

Supersymmetric Yang–Mills theories with several times reduce to ten dimensions

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Abstract

Supersymmetric Yang–Mills theories in signatures with several times are built by restricting the gaugino to the kernel of k mutually orthogonal null vectors and letting the supersymmetry close on translations contracted with their product. We prove that every theory of this kind, defined by five axioms that Nishino’s construction satisfies, has residual dimension $s + t - 2k$ equal to ten (or to two, a degenerate case), and that its null vectors can be real only if $k \leq \min(s, t)$. The cubic fermion terms reduce to a Fierz identity on the residual spinors, and that identity forces the dimension of the spinors to be $2(d - 2)$ on a d -dimensional vector space, which happens only for $d = 3, 4, 6, 10$. With Majorana–Weyl spinors the signature must therefore satisfy $s - t \in \{8, 0, -8\}$. There is no such theory in $(9 + 8n) + 1$, $(10 + 8n) + 2$, $(11 + 8n) + 3$ or $(12 + 8n) + 4$ for $n \geq 1$, although it has been claimed that the existence proofs for the signatures $(9 + m) + (1 + m)$ carry over to these; the signatures $27 + 3$ and $19 + 3$ are among those excluded. For a single time the result does not depend on the construction: in $(9 + 8n) + 1$ with $n \geq 1$ every massless multiplet without tensor charges has at least $2^{2^{2+4n}}$ states, so a unitary theory with no tensor charges on its massless states has no massless vector multiplet.

Keywords: Supersymmetric Gauge Theory, Extended Supersymmetry, Space-Time Symmetries, Field Theories in Higher Dimensions

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1 Introduction

Supersymmetric Yang–Mills theory, with a gauge field and a gaugino as its only fields, exists for a single time in three, four, six and ten dimensions and in no others [1, 2, 3, 4]. The reason is one algebraic fact. Varying the gauge field inside the covariant derivative of the gaugino kinetic term produces a term cubic in the gaugino, and no other term in the variation of the Lagrangian can cancel it. This term vanishes precisely when the symmetric gamma matrices obey the Fierz identity

$$\gamma_{\alpha\beta}^{\mu}\gamma_{\mu\gamma\delta} + \gamma_{\alpha\gamma}^{\mu}\gamma_{\mu\beta\delta} + \gamma_{\alpha\delta}^{\mu}\gamma_{\mu\beta\gamma} = 0, \quad (1.1)$$

which holds for the Majorana spinors of three and four dimensions, the Weyl spinors of six and the Majorana–Weyl spinors of ten, the spinors attached to the four normed division algebras [2, 3, 4]. The same identity makes the Green–Schwarz superstring κ -symmetric [5], its analogues for forms of higher degree produce the scan of supersymmetric branes [6, 7], and Blencowe and Duff extended that scan to spacetimes with several times [8]. Above ten dimensions the spinors are too large: a Majorana–Weyl spinor in D dimensions has $2^{D/2-1}$ components, while (1.1) allows $2(D-2)$ of them (Theorem 3.3).

A supersymmetric Yang–Mills theory in more dimensions therefore needs more than one time. Signatures with several times have been studied for several reasons. F-theory points to a twelve-dimensional origin of type IIB string theory [9]; Bars proposed an S-theory in $11 + 2$ dimensions and made a case for fourteen [10, 11]; in two-time physics an $Sp(2, \mathbb{R})$ gauge symmetry removes one time and one space direction [12, 13, 14]; and the dualities of string theory and M-theory relate theories of different signatures [15, 16]. Superparticles beyond eleven dimensions were studied by Rudychev and Sezgin [17], and supergravity in $10 + 2$ dimensions by Nishino [18]. Supersymmetric Yang–Mills equations in $10 + 2$ dimensions were written by Nishino and Sezgin [19], a theory in $11 + 3$ dimensions was given by Sezgin [20], and the mechanism behind these constructions was set out by Rudychev, Sezgin and Sundell [21]. Nishino then built supersymmetric Yang–Mills theories in every signature $(9 + m) + (1 + m)$ [22]. His construction introduces m mutually orthogonal null vectors $n_{(1)}, \dots, n_{(m)}$, constrains the gaugino by $\not{n}_{(i)}\lambda = 0$, and lets the supersymmetry algebra close on translations contracted with $n_{(1)} \wedge \dots \wedge n_{(m)}$. The cubic gaugino terms then need not vanish. They only have to take a form that can be absorbed into the field equation of the gauge field, and in Nishino’s construction they do.

Rios, Marrani and Chester observed that the reality and symmetry properties of spinors and gamma matrices in $\mathbb{R}^{s,t}$ depend only on $D = s + t$ and on $s - t$ modulo eight [23]. Moving $4n$ timelike directions of Nishino’s signatures into space changes neither, and they concluded that “the proofs given in [6] (see also [5]) for the existence of SYM’s with superalgebras (3.1)–(3.2)

can be successfully repeated to show the existence of SYM's" in the four families¹

$$(9 + 8n) + 1, \quad (10 + 8n) + 2, \quad (11 + 8n) + 3, \quad (12 + 8n) + 4, \quad (1.2)$$

with the chiral superalgebras of their eqs. (3.5)–(3.8). They called these theories exceptional super Yang–Mills theories. The signature $27 + 3$, the case $n = 2$ of the third family, was then proposed as the home of a theory whose fields fill the exceptional Jordan algebra and whose worldvolume would carry M-theory [24]. The same theory was used to model dark matter [25], it entered the construction of “monstrous” M-theory [26] and the programme of exceptional periodicity [27], and the signatures $19 + 3$ and $27 + 3$ were taken as starting points for higher M-theories [28]. No construction of these theories has been published, and we found no paper that tests the claim.²

The claim is false. Reality and symmetry of spinors are not the only data that enter Nishino’s construction. The null vectors must be mutually orthogonal, and if they annihilate a real gaugino they lie in the complexification of a real totally isotropic subspace, whose dimension cannot exceed $\min(s, t)$ (Lemmas 2.2 and 2.3). Moving timelike directions into space lowers this number, and with it the number of null directions that can be projected out. What remains after the projection is a theory on the orthogonal complement of the null frame, of dimension $D' = s + t - 2k$ for k null vectors. We show that the cubic terms can be absorbed only if the residual spinors satisfy (1.1) themselves, and this forces $D' = 10$. The main result, proved in Sec. 5.4, reads as follows.

Main Theorem. *Let a supersymmetric Yang–Mills theory with a Majorana–Weyl gaugino in signature (s, t) be built on a frame of k null vectors in the manner of Nishino, in the sense of the axioms (N1)–(N5) of Definition 5.3. Then $k \leq \min(s, t)$ and $s + t - 2k \in \{2, 10\}$, so that $s - t \in \{8, 0, -8\}$. In particular there is no such theory in any of the signatures (1.2) with $n \geq 1$, and none in $27 + 3$ or in $19 + 3$.*

Figure 1 shows the geometry. Nishino’s signatures lie on the plane $s + t - 2k = 10$, while every signature of the families (1.2) with $n \geq 1$ has residual dimension at least 16 for every admissible k . What decides existence is $s - t$ itself, not its residue modulo eight.³

For a single time we can say more, without any assumption about how the theory is built. In $(9 + 8n) + 1$ the anticommutator of two supercharges contains the momentum, and in a unitary theory the supercharges that act nontrivially on a massless state generate a Clifford algebra with 2^{3+4n} generators. A massless multiplet without tensor charges then has at least 2^{2+4n} states, half of them bosonic (Theorem 6.1). For $n = 1$ that is 2^{64} states, against the 16 bosonic states of a vector field in eighteen dimensions.

The paper is organized as follows. Section 2 collects what we need about spinors in signature (s, t) : the decomposition of the symmetric square of a Weyl spinor, which shows that the superalgebras of Ref. [23] are the most general ones and carry no information about dynamics, and two lemmas on isotropic subspaces. Section 3 proves the theorem on which everything else rests: if symmetric gamma matrices obey an identity of the type that absorption requires, the spinor dimension is $2(d - 2)$. Section 4 applies it to ordinary super Yang–Mills theory. Section 5 describes null frames by fermionic oscillators, states the axioms, and proves the reduction to ten residual dimensions and the Main Theorem. Section 6 treats the single-time family by

¹References [6] and [5] of Ref. [23] are our Refs. [22] and [20], and its superalgebras (3.1)–(3.4) belong to the signatures $(9 + 4n) + (1 + 4n)$, $(10 + 4n) + (2 + 4n)$, $(11 + 4n) + (3 + 4n)$, $(12 + 4n) + (4 + 4n)$. Equation numbers in Refs. [22], [23] and [24] are those of arXiv:hep-th/9708064v1, arXiv:1811.06101v3 and arXiv:1906.10709v4.

²We read every record in the INSPIRE database that cites Ref. [23] or Ref. [24], thirteen and eleven records in October 2026. All of them use or quote the existence of the theories; none constructs one, and none asks whether Nishino’s argument carries over.

³Bott periodicity makes the real Clifford algebras of $\mathbb{R}^{s,t}$ and $\mathbb{R}^{s+4,t-4}$ isomorphic, but no isomorphism carries the space of generators $\mathbb{R}^{s,t}$ onto $\mathbb{R}^{s+4,t-4}$. Null vectors, their orthogonality and the bound $\min(s, t)$ are properties of the generators.

