

A TRUNCATED THETA-TYPE SERIES WITH TRIANGULAR WEIGHTS

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ABSTRACT. Merca (2026) recently introduced the two-block odd partition function $a(n)$, proved that its generating function factors as $A(q) = P(q)S(q)$, where $P(q)$ is the generating function of partitions whose odd parts are distinct and whose even parts are unrestricted and $S(q)$ is a theta-type series with triangular-number weights, and conjectured that for $N \equiv 1, 2 \pmod{4}$ the truncation $P(q)S_N(q)$ of this product has nonnegative coefficients, positive from q^3 onward. We prove the conjecture in the stronger form $P(q)S_N(q) - A(q) \in \mathbb{N}[[q]]$.

1. INTRODUCTION

Recently Merca [Mer26] defined the *two-block odd partition function* $a(n)$ as the signed number of partitions of n into exactly two distinct part sizes, each occurring an odd number of times: $a(n)$ counts the representations $n = (2r_1 + 1)x + (2r_2 + 1)y$ with $1 \leq x < y$, with sign $(-1)^{r_1 + r_2}$. Its generating function is the double Lambert series

$$A(q) := \sum_{n \geq 1} a(n)q^n = \sum_{1 \leq n_1 < n_2} \frac{q^{n_1}}{1 + q^{2n_1}} \frac{q^{n_2}}{1 + q^{2n_2}}. \quad (1.1)$$

We set some notations first. Throughout, q is a formal variable, $(a; q)_\infty = \prod_{i \geq 0} (1 - aq^i)$, and $\mathbb{N}[[q]]$ denotes the set of formal power series with nonnegative integer coefficients. For a series $X(q) = \sum_n x(n)q^n$ we write $[q^n]X = x(n)$, and for two series we write $X \geq Y$ if $X - Y \in \mathbb{N}[[q]]$. We write $T_j = j(j+1)/2$ for the triangular numbers. Let χ_4 be the nontrivial character modulo 4, $\lambda(n) = \sum_{d|n} \chi_4(d)$, $\sigma(n) = \sum_{d|n} d$, with $\sigma(x) = 0$ if $x \notin \mathbb{Z}_{>0}$.

Among other things, Merca [Mer26] proved the following results.

Theorem 1.1 ([Mer26, Theorem 1.1 and Corollary 1.2]). *For every $n \geq 1$,*

$$a(n) = \frac{\sigma(n) - \lambda(n)}{4} - \frac{\sigma(n/2)}{2} + \sigma(n/4). \quad (1.2)$$

Consequently $a(n) \geq 0$ for all $n \geq 1$, with equality if and only if $n \in \{1, 2\}$.

Theorem 1.2 ([Mer26, Theorem 1.3]). *Let*

$$P(q) := \sum_{n \geq 0} \text{pod}(n)q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{(q^2; q^4)_\infty}{(q; q)_\infty} \quad (1.3)$$

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be the generating function of the partitions in which odd parts are distinct and even parts are unrestricted, and let

$$S(q) := \sum_{r \geq 1} (-1)^{T_{r-1}} T_{\lceil r/2 \rceil} q^{T_{r+1}} = q^3 - q^6 - 3q^{10} + 3q^{15} + 6q^{21} - 6q^{28} - 10q^{36} + 10q^{45} + \dots \quad (1.4)$$

Then $A(q) = P(q)S(q)$.

For $N \geq 1$ let $S_N(q)$ be the N -th partial sum of (1.4) and let

$$F_N(q) := P(q) S_N(q) = \frac{(q^2; q^4)_\infty}{(q; q)_\infty} \sum_{r=1}^N (-1)^{T_{r-1}} T_{\lceil r/2 \rceil} q^{T_{r+1}}. \quad (1.5)$$

Since $A = PS \in \mathbb{N}[[q]]$ by Theorem 1.1, it is natural to ask whether the truncations F_N are nonnegative as well. This is what Merca conjectured.

Conjecture 1.3. [Mer26] *If $N \equiv 1, 2 \pmod{4}$, then $F_N(q)$ has nonnegative coefficients. Moreover, for all $n \geq 3$ the coefficient of q^n in $F_N(q)$ is positive.*

Our goal here is to prove this conjecture¹.

We write $R_N := S - S_N$ for the tail of the series (1.4), so that

$$F_N(q) = P(q)S_N(q) = A(q) - P(q)R_N(q). \quad (1.6)$$

The main result of this note is that the second term is itself a series with nonpositive coefficients whenever $N \equiv 1, 2 \pmod{4}$. This is the exact analogue of the phenomenon behind the truncated pentagonal number theorem of Andrews and Merca [AM12], where the tail of the pentagonal series divided by $(q; q)_\infty$ has coefficients of constant sign.

Theorem 1.4. *Let $N \equiv 1, 2 \pmod{4}$. Then*

$$-P(q)R_N(q) \in \mathbb{N}[[q]], \quad \text{equivalently} \quad P(q)S_N(q) - A(q) \in \mathbb{N}[[q]].$$

In particular Conjecture 1.3 holds, that is, we have $F_N(q) \in \mathbb{N}[[q]]$, and the coefficients of q^0, q^1, q^2 in F_N vanish, and $[q^n]F_N \geq a(n) > 0$ for every $n \geq 3$.

Theorem 1.4 is a consequence of two results which we state below, where the tail is divided by a finite product rather than by $P(q)^{-1}$. Both of these results are proved by the method of [SS26]: explicit quasi-polynomial estimates for the number of partitions into a fixed finite set of parts, an exact evaluation of the resulting alternating polynomial sums, and a case analysis whose finitely many polynomial inequalities are certified by computer algebra. This method has been applied successfully to several open problems in the recent years: see for example the work of Zhou [Zho24], Yao [Yao24], Chen and Yao [CY25] and Liu [Liu25] for a sampler of this.

Theorem 1.5. *Let $k \geq 0$ be even and $N = 2k + 1$ (that is, $N \equiv 1 \pmod{4}$). Then*

$$\frac{-R_N(q)}{(1-q)(1-q^3)(1-q^5)} \in \mathbb{N}[[q]].$$

¹We note that Jha [J26] has also announced a different proof of Conjecture 1.3. The techniques used in Jha seems to be different from those used in the present work.

