rightarrow H^b(Y_{L_0}) \rightarrow H^b(B) \rightarrow H_c^{b+1}(W_{\text{tot}}^\circ) \rightarrow \cdots$ is Γ_{tot} -equivariant, and taking $\bar{\psi}$ -parts is exact; since $j_!$ is an isomorphism on $\bar{\psi}$ -parts in every degree, $H^*(B)_{\bar{\psi}} = 0$.

For F_{tot} , stratify $F_{\text{tot}} - F_{\text{tot}}^\circ$ by the locally closed sets $\prod_k F_{k,J_k}^\circ$, where $F_{k,J}^\circ \subseteq F_k$ is the locus on which exactly the coordinates in J vanish, and $J_k \neq \emptyset$ for at least one k . On such a stratum the factor $\mu_{e'_k} \subseteq \Gamma_{F_k}$ in a coordinate $j \in J_k$ acts trivially, since it rescales $Y_j = 0$, while the character

$-a$ is nontrivial on it because $a_{k,j} \neq 0$. So the $(-a)$ -part of the compactly supported cohomology of every stratum vanishes. Filtering by the closed unions of strata of dimension at most c and using the open-closed long exact sequences gives $H_c^*(F_{\text{tot}} - F_{\text{tot}}^\circ)_{-a} = 0$, and the open-closed sequence for $F_{\text{tot}}^\circ \subseteq F_{\text{tot}}$ gives the last claim. $\square$

The cycle. Choose a nonzero $v \in \mathcal{V}$ and, by Lemma 5.11, a rational algebraic cycle $z = \sum_\lambda n_\lambda z_\lambda$ on F_{tot} with $[z] = v$, the z_λ irreducible of dimension ν . Every component of v on a distinct conjugate line $\bigotimes_k V(ta_k)$ is nonzero: the Galois action on coefficients permutes these lines transitively and fixes v . Put

$$Z = p_* z = \sum_\lambda n_\lambda \deg(z_\lambda \rightarrow p(z_\lambda)) p(z_\lambda),$$

a rational cycle of dimension ν on S'_{tot} , and Z° its restriction to S'_{tot}° , so that $Z^\circ = p_*^\circ(z^\circ)$ with z° the restriction of z to F_{tot}° . In each slot with $s_k \geq 3$ fix one connected component $\mathcal{E}_k$ of W_k° , the total space of a rank- g_X vector bundle over $S_k'^\circ$ (Lemma 5.8(a)); in a slot with $s_k = 2$ put $\mathcal{E}_k = W_k^\circ$, the trivial bundle over $S_k'^\circ = F_k$ (Subsection 5.4). Let $\pi_\mathcal{E}: \mathcal{E} = \prod_k \mathcal{E}_k \rightarrow S_{\text{tot}}'^\circ$, a vector bundle of rank q_{tot} . It is open and closed in W_{tot}° . For each λ with $p(z_\lambda)$ meeting $S_{\text{tot}}'^\circ$, the set $\pi_\mathcal{E}^{-1}(p(z_\lambda) \cap S_{\text{tot}}'^\circ)$ is irreducible of dimension $\nu + q_{\text{tot}}$ and closed in W_{tot}° ; let C_λ be its Zariski closure in Y , which is irreducible of the same dimension and contained in the closed set Y_{L_0} . Define

$$C(Z) = \sum_\lambda n_\lambda \deg(z_\lambda \rightarrow p(z_\lambda)) C_\lambda,$$

the sum over those λ . It is a rational algebraic cycle on Y of pure dimension $\nu + q_{\text{tot}} = h_{\text{tot}}/2$, and its restriction to W_{tot}° is the cycle $\pi_\mathcal{E}^{-1}(Z^\circ)$. The closure of an irreducible locally closed set is irreducible of the same dimension. We write $[C(Z)] \in H_{h_{\text{tot}}}(Y, \mathbb{Q})$ for its fundamental class in homology, and $\langle [C(Z)], u \rangle$ for the evaluation of a class $u \in H^{h_{\text{tot}}}(Y)$ on it.

Map of the argument. The spaces involved in Lemma 5.13, Corollary 5.14 and Proposition 5.15 are

$$P \xrightarrow{\varpi} Y \supseteq Y_{L_0} \supseteq W_{\text{tot}}^\circ \supseteq \mathcal{E} \xrightarrow{\pi_\mathcal{E}} S_{\text{tot}}'^\circ \xleftarrow{p^\circ} F_{\text{tot}}^\circ \subseteq F_{\text{tot}}.$$

Here ϖ is the quotient by $G_{0,\text{tot}}$, $\mathcal{E}$ is open and closed in W_{tot}° , $\pi_\mathcal{E}$ is a vector bundle of rank q_{tot} , and p° is finite étale. The group Γ_{tot} acts on Y , Y_{L_0} and W_{tot}° , and $\Gamma_{F,\text{tot}}$ acts on F_{tot} and F_{tot}° . The cycles move from right to left: z on F_{tot} , then $Z = p_* z$, then $C(Z)$, the closure of $\pi_\mathcal{E}^{-1} Z^\circ$ in Y , then $\varpi^! C(Z)$ on P , then its projection by $e_{[\psi]}$. The cohomology classes move from left to right, as follows.

step	class	group	character
1	$u \in H^{h_{\text{tot}}}(Y), i^*u = j_!u_c, u_c \in H_c^{h_{\text{tot}}}(W_{\text{tot}}^\circ)$	Γ_{tot}	$\bar{\psi}$
2–3	$u_c _{\mathcal{E}} \in H_c^{h_{\text{tot}}}(\mathcal{E}), \text{ nonzero}$		
4	$u_c _{\mathcal{E}} = \pi_{\mathcal{E}}^*w \cup \tau_{\mathcal{E}}, w \in H_c^{2\nu}(S_{\text{tot}}'^\circ)$		
5	$(p^\circ)^*w \in H_c^{2\nu}(F_{\text{tot}}^\circ)$	$\Gamma_{F, \text{tot}}$	$-a$
6	$\xi = j'_!(p^\circ)^*w \in H^{2\nu}(F_{\text{tot}})$	$\Gamma_{F, \text{tot}}$	$-a$
7	paired with $[z] = v$, whose component of character a is nonzero	$\Gamma_{F, \text{tot}}$	a

Here $2\nu = \dim F_{\text{tot}}$ and $h_{\text{tot}} = 2\nu + 2q_{\text{tot}}$; the pairing in step 7 is Poincaré duality on F_{tot} , and steps 1–6 identify it with $\langle [C(Z)], u \rangle$ up to sign. In Corollary 5.14 the nonvanishing is pulled back to P , and in Proposition 5.15 the $\mathbb{Q}[\Gamma_{\text{tot}}]$ -module generated by the projection gives all of $\mathcal{U}$.

Lemma 5.13 (The cone pairing). *Assume (H1)–(H4). Let u generate the line $H^{h_{\text{tot}}}(Y)_{\bar{\psi}}$ (Lemma 5.9). Then there is a generator ξ of $\bigotimes_k V(-a_k)$ such that*

$$\langle [C(Z)], u \rangle = \pm \int_{F_{\text{tot}}} [z] \cup \xi \neq 0.$$

Proof. Step 1. By Lemma 5.12 there is a unique $u_c \in H_c^{h_{\text{tot}}}(W_{\text{tot}}^\circ)_{\bar{\psi}}$ with $j_!u_c = i^*u$, and $u_c \neq 0$. Since $C(Z)$ is supported in Y_{L_0} , $\langle [C(Z)], u \rangle_Y = \langle [C(Z)], i^*u \rangle_{Y_{L_0}} = \langle [C(Z)], j_!u_c \rangle_{Y_{L_0}}$.

Step 2 (no boundary terms). By fact (ii), this equals $\langle [C(Z)]|_{W_{\text{tot}}^\circ}, u_c \rangle = \langle [\pi_{\mathcal{E}}^{-1}Z^\circ], u_c \rangle_{W_{\text{tot}}^\circ}$, so whatever part of $C(Z)$ lies in B does not enter.

Step 3 (one component). The cycle $\pi_{\mathcal{E}}^{-1}Z^\circ$ lies in the open and closed subset $\mathcal{E}$, so the pairing is $\langle [\pi_{\mathcal{E}}^{-1}Z^\circ], u_c|_{\mathcal{E}} \rangle_{\mathcal{E}}$. The components of W_{tot}° are permuted transitively by Γ_{tot} (Lemma 5.8(a) in each slot), and $H_c^*(W_{\text{tot}}^\circ) = \bigoplus_{\gamma} H_c^*(\gamma\mathcal{E})$ over the distinct translates. If $\gamma^*u_c = \bar{\psi}(\gamma)u_c$ for all γ , then restricting to $\mathcal{E}$ gives $\gamma^*(u_c|_{\gamma\mathcal{E}}) = \bar{\psi}(\gamma)u_c|_{\mathcal{E}}$, where $\gamma: \mathcal{E} \rightarrow \gamma\mathcal{E}$. So if $u_c|_{\mathcal{E}} = 0$ then u_c vanishes on every component, hence $u_c = 0$. Since $u_c \neq 0$, $u_c|_{\mathcal{E}} \neq 0$.

Step 4 (Thom). By fact (iii) there is a unique $w \in H_c^{2\nu}(S_{\text{tot}}'^\circ)$ with $\pi_{\mathcal{E}}^*w \cup \tau_{\mathcal{E}} = u_c|_{\mathcal{E}}$, it is nonzero, and the pairing equals $\pm \langle [Z^\circ], w \rangle_{S_{\text{tot}}'^\circ}$.

Step 5 (finite quotient). Since $Z^\circ = p_*^\circ z^\circ$ and p° is proper, the projection formula gives $\langle [Z^\circ], w \rangle = \langle [z^\circ], (p^\circ)^*w \rangle_{F_{\text{tot}}^\circ}$. The class $(p^\circ)^*w$ is nonzero: p° is finite étale of degree $|H'_{F, \text{tot}}|$, and $p_*^\circ(p^\circ)^*w = |H'_{F, \text{tot}}|w \neq 0$ for the trace map p_*° on compactly supported cohomology. Moreover, by Lemma 5.8(b) for the opposite characters, slot by slot with Künneth, it lies in $H_c^{2\nu}(F_{\text{tot}}^\circ)_{-a}$.

Step 6 (back to F_{tot}). Put $\xi = j'_!(p^\circ)^*w$. By Lemma 5.12 it is a nonzero element of $H^{2\nu}(F_{\text{tot}})_{-a} = \bigotimes_k V(-a_k)$, a line by Lemma 5.8(b), so it generates it. By fact (ii) for j' , $\langle [z^\circ], (p^\circ)^*w \rangle_{F_{\text{tot}}^\circ} = \langle [z], \xi \rangle_{F_{\text{tot}}} = \int_{F_{\text{tot}}} [z] \cup \xi$; components of z inside $F_{\text{tot}} - F_{\text{tot}}^\circ$ contribute zero to both sides.

Step 7 (non-vanishing). Poincaré duality on the smooth projective F_{tot} is invariant under $\Gamma_{F, \text{tot}}$, so it pairs the b -part with the b' -part trivially unless $b + b' = 0$, and it is perfect; hence it pairs $\bigotimes_k V(a_k)$ perfectly with $\bigotimes_k V(-a_k)$. The component of $[z] = v$ on $\bigotimes_k V(a_k)$ is nonzero, so

$\int_{F_{\text{tot}}} [z] \cup \xi \neq 0$. Signs from orientation conventions affect only the sign in the statement. $\square$

In the simplest case, one slot with $s = 2$, $C(Z)$ is the closure of one copy of $\mathbb{A}^{g_X}$ in W_0 , a component of the inverse image of $|L_0|$, and the lemma recovers the nonvanishing of Schoen's cycle [18, Lemma 2.6].

Let $P = \prod_k D_k^{h_k}$, a smooth projective variety, $G_{0,\text{tot}} = \prod_k G_{0,k}$, and $\varpi: P \rightarrow Y = P/G_{0,\text{tot}}$ the quotient map. For an irreducible subvariety $V \subseteq Y$, choose an irreducible component V' of $\varpi^{-1}(V)$; it maps onto V , and the components of $\varpi^{-1}(V)$ are its translates. Put

$$\varpi^! V = \frac{|G_{0,\text{tot}}|}{|G_{0,\text{tot}} \cdot V'| \deg(V' \rightarrow V)} \sum_{V'' \in G_{0,\text{tot}} \cdot V'} V'',$$

a rational algebraic cycle on P , invariant under $G_{0,\text{tot}}$, with $\varpi_* \varpi^! V = |G_{0,\text{tot}}| V$; extend $\varpi^!$ linearly to cycles.