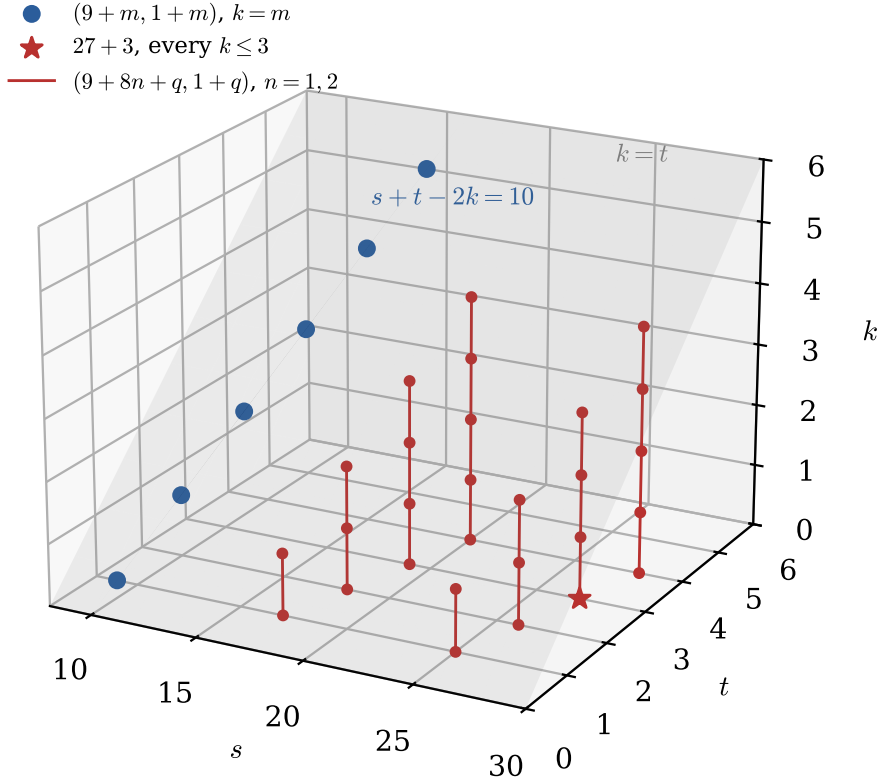


Figure 1: Signatures (s, t) and numbers k of null vectors. Blue points: Nishino's signatures $(9 + m) + (1 + m)$ with $k = m$, on the plane $s + t - 2k = 10$ (blue). Red segments: the signatures $(9 + 8n + q) + (1 + q)$ of the families (1.2) for $n = 1, 2$ and $q = 0, 1, 2, 3$, with every $k \leq t$; the star marks $27 + 3$. The grey surface is the wall $k = t$, beyond which real null frames do not exist. No red point reaches the blue plane.

representation theory, and Sec. 7 spells out what the results mean for $27 + 3$ and $19 + 3$. Appendix A proves the oscillator decomposition, Appendix B describes numerical checks in explicit gamma matrices, and Appendix C lists the symmetric squares for the families (1.2). Figure 2 shows how the results depend on one another.

2 Spinors in signature (s, t)

2.1 Conventions

Let $V = \mathbb{R}^{s,t}$ with metric $\eta = \text{diag}(+1, \dots, +1, -1, \dots, -1)$, s plus signs and t minus signs, and $D = s + t$. Except in Remark 5.11 we take $D = 2r$ even. The Clifford relations are $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$. The Dirac module Δ of the complex Clifford algebra $\text{Cl}(V_{\mathbb{C}})$ has dimension 2^r and splits into the Weyl modules $\Delta_{\pm}$ of dimension 2^{r-1} , the eigenspaces of the chirality operator Γ_* , normalized so that $\Gamma_*^2 = 1$. For a vector u we write $\psi = u_\mu \gamma^\mu$, and $\gamma^{\mu_1 \dots \mu_p}$ is the antisymmetrized product of unit weight, so that $\gamma^{\mu_1 \dots \mu_p} = \gamma^{\mu_1} \dots \gamma^{\mu_p}$ when the indices are distinct and the basis is orthonormal.

There is a nondegenerate bilinear form B on Δ , unique up to a factor once a sign $\theta = \pm 1$ is chosen, such that

$$B(\gamma^\mu \psi, \chi) = \theta B(\psi, \gamma^\mu \chi), \quad B(\psi, \chi) = \epsilon_B B(\chi, \psi), \quad (2.1)$$

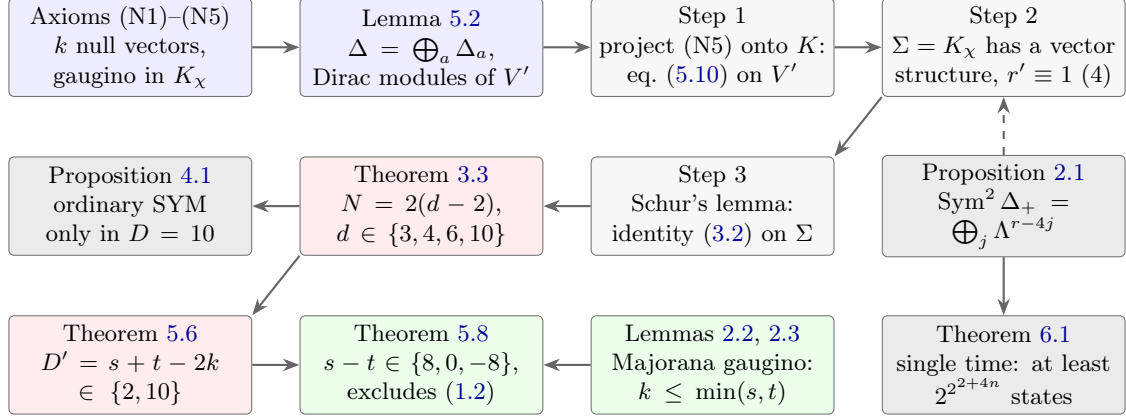


Figure 2: The logical structure of the paper. The axioms pass through the oscillator decomposition and Steps 1–3 to the identity (3.2) on the residual spinors; Theorem 3.3 turns that identity into $D' \in \{2, 10\}$, and the bound on the number of null vectors converts this into a condition on $s - t$. Proposition 2.1 enters Step 2 (dashed) and the single-time theorem. Proposition 4.1 is the case without null vectors.

with $\epsilon_B = \pm 1$ determined by r and θ [29, 30, 31]. We write $\bar{\psi}\chi = B(\psi, \chi)$ and $\bar{\psi}\gamma^{\mu_1 \cdots \mu_p}\chi = B(\psi, \gamma^{\mu_1 \cdots \mu_p}\chi)$. The form B pairs Δ_+ with Δ_+ when r is even and with Δ_- when r is odd.

A real structure on Δ is an antilinear map $C : \Delta \rightarrow \Delta$ with $C^2 = 1$ that commutes with every γ^μ . It exists exactly when $s - t \equiv 0, 1, 2 \pmod{8}$, and it preserves $\Delta_\pm$ exactly when $s - t \equiv 0 \pmod{8}$ [32, 29, 30]. A Majorana spinor is a spinor with $C\psi = \psi$. Majorana–Weyl spinors, which are the spinors of all the theories in this paper, therefore exist if and only if $s - t \equiv 0 \pmod{8}$, and every signature in (1.2) and in Nishino’s list has this property.

2.2 The symmetric square of a Weyl spinor

The anticommutator of two supercharges in the Weyl module Δ_+ is a symmetric element of $\Delta_+ \otimes \Delta_+$, so the superalgebras that can be written down in signature (s, t) are classified by the decomposition of $\text{Sym}^2 \Delta_+$ [33, 31]. We write $\Lambda^p = \Lambda^p V_{\mathbb{C}}$, and Λ^r stands for the half Λ_+^r of the middle-dimensional forms selected by Δ_+ .

Proposition 2.1. *For $D = 2r$,*

$$\text{Sym}^2 \Delta_+ \cong \bigoplus_{j \geq 0, r-4j \geq 0} \Lambda^{r-4j}, \quad \Lambda^2 \Delta_+ \cong \bigoplus_{j \geq 0, r-2-4j \geq 0} \Lambda^{r-2-4j}. \quad (2.2)$$

In particular $\Lambda^1 \subset \text{Sym}^2 \Delta_+$ if and only if $r \equiv 1 \pmod{4}$, that is $D \equiv 2 \pmod{8}$, and $\Lambda^1 \subset \Lambda^2 \Delta_+$ if and only if $r \equiv 3 \pmod{4}$.

Proof. The forms $(\psi, \chi) \mapsto \bar{\psi}\gamma^{\mu_1 \cdots \mu_p}\chi$ restricted to $\Delta_+ \times \Delta_+$ vanish unless $p \equiv r \pmod{2}$, and those with $p \equiv r \pmod{2}$, $0 \leq p \leq r$, give an isomorphism $\Delta_+ \otimes \Delta_+ \cong \bigoplus_p \Lambda^p$ [29, 30]. (As a check, $\sum_{p \equiv r(2), p < r} \binom{2r}{p} + \frac{1}{2} \binom{2r}{r} = 2^{2r-2} = (\dim \Delta_+)^2$.) It remains to decide which summands are symmetric. By (2.1),

$$\bar{\psi}\gamma^{\mu_1 \cdots \mu_p}\chi = \theta^p B(\gamma^{\mu_p \cdots \mu_1} \psi, \chi) = \theta^p (-1)^{p(p-1)/2} \epsilon_B \bar{\chi} \gamma^{\mu_1 \cdots \mu_p} \psi,$$

so the p -th form has symmetry $\sigma_p = \epsilon_B \theta^p (-1)^{p(p-1)/2}$, and $\sigma_{p+2}/\sigma_p = \theta^2 (-1)^{2p+1} = -1$. The symmetric summands are therefore either those with $p \equiv r \pmod{4}$ or those with $p \equiv r - 2 \pmod{4}$. Using $\binom{2r}{p} = \binom{2r}{2r-p}$ and summing over the fourth roots of unity, the first set has total

dimension

$$\frac{1}{2} \sum_{\substack{0 \leq p \leq 2r \\ p \equiv r \pmod{4}}} \binom{2r}{p} = \frac{1}{8} \sum_{\omega^4=1} \omega^{-r} (1 + \omega)^{2r} = \frac{1}{8} (2^{2r} + 2^{r+1}) = 2^{2r-3} + 2^{r-2},$$

because $i^{-r}(1+i)^{2r} = i^r(1-i)^{2r} = 2^r$ and the term with $\omega = -1$ vanishes. This is $N(N+1)/2$ for $N = 2^{r-1}$, while the second set has dimension $N(N-1)/2$. Hence the symmetric summands are those with $p \equiv r \pmod{4}$. $\square$

Appendix C lists the decomposition for the families (1.2). For $(9+8n)+1$ one has $r = 5+4n$, and $\text{Sym}^2 \Delta_+$ consists of the forms of degrees $1, 5, \dots, 5+4n$; for $(10+8n)+2$ the degrees are $2, 6, \dots, 6+4n$; for $(11+8n)+3$ they are $3, 7, \dots, 7+4n$; and for $(12+8n)+4$ they are $0, 4, \dots, 8+4n$. These are exactly the right-hand sides of eqs. (3.5)–(3.8) of Ref. [23]. The superalgebras proposed there are therefore the free ones: the anticommutator of two supercharges is an arbitrary element of $\text{Sym}^2 \Delta_+$, with a central charge for every summand. Such an algebra exists for every Weyl spinor in every signature, and its existence says nothing about field theories.⁴ It contains the momentum only for $r \equiv 1 \pmod{4}$.

2.3 Isotropic subspaces

A subspace U of V or of $V_{\mathbb{C}}$ is totally isotropic if $u \cdot u' = 0$ for all $u, u' \in U$.