Theorem 1.6. *Let $k \geq 0$ be even and $N = 2k + 2$ (that is, $N \equiv 2 \pmod{4}$). Then*

$$\frac{-R_N(q)}{(1-q)(1-q^3)(1-q^4)(1-q^5)(1-q^7)} \in \mathbb{N}[[q]].$$

The rest of the paper is organized as follows: Section 2 rewrites S , its truncations and its tails in a convenient paired form, records the elementary factorisation of $P(q)$ that we need, and proves the reduction of Theorem 1.4 to Theorems 1.5 and 1.6. Section 3 collects all the necessary tools that we need: quasi-polynomials for partitions into the parts of $\{1, 3, 5\}$ and of $\{1, 3, 4, 5, 7\}$, an exact summation formula for alternating sums of polynomials, and two elementary positivity criteria for polynomials on an interval. Sections 4 and 5 prove Theorems 1.5 and 1.6 respectively. The explicit polynomials needed in Section 4 are displayed there; those needed in Section 5 are given in Appendix A.

2. PRELIMINARY RESULTS AND A REDUCTION

We use the following notation throughout. For $m \geq 1$ put

$$a_m := m(2m + 1) = T_{2m}, \quad b_m := (m + 1)(2m + 1) = T_{2m+1}, \quad (2.1)$$

so that

$$b_m - a_m = 2m + 1, \quad a_{m+1} - b_m = 2m + 2, \quad a_{m+1} - a_m = 4m + 3, \quad (2.2)$$

and $a_1 < b_1 < a_2 < b_2 < \dots$.

We have the following easy to prove lemmas.

Lemma 2.1. *We have*

$$S(q) = \sum_{m \geq 1} (-1)^{m-1} T_m (q^{a_m} - q^{b_m}), \quad (2.3)$$

and, for every even $k \geq 0$,

$$-R_{2k+1} = T_{k+1} q^{b_{k+1}} + \sum_{m \geq k+2} (-1)^m T_m (q^{a_m} - q^{b_m}), \quad (2.4)$$

$$-R_{2k+2} = \sum_{m \geq k+2} (-1)^m T_m (q^{a_m} - q^{b_m}) = -R_{2k+1} - T_{k+1} q^{b_{k+1}}. \quad (2.5)$$

Proof. Pair the terms $r = 2m - 1$ and $r = 2m$ in (1.4). For $r = 2m - 1$ we have $T_{r-1} = (m - 1)(2m - 1) \equiv m - 1 \pmod{2}$ and $T_{r+1} = T_{2m} = a_m$, and for $r = 2m$ we have $T_{r-1} = m(2m - 1) \equiv m \pmod{2}$ and $T_{r+1} = T_{2m+1} = b_m$; in both cases $T_{\lceil r/2 \rceil} = T_m$. This gives (2.3).

Removing the first $2k + 1$, respectively $2k + 2$, terms of (1.4) from (2.3) gives the tail formulas (2.4) and (2.5). $\square$

Lemma 2.2. *Let $D := \{m \geq 1 : m \text{ odd}\} \cup \{m \geq 1 : 4 \mid m\}$. Then*

$$P(q) = \prod_{m \in D} \frac{1}{1 - q^m}. \quad (2.6)$$

Proof. In $P(q) = \frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^4; q^4)_\infty}$ the factors $(1 - q^m)$ with m even in $(q; q)_\infty$ cancel against $(q^2; q^2)_\infty$, and what remains in the denominator is $\prod_{m \text{ odd}} (1 - q^m) \prod_{4 \mid m} (1 - q^m)$. $\square$

We now reduce proving Theorem 1.4 to proving Theorems 1.5 and 1.6.

Proof of Theorem 1.4, assuming Theorems 1.5 and 1.6. Let $S \subset D$ be a finite set. By Lemma 2.2,

$$P(q) \prod_{s \in S} (1 - q^s) = \prod_{m \in D \setminus S} \frac{1}{1 - q^m} \in \mathbb{N}[[q]], \quad (2.7)$$

and hence

$$-P(q)R_N(q) = \left(P(q) \prod_{s \in S} (1 - q^s) \right) \cdot \frac{-R_N(q)}{\prod_{s \in S} (1 - q^s)} \in \mathbb{N}[[q]]$$

whenever the second factor lies in $\mathbb{N}[[q]]$.

For $N \equiv 1 \pmod{4}$ this is Theorem 1.5 with $S = \{1, 3, 5\} \subset D$, and for $N \equiv 2 \pmod{4}$ it is Theorem 1.6 with $S = \{1, 3, 4, 5, 7\} \subset D$ (note that $4 \in D$). This proves $-P(q)R_N(q) \in \mathbb{N}[[q]]$, that is, $P(q)S_N(q) - A(q) \in \mathbb{N}[[q]]$ by (1.6).

The remaining assertions follow: $[q^n]F_N \geq a(n)$ for all n , which is positive for $n \geq 3$ by Theorem 1.1, and $[q^n]F_N = 0$ for $n \leq 2$ because S_N has no term of degree less than 3. $\square$

Results such as Theorem 1.4 comes with some partition theoretic linear inequalities as easy corollaries. We mention the one associated with Theorem 1.4 giving us linear inequalities for $\text{pod}(n)$ below.

Corollary 2.3. *For every even $k \geq 0$ and every $n \geq 1$,*

$$\begin{aligned} \sum_{m=1}^k (-1)^{m-1} T_m (\text{pod}(n - a_m) - \text{pod}(n - b_m)) + T_{k+1} \text{pod}(n - a_{k+1}) &\geq a(n), \\ \sum_{m=1}^{k+1} (-1)^{m-1} T_m (\text{pod}(n - a_m) - \text{pod}(n - b_m)) &\geq a(n), \end{aligned}$$

where $\text{pod}(x) = 0$ for $x < 0$.

Proof. These are the coefficient inequalities $[q^n](PS_N) \geq a(n)$ for $N = 2k + 1$ and $N = 2k + 2$, where S_N is written out using the paired form (2.3) of S . $\square$

3. THE TOOLKIT

In this section, we collect results which are essential for the proofs of Theorems 1.5 and 1.6.

For a finite set S of positive integers let $\alpha_S(n)$ denote the number of partitions of n into parts belonging to S , with $\alpha_S(n) = 0$ for $n < 0$, so that $\sum_{n \geq 0} \alpha_S(n) q^n = \prod_{s \in S} (1 - q^s)^{-1}$. We need two instances of S for our results.

