

Gluing defects for projected coarse graining in finite spin-foam models

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Abstract

Orthogonal truncation can discard states that contribute to a later boundary amplitude. For a self-adjoint contraction T on a finite-dimensional Hilbert space and an isometric inclusion s of retained states, we prove that the two-step residual $\Delta = s^*T^2s - (s^*Ts)^2$ controls every finite placement of intermediate projections in every Schatten norm. For blocks of lengths $n_1, \dots, n_k$, the error is at most $\sum_{i < j} n_i n_j \|\Delta\|$, and the constant is optimal even for positive contractions. The residual vanishes exactly when the retained subspace is invariant. Two-dimensional finite-group gauge sums realize an idempotent projected transfer operator with nonzero residual, as well as positive-semidefinite transfer operators with identical one-step retained data and different residuals. We derive their gluing and subdivision identities with explicit boundary measures and BF normalization. We also bound the accumulation of local errors under tensor contraction and the distance to compatible linear amplitude data. Colimits extend compatible amplitude families, and quotients by invariant subspaces preserve the comparison maps. A measurement map that is injective on the retained ansatz gives a unique normalized least-squares fit. Exact rational calculations reproduce the finite-group examples and the blocking comparisons.

Keywords: spin foams; finite gauge theory; coarse graining; cylindrical consistency; tensor gluing; truncation error.

1 Introduction

Coarse graining integrates microscopic variables to obtain effective amplitudes on a coarser discretization. In spin-foam models, both the boundary state space and the bulk discretization can change during this process. Cylindrical consistency compares amplitudes along specified boundary embeddings, while composition compares the amplitude of an assembled region with a contraction of its constituent amplitudes. These questions motivate background-independent renormalization and the consistent-boundary approach [1–4].

Truncation creates a concrete obstruction to combining these requirements. A state removed at one interface can propagate through a discarded sector and return to a retained state at a later interface. Matching the coefficients of a single retained block does not determine this return contribution. The relevant comparison is between exact blocking followed by projection and blocking with projections inserted at intermediate interfaces. We quantify this difference and realize it in exactly computable finite-group models.

Let T be a self-adjoint contraction on a finite-dimensional boundary Hilbert space and let $s : V \rightarrow \mathcal{H}$ be an isometric inclusion of retained states. Set

$$S_n = s^*T^n s, \quad P = ss^*, \quad \Delta = S_2 - S_1^2.$$

The two-step residual satisfies

$$\Delta = ((I - P)Ts)^*((I - P)Ts). \tag{1.1}$$

For every Schatten norm, Theorem 2.1 bounds the error of an arbitrary finite blocking schedule by $\|\Delta\|$ times an explicit quadratic function of its block lengths. In this class of operators,

vanishing of the two-step residual is equivalent to invariance of the retained subspace and to exact composition at every length. The positivity of this residual is a special case of the Schwarz inequality for unital completely positive maps [5].

An exact finite-group state sum realizes this residual. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, a strictly positive plaquette weight yields a transfer operator with eigenvalues $(1, a, -a, 0)$, where $0 < a < 1/2$. Retaining the vacuum and an equal superposition of two character modes gives

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}.$$

The one-step retained operator is fixed under $K \mapsto K^2$, although the exact two-step result differs by a^2 . A second example has a positive-semidefinite transfer operator and the same one-step retained data as a model with zero defect. Thus a fixed point of the projected update can have a nonzero blocking residual, and one-step retained data can fail to determine this residual even for positive-semidefinite transfer operators. Section 4 gives both constructions and proves that an idempotent compression of a positive contraction has zero residual.

The finite state sums are organized by configuration spaces, boundary projections, and measure-preserving maps that forget fine variables. Boundary gluing contracts kernels along a common interface. This description identifies the measures and state spaces needed to compare exact and truncated amplitudes. For general finite diagrams, we bound the propagation of local gluing errors and give the nearest solution of a compatible system of linear refinement equations. The bounds depend on intermediate kernel norms, interface dimensions, and comparison-map singular values. Kan extensions and quotients by invariant subspaces describe retained spaces on which the linear comparison maps compose exactly; a measurement map that is injective on the retained ansatz determines a unique normalized least-squares fit.

1.1 Relation to existing constructions

Finite-group lattice gauge models and their spin-foam transforms provide a setting in which sums, measures, boundary states, and truncations can all be evaluated exactly [6]. Finite gauge theory also has an intrinsic formulation using groupoids of bundles and automorphism weights, with precise gluing rules [7]. We specify the normalization relating that formulation to the two-dimensional normalized-Haar lattice convention used here. This distinction matters already for the partition function of a sphere.

Tensor-network coarse graining repeatedly contracts tensors and truncates the resulting boundary spaces. Decorated tensor-network renormalization retains geometric and gauge information during blocking [8]. Holonomy formulations and later spin-foam coarse-graining calculations likewise track effective amplitudes beyond their initial local form [9, 10]. The estimates below apply once the algorithm specifies its boundary spaces, embeddings, norms, and projections. Our finite-group examples use convolution and a fixed orthogonal projection, so both sides of each blocking comparison can be computed exactly.

Lu's causal-net category includes classes of coarse-graining, merging, and contraction morphisms that are closed under composition [11]. For state-sum comparisons we also specify probability measures, boundary maps, amplitude kernels, and norms. The categorical treatment of Hamiltonian renormalization by Rodriguez Zarate also emphasizes embeddings, projections, and coarse-graining structures [12]. The present construction supplies a finite state-sum realization and estimates for the failure of compressed composition.

Bruno, Colafranceschi, Mele, and Rovelli formulate spin-foam continuum limits with compatible boundary embeddings and gluing, and prove a topological-field-theory conclusion under their norm-convergence assumptions [13, Theorem 5.1]. Our results concern finite diagrams and finite blocking schedules. Passing to an infinite family requires additional uniform estimates and an appropriate topology. In particular, a compatible functional on an algebraic inductive limit of

isometric boundary spaces extends continuously to its Hilbert completion precisely when the norms of its restrictions are uniformly bounded. Necessity follows by restriction, and sufficiency follows from density and the common bound.

The EPRL and FK models provide four-dimensional spin-foam amplitudes to which finite truncation estimates may be applied [14–16]. Hypercuboidal and curvature-restricted models have been studied by numerical coarse graining [17, 18]. Section 7 describes the boundary measurements needed for such an application.

Section 2 proves the compression bound and its optimality. Section 3 defines the finite boundary comparisons and gluing operations, and Section 4 constructs the gauge-theory examples. Section 5 treats general finite comparison diagrams. Section 6 gives compatible extensions and retained fits. Section 7 describes the computations and further applications.

1.2 Notation and scope

All vector spaces are over $\mathbb{C}$. Finite configuration spaces carry the stated strictly positive reference probability measure; interaction weights may be complex. Amplitudes are linear functionals, and kernel composition uses a bilinear pairing with the chosen measure. Hilbert inner products are conjugate-linear in the first argument. Adjoints and Schatten norms refer to those Hilbert structures. The identity on a retained space is denoted by I ; context distinguishes a quotient map from an orthogonal projection.

A residual is the difference between prescribed maps or kernels with a common domain and codomain. A seminorm defined by selected observables measures this difference modulo its null space. The worked examples have fixed vacuum normalization. Residuals and kernel norms are evaluated with this same normalization.

2 Compression of transfer operators

Orthogonal truncation removes intermediate states that may contribute to a later retained amplitude. For a self-adjoint contraction, the two-step compression residual measures this contribution and bounds the error caused by projections at any finite set of intermediate steps.

Let $\mathcal{H}$ be a finite-dimensional boundary Hilbert space and let $s: V \rightarrow \mathcal{H}$ be an isometric inclusion of a retained sector. Write

$$P = ss^*, \quad Q = I - P, \quad S_n = s^* T^n s \quad (n \geq 0), \quad S_0 = I_V,$$

where $T: \mathcal{H} \rightarrow \mathcal{H}$ is a transfer operator. Thus S_n is obtained by performing n elementary steps in the full space and projecting at the two external boundaries. An ordered partition $\pi = (n_1, \dots, n_k)$ of n specifies the retained product

$$S_\pi = S_{n_1} \cdots S_{n_k}.$$

It inserts a projection between successive blocks. Different partitions therefore specify different locations for discarding intermediate states. Write $R_{m,n} = S_{m+n} - S_m S_n$ for the residual of a binary blocking step.

For an operator X between finite-dimensional Hilbert spaces, let $\|X\|_p = (\text{tr } |X|^p)^{1/p}$ for $1 \leq p < \infty$, and let $\|X\|_\infty = \|X\|_{\text{op}}$. In particular, $p = 2$ is the Hilbert–Schmidt norm.

Theorem 2.1 (Compression residuals and blocking schedules). *Suppose that $T = T^*$ and $\|T\|_{\text{op}} \leq 1$. Set*

$$B = QTs, \quad \Delta = S_2 - S_1^2, \quad \delta_p = \|\Delta\|_p.$$

We also write $\delta_{\text{op}} = \delta = \delta_\infty$ and $\delta_{\text{HS}} = \delta_2$. Then, for every $m, n \geq 1$,

$$\Delta = B^*B, \quad \delta_p = \|B\|_{2p}^2, \quad (2.1)$$

$$S_{m+n} - S_m S_n = (QT^m s)^* (QT^n s), \quad (2.2)$$

$$\|S_{m+n} - S_m S_n\|_p \leq mn \delta_p. \quad (2.3)$$

Here $2p = \infty$ when $p = \infty$. For every ordered partition $\pi = (n_1, \dots, n_k)$ of n ,

$$\|S_n - S_\pi\|_p \leq C(\pi) \delta_p, \quad C(\pi) = \sum_{i < j} n_i n_j = \frac{n^2 - \sum_i n_i^2}{2}. \quad (2.4)$$

Consequently, two partitions π, π' of the same n satisfy

$$\|S_\pi - S_{\pi'}\|_p \leq (C(\pi) + C(\pi')) \delta_p, \quad \|S_n - S_1^n\|_p \leq \binom{n}{2} \delta_p. \quad (2.5)$$

The following conditions are equivalent:

- (i) $\Delta = 0$;
- (ii) $QTs = 0$;
- (iii) $T(sV) \subseteq sV$;
- (iv) $S_n = S_1^n$ for every $n \geq 0$;
- (v) $S_{m+n} = S_m S_n$ for every $m, n \geq 0$.