Lemma 2.2. *A real totally isotropic subspace $U \subset \mathbb{R}^{s,t}$ has $\dim U \leq \min(s, t)$, and the bound is attained.*⁵

Proof. Write $V = V_+ \oplus V_-$ with V_+ positive definite of dimension s and V_- negative definite of dimension t . If $u \in U \cap V_-$, then $u \cdot u = 0$ forces $u = 0$, so the projection $U \rightarrow V_+$ is injective and $\dim U \leq s$. Exchanging the roles gives $\dim U \leq t$. With orthonormal bases $e_1, \dots, e_s$ of V_+ and $f_1, \dots, f_t$ of V_- , the vectors $e_i + f_i$, $i \leq \min(s, t)$, span a totally isotropic subspace of dimension $\min(s, t)$. $\square$

The next lemma says that a Majorana spinor cannot distinguish a complex null vector from its complex conjugate.

Lemma 2.3. *Let $s - t \equiv 0, 1, 2 \pmod{8}$ and let $\lambda \neq 0$ be a Majorana spinor. The set $U_\lambda = \{u \in V_{\mathbb{C}} : \psi\lambda = 0\}$ is a complex totally isotropic subspace invariant under complex conjugation. Hence $U_\lambda = U_{\mathbb{R}} \otimes \mathbb{C}$ with $U_{\mathbb{R}} \subset V$ real and totally isotropic, and $\dim_{\mathbb{C}} U_\lambda \leq \min(s, t)$.*

Proof. If $\psi\lambda = 0$ and $\psi'\lambda = 0$, then $2(u \cdot u')\lambda = (\psi\psi' + \psi'\psi)\lambda = 0$, so $u \cdot u' = 0$. Since C is antilinear and commutes with the γ^μ , $C(\psi\lambda) = \bar{u}_\mu \gamma^\mu C\lambda = \bar{\psi}\lambda$, so $\psi\lambda = 0$ implies $\bar{\psi}\lambda = 0$. A conjugation-invariant complex subspace is the complexification of its real points, and the real points of a totally isotropic subspace form a totally isotropic subspace of V . Lemma 2.2 finishes the proof. $\square$

Figure 3 illustrates the lemma in $\mathbb{R}^{2,1}$, where $\min(s, t) = 1$: two independent real null vectors are never orthogonal.

⁴For $27+3$ one has $r = 15$ and $\text{Sym}^2 \Delta_+ = \Lambda_+^{15} \oplus \Lambda_+^{11} \oplus \Lambda_+^7 \oplus \Lambda_+^3$, of dimension $77\,558\,760 + 54\,627\,300 + 2\,035\,800 + 4\,060 = 134\,225\,920 = 2^{13}(2^{14} + 1)$. These are the 3-, 7-, 11- and 15-form charges of eq. (2.1) of Ref. [24].

⁵The number $\min(s, t)$ is the Witt index of η [34, 35].

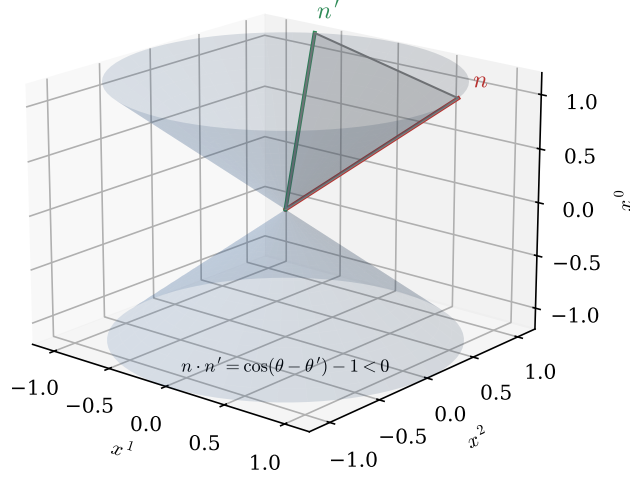


Figure 3: The light cone of $\mathbb{R}^{2,1}$. Two independent real null vectors $n = (1, \cos \theta, \sin \theta)$ and $n' = (1, \cos \theta', \sin \theta')$ satisfy $n \cdot n' = \cos(\theta - \theta') - 1 < 0$, so the plane they span (grey) is not isotropic. In $\mathbb{R}^{s,t}$ the largest isotropic subspaces have dimension $\min(s, t)$ (Lemma 2.2).

3 Vector structures and the cubic identity

Every condition on cubic fermion terms that appears in this paper is a statement about the following structure.

Definition 3.1. Let W be a complex vector space of dimension d with a nondegenerate symmetric bilinear form η , and let Σ be a complex vector space of dimension $N \geq 1$ with dual Σ^* . A *vector structure* on Σ is a pair of linear maps, γ from W to the symmetric maps $\Sigma \rightarrow \Sigma^*$ and $\tilde{\gamma}$ from W to the maps $\Sigma^* \rightarrow \Sigma$, such that in a basis of W

$$\tilde{\gamma}^\mu \gamma^\nu + \tilde{\gamma}^\nu \gamma^\mu = 2\eta^{\mu\nu} 1_\Sigma, \quad \gamma^\mu \tilde{\gamma}^\nu + \gamma^\nu \tilde{\gamma}^\mu = 2\eta^{\mu\nu} 1_{\Sigma^*}. \quad (3.1)$$

In a basis e_α of Σ we write $\gamma_{\alpha\beta}^\mu$ and $\tilde{\gamma}^{\mu\alpha\beta}$ for the matrix entries; $\gamma_{\alpha\beta}^\mu = \gamma_{\beta\alpha}^\mu$.

The standard example is $\Sigma = \Delta_+$ in $D = 2r$ with r odd, $\Sigma^* \cong \Delta_-$ through B , $\gamma_{\alpha\beta}^\mu = B(e_\alpha, \gamma^\mu e_\beta)$, and $\tilde{\gamma}^\mu$ the Clifford action $\Delta_- \rightarrow \Delta_+$; by Proposition 2.1 the forms are symmetric when $r \equiv 1 \pmod{4}$. In ten dimensions these are the familiar 16×16 matrices $\sigma_{\alpha\beta}^\mu$ and $\bar{\sigma}^{\mu\alpha\beta}$ [36].

Lemma 3.2. Let $(\Sigma, \gamma, \tilde{\gamma})$ be a vector structure with $d \geq 3$. Then

- (a) each $\tilde{\gamma}^\mu$ is symmetric, and in an orthonormal basis $\tilde{\gamma}^\mu = \eta^{\mu\mu}(\gamma^\mu)^{-1}$ (no sum);
- (b) $\gamma_\mu \tilde{\gamma}^\mu = d 1_{\Sigma^*}$ and $\gamma_\mu \tilde{\gamma}^\nu \gamma^\mu = -(d-2)\gamma^\nu$;
- (c) $\text{tr}_{\Sigma^*}(\gamma^\mu \tilde{\gamma}^\nu) = \gamma_{\alpha\beta}^\mu \tilde{\gamma}^{\nu\beta\alpha} = N\eta^{\mu\nu}$.

Proof. (a) Setting $\mu = \nu$ in (3.1) gives $\gamma^\mu \tilde{\gamma}^\mu = \eta^{\mu\mu} 1_{\Sigma^*}$ and $\tilde{\gamma}^\mu \gamma^\mu = \eta^{\mu\mu} 1_\Sigma$, and the inverse of a symmetric map is symmetric. (b) Contracting the second relation in (3.1) gives the first identity, and then $\gamma_\mu \tilde{\gamma}^\nu \gamma^\mu = \gamma_\mu (2\eta^{\nu\mu} - \tilde{\gamma}^\mu \gamma^\nu) = 2\gamma^\nu - d\gamma^\nu$. (c) On $\Sigma \oplus \Sigma^*$ the operators $\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}$ obey $\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\eta^{\mu\nu}$, and $\Gamma_* = \text{diag}(1, -1)$ anticommutes with every Γ^μ . In an orthonormal basis take $\mu \neq \nu$ and choose $\rho \notin \{\mu, \nu\}$, which is possible because $d \geq 3$. The

invertible operator Γ^ρ commutes with $\Gamma^\mu \Gamma^\nu$ and anticommutes with $\Gamma_* \Gamma^\mu \Gamma^\nu$, so $\text{tr}(\Gamma_* \Gamma^\mu \Gamma^\nu) = 0$; and $\text{tr}(\Gamma^\mu \Gamma^\nu) = -\text{tr}(\Gamma^\nu \Gamma^\mu) = 0$. The restriction of $\Gamma^\mu \Gamma^\nu = \text{diag}(\tilde{\gamma}^\mu \gamma^\nu, \gamma^\mu \tilde{\gamma}^\nu)$ to Σ^* is $\gamma^\mu \tilde{\gamma}^\nu$, and its trace $\frac{1}{2} \text{tr}((1 - \Gamma_*) \Gamma^\mu \Gamma^\nu)$ therefore vanishes for $\mu \neq \nu$. For $\mu = \nu$ it is $N \eta^{\mu\mu}$ by (a). $\square$

The condition that absorption of cubic terms will produce is the following. For a constant $j \in \mathbb{C}$ consider

$$\frac{1}{2}(\gamma_{\alpha\beta}^\mu \gamma_{\mu\gamma\delta} + \gamma_{\alpha\gamma}^\mu \gamma_{\mu\beta\delta}) = j \gamma_{\alpha\delta}^\sigma \gamma_{\sigma\beta\gamma} \quad \text{for all } \alpha, \beta, \gamma, \delta. \quad (3.2)$$

For $j = -\frac{1}{2}$ this is the identity (1.1).

Theorem 3.3. *Let $(\Sigma, \gamma, \tilde{\gamma})$ be a vector structure with $d \geq 3$, and suppose that (3.2) holds for some $j \in \mathbb{C}$. Then*

$$N = 2(d-2), \quad j = -\frac{1}{2}, \quad (3.3)$$

so the identity is (1.1). Moreover N is divisible by $2^{\lceil d/2 \rceil - 1}$, and therefore $(d, N) \in \{(3, 2), (4, 4), (6, 8), (10, 16)\}$.

Proof. Contract (3.2) with $\tilde{\gamma}^{\nu\delta\gamma}$, which is symmetric by Lemma 3.2(a), and sum over γ, δ . Inside the bracket on the left, the first term gives $\gamma_{\alpha\beta}^\mu \text{tr}(\gamma_\mu \tilde{\gamma}^\nu) = N \gamma_{\alpha\beta}^\nu$ and the second gives $(\gamma^\mu \tilde{\gamma}^\nu \gamma_\mu)_{\alpha\beta} = -(d-2) \gamma_{\alpha\beta}^\nu$, by Lemma 3.2(b),(c). The right-hand side gives $j(\gamma^\sigma \tilde{\gamma}^\nu \gamma_\sigma)_{\alpha\beta} = -j(d-2) \gamma_{\alpha\beta}^\nu$. Since $\gamma^\nu \neq 0$,

$$N - (d-2) = -2j(d-2). \quad (3.4)$$

Next contract with $\tilde{\gamma}^{\nu\beta\gamma}$ and sum over β, γ . Both terms in the bracket give $(\gamma^\mu \tilde{\gamma}^\nu \gamma_\mu)_{\alpha\delta} = -(d-2) \gamma_{\alpha\delta}^\nu$, and the right-hand side gives $j \gamma_{\alpha\delta}^\sigma \text{tr}(\gamma_\sigma \tilde{\gamma}^\nu) = jN \gamma_{\alpha\delta}^\nu$, so

$$jN = -(d-2). \quad (3.5)$$

Substituting $j = -(d-2)/N$ into (3.4) gives $N^2 - (d-2)N - 2(d-2)^2 = 0$, that is $(N - 2(d-2))(N + d - 2) = 0$. As $N + d - 2 > 0$, we get $N = 2(d-2)$ and $j = -\frac{1}{2}$.