Lemma 3.1 ([SS26, Lemma 2.1]). *Let $o_3(n) := \alpha_{\{1, 3, 5\}}(n)$ and $Q(x) := (2x + 9)^2/120$. Then for all $n \geq 0$*

$$o_3(n) = Q(n) + \varepsilon(n), \quad -\frac{49}{120} \leq \varepsilon(n) \leq \frac{39}{120} = \frac{13}{40}, \quad (3.1)$$

where ε is periodic with period 15.

The next lemma is a particular instance of what Liu has observed [Liu25, Eq. (3.7)]. The proof is similar to other such results in the literature, so we only sketch the essential parts here.

Lemma 3.2. *Let $\alpha(n) := \alpha_{\{1,3,4,5,7\}}(n)$ and*

$$P_4(x) := \frac{x^4}{10080} + \frac{x^3}{252} + \frac{55x^2}{1008} + \frac{25x}{84} + \frac{157091}{302400}. \quad (3.2)$$

Then for all $n \geq 0$

$$\alpha(n) = P_4(n) + \eta(n), \quad \eta_{\min} := -\frac{110771}{302400} \leq \eta(n) \leq \frac{145309}{302400} =: \eta_{\max}, \quad (3.3)$$

where η is periodic with period 420, and both bounds are attained. We put

$$E := \eta_{\max} - \eta_{\min} = \frac{256080}{302400} = \frac{1067}{1260}.$$

Proof. The parts 1, 3, 4, 5, 7 are pairwise coprime, so the five-part case of the classical quasi-polynomial theorem of Pólya and Szegő [PS98, Part I, Problem 27.1] applies, in the explicit form given by Liu [Liu25, Eq. (3.7)] and Liu and Liao [LL25, Lemma 3.2]. For the parts 1, 3, 4, 5, 7 it reads

$$\alpha(n) = \frac{n^4 + 40n^3 + 550n^2 + 3000n}{10080} + t(n), \quad t(n+420) = t(n),$$

so that (3.3) holds with $\eta(n) := t(n) - \frac{157091}{302400}$, which is periodic with period 420. Its extreme values can therefore be read off from one period, exactly as in the proofs of [CY25, Lemmas 2.2, 3.1 and 3.2]: evaluating $\eta(n) = \alpha(n) - P_4(n)$ for $0 \leq n \leq 419$ gives

$$\begin{aligned} \max_n \eta(n) &= \eta(0) = 1 - \frac{157091}{302400} = \frac{145309}{302400}, \\ \min_n \eta(n) &= \eta(2) = 1 - \frac{8536}{10080} - \frac{157091}{302400} = -\frac{110771}{302400}, \end{aligned}$$

where we used $\alpha(0) = \alpha(2) = 1$. □

We need the following fact about polynomials over $\mathbb{Q}$.

Lemma 3.3. *Let f be a polynomial in m , then there exists a unique polynomial g of the same degree with*

$$g(m) + g(m+1) = f(m), \quad (3.4)$$

and for all integers $A \leq B$, we have

$$\sum_{m=A}^B (-1)^m f(m) = (-1)^A g(A) + (-1)^B g(B+1). \quad (3.5)$$

Proof. The linear map $g \mapsto g(m) + g(m+1)$ on polynomials of degree $\leq d$ is injective (its leading coefficient is 2 times that of g), hence bijective, which gives existence and uniqueness. For (3.5),

$$\begin{aligned} \sum_{m=A}^B (-1)^m (g(m) + g(m+1)) &= \sum_{m=A}^B (-1)^m g(m) - \sum_{m=A+1}^{B+1} (-1)^m g(m) \\ &= (-1)^A g(A) - (-1)^{B+1} g(B+1). \end{aligned} \quad \square$$

We shall also use the following elementary identity:

$$\sum_{m=A}^B T_m = \binom{B+2}{3} - \binom{A+1}{3}. \quad (3.6)$$

Lemma 3.4. Let $\varphi(u) = A_2u^2 + A_1u + A_0$ with $A_2 > 0$, and let $L > 0$. If

$$\varphi(0) \geq 0, \quad \varphi(L) \geq 0, \quad A_0 - A_2L^2 \geq 0,$$

then $\varphi(u) \geq 0$ for all $u \in [0, L]$. The same conclusion holds if $\varphi(0) \geq 0$ and $A_1 \geq 0$.

Proof. Put $u^* := -A_1/(2A_2)$. Completing the square gives us

$$\varphi(u) = A_2(u - u^*)^2 + \varphi(u^*), \quad \varphi(u^*) = A_0 - \frac{A_1^2}{4A_2},$$

so φ is decreasing on $(-\infty, u^*]$ and increasing on $[u^*, \infty)$. We distinguish three cases according to the position of u^* .

Case $u^* \leq 0$. Then φ is increasing on $[0, L]$, so $\varphi(u) \geq \varphi(0) \geq 0$ for $u \in [0, L]$.

Case $u^* \geq L$. Then φ is decreasing on $[0, L]$, so $\varphi(u) \geq \varphi(L) \geq 0$ for $u \in [0, L]$.

Case $0 < u^* < L$. Then the minimum of φ on $[0, L]$ is the global minimum $\varphi(u^*)$. Since $A_2 > 0$ and $0 < u^* < L$, we have $|A_1| = 2A_2u^* < 2A_2L$, hence $A_1^2/(4A_2) < A_2L^2$ and

$$\varphi(u) \geq \varphi(u^*) = A_0 - \frac{A_1^2}{4A_2} > A_0 - A_2L^2 \geq 0 \quad (u \in [0, L]).$$

This proves the first assertion.

For the second, note that $A_1 \geq 0$ is equivalent to $u^* \leq 0$, so we are in the first case and $\varphi(u) \geq \varphi(0) \geq 0$ for all $u \geq 0$. $\square$

Lemma 3.5. Let $p(u) = \sum_{j=0}^d c_j u^j$ be a polynomial of degree at most d and let $L > 0$. Define

$$\beta_i := \sum_{j=0}^i \frac{\binom{i}{j}}{\binom{d}{j}} c_j L^j \quad (0 \leq i \leq d). \quad (3.7)$$

Then

$$p(Lx) = \sum_{i=0}^d \beta_i \binom{d}{i} x^i (1-x)^{d-i},$$

and consequently $\beta_0, \dots, \beta_d \geq 0$ implies $p(u) \geq 0$ for all $u \in [0, L]$.