Corollary 5.14. *Under the hypotheses of Lemma 5.13, the class $[\varpi^! C(Z)] \in H^{h_{\text{tot}}}(P, \mathbb{Q})$ is algebraic and has a nonzero component on the line $\bigotimes_k U_{\psi_k}$.*

Proof. By the projection formula for ϖ (naturality of the evaluation pairing) and Lemma 5.13,

$$\int_P [\varpi^! C(Z)] \cup \varpi^* u = \langle \varpi_* [\varpi^! C(Z)], u \rangle_Y = |G_{0,\text{tot}}| \langle [C(Z)], u \rangle_Y \neq 0.$$

The class $[\varpi^! C(Z)]$ is $G_{0,\text{tot}}$ -invariant, so it lies in $H^{h_{\text{tot}}}(P)^{G_{0,\text{tot}}} = \varpi^* H^{h_{\text{tot}}}(Y)$, on which Γ_{tot} acts through any lift to $\prod_k \hat{G}_k$. These lifts are automorphisms of P and preserve its Poincaré pairing, so a class in $H^{h_{\text{tot}}}(P)^{G_{0,\text{tot}}}$ of character χ pairs to zero with $\varpi^* u$ unless $\chi = \psi$. Hence $[\varpi^! C(Z)]$ has a nonzero ψ -component, and by Lemma 5.4 and Künneth the ψ -part of $H^{h_{\text{tot}}}(P)^{G_{0,\text{tot}}}$ is the line $\bigotimes_k U_{\psi_k}$. $\square$

Proposition 5.15. *Assume (H1)–(H4). Let $\mathcal{U}$ be the rational subspace of $H^{h_{\text{tot}}}(\prod_k D_k^{h_k}, \mathbb{Q})$ whose complexification is the sum of the distinct lines $\bigotimes_k U_{\psi_k^t}$. Then $\mathcal{U}$ is spanned by classes of algebraic cycles.*

Proof. By Corollary 5.14, the rational algebraic class $x = [\varpi^! C(Z)]$ on P lies in $H^{h_{\text{tot}}}(P, \mathbb{Q})^{G_{0,\text{tot}}}$ and has a nonzero component on $\bigotimes_k U_{\psi_k}$. The group Γ_{tot} acts on $H^*(P)^{G_{0,\text{tot}}}$ through automorphisms of P in $\prod_k \hat{G}_k$, which preserve algebraic classes. Let $e_{[\psi]} \in \mathbb{Q}[\Gamma_{\text{tot}}]$ be the central idempotent belonging to the Galois orbit O of ψ ,

$$e_{[\psi]} = \frac{1}{|\Gamma_{\text{tot}}|} \sum_{\gamma \in \Gamma_{\text{tot}}} c_{\gamma} \gamma, \quad c_{\gamma} = \sum_{\chi \in O} \chi(\gamma)^{-1} \in \mathbb{Q},$$

the coefficients being rational because O is Galois stable. Each γ acts on $H^*(P)^{G_{0,\text{tot}}}$ as the pullback by a lift $\hat{\gamma} \in \prod_k \hat{G}_k$, that is, by the algebraic correspondence given by the graph of $\hat{\gamma}$. So $e_{[\psi]} x = \frac{1}{|\Gamma_{\text{tot}}|} \sum_{\gamma} c_{\gamma} \hat{\gamma}^* x$ is a

rational combination of algebraic classes, a rational algebraic class, and it is nonzero, because its component on $\bigotimes_k U_{\psi_k}$ is that of x . By Lemma 5.4 and Künneth, the ψ^t -part of $H^{h_{\text{tot}}}(P)^{G_{0,\text{tot}}}$ is the line $\bigotimes_k U_{\psi_k^t}$, so $e_{[\psi]}x \in \mathcal{U}$. It remains to see that $\mathcal{U}$ is generated by any nonzero vector as a $\mathbb{Q}[\Gamma_{\text{tot}}]$ -module; this is Lemma 5.16 below, applied to $M = H^{h_{\text{tot}}}(P, \mathbb{Q})^{G_{0,\text{tot}}}$ and $\vartheta_0 = \psi$, whose isotypic parts in the orbit are the one-dimensional lines just named. The $\mathbb{Q}[\Gamma_{\text{tot}}]$ -span of $e_{[\psi]}x$ consists of algebraic classes, so it is all of $\mathcal{U}$. $\square$

Lemma 5.16. *Let Γ be a finite abelian group, ϑ_0 a character of Γ with Galois orbit $O = \{\vartheta_0^t\}$, and M a finite-dimensional $\mathbb{Q}[\Gamma]$ -module with $\dim_{\mathbb{C}}(M \otimes \mathbb{C})_{\chi} = 1$ for every $\chi \in O$. Then $\mathcal{U}_O = e_O M$, whose complexification is $\bigoplus_{\chi \in O} (M \otimes \mathbb{C})_{\chi}$, is a simple $\mathbb{Q}[\Gamma]$ -module, and any nonzero vector of it generates it.*

Proof. By Wedderburn's theorem for the commutative semisimple algebra $\mathbb{Q}[\Gamma]$, $\mathbb{Q}[\Gamma] = \prod_{O'} \mathbb{Q}[\Gamma]e_{O'}$ over the Galois orbits O' of characters, with $\mathbb{Q}[\Gamma]e_{O'} \cong \mathbb{Q}(\chi)$ for $\chi \in O'$, a field of degree $|O'|$. The factor for O acts on $\mathcal{U}_O$ through the field $\mathbb{Q}(\vartheta_0)$ and the other factors act by 0. Since $\dim_{\mathbb{Q}} \mathcal{U}_O = |O| = [\mathbb{Q}(\vartheta_0) : \mathbb{Q}]$, $\mathcal{U}_O$ is a one-dimensional $\mathbb{Q}(\vartheta_0)$ -vector space; so it is simple, and any nonzero vector spans it over $\mathbb{Q}(\vartheta_0) = \mathbb{Q}[\Gamma]e_O$. $\square$

In our situation ψ has order $\text{lcm}_k e_k = M_0$, because its restriction to Γ_k is the faithful ψ_k ; so $t \mapsto \psi^t$ is injective on $(\mathbb{Z}/M_0)^{\times}$ and the lines $\bigotimes_k U_{\psi_k^t}$, $t \in (\mathbb{Z}/M_0)^{\times}$, are pairwise distinct.

5.8. Abel–Prym. For each slot let $J_k = J(D_k)$, of dimension g_k , with theta class Θ_k , and let $\Xi_k : D_k^{h_k} \rightarrow J_k$ be the sum map after a choice of base point.

Lemma 5.17 (After Schoen [18, proof of Corollary 3.1, pp. 24–25]). *There is an algebraic correspondence T_k with rational coefficients from $D_k^{h_k}$ to J_k such that $T_k \circ \Xi_k^*$ is the identity on $H^{h_k}(J_k, \mathbb{Q})$. In particular T_k maps $U_{\psi_k^t}$ isomorphically onto $\det H^1(D_k, \mathbb{C})_{\psi_k^t}$ for every t .*

Proof. First, $h_k \leq g_k$: ψ_k and $\bar{\psi}_k$ are distinct because $e_k \geq 3$, both eigenspaces have dimension h_k , and their sum lies in $H^1(D_k)$, of dimension $2g_k$. For $a \in H^{h_k}(J_k)$ the projection formula gives $\Xi_{k*} \Xi_k^* a = a \cup \Xi_{k*}(1)$. The image of Ξ_k is the subvariety W_{h_k} of effective divisor classes of degree h_k , onto which Ξ_k has degree $h_k!$, and Poincaré's formula gives $[W_{h_k}] = \Theta_k^{g_k - h_k} / (g_k - h_k)!$ (Griffiths–Harris [6, p. 350], as used by Schoen); so $\Xi_{k*}(1) = c_k \Theta_k^{g_k - h_k}$ with $c_k = h_k! / (g_k - h_k)!$. Cup product with $\Theta_k^{g_k - h_k}$ is an isomorphism $L_k : H^{h_k}(J_k) \rightarrow H^{2g_k - h_k}(J_k)$ by hard Lefschetz. Its inverse is a polynomial in the Lefschetz operator and its adjoint Λ , and Lieberman's Theorem 3 [11, p. 372] states that on an abelian variety Λ maps algebraic classes to algebraic classes, through an algebraic correspondence; so L_k^{-1} is induced by an algebraic correspondence with rational coefficients, as in

Schoen's proof. Put $T_k = c_k^{-1} L_k^{-1} \circ \Xi_{k*}$. The last statement follows because $U_{\psi^t} = \Xi_k^* \det H^1(D_k)_{\psi^t}$ (Schoen's Remark 1.4). $\square$

Proof of Theorem 5.1. The external product $\boxtimes_k T_k$ is an algebraic correspondence with rational coefficients from $\prod_k D_k^{h_k}$ to $\prod_k J_k$. By Lemma 5.17 it maps each line $\bigotimes_k U_{\psi_k^t}$ isomorphically onto $\ell^{(t)}$, so it maps $\mathcal{U}$ onto W . By Proposition 5.15, $\mathcal{U}$ is spanned by algebraic classes, hence so is W . $\square$

Remark 5.18. The conclusion holds at every cover $D \rightarrow X$ with the given data. The theorem does not cover slots with $s_k \leq 1$, for which there is no special fibre to localise to. It says nothing about whether every balanced combination of characters of every group satisfies (H2) and (H3); in the application these are certified (Proposition 8.4).

6. THE HERMITIAN FORM AND DOMINANCE

6.1. The Hermitian form as a parabolic cup product. To compute the discriminant of the family we need its first cohomology and Hermitian form in terms of the monodromy tuple alone. Let $\pi: C \rightarrow \mathbb{P}^1$ be a cover in Hur, $U_C = \mathbb{P}^1 - \{p_j\}$, and $\mathcal{R} = \pi_* \mathbb{Q}|_{U_C}$, a local system with fibre $\mathbb{Q}[G]$. Put $L' = \text{Hom}_G(V_\sigma, \mathcal{R} \otimes K)$, a local system of K -vector spaces of rank d . The fibre pairing of $\mathcal{R}$ (multiplication followed by the sum over the fibre), made conjugate-linear in the first variable, restricts on $V_\sigma \otimes L'$ to $h_V \otimes h_{L'}$ for a flat positive definite Hermitian form $h_{L'}$ on L' , unique up to a positive rational factor. Let

$$V_{\text{par}} = H_{\text{par}}^1(U_C, L'), \quad \mathcal{S}(x, y) = \int_{U_C} h_{L'}(x \cup y), \quad h = \vartheta \mathcal{S},$$

the parabolic cohomology (the image of compactly supported cohomology), a K -space of dimension m_σ , with its skew-Hermitian cup product $\mathcal{S}$ and the Hermitian form h .

Proposition 6.1 ([21, Proposition 7.1]). *There is a K -linear or K -antilinear isomorphism $H_1(A_\sigma(C), \mathbb{Q}) \cong V_{\text{par}}$ under which the Hermitian form H of $A_\sigma(C)$ is a nonzero rational multiple of h , or of $\bar{h}$. In particular H and h have the same discriminant.*

Proof. Step 1. On $H_1(A, \mathbb{Q})$ the form $\mathcal{T} = H/2\vartheta$ is the unique K -sesquilinear skew-Hermitian form with $\text{Tr}_{K/\mathbb{Q}} \mathcal{T} = E$.

Step 2. Let $N' = H_1(J(C), \mathbb{Q})_\varepsilon$ and $M = \text{Hom}_{K[G]}(V_\sigma, N')$. Evaluation $V_\sigma \otimes_K M \rightarrow N'$ is a G -equivariant isomorphism, because $\text{End}_{K[G]}(V_\sigma) = K$. A G -invariant sesquilinear form on N' pulls back to $h_V \otimes \psi_M$ for a unique sesquilinear form ψ_M on M , since the G -invariant sesquilinear forms on V_σ are the multiples of h_V . Applied to $\mathcal{T}$, which is G -invariant by the uniqueness in Step 1, this gives $\mathcal{T} = h_V \otimes \psi_M$, and $\mathcal{T}|_{A_\sigma} = h_V(v, v) \psi_M$ under $m \mapsto v \otimes m$, because e is Rosati-fixed.

Step 3: cohomology of C as parabolic cohomology. Let $j: U_C \hookrightarrow \mathbb{P}^1$. Since π is finite and C is smooth, $\pi_* K_C = j_*(\mathcal{R} \otimes K)$: at a branch point

the stalk of $\pi_* K_C$ is the space of functions on the fibre, which is the space of invariants of the local monodromy on $\mathcal{R} \otimes K$. As π is finite, $H^1(C, K) = H^1(\mathbb{P}^1, j_*(\mathcal{R} \otimes K))$. For a local system $\mathbb{L}$ on U_C , the sequences $0 \rightarrow j_! \mathbb{L} \rightarrow j_* \mathbb{L} \rightarrow \bigoplus_j (\mathbb{L}^{T_j})_{p_j} \rightarrow 0$ and $j_* \mathbb{L} \rightarrow Rj_* \mathbb{L} \rightarrow (R^1 j_* \mathbb{L})[-1]$, whose third terms are skyscrapers, show that $H_c^1(U_C, \mathbb{L}) \rightarrow H^1(\mathbb{P}^1, j_* \mathbb{L})$ is surjective and $H^1(\mathbb{P}^1, j_* \mathbb{L}) \rightarrow H^1(U_C, \mathbb{L})$ is injective; so $H^1(\mathbb{P}^1, j_* \mathbb{L})$ is the parabolic cohomology $H_{\text{par}}^1(U_C, \mathbb{L})$. Under these identifications the cup product of C followed by integration is the cup product $H_c^1(U_C, \mathcal{R}) \times H^1(U_C, \mathcal{R}) \rightarrow H_c^2(U_C, \mathbb{Q}) = \mathbb{Q}$ with the fibre pairing of $\mathcal{R}$, which is the pushforward of the multiplication on $\mathbb{Q}_C$ followed by the trace $\pi_* \mathbb{Q}_C \rightarrow \mathbb{Q}$. The functor $\text{Hom}_{K[G]}(V_\sigma, -)$ is exact and commutes with these operations, so it identifies the σ -part of the skew-Hermitian lift of the cup product with $h_V \otimes \mathcal{S}$.

Step 4. Poincaré duality on C and the identification of H_1 with H^1 by E are G -equivariant and carry $\mathcal{T}$ to $\pm$ the lift of Step 3. Depending on the normalisation of the K -action on cohomology (through ι or through its transpose), they carry the σ -component to the σ -component or to the $\bar{\sigma}$ -component; accordingly the induced isomorphism $M \cong V_{\text{par}}$ is K -linear or K -antilinear, and $\psi_M = c\mathcal{S}$ or $\psi_M = c\bar{\mathcal{S}}$ with $c \in K^\times$. Both forms are skew-Hermitian, so $c = \bar{c}$ is rational. In either case the discriminants agree: scaling a Hermitian form of even dimension by a rational number multiplies its determinant by a norm, and $\det \bar{h} = \det h$. So the choice of normalisation does not matter for the discriminant. $\square$

The form $\mathcal{S}$ is computable from the tuple. Let W be a local system of K -vector spaces on $\mathbb{P}^1$ minus r points with a flat Hermitian form h_W and unitary monodromy $T_1, \dots, T_r$ along standard loops with $\ell_1 \cdots \ell_r = 1$, generating a finite group. Put $Q_i = T_1 \cdots T_{i-1}$. A parabolic cocycle is given by vectors a_i with $\sum_i Q_i(1 - T_i)a_i = 0$; its value on ℓ_i is $c_i = (1 - T_i)a_i$. Put $F_i = \sum_{j < i} Q_j c_j$.