Proof. Insert $I = P + Q$ between T^m and T^n . Since $P = ss^*$,

$$s^* T^{m+n} s - s^* T^m s s^* T^n s = s^* T^m Q T^n s.$$

Self-adjointness gives (2.2), and $m = n = 1$ gives $\Delta = B^*B$. The singular values of B^*B are the squares of the singular values of B , proving (2.1).

Let $b_j = QT^j s: V \rightarrow Q\mathcal{H}$ and $D = QTQ|_{Q\mathcal{H}}$. Both D and every S_j are contractions in operator norm. Decomposing $T^j s = sS_j + b_j$ gives

$$b_0 = 0, \quad b_{j+1} = BS_j + Db_j, \quad b_j = \sum_{\ell=0}^{j-1} D^{j-1-\ell} BS_\ell.$$

The ideal property and triangle inequality for Schatten norms imply

$$\|b_j\|_{2p} \leq j \|B\|_{2p}.$$

The Schatten Hölder inequality gives $\|X^*Y\|_p \leq \|X\|_{2p} \|Y\|_{2p}$. For $p = \infty$ this is submultiplicativity. For $p = 2$ it follows from Hilbert–Schmidt Cauchy–Schwarz applied to XX^* and YY^* ; the other values are the usual Schatten Hölder inequality. Applying it to $b_m^* b_n$ proves (2.3).

To prove (2.4), put $r = n_2 + \dots + n_k$ and write

$$S_n - S_{n_1} \cdots S_{n_k} = (S_n - S_{n_1} S_r) + S_{n_1} (S_r - S_{n_2} \cdots S_{n_k}).$$

The first term has norm at most $n_1 r \delta_p$; multiplication by S_{n_1} does not increase a Schatten norm. Induction on k yields $\sum_{i < j} n_i n_j \delta_p$. The comparison of two schedules follows by the triangle inequality through S_n .

Finally, $\Delta = 0$ is equivalent to $B = 0$, hence to invariance of sV . In that case $Ts = sS_1$ and induction gives $T^n s = sS_1^n$, hence $S_{m+n} = S_1^{m+n} = S_m S_n$. Conversely, $S_2 = S_1^2$ follows from either (iv) or (v), completing the equivalence. $\square$

The inequality $\Delta \geq 0$ is the Schwarz inequality for the unital completely positive compression map $X \mapsto s^* X s$ [5, Corollary 2.8]. The theorem uses the identity $\Delta = B^* B$ to control every finite placement of intermediate truncations by a single two-step residual. The constant $C(\pi)$ counts pairs of elementary steps lying in different retained blocks. It is independent of the boundary dimension; the measured residual δ_p can itself depend on that dimension. The norm choice is consequently part of the quantitative statement.

Corollary 2.2 (Sharpness of the schedule constant). *For each fixed ordered partition π , the constant $C(\pi)$ in (2.4) is optimal over the self-adjoint contractions and isometric inclusions in Theorem 2.1. It remains optimal when $0 \leq T \leq I$ is required.*

Proof. For $0 < \varepsilon < 1/2$, take

$$T_\varepsilon = \begin{pmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{pmatrix}, \quad s : \mathbb{C} \rightarrow \mathbb{C}^2, \quad sz = (z, 0).$$

The eigenvalues of T_ε are 1 and $1 - 2\varepsilon$, and

$$S_j = \frac{1 + (1 - 2\varepsilon)^j}{2}, \quad \Delta = \varepsilon^2.$$

For a fixed positive integer j , $S_j = 1 - j\varepsilon + j(j-1)\varepsilon^2 + O(\varepsilon^3)$. Multiplying these finite expansions for $\pi = (n_1, \dots, n_k)$, with $\sum_i n_i = n$, yields

$$S_n - \prod_i S_{n_i} = \left(n(n-1) - \sum_i n_i(n_i-1) - \sum_{i < j} n_i n_j \right) \varepsilon^2 + O(\varepsilon^3) = C(\pi) \varepsilon^2 + O(\varepsilon^3).$$

The remainder constant may depend on the fixed partition. All Schatten norms of a scalar equal its absolute value. Dividing by $\delta_p = \varepsilon^2$ and letting $\varepsilon \downarrow 0$ proves optimality. $\square$

Self-adjointness gives $\Delta = B^* B$ and the adjoint factorization in (2.2). For an arbitrary T , the algebraic identity

$$S_{m+n} - S_m S_n = s^* T^m Q T^n s \tag{2.6}$$

still holds, and

$$\|S_{m+n} - S_m S_n\|_{\text{op}} \leq \|s^* T^m Q\|_{\text{op}} \|Q T^n s\|_{\text{op}}.$$

For example, if $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $sz = (z, 0)$, then $S_j = 0$ for every $j \geq 1$, although $Ts \neq 0$ lies in the discarded subspace. Thus the equivalence between zero residual and invariance uses self-adjointness.

3 Finite boundary comparisons and tensor gluing

We work with finite state sums, including finite-group models and fixed finite-dimensional truncations. Refinement and composition act on the amplitude data in two ways. A comparison category describes changes of configuration variables on a fixed region. Tensor diagrams describe the composition of regions along boundary channels. When a comparison integrates the variables of an interface contraction, the two constructions agree.

3.1 Configuration presentations and boundary maps

A finite probability space is a nonempty finite set S with strictly positive weights $\nu_S(s)$ satisfying $\sum_s \nu_S(s) = 1$. A map $q : S' \rightarrow S$ is measure preserving if $q_* \nu_{S'} = \nu_S$. Such a map is surjective. Write FP for the category of finite probability spaces and measure-preserving maps, using finite subsets of a fixed universe so that the category is small.

Definition 3.1 (Finite configuration presentation). A configuration presentation is a tuple

$$D = (\Omega_D, B_D, p_D, \lambda_D, \nu_D), \quad (p_D)_* \lambda_D = \nu_D,$$

where (Ω_D, λ_D) and (B_D, ν_D) are finite probability spaces and $p_D : \Omega_D \rightarrow B_D$ records the boundary configuration. A comparison arrow $r : D \rightarrow D'$ is a pair of measure-preserving maps

$$q_r : \Omega_{D'} \longrightarrow \Omega_D, \quad q_r^\partial : B_{D'} \longrightarrow B_D$$

such that

$$p_D q_r = q_r^\partial p_{D'}. \quad (3.1)$$

The category of these presentations is denoted by $\mathcal{D}$. A chosen model uses a small subcategory closed under the comparisons under consideration.

Thus the arrow $D \rightarrow D'$ points from coarse variables to fine variables, while q_r forgets fine information. If $s : D' \rightarrow D''$, then

$$q_{sr} = q_r q_s, \quad q_{sr}^\partial = q_r^\partial q_s^\partial.$$

The identity consists of the two identity maps. Equation (3.1) is preserved by composition, so these rules define a category. Presentations may carry a region label, with comparisons restricted to a fixed label.

The boundary assignment is the functor

$$\partial : \mathcal{D} \longrightarrow \mathbf{FP}^{\text{op}}, \quad \partial D = (B_D, \nu_D), \quad \partial r = (q_r^\partial)^{\text{op}}. \quad (3.2)$$

An interface is a boundary object of $\mathbf{FP}$, together with the channel identifications required for a specified gluing. Its role in composition will be defined in Section 3.3.

For a finite probability space B , put

$$H(B) = \{\psi : B \rightarrow \mathbb{C}\}, \quad \langle \psi, \phi \rangle_B = \sum_{b \in B} \nu_B(b) \overline{\psi(b)} \phi(b).$$

Pullback along a measure-preserving map gives an isometric embedding. Consequently

$$H : \mathcal{D} \longrightarrow \mathbf{FinHilb}, \quad H(D) = H(B_D), \quad H(r)\psi = \psi \circ q_r^\partial, \quad (3.3)$$

is a covariant functor. Here $\mathbf{FinHilb}$ has all linear maps as morphisms; the maps selected by (3.3) are isometries.

3.2 Amplitudes and exact integration

A complex weight $w_D : \Omega_D \rightarrow \mathbb{C}$ determines the linear amplitude

$$A_D(\psi) = \sum_{\omega \in \Omega_D} \lambda_D(\omega) w_D(\omega) \psi(p_D \omega). \quad (3.4)$$

The weight may include local interaction factors, gauge fixing, and normalization factors. Its boundary density is

$$K_D(b) = \frac{1}{\nu_D(b)} \sum_{p_D \omega = b} \lambda_D(\omega) w_D(\omega), \quad A_D(\psi) = \sum_b \nu_D(b) K_D(b) \psi(b). \quad (3.5)$$

The last expression uses a complex-bilinear pairing. The Hilbert norm above supplies the associated norm on linear functionals.

Definition 3.2 (Refinement compatibility). A family $A = (A_D)$ is compatible with a comparison arrow $r : D \rightarrow D'$ if

$$A_{D'} H(r) = A_D. \quad (3.6)$$

Its unnormalized refinement residual is the functional

$$R_r(A) = A_{D'} H(r) - A_D \in H(D)^*.$$

Compatibility on a comparison subcategory means that A is a cocone from H to the constant functor with value $\mathbb{C}$.

For a measure-preserving map $q : S' \rightarrow S$, define weighted pushforward of densities by

$$(q!f)(s) = \frac{1}{\nu_S(s)} \sum_{q(s')=s} \nu_{S'}(s') f(s'). \quad (3.7)$$

This is conditional expectation onto the coarse variables. It satisfies $(qq')! = q!q'!$.

Proposition 3.3. *For a comparison arrow $r : D \rightarrow D'$, the following conditions are equivalent:*

$$A_{D'} H(r) = A_D, \quad (q_r^\partial)! K_{D'} = K_D.$$

In particular, choosing $w_D = (q_r)!w_{D'}$ gives refinement compatibility. Exact integration along a composite comparison agrees with successive integration along its factors.

Proof. For every $\psi \in H(B_D)$,

$$A_{D'} H(r)(\psi) = \sum_{b \in B_D} \nu_D(b) ((q_r^\partial)! K_{D'})(b) \psi(b).$$

Testing the coordinate functions proves the equivalence. If $w_D = (q_r)!w_{D'}$, then (3.1) gives

$$\sum_{\omega'} \lambda_{D'}(\omega') w_{D'}(\omega') \psi(p_D q_r \omega') = \sum_{\omega} \lambda_D(\omega) ((q_r)!w_{D'})(\omega) \psi(p_D \omega).$$

This is (3.6). The composite statement follows by regrouping the finite sums in (3.7). $\square$

When $H(r)$ is a proper embedding, refinement compatibility fixes the fine amplitude on $\text{im } H(r)$; its values on a complementary subspace remain free.