For the divisibility, the operators Γ^μ and Γ_* of the proof of Lemma 3.2(c) make $\Sigma \oplus \Sigma^*$ a module over the complex Clifford algebra on $d+1$ generators. This algebra is semisimple and its irreducible modules have dimension $2^{\lfloor (d+1)/2 \rfloor} = 2^{\lceil d/2 \rceil}$, so $2^{\lceil d/2 \rceil}$ divides $2N = 4(d-2)$. For $d = 5, 7, 8, 9$ it does not, and for $d \geq 11$ one has $4(d-2) < 2^{\lceil d/2 \rceil}$. This leaves $d = 3, 4, 6, 10$. $\square$

Remark 3.4. The four solutions are the Majorana spinors of three and four dimensions, the Weyl spinors of six (with their $Sp(1)$ doublet) and the Majorana–Weyl spinors of ten, and in each case (1.1) holds [2, 3, 4]. Theorem 3.3 is a converse that makes no reference to division algebras and allows a general constant j : the dimension count alone fixes both N and j . The constant j is left free because absorbing the cubic terms into a modified field equation, the mechanism of Nishino’s construction, produces (3.2) with a constant that is not known in advance (Sec. 5). Figure 4 compares $2(d-2)$ with the smallest admissible N .

4 Ordinary super Yang–Mills theory

Before turning to null projections we record what Theorem 3.3 gives without them. Let the gauge algebra $\mathfrak{g}$ be a compact non-abelian Lie algebra with structure constants f^{IJK} , totally antisymmetric in an orthonormal basis. Let the gaugino λ^I and the parameter ϵ be Grassmann-odd Weyl spinors in Δ_+ (Majorana–Weyl when $s - t \equiv 0 \pmod{8}$), and consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^I F^{I\mu\nu} + \frac{c}{2} \bar{\lambda}^I \gamma^\mu D_\mu \lambda^I, \quad \delta A_\mu^I = a \bar{\epsilon} \gamma_\mu \lambda^I, \quad \delta \lambda^I = b \gamma^{\mu\nu} F_{\mu\nu}^I \epsilon, \quad (4.1)$$

with $D_\mu \lambda^I = \partial_\mu \lambda^I + g f^{IJK} A_\mu^J \lambda^K$ and nonzero constants a, b, c, g .

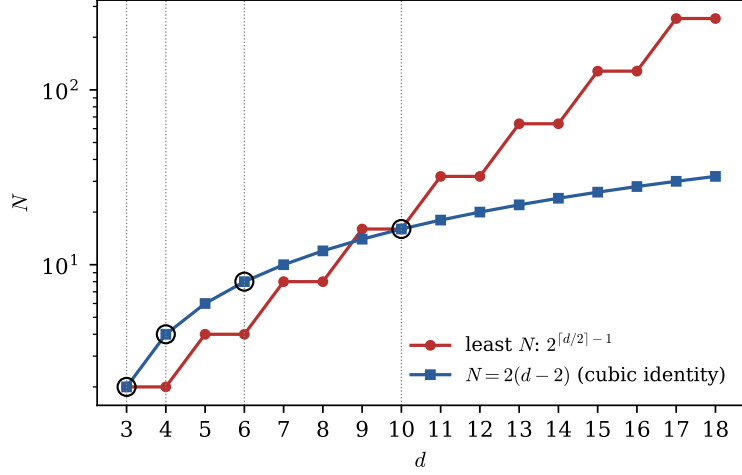


Figure 4: The spinor dimension $N = 2(d - 2)$ forced by the cubic identity (blue) against the smallest dimension $2^{\lfloor d/2 \rfloor - 1}$ that a vector structure can have (red). A solution requires the blue value to be a multiple of the red one. This happens for $d = 3, 4, 6, 10$ (circles) and never again, because the red curve grows exponentially.

Proposition 4.1. *If $\delta\mathcal{L}$ is a total derivative, then the forms $\gamma_{\alpha\beta}^\mu = B(e_\alpha, \gamma^\mu e_\beta)$ on Δ_+ define a vector structure satisfying (1.1). Consequently $D = 10$, and with Majorana–Weyl spinors the signature is $(9, 1)$, $(5, 5)$ or $(1, 9)$.*

Proof. The kinetic term is nonzero only if B pairs Δ_+ with $\gamma^\mu \Delta_+ = \Delta_-$, that is for r odd, and it is not a total derivative only if the form $\gamma_{\alpha\beta}^\mu$ is symmetric: for Grassmann-odd λ and antisymmetric G , $\lambda^\alpha G_{\alpha\beta} \partial \lambda^\beta = \frac{1}{2} \partial (\lambda^\alpha G_{\alpha\beta} \lambda^\beta)$. With $\Sigma = \Delta_+$, $\Sigma^* \cong \Delta_-$ and $\tilde{\gamma}^\mu$ the Clifford action $\Delta_- \rightarrow \Delta_+$, the relations (3.1) are those of $\text{Cl}(V)$, so $(\Delta_+, \gamma, \tilde{\gamma})$ is a vector structure with $d = D$ and $N = 2^{r-1}$.

The terms of $\delta\mathcal{L}$ cubic in λ come only from δA inside the covariant derivative, since $\delta\lambda$ is linear in F and δF is linear in λ . They are

$$\frac{c}{2} ag f^{IJK} (\bar{\lambda}^I \gamma^\mu \lambda^K) (\bar{\epsilon} \gamma_\mu \lambda^J) = \pm \frac{c}{2} ag \gamma_{\alpha\gamma}^\mu \gamma_{\mu\delta\beta} \epsilon^\delta T^{\alpha\beta\gamma}, \quad T^{\alpha\beta\gamma} = f^{IJK} \lambda^{I\alpha} \lambda^{J\beta} \lambda^{K\gamma}, \quad (4.2)$$

where the sign depends only on the ordering convention for Grassmann variables. They contain no derivative, so they must vanish. The tensor T is totally symmetric: exchanging two of its indices and relabelling the gauge indices produces one sign from f and one from the anticommuting λ . It spans $\text{Sym}^3 \Sigma$. Indeed $\mathfrak{g}$ contains a subalgebra $\mathfrak{su}(2)$, and for λ with values in it $T^{\alpha\beta\gamma} c_{\alpha\beta\gamma} = 6c_{\alpha\beta\gamma} \lambda^{1\alpha} \lambda^{2\beta} \lambda^{3\gamma}$ for every symmetric c , after normalizing $f^{123} = 1$. Since the monomials $\lambda^{1\alpha} \lambda^{2\beta} \lambda^{3\gamma}$ are linearly independent, only $c = 0$ annihilates every T . Hence, for every δ , the part of $\gamma_{\alpha\gamma}^\mu \gamma_{\mu\delta\beta}$ symmetric in α, β, γ vanishes. Because each factor is symmetric in its own pair of indices, that symmetrization is one third of the sum over the three ways of pairing δ with α, β or γ , which is (1.1). Theorem 3.3 with $N = 2^{r-1}$ and $d = 2r$ gives $2^{r-1} = 4(r-1)$, whose only solution with $r \geq 2$ is $r = 5$. Majorana–Weyl spinors in $D = 10$ require $s - t \in \{8, 0, -8\}$. $\square$

The proposition is the familiar statement that ten is the largest dimension of super Yang–Mills theory [1], obtained here from the dimension count of Theorem 3.3 rather than from an explicit Fierz rearrangement. Several times do not help as long as the gaugino is unconstrained, because the proof uses only D . This is why every construction beyond ten dimensions constrains the gaugino.

5 Null-projected super Yang–Mills theories

5.1 Null frames as fermionic oscillators

Definition 5.1. A *null frame of rank k* is a set of vectors $n_1, \dots, n_k, m_1, \dots, m_k \in V_{\mathbb{C}}$ with

$$n_i \cdot n_j = 0, \quad m_i \cdot m_j = 0, \quad n_i \cdot m_j = \delta_{ij}. \quad (5.1)$$

It is *real* if all the vectors lie in V . Its *residual space* is the orthogonal complement V' of $\text{span}(n_i, m_i)$, of dimension $D' = D - 2k$, and we set $v = n_1 \wedge \dots \wedge n_k$.

The span of the n_i and m_i is a sum of k hyperbolic planes, so for a real frame V' is real of signature $(s - k, t - k)$. By Lemma 2.2, a real null frame of rank k exists exactly when $k \leq \min(s, t)$. In Nishino's notation $n_i = n_{(i)}$ and $m_i = m_{(i)}$; his signatures $(9 + m) + (1 + m)$ carry real frames of rank m with residual space $\mathbb{R}^{9,1}$.

The relations (5.1) say that the operators

$$b_i = \frac{1}{\sqrt{2}} \not{n}_i, \quad b_i^\dagger = \frac{1}{\sqrt{2}} \not{m}_i \quad (5.2)$$

obey the canonical anticommutation relations $\{b_i, b_j^\dagger\} = \delta_{ij}$, $\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0$. (The dagger is notation; no adjoint is implied.) The number operators

$$P_i^\uparrow = b_i^\dagger b_i = \frac{1}{2} \not{m}_i \not{n}_i, \quad P_i^\downarrow = b_i b_i^\dagger = \frac{1}{2} \not{n}_i \not{m}_i = 1 - P_i^\uparrow \quad (5.3)$$

are Nishino's projectors [22], and they commute with each other. For $a \in \{0, 1\}^k$ let Δ_a be the joint eigenspace on which $P_i^\uparrow = a_i$; mode i is occupied in Δ_a if $a_i = 1$. Write $\mathbf{1} = (1, \dots, 1)$, $\mathbf{0} = (0, \dots, 0)$, $\bar{a} = \mathbf{1} - a$, and e_i for the i -th unit vector of $\{0, 1\}^k$. For vectors of V' we use primed indices, $\gamma^{\mu'}$.

Lemma 5.2. (i) $\Delta = \bigoplus_a \Delta_a$. Each $\gamma^{\mu'}$ preserves every Δ_a , and each Δ_a is a Dirac module of $\text{Cl}(V'_{\mathbb{C}})$, of dimension 2^{r-k} .