Lemma 6.2 ([21, Lemma 7.2]). *For parabolic cocycles a, b , the cup product of their classes with the fibre pairing h_W , on the fundamental class oriented so that the loops run clockwise, is*

$$\mathcal{S}(a, b) = \sum_{i=1}^r h_W(Q_i^{-1} F_i^a + a_i, c_i^b).$$

It does not depend on the choice of the vectors a_i , which are determined by the cocycle only modulo $\ker(1 - T_i)$; it vanishes on coboundaries and is skew-Hermitian.

Proof. We use the following Δ -complex structure on $\mathbb{P}^1$ (Figure 1). Its vertices are a base point b_0 , a second point b_∞ , and the punctures $p_1, \dots, p_r$. Its edges are $e_0, \dots, e_{r-1}$ from b_0 to b_∞ , cutting the sphere into r sectors, with the sector between e_{i-1} and e_i containing p_i (indices of e modulo r , so $e_r = e_0$); and, for each i , an edge f_i from b_0 to p_i and an edge g_i from p_i to

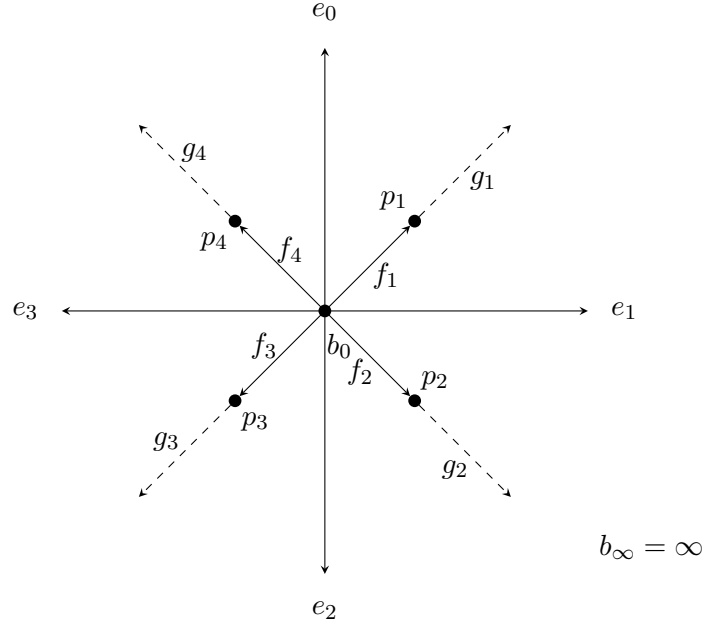


FIGURE 1. The Δ -complex of Lemma 6.2 for $r = 4$, drawn in the plane $\mathbb{P}^1 - \{\infty\}$. The edges e_i and g_i end at $b_\infty = \infty$. The sector between e_{i-1} and e_i contains p_i and is cut by the path $f_i \cup g_i$ into the two triangles at p_i . The loop ℓ_i , out along e_{i-1} and back along e_i , runs clockwise around p_i .

b_∞ inside that sector. Its faces are the $2r$ triangles with vertices (b_0, p_i, b_∞) in this order: for each i one triangle with edges f_i, g_i, e_{i-1} and one with edges f_i, g_i, e_i . (The Euler characteristic is $(r+2) - 3r + 2r = 2$.) The loop ℓ_i goes out along e_{i-1} and back along e_i , and we orient $\mathbb{P}^1$ so that it runs clockwise around p_i ; then $\ell_1 \cdots \ell_r$ is out along e_0 and back along $e_r = e_0$, which is trivial.

The monodromy group is finite, so there is a finite Galois cover branched over the punctures on which W becomes trivial; lift the complex to it. There the cup product of 1-cochains followed by h_W , evaluated on the fundamental class, is the simplicial cup product $(f \cup f')[v_0, v_1, v_2] = h_W(f[v_0, v_1], f'[v_1, v_2])$ [7, §3.2]. Equivariant cochains contribute the same term on every lift of a triangle, so dividing by the degree gives a sum over the triangles of $\mathbb{P}^1$. Choose the lifts of the two triangles at p_i so that they share their edge g_i ; their edges f_i are then two lifts that differ by the local monodromy at p_i , and the two triangles enter the fundamental class with opposite signs. The terms $h_W(f[b_0, p_i], f'[p_i, b_\infty])$ of the two triangles combine, by the cocycle condition on each triangle, into

$$\mathcal{S}_{\text{simp}}(f, f') = \sum_i h_W(v_{e_i}^f - v_{e_{i-1}}^f, v_{g_i}^{f'}),$$

with v_ϵ^f the value of f on the chosen lift of ϵ . The cocycle conditions on the two triangles at the i -th puncture give $a_i = Q_i^{-1}v_{f_i}$ and $v_{e_{i-1}} = v_{e_0} - F_i$. Substituting these into $\mathcal{S}_{\text{simp}}$ and summing gives

$$\mathcal{S}_{\text{simp}}(f, f') = - \sum_i [h_W(c_i, a'_i) + h_W(Q_i c_i, F'_i)] + h_W\left(\sum_i Q_i c_i, v_{e_0}^{f'}\right),$$

and the last term vanishes by the cocycle relation $\sum_i Q_i c_i = 0$. By unitarity of Q_i , $h_W(c_i, Q_i^{-1}F'_i) = h_W(Q_i c_i, F'_i)$, so

$$\overline{\mathcal{S}(b, a)} = \sum_i h_W(c_i, Q_i^{-1}F'_i + a'_i) = \sum_i [h_W(c_i, a'_i) + h_W(Q_i c_i, F'_i)],$$

and $\mathcal{S}_{\text{simp}}(f, f') = -\overline{\mathcal{S}(b, a)}$. It remains to show $\mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} = 0$. Since T_i is unitary and $c_i = (1 - T_i)a_i$, one has $h_W(c_i, c'_i) = h_W(a_i, c'_i) + h_W(c_i, a'_i)$, and since $F_{i+1} = F_i + Q_i c_i$,

$$h_W(F_{i+1}, F'_{i+1}) - h_W(F_i, F'_i) = h_W(F_i, Q_i c'_i) + h_W(Q_i c_i, F'_i) + h_W(c_i, c'_i).$$

Hence

$$\begin{aligned} \mathcal{S}(a, b) + \overline{\mathcal{S}(b, a)} &= \sum_i [h_W(F_{i+1}, F'_{i+1}) - h_W(F_i, F'_i)] \\ &= h_W(F_{r+1}, F'_{r+1}) - h_W(F_1, F'_1) = 0, \end{aligned}$$

because $F_1 = 0$ and $F_{r+1} = \sum_i Q_i c_i = 0$. So $\mathcal{S}_{\text{simp}} = \mathcal{S}$, and $\mathcal{S}$ is skew-Hermitian. A coboundary pairs to zero by the same telescoping. Finally, replacing a_i by $a_i + k$ with $T_i k = k$ changes neither c_i nor any F_j , and changes $\mathcal{S}(a, b)$ by $h_W(k, c'_i) = h_W(k, a'_i) - h_W(T_i^{-1}k, a'_i) = 0$, by unitarity; the second variable enters only through the c'_i . $\square$

6.2. The codifferential and dominance. Let $C \in \text{Hur}$ with branch set B_C , $|B_C| = r$, and put $\mathcal{D}_\sigma(C) = \text{Hom}_G(\sigma, H^0(C, K_C))$ and $\mathcal{D}_{\bar{\sigma}}(C) = \text{Hom}_G(\bar{\sigma}, H^0(C, K_C))$, each of dimension n by Proposition 3.1. The tangent space to Hur at C modulo PGL_2 is Serre dual to $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$, of dimension $r - 3$. Let

$$\mu_C: \mathcal{D}_\sigma(C) \otimes \mathcal{D}_{\bar{\sigma}}(C) \rightarrow H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$$

be multiplication of differentials followed by contraction of $\sigma \otimes \bar{\sigma}$ against its invariant line, with values in $H^0(C, K_C^{\otimes 2})^G = H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B_C))$.

Lemma 6.3 (Codifferential). *The cotangent space of $\mathcal{H}_n$ at the period point of $A_\sigma(C)$ is $\mathcal{D}_\sigma(C) \otimes \mathcal{D}_{\bar{\sigma}}(C)$, of dimension n^2 , and μ_C is, up to a nonzero normalization, the codifferential at C of the period map of the family A_σ .*

Proof. The codifferential of the period map of $J(C)$ along the Hurwitz family is multiplication of differentials $H^0(K_C) \otimes H^0(K_C) \rightarrow H^0(K_C^{\otimes 2})$ followed by restriction to G -invariants [3, §§2–3]. Under $H^0(C, K_C) = \bigoplus_\tau \tau \otimes \text{Hom}_G(\tau, H^0(K_C))$, the part that pairs the σ - and $\bar{\sigma}$ -components and lands in the invariants is μ_C . The domain $\mathcal{H}_n$ parametrizes the σ -eigenspace of $H^{1,0}$, which is $\mathcal{D}_\sigma(C)$ with compatible conventions, and its

tangent space is $\mathrm{Hom}(H_\sigma^{1,0}, H_\sigma^{0,1}) \cong \mathcal{D}_\sigma(C)^\vee \otimes \overline{\mathcal{D}_{\bar{\sigma}}(C)}$, whose dual, via the polarization, is $\mathcal{D}_\sigma(C) \otimes \mathcal{D}_{\bar{\sigma}}(C)$. $\square$

Proposition 6.4 (Rank at one point). *Let S be a connected component of Hur and suppose $\mathrm{rk} \mu_{C_1} = n^2$ for one $C_1 \in S$. Then the period map of the family A_σ over S is a submersion at C_1 and has generic rank n^2 .*

Proof. The locus where $\mathrm{rk} \mu_C = n^2$ is Zariski open in the irreducible S and contains C_1 ; $n^2 = \dim \mathcal{D}_\sigma(C) \otimes \mathcal{D}_{\bar{\sigma}}(C)$ is the maximum. $\square$

A dimension count $r - 3 \geq n^2$ is necessary for this and not sufficient; the rank must be certified at a point.

7. SPREADING

Let $\mathcal{M} = \mathcal{M}(K, n, \mathfrak{d})$ be the moduli space of Subsection 2.1.

Lemma 7.1. *Let $\mathcal{W} = \bigwedge_K^{2n} R^1 f_* \mathbb{Q} \subset R^{2n} f_* \mathbb{Q}$ on $\mathcal{M}$. The set $\mathcal{M}^{\mathrm{alg}}$ of points t at which $\mathcal{W}_t$ is spanned by classes of algebraic cycles is a countable union of closed algebraic subsets of $\mathcal{M}$.*

Proof. This is the standard argument with relative Hilbert schemes [26, §3.3.2, pp. 90–94, and the proofs of Theorems 7.19 and 8.26]; we give the details. Fix a relatively ample line bundle on the smooth projective family $f: \mathcal{A} \rightarrow \mathcal{M}$. For each Hilbert polynomial, the relative Hilbert scheme of n -dimensional closed subschemes of the fibres is projective over $\mathcal{M}$; there are countably many Hilbert polynomials, and each relative Hilbert scheme has finitely many irreducible components [26, pp. 90–91]. A rational algebraic class in $H^{2n}(\mathcal{A}_t, \mathbb{Q})$ is a rational combination of classes of n -dimensional subschemes of $\mathcal{A}_t$, so it suffices to consider finite tuples of such subschemes.

Fix a finite tuple $\mathcal{H}_1, \dots, \mathcal{H}_k$ of components, let T be an irreducible component of $\mathcal{H}_1 \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathcal{H}_k$, let $\tilde{T} \rightarrow T$ be a resolution of singularities, and let $g: \tilde{T} \rightarrow \mathcal{M}$ be the composite. Then $\tilde{T}$ is smooth and connected, and g is proper, so $g(\tilde{T})$ is a closed algebraic subset of $\mathcal{M}$. The universal subschemes pull back to subschemes $\mathcal{Z}_l \subset \mathcal{A} \times_{\mathcal{M}} \tilde{T}$, flat over $\tilde{T}$, and the classes $[\mathcal{Z}_{l,u}] \in H^{2n}(\mathcal{A}_{g(u)}, \mathbb{Q})$, $u \in \tilde{T}$, are global flat sections of the local system $g^* R^{2n} f_* \mathbb{Q}$ on $\tilde{T}$: the fibre classes of a flat family of subschemes of a smooth projective family are locally constant, as used in [26, proof of Theorem 3.33, p. 94]. Let $\mathcal{F}_u$ be the span of $[\mathcal{Z}_{1,u}], \dots, [\mathcal{Z}_{k,u}]$. Parallel transport along a path from u to u' is an isomorphism carrying each of these global sections to itself, so it carries $\mathcal{F}_u$ onto $\mathcal{F}_{u'}$ and $g^* \mathcal{W}_u$ onto $g^* \mathcal{W}_{u'}$; in particular $\dim \mathcal{F}_u$ does not jump. Hence the set of u with $g^* \mathcal{W}_u \subseteq \mathcal{F}_u$ is either all of $\tilde{T}$ or empty. Call T *good* in the first case; then $\mathcal{W}_t$ is algebraic for every $t \in g(\tilde{T})$.

Conversely, if $\mathcal{W}_t$ is spanned by algebraic classes, choose subschemes $Z_1, \dots, Z_k$ of $\mathcal{A}_t$ whose classes span a space containing $\mathcal{W}_t$; the point $(Z_1, \dots, Z_k)$ lies on some component T , which is then good, and $t \in g(\tilde{T})$. So $\mathcal{M}^{\mathrm{alg}}$ is the union of the closed sets $g(\tilde{T})$ over the good T , of which there are countably many. $\square$

Theorem 7.2. *Let S be a connected component of Hur such that $W_K(A_\sigma(C))$ is spanned by algebraic classes for every $C \in S$, and such that $\text{rk } \mu_{C_1} = n^2$ at some $C_1 \in S$. Then the Weil classes of every polarized abelian $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}(S)$ are algebraic.*

Proof. Choose a small polydisc $\Delta \ni C_1$ over which the covers form an analytic family. Over the simply connected Δ the spaces $H_1(A_\sigma(C), \mathbb{Q})$ with ι and H form a trivial flat bundle of Hermitian spaces of discriminant $\mathfrak{d}(S)$, so Landherr's theorem gives a K -isometry $m: H_1(A_\sigma(C_1), \mathbb{Q}) \rightarrow (V, H)$, which we transport over Δ . Sending C to the complex structure of $H_1(A_\sigma(C), \mathbb{R})$, transported by m , gives the period map $\Phi: \Delta \rightarrow \mathcal{H}_n$, a submersion at C_1 by Lemma 6.3 and Proposition 6.4. Shrinking Δ , $\Phi(\Delta)$ contains a ball $\mathcal{B}$, whose image in $\mathcal{M}$ is open, since $\Gamma(\ell)$ acts properly and freely.