Proof. We have

$$\begin{aligned} x^j &= x^j (x + (1-x))^{d-j} \\ &= \sum_{i=j}^d \binom{d-j}{i-j} x^i (1-x)^{d-i} \\ &= \sum_{i=j}^d \frac{\binom{i}{j}}{\binom{d}{j}} \binom{d}{i} x^i (1-x)^{d-i}, \end{aligned}$$

where in the last step, we used $\binom{d-j}{i-j} \binom{d}{j} = \binom{d}{i} \binom{i}{j}$. Multiplying the j -th identity by $c_j L^j$ and summing over $0 \leq j \leq d$, the left-hand side becomes $p(Lx)$, while on the right-hand side the coefficient of $\binom{d}{i} x^i (1-x)^{d-i}$ is $\sum_{j \leq i} \binom{i}{j} \binom{d}{j}^{-1} c_j L^j = \beta_i$; this proves, for all real x , we have

$$p(Lx) = \sum_{i=0}^d \beta_i \binom{d}{i} x^i (1-x)^{d-i}.$$

Now let $u \in [0, L]$ and put $x = u/L \in [0, 1]$. Then $x \geq 0$ and $1-x \geq 0$, so each $\binom{d}{i} x^i (1-x)^{d-i}$ is nonnegative, and if $\beta_0, \dots, \beta_d \geq 0$ then $p(u) = p(Lx) \geq 0$. $\square$

4. PROOF OF THEOREM 1.5: THE CASE $N \equiv 1 \pmod{4}$

Throughout this section, $k \geq 0$ is an even integer and $N = 2k + 1$. By (2.4) and Lemma 3.1, we can write

$$\frac{-R_N(q)}{(1-q)(1-q^3)(1-q^5)} = \sum_{n \geq 0} b_k(n) q^n, \quad (4.1)$$

where

$$b_k(n) := T_{k+1} o_3(n - b_{k+1}) + \sum_{m \geq k+2} (-1)^m T_m (o_3(n - a_m) - o_3(n - b_m)), \quad (4.2)$$

with the convention $o_3(x) = 0$ for $x < 0$. Thus, to prove Theorem 1.5, it suffices to show that $b_k(n) \geq 0$ for all integers $n \geq 0$. Note that, since o_3 is nondecreasing, the m -th summand in (4.2) is nonnegative for even m and nonpositive for odd m .

For $m_0 \geq k + 2$ we define the partial sums

$$B(n, k, m_0) := T_{k+1} o_3(n - b_{k+1}) + \sum_{m=k+2}^{m_0-1} (-1)^m T_m (o_3(n - a_m) - o_3(n - b_m)), \quad (4.3)$$

so that $B(n, k, k+2) = T_{k+1} o_3(n - b_{k+1})$. The following lemma, which is the analogue of [SS26, Lemma 2.2] and of [CY25, Lemma 2.3], gives a polynomial lower bound for these partial sums.

Lemma 4.1. *Let $k \geq 0$ be even, $m_0 \geq k + 2$ and $n \geq b_{m_0-1}$ be integers. Then*

$$B(n, k, m_0) \geq \Phi(n, k, m_0) := \Phi_0(n, k) + (m_0^3 - m_0) \left((-1)^{m_0} \frac{2m_0^2 - n - 8}{30} - \frac{11}{90} \right), \quad (4.4)$$

where

$$\begin{aligned} \Phi_0(n, k) := & \frac{(k+1)(k+2)}{60} n^2 - \left(\frac{k^4}{15} + \frac{2k^3}{5} + \frac{41k^2}{60} + \frac{k}{4} - \frac{1}{10} \right) n \\ & + \frac{k^6}{15} + \frac{3k^5}{5} + \frac{107k^4}{60} + \frac{82k^3}{45} + \frac{7k^2}{60} - \frac{k}{18} + \frac{2}{5}. \end{aligned} \quad (4.5)$$

Proof. Since $n \geq b_{m_0-1}$, all the arguments of o_3 in (4.3) are nonnegative, so we may use Lemma 3.1 for each of them. By (3.1), for all $x, y \geq 0$ we have $o_3(x) \geq Q(x) - \frac{49}{120}$ and $o_3(x) - o_3(y) \geq Q(x) - Q(y) - \frac{49}{120} - \frac{39}{120} = Q(x) - Q(y) - \frac{11}{15}$, and the same with the roles of x and y interchanged. Hence

$$\begin{aligned} B(n, k, m_0) \geq & T_{k+1} \left(Q(n - b_{k+1}) - \frac{49}{120} \right) \\ & + \sum_{m=k+2}^{m_0-1} \left((-1)^m T_m (Q(n - a_m) - Q(n - b_m)) - \frac{11}{15} T_m \right). \end{aligned} \quad (4.6)$$

A short computation using (2.1) gives

$$Q(n - a_m) - Q(n - b_m) = \frac{(b_m - a_m)(2n - a_m - b_m + 9)}{30} = \frac{(2m+1)(n - 2m^2 - 2m + 4)}{15},$$

so that $T_m(Q(n - a_m) - Q(n - b_m)) = \frac{f(m)}{30}$ with

$$f(m) := m(m+1)(2m+1)(n - 2m^2 - 2m + 4).$$

One checks directly that the polynomial $g(m) := -(m^3 - m)(2m^2 - n - 8)$ satisfies $g(m) + g(m+1) = f(m)$. Therefore, by Lemma 3.3 (recall that $k+2$ is even) and by (3.6), we have

$$\sum_{m=k+2}^{m_0-1} (-1)^m f(m) = g(k+2) + (-1)^{m_0-1} g(m_0),$$

and

$$\sum_{m=k+2}^{m_0-1} T_m = \binom{m_0+1}{3} - \binom{k+3}{3}.$$

Substituting these into (4.6), we obtain

$$\begin{aligned} B(n, k, m_0) &\geq \frac{T_{k+1}((2(n - b_{k+1}) + 9)^2 - 49)}{120} + \frac{g(k+2) - (-1)^{m_0} g(m_0)}{30} \\ &\quad - \frac{11}{15} \left(\binom{m_0+1}{3} - \binom{k+3}{3} \right) \\ &= \Phi_0(n, k) + (-1)^{m_0} \frac{(m_0^3 - m_0)(2m_0^2 - n - 8)}{30} - \frac{11}{90} (m_0^3 - m_0), \end{aligned}$$

where $\Phi_0(n, k)$ collects all the terms which do not involve m_0 , namely

$$\Phi_0(n, k) = \frac{T_{k+1}((2(n - b_{k+1}) + 9)^2 - 49)}{120} + \frac{g(k+2)}{30} + \frac{11}{15} \binom{k+3}{3}.$$