(ii) $\not{n}_i$ maps Δ_a with $a_i = 0$ isomorphically onto Δ_{a+e_i} and annihilates Δ_a with $a_i = 1$; $\not{m}_i$ does the reverse. Hence

$$K := \bigcap_i \ker \not{n}_i = \Delta_{\mathbf{1}}, \quad \widehat{K} := \bigcap_i \ker \not{m}_i = \Delta_{\mathbf{0}}. \quad (5.4)$$

(iii) $T = \not{n}_1 \not{n}_2 \dots \not{n}_k$ maps $\widehat{K}$ isomorphically onto K , annihilates Δ_a for $a \neq \mathbf{0}$, and satisfies $T \gamma^{\mu'} = (-1)^k \gamma^{\mu'} T$; in particular T commutes with $\text{Spin}(V')$.

(iv) $B(\Delta_a, \Delta_b) = 0$ unless $b = \bar{a}$.

(v) On each Δ_a the chirality Γ_* equals $\pm(-1)^{|a|} \Gamma'_*$, where Γ'_* is the chirality of V' , $|a|$ is the number of occupied modes, and the overall sign does not depend on a . Thus $\Delta_a \cap \Delta_+$ and $\Delta_a \cap \Delta_-$ are the two Weyl modules of V' .

The proof is in Appendix A. Figure 5 shows the case $k = 3$. The kernel K of all the $\not{n}_i$ is the fully occupied corner, the kernel $\widehat{K}$ of all the $\not{m}_i$ is the empty corner, T climbs from one to the other, and B pairs opposite corners.

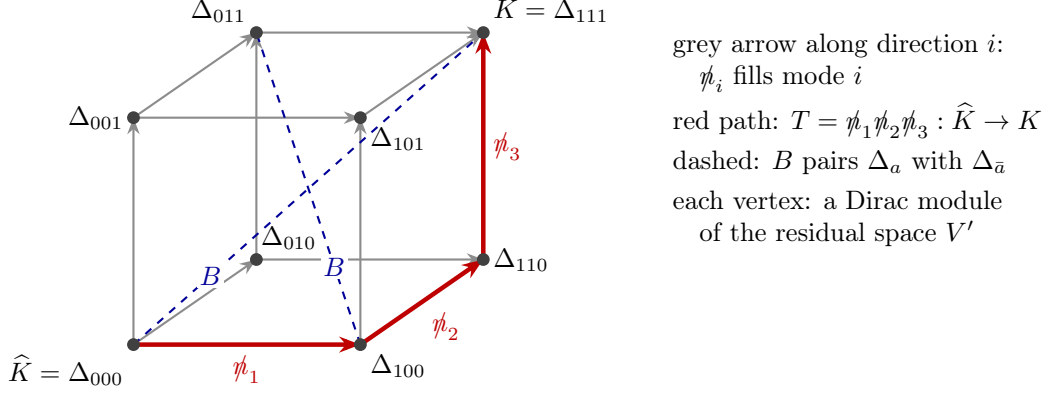


Figure 5: The decomposition of the Dirac module for a null frame of rank $k = 3$. The vertices are the joint eigenspaces Δ_a of the number operators $P_i^\dagger$, each a Dirac module of the residual space V' . The gaugino lives in the fully occupied corner K ; the supersymmetry parameters that act on the residual components of the gauge field live in the opposite corner $\widehat{K}$.

5.2 The axioms

Definition 5.3. A null-projected super Yang–Mills theory of rank k in signature (s, t) , $D = 2r$, consists of a null frame of rank k , a compact non-abelian gauge algebra $\mathfrak{g}$, a gauge field A_μ^I , a Grassmann-odd gaugino λ^I and Grassmann-odd supersymmetry parameters ϵ , subject to the following conditions.

- (N1) *Constraint.* λ^I takes values in $K_\chi = K \cap \Delta_\chi$ for a fixed chirality χ ; that is, $\not\eta_i \lambda^I = 0$ for all i . The parameter ϵ ranges over a Weyl module $\Delta_{\chi'}$.⁶
- (N2) *Transformations.* $\delta A_\mu^I = a \bar{\epsilon} \gamma_\mu \lambda^I$ with $a \neq 0$, and $\delta \lambda^I = \Phi(F^I) \epsilon$ for a linear map Φ from two-forms to $\text{End } \Delta$ that does not depend on λ . The covariant derivative is $D_\mu \lambda^I = \partial_\mu \lambda^I + g f^{IJK} A_\mu^J \lambda^K$ with $g \neq 0$.
- (N3) *Residual gauge field.* The components δA_μ^I along V' do not vanish identically.
- (N4) *Closure.* The commutator of two supersymmetries contains the translation by

$$\xi^\mu = c \bar{\epsilon}_2 \gamma^{\mu\nu_1 \dots \nu_k} \epsilon_1 v_{\nu_1 \dots \nu_k}, \quad c \neq 0, \quad (5.5)$$

and its components $\xi^{\mu'}$ along V' do not vanish identically.

- (N5) *Absorption.* For every λ with values in $K_\chi \otimes \mathfrak{g}$ there are J_σ^I , depending on λ but not on ϵ , such that for every ϵ the part of $\delta(\gamma^\mu D_\mu \lambda^I)$ quadratic in λ equals

$$\gamma^{\sigma\nu_1 \dots \nu_k} \epsilon v_{\nu_1 \dots \nu_k} J_\sigma^I. \quad (5.6)$$

Axiom (N5) is what makes the theory consistent: a term of the form (5.6) is cancelled by a source term in the field equation of the gauge field, which becomes $D^\mu F_{\mu[\sigma} v_{\nu_1 \dots \nu_k]} \propto J_{[\sigma} v_{\nu_1 \dots \nu_k]}$. For $k = 0$ the axiom says that the cubic terms have the form $\gamma^\sigma \epsilon J_\sigma$.

Example 5.4. Nishino's theories [22] in $(9 + m) + (1 + m)$ satisfy (N1)–(N5) with $k = m$. His constraint (2.12), the contraction of v with $\gamma^\mu \lambda$, is equivalent to $\not\eta_{(i)} \lambda = 0$ for every i ,

⁶As in Nishino's construction, ϵ is not constrained. The proof uses only the component of ϵ in $\widehat{K}$, which by Lemma 5.2(iv) is the only component that enters the residual components of δA . Any family of parameters whose $\widehat{K}$ -components form a nonzero $\text{Spin}(V')$ -invariant subspace leads to the same conclusions.

and he rewrites it as $\lambda = P^\dagger \lambda$ in eq. (2.22). The transformations (N2) are his eq. (2.3), with $\Phi(F)\epsilon = \frac{1}{4}\gamma^{\mu\nu\rho_1\cdots\rho_m}\epsilon F_{\mu\nu}v_{\rho_1\cdots\rho_m}$, and the closure (N4) is his eqs. (2.16)–(2.18). Axiom (N5) is his eq. (2.29): the surviving cubic term is $\gamma^{[m+1]}\epsilon$ contracted with the bilinear $\bar{\lambda}^K\gamma_{[m+1]}\lambda^J$, and for gauginos in K this $(m+1)$ -form is the wedge product of v with a vector, so the term has the form (5.6); it is absorbed by the source in his field equation (2.15).⁷ The residual space is $\mathbb{R}^{9,1}$, so $D' = 10$.

The variation of the gaugino equation follows directly from the axioms.

Lemma 5.5. *Under (N2), the part of $\delta(\gamma^\mu D_\mu \lambda^I)$ quadratic in λ is $ag f^{IJK} \gamma^\mu \lambda^K (\bar{\epsilon} \gamma_\mu \lambda^J)$.*

Proof. $\delta(D_\mu \lambda^I) = D_\mu(\delta \lambda^I) + g f^{IJK} \delta A_\mu^J \lambda^K$. The first term is independent of λ because Φ is, and the second is $ag f^{IJK} (\bar{\epsilon} \gamma_\mu \lambda^J) \lambda^K$. $\square$

5.3 Reduction to the residual space

Theorem 5.6. *In a null-projected super Yang–Mills theory of rank k the residual dimension $D' = D - 2k$ equals 10 or 2.*

Proof. D' is even, and $D' \geq 1$ because (N3) requires residual directions. Assume $D' \geq 3$; we prove $D' = 10$. The proof has three steps.

Step 1: projection onto the occupied corner. Since $n_i \cdot m_j = \delta_{ij}$, the inverse metric decomposes as $\eta^{-1} = \eta'^{-1} + \sum_i (n_i \otimes m_i + m_i \otimes n_i)$, so

$$\gamma^\mu \lambda (\bar{\epsilon} \gamma_\mu \lambda') = \gamma^{\mu'} \lambda (\bar{\epsilon} \gamma_{\mu'} \lambda') + \sum_i [\not{n}_i \lambda (\bar{\epsilon} \not{n}_i \lambda') + \not{m}_i \lambda (\bar{\epsilon} \not{m}_i \lambda')]. \quad (5.7)$$

For $\lambda, \lambda' \in K$ the bracket vanishes, because $\not{n}_i$ annihilates K . By Lemma 5.5 the left-hand side of (N5) is therefore

$$ag f^{IJK} \gamma^{\mu'} \lambda^K (\bar{\epsilon} \gamma_{\mu'} \lambda^J), \quad (5.8)$$

which lies in the occupied corner Δ_1 because $\gamma^{\mu'}$ preserves every Δ_a . On the right-hand side of (5.6), write $\gamma(\omega)$ for the antisymmetrized product of gamma matrices attached to a $(k+1)$ -vector ω , normalized so that $\gamma(u_0 \wedge \cdots \wedge u_k) = \not{u}_0 \cdots \not{u}_k$ for mutually orthogonal u_i . Up to a nonzero factor, which we absorb into J , the right-hand side is $\gamma(J \wedge v)\epsilon$. Decompose $J = J' + \sum_i [(J \cdot n_i) m_i + (J \cdot m_i) n_i]$ with $J' \in V'_\mathbb{C}$. Since $n_i \wedge v = 0$, the vector $u' \in V'_\mathbb{C}$ is orthogonal to all the n_l , and m_i is orthogonal to all n_l with $l \neq i$,

$$\gamma(u' \wedge v) = \not{u}' T, \quad \gamma(n_i \wedge v) = 0, \quad \gamma(m_i \wedge v) = \pm \left(\prod_{l \neq i} \not{n}_l \right) (P_i^\downarrow - P_i^\uparrow), \quad (5.9)$$

where the last formula uses $\frac{1}{2}(\not{n}_i \not{n}_i - \not{n}_i \not{n}_i) = P_i^\downarrow - P_i^\uparrow$. Now take $\epsilon \in \widehat{K}_{\chi'} = \widehat{K} \cap \Delta_{\chi'}$. Then $\gamma^{\mu'} T \epsilon \in \Delta_1$, while $(\prod_{l \neq i} \not{n}_l)(P_i^\downarrow - P_i^\uparrow)\epsilon$ lies in Δ_{1-e_i} , in which mode i is empty. Applying the projection $\prod_i P_i^\uparrow$ onto Δ_1 to both sides of (N5), we obtain for all $\epsilon \in \widehat{K}_{\chi'}$

$$ag f^{IJK} \gamma^{\mu'} \lambda^K (\bar{\epsilon} \gamma_{\mu'} \lambda^J) = \gamma^{\mu'} T \epsilon J_{\mu'}^I. \quad (5.10)$$

Step 2: the residual vector structure. By Lemma 5.2(iii), $x = T\epsilon$ ranges over $K_{\chi''}$ with $\chi'' = (-1)^k \chi'$ as ϵ ranges over $\widehat{K}_{\chi'}$. Define

$$\beta(x, y) = B(T^{-1}x, y) \quad (x \in K_{\chi''}, y \in \Delta_1). \quad (5.11)$$

⁷Nishino writes $v_{[4n]}$ for the class $m = 4n$ and treats $m = 4n+1, 4n+2, 4n+3$ in the same way. We write k for the number of null vectors in all cases.