Let $C \in \Delta$ with $z = \Phi(C) \in \mathcal{B}$, and let A_z be the fibre of $\mathcal{A}$ over the image of z : $H_1(A_z, \mathbb{Z}) = \Lambda$, with complex structure z and polarization $\text{Tr}_{K/\mathbb{Q}}(H/2\vartheta)$ scaled as in Subsection 2.1. The lattice $m(H_1(A_\sigma(C), \mathbb{Z}))$ is commensurable with Λ ; if $N_1 m(H_1(A_\sigma(C), \mathbb{Z})) \subseteq \Lambda$ for an integer $N_1 \geq 1$, then $N_1 m$ is a K -linear map of lattices compatible with the complex structures, so it is a K -linear isogeny $A_\sigma(C) \rightarrow A_z$. Since m is an isometry of Hermitian forms, it pulls the polarization of A_z back to a positive rational multiple of that of $A_\sigma(C)$, which changes neither the Weil plane nor the class of the discriminant. By hypothesis and Lemma 2.2, the Weil classes of A_z are algebraic. So $\mathcal{M}^{\text{alg}}$ contains a nonempty open set. A nonempty open subset of $\mathcal{M}$ is a Baire space, so by Lemma 7.1 one of the closed sets has nonempty interior, and since $\mathcal{M}$ is irreducible it is all of $\mathcal{M}$. Every polarized $2n$ -fold of Weil type for K with discriminant $\mathfrak{d}(S)$ is K -isogenous to a fibre of $\mathcal{A}$ (Lemma 2.2). $\square$

8. THE GROUP $W(E_8) \cdot \mu_4$ AND THE PROOF OF THEOREM A

8.1. The group and the class datum. Let $K = \mathbb{Q}(i)$ and $W = W(E_8)$, the Weyl group of E_8 , of order 696 729 600, acting on $\mathbb{Q}^8$; we use the basis of fundamental weights in Bourbaki's numbering (Appendix A). Let

$$G = W \cdot \mu_4 = W \cup iW \subset GL_8(K),$$

of order 1 393 459 200, the central product of W and μ_4 over $\{\pm 1\}$, and let σ be its natural representation on K^8 , $d = 8$. It is absolutely irreducible, because W is, and its character field is K , since $\text{tr } \sigma(iw) = i \text{tr}(w)$. So σ satisfies the assumptions of Subsection 3.2.

The class datum has $r = 12$ points: ten reflections of W , an element $g_{11} = ix_1$ with $x_1 \in W$ of characteristic polynomial Φ_4^4 (so $x_1^2 = -1$), and an element $g_{12} = ix_2$ with $x_2 \in W$ of characteristic polynomial $\Phi_2^2 \Phi_6 \Phi_{12}$. Elements ix and iy of G are conjugate exactly when x and y are conjugate in W ; the classes of x_1 and x_2 are those of the elements a^5 and $-a^{-1}d^5a$ of Appendix A. Let $\mathbf{g} = (g_1, \dots, g_{12})$ be the tuple of Appendix A: it has product 1, it generates G , and its entries lie in these classes. Let C_0 be the

cover of Appendix B, whose monodromy tuple is $\mathbf{g}$, and S the connected component of the Hurwitz space containing C_0 , of dimension $12 - 3 = 9$.

Why this datum. For sixfolds ($n = 3$), Proposition 3.1 gives $\sum_j c_\sigma(g_j) = 2d + 6 = 22$, and dominance needs at least $n^2 + 3 = 12$ branch points with $c_\sigma \geq 1$. A reflection has $c_\sigma = 1$; the two non-reflection points have $c_\sigma = 4$ and 8, and $10 + 4 + 8 = 22$, with exactly 12 points. So this datum is at the edge of what the count allows.

Lemma 8.1. *For every $C \in \text{Hur}$, σ occurs in $H^1(C)$ with multiplicity 6 and Hodge numbers $(3, 3)$, so $A_\sigma(C)$ is a polarized abelian sixfold of Weil type for K .*

Proof. A reflection has $c_\sigma = 1$ and eigenvalue angles $(0^7, \frac{1}{2})$, so age $\frac{1}{2}$. The element ix_1 has eigenvalues $i \cdot (\pm i) = \mp 1$, four of each, so $c_\sigma = 4$ and age 2. The eigenvalues of x_2 are -1 (twice), the primitive sixth roots of unity and the primitive twelfth roots of unity; multiplying by i never gives 1, so $c_\sigma(ix_2) = 8$, and the eight angles of ix_2 sum to 4. Proposition 3.1 gives $m_\sigma = -16 + 10 + 4 + 8 = 6$. The ages of the inverses are c_σ minus the ages, with the same total $22 - 11 = 11$, so both Hodge numbers equal $11 - 8 = 3$. Conclude with Lemma 3.3. $\square$

The characters of G : the irreducible characters of W are all real and orthogonal, and those that are -1 on the central element $-1 \in W$ extend to G in two complex conjugate ways, $i \mapsto \pm i$. Exactly 45 of the resulting pairs $\{\tau, \bar{\tau}\}$ have $m_\tau > 0$, and there are no quaternionic τ , so $X^*(T) = \mathbb{Z}^{45}$, with e_σ a coordinate vector (Appendix C). The representation σ is the only one with $m_\tau = 6$.

8.2. Dominance and the discriminant.

Proposition 8.2. $\text{rk } \mu_{C_0} = 9$, and $\mathfrak{d}(S) = [-3]$.

Proof. The rank is the certificate of Appendix B: a 9×9 minor of μ_{C_0} , in residue coordinates, is enclosed in a complex ball that excludes 0.

For the discriminant, Proposition 6.1 reduces $\mathfrak{d}(S)$ to $\det h$ for $V_{\text{par}} = H_{\text{par}}^1(U_{C_0}, L')$, computed from the tuple by Lemma 6.2 as exact linear algebra over K .

The tuple used. The computation uses the tuple $T_j = g_{13-j}^T$, $j = 1, \dots, 12$: the transposes in reverse order (the two scalar points first, then the ten reflections), as stored in the supplementary files. Since every $g \in W$ preserves the Gram matrix H^{-1} of the fundamental weights, with H the Cartan matrix, we have $g^T = H^{-1}g^{-1}H$, so (T_j) is the conjugate by H of $(g_{12}^{-1}, \dots, g_1^{-1})$, and each T_j preserves H . The tuple $(g_{12}^{-1}, \dots, g_1^{-1})$ is the monodromy of the same local system with respect to the opposite orientation of $\mathbb{P}^1$ (the standard generators read backwards). Reversing the orientation identifies the two parabolic cohomology groups and changes the sign of the intersection pairing; for a 6×6 Hermitian matrix this changes neither the determinant

nor the signature $(3, 3)$. So the computation below gives $\det h$ and the signature of h for $\mathbf{g}$.

The basis. Put $Q_j = T_1 \cdots T_{j-1}$ and let E_j be the matrix of pivot columns of $1 - T_j$ (a basis of $\text{im}(1 - T_j)$), of $8, 4, 1, \dots, 1$ columns. A parabolic cocycle is recorded by vectors a_j with $c_j = E_j a_j$, and the cocycle relation is $\sum_j Q_j E_j a_j = 0$; the solutions form the right kernel Z^1 of the 8×22 matrix $(Q_1 E_1 \mid \cdots \mid Q_{12} E_{12})$, of dimension 14. We take for Z^1 the reduced row echelon basis of this kernel (in Sage, `right_kernel_matrix()`). The form of Lemma 6.2, with $h_W = H$ and times $\vartheta = i$, is Hermitian of rank 6 on Z^1 , its radical being the coboundaries, and the first six basis vectors span a complement of the radical. On them its Gram matrix is

$$h_0 = \begin{pmatrix} -2 & 1+i & 0 & -1-i & 2+i & -1+i \\ 1-i & 2 & 2i & 0 & -1-3i & 2 \\ 0 & -2i & 0 & i & 0 & -1-i \\ -1+i & 0 & -i & 0 & i & 0 \\ 2-i & -1+3i & 0 & -i & -1 & \frac{3}{2} + \frac{1}{2}i \\ -1-i & 2 & -1+i & 0 & \frac{3}{2} - \frac{1}{2}i & 0 \end{pmatrix},$$

of determinant -3 and signature $(3, 3)$. The class of -3 in $\mathbb{Q}^\times / \text{Nm } K^\times$ is $[-3]$. The script is `independent-det/independent_det.py` in the supplementary files; it asserts each step above (that each T_j preserves H , that the product is 1, that $\dim Z^1 = 14$, that the form is Hermitian of rank 6) and prints this block. The six basis vectors are printed in Appendix D. A second program, written separately, computes $\det h = -6$ for the same tuple in another basis; $(-6)/(-3) = 2 = \text{Nm}(1+i)$, so the classes agree. $\square$

The discriminant is computed on the component S of C_0 , where the rank is certified, so no comparison between components is needed. We do not claim that the class datum has a single Hurwitz component.

8.3. The supplier identity.

Proposition 8.3 (The integer witness). *In $X^*(T) = \mathbb{Z}^{45}$,*

$$e_\sigma = \sum_{i=1}^{95} n_i a(\chi_i) + \sum_{l=1}^3 a_{H'_l}(\mathbf{n}_l)$$

with integers n_i , where the (H'_i, χ_i) are Schoen-admissible suppliers and the $(H'_l, \mathbf{n}_l)$ are balanced product suppliers with $H'_l = \langle i \rangle C_W(w_l)$, for w_1 of order 3 with $|C_W(w_1)| = 311\,040$, w_2 of order 12 with $|C_W(w_2)| = 288$, and w_3 of order 6 with $|C_W(w_3)| = 864$. Of the Schoen suppliers, 94 come from a search pool described in Appendix C, and one comes from $H' = \langle i \rangle N_W(P_3)$, P_3 a Sylow 3-subgroup of W , $|H'| = 7776$.

Proof. This is the certificate of Appendix C: an integer matrix with 45 columns whose rows are characters of explicit suppliers, and an integer coefficient vector whose product with it is e_σ . Schoen-admissibility of each Schoen row is checked from its exponent counts (positive even h , simplicity

of the exponent tuple in the sense of Subsection 4.3, order $m \geq 3$), and balance of each product row at $t = 1, 5, 7, 11$. $\square$

Proposition 8.4 (Slot data). *The three product suppliers of Proposition 8.3 are product-admissible (Definition 4.10): the curves C/H'_l have genus 3259, 5 341 793 and 1 730 347; every slot character has order 4 or 12; every slot has at least 20 branch points; and $e'_k \in \{4, 12\}$. This holds for every integral lift of the given rows.*

Proof. The rows of the witness are characters $a_{H'_l}(\mathbf{n})$; the integral lift $\mathbf{n}$ is not unique, and different lifts give different slots. The certificate of Appendix C, Table 4, computes, for every linear character χ of H'_l of order at least 3 with $\dim V_\chi > 0$ (so for every character that can occur in any integral lift), the branch data of $C/\ker \chi \rightarrow C/H'_l$ from the double cosets $H'_l \backslash G / \langle g_j \rangle$, and checks $s \geq 2$ and $e' \mid 12$. It also checks that the imbalance $I_\chi(t) = p_{\chi^t} - q_{\chi^t}$ satisfies $I_\chi(t) = \varepsilon(t)I_\chi(1)$ with $\varepsilon = (+, +, -, -)$ at $t = 1, 5, 7, 11$. Hence $\delta_t(\mathbf{n}) = \varepsilon(t)\delta_1(\mathbf{n})$ for every lift, and $\delta_1(\mathbf{n}) = \delta_T(a_{H'_l}(\mathbf{n}))$ depends only on the row; the certificate checks that it vanishes for each row. Since every character that occurs has order dividing 12, the units t modulo 12 cover every simultaneous Galois conjugate. $\square$

Example 8.5 (A product supplier). The smallest of the three is the row for w_1 , of order 3, with $|C_W(w_1)| = 311\,040$ and $H' = \langle i \rangle C_W(w_1)$ of order 622 080. The quotient $X = C/H'$ has genus $g_X = 3259$. One integral lift has two slots, both with negative coefficient, so the slot characters are the inverses $\psi_1 = \bar{\chi}_2$ and $\psi_2 = \bar{\chi}_7$ of two linear characters of H' (numbered as in the certificate):

slot	order e	branch pts. s	$h = 2g_X - 2 + s$	d	$2p_t - h$ at $t = 1, 5, 7, 11$
ψ_1	4	6732	13 248	1	$(-4, -4, 4, 4)$
ψ_2	12	20	6536	1	$(4, 4, -4, -4)$

The imbalances are given in the orientation convention of the certificate, $p_t = g_X - 1 + \sum_b \{t\alpha_b/e\}$ with $\{\cdot\}$ the fractional part; this is the opposite of the convention of Section 5, which negates all of them. Neither slot is of Hodge type, and the exponent tuples of the cyclic covers are not simple (Subsection 4.3); their product is of Hodge type at every t . In the orientation convention of the certificate the line $\det H^1(D)_{\psi_1}$ has Hodge type $(6622, 6626)$ and $\det H^1(D)_{\psi_2}$ type $(3270, 3266)$; reversing the convention swaps each pair. The line $\ell_{\mathbf{n}}$ has degree 19 784 and type $(9892, 9892)$.