3.3 Tensor diagrams and boundary gluing

Fix a collection of channel types, each specified by a finite probability space. An object is a finite ordered list of channel types, with concatenation as tensor product and the empty list as tensor unit. A generating vertex has an ordered input list, an ordered output list, and a linear map between the corresponding tensor products of state spaces. Morphisms are finite directed acyclic string diagrams of these vertices, modulo graph isomorphisms preserving types, vertex labels, port orders, and external boundary orders. Identity wires and wire permutations are included. Composition joins matching outputs to inputs, and tensor product is ordered disjoint union. These operations define the free symmetric monoidal category on the chosen typed vertices.

Evaluation sends a channel X to $H(X)$, a vertex to its assigned linear map, and wire permutations to the corresponding permutations of tensor factors. A linear extension of the vertex order gives a composite linear map. Interchanging two independent vertices leaves this map unchanged by the interchange law. Thus evaluation is well defined and is a symmetric monoidal functor to $\mathbf{FinVect}_{\mathbb{C}}$. This is the usual string-diagram construction [19].

A kernel $K : Y \times X \rightarrow \mathbb{C}$ represents the map

$$(T_K \psi)(y) = \sum_{x \in X} \nu_X(x) K(y, x) \psi(x). \quad (3.8)$$

The identification $H(Y \times X) \cong H(Y) \otimes H(X)^*$ uses the bilinear pairing $\sum_x \nu_X(x) f(x) g(x)$. In these coordinates the identity kernel is

$$K_{\text{id},X}(x', x) = \frac{\mathbf{1}_{\{x'=x\}}}{\nu_X(x)}. \quad (3.9)$$

Definition 3.4 (Interface contraction). Let $K_1 : U \times X \rightarrow \mathbb{C}$ and $K_2 : Y \times U \rightarrow \mathbb{C}$. Their gluing along the boundary channel U is

$$\mathcal{C}_U(K_2, K_1)(y, x) = \sum_{u \in U} \nu_U(u) K_2(y, u) K_1(u, x). \quad (3.10)$$

For a prescribed kernel K_D of the assembled region, the gluing residual is

$$G_{D;2,1} = K_D - \mathcal{C}_U(K_2, K_1). \quad (3.11)$$

Orientation conventions, including inversion of finite-group holonomies on the second copy of an interface, are included in the chosen identification with the dual state space $H(U)^*$.

Proposition 3.5. *Interface contraction satisfies*

$$T_{\mathcal{C}_U(K_2, K_1)} = T_{K_2} T_{K_1}.$$

It is associative, has the kernels (3.9) as units, and is compatible with parallel tensor product. Let $\tilde{K}$ be the kernel of a linear map

$$\tilde{T} : H(X) \otimes H(U) \longrightarrow H(Y) \otimes H(U).$$

Closing its two U ports gives the partial trace, with kernel

$$(\text{Tr}^U \tilde{K})(y, x) = \sum_{u \in U} \nu_U(u) \tilde{K}(y, u; x, u). \quad (3.12)$$

Proof. The composition identity follows by substituting (3.8) twice. Associativity and compatibility with parallel composition follow from interchange of finite sums. Formula (3.9) gives the unit identities directly. The vectors $\mathbf{1}_{\{u\}}/\sqrt{\nu_U(u)}$ form an orthonormal basis; evaluating the ordinary finite-dimensional partial trace in this basis gives (3.12). $\square$

The trace in Proposition 3.5 has both an input and an output U factor. Its coherence properties are the standard traced-monoidal identities [20]. A closure can also be evaluated directly as a finite tensor-network sum.

Two factorizations with a common domain and codomain can be compared after closing a shared channel. Consider

$$\begin{aligned} a : H(X) \otimes H(U) &\longrightarrow H(A) \otimes H(U), & b : H(A) \otimes H(U) &\longrightarrow H(Y) \otimes H(U), \\ c : H(X) \otimes H(U) &\longrightarrow H(B) \otimes H(U), & d : H(B) \otimes H(U) &\longrightarrow H(Y) \otimes H(U). \end{aligned}$$

Their difference after partial trace is

$$R_\diamond = \text{Tr}^U(ba) - \text{Tr}^U(dc) : H(X) \longrightarrow H(Y). \quad (3.13)$$

Proposition 3.6 (A refinement–gluing square). *Let $q_X : X' \rightarrow X$, $q_U : U' \rightarrow U$, and $q_Y : Y' \rightarrow Y$ be measure-preserving maps. For kernels K_1 and K_2 as in Definition 3.4, put*

$$K'_1(u', x') = K_1(q_U u', q_X x'), \quad K'_2(y', u') = K_2(q_Y y', q_U u').$$

Then

$$\mathcal{C}_{U'}(K'_2, K'_1)(y', x') = \mathcal{C}_U(K_2, K_1)(q_Y y', q_X x'). \quad (3.14)$$

Writing $J_X = q_X^*$, and similarly for the other channels, the individual refined operators satisfy

$$T_{K'_1} = J_U T_{K_1} J_X^*, \quad T_{K'_2} = J_Y T_{K_2} J_U^*.$$

Proof. The adjoint J_X^* is the weighted averaging map $(q_X)_!$. Substitution in (3.8) gives the two operator identities. Since $J_U^* J_U = I$, their product is $J_Y T_{K_2} T_{K_1} J_X^*$, which is (3.14). Equivalently, the equality follows by pushing the sum over U' forward to U . $\square$

3.4 Realization by finite state sums

A finite tensor network gives a configuration presentation as follows. Assign an independent configuration variable to every vertex port and every external port. Take Ω_D to be their product, B_D to be the product over external ports, and p_D to be coordinate restriction, all with product reference measures. The weight w_D is the product of the vertex kernels and one identity kernel (3.9) for every wire joining two ports. Summing the internal port variables identifies each connected pair with the correct normalization. Formula (3.5) therefore evaluates the tensor network, including identity wires. This presentation keeps the boundary projection surjective even when a kernel is supported on a proper subset of boundary configurations.

For example, let G be a finite group with uniform measure. A chain with $n \geq 1$ links has

$$\Omega_n = G^{n+1}, \quad B_n = G^2, \quad p_n(g_0, \dots, g_n) = (g_0, g_n).$$

For link kernels $K_1, \dots, K_n$, choose

$$w_n(g_0, \dots, g_n) = \prod_{j=1}^n K_j(g_j, g_{j-1}).$$

Its boundary density is the kernel of $T_{K_n} \cdots T_{K_1}$. Forgetting intermediate group variables is a measure-preserving surjection and hence defines a comparison arrow in $\mathcal{D}$ in the coarse-to-fine direction. Proposition 3.3 identifies its exact coarse weight with the corresponding contracted kernel.

When integration over forgotten variables contracts an interface, the comparison map and the tensor composition give the same kernel. More generally, a comparison arrow specifies the variables being integrated, and a tensor decomposition specifies the factors being contracted. Their results can be compared on a common external boundary.

3.5 Linear comparison relations

Linear comparison relations can be imposed by taking a quotient of a direct sum. For a span of vector spaces

$$V_1 \xleftarrow{\alpha_1} W \xrightarrow{\alpha_2} V_2,$$

put

$$Q = (V_1 \oplus V_2) / \text{im}(\alpha_1, -\alpha_2).$$

If $a_i : V_i \rightarrow \mathbb{C}$ satisfy $a_1\alpha_1 = a_2\alpha_2$, then

$$[v_1, v_2] \mapsto a_1(v_1) + a_2(v_2)$$

is the unique functional on Q with the prescribed restrictions. This follows directly from the definition of the quotient. Such quotients will be used for auxiliary cocones in Section 6. Physical boundary composition is defined by (3.10), on the tensor product of the local amplitude spaces.

4 Finite-group gauge kernels and orthogonal truncation

Finite-group lattice gauge theory realizes the preceding composition and compression maps by exact sums. With normalized Haar measure, BF kernels satisfy subdivision invariance and exact gluing. A strictly positive plaquette weight on $\mathbb{Z}_2 \times \mathbb{Z}_2$ gives an idempotent compressed transfer operator with nonzero two-step residual.

4.1 The lattice measure and boundary pairing

Let G be a finite group of order q . Throughout this section,

$$\int_G f(g) dg := \frac{1}{q} \sum_{g \in G} f(g), \quad \Delta_G(g) := q \mathbf{1}_{\{e\}}(g). \quad (4.1)$$

Thus $\int_G \Delta_G(g) f(g) dg = f(e)$. Convolution is defined by

$$(f * g)(x) := \int_G f(xh^{-1})g(h) dh, \quad f * \Delta_G = \Delta_G * f = f. \quad (4.2)$$

These conventions are standard in finite-group lattice gauge theory [6].

Let D be a finite oriented cell decomposition of a compact surface with boundary b . Faces are specified by oriented polygonal attaching maps; side identifications are allowed. Write $E_{\text{int}}(D)$ and $E_b(D)$ for its internal and boundary edges. Each edge carries $g_e \in G$, and $H_f(g)$ is the ordered holonomy around the face f . Assign to each face a central function $w_f : G \rightarrow \mathbb{C}$. The boundary kernel is

$$K_D(g_b) := \int_{G^{E_{\text{int}}(D)}} \prod_{f \in F(D)} w_f(H_f(g)) \prod_{e \in E_{\text{int}}(D)} dg_e. \quad (4.3)$$

All integrals in this expression use (4.1). To obtain a presentation in $\mathcal{D}$ as in Definition 3.1, take

$$\Omega_D = G^{E(D)}, \quad B_D = G^{E_b(D)}, \quad p_D(g) = g|_{E_b(D)}, \quad w_D(g) = \prod_{f \in F(D)} w_f(H_f(g)),$$

with product Haar measures λ_D and ν_D . Coordinate restriction satisfies $(p_D)_* \lambda_D = \nu_D$, and its conditional boundary density (3.5) is exactly (4.3).