By Lemma 5.2(iv) and the nondegeneracy of B , β is a nondegenerate pairing of $K_{\chi''}$ with the Weyl half of Δ_1 that B pairs with $\widehat{K}_{\chi'}$, and it is $\text{Spin}(V')$ -invariant by Lemma 5.2(iii). In this notation

$$\delta A_{\mu'}^I = a \beta(T\epsilon, \gamma_{\mu'} \lambda^I), \quad \xi^{\mu'} = c \beta(T\epsilon_2, \gamma^{\mu'} T\epsilon_1) \quad (\epsilon, \epsilon_1, \epsilon_2 \in \widehat{K}_{\chi'}), \quad (5.12)$$

the first because only the $\widehat{K}$ -component of ϵ pairs with $\gamma_{\mu'} \lambda \in \Delta_1$, the second by (5.9), with the normalization of v absorbed into c . Thus $G^{\mu'}(x, y) = \beta(x, \gamma^{\mu'} y)$ is a nonzero equivariant pairing $K_{\chi''} \times K_{\chi''} \rightarrow V'_{\mathbb{C}}$ by (N4), and a nonzero equivariant pairing $K_{\chi''} \times K_{\chi} \rightarrow V'_{\mathbb{C}}$ by (N3). By Lemma 5.2(v), $K_{\chi''}$ and K_{χ} are Weyl modules of V' . With $D' = 2r'$, Proposition 2.1 and its analogue for opposite chiralities, $\Delta'_+ \otimes \Delta'_- \cong \bigoplus_j \Lambda^{r'-1-2j}$, show that the tensor square of a Weyl module contains Λ^1 only for r' odd, and the product of the two Weyl modules only for r' even. Hence r' is odd and $K_{\chi''}$ and K_{χ} have the same V' -chirality. Both lie in Δ_1 , so by Lemma 5.2(v) they coincide: $K_{\chi''} = K_{\chi} =: \Sigma$.

Next, $G^{\mu'}$ is symmetric on Σ . The commutator of two supersymmetries is antisymmetric under $\epsilon_1 \leftrightarrow \epsilon_2$, and so is the parameter ξ of the translation in it. For Grassmann-odd $x_i = T\epsilon_i$ one has $x_2^\alpha G_{\alpha\beta} x_1^\beta = -x_1^\beta G_{\alpha\beta} x_2^\alpha$, so in (5.12) the symmetric part of G contributes a term that is odd under the exchange and the antisymmetric part a term that is even. The even term must vanish, and since x_1, x_2 are independent Grassmann variables the antisymmetric part of G is zero.⁸ By (N4), $G \neq 0$. So G is a nonzero symmetric equivariant map $\text{Sym}^2 \Sigma \rightarrow V'_{\mathbb{C}}$, and Proposition 2.1 gives $r' \equiv 1 \pmod{4}$. Identify Σ^* with $\gamma^{\mu'} \Sigma \subset \Delta_1$ through β , and let $\gamma_{\alpha\beta}^{\mu'} = G^{\mu'}(e_\alpha, e_\beta)$ and $\tilde{\gamma}^{\mu'}$ be the Clifford action back to Σ . The relations (3.1) are then those of $\text{Cl}(V')$, so $(\Sigma, \gamma, \tilde{\gamma})$ is a vector structure on $W = V'_{\mathbb{C}}$ with $d = D'$ and $N = \dim \Sigma = 2^{r'-1}$.

Step 3: the identity. Write $x = T\epsilon$ and $S^{I\beta\delta} = f^{IJK} \lambda^{J\beta} \lambda^{K\delta}$. Exchanging β and δ and relabelling $J \leftrightarrow K$ gives one sign from f and one from the anticommuting λ , so S^I is symmetric. In components, (5.10) reads

$$\kappa \gamma_{\alpha\delta}^{\mu'} \gamma_{\mu' \gamma\beta} x^\gamma S^{I\beta\delta} = \gamma_{\alpha\gamma}^{\mu'} x^\gamma J_{\mu'}^I, \quad (5.13)$$

with $\kappa = \pm ag \neq 0$, the sign fixed by the ordering convention. Since (5.13) holds for all x , it is an identity between the matrices that multiply x^γ . Contracting with $\tilde{\gamma}^{\nu\gamma\alpha}$ and using Lemma 3.2(c),

$$J^{I\nu} = \frac{\kappa}{N} \tilde{\gamma}^{\nu\gamma\alpha} \gamma_{\alpha\delta}^{\mu'} \gamma_{\mu' \gamma\beta} S^{I\beta\delta}. \quad (5.14)$$

So J^I is a linear, $\text{Spin}(V')$ -equivariant function of $S^I \in \text{Sym}^2 \Sigma$ with values in $V'_{\mathbb{C}}$. Because $r' \equiv 1 \pmod{4}$, Λ^1 occurs exactly once in $\text{Sym}^2 \Sigma$, and Schur's lemma gives $J^{I\nu} = \kappa j \gamma_{\beta\delta}^{\nu} S^{I\beta\delta}$ for a constant j . Substituting into (5.13),

$$(\gamma_{\alpha\delta}^{\mu'} \gamma_{\mu' \gamma\beta} - j \gamma_{\alpha\gamma}^{\mu'} \gamma_{\mu' \beta\delta}) S^{I\beta\delta} = 0. \quad (5.15)$$

The tensors S^I span $\text{Sym}^2 \Sigma$: for λ with values in an $\mathfrak{su}(2) \subset \mathfrak{g}$, $S^3 = \lambda^1 \otimes \lambda^2 + \lambda^2 \otimes \lambda^1$ up to normalization, and the monomials $\lambda^{1\beta} \lambda^{2\delta}$ are independent.⁹ Hence the part of the bracket in (5.15) symmetric in β, δ vanishes:

$$\frac{1}{2} (\gamma_{\alpha\delta}^{\mu'} \gamma_{\mu' \gamma\beta} + \gamma_{\alpha\beta}^{\mu'} \gamma_{\mu' \gamma\delta}) = j \gamma_{\alpha\gamma}^{\mu'} \gamma_{\mu' \beta\delta}.$$

Exchanging the names of γ and δ and using the symmetry of $\gamma_{\mu'}$ gives exactly (3.2). Theorem 3.3 now gives $2^{r'-1} = N = 2(D' - 2) = 4(r' - 1)$, whose only solution is $r' = 5$, that is $D' = 10$. $\square$

⁸This is the usual reason why the anticommutator of supercharges is symmetric: with Grassmann parameters, $[\delta(\epsilon_1), \delta(\epsilon_2)] = \epsilon_2^\alpha \epsilon_1^\beta \{Q_\alpha, Q_\beta\}$ up to convention-dependent factors.

⁹For Majorana gauginos the components $\lambda^{J\beta}$ are real Grassmann generators in a real basis of Σ . The identity (5.15) is linear in S^I , so it extends to all complex symmetric tensors.

Step 1 is where the transplanted constructions fail. In Nishino's signatures the occupied corner K is a sixteen-dimensional Weyl module of the residual $\mathbb{R}^{9,1}$, and (5.10) holds by the ten-dimensional identity (1.1): contracting (1.1) with a symmetric $S^{\beta\delta}$ gives $\gamma_{\alpha\delta}^{\mu'}\gamma_{\mu'}\gamma_{\beta}S^{\beta\delta} = -\frac{1}{2}\gamma_{\alpha\gamma}^{\mu'}\gamma_{\mu'}\gamma_{\beta\delta}S^{\beta\delta}$, so $J_{\mu'}$ is a multiple of the residual fermion current, the source that Nishino places in the field equation (2.15) of the gauge field. In $(9+8n+q)+(1+q)$ with $n \geq 1$ the frame has at most $1+q$ null vectors, so the residual space has dimension at least $s-t = 8+8n \geq 16$, and its Weyl spinors are too large for any identity (3.2).

Remark 5.7. The case $D' = 2$ escapes the argument because Lemma 3.2(c) needs $d \geq 3$. There Σ is one-dimensional, $\gamma^{\mu'}$ is a null vector, and (3.2) holds trivially because both sides vanish. A theory with $D' = 2$ has a residual gauge field in two dimensions, without propagating degrees of freedom. By Theorem 5.8(iii) below it can occur only for $s = t$.

5.4 The Main Theorem

Theorem 5.8 (Main Theorem). *Let a null-projected super Yang–Mills theory of rank k in signature (s, t) have a Majorana–Weyl gaugino, so that $s - t \equiv 0 \pmod{8}$. Then*

- (i) $k \leq \min(s, t)$, and $n_1, \dots, n_k$ lie in the complexification of a real totally isotropic subspace of V ;
- (ii) $s + t - 2k \in \{2, 10\}$;
- (iii) $s - t \in \{8, 0, -8\}$, and $s = t$ if $s + t - 2k = 2$.