The proof of Theorem 5.1 turns this into a Fermat statement. The first slot gives the Fermat hypersurface of degree 4 and dimension $s - 2 = 6730$, the second that of degree 12 and dimension 18. By Lemma 5.8(c) the two Fermat lines have types $(3363, 3367)$ and $(11, 7)$, and their product, on the product of dimension $6748 = 2\nu$, has type $(3374, 3374)$: a Hodge class. Pulled back along $[Y_j] \mapsto [Y_j^3]$ from X_{12}^{6730} on the first factor, it is a Hodge class on $X_{12}^{6730} \times X_{12}^{18}$, algebraic by Shioda's theorem. The cone over a

representing cycle in the product of the two rank-3259 vector bundles has dimension $3374 + 2 \cdot 3259 = 9892$, half of $h_{\text{tot}} = 19784$, and Lemma 5.13 and the Abel–Prym correspondence give the algebraicity of $W_{\mathbf{n}}$ on $J(D)^2$. No integral lift of this row uses characters from a single cyclic group, so the statement is about a non-cyclic abelian cover.

Example 8.6 (A Schoen supplier that needs Lemma 4.7). The Schoen supplier from $H' = \langle i \rangle N_W(P_3)$, $|H'| = 7776$, has a character χ of order $m = 4$. The quotient C/H' has genus $q = 384\,479$, the cyclic cover $C/\ker \chi \rightarrow C/H'$ has $s = 88$ branch points, all with exponent $2 = m/2$, and $h = 769\,044$. The exponent tuple is simple ($2 \equiv -2$), and h is positive and even, so the supplier is Schoen-admissible. Since $4 \mid m$ and every exponent is $m/2$, Schoen’s Sublemma 2.9 finds no prime $p \mid m$ ramified at the branch points, and Lemma 4.7 is used. This supplier removes an index-3 obstruction in the lattice generated by the search pool.

8.4. Proof of Theorem A. By Lemma 8.1, $A_\sigma(C)$ is a polarized abelian sixfold of Weil type for $K = \mathbb{Q}(i)$ for every $C \in \text{Hur}$. By Propositions 8.3 and 8.4, condition (L) of Definition 4.13 holds, so by Corollary 4.16 the Weil classes of $A_\sigma(C)$ are algebraic for every $C \in \text{Hur}$; here Corollary 4.12, that is Theorem C, supplies the three product suppliers at every C , and Lemma 4.6, with Lemma 4.7, the 95 Schoen suppliers. By Proposition 8.2, $\text{rk } \mu_{C_0} = 9 = n^2$ and $\mathfrak{d}(S) = [-3]$. Theorem 7.2 gives the Weil classes of every polarized abelian sixfold of Weil type for $\mathbb{Q}(i)$ with discriminant $[-3]$. $\square$

9. POWERS: PROOF OF THEOREM B

Let W be a complex vector space of dimension N , and U_1, U_2 complex vector spaces of dimension m . Put $\Lambda = \bigwedge(W \otimes U_1) \otimes \bigwedge(W^* \otimes U_2)$, with the commuting actions of $GL(W)$ and $GL(U_1) \times GL(U_2)$. For $u \in U_1$, $u' \in U_2$ let $\theta(u, u') = \sum_i (w_i \otimes u) \wedge (w_i^* \otimes u')$ for dual bases, and for $x \in U_1$ let $\det(W \otimes x) \in \bigwedge^N(W \otimes U_1)$ be the image of a generator of $\bigwedge^N W$ under $w \mapsto w \otimes x$; define $\det(W^* \otimes x')$ similarly.

Lemma 9.1 ([21, Lemma 11.1]). $\Lambda^{SL(W)}$ is generated by the $\theta(u, u')$, the $\det(W \otimes x)$ and the $\det(W^* \otimes x')$.

Proof. We deduce it from the first fundamental theorem of invariant theory for $SL(W)$, in the form of Kraft and Procesi [9, §8.4, Theorem (FFT for SL_n), p. 90]: *the ring of invariants $K[V^p \oplus V^{*q}]^{SL(V)}$ is generated by the scalar products $\langle j \mid i \rangle$ and the determinants $[i_1, \dots, i_n]$ and $[j_1, \dots, j_n]^*$, for a vector space V of dimension n over a field of characteristic zero (see also [28, Ch. II]).* We use it in multilinear form. An $SL(W)$ -invariant element of $W^{\otimes p} \otimes W^{*\otimes q}$ is an invariant multilinear function on $W^{*p} \times W^q$, that is, an invariant polynomial on $W^{*p} \oplus W^q$ that is homogeneous of degree one in each of the $p + q$ vector variables. Apply the theorem with $V = W^*$. A polynomial in the scalar products and determinants that is homogeneous

of degree one in each variable is a linear combination of products of scalar products and determinants in which every variable occurs exactly once. So the invariants in $W^{\otimes p} \otimes W^{*\otimes q}$ are spanned by tensor products of three kinds of elementary invariants, each occupying some of the tensor factors: the contraction $\sum_i w_i \otimes w_i^*$ in one factor W and one factor W^* ; the determinant tensor $\sum_{\pi \in S_N} \text{sgn}(\pi) w_{\pi(1)} \otimes \cdots \otimes w_{\pi(N)}$ in N factors W ; and the dual determinant tensor in N factors W^* .

The homogeneous piece $\bigwedge^p(W \otimes U_1) \otimes \bigwedge^q(W^* \otimes U_2)$ of Λ is the image of $W^{\otimes p} \otimes W^{*\otimes q} \otimes U_1^{\otimes p} \otimes U_2^{\otimes q}$ under antisymmetrization, which is $SL(W)$ -equivariant and surjective. Since $SL(W)$ is reductive, the invariants of the image are the image of the invariants, so they are spanned by the images of $I \otimes u_1 \otimes \cdots \otimes u_p \otimes u'_1 \otimes \cdots \otimes u'_q$ with I a product of elementary invariants. Antisymmetrization turns tensor products into wedge products, up to sign. The image of a contraction in the factors $(w \otimes u_k)$ and $(w^* \otimes u'_l)$ is $\theta(u_k, u'_l)$. The image of a determinant tensor in the factors $u_{k_1}, \dots, u_{k_N}$ is $\sum_{\pi} \text{sgn}(\pi) (w_{\pi(1)} \otimes u_{k_1}) \wedge \cdots \wedge (w_{\pi(N)} \otimes u_{k_N})$, which is the full polarization of the homogeneous polynomial map $x \mapsto N! \det(W \otimes x)$ of degree N , evaluated at $(u_{k_1}, \dots, u_{k_N})$; by the polarization identity it is a linear combination of values $\det(W \otimes x)$. The same holds for the dual determinant tensors. So every invariant is a combination of products of the stated generators. $\square$

Theorem 9.2. *Let (A, ι, E) be a polarized abelian $2n$ -fold of Weil type for K , $n \geq 2$, whose Weil classes are algebraic and whose Hodge group is SU_H . Then every Hodge class on every power A^m is algebraic.*

Proof. Let $H^1(A, \mathbb{C}) = W \oplus W'$ be the decomposition into the two eigenspaces of K . The polarization identifies W' with W^* as representations of $SU_H(\mathbb{C}) = SL(W)$. The Hodge group of A^m is SU_H acting diagonally, so the Hodge classes on A^m , tensored with $\mathbb{C}$, are $\Lambda^{SL(W)}$ with $N = 2n$ and $U_1 = U_2 = \mathbb{C}^m$. The $\theta(u, u')$ are Hodge classes of degree two, hence combinations of divisor classes. Let $\mathcal{O} = \{y \in K : \iota(y) \in \text{End}(A)\}$, an order of K . For $\mathbf{y} = (y_1, \dots, y_m) \in \mathcal{O}^m$ the homomorphism $f_{\mathbf{y}}: A^m \rightarrow A$, $(a_k) \mapsto \sum_k \iota(y_k) a_k$, pulls the Weil classes of A back to $\det(W \otimes \tilde{\mathbf{y}})$ and $\det(W^* \otimes \tilde{\mathbf{y}})$, with $\tilde{\mathbf{y}} \in \mathbb{C}^m = U_1$ the image of $\mathbf{y}$ under the given embedding of K ; these are algebraic. The set of all $\tilde{\mathbf{y}}, \mathbf{y} \in \mathcal{O}^m$, contains $\mathbb{Z}^m$, which is Zariski dense in $\mathbb{C}^m$: a polynomial vanishing on $\mathbb{Z}^m$ is zero, by induction on m . The map $x \mapsto \det(W \otimes x)$ is polynomial, so a linear form vanishing on its values at a Zariski dense set vanishes on all its values. Hence these classes span all $\det(W \otimes x)$, $x \in U_1$, and likewise for W^* . By Lemma 9.1, products of algebraic classes span all Hodge classes over $\mathbb{C}$, and rational descent as in Corollary 4.16 finishes the proof. $\square$

Proof of Theorem B. Theorem A and Theorem 9.2 with $n = 3$. $\square$

APPENDIX A. THE MONODROMY TUPLE

Matrices act on column vectors of coordinates with respect to the fundamental weights $\omega_1, \dots, \omega_8$ of E_8 in Bourbaki's numbering (the simple roots $\alpha_1, \alpha_3, \alpha_4, \dots, \alpha_8$ form a chain and α_2 is joined to α_4); column j of a matrix is the image of ω_j . Since the E_8 lattice is unimodular, this is a $\mathbb{Z}$ -basis of it, and W preserves the Gram matrix of the fundamental weights, the inverse of the Cartan matrix. For example the simple reflection s_{α_1} maps ω_1 to $\omega_1 - \alpha_1 = -\omega_1 + \omega_3$ and fixes the other ω_j ; it is the matrix c below. Let

$$a = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -2 & -3 & -3 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & -2 & -3 & -5 & -4 & -3 & -2 & -1 \\ 2 & 3 & 3 & 5 & 4 & 3 & 2 & 1 \\ 2 & 2 & 4 & 5 & 4 & 3 & 2 & 1 \\ -2 & -2 & -3 & -4 & -4 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $d = (abc)^{-1}$. Then a has characteristic polynomial Φ_{20} and centralizer of order 20 in W , and a^5 has characteristic polynomial Φ_4^4 and centralizer of order 46080; b and c are reflections; d has characteristic polynomial $\Phi_1^2\Phi_3\Phi_{12}$; and a, b, c generate W . The tuple is

$$\mathbf{g} = (b, c, aba^{-1}, aca^{-1}, a^2ba^{-2}, a^2ca^{-2}, a^3ba^{-3}, a^3ca^{-3}, a^4ba^{-4}, a^4ca^{-4}, ia^5, -ia^{-1}d^5a).$$

Its product is 1: the first ten entries multiply to $(bca)^5a^{-5}$, and $bca = a^{-1}d^{-1}a$ because $abcd = 1$, so the product is $a^{-1}d^{-5}a \cdot a^{-5} \cdot a^5 \cdot a^{-1}d^5a = 1$. So $g_{11} = ix_1$ with $x_1 = a^5$, and $g_{12} = ix_2$ with $x_2 = -a^{-1}d^5a$, of characteristic polynomial $\Phi_2^2\Phi_6\Phi_{12}$. The twelve matrices generate a group of order 1393459200, that is all of G ; this was checked in GAP [5] from their permutation action on the 480 vectors formed by the roots of E_8 and their multiples by i . The ten reflections alone generate a subgroup of order 51840, so the two other entries are needed for generation.

The matrices a, b, c were found by a search in W for a with characteristic polynomial Φ_{20} and pairs of reflections (b, c) with $d = (abc)^{-1}$ of characteristic polynomial $\Phi_1^2\Phi_3\Phi_{12}$ and $\langle a, b, c \rangle = W$. Replacing d by a solution with characteristic polynomial $\Phi_2^2\Phi_6\Phi_{12}$ gives the conjugate class datum, exchanged with this one by the automorphism of G that fixes W and maps i to $-i$.

APPENDIX B. THE RANK CERTIFICATE

The cover. Let M be a Fuchsian system $dY = \left(\frac{B_0}{z} + \frac{B_1}{z-1} + \frac{B_c}{z-c_0}\right)Y dz$ of rank 8 on $\mathbb{P}_z^1$, with $c_0 = 2 + i$ and residues $B_0, B_1, B_c \in M_8(\mathbb{C})$, whose

monodromy around $0, 1, c_0, \infty$ is exactly (a, b, c, d) up to simultaneous conjugation. Its pullback along $s \mapsto s^5 = z$, tensored with the rank-one system $d\phi = \frac{1}{4}(\frac{ds}{s})\phi$ (monodromy i at $s = 0$ and $-i$ at $s = \infty$), has twelve singular points: $s = 0$, the fifth roots of 1, the fifth roots of c_0 , and $s = \infty$, with monodromy tuple $\mathbf{g}$. Let C_0 be the G -cover of $\mathbb{P}_s^1$ branched at these twelve points with monodromy tuple $\mathbf{g}$. We know of no rigid local system that pulls back to this datum, so a non-rigid system on four points, whose character variety has dimension 12, is used instead.

Existence of M . The system M is found numerically, by continuation from a system with abelian monodromy, and then shown to exist exactly near the numerical solution. The residues at 1 and c_0 have rank one; after conjugating so that some entries are fixed (this removes the action of simultaneous conjugation), they and an invertible matrix Q that should conjugate the monodromy of M to (a, b, c) form 87 unknowns, and 87 polynomial-in-the-monodromy equations express this conjugacy: two trace conditions making the residues at 1 and c_0 of trace $\frac{1}{2}$, so that their monodromies are reflections; seven rows of the eigenvector equation for each of the eight eigenvectors of a (56 equations); 14 equations for each of b and c matching the reflecting vectors; and $\det Q = 1$. Here $B_0 = \text{diag}(\beta_1, \dots, \beta_8)$ with $\beta_k = m_k/20$, $m_k \in \{1, 3, 7, 9, 11, 13, 17, 19\}$, and the residues at 1 and c_0 are $u_1 v_1^T$ and $u_2 v_2^T$, with u_1 and one entry of u_2 fixed. Each eigenvector equation drops one row, and each reflection equation is imposed on all but one row and on one pivot row of $M - 1$. A zero of this system gives the exact conjugacy $M_k Q = Q t_k$ at all three finite poles provided the following side conditions hold: for each eigenvalue λ_j of a , the seven kept rows of $M_0 - \lambda_j$ have rank seven; for the poles 1 and c_0 , the seven kept rows of $M_k + 1$ have rank seven and the chosen pivot row of $M_k - 1$ is nonzero. The certificate checks these minors and entries in balls over the whole ball of radius ρ below, and none of the enclosures contains 0. The monodromy is computed by Taylor expansion of solutions in ball arithmetic [1] at 256 bits, with a rigorous bound on the tail of each series, along 115 steps of a path whose endpoints match exactly. Let F be the map of the 87 equations, $\tilde{x}$ a refined numerical zero with $|F(\tilde{x})|$ of order 10^{-37} , and Y_0 an approximate inverse of the Jacobian at $\tilde{x}$. The map $w \mapsto w - Y_0 F(w)$ sends the ball of radius ρ about $\tilde{x}$ into itself and is a contraction there, by the inequalities $\eta + \kappa \rho \leq \rho$ and $\kappa < 1$, where η bounds $\|Y_0 F(\tilde{x})\|$ and κ bounds the deviation of Y_0 times the Jacobian from the identity on the ball (with finite-difference and Cauchy-estimate errors included, on a polydisc of radius 2^{-40}). With every bound rounded outward:

$$\eta \approx 1.05 \cdot 10^{-35}, \quad \kappa \approx 0.0056, \quad \rho \approx 1.05 \cdot 10^{-34}.$$

So F has a unique zero in this ball, and M exists exactly, with residues within ρ of $\tilde{x}$. Gershgorin discs then pin the exponents at ∞ to exact rational values. The residues of M , to 256 bits, are in the supplementary files.