The extended boundary space and its gauge-invariant subspace are

$$\mathcal{K}_b = L^2(G^{E_b(D)}, dg_b), \quad \mathcal{H}_b = \mathcal{K}_b^{G^{V_b(D)}}. \quad (4.4)$$

Here a vertex assignment (k_v) acts by $g_e \mapsto k_{t(e)} g_e k_{s(e)}^{-1}$, with the oriented product defining H_f chosen in the same convention. Centrality of the face weights and Haar invariance imply that K_D is gauge invariant. The associated linear amplitude is

$$A_D(\psi) := \int_{G^{E_b(D)}} K_D(g_b) \psi(g_b) dg_b, \quad \psi \in \mathcal{H}_b. \quad (4.5)$$

The bilinear pairing in this formula includes the orientation convention for an incoming boundary. The Hilbert inner product on $\mathcal{K}_b$ is $\langle \phi, \psi \rangle = \int \bar{\phi} \psi dg_b$.

Proposition 4.1 (Gluing and subdivision). *The kernels (4.3) have the following properties.*

- (i) *Suppose $D = D_1 \cup_I D_2$, where I is a boundary edge subcomplex, the face interiors of D_1 and D_2 are disjoint, and the edges of I become internal edges of D . Then*

$$K_D(x, y) = \int_{G^{E(I)}} K_{D_1}(x, h) K_{D_2}(h^{-1}, y) dh. \quad (4.6)$$

The inverse is applied edgewise and records the opposite orientations of the two copies of I .

- (ii) *Subdividing an internal edge leaves the kernel unchanged. A boundary edge subdivision induces the isometric pullback*

$$j : \mathcal{H}_b \longrightarrow \mathcal{H}_{b'}, \quad j\psi = \psi \circ \pi, \quad (4.7)$$

where π multiplies the new holonomies in their oriented order. Under this subdivision, $K_{D'} = K_D \circ \pi$ and $A_{D'} j = A_D$.

- (iii) In the BF case $w_f = \Delta_G$, replacing any face by a finite cell decomposition of its characteristic disk relative to its boundary preserves the boundary kernel. Consequently, the BF amplitudes are cylindrically consistent under finite compositions of these subdivisions and obey (4.6) on every admissible iterated gluing.

Proof. For (i), the internal edges of the glued complex form the disjoint union of the internal edges of the two pieces and the identified interface edges. Substituting (4.3) on the right-hand side of (4.6) therefore gives exactly the integral defining K_D . This is a finite Fubini identity. Interface vertices, including endpoints that remain on the external boundary, introduce no additional factor in this edge-measure convention. The identity holds for the extended edge kernels and remains valid upon restriction to gauge-invariant external states.

For (ii), normalized Haar measure satisfies

$$\int_{G^2} F(g_2 g_1) dg_1 dg_2 = \int_G F(g) dg. \quad (4.8)$$

Apply this identity to the subdivided internal edge. On the boundary, the same identity applied to $|\psi|^2$ proves that j is isometric. New vertex gauge transformations cancel inside $g_2 g_1$, so j preserves the gauge-invariant subspaces. The identity applied to $K_D \psi$ gives $A_{D'} j = A_D$.

For (iii), consider a cellulated disk with fixed boundary edge holonomies and let H denote their oriented product. Flat internal assignments exist precisely when $H = e$. In that case their number is $q^{V_{\text{int}}}$. Indeed, fix the group frame at one boundary vertex. Flatness on a disk expresses every edge holonomy through vertex frames; the remaining boundary frames are determined by the prescribed boundary holonomies, and the internal vertex frames are independent choices in G . Every flat internal assignment is obtained uniquely in this way. Hence the BF kernel on the disk equals

$$q^{F - E_{\text{int}} + V_{\text{int}}} \mathbf{1}_{\{H=e\}} = q \mathbf{1}_{\{H=e\}} = \Delta_G(H). \quad (4.9)$$

The middle equality follows from the Euler relation for a disk, since its boundary has equally many vertices and edges. This is the kernel of the unsubdivided face. The identity holds before any identifications among the boundary sides and therefore also after substituting the identifications in a face attaching map. Replacing faces one at a time proves subdivision invariance. Repeated finite integration proves the assertion for iterated gluings. $\square$

For a weighted face, the corresponding elementary blocking identity is

$$\int_G w_1(Ah) w_2(h^{-1}B) dh = (w_1 * w_2)(AB). \quad (4.10)$$

To verify the formula, set $k = h^{-1}B$, so that $Ah = ABk^{-1}$, and use Haar invariance. Thus subdivision of a face with weight w into two weighted faces is exact when $w = w_1 * w_2$. Inserting a BF face corresponds to multiplication by the convolution unit Δ_G .

Remark 4.2 (Closed-surface normalization). For a connected closed surface Σ with a finite cell decomposition, the number of flat edge assignments is $q^{V-1} |\text{Hom}(\pi_1(\Sigma), G)|$. Therefore the BF partition function in (4.3) is

$$Z_\Sigma = q^{\chi(\Sigma)-1} |\text{Hom}(\pi_1(\Sigma), G)| = q^{\chi(\Sigma)} Z_\Sigma^{\text{grp d}}, \quad (4.11)$$

where $Z_\Sigma^{\text{grp d}}$ is the groupoid cardinality of flat G -bundles. The latter weights each isomorphism class by the reciprocal of its automorphism-group order [7, Eq. (2.1)]. Formula (4.11) specifies the Euler factor relating the two conventions. For example, gluing two BF disks gives $Z_{S^2} = \int_G \Delta_G(h) \Delta_G(h^{-1}) dh = q$, whereas $Z_{S^2}^{\text{grp d}} = q^{-1}$.

4.2 Annular kernels and exact blocking

The transfer-operator form follows from an annulus with boundary holonomies x, y and a radial edge holonomy t . With one central plaquette weight w , its kernel is

$$K_w(y, x) = \int_G w(ytx^{-1}t^{-1}) dt. \quad (4.12)$$

The boundary Hilbert space is the space of class functions $\mathcal{H} = L^2(G, dg)^{\text{Ad}G}$. Integrating (4.12) against $f \in \mathcal{H}$ gives

$$(T_w f)(y) = \int_G w(yx^{-1}) f(x) dx. \quad (4.13)$$

Indeed, the change of variable $x \mapsto txt^{-1}$ removes t , and f is invariant under that change. For an abelian group every function is a class function and $K_w(y, x) = w(yx^{-1})$.

Gluing two annuli gives

$$T_{w_2} T_{w_1} = T_{w_2 * w_1}, \quad T_{\Delta_G} = I_{\mathcal{H}}. \quad (4.14)$$

Both identities follow directly from normalized Haar integration. In particular, a strip of n identical plaquettes has transfer operator $T_w^n = T_{w*}^n$. This is exact integration of the intermediate boundary holonomies. If w is real, $w(g^{-1}) = w(g)$, $w \geq 0$, and $\int w = 1$, then T_w is a self-adjoint Markov contraction and fixes the unit vector $\mathbf{1}$. These properties follow, respectively, from changing $x \leftrightarrow y$, positivity of the kernel, Jensen's inequality, and Haar invariance.

4.3 A two-mode orthogonal truncation

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, with characters

$$\chi_1(x, y) = (-1)^x, \quad \chi_2(x, y) = (-1)^y, \quad \chi_{12} = \chi_1 \chi_2.$$

The four functions $\mathbf{1}, \chi_1, \chi_2, \chi_{12}$ are an orthonormal basis of $\mathcal{H} = L^2(G, dg)$. Consider the plaquette weight

$$w_{\alpha, \beta} = \mathbf{1} + \alpha \chi_1 + \beta \chi_2, \quad \alpha, \beta \in \mathbb{R}, \quad |\alpha| + |\beta| < 1. \quad (4.15)$$

This weight is strictly positive and has Haar integral one. Character orthogonality gives

$$T_w \mathbf{1} = \mathbf{1}, \quad T_w \chi_1 = \alpha \chi_1, \quad T_w \chi_2 = \beta \chi_2, \quad T_w \chi_{12} = 0. \quad (4.16)$$

Set

$$u = \frac{\chi_1 + \chi_2}{\sqrt{2}}, \quad v = \frac{\chi_1 - \chi_2}{\sqrt{2}}, \quad s : \mathbb{C}^2 \longrightarrow \mathcal{H}, \quad se_0 = \mathbf{1}, \quad se_1 = u. \quad (4.17)$$

The map s is isometric and $P = ss^*$ is the orthogonal projection onto the retained space. Thus v and χ_{12} are discarded. Writing $m = (\alpha + \beta)/2$ and $\ell = (\alpha - \beta)/2$, one obtains

$$T_w u = mu + \ell v, \quad T_w v = \ell u + mv. \quad (4.18)$$

The compressed strip amplitudes are

$$S_n := s^* T_w^n s = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\alpha^n + \beta^n}{2} \end{pmatrix}, \quad n \geq 1, \quad S_0 = I_{\mathbb{C}^2}. \quad (4.19)$$

The two-step comparison is consequently

$$\Delta = S_2 - S_1^2 = \frac{(\alpha - \beta)^2}{4} |e_1\rangle\langle e_1|, \quad \delta_{\text{op}} = \delta_{\text{HS}} = \frac{(\alpha - \beta)^2}{4}. \quad (4.20)$$

Here $T_w u$ has a discarded component ℓv , which contributes $\ell^2 u$ after the next plaquette. The intermediate projection in S_1^2 removes this contribution, as in (2.1). Both computations use the same plaquette weight, normalized boundary pairing, and orthogonal projection.

Proposition 4.3 (Matched positive models with different defects). *Let T^A and T^B have parameters $(\alpha, \beta) = (3/8, 1/8)$ and $(1/4, 1/4)$, respectively, and use the same isometry (4.17). Both are positive semidefinite Markov contractions, and*

$$S_1^A = S_1^B = \text{diag}(1, 1/4), \quad \Delta^A = \frac{1}{64}|e_1\rangle\langle e_1|, \quad \Delta^B = 0. \quad (4.21)$$

Every matching observable that depends only on S_1 therefore agrees for these models, whereas their two-plaquette matrix elements on e_1 differ by $1/64$.