Expanding this expression gives (4.5) which proves the lemma. $\square$

We are now ready to prove Theorem 1.5. The argument follows the pattern of [SS26, Lemma 2.4] and [CY25, Section 2.1]: we locate n in one of the intervals $[a_M, a_{M+1})$, $M \geq k+2$, which partition $\{n : n \geq a_{k+2}\}$, and in each of four cases we bound $b_k(n)$ from below by an explicit polynomial using Lemma 4.1 and Lemma 3.1. The resulting polynomials are quadratic in the position of n inside the interval, and their nonnegativity is established by Lemma 3.4 after expanding them in the variables $\kappa, s \geq 0$ defined by

$$k = 2\kappa, \quad M = \begin{cases} 2\kappa + 2 + s, & M \text{ even}, \\ 2\kappa + 3 + s, & M \text{ odd}. \end{cases} \quad (4.7)$$

Proof of Theorem 1.5. We show that $b_k(n) \geq 0$ for all $n \geq 0$. If $n < b_{k+1}$, then $b_k(n) = 0$, and if $b_{k+1} \leq n < a_{k+2}$, then $b_k(n) = T_{k+1} o_3(n - b_{k+1}) \geq 0$ by (4.2). So let $n \geq a_{k+2}$. Since $a_{k+2} < a_{k+3} < \dots$, there exists a unique integer $M \geq k+2$ such that

$$a_M \leq n \leq a_{M+1} - 1 = a_M + 4M + 2,$$

and we put $t := n - a_M$, so that $0 \leq t \leq 4M + 2$. By (4.2) and (4.3), all the terms with $m \geq M+1$ vanish, and

$$b_k(n) = \begin{cases} B(n, k, M) + (-1)^M T_M o_3(t), & 0 \leq t \leq 2M \text{ (i.e. } a_M \leq n < b_M), \\ B(n, k, M+1), & 2M+1 \leq t \leq 4M+2 \text{ (i.e. } b_M \leq n < a_{M+1}), \end{cases} \quad (4.8)$$

because $o_3(n - b_M) = 0$ in the first range. Note that $n \geq a_M \geq b_{M-1} + 2M$ and, in the second range, $n \geq b_M$, so Lemma 4.1 is applicable to $B(n, k, M)$ and to $B(n, k, M+1)$ respectively.

We now consider four cases.

Case 1. $a_M \leq n \leq b_M - 1$ and M is even.

Write $M = 2\kappa + 2 + s$ as in (4.7). By (4.8), Lemma 4.1 and (3.1),

$$b_k(n) = B(n, k, M) + T_M o_3(t) \geq \Phi(n, k, M) + T_M \left(Q(t) - \frac{49}{120} \right) =: A_2 t^2 + A_1 t + A_0,$$

where, after substituting $n = a_M + t$ and (4.7),

$$\begin{aligned} 60 A_2 &= 8\kappa^2 + 4\kappa s + 16\kappa + s^2 + 5s + 8, \\ 60 A_1 &= 64\kappa^3 s + 32\kappa^3 + 16\kappa^2 s^2 + 144\kappa^2 s + 152\kappa^2 + 12\kappa s^2 + 128\kappa s \\ &\quad + 208\kappa - 2s^3 + 5s^2 + 59s + 88, \\ 90 A_0 &= 384\kappa^4 s^2 + 576\kappa^4 s + 96\kappa^4 + 192\kappa^3 s^3 + 1584\kappa^3 s^2 + 2520\kappa^3 s \\ &\quad + 552\kappa^3 + 24\kappa^2 s^4 + 504\kappa^2 s^3 + 2394\kappa^2 s^2 + 3450\kappa^2 s + 1068\kappa^2 \\ &\quad + 36\kappa s^4 + 396\kappa s^3 + 1293\kappa s^2 + 1671\kappa s + 864\kappa + 9s^4 + 49s^3 \\ &\quad + 114s^2 + 212s + 252. \end{aligned} \tag{4.9}$$

Clearly $A_2 > 0$ and $A_0 \geq 0$. We apply Lemma 3.4 on the interval $[0, L]$ with $L = 2M = 4\kappa + 2s + 4$. The value at $t = L$ is

$$\begin{aligned} 90 (A_2 L^2 + A_1 L + A_0) &= 384\kappa^4 s^2 + 960\kappa^4 s + 480\kappa^4 + 192\kappa^3 s^3 + 1872\kappa^3 s^2 + 4152\kappa^3 s \\ &\quad + 2424\kappa^3 + 24\kappa^2 s^4 + 552\kappa^2 s^3 + 3162\kappa^2 s^2 + 6426\kappa^2 s + 4380\kappa^2 \\ &\quad + 36\kappa s^4 + 468\kappa s^3 + 2139\kappa s^2 + 4329\kappa s + 3408\kappa + 9s^4 + 106s^3 \\ &\quad + 513s^2 + 1142s + 972 \geq 0, \end{aligned}$$

and

$$\begin{aligned} 90 (A_0 - A_2 L^2) &= 384\kappa^4 s^2 + 576\kappa^4 s - 96\kappa^4 + 192\kappa^3 s^3 + 1584\kappa^3 s^2 + 2232\kappa^3 s \\ &\quad - 216\kappa^3 + 24\kappa^2 s^4 + 504\kappa^2 s^3 + 2226\kappa^2 s^2 + 2562\kappa^2 s - 84\kappa^2 \\ &\quad + 36\kappa s^4 + 348\kappa s^3 + 933\kappa s^2 + 759\kappa s + 96\kappa + 3s^4 - 5s^3 - 78s^2 \\ &\quad - 100s + 60. \end{aligned} \tag{4.10}$$

We distinguish three sub-cases.