Proof. (i) The values of a Majorana–Weyl gaugino are real spinors in K_χ , and by (N3) the gaugino does not vanish identically, so K_χ contains a Majorana spinor $\psi \neq 0$. By Lemma 2.3, $U_\psi \supset \text{span}(n_1, \dots, n_k)$ is the complexification of a real totally isotropic subspace $U_\mathbb{R}$, so $k \leq \dim U_\mathbb{R} \leq \min(s, t)$. (ii) is Theorem 5.6. (iii) From (i) and (ii), $|s - t| = s + t - 2\min(s, t) \leq s + t - 2k \leq 10$, and $s - t \equiv 0 \pmod{8}$ leaves $s - t \in \{8, 0, -8\}$; if $s + t - 2k = 2$, then $|s - t| \leq 2$ forces $s = t$. $\square$

Corollary 5.9. *There is no null-projected super Yang–Mills theory with a Majorana–Weyl gaugino in any of the signatures (1.2) with $n \geq 1$. In particular there is none in $17+1$, $19+3$, $25+1$ or $27+3$.*

Proof. In the families (1.2), $s - t = 8 + 8n \geq 16$ for $n \geq 1$. $\square$

Theorem 5.8 leaves exactly the signatures in which Nishino's mechanism has room to work. For $s - t = 8$ these are Nishino's own signatures with $k = t - 1$, and the reflected signatures $(1+m) + (9+m)$ give $s - t = -8$. For $s - t = 0$ the signatures $(5+k) + (5+k)$ have residual space $\mathbb{R}^{5,5}$, whose Majorana–Weyl spinors obey (1.1), and the absorption equation (5.10) has a solution there as well (Appendix B); we do not write out the full theory in these signatures. The second degenerate family of Ref. [23], $9 + (1+8n)$ and its companions, has $s - t = 8 - 8n$. Theorem 5.8 excludes it for $n \geq 3$ and does not exclude it for $n = 1, 2$.

Remark 5.10. The reality of the gaugino is essential for (i). With a complex Weyl gaugino a complex frame of rank $k > \min(s, t)$ is allowed, and Theorem 5.6 still gives $D' = 10$. For instance $(17, 1)$ admits a complex frame of rank four with residual dimension ten, and the absorption equation (5.10) can then be solved, as the numerical check of Appendix B confirms. But the occupied corner of such a frame contains no Majorana spinor (Lemma 2.3), so the theory has no real form: it is ten-dimensional super Yang–Mills theory on a complex subspace. The superalgebras of Ref. [23] are built on Majorana–Weyl generators.

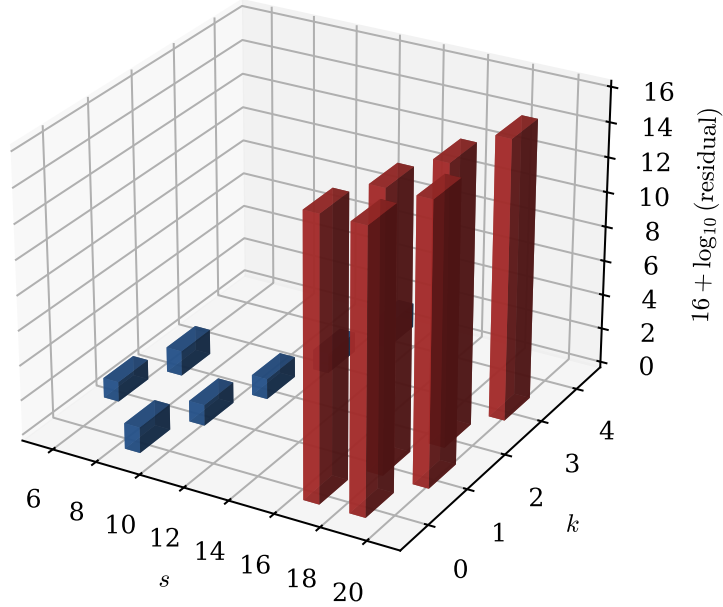


Figure 6: Relative residual of the least-squares solution of the absorption equation (5.10) for real frames, plotted as $16 + \log_{10}(\text{residual})$ against s and k . Blue: $(9, 1)$, $(10, 2)$, $(11, 3)$, $(12, 4)$, $(13, 5)$ with $k = t - 1$, and $(6, 6)$, $(7, 7)$ with $k = 1, 2$, all with $D' = 10$; the residuals are below 3×10^{-15} . Red: $(17, 1)$, $(18, 2)$, $(19, 3)$, $(20, 4)$ with every tested k , $D' \in \{16, 18, 20, 22\}$; the residuals lie between 0.97 and 1, so the equation has no solution.

Remark 5.11. For odd D there are no Weyl spinors, so there are no Majorana–Weyl gauginos and the Main Theorem has no odd-dimensional counterpart; the families (1.2) all have even D . Theorem 3.3 itself does not assume that d is even.

Figure 6 summarizes the numerical test of Step 1 in explicit gamma matrices, described in Appendix B: the absorption equation (5.10) has a solution, to machine precision, exactly for the signatures and ranks with $D' = 10$.

6 A single time: counting states

For the first family in (1.2) the time is single, and representation theory gives a bound that does not depend on how the theory is constructed. Consider a unitary, Poincaré-invariant quantum theory in $\mathbb{R}^{9+8n,1}$ with $n \geq 1$, whose supercharges Q_α form one Majorana–Weyl spinor, so that there are $S = 2^{4+4n}$ real supercharges. By Proposition 2.1 the most general anticommutator is

$$\{Q_\alpha, Q_\beta\} = (\gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu + \sum_{p \in \{5, 9, \dots, 5+4n\}} (\gamma^{\mu_1 \dots \mu_p} \mathcal{C})_{\alpha\beta} Z_{\mu_1 \dots \mu_p}, \quad (6.1)$$

with $\mathcal{C}$ the charge conjugation matrix and $Z_{[p]}$ tensor charges, in accordance with the general structure of supersymmetry algebras [37, 33]. Tensor charges of this kind are carried by extended objects [38]; in ten dimensions the five-form charge is carried by five-branes. We state their absence on the massless multiplet as hypothesis (H).

Theorem 6.1. *Assume (H) that the tensor charges vanish on a massless supermultiplet. Then the multiplet has at least $2^{S/4} = 2^{2+4n}$ states for each momentum, half of them bosonic. In*

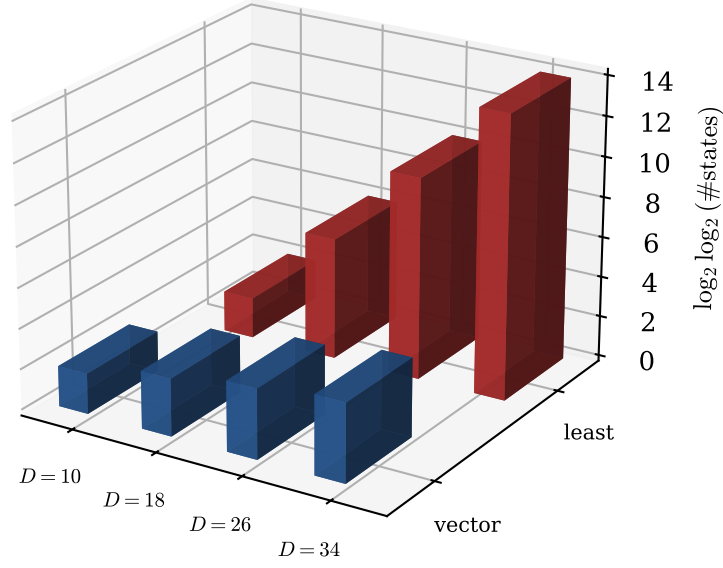


Figure 7: Massless multiplets in $(9+8n)+1$, plotted as $\log_2 \log_2$ of the number of states. Blue: a vector field and a Majorana–Weyl gaugino, $(8+8n)+2^{3+4n}$ states. Red: the least number $2^{2^{2+4n}}$ of states of a massless multiplet without tensor charges (Theorem 6.1). The two agree only for $D=10$.

particular the theory has no massless vector multiplet: such a multiplet would have $D-2=8+8n$ bosonic states for each generator of the gauge algebra.

Proof. On the multiplet, with momentum p , $p^2=0$, $p^0>0$, the anticommutator is $M_{\alpha\beta}=(\not{p}C)_{\alpha\beta}$, a real symmetric matrix, positive semidefinite because the Q_α are Hermitian and the norm is positive. Its rank is $S/2$: choosing a real null frame of rank one with $n_1=p$, Lemma 5.2 shows that $\not{p}$ annihilates exactly the half K of each Weyl module. A supercharge Q in the kernel of M has $\{Q, Q\}=0$, so $\|Q\psi\|^2=0$ and Q vanishes on every state ψ of the multiplet. The remaining $S/2=2^{3+4n}$ combinations, rescaled, satisfy $\{\hat{Q}_a, \hat{Q}_b\}=2\delta_{ab}$ and generate a complex Clifford algebra on 2^{3+4n} generators, whose irreducible representations have dimension $2^{2^{2+4n}}$ and are split evenly by $(-1)^F$. The multiplet is a direct sum of such representations. $\square$

For $n=0$ the bound is $2^4=16$: the eight bosonic and eight fermionic states of ten-dimensional super Yang–Mills theory. For $n=1$, $D=18$, it is 2^{64} , against sixteen bosonic and 128 fermionic states of a vector and a Majorana–Weyl gaugino. Figure 7 shows the gap for $n\leq 3$. The argument is the one that bounds the helicity of massless supermultiplets in four dimensions by one for sixteen supercharges and by two for thirty-two [39, 40]. Here the number of supercharges is so large that no field theory with finitely many vector, spinor and scalar fields can fill a single multiplet.

7 What this means for $27+3$ and $19+3$

In $27+3$ the Weyl spinor has $2^{14}=16384$ components, and its symmetric square contains forms of degrees 3, 7, 11 and 15 only. The anticommutator of two supercharges contains no momentum, and translations can be obtained from it only by contracting a form charge with null vectors, as in (N4); this is how all the constructions of Refs. [19, 20, 22] proceed. With three

times $k \leq 3$, so the residual space has dimension at least 24, and Corollary 5.9 applies: the cubic gaugino terms cannot be absorbed. The same holds in $19 + 3$, where the residual dimension is at least 16, and in every other signature of the families (1.2) with $n \geq 1$. Taking a gauge group built from the exceptional Jordan algebra, or arranging the fields in representations of $E_{8(-24)}$ or of the algebras of exceptional periodicity, does not change this, because the obstruction (5.10) already arises for gauginos with values in any $\mathfrak{su}(2)$ subalgebra of the gauge algebra.

The results leave the algebraic structures of Refs. [23, 27, 26] untouched. The superalgebras (3.5)–(3.8) of Ref. [23] exist by Proposition 2.1, and so do the cosets and Jordan-algebraic constructions built on them. What fails is the step from these superalgebras to field theories. The physical proposals that rest on an exceptional super Yang–Mills theory in $27 + 3$, namely a worldvolume theory for M-theory [24], the dark matter model of Ref. [25] and the higher M-theories of Ref. [28], need a different starting point. The theories that do exist in many dimensions with several times are, by Theorem 5.8, those with $s - t \in \{8, 0, -8\}$, and their cubic fermion structure is that of ten-dimensional super Yang–Mills theory on the residual space. This agrees with the view, expressed already in Ref. [21], that the null projections of super Yang–Mills theories beyond ten dimensions encode ten-dimensional theories with a covariant presentation of the extra directions.