Proof. The spectra in (4.16) lie in $[0, 1]$. Equations (4.19) and (4.20) give the stated identities and $\langle e_1, S_2^A e_1 \rangle = 5/64$, $\langle e_1, S_2^B e_1 \rangle = 1/16$. $\square$

4.4 A fixed retained operator with nonzero blocking defect

Choose $\alpha = a$, $\beta = -a$, where $0 < a < 1/2$. Then

$$w = \mathbf{1} + a\chi_1 - a\chi_2 > 0, \quad T_w u = av, \quad T_w v = au. \quad (4.22)$$

The full retained operator is

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_1^2 = S_1. \quad (4.23)$$

Thus repeated composition leaves S_1 unchanged. Exact integration of two microscopic plaquettes, followed by the same orthogonal compression, gives

$$w * w = \mathbf{1} + a^2\chi_1 + a^2\chi_2, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}, \quad \delta_{\text{op}} = \delta_{\text{HS}} = a^2. \quad (4.24)$$

The vacuum normalization remains $\langle e_0, S_n e_0 \rangle = 1$ at every length. The nonzero defect is detected by the retained boundary state e_1 , since $\langle e_1, S_2 e_1 \rangle = a^2$ while $\langle e_1, S_1^2 e_1 \rangle = 0$.

All binary comparisons can be evaluated explicitly. For positive integers m, n , let $R_{m,n} = S_{m+n} - S_m S_n$. Formula (4.19) gives

$$R_{m,n} = \begin{cases} a^{m+n}|e_1\rangle\langle e_1|, & \text{if } m \text{ and } n \text{ are both odd,} \\ 0, & \text{otherwise.} \end{cases} \quad (4.25)$$

It also verifies the general return identity (2.2). The intermediate discarded channel determines every failure of composition in (4.25).

At $a = 1/4$, order the holonomies as $(0, 0), (0, 1), (1, 0), (1, 1)$. The local weight has values $(1, 3/2, 1/2, 1)$, and the transfer and projection matrices in value coordinates are

$$T_w = \frac{1}{8} \begin{pmatrix} 2 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \end{pmatrix}, \quad P = \frac{1}{4} \begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix}. \quad (4.26)$$

The inner product on these coordinates is one quarter of the Euclidean inner product. Direct multiplication checks $P = P^* = P^2$, $S_1 = \text{diag}(1, 0)$, and $S_2 = \text{diag}(1, 1/16)$. Both defect norms therefore equal $1/16$.

The plaquette weight is pointwise positive, and T_w is a Markov operator with eigenvalue $-a$. Positive-semidefinite transfer operators satisfy an additional restriction at an idempotent compression.

Proposition 4.4 (Positive transfer operators at a retained fixed point). *Let $0 \leq T \leq I$ be an operator on a finite-dimensional Hilbert space, let s be an isometry, and set $P = ss^*$ and $K = s^*Ts$. If $K^2 = K$, then*

$$(I - P)Ts = 0, \quad s^*T^n s = K^n \quad \text{for every } n \geq 0. \quad (4.27)$$

Proof. The leakage identity and $T^2 \leq T$ imply

$$0 \leq s^*T(I - P)Ts = s^*T^2s - K^2 \leq K - K^2 = 0.$$

Thus $(I - P)Ts = 0$. Since $PTs = sK$, this gives $Ts = sK$; iteration proves the second identity. $\square$

Positive-semidefinite transfer operators can have nonzero compression residuals, as in (4.20). An idempotent compression of a positive contraction has zero residual. For BF theory, $T_{\Delta_G} = I$, so every retained subspace is invariant and all the compression defects vanish. These explicit kernels provide finite tests for orthogonal truncations of the kind used in tensor-network coarse graining [8].

Proposition 4.5 (Independent interface channels). *For $k \geq 1$ independent copies of (4.22), set $T^{(k)} = T_w^{\otimes k}$ and $s^{(k)} = s^{\otimes k}$, with product Haar measure and product plaquette weight. The retained dimension is 2^k , $S_n^{(k)} = S_n^{\otimes k}$, and $S_1^{(k)}$ is exactly idempotent. The two-step defect $\Delta^{(k)} = S_2^{(k)} - (S_1^{(k)})^2$ satisfies*

$$\|\Delta^{(k)}\|_{\text{op}} = a^2, \quad \|\Delta^{(k)}\|_{\text{HS}} = \sqrt{(1 + a^4)^k - 1}. \quad (4.28)$$

Proof. Tensor factorization gives $S_n^{(k)} = S_n^{\otimes k}$. The all-vacuum vector has defect eigenvalue zero. A tensor basis vector with $r \geq 1$ factors equal to e_1 has eigenvalue a^{2r} , of multiplicity $\binom{k}{r}$. Since $0 < a < 1/2$, the largest eigenvalue is a^2 , while the squared Hilbert–Schmidt norm is $\sum_{r=1}^k \binom{k}{r} a^{4r} = (1 + a^4)^k - 1$. $\square$

Thus the Hilbert–Schmidt defect grows with the number of interface channels even when the operator defect and retained fixed point remain unchanged.

5 Error estimates for finite comparison diagrams

The compression bound in Section 2 concerns repeated powers of one transfer operator. We now consider compositions of different kernels, linear refinement equations, and symmetry constraints. All state spaces are finite-dimensional Hilbert spaces. Amplitude functionals carry the operator norm, and kernel norms use the stated boundary measures. Comparison maps and normalizations remain fixed within each estimate.

5.1 Interface contractions and assembly trees

For finite probability spaces X, I, Y , define the interface contraction

$$\mathcal{C}_I(f, g)(x, y) = \int_I f(x, h)g(h, y) d\nu_I(h), \quad f \in L^2(X \times I), \quad g \in L^2(I \times Y). \quad (5.1)$$

On a finite group, ν_I is normalized counting measure. Reversing the interface orientation replaces one argument by h^{-1} and preserves the norm.

Lemma 5.1 (Norm of interface contraction). *The contraction (5.1) satisfies*

$$\|\mathcal{C}_I(f, g)\|_{L^2(X \times Y)} \leq \|f\|_{L^2(X \times I)} \|g\|_{L^2(I \times Y)}. \quad (5.2)$$

The constant is one and is sharp.

Proof. For each (x, y) , the Cauchy–Schwarz inequality on I bounds $|\mathcal{C}_I(f, g)(x, y)|^2$ by

$$\left(\int_I |f(x, h)|^2 d\nu_I(h) \right) \left(\int_I |g(h, y)|^2 d\nu_I(h) \right).$$

Integrating over x, y gives the product of the squared norms. Constant unit kernels give equality. $\square$

Kernel L^2 norms in this convention equal the Hilbert–Schmidt norms of the associated integral operators. Thus the $p = 2$ case of Theorem 2.1 applies directly to kernel errors. Kernel norms depend on the boundary spaces. For example, the identity kernel on a finite set of size d has L^2 norm $\sqrt{d}$. If an operation closes a free tensor channel by the ordinary partial trace, its operator norm between Hilbert–Schmidt spaces is $\sqrt{d}$ for a channel of dimension d . Indeed, the Hilbert–Schmidt adjoint of the partial trace is $X \mapsto X \otimes I_d$, whose norm is $\sqrt{d}$. These factors enter the contraction constants in the next theorem.

Consider a finite rooted ordered tree $\mathcal{T}$. At a binary vertex v with children v_1, v_2 , let $\mathcal{C}_v$ be a bilinear map satisfying

$$\mathcal{C}_v : E_{v_1} \times E_{v_2} \rightarrow E_v, \quad \|\mathcal{C}_v(x, y)\| \leq \beta_v \|x\| \|y\|.$$

A unary vertex has a linear map $L_v : E_{v_1} \rightarrow E_v$. It represents a refinement comparison or a fixed linear boundary operation. Let $k_v \in E_v$ be the prescribed amplitude data. Starting from specified leaf data $\hat{k}_v$, recursively construct $\hat{k}_v$ at internal vertices using the same $\mathcal{C}_v$ or L_v . Define local residuals by

$$r_v = \begin{cases} k_v - \hat{k}_v, & v \text{ a leaf,} \\ k_v - L_v k_{v_1}, & v \text{ unary,} \\ k_v - \mathcal{C}_v(k_{v_1}, k_{v_2}), & v \text{ binary.} \end{cases}$$

In particular, a binary residual is the tensor gluing error. A unary refinement residual has the form $A_D - A_{D'} H(r)$ for $r : D \rightarrow D'$ after orienting the comparison from fine amplitudes to coarse amplitudes.

Theorem 5.2 (Propagation through an assembly tree). *Put $M_v = \max\{\|k_v\|, \|\hat{k}_v\|\}$. Give a parent–child edge the weight*

$$w(v, v_1) = \|L_v\| \quad \text{at a unary parent,} \quad w(v, v_i) = \beta_v M_{v_{3-i}} \quad \text{at a binary parent.}$$

Let $W(v)$ be the product of these weights along the unique path from v to the root o , with $W(o) = 1$. Then

$$\|k_o - \hat{k}_o\| \leq \sum_{v \in \mathcal{T}} W(v) \|r_v\|. \quad (5.3)$$

If the leaves agree, there are N leaves and p unary vertices, the height is h , and every internal residual is at most ε , then

$$\|k_o - \hat{k}_o\| \leq (N - 1 + p) q^h \varepsilon, \quad q = \max\{1, \max_{v \text{ unary}} \|L_v\|, \max_{v \text{ binary}} \beta_v M_{v_1}, \max_{v \text{ binary}} \beta_v M_{v_2}\}. \quad (5.4)$$

An empty maximum is omitted.

Proof. Set $e_v = \|k_v - \hat{k}_v\|$. At a unary vertex, $e_v \leq \|r_v\| + \|L_v\| e_{v_1}$. At a binary vertex, bilinearity gives

$$\begin{aligned} k_v - \hat{k}_v &= r_v + \mathcal{C}_v(k_{v_1} - \hat{k}_{v_1}, k_{v_2}) \\ &\quad + \mathcal{C}_v(\hat{k}_{v_1}, k_{v_2} - \hat{k}_{v_2}). \end{aligned}$$

Hence $e_v \leq \|r_v\| + \beta_v M_{v_2} e_{v_1} + \beta_v M_{v_1} e_{v_2}$. Expanding these inequalities down the finite tree proves (5.3). A tree with N leaves and p unary vertices has $N - 1 + p$ internal vertices, and each path weight is at most q^h , proving (5.4). $\square$

The reconstructed norms in M_v can be bounded recursively from the leaf norms and the displayed operator bounds. If all intermediate norms and contraction norms are at most one, (5.4) becomes $(N - 1 + p)\varepsilon$. For two trees evaluating the same finite tensor network, finite summation and associativity identify their exact root values. The difference of their prescribed root values is therefore bounded by the sum of their two expressions in (5.3).