- If $s = 0$, then by (4.9) $A_1 = \frac{2}{15}(\kappa + 1)^2(4\kappa + 11) > 0$, and the last assertion of Lemma 3.4 gives $b_k(n) \geq A_0 \geq 0$.
- If $\kappa \geq 1$ and $s \geq 1$, we replace κ by $\kappa + 1$ and s by $s + 1$ in (4.10); the resulting polynomial in $\kappa, s \geq 0$ has only nonnegative coefficients, so $A_0 - A_2 L^2 \geq 0$ and Lemma 3.4 gives $b_k(n) \geq 0$.
- If $\kappa = 0$ and $s \geq 1$, then $60A_1 = -2s^3 + 5s^2 + 59s + 88$, which is positive for $1 \leq s \leq 7$ (its value at $s = 7$ is 60), so again $b_k(n) \geq A_0 \geq 0$ for these s . For $s \geq 8$, (4.10) reduces to $3s^4 - 5s^3 - 78s^2 - 100s + 60$, and replacing s by $s + 8$ gives $3s^4 + 91s^3 + 954s^2 + 3836s + 3996 > 0$; hence $A_0 - A_2 L^2 \geq 0$ and Lemma 3.4 applies.

Thus $b_k(n) \geq 0$ in Case 1.

Case 2. $a_M \leq n \leq b_M - 1$ and M is odd.

Write $M = 2\kappa + 3 + s$ as in (4.7). By (4.8), Lemma 4.1 and (3.1),

$$b_k(n) = B(n, k, M) - T_M o_3(t) \geq \Phi(n, k, M) - T_M \left(Q(t) + \frac{39}{120} \right) =: \psi(t).$$

Here ψ is a quadratic polynomial in t with leading coefficient $-\frac{1}{60}(4\kappa s + 8\kappa + s^2 + 7s + 10) < 0$, so ψ is concave and $\psi(t) \geq \min\{\psi(0), \psi(2M)\}$ for $0 \leq t \leq 2M$. Now

$$\begin{aligned} 90 \psi(0) &= 384\kappa^4 s^2 + 1344\kappa^4 s + 1152\kappa^4 + 192\kappa^3 s^3 + 2160\kappa^3 s^2 \\ &\quad + 6456\kappa^3 s + 5808\kappa^3 + 24\kappa^2 s^4 + 600\kappa^2 s^3 + 4194\kappa^2 s^2 \\ &\quad + 11286\kappa^2 s + 10212\kappa^2 + 36\kappa s^4 + 588\kappa s^3 + 3417\kappa s^2 + 8361\kappa s \\ &\quad + 7182\kappa + 15s^4 + 205s^3 + 1008s^2 + 2084s + 1536, \\ 90 \psi(2M) &= 384\kappa^4 s^2 + 1728\kappa^4 s + 1920\kappa^4 + 192\kappa^3 s^3 + 2448\kappa^3 s^2 \\ &\quad + 8664\kappa^3 s + 9072\kappa^3 + 24\kappa^2 s^4 + 648\kappa^2 s^3 + 5106\kappa^2 s^2 \\ &\quad + 15150\kappa^2 s + 14676\kappa^2 + 36\kappa s^4 + 660\kappa s^3 + 4083\kappa s^2 + 10287\kappa s \\ &\quad + 8946\kappa + 15s^4 + 196s^3 + 918s^2 + 1805s + 1266, \end{aligned}$$

both of which are clearly positive. Thus $b_k(n) \geq 0$ in Case 2.

Case 3. $b_M \leq n \leq a_{M+1} - 1$ and M is even.

Write $M = 2\kappa + 2 + s$ and $t = 2M + 1 + u$, so that $0 \leq u \leq 2M + 1$. By (4.8) and Lemma 4.1 (applied with $m_0 = M + 1$, which is odd),

$$\begin{aligned} 90 b_k(n) &\geq 90 \Phi(n, k, M + 1) \\ &= 384\kappa^4 s^2 + 960\kappa^4 s + 480\kappa^4 + 192\kappa^3 s^3 + 1872\kappa^3 s^2 + 96\kappa^3 s u \\ &\quad + 4248\kappa^3 s + 144\kappa^3 u + 2568\kappa^3 + 24\kappa^2 s^4 + 552\kappa^2 s^3 + 24\kappa^2 s^2 u \\ &\quad + 3186\kappa^2 s^2 + 312\kappa^2 s u + 6738\kappa^2 s + 6\kappa^2 u^2 + 486\kappa^2 u + 4728\kappa^2 \\ &\quad + 36\kappa s^4 + 468\kappa s^3 + 54\kappa s^2 u + 2193\kappa s^2 + 354\kappa s u + 4551\kappa s \\ &\quad + 9\kappa u^2 + 513\kappa u + 3582\kappa + 9s^4 + 3s^3 u + 109s^3 + 39s^2 u + 519s^2 \\ &\quad + 144s u + 1121s + 3u^2 + 171u + 942, \end{aligned}$$

which is clearly positive for all $\kappa, s, u \geq 0$. Thus $b_k(n) \geq 0$ in Case 3.

Case 4. $b_M \leq n \leq a_{M+1} - 1$ and M is odd.

Write $M = 2\kappa + 3 + s$ and $t = 2M + 1 + u$ with $0 \leq u \leq L := 2M + 1 = 4\kappa + 2s + 7$. By (4.8) and Lemma 4.1 (applied with $m_0 = M + 1$, which is even),

$$b_k(n) \geq \Phi(n, k, M + 1) =: A_2 u^2 + A_1 u + A_0,$$

where

$$\begin{aligned}
30 A_2 &= (\kappa + 1)(2\kappa + 1), \\
30 A_1 &= 32\kappa^3 s + 64\kappa^3 + 8\kappa^2 s^2 + 96\kappa^2 s + 178\kappa^2 + 6\kappa s^2 + 58\kappa s + 119\kappa - s^3 \\
&\quad - 8s^2 - 17s - 1, \\
90 A_0 &= 384\kappa^4 s^2 + 1728\kappa^4 s + 1920\kappa^4 + 192\kappa^3 s^3 + 2448\kappa^3 s^2 \\
&\quad + 8760\kappa^3 s + 9264\kappa^3 + 24\kappa^2 s^4 + 648\kappa^2 s^3 + 5130\kappa^2 s^2 \\
&\quad + 15438\kappa^2 s + 15204\kappa^2 + 36\kappa s^4 + 660\kappa s^3 + 4101\kappa s^2 + 10461\kappa s \\
&\quad + 9294\kappa + 15s^4 + 193s^3 + 894s^2 + 1754s + 1260.
\end{aligned}$$