Differentials from the system. Let L be the pulled-back system on $\mathbb{P}_s^1$, after a change of the coordinate s by a Möbius transformation that makes ∞ a nonsingular point and a scalar gauge transformation by a power of the coordinate (neither changes the cover), and let Y be a fundamental solution. At a pole p with residue $A_p = P_p \Lambda_p P_p^{-1}$, $\Lambda_p = \text{diag}(\lambda_{p,1}, \dots, \lambda_{p,8})$, the local monodromy has finite order, hence is semisimple, and by Levelt's normal form [10] there is a local solution $Y = P_p H_p (s - p) (s - p)^{\Lambda_p}$ with H_p holomorphic and $H_p(0) = 1$, also where eigenvalues of A_p differ by integers (as at $s = 0$, where the exponents are $(m_k + 1)/4$). Where no two eigenvalues differ by a positive integer, H_p is the unique solution of the Frobenius recursion.

Lemma B.1 (Differentials from the system). *$\mathcal{D}_\sigma(C_0)$ is identified with the space of $r \in \mathbb{C}(s)^8$ such that the vector of forms $Y^{-1}r ds$ is holomorphic on C_0 , and $\mathcal{D}_{\bar{\sigma}}(C_0)$ with the space of $q \in \mathbb{C}(s)^8$ such that $Y^T q ds$ is holomorphic on C_0 (up to exchanging σ and $\bar{\sigma}$, which does not affect μ_{C_0}). Concretely the conditions are*

$$\text{ord}_p [(P_p H_p)^{-1} r]_j \geq \lfloor \lambda_{p,j} \rfloor, \quad \text{ord}_p [(P_p H_p)^T q]_j \geq \lfloor -\lambda_{p,j} \rfloor$$

at each pole and each j , holomorphy elsewhere, and vanishing to order 2 at ∞ . Under these identifications $\mu_{C_0}(r \otimes q) = q^T r ds^2$.

Proof. Pulled back to C_0 minus the ramification points, Y is single-valued, because its monodromy factors through σ and the loops that lift to closed loops on C_0 map to 1 in G ; and $g^* Y = Y \sigma'(g)$ for $g \in G$, with $\sigma' \in \{\sigma, \bar{\sigma}\}$ depending on conventions. For a rational vector r , the vector $\omega = Y^{-1} r ds$ of meromorphic forms then transforms by $g^* \omega = \sigma'(g)^{-1} \omega$, so its components span the image of a G -map from $\sigma'^\vee$ to the meromorphic forms on C_0 . Conversely a G -map from $\sigma'^\vee$ to $H^0(K_{C_0})$ is a vector ω of holomorphic forms with this transformation law, and $r = Y \omega / ds$ is G -invariant, hence descends; it is meromorphic at the poles because Y and Y^{-1} have moderate growth at a Fuchsian singularity, and at the regular point ∞ , where ds has a double pole, it vanishes to order at least two. So r is rational. Since σ is unitary, $\sigma'^\vee \cong \bar{\sigma}'$; the same argument with Y^T gives the second space, and $q^T Y \cdot Y^{-1} r = q^T r$ is the invariant contraction. For the order condition: the local monodromy at p has finite order e , so $e \lambda_{p,j} \in \mathbb{Z}$; in a local coordinate y on C_0 with $s - p = y^e$, the j -th component of $Y^{-1} r ds$ is $[(P_p H_p)^{-1} r]_j (s - p)^{-\lambda_{p,j}} ds$, of order $e \text{ord}_p [(P_p H_p)^{-1} r]_j - e \lambda_{p,j} + e - 1$ in y , which is ≥ 0 exactly when $\text{ord}_p [(P_p H_p)^{-1} r]_j \geq \lfloor \lambda_{p,j} \rfloor$. $\square$

The following two lemmas turn these conditions into finite linear algebra.

Lemma B.2. *Put $M_p = \max(0, -\min_j \lfloor \lambda_{p,j} \rfloor)$ for $\mathcal{D}_\sigma$, and the same with $-\lambda_{p,j}$ for $\mathcal{D}_{\bar{\sigma}}$.*

- (1) *Every $r \in \mathcal{D}_\sigma(C_0)$ is uniquely of the form $r = \sum_p \sum_{m=1}^{M_p} c_{p,m} (s - p)^{-m}$ with $c_{p,m} \in \mathbb{C}^8$, and likewise for $\mathcal{D}_{\bar{\sigma}}(C_0)$.*

- (2) $\mathcal{D}_\sigma(C_0)$ is the kernel of the linear map Φ_A from the space of these coefficients to the Laurent coefficients that the conditions require to vanish, and similarly for Φ_B ; both kernels have dimension 3.
- (3) If a square submatrix $\Phi_{R,F'}$ of size $n_A - 3$, on a row set R and the complement F' of three columns F , is invertible, then the vectors x_f , $f \in F$, with coordinate 1 at f , 0 at the other columns of F , and $-\Phi_{R,F'}^{-1}\Phi_{R,f}$ on F' , form a basis of $\ker \Phi_A$; if $\Phi_{R,F'}$ is certified invertible in ball arithmetic, the same formula computed in balls encloses this basis.

Proof. (1) Away from the poles Y is holomorphic and invertible, so r is holomorphic there. At a pole, $P_p H_p$ is holomorphic and invertible, so $\text{ord}_p r \geq \min_j [\lambda_{p,j}] \geq -M_p$. Since r vanishes at ∞ , it is the sum of its principal parts, and principal parts at distinct points are independent. (2) Each condition is the vanishing of finitely many Laurent coefficients, which are linear in the unknowns, and involve the Taylor coefficients of H_p only to bounded order. By (1) the kernel maps isomorphically onto $\mathcal{D}_\sigma(C_0)$, which has dimension 3 by Proposition 3.1. (3) The x_f lie in the kernel $\ker \Phi_R$ of the rows R and are independent. Since $\Phi_{R,F'}$ is invertible, the rows R have rank $n_A - 3$, so $\ker \Phi_R$ has dimension 3 and is spanned by the x_f ; it contains $\ker \Phi_A$, of dimension 3, so they are equal. Ball inversion encloses the inverse of every matrix in the ball. $\square$

Lemma B.3. *Let $B \subset \mathbb{C}$ be a set of 12 points. The map sending $g(s) ds^2 \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B))$ to its residues $(\text{Res}_p g)_{p \in B} \in \mathbb{C}^{12}$ is injective, and its image is the 9-dimensional subspace cut out by $\sum_p \rho_p p^k = 0$ for $k = 0, 1, 2$.*

Proof. The function g has at most simple poles on B and is otherwise holomorphic on $\mathbb{C}$; at ∞ , with $s = 1/t$, $ds^2 = t^{-4} dt^2$, so $g = O(s^{-4})$. Hence $g = \sum_p \rho_p / (s - p)$, which proves injectivity, and expanding, $g = \sum_{k \geq 0} (\sum_p \rho_p p^k) s^{-k-1}$, so $g = O(s^{-4})$ is exactly the three relations. $\square$

So μ_{C_0} , in *residue coordinates*, is the 9×12 matrix of residues of $q_b^T r_a ds^2$ at the twelve poles, for bases r_a of $\mathcal{D}_\sigma(C_0)$ and q_b of $\mathcal{D}_{\bar{\sigma}}(C_0)$, and its rank is that of μ_{C_0} .

The rank. The Frobenius expansions are computed to order 12, with exact rational exponents. Here Φ_A is 45×48 and Φ_B is 117×120 , and square pivot blocks of sizes 45 and 117 are certified invertible by bounds $\|I - RS\|_\infty < 1$ for an approximate inverse R . The residue relations of Lemma B.3 hold within the balls, which is a consistency check. The 9×9 minor of the residue matrix on columns 4 to 12 is

$$0.105574412595065626 - 0.044099913684798359i$$

with coordinate radii below $7 \cdot 10^{-18}$. It is not zero, so $\text{rk } \mu_{C_0} = 9$. For the conjugate class datum the same procedure gives a minor of modulus greater than $5.4 \cdot 10^{-12}$, with radii below $6 \cdot 10^{-27}$.

APPENDIX C. THE SUPPLIER CERTIFICATES

The character group. G is built in GAP 4.14 [5] with its character table library, from the simple reflections of W in the basis of Appendix A and the scalar i . Its 45 pairs of non-real irreducible characters with $m_\tau > 0$ give $X^*(T) = \mathbb{Z}^{45}$, and e_σ is the second coordinate vector. A supplier (H', χ) contributes the row $a(\chi)$, computed by Frobenius reciprocity from the restrictions $\tau|_{H'}$; the program checks each row's h against the Chevalley–Weil count $h = \sum_\tau m_\tau \langle \tau|_{H'}, \chi \rangle$ and against $2q - 2 + s$.

The witness. The witness consists of an integer matrix with 3296 rows and 45 columns, an integer coefficient vector with 98 nonzero entries, and the target e_σ . The rows are: 3214 Schoen-admissible suppliers from a search pool; 16 Schoen-admissible suppliers of order 4 on $\langle i \rangle N_W(P_3)$; and 66 balanced product rows on the three groups $\langle i \rangle C_W(w_l)$. The 98 nonzero coefficients sit on 94 pool rows, on one row from $\langle i \rangle N_W(P_3)$ (row 3215, Example 8.6), and on one product row for each group (rows 3231, 3269 and 3244 for w_1 , w_2 and w_3 ; Example 8.5 is row 3231). Table 1 lists them with their signs and their lengths in decimal digits; the integers themselves fill about 72 000 digits and are in the supplementary files. Checking the identity takes seconds in GAP.

TABLE 1. The nonzero coefficients of the witness: row index, sign, number of decimal digits. Rows 1 to 1125 are pool rows; the last four are the supplier on $\langle i \rangle N_W(P_3)$ and the product rows for w_1 , w_3 , w_2 .

row	dig.	row	dig.	row	dig.	row	dig.	row	dig.					
1	+	753	21	+	752	41	+	753	226	+	753	459	+	753
2	−	753	22	+	753	42	−	752	236	+	751	463	−	753
3	+	753	23	+	753	43	+	752	243	−	753	467	+	752
4	+	753	24	+	753	44	−	753	246	−	1	469	+	753
5	−	752	25	+	753	45	+	753	290	+	752	471	−	753
6	−	753	26	−	753	46	−	752	313	+	753	473	+	751
7	−	753	27	−	753	76	−	752	314	+	753	477	−	753
8	+	753	28	+	753	78	+	753	330	−	753	478	−	752
9	−	753	29	−	753	79	+	753	332	−	753	492	+	753
10	+	752	30	−	753	88	+	753	338	+	753	493	−	753
11	+	753	31	+	753	105	+	753	345	+	752	536	−	752
12	−	753	32	−	753	116	−	753	364	−	753	574	+	752
13	−	753	33	−	753	126	+	753	366	+	752	857	+	753
14	+	752	34	−	753	129	−	753	403	−	753	1125	+	752
15	−	753	35	+	752	156	+	752	404	+	753	3215	−	752
16	−	753	36	+	753	195	−	753	425	−	753	3231	−	753
17	+	752	37	+	752	208	+	752	426	−	753	3244	+	752
18	+	752	38	−	1	217	+	752	434	−	752	3269	−	753
19	+	753	39	−	753	221	−	753	437	−	753			
20	−	753	40	+	753	224	+	752	457	−	753			

The search pool consists of the Schoen-admissible suppliers on: cyclic subgroups $\langle iw \rangle$ and subgroups $\langle i, w \rangle$ for all 112 classes of W ; subgroups of the normalizers of the cyclic groups generated by g_{12} and its square; the groups $\langle i \rangle C_W(w)$ and $\langle i \rangle N_W(\langle w \rangle)$ for all classes; subgroups of $C_G(g_{12}^3)$ meeting the classes of order 12; overgroups of a fixed 3-subgroup inside 3-local subgroups; subgroups of the normalizers of $\langle y \rangle$ and $\langle y^2 \rangle$ for $y = iw$ with w in two classes of elements of W ; and the groups $\langle i \rangle W_S \langle -1 \rangle$ and $\langle i \rangle N_W(W_S)$ for reflection subgroups W_S of rank 7 and 8. The pool spans a sublattice of rank 42 with an elementary divisor 3, and the functional of Appendix E vanishes on it. The 16 suppliers on $\langle i \rangle N_W(P_3)$ remove the elementary divisor, and the three product rows raise the rank to 44; the target lies in the integer span.

The suppliers. Table 2 identifies the supplier (H', χ) of each of the 94 pool rows with a nonzero coefficient and of row 3215, with its branch data m , q , s and $h = 2q - 2 + s$. Except in the three rows discussed below, each identification is an exact match of the row of the witness with the vector $a(\chi)$ computed from (H', χ) .