5.2 Correction of linear refinement equations

Fix a finite comparison diagram. At each vertex v let E_v be a finite-dimensional Hilbert space of amplitude functionals or retained linear data, and for an oriented edge $e : u \rightarrow v$ let $T_e : E_u \rightarrow E_v$ be a fixed linear comparison map. Refinement embeddings $\iota : H_v \rightarrow H_u$ give $T_e(A) = A\iota$ on amplitude duals. For positive edge weights w_e , define

$$\mathcal{D} : \bigoplus_v E_v \longrightarrow \bigoplus_{e:u \rightarrow v} E_v, \quad (\mathcal{D}a)_e = \sqrt{w_e}(a_v - T_e a_u). \quad (5.5)$$

Its kernel consists exactly of compatible amplitude cocones for the selected arrows. Additional linear conditions, including a fixed value on a reference state, can be appended as rows, giving an affine system $\mathcal{D}a = b$.

Theorem 5.3 (Exact linear correction). *Suppose $b \in \text{ran } \mathcal{D}$ and $\mathcal{D} \neq 0$. Let $\mathcal{D}^\dagger$ be the Moore–Penrose inverse and $\sigma_+(\mathcal{D}) > 0$ its smallest nonzero singular value. Then*

$$a_{\text{corr}} = a - \mathcal{D}^\dagger(\mathcal{D}a - b) \quad (5.6)$$

is the unique nearest solution of $\mathcal{D}x = b$ in the direct-sum Hilbert norm, and

$$\text{dist}(a, \{x : \mathcal{D}x = b\}) = \|\mathcal{D}^\dagger(\mathcal{D}a - b)\| \leq \frac{\|\mathcal{D}a - b\|}{\sigma_+(\mathcal{D})}. \quad (5.7)$$

The constant $1/\sigma_+(\mathcal{D})$ is sharp.

Proof. Since $\mathcal{D}a - b \in \text{ran } \mathcal{D}$, $\mathcal{D}\mathcal{D}^\dagger(\mathcal{D}a - b) = \mathcal{D}a - b$. Thus $\mathcal{D}a_{\text{corr}} = b$. Moreover, $a - a_{\text{corr}} \in \text{ran } \mathcal{D}^\perp = (\ker \mathcal{D})^\perp$. This is the orthogonality characterization of the nearest point in the affine solution space. The singular-value decomposition gives $\|\mathcal{D}^\dagger\| = 1/\sigma_+(\mathcal{D})$ and equality along a left singular vector for $\sigma_+(\mathcal{D})$. $\square$

The factor $1/\sigma_+(\mathcal{D})$ converts a linear residual into distance from a compatible cocone. Tensor gluing and positivity impose further constraints on the amplitude data; a simultaneous correction must preserve these constraints as well.

Proposition 5.4 (Comparison graphs and their spectral gaps). *Suppose the comparison graph G is connected and there are unitary maps $J_v : E \rightarrow E_v$ such that $T_e = J_v J_u^*$ for every edge $e : u \rightarrow v$. Let L_G be the weighted graph Laplacian, with weights w_e , and let $\lambda_2(L_G) > 0$ be its first positive eigenvalue. For $z_v = J_v^* a_v$ and $\bar{z} = |V(G)|^{-1} \sum_v z_v$, the corrected cocone is $a_{\text{corr},v} = J_v \bar{z}$, and*

$$\sum_v \|z_v - \bar{z}\|^2 \leq \frac{1}{\lambda_2(L_G)} \sum_{e:u \rightarrow v} w_e \|z_v - z_u\|^2. \quad (5.8)$$

If $\mathcal{S}$ is a spanning tree and $\|z_v - z_u\| \leq \varepsilon$ on its edges, then every comparison edge $e : u \rightarrow v$ satisfies

$$\|z_v - z_u\| \leq d_{\mathcal{S}}(u, v)\varepsilon, \quad \max_v \|z_v - \bar{z}\| \leq \text{diam}(\mathcal{S})\varepsilon. \quad (5.9)$$

Proof. Using J_v^* to identify each vertex space and each edge target space with E transforms $\mathcal{D}$ into the weighted incidence operator tensored with I_E . Its squared singular values are the eigenvalues of L_G , with the same multiplicity factor. Its kernel is the constant families, whose orthogonal projection is averaging. Expanding each coordinate of $z - \bar{z}$ in a Laplacian eigenbasis gives (5.8). Along the unique tree path from u to v , the difference $z_v - z_u$ is the signed sum of the edge differences. The first inequality in (5.9) follows. Averaging it over u gives the second. $\square$

For a family of comparison graphs with $\lambda_2(L_G) \geq \gamma > 0$, the linear correction constants are uniformly bounded. An unweighted path on N vertices has $\lambda_2 = 2(1 - \cos(\pi/N))$, so the correction constant grows linearly in N . Adding edges to a connected graph with these same unitary identifications preserves the consistency subspace and increases its quadratic form, hence cannot decrease λ_2 . More generally this monotonicity holds when new linear tests leave the kernel unchanged.

5.3 Truncating comparison maps and transporting Ward identities

Let F be a functor from a finite comparison category to finite-dimensional Hilbert spaces and linear maps. Choose $P_D : F(D) \rightarrow V_D$ and $s_D : V_D \rightarrow F(D)$ with $P_D s_D = I$, and put $\hat{F}(f) = P_{D'} F(f) s_D$ for $f : D \rightarrow D'$. For composable $f : D \rightarrow D'$ and $g : D' \rightarrow D''$, direct multiplication gives

$$\hat{F}(g)\hat{F}(f) - \hat{F}(gf) = P_{D''} F(g)(s_{D'} P_{D'} - I) F(f) s_D. \quad (5.10)$$

The right-hand side quantifies the intermediate discarded component that returns to the retained sector. It must vanish if the retained maps are to define a functor. For orthogonal compression of repeated transfer maps, (5.10) reduces, up to the sign convention, to (2.6).

There is also a direct perturbation estimate for amplitudes. Given $A_D : F(D) \rightarrow \mathbb{C}$, define $A_D^{\text{tr}} = A_D s_D$ and write

$$r_f = A_{D'} F(f) - A_D, \quad E_f = s_{D'} \hat{F}(f) - F(f) s_D.$$

Then

$$A_{D'}^{\text{tr}} \hat{F}(f) - A_D^{\text{tr}} = A_{D'} E_f + r_f s_D, \quad \|A_{D'}^{\text{tr}} \hat{F}(f) - A_D^{\text{tr}}\| \leq \|A_{D'}\| \|E_f\| + \|r_f\| \|s_D\|. \quad (5.11)$$

The first term is caused by truncation of the comparison map; the second is the original refinement residual restricted to the retained space. If $G_D \subseteq F(D)$ is the span of symmetry variations, the Ward identities are $A_D|_{G_D} = 0$. For a retained variation space $G_D^{\text{tr}} \subseteq V_D$, the inclusion $s_D(G_D^{\text{tr}}) \subseteq G_D$ therefore preserves these identities.

For a specified linear symmetry variation X , a Ward identity takes the form $AX = 0$. A discrete symmetry g gives $X = g - I$. The following estimate compares local and assembled Ward identities.

Proposition 5.5 (Ward identities under assembly). *Let $m : E_{\text{loc}} \rightarrow E$ be a surjective linear assembly map between finite-dimensional Hilbert spaces and let $r : E \rightarrow E_{\text{loc}}$ be a linear right inverse. Let $A : E \rightarrow \mathbb{C}$, $\sigma : E_{\text{loc}} \rightarrow \mathbb{C}$ and let $X : E \rightarrow E$, $X_{\text{loc}} : E_{\text{loc}} \rightarrow E_{\text{loc}}$ be linear maps. With*

$$E_A = Am - \sigma, \quad E_X = Xm - mX_{\text{loc}},$$

one has

$$\|AX\| \leq \|r\| (\|\sigma X_{\text{loc}}\| + \|E_A\| \|X_{\text{loc}}\| + \|A\| \|E_X\|). \quad (5.12)$$

In particular, exact assembly, exact compatibility of the symmetry generators, and local Ward identities imply the global Ward identity.

Proof. Since $mr = I$, the identities defining E_A, E_X give

$$AX = AXmr = ((\sigma + E_A)X_{\text{loc}} + AE_X)r.$$

Taking norms proves the claim. $\square$

For $E \neq \{0\}$, a right inverse of minimum operator norm is $r = m^\dagger$, with $\|r\| = 1/\sigma_+(m)$. Uniform propagation across larger diagrams therefore requires control of this smallest singular value as well as of the local residuals.

5.4 Local stability of an iterative map

A smooth, single-valued matching rule defines an iterative map on a fixed retained space. The following proposition gives geometric decay of locally Lipschitz residuals near a fixed point at which they vanish, provided that the derivative has spectral radius less than one.

Proposition 5.6 (Decay near a compatible fixed point). *Let U be a neighborhood in a finite-dimensional normed space and let $R : U \rightarrow U$ be differentiable at x_* , with $R(x_*) = x_*$. Suppose the spectral radius of $DR(x_*)$ is less than one. Let $\Phi : U \rightarrow W$ be a locally Lipschitz residual map with $\Phi(x_*) = 0$. For every q strictly between that spectral radius and one, there are an equivalent norm, a neighborhood U_q of x_* , and a constant C_q such that*

$$\|\Phi(R^n x)\| \leq C_q q^n \|x - x_*\| \quad (x \in U_q, n \geq 0). \quad (5.13)$$

The same conclusion holds on a quotient by redundant directions when both R and Φ descend to that quotient and satisfy the stated hypotheses there.

Proof. Choose an adapted norm and a number $\rho < q$ such that $\|DR(x_*)\| \leq \rho$. Differentiability at x_* gives a ball on which

$$\|R(x) - x_*\| \leq \rho \|x - x_*\| + (q - \rho) \|x - x_*\| = q \|x - x_*\|.$$

This ball is forward invariant. Iteration and the local Lipschitz bound $\|\Phi(x)\| \leq C_q \|x - x_*\|$ give (5.13). Applying the same argument in the quotient proves the last assertion. $\square$

For a fixed finite set of matrices, refinement residuals are linear and tensor gluing residuals are polynomial, so the required local Lipschitz property follows directly. A conclusion uniform in an enlarging test family requires uniform contraction and Lipschitz constants.