Note that A_1 may be negative (for $\kappa = 0$ it equals $-(s^3 + 8s^2 + 17s + 1)/30$), so the minimum of this quadratic over the real line need not be nonnegative, and we use Lemma 3.4 instead. We have $A_2 > 0$, $A_0 > 0$,

$$\begin{aligned}
90 (A_2 L^2 + A_1 L + A_0) &= 384\kappa^4 s^2 + 2112\kappa^4 s + 2784\kappa^4 + 192\kappa^3 s^3 + 2736\kappa^3 s^2 \\
&\quad + 11064\kappa^3 s + 13224\kappa^3 + 24\kappa^2 s^4 + 696\kappa^2 s^3 + 5970\kappa^2 s^2 \\
&\quad + 19530\kappa^2 s + 21216\kappa^2 + 36\kappa s^4 + 684\kappa s^3 + 4515\kappa s^2 + 12489\kappa s \\
&\quad + 12390\kappa + 9s^4 + 124s^3 + 636s^2 + 1475s + 1386 > 0,
\end{aligned}$$

and

$$\begin{aligned}
90 (A_0 - A_2 L^2) &= 384\kappa^4 s^2 + 1728\kappa^4 s + 1824\kappa^4 + 192\kappa^3 s^3 + 2448\kappa^3 s^2 \\
&\quad + 8664\kappa^3 s + 8784\kappa^3 + 24\kappa^2 s^4 + 648\kappa^2 s^3 + 5106\kappa^2 s^2 \\
&\quad + 15126\kappa^2 s + 14358\kappa^2 + 36\kappa s^4 + 660\kappa s^3 + 4065\kappa s^2 + 10161\kappa s \\
&\quad + 8685\kappa + 15s^4 + 193s^3 + 882s^2 + 1670s + 1113 > 0.
\end{aligned}$$

By Lemma 3.4, $b_k(n) \geq 0$ for $0 \leq u \leq L$, which is Case 4.

We have shown that $b_k(n) \geq 0$ for all $n \geq 0$, and Theorem 1.5 now follows from (4.1). $\square$

5. PROOF OF THEOREM 1.6: THE CASE $N \equiv 2 \pmod{4}$

Throughout this section, $k \geq 0$ is an even integer, $N = 2k + 2$, and $\alpha = \alpha_{\{1,3,4,5,7\}}$ is as in Lemma 3.2. By (2.5),

$$\frac{-R_N(q)}{(1-q)(1-q^3)(1-q^4)(1-q^5)(1-q^7)} = \sum_{n \geq 0} c_k(n) q^n, \quad (5.1)$$

where

$$c_k(n) := \sum_{m \geq k+2} (-1)^m T_m (\alpha(n - a_m) - \alpha(n - b_m)), \quad (5.2)$$

with $\alpha(x) = 0$ for $x < 0$, and it suffices to show that $c_k(n) \geq 0$ for all $n \geq 0$. The proof runs parallel to that of Theorem 1.5, with Lemma 3.2 in place of Lemma 3.1: there is no leading term, the main terms are now of degree 4 in n , and the final positivity check uses Lemma 3.5 instead of Lemma 3.4.

For $m_0 \geq k + 2$ let

$$C(n, k, m_0) := \sum_{m=k+2}^{m_0-1} (-1)^m T_m \left(\alpha(n - a_m) - \alpha(n - b_m) \right), \quad (5.3)$$

an empty sum when $m_0 = k + 2$. Recall $E = \eta_{\max} - \eta_{\min} = \frac{1067}{1260}$ from Lemma 3.2.

Lemma 5.1. *Let $k \geq 0$ be even, $m_0 \geq k + 2$ and $n \geq b_{m_0-1}$ be integers. Then*

$$C(n, k, m_0) \geq \Phi_2(n, k, m_0) := \Phi_{2,0}(n, k) + (m_0^3 - m_0) \left((-1)^{m_0} \frac{W(m_0, n)}{20160} - \frac{1067}{7560} \right), \quad (5.4)$$

where

$$W(m, n) := 32m^6 - 48m^4n - 736m^4 + 24m^2n^2 + 692m^2n + 5322m^2 - 4n^3 - 162n^2 - 2272n - 11725 \quad (5.5)$$

and

$$\begin{aligned} \Phi_{2,0}(n, k) := & \left(\frac{k^3}{5040} + \frac{k^2}{840} + \frac{11k}{5040} + \frac{1}{840} \right) n^3 \\ & - \left(\frac{k^5}{840} + \frac{k^4}{84} + \frac{43k^3}{1120} + \frac{67k^2}{1680} - \frac{5k}{672} - \frac{11}{560} \right) n^2 \\ & + \left(\frac{k^7}{420} + \frac{k^6}{30} + \frac{823k^5}{5040} + \frac{151k^4}{504} + \frac{61k^3}{5040} - \frac{229k^2}{504} - \frac{55k}{252} + \frac{17}{210} \right) n \\ & - \frac{k^9}{630} - \frac{k^8}{35} - \frac{4k^7}{21} - \frac{8k^6}{15} - \frac{3029k^5}{10080} + \frac{6359k^4}{5040} \\ & + \frac{106861k^3}{60480} + \frac{559k^2}{3360} + \frac{29501k}{60480} + \frac{9031}{10080}. \end{aligned} \quad (5.6)$$

Proof. As in the proof of Lemma 4.1, all arguments of α are nonnegative, and (3.3) gives $\pm(\alpha(x) - \alpha(y)) \geq \pm(P_4(x) - P_4(y)) - E$ for $x, y \geq 0$. Hence

$$C(n, k, m_0) \geq \sum_{m=k+2}^{m_0-1} \left((-1)^m f_2(m) - E T_m \right),$$

and

$$f_2(m) := T_m \left(P_4(n - a_m) - P_4(n - b_m) \right).$$

Here f_2 is a polynomial of degree 9 in m (and of degree 3 in n , the terms in n^4 cancelling), and one verifies that the polynomial

$$g_2(m) := -\frac{(m^3 - m) W(m, n)}{20160}$$

satisfies $g_2(m) + g_2(m+1) = f_2(m)$. By Lemma 3.3 and (3.6),

$$\begin{aligned} C(n, k, m_0) & \geq g_2(k+2) + (-1)^{m_0-1} g_2(m_0) - E \left(\binom{m_0+1}{3} - \binom{k+3}{3} \right) \\ & = \Phi_{2,0}(n, k) + (-1)^{m_0} \frac{(m_0^3 - m_0) W(m_0, n)}{20160} - \frac{1067}{7560} (m_0^3 - m_0), \end{aligned}$$

where $\Phi_{2,0}(n, k) = g_2(k+2) + E \binom{k+3}{3}$ collects the terms free of m_0 ; expanding gives (5.6). $\square$