- *Notation.* $\mathcal{C}_c$ is the c -th conjugacy class of W in the ordering of the character table of $W(E_8)$ in the GAP character table library [5], and $\zeta_n = e^{2\pi i/n}$. Recall that g_{12} is the last entry of the tuple $\mathbf{g}$ (Appendix A). In two rows, $y = iw$ with $w \in \mathcal{C}_{53}$.
- *Abelian rows* (18 rows). Here $H' = \langle i, w \rangle$ or $H' = \langle iw \rangle$, and χ is given by its values on the generators.
- *Rows $\leq N_G(\langle g_{12} \rangle)$, $N_G(\langle g_{12}^2 \rangle)$, $C_G(g_{12}^3)$ or $N_G(\langle y^2 \rangle)$.* H' is a subgroup of the named group, given by its isomorphism type. The χ column gives the order of χ , then $\chi(i)$ (or $i \notin H'$), then $|\ker \chi|$.
- *Rows $\langle i \rangle \cdot H_0$.* Here $H' = \langle i \rangle H_0$ with $H_0 \leq W$ as named and $|H'| = 2|H_0|$, and χ is given by $\chi(i) = -i$ and $\chi_0 = \chi|_{H_0}$, with $\chi_0(-1) = -1$, its order and $|\ker \chi_0|$. In the rows $W_S \langle -1 \rangle$, S is a set of nodes of the extended Dynkin diagram of E_8 , node 9 being the negative of the highest root.

In the three rows 492, 493 and 857, whose groups H_0 have orders 161 280 and 5 806 080, the character χ_0 was not singled out among the linear characters of H_0 . For these rows the vector is the one recorded with the pool, together with its q , s and h .

Row 3215, which is not a pool row, is the single Schoen supplier $(\langle i \rangle N_W(P_3), \chi)$ of Proposition 8.3 and Example 8.6. Here $\chi = \chi_0 \cdot (i \mapsto i)$, with χ_0 the linear character of order 2 of $N_W(P_3)$ with $\chi_0(-1) = -1$; it is listed in the last line of Table 2. The three remaining nonzero rows (3231, 3244, 3269) are not single suppliers. Each is a δ -balanced integer combination of the linear characters of $\langle i \rangle C_W(w_l)$, the product supplier of Proposition 8.4.

Table 2: The suppliers of the 94 nonzero pool rows and of row 3215: row, sign of the coefficient, H' , $|H'|$, χ , m , q , s , h .

row	H'	$ H' $	χ	m	q	s	h
1	$+$	$\leq N_G(\langle g_{12} \rangle)$, C_4	4	4; $-i$; 1	4 769 305 601	0	1 538 611 200
2	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2$	8	4; $-i$; 2	4 381 024 001	14 515 200	776 563 200
3	$+$	$\langle i, w \rangle$, $w \in C_4$	8	$\chi(i) = -i$, $\chi(w) = 1$	4 384 652 801	0	769 305 600
4	$+$	$\leq N_G(\langle g_{12} \rangle)$, C_{12}	12	4; $-i$; 3	4 256 435 201	0	512 870 400
5	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times S_3$	24	4; $-i$; 6	4 124 588 801	14 515 200	263 692 800
6	$-$	$\leq N_G(\langle g_{12} \rangle)$, C_4	4	4; $i \notin H'$; 1	4 769 305 601	0	1 538 611 200
7	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2$	8	4; $-i$; 2	4 384 652 801	0	769 305 600
8	$+$	$\langle i, w \rangle$, $w \in C_{70}$	16	$\chi(i) = -i$, $\chi(w) = 1$	4 192 326 401	0	384 652 800
9	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 127 008 001	4 838 400	258 854 400
10	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2$	8	4; $-i$; 2	4 384 648 193	9 216	769 305 600
11	$+$	$\leq N_G(\langle g_{12} \rangle)$, C_{12}	12	12; $-i$; 1	12 256 435 201	0	512 870 400
12	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, C_{12}	12	4; $-i$; 3	4 256 435 201	0	512 870 400
13	$-$	$\langle i, w \rangle$, $w \in C_{19}$	16	$\chi(i) = -i$, $\chi(w) = 1$	4 192 326 401	0	384 652 800
14	$+$	$\langle i, w \rangle$, $w \in C_{30}$	24	$\chi(i) = -i$, $\chi(w) = 1$	4 128 217 601	0	256 435 200
15	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 128 216 065	3 072	256 435 200
16	$-$	$\langle iw \rangle$, $w \in C_{23}$	20	$\chi(iw) = \zeta_{20}^{15}$	4 153 861 121	0	307 722 240
17	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_3$	36	4; $-i$; 9	4 85 478 401	0	170 956 800
18	$+$	$\langle iw \rangle$, $w \in C_4$	4	$\chi(iw) = \zeta_4^3$	4 769 305 601	0	1 538 611 200
19	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times S_3$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
20	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2 \times C_2$	16	4; $-i$; 4	4 190 509 697	7 262 208	388 281 600
21	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_2 \times C_4 \times S_3$	48	4; $-i$; 12	4 62 293 633	7 259 136	131 846 400
22	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times S_3$	72	4; $-i$; 18	4 41 529 601	4 838 400	87 897 600
23	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times S_3$	24	4; $i \notin H'$; 6	4 128 217 601	0	256 435 200
24	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2$	8	4; $i \notin H'$; 2	4 384 648 193	9 216	769 305 600
25	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, C_{12}	12	4; $i \notin H'$; 3	4 256 435 201	0	512 870 400
26	$-$	$\langle iw \rangle$, $w \in C_{70}$	4	$\chi(iw) = \zeta_4^3$	4 769 305 601	0	1 538 611 200
27	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2$	8	4; $-i$; 2	4 384 652 801	0	769 305 600
28	$+$	$\leq N_G(\langle g_{12} \rangle)$, C_{12}	12	12; $-i$; 1	12 256 434 985	648	512 870 616
29	$-$	$\langle i, w \rangle$, $w \in C_{72}$	16	$\chi(i) = -i$, $\chi(w) = 1$	4 192 326 401	0	384 652 800
30	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
31	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_4 \times S_3$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
32	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
33	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_{12} \times C_3$	36	12; $-i$; 3	12 85 478 329	216	170 956 872
34	$-$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_2 \times C_2$	16	4; $-i$; 4	4 190 509 697	7 262 208	388 281 600
35	$+$	$\leq N_G(\langle g_{12} \rangle)$, $C_4 \times C_4$	16	4; $-i$; 4	4 192 324 097	4 608	384 652 800
36	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 127 008 001	4 838 400	258 854 400
37	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_4 \times C_2 \times C_2$	16	4; $-i$; 4	4 192 326 401	0	384 652 800
38	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
39	$-$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_4 \times S_3$	24	4; $-i$; 6	4 128 217 601	0	256 435 200
40	$+$	$\leq N_G(\langle g_{12}^2 \rangle)$, $C_{12} \times C_2 \times C_2$	48	4; $-i$; 12	4 63 503 233	2 420 736	129 427 200
41	$+$	$\langle i, w \rangle$, $w \in C_{97}$	48	$\chi(i) = -i$, $\chi(w) = 1$	4 64 108 801	0	128 217 600
42	$-$	$\langle i, w \rangle$, $w \in C_{93}$	40	$\chi(i) = -i$, $\chi(w) = 1$	4 76 204 801	2 903 040	155 312 640

row	H'	$ H' \chi$	m	q	s	h
43	$+\langle i, w \rangle, w \in \mathcal{C}_{95}$	$48 \chi(i) = -i, 4$ $\chi(w) = \zeta_{12}^6$	64	108 801	0	128 217 600
44	$-\leq N_G(\langle g_{12}^2 \rangle), C_2 \times C_4 \times S_3$	$48 \ 4; -i; 12$	4	64 108 033	1 536	128 217 600
45	$+\leq N_G(\langle g_{12} \rangle), C_{12} \times C_2$	$24 \ 12; -i; 2$	12	127 007 893	4 838 724	258 854 508
46	$-\leq N_G(\langle g_{12}^2 \rangle), C_2 \times C_4 \times S_3$	$48 \ 4; -i; 12$	4	62 294 401	7 257 600	131 846 400
76	$-\langle i, w \rangle, w \in \mathcal{C}_{44}$	$32 \chi(i) = -i, 4$ $\chi(w) = 1$	96	162 049	2 304	192 326 400
78	$+\langle i, w \rangle, w \in \mathcal{C}_{59}$	$48 \chi(i) = -i, 4$ $\chi(w) = 1$	64	108 801	0	128 217 600
79	$+\langle i, w \rangle, w \in \mathcal{C}_{50}$	$40 \chi(i) = -i, 4$ $\chi(w) = 1$	76	930 561	0	153 861 120
88	$+\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times C_6$	$72 \ 4; -i; 18$	4	42 738 609	1 056	85 478 272
105	$+\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times C_2$	$24 \ 12; -i; 2$	12	127 008 001	4 838 400	258 854 400
116	$-\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times C_6$	$72 \ 12; -i; 6$	12	42 335 929	1 613 016	86 284 872
126	$+\leq N_G(\langle g_{12}^2 \rangle), C_2 \times C_4 \times S_3$	$48 \ 4; -i; 12$	4	64 108 801	0	128 217 600
129	$-\leq N_G(\langle g_{12}^2 \rangle), C_2 \times C_2 \times C_4 \times S_3$	$96 \ 4; -i; 24$	4	31 751 617	1 210 368	64 713 600
156	$+\leq N_G(\langle g_{12} \rangle), C_8 \times C_4$	$32 \ 4; i \notin H'; 8$	4	96 162 049	2 304	192 326 400
195	$-\langle iw \rangle, w \in \mathcal{C}_{23}$	$20 \chi(iw) = \zeta_{20}^3$	20	153 861 121	0	307 722 240
208	$+\langle iw \rangle, w \in \mathcal{C}_{41}$	$28 \chi(iw) = \zeta_{28}$	28	109 900 801	0	219 801 600
217	$+\langle iw \rangle, w \in \mathcal{C}_{48}$	$36 \chi(iw) = \zeta_{36}^3$	12	85 478 329	0	170 956 656
221	$-\leq N_G(\langle g_{12} \rangle), C_{12} \times C_4$	$48 \ 4; -i; 12$	4	64 107 973	1 560	128 217 504
224	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{52}$	$40 \chi(i) = -i; 20$ $\chi_0: 20; 1$	76	929 409	2 304	153 861 120
226	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{61}$	$48 \chi(i) = -i; 24$ $\chi_0: 24; 1$	64	107 841	1 920	128 217 600
236	$+\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times C_6$	$72 \ 4; -i; 18$	4	42 739 129	0	85 478 256
243	$-\leq C_G(\langle g_{12}^2 \rangle), C_{12} \times C_4$	$48 \ 12; -i; 4$	12	64 107 841	1 980	128 217 660
246	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{64}$	$60 \chi(i) = -i; 60$ $\chi_0: 30; 1$	51	287 041	0	102 574 080
290	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{49}$	$108 \chi(i) = -i; 12$ $\chi_0: 6; 9$	28	492 777	0	56 985 552
313	$+\leq N_G(\langle y^2 \rangle), C_4 \times (C_3 \rtimes (C_4 \times D_8))$	$384 \ 4; -i; 96$	4	8 013 281	640	16 027 200
314	$+\leq C_G(\langle g_{12}^3 \rangle), C_4 \times ((C_2 \times C_2 \times Q_8) \rtimes C_3)$	$384 \ 4; i \notin H'; 96$	4	8 013 361	288	16 027 008
330	$-\langle i, w \rangle, w \in \mathcal{C}_{93}$	$40 \chi(i) = -i, 20$ $\chi(w) = \zeta_{10}^2$	76	204 801	2 903 040	155 312 640
332	$-\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times C_2 \times C_2$	$48 \ 12; -i; 4$	12	63 503 233	2 420 736	129 427 200
338	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{41}$	$56 \chi(i) = -i; 28$ $\chi_0: 14; 2$	54	432 001	2 073 600	110 937 600
345	$+\leq N_G(\langle g_{12} \rangle), C_8 \times C_4 \times C_2$	$64 \ 8; -i; 8$	8	47 627 329	1 815 744	97 070 400
364	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{102}$	$96 \chi(i) = -i; 12$ $\chi_0: 12; 4$	31	751 521	1 210 590	64 713 630
366	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{40}$	$96 \chi(i) = -i; 12$ $\chi_0: 6; 8$	31	449 217	2 419 968	65 318 400
403	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{51}$	$160 \chi(i) = -i; 20$ $\chi_0: 10; 8$	18	869 761	1 451 520	39 191 040
404	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{94}$	$240 \chi(i) = -i; 20$ $\chi_0: 10; 12$	12	337 921	1 935 360	26 611 200
425	$-\leq N_G(\langle y^2 \rangle), C_4 \times (((C_4 \times C_2) \rtimes C_2) \times S_3)$	$384 \ 4; -i; 96$	4	7 635 473	1 512 256	16 783 200