6 Compatible extensions and retained spaces

The estimates above measure the failure of retained maps to compose. When microscopic amplitudes are compatible, colimits give extensions that preserve their linear comparison equations exactly. Quotients by invariant subspaces then give retained functors. Injectivity of the measurement map on the retained ansatz ensures uniqueness of the normalized least-squares fit.

6.1 Extension of a compatible cocone

Let $i : \mathcal{C} \hookrightarrow \mathcal{E}$ be a full inclusion of small comparison categories and let $F : \mathcal{C} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ be a functor. The pointwise left Kan extension is

$$(\mathrm{Lan}_i F)(D) = \mathrm{colim}_{(C, \alpha) \in (i \downarrow D)} F(C), \quad (6.1)$$

where an object of $(i \downarrow D)$ is an arrow $\alpha : i(C) \rightarrow D$ and a morphism $(C, \alpha) \rightarrow (C', \alpha')$ is an arrow $h : C \rightarrow C'$ with $\alpha' i(h) = \alpha$. Postcomposition by $f : D \rightarrow D'$ defines the comparison map of the extension.

Lemma 6.1. *Every natural transformation $a : F \Rightarrow \mathbb{C}$ induces a unique natural transformation*

$$a^{\mathrm{Lan}} : \mathrm{Lan}_i F \Rightarrow \mathbb{C}$$

whose composite with each canonical colimit map $F(C) \rightarrow (\mathrm{Lan}_i F)(D)$ is a_C .

Proof. For fixed D , the maps a_C form a cocone with vertex $\mathbb{C}$ on the diagram in (6.1). The universal property of its colimit gives the required functional. If $f : D \rightarrow D'$, its two possible composites with every canonical map from $F(C)$ agree with a_C , so uniqueness gives naturality. This argument also applies to an empty indexing category, whose colimit is the zero vector space. $\square$

For a disconnected indexing category, the colimit separates into a direct sum over connected components, and the induced amplitude is the sum of the component amplitudes. For an empty indexing category it is the zero functional.

Definition 6.2 (Final subcategory). An inclusion $j_D : \mathcal{J}_D \hookrightarrow (i \downarrow D)$ is final if, for every object β of $(i \downarrow D)$, the comma category $(\beta \downarrow j_D)$ is nonempty and connected. Its objects are arrows $\beta \rightarrow j_D(J)$ in $(i \downarrow D)$.

Finality gives a canonical isomorphism [21, Lemma 4.17.2]

$$\operatorname{colim}_{\mathcal{J}_D} F \cong (\operatorname{Lan}_i F)(D). \quad (6.2)$$

Indeed, a cocone on the smaller diagram extends to an object β by choosing an arrow $\beta \rightarrow j_D(J)$. Connectedness of $(\beta \downarrow j_D)$ makes this extension independent of the choice. The construction gives mutually inverse correspondences between cocones, which proves (6.2) by their universal properties.

6.2 Functorial finite diagrams and quotients

To obtain finite-dimensional objects, choose finite full subcategories $\mathcal{J}_D \subseteq (i \downarrow D)$ such that postcomposition by every retained comparison $f : D \rightarrow D'$ sends $\mathcal{J}_D$ into $\mathcal{J}_{D'}$. Define

$$T(D) = \operatorname{colim}_{\mathcal{J}_D} F(C). \quad (6.3)$$

If all $F(C)$ are finite dimensional, then so is $T(D)$. Universal properties induce maps $T(f) : T(D) \rightarrow T(D')$, and uniqueness gives

$$T(gf) = T(g)T(f), \quad T(\operatorname{id}_D) = \operatorname{id}_{T(D)}.$$

Thus T is a functor. Lemma 6.1 supplies a natural amplitude $a^T : T \Rightarrow \underline{\mathbb{C}}$.

This closure condition has an explicit finite realization. Suppose $\mathcal{E}$ is a finite category and $\mathcal{C}$ is a full subcategory. Start with finitely many arrows $i(C) \rightarrow D$ and close this collection under postcomposition in $\mathcal{E}$. For each D , take the full subcategory on the resulting arrows with codomain D . There are finitely many such arrows, and the required closure holds by construction.

Proposition 6.3 (Natural retained quotients). *Let $T : \mathcal{E} \rightarrow \mathbf{FinVect}_{\mathbb{C}}$ be the finite extension above. Suppose subspaces $N_D \subseteq T(D)$ satisfy*

$$T(f)N_D \subseteq N_{D'} \quad \text{for every } f : D \rightarrow D'. \quad (6.4)$$

Then $E(D) = T(D)/N_D$ is a functor and the quotient maps $P_D : T(D) \rightarrow E(D)$ form a natural transformation. The amplitude a^T descends to a natural amplitude a^E on E if and only if

$$a_D^T(N_D) = 0 \quad \text{for every } D. \quad (6.5)$$

Proof. Condition (6.4) makes

$$E(f)[v] = [T(f)v]$$

well defined. It also gives the functor laws and $E(f)P_D = P_{D'}T(f)$. Condition (6.5) is exactly the criterion for a functional to factor through P_D . The resulting functionals are natural because their two composites with the surjection P_D agree by naturality of a^T . $\square$

Proposition 6.4 (Stable closure of discarded subspaces). *Suppose $\mathcal{E}$ is finite. Given subspaces $N_D^0 \subseteq T(D)$, define*

$$N_D = \sum_{\substack{C \in \operatorname{Ob} \mathcal{E} \\ f : C \rightarrow D}} T(f)N_C^0. \quad (6.6)$$

This is the smallest family containing N_D^0 and satisfying (6.4). It is computable by taking the column span of the matrices $T(f)$ applied to bases of the N_C^0 . For the natural amplitude a^T , condition (6.5) holds for this family if and only if $a_D^T(N_D^0) = 0$ for all D .

Proof. The identity arrow shows $N_D^0 \subseteq N_D$. For $g : D \rightarrow D'$, each summand satisfies $T(g)T(f)N_C^0 = T(gf)N_C^0 \subseteq N_{D'}$, proving stability. Every stable family containing the initial subspaces contains each summand in (6.6), which proves minimality. Finally, naturality gives $a_D^T T(f) = a_C^T$. It therefore suffices that all initial subspaces are annihilated; necessity follows from their inclusion in the closure. $\square$

The closure can fill an entire space $T(D)$, giving the zero retained quotient at that object. A specified collection of linear observables $\mathcal{O}_D \subseteq T(D)^*$ is preserved exactly when

$$N_D \subseteq \bigcap_{\ell \in \mathcal{O}_D} \ker \ell.$$

The retained dimension can be computed and the inclusion checked by finite-dimensional linear algebra.

A retained functional can also be specified using sections. Suppose maps $S_D : E(D) \rightarrow T(D)$ satisfy

$$P_D S_D = \text{id}_{E(D)}, \quad T(f) S_D = S_{D'} E(f). \quad (6.7)$$

Then $a_D^{\text{ret}} = a_D^T S_D$ is natural. The second equation in (6.7) ensures that the sections commute with comparison maps. If (6.5) holds, every section gives the same retained functional.

6.3 Recovery from the microscopic restriction

For a functor $G : \mathcal{E} \rightarrow \mathbf{Vect}_{\mathbb{C}}$, the canonical comparison is

$$\epsilon_G : \text{Lan}_i(G|_{\mathcal{C}}) \rightarrow G.$$

This map compares the colimit of the microscopic data with the prescribed value of G at each object.

Proposition 6.5. *For a full inclusion i , the canonical map ϵ_G is an isomorphism if and only if G is isomorphic to $\text{Lan}_i F$ for some $F : \mathcal{C} \rightarrow \mathbf{Vect}_{\mathbb{C}}$.*

Proof. For $C \in \mathcal{C}$, the object $(C, \text{id}_{i(C)})$ is terminal in $(i \downarrow i(C))$ because i is fully faithful. Hence $i^* \text{Lan}_i F \cong F$. The unit of the adjunction $\text{Lan}_i \dashv i^*$ is therefore an isomorphism. The triangle identity implies that the counit ϵ_G is an isomorphism for $G = \text{Lan}_i F$, and hence for every functor isomorphic to such an extension. Conversely, if ϵ_G is an isomorphism, then $G \cong \text{Lan}_i(G|_{\mathcal{C}})$ by definition. $\square$

For functors in this essential image, these colimits recover the functor from its microscopic comparison data. Compatibility with boundary gluing is expressed by the tensor composition law of Section 3.3.

6.4 Normalized least-squares fitting

Fix a finite-dimensional space V of amplitude or kernel coefficients. For positive integers p, m , let $S : \mathbb{C}^p \rightarrow V$ be an injective coefficient map for a retained linear ansatz and let $L : V \rightarrow \mathbb{C}^m$ be the linear map collecting the selected measurements. Put $J = LS$ and fix a positive-definite Hermitian matrix W on $\mathbb{C}^m$. For measured target data $y \in \mathbb{C}^m$, define

$$\theta(y) = \underset{\theta \in \mathbb{C}^p}{\text{argmin}} (J\theta - y)^* W (J\theta - y). \quad (6.8)$$

Proposition 6.6. *If J is injective, then (6.8) has the unique solution*

$$\theta(y) = (J^* W J)^{-1} J^* W y. \quad (6.9)$$

The solution depends linearly, and hence smoothly, on y . The map

$$\Pi_{\text{fit}} = S(J^*WJ)^{-1}J^*WL : V \longrightarrow V \quad (6.10)$$

is a bounded projection onto $\text{im } S$. Moreover,

$$\|\theta(y) - \theta(y')\|_2 \leq \frac{\|W^{1/2}(y - y')\|_2}{\sigma_{\min}(W^{1/2}J)}. \quad (6.11)$$

Proof. For every nonzero v , one has $v^*J^*WJv = \|W^{1/2}Jv\|_2^2 > 0$, so the normal matrix is positive definite. Completing the square gives (6.9) and uniqueness. Formula (6.10) acts as the identity on $S\mathbb{C}^p$, so it is a projection onto that subspace. The operator taking $W^{1/2}y$ to $\theta(y)$ is the left pseudoinverse of the full-column-rank matrix $W^{1/2}J$; its norm is the reciprocal of the smallest singular value, proving (6.11). $\square$

Corollary 6.7 (Affine normalized ansatz). *Let $\mathcal{N} : V \rightarrow Z$ be a linear constraint map, and fix $c \in Z$. Choose $v_0 \in V$ and an injective $S : \mathbb{C}^p \rightarrow V$ with $\mathcal{N}v_0 = c$ and $\mathcal{N}S = 0$. If $J = LS$ is injective, the least-squares fit in the affine ansatz $v_0 + S\mathbb{C}^p$ is unique and equals*

$$v_{\text{fit}}(y) = v_0 + S(J^*WJ)^{-1}J^*W(y - Lv_0). \quad (6.12)$$

It satisfies $\mathcal{N}v_{\text{fit}}(y) = c$, depends affinely and smoothly on y , and has the same coefficient stability bound (6.11).