Two remarks fix the scope of the results. The Main Theorem concerns theories with a single gaugino of definite chirality and the transformation rules (N2). Theories with several gauginos, with new auxiliary fields, or with transformation rules nonlinear in λ lie outside it, and no construction of that kind is known in these signatures. Theorem 6.1 assumes unitarity and hypothesis (H), which is why it covers only the single-time family: with several times there is no positive Hilbert space on which the argument could run, and Theorem 5.8 is the relevant statement.

8 Conclusions

Supersymmetric Yang–Mills theories built with null projections in signatures with several times are governed by an invariant that the Clifford algebra does not see. The bound $\min(s, t)$ limits the number of null directions that can be projected out of a real gaugino, and the residual theory must be ten-dimensional because its cubic fermion terms satisfy the ten-dimensional Fierz identity or none at all. The dimension count $N = 2(d - 2)$ of Theorem 3.3 makes the second statement rigid, and together the two leave only the signatures with $s - t \in \{8, 0, -8\}$. The families $(9 + 8n) + 1$, $(10 + 8n) + 2$, $(11 + 8n) + 3$ and $(12 + 8n) + 4$ with $n \geq 1$, including $27 + 3$ and $19 + 3$, are excluded. For $(9 + 8n) + 1$ the exclusion holds for every unitary theory without tensor charges on its massless states, however it is constructed. A supersymmetric gauge theory in these signatures would have to violate the axioms (N1)–(N5) in an essential way, and it would then no longer be a repetition of Nishino’s proof.

Data availability

All results of this paper are proved in the text, and no argument depends on a numerical computation. The Python programs that check the statements in explicit gamma matrices (Appendix B, 34 checks) and draw the plotted figures are archived on Zenodo together with the L^AT_EX source of the paper [41]. The schematics of Figs. 2 and 5 are drawn in the L^AT_EX source with TikZ.

Use of AI tools

AI tools (Claude Code, Anthropic) were used for the literature search, for developing and checking the proofs, for drafting and editing the text, and for writing the verification code and the scripts that draw the figures. The author directed the work and takes full responsibility for the content of the paper.

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A The oscillator decomposition

We prove Lemma 5.2. The $P_i^\uparrow$ are commuting projections, so the products $\Pi_a = \prod_i (a_i P_i^\uparrow + (1 - a_i) P_i^\downarrow)$ are orthogonal idempotents with $\sum_a \Pi_a = 1$, and $\Delta = \bigoplus_a \Delta_a$ with $\Delta_a = \Pi_a \Delta$. For $j \neq i$, η_i anticommutes with η_j and $\eta_j^\dagger$ and therefore commutes with $P_j^\uparrow$. From $\eta_i^2 = 0$ and $\eta_i \eta_i^\dagger + \eta_i^\dagger \eta_i = 2$ we get $\eta_i \eta_i^\dagger \eta_i = 2\eta_i$, that is $P_i^\uparrow \eta_i = \eta_i$ and $\eta_i P_i^\uparrow = 0$. Hence η_i annihilates Δ_a when $a_i = 1$ and maps Δ_a into Δ_{a+e_i} when $a_i = 0$; on the latter $\eta_i^\dagger \eta_i = 2P_i^\downarrow = 2$, so the map is injective. The same argument with $\eta_i^\dagger$ gives an injection $\Delta_{a+e_i} \rightarrow \Delta_a$, so both maps are isomorphisms. This proves (ii), and all the Δ_a have the same dimension $2^r/2^k$.

Each $\gamma^{\mu'}$ anticommutes with every b_i and $b_i^\dagger$, hence commutes with every $P_i^\uparrow$ and preserves every Δ_a . Therefore Δ_a is a module over $\text{Cl}(V_\mathbb{C}')$, which for even D' is a simple algebra of dimension 2^{2r-2k} with a unique irreducible module, of dimension 2^{r-k} . As $\dim \Delta_a = 2^{r-k}$, Δ_a is that module. This proves (i).

For (iii), T is the composition of the isomorphisms of (ii) along the path $\mathbf{0} \rightarrow e_1 \rightarrow e_1 + e_2 \rightarrow \dots \rightarrow \mathbf{1}$, it annihilates every Δ_a with $a \neq \mathbf{0}$ by (ii), and $\gamma^{\mu'}$ anticommutes with each of its factors. For (iv), by (2.1), $B(P_i^\uparrow \psi, \chi) = \frac{1}{2} \theta^2 B(\psi, \eta_i^\dagger \eta_i \chi) = B(\psi, P_i^\downarrow \chi)$. So if $\psi \in \Delta_a$, $\chi \in \Delta_b$ and $a_i = b_i = 1$, then $B(\psi, \chi) = B(P_i^\uparrow \psi, \chi) = B(\psi, P_i^\downarrow \chi) = 0$, and similarly if $a_i = b_i = 0$. For (v), choose an orthonormal basis of $V_\mathbb{C}$ consisting of a basis of $V_\mathbb{C}'$ and the vectors $e_i = (n_i + m_i)/\sqrt{2}$, $f_i = (n_i - m_i)/\sqrt{2}$, with $e_i^2 = 1$ and $f_i^2 = -1$. Then $\epsilon_i f_i = \frac{1}{2}(\eta_i^\dagger \eta_i - \eta_i \eta_i^\dagger) = P_i^\downarrow - P_i^\uparrow$, and the chirality operator, proportional to the product of all basis gamma matrices, is $\Gamma_* = \pm \Gamma'_* \prod_i (P_i^\downarrow - P_i^\uparrow)$, with the sign fixed by $\Gamma_*^2 = 1$. On Δ_a the product equals $(-1)^{|a|}$.

B Numerical checks in explicit gamma matrices

The program `verify.py` in the repository builds the gamma matrices of $\mathbb{R}^{s,t}$ as Kronecker products of Pauli matrices, multiplying the first t Euclidean generators by i , and checks the Clifford relations. It finds a charge conjugation matrix $\mathcal{C}$ with $\mathcal{C} \gamma^\mu \mathcal{C}^{-1} = \pm (\gamma^\mu)^T$ among the products of the real or of the imaginary Euclidean generators, and builds the vector forms of Definition 3.1 from it. It then runs the following checks, all of which pass (34 checks, no failures; the run takes about forty seconds).

- (a) Proposition 2.1: the dimension count for $r = 1, \dots, 16$, and the form degrees of the four families (1.2) for $n = 0, 1, 2$.

- (b) The identity (1.1) on random spinors: it holds in (9, 1), (5, 5) and (1, 9) (relative residual at most 1.3×10^{-16}) and fails in (17, 1), (13, 5), (9, 9) and (25, 1) (relative residual between 0.09 and 0.36).
- (c) The identity (3.2) with j fitted by least squares: $j = -0.5000$ in (9, 1) and (5, 5); no j fits in (17, 1) and (25, 1) (residual at least 0.99).
- (d) Lemma 3.2(b),(c) in (9, 1) and (17, 1), exactly.
- (e) The absorption equation (5.10), solved by least squares for J over random $\lambda \in K$ and $\epsilon \in \hat{K}$ with explicit real frames $\not{p}_i = (\gamma_{\text{space}} + \gamma_{\text{time}})/\sqrt{2}$; see Table 1 and Fig. 6.
- (f) Lemma 2.3 in (17, 1) with a complex frame of rank four: the complex conjugate of each n_i is m_i , and $\not{p}_i \not{p}_i + \not{p}_i \not{p}_i = 2$, so the occupied corner contains no Majorana spinor.
- (g) Lemma 2.2: real null frames of rank $\min(s, t)$ satisfying (5.1).
- (h) Theorem 6.1: the counts of Fig. 7 for $n = 0, 1, 2, 3$.

(s, t)	k	frame	D'	$\dim K_\chi$	residual	solvable
(9, 1)	0	–	10	16	3.0×10^{-15}	yes
(10, 2)	1	real	10	16	1.6×10^{-15}	yes
(11, 3)	2	real	10	16	1.5×10^{-15}	yes
(12, 4)	3	real	10	16	1.4×10^{-15}	yes
(13, 5)	4	real	10	16	2.6×10^{-15}	yes
(6, 6)	1	real	10	16	1.4×10^{-15}	yes
(7, 7)	2	real	10	16	2.8×10^{-15}	yes
(17, 1)	0	–	18	256	0.98	no
(18, 2)	1	real	18	256	0.99	no
(19, 3)	0	–	22	1024	1.0	no
(19, 3)	2	real	18	256	0.99	no
(20, 4)	1	real	22	1024	1.0	no
(20, 4)	3	real	18	256	0.98	no
(17, 1)	4	complex	10	16	1.7×10^{-15}	yes (no real form)
(19, 3)	2	complex	18	256	0.97	no

Table 1: The absorption equation (5.10) in explicit gamma matrices. The residual is the relative norm of the least-squares remainder over three random draws. The equation is solvable exactly when $D' = 10$, as Theorem 5.6 requires; the complex frame in (17, 1) is the case of Remark 5.10.

C Symmetric squares for the four families

Table 2 lists $\text{Sym}^2 \Delta_+$, by Proposition 2.1, for the families (1.2) with $n = 0, 1, 2$, together with the residual dimension $s - t$ of a real null frame of the largest rank $k = t$. The momentum Λ^1 occurs only in the first family. Since D' decreases as k grows, a residual dimension of ten is reachable only when $s - t \leq 10$, that is only for $n = 0$.

(s, t)	D	$\dim \Delta_+$	degrees in $\text{Sym}^2 \Delta_+$	$\dim \text{Sym}^2 \Delta_+$	D' at $k = t$
(9, 1)	10	16	5, 1	136	8
(10, 2)	12	32	6, 2	528	8
(11, 3)	14	64	7, 3	2080	8
(12, 4)	16	128	8, 4, 0	8256	8
(17, 1)	18	256	9, 5, 1	32896	16
(18, 2)	20	512	10, 6, 2	131328	16
(19, 3)	22	1024	11, 7, 3	524800	16
(20, 4)	24	2048	12, 8, 4, 0	2098176	16
(25, 1)	26	4096	13, 9, 5, 1	8390656	24
(26, 2)	28	8192	14, 10, 6, 2	33558528	24
(27, 3)	30	16384	15, 11, 7, 3	134225920	24
(28, 4)	32	32768	16, 12, 8, 4, 0	536887296	24

Table 2: Symmetric squares of the Weyl spinor for the families (1.2), $n = 0, 1, 2$. The last column is $s - t = s + t - 2 \min(s, t)$, the residual dimension of a real null frame of rank $k = t$. For $n = 0$ Nishino's frames have $k = t - 1$ and $D' = 10$; a frame with $k = t$ gives $D' = 8$, which Theorem 5.6 excludes. For $n \geq 1$ every admissible frame has $D' \geq 16$.

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