row	H'	$ H' \chi$	m	q	s	h
426	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{59}$	192 $\chi(i) = -i; 12$ $\chi_0: 6; 16$	12	15 724 609	1 209 984	32 659 200
434	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{23}$	2 400 $\chi(i) = -i; 20$ $\chi_0: 10; 120$	20	1 161 217	483 840	2 806 272
437	$-\leq C_G(g_{12}^3), C_4 \times ((C_2 \times C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_3)$	768 $4; i \notin H'; 192$	4	4 006 681	144	8 013 504
457	$-\langle i \rangle \cdot N_W(\langle w \rangle), w \in \mathcal{C}_{106}$	432 $\chi(i) = -i; 4$ $\chi_0: 2; 108$	4	7 055 995	268 800	14 380 788
459	$+\langle i \rangle \cdot N_W(\langle w \rangle), w \in \mathcal{C}_{106}$	432 $\chi(i) = -i; 4$ $\chi_0: 2; 108$	4	7 055 995	0	14 111 988
463	$-\langle i \rangle \cdot N_W(\langle w \rangle), w \in \mathcal{C}_{59}$	768 $\chi(i) = -i; 4$ $\chi_0: 2; 192$	4	3 742 057	1 058 688	8 542 800
467	$+\langle i \rangle \cdot N_W(\langle w \rangle), w \in \mathcal{C}_{59}$	768 $\chi(i) = -i; 4$ $\chi_0: 2; 192$	4	3 742 057	907 488	8 391 600
469	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{19}$	1 536 $\chi(i) = -i; 4$ $\chi_0: 4; 192$	4	1 852 009	605 184	4 309 200
471	$-\langle i \rangle \cdot W_S(-1), S = \{1, 2, 3, 4, 5, 7, 8, 9\}$	4 608 $\chi(i) = -i; 4$ $\chi_0: 2; 1152$	4	585 829	327 744	1 499 400
473	$+\langle i \rangle \cdot N_W(\langle w \rangle), w \in \mathcal{C}_{51}$	640 $\chi(i) = -i; 4$ $\chi_0: 2; 160$	4	4 717 441	362 880	9 797 760
477	$-\langle i \rangle \cdot W_S(-1), S = \{1, 2, 3, 4, 6, 7, 8\}$	11 520 $\chi(i) = -i; 4$ $\chi_0: 2; 2880$	4	226 801	161 280	614 880
478	$-\langle i \rangle \cdot W_S(-1), S = \{1, 3, 4, 5, 6, 8, 9\}$	17 280 $\chi(i) = -i; 4$ $\chi_0: 2; 4320$	4	147 835	120 960	416 628
492	$+\langle i \rangle \cdot W_S(-1), S = \{1, 2, 4, 5, 6, 7, 8, 9\}$	322 560 $\chi(i) = -i; 4$ χ_0 not singled out	4	6 919	10 104	23 940
493	$-\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{67}$	11 612 160 $\chi(i) = -i; 4$ χ_0 not singled out	4	103	636	840
536	$-\leq N_G(\langle g_{12} \rangle), C_8 \times (C_3 \rtimes C_4)$	96 $4; i \notin H'; 24$	4	32 054 017	768	64 108 800
574	$+\leq N_G(\langle g_{12}^2 \rangle), C_{12} \times Q_8$	96 $12; i \notin H'; 8$	12	32 053 987	852	64 108 824
857	$+\langle i \rangle \cdot C_W(w), w \in \mathcal{C}_{67}$	11 612 160 $\chi(i) = -i; 4$ χ_0 not singled out	4	103	16	220
1125	$+\leq C_G(g_{12}^3), C_{12} \times ((C_4 \times C_4) \rtimes C_2)$	384 $12; -i; 32$	12	8 013 441	320	16 027 200
3215	$-\langle i \rangle \cdot N_W(P_3)$	7 776 $\chi(i) = i; \chi_0: 2; 1944$	4	384 479	88	769 044

Independent computations. The product rows, the suppliers on $\langle i \rangle N_W(P_3)$ and the combination step were rebuilt by a second program, written separately, which found its own integer witness with 98 terms (with different support), confirmed that every row of the original matrix belongs to the corresponding family, and recomputed the product. That program used the same realization of G , the same class data and the same definition and admissibility test of the search pool, so it confirms the product rows and the combination step but not every prerequisite. A third computation reconstructed the three product rows and the row from $\langle i \rangle N_W(P_3)$ and verified balance at $t = 1, 5, 7, 11$. Table 3 lists every computer-checked claim of the paper.

TABLE 3. The computer-checked claims.

claim	software	independent check
$\mathbf{g}$ generates G ; the ten reflections generate a group of order 51 840 (App. A)	GAP	none
$X^*(T) = \mathbb{Z}^{45}$ and the rows $a(\chi)$ (App. C)	GAP	rows checked against $h = 2q - 2 + s$
$W_{\text{coef}} \cdot W_{\text{gen}} = e_\sigma$ (Prop. 8.3)	GAP	second witness, same group data
admissibility and balance of the product rows; slot data for every lift (Prop. 8.4)	GAP	third computation of balance
$\det h = -3$, signature $(3, 3)$ (Prop. 8.2)	Sage, exact over K	second program, $\det h = -6$
existence of M and $\text{rk } \mu_{C_0} = 9$ (App. B)	Python, Arb ball arithmetic	conjugate datum; rerun of the verification
Shioda's condition (P_{12}) (Subsec. 5.6)	Python, Normaliz	Shioda's verification [23, p. 182]
Schoen suppliers alone fail (App. E; not used)	GAP	separate recomputation

Slot data. For each product row, Table 4 lists one integral lift (the characters are numbered as in the certificate's enumeration of the linear characters of H'_l , and a negative coefficient means that the slot uses the inverse character), the genus of C/H'_l , and the slot data. The certificate covers every integral lift (Proposition 8.4), so the choice of lift does not matter.

TABLE 4. Slot data of the three product suppliers.

w_l	genus g_X	lift	slots (e, s, d, e')
w_1 (order 3)	3259	$-\chi_2 - \chi_7$	$(4, 6732, 1, 4); (12, 20, 1, 12)$
w_2 (order 12)	5 341 793	$\chi_1 + \chi_3 - \chi_5 - \chi_9$	$(4, 1176, 1, 4);$ $(12, 1236, 1, 12)$ twice; $(12, 1176, 3, 4)$
w_3 (order 6)	1 730 347	$-\chi_5 + 9\chi_8 + 12\chi_9 + 11\chi_{10} - \chi_{13} + 11\chi_{19}$	45 slots, all with $e = 12$ and $d = 1$; $s \in \{112, 124, 201\,688, 201\,712\}$

For w_3 the six characters have (s, h) equal to $(201\,712, 3\,662\,404)$ for $\chi_5, \chi_9, \chi_{10}$, $(112, 3\,460\,804)$ for χ_8 , $(201\,688, 3\,662\,380)$ for χ_{13} and $(124, 3\,460\,816)$ for χ_{19} , with imbalances at $t = 1$ of absolute value 22, except 20 for χ_{19} ; the combination has imbalance 0 at $t = 1, 5, 7, 11$. For w_2 every slot has imbalance ± 64 at $t = 1$, and the signs cancel. In the slot with $d = 3$ the cyclic cover has reduced order $e' = 4$, and Lemma 5.8(a) applies with three components. The slot program was also run on three cases with known answers: a slot with all exponents 0 fails (H2), the supplier of

Example 8.6 is a single slot of Hodge type with $d = 2$ and $e' = 2$, and a cyclic cover of a genus-one curve with exponents $(2, 2, 4, 4)$ modulo 6 has $d = 2$ and $e' = 3$.

Supplementary files. The supplementary files are the certificate archive [22], doi:10.5281/zenodo.23549514. Its README lists every file named below with its path in the archive. The integer witness and a GAP script that checks the identity are in `witness/`; the witness file `witness/ncb/ncb_witness.g` has SHA-256 hash

93a68c0760a10eb84f2d0760a258f66075c25aaa230c82ae33c395ded6fcf3ba.

The rank certificate (residues, enclosures and programs) is in `dominance/`, the discriminant computation in `independent-det/`, the slot data for every integral lift in `gale3-slots/`, the files of pool vectors read by the witness in `pool-scripts/` (byte-identical, by SHA-256, to the copies the witness read when it was built), and the recomputation of Shioda's condition for $m = 12$ (Subsection 5.6) in `shioda-p12/`. The archive also contains the scripts that rebuild the witness from the group (`qi-gale/`). The full rebuild of the witness takes several hours and about 10 GB of memory; the other checks take seconds to minutes.

APPENDIX D. THE COCYCLE BASIS OF PROPOSITION 8.2

This appendix prints the basis vectors used in the discriminant computation of Proposition 8.2. A vector z of the reduced row echelon basis of Z^1 is the tuple $(a_1, \dots, a_{12})$ of Proposition 8.2, of lengths $8, 4, 1, \dots, 1$, in the tuple (T_j) used there. We write it as $(a_1) \mid (a_2) \mid (a_3, \dots, a_{12})$, with entries in $K = \mathbb{Q}(i)$.

For the tuple $\mathbf{g}$ the first six vectors $z_0^g, \dots, z_5^g$ span a complement of the radical; their Gram matrix is h_0 . For the conjugate class datum (Appendix A) the vectors used are $z_0^h, z_1^h, z_3^h, z_4^h, z_5^h, z_6^h$; their Gram matrix also has determinant -3 and signature $(3, 3)$. The vector z_2^h is printed because it

precedes z_6^h in the echelon basis.

$$\begin{aligned}
z_0^g &= (1, 0, 0, 0, 0, 0, 0, 0) \mid (0, 0, 0, 0) \mid \\
&\quad (0, 0, 0, 0, 0, 0, 0, \frac{1}{2} + \frac{1}{2}i, 1 + i) \\
z_1^g &= (0, 1, 0, 0, 0, 0, 0, 0) \mid (0, 0, i, -1 + i) \mid \\
&\quad (0, 1 - i, 0, 0, 0, -1 + i, 0, 2, \frac{1}{2}, 0) \\
z_2^g &= (0, 0, 1, 0, 0, 0, 0, 0) \mid (0, 0, 1, 0) \mid \\
&\quad (0, 0, 0, 0, 0, 1, 0, 0, -\frac{1}{2}, -1) \\
z_3^g &= (0, 0, 0, 1, 0, 0, 0, 0) \mid (0, 0, 0, 1) \mid \\
&\quad (0, 0, 0, 0, 0, 0, 0, -1, -\frac{1}{2}, 0) \\
z_4^g &= (0, 0, 0, 0, 1, 0, 0, 0) \mid (0, 0, -1 - i, -\frac{1}{2} - \frac{1}{2}i) \mid \\
&\quad (0, -1 + i, 0, \frac{1}{2} + \frac{1}{2}i, 0, -i, 0, 0, \frac{3}{4} - \frac{1}{4}i, 1) \\
z_5^g &= (0, 0, 0, 0, 0, 1, 0, 0) \mid (0, 0, 1, 0) \mid \\
&\quad (0, -2i, 0, -1, 0, i, 0, 1 - i, \frac{1}{2}i, 0) \\
z_0^h &= (1, 0, 0, 0, 0, 0, 0, 0) \mid (0, 0, -1, i) \mid \\
&\quad (0, 2i, 0, 1, 0, i, 0, 2 + i, 0, 1 - i) \\
z_1^h &= (0, 1, 0, 0, 0, 0, 0, 0) \mid (0, 0, 0, -1) \mid \\
&\quad (0, -1 - i, 0, i, 0, -1 - i, 0, -1, 1, i) \\
z_2^h &= (0, 0, 1, 0, 0, 0, 0, 0) \mid (0, 0, 1, 0) \mid \\
&\quad (0, 0, 0, 0, 0, 0, 0, -1, 1, -1) \\
z_3^h &= (0, 0, 0, 1, 0, 0, 0, 0) \mid (0, 0, 0, 1) \mid \\
&\quad (0, 1, 0, 0, 0, 0, 0, 1, -1, 1) \\
z_4^h &= (0, 0, 0, 0, 1, 0, 0, 0) \mid (0, 0, -\frac{1}{2} - \frac{1}{2}i, -1) \mid \\
&\quad (0, -2, 0, -\frac{1}{2} - \frac{1}{2}i, 0, 0, 0, -1, -\frac{1}{2} + \frac{1}{2}i, -\frac{1}{2} - \frac{1}{2}i) \\
z_5^h &= (0, 0, 0, 0, 0, 1, 0, 0) \mid (0, 0, 0, -i) \mid \\
&\quad (0, 1, 0, i, 0, 1, 0, 1, -1, i) \\
z_6^h &= (0, 0, 0, 0, 0, 0, 1, 0) \mid (0, 0, \frac{1}{2} + \frac{1}{2}i, 1) \mid \\
&\quad (0, 1 - i, 0, -\frac{1}{2} - \frac{3}{2}i, 0, 0, 0, -1 - i, \frac{3}{2} - \frac{3}{2}i, -\frac{1}{2} - \frac{1}{2}i)
\end{aligned}$$

To check h_0 from these vectors, set

$$F_1 = 0, \quad F_{j+1} = F_j + Q_j E_j a_j,$$

with T_j , Q_j , E_j as in Proposition 8.2. Then evaluate the form of Lemma 6.2,

$$h(z, z') = i \sum_{j=1}^{12} \overline{(Q_j^{-1} F_j + U_j a_j)}^T H E_j a'_j,$$

where H is the Cartan matrix of E_8 and U_j is the matrix of the same pivot columns of the identity, so that $E_j a_j = (1 - T_j)U_j a_j$. Recomputing h on $z_0^g, \dots, z_5^g$ gives the matrix h_0 of Proposition 8.2. Recomputing it on $z_0^h, z_1^h, z_3^h, \dots, z_6^h$ gives the block for the conjugate datum. The script `independent-det/independent_det.py` in the supplementary files does this; these vectors are the first rows of the matrix Z that it computes.

APPENDIX E. SCHOEN SUPPLIERS ALONE DO NOT SUFFICE

This appendix is not used in the proof. It records why a second source of cycles is needed: for this class datum, condition (L) with Schoen-admissible suppliers alone fails.

Let $y = iw$ for a suitable element $w \in W$ of order 3. For a subgroup H' and a linear character χ , Frobenius reciprocity gives a linear functional f on $X^*(T)$ with $f(e_\sigma) = 1$ and

$$f(a(\chi)) = \frac{|C_G(y)|}{|H'|} \sum_{z \in H' \cap y^G} \text{Im } \chi(z).$$

A local analysis in the centralizers $C_{H'}(z)$, $z \in H' \cap y^G$, shows that a Schoen-admissible supplier with $f(a(\chi)) \neq 0$ must contain one of finitely many configurations of three elements; up to conjugation there are 1248 of them. A necessary condition that is inherited by overgroups rules out all but 72, and the subgroups containing these 72 were enumerated exhaustively: they carry 4496 characters of the relevant kind, of which 626 give Schoen-admissible suppliers, and f vanishes on all of them. The configurations, the necessary conditions and all 4496 characters were recomputed by a separate program; the enumeration itself was checked from its saved state. So e_σ is not in the subgroup generated by the characters of Schoen-admissible suppliers. Brauer's induction theorem, which supplies (L) in [21, §5], does not apply: the class of ix_2 is not conjugate to its inverse. The programs and their outputs are in the certificate archive [22], directory `qi-schoen-negative` (subdirectories `LS1`, `LS2`, `LS2R`).

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