Proof. Apply Proposition 6.6 to the shifted target $y - Lv_0$. The constraint follows from $\mathcal{N}S = 0$. $\square$

For amplitude data one can impose $A(\mathbf{1}) = 1$. For kernel data one can impose the vacuum condition $T_K\mathbf{1} = \mathbf{1}$ by setting $\mathcal{N}(K) = T_K\mathbf{1}$. Choosing the affine ansatz within either constraint fixes the amplitude scale throughout fitting.

For the linear projections Π_{fit} of (6.10) on a functor V , compatibility with comparison maps is expressed by

$$\Pi_{\text{fit}, D'}V(f) = V(f)\Pi_{\text{fit}, D}, \quad f : D \rightarrow D'.$$

When these equations hold, the retained images form a subfunctor.

7 Computations and further applications

7.1 Boundary measurements

A blocking comparison requires a common boundary space and normalization. Given the state embeddings and interface pairings, one computes the exact block kernel and the kernel obtained by inserting intermediate truncations. For a self-adjoint contraction, Theorem 2.1 bounds their difference in terms of the two-step residual. For a general tensor network, Theorem 5.2 bounds the total error in terms of the local residuals, contraction norms, and intermediate kernel norms. Theorem 5.3 gives the distance to a compatible family of linear amplitude data.

The choice of measurements determines which discrepancies can be observed. In the two-character family of Section 4, S_1 depends on $(\alpha + \beta)/2$, while Δ depends on $(\alpha - \beta)^2/4$. Models with the same sum $\alpha + \beta$ therefore have identical one-step retained data. Measuring s^*T^2s distinguishes models with different values of $(\alpha - \beta)^2$, as Proposition 4.3 shows. Taking V to contain the coefficients of both one-step and two-step kernels makes these matrix elements linear measurements for the normalized fit of Section 6.4.

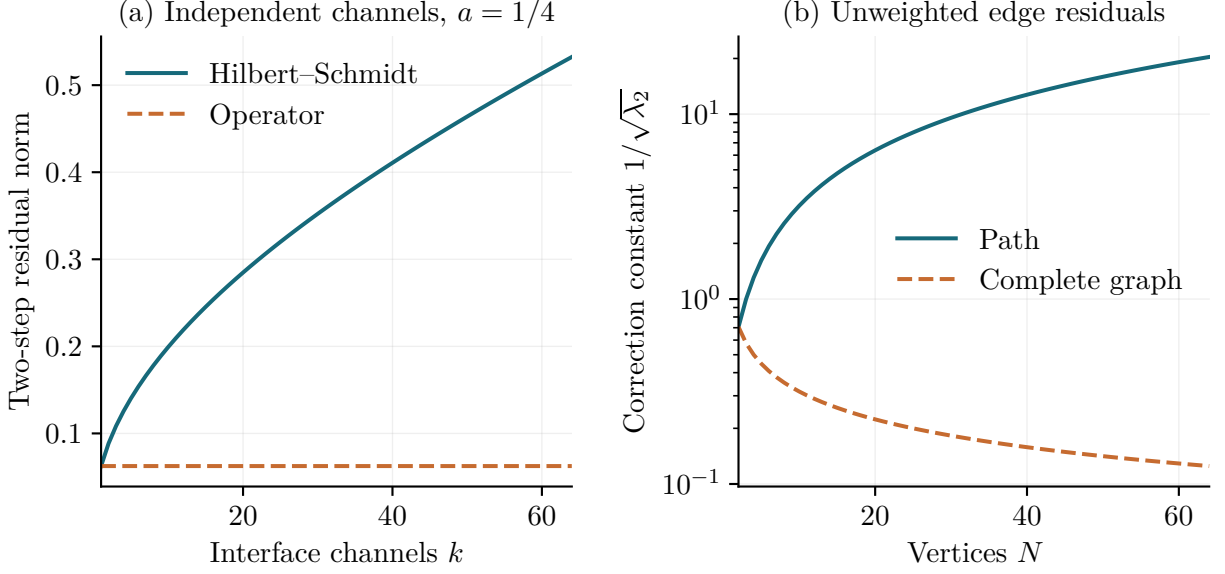


Figure 1: Dependence of the error bounds on the number of interface channels and comparison vertices. (a) For k independent copies of the example with $S_1^2 = S_1$ and $a = 1/4$, the operator norm of the two-step residual remains $1/16$, while its Hilbert–Schmidt norm is $\sqrt{(1 + 1/256)^k - 1}$. The retained dimension is 2^k . (b) For a connected diagram with common unitary identifications and unweighted edges, the linear correction constant is $1/\sqrt{\lambda_2}$. A path has $\lambda_2 = 2(1 - \cos(\pi/N))$, and the complete graph has $\lambda_2 = N$. The residual norm uses the sum of squared edge residuals; replacing the sum by its mean multiplies the correction constant by the square root of the number of edges.

7.2 Dependence on tests and interfaces

Figure 1 shows how interface size and the choice of comparisons affect the estimates. In the independent-channel example, the operator-norm residual is constant and the Hilbert–Schmidt residual grows with the number of channels. Although the schedule constant $C(\pi)$ is independent of dimension, a uniform error bound also requires control of the measured residual in the chosen norm. For general assemblies, partial traces contribute factors $\sqrt{d}$, and the intermediate kernel norms enter the weights in (5.3).

Under the unitary-identification hypothesis of Proposition 5.4, the linear correction constant is $1/\sqrt{\lambda_2(L_G)}$. On an unweighted path it grows asymptotically as N/π ; on a complete graph it equals $N^{-1/2}$. These constants use the sum of squared edge residuals. Replacing that sum by its average multiplies the correction constant by the square root of the number of edges. The comparison norm and graph must therefore be specified together.

For a sequence of self-adjoint contractions and retained spaces, let $\delta_{p,j}$ be the two-step residual norm and let n_j be the number of elementary steps. The condition $n_j^2 \delta_{p,j} \rightarrow 0$ implies that projection after every step approximates exact blocking through length n_j in Schatten p norm. A continuum construction would also require compatible boundary embeddings and convergence of the resulting amplitudes to a specified limit.

7.3 Four-dimensional spin-foam amplitudes

The finite-group calculations suggest a further application to hypercuboidal and curvature-restricted spin-foam amplitudes [17, 18]. At a fixed spin cutoff, one could choose a boundary basis and inner product, express the coarse-to-fine embeddings as matrices, and compare subdivisions of a coarse block along two coordinate directions. The face and edge normalizations would

determine the interface contractions, while a fixed ansatz and a measurement map injective on that ansatz would determine the retained fit.

The required quantities are the directly blocked boundary kernel, the kernel assembled from separately truncated subblocks, and the kernels obtained from the two subdivision orders. Their differences would measure the effects of intermediate truncation and subdivision order on the same external states. Matrix elements beyond those used to fit one-step amplitudes would test the return contribution identified in the finite-group examples. This proposed calculation would require estimates of truncation tails, fit singular values, and contraction norms at each cutoff. The general identity (2.6) and the assembly estimate apply to transfer operators without self-adjointness; the sharper bounds of Theorem 2.1 require its stated contraction and self-adjointness assumptions.

7.4 Reproducible calculations

The accompanying program `anc/code/reproduce.py` constructs finite-group transfer matrices from their configuration-space weights, performs the projections, and checks the formulas using exact rational arithmetic. It checks BF normalization and gluing for $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the nonabelian group S_3 , including a disk triangulated around one interior vertex and a sphere obtained by gluing two disks. There are 400 BF equalities in this enumeration.

The file `anc/data/matched_models.csv` records the exact one-step coefficients, two-step coefficients, and residuals of the two positive-semidefinite models in Proposition 4.3. For each of the two nonzero-residual transfer models, the program evaluates all 1,023 ordered partitions of lengths one through ten and compares their projected matrix products with the exact compressed block. It also checks 36 pairs (m, n) with $1 \leq m, n \leq 6$. CSV files contain the exact residuals and schedule bounds. Independent-channel formulas are evaluated through 64 channels, with direct tensor-matrix checks for the first seven; the gaps of both path and complete-graph Laplacians are checked numerically through 64 vertices. Numerical eigenvalue computations and plotting use NumPy and Matplotlib, while the central state sums and schedule equalities use rational numbers. Fifteen exact matrix evaluations check the example establishing optimality of the constant at decreasing positive parameters. The recorded software versions and execution results are in `anc/data/verification.json`.

8 Conclusion

The two-step compression residual measures the contribution of discarded modes that return to the retained space. For self-adjoint contractions it bounds the error of every finite blocking schedule, with an optimal constant determined by the block lengths, and it vanishes exactly when the retained subspace is invariant. Finite-group gauge kernels give exact examples in which one-step retained data agree while the two-step residuals differ. They also exhibit a fixed projected operator with nonzero residual; an idempotent compression of a positive contraction has zero residual.

For finite comparison diagrams, the assembly and linear correction estimates identify the norms and singular values that govern error propagation. Compatible amplitude families extend to colimits and descend to quotients by invariant subspaces contained in their kernels. A measurement matrix of full column rank makes the normalized least-squares fit unique. Four-dimensional applications require the corresponding spin-foam kernels and truncation estimates.

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Data and code availability. The accompanying source package contains the reproduction program, generated CSV data, computational results, and figure. The calculations can be regenerated using the instructions in `README.txt`.

Author contribution. Zhenyu Sun contributed to conceptualization, methodology, formal analysis, investigation, writing the original draft, revision, and funding acquisition.

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