Proof of Theorem 1.6. If $n < a_{k+2}$, then $c_k(n) = 0$. For $n \geq a_{k+2}$ let $M \geq k + 2$ be the unique integer with $a_M \leq n \leq a_{M+1} - 1$ and put $t := n - a_M \in [0, 4M + 2]$. Exactly as in (4.8),

$$c_k(n) = \begin{cases} C(n, k, M) + (-1)^M T_M \alpha(t), & 0 \leq t \leq 2M, \\ C(n, k, M + 1), & 2M + 1 \leq t \leq 4M + 2, \end{cases} \quad (5.7)$$

and Lemma 5.1 applies to the partial sums occurring here.

We use the substitution (4.7) and again consider four cases; in each case we obtain, from (5.7), Lemma 5.1 and (3.3), a lower bound $c_k(n) \geq p(u)$ by a polynomial p of degree at most 4 in a variable $u \in [0, L]$, whose coefficients are polynomials in κ and s :

- **Case 1.** $a_M \leq n < b_M$, $M = 2\kappa + 2 + s$ even: $u = t$, $L = 2M$, and $p(u) = \Phi_2(a_M + u, k, M) + T_M(P_4(u) + \eta_{\min})$.
- **Case 2.** $a_M \leq n < b_M$, $M = 2\kappa + 3 + s$ odd: $u = t$, $L = 2M$, and $p(u) = \Phi_2(a_M + u, k, M) - T_M(P_4(u) + \eta_{\max})$.
- **Case 3.** $b_M \leq n < a_{M+1}$, $M = 2\kappa + 2 + s$ even: $u = t - 2M - 1$, $L = 2M + 1$, and $p(u) = \Phi_2(b_M + u, k, M + 1)$.
- **Case 4.** $b_M \leq n < a_{M+1}$, $M = 2\kappa + 3 + s$ odd: $u = t - 2M - 1$, $L = 2M + 1$, and $p(u) = \Phi_2(b_M + u, k, M + 1)$.

(In Cases 1 and 2 we used $T_M \alpha(t) \geq T_M(P_4(t) + \eta_{\min})$ and $-T_M \alpha(t) \geq -T_M(P_4(t) + \eta_{\max})$ respectively.) In each case we compute the five coefficients $\beta_0, \dots, \beta_4$ of p on $[0, L]$ according to (3.7) with $d = 4$; they are polynomials in (κ, s) and are listed in Appendix A.

One finds:

- In Cases 2, 3 and 4, all five β_i have nonnegative coefficients as polynomials in κ and s , hence $\beta_i \geq 0$ for all $\kappa, s \geq 0$, and Lemma 3.5 gives $c_k(n) \geq p(u) \geq 0$.
- In Case 1, all five β_i have nonnegative coefficients after replacing s by $s + 1$, hence $c_k(n) \geq 0$ whenever $s \geq 1$. If $s = 0$, then $M = k + 2$, the sum (5.3) is empty, and (5.7) gives directly $c_k(n) = T_{k+2} \alpha(t) \geq 0$.

Therefore $c_k(n) \geq 0$ for all $n \geq 0$, and Theorem 1.6 follows from (5.1). $\square$

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APPENDIX A. EXPLICIT POLYNOMIALS FOR THEOREM 1.6

The following expressions are the coefficients $\beta_0, \dots, \beta_4$ defined in (3.7) (with $d = 4$) of the lower bounds $p(u)$ on $[0, L]$ of Cases 1–4 in the proof of Theorem 1.6, as polynomials in the variables (κ, s) of (4.7); for Case 1 they are printed in the variables (κ, v) , where $s = v + 1$. Every coefficient of every polynomial below is nonnegative.

Case 1: β_0 , in the variables (κ, v) with $s = v + 1$ (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 v^3 + 184320\kappa^6 v^2 + 196608\kappa^6 v + 61440\kappa^6 + 36864\kappa^5 v^4 + 479232\kappa^5 v^3 \\
& + 1689600\kappa^5 v^2 + 2085888\kappa^5 v + 838656\kappa^5 + 9216\kappa^4 v^5 + 241920\kappa^4 v^4 \\
& + 1913856\kappa^4 v^3 + 6308160\kappa^4 v^2 + 8755392\kappa^4 v + 4128384\kappa^4 + 768\kappa^3 v^6 \\
& + 42624\kappa^3 v^5 + 623232\kappa^3 v^4 + 3926304\kappa^3 v^3 + 12362256\kappa^3 v^2 + 18407616\kappa^3 v \\
& + 9390288\kappa^3 + 2304\kappa^2 v^6 + 70272\kappa^2 v^5 + 779616\kappa^2 v^4 + 4386720\kappa^2 v^3 \\
& + 13306944\kappa^2 v^2 + 19691484\kappa^2 v + 10078140\kappa^2 + 2112\kappa v^6 + 49536\kappa v^5 \\
& + 490224\kappa v^4 + 2554776\kappa v^3 + 7033218\kappa v^2 + 9245076\kappa v + 4370154\kappa + 696v^6 \\
& + 15240v^5 + 132888v^4 + 558161v^3 + 1139085v^2 + 1011874v + 340398
\end{aligned}$$

Case 1: β_1 , in the variables (κ, v) with $s = v + 1$ (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 v^3 + 202752\kappa^6 v^2 + 242688\kappa^6 v + 89088\kappa^6 + 36864\kappa^5 v^4 + 497664\kappa^5 v^3 \\
& + 1867008\kappa^5 v^2 + 2511744\kappa^5 v + 1105536\kappa^5 + 9216\kappa^4 v^5 + 247680\kappa^4 v^4 \\
& + 2033664\kappa^4 v^3 + 7021536\kappa^4 v^2 + 10313568\kappa^4 v + 5087232\kappa^4 + 768\kappa^3 v^6 \\
& + 43200\kappa^3 v^5 + 649728\kappa^3 v^4 + 4240272\kappa^3 v^3 + 13814640\kappa^3 v^2 + 21166992\kappa^3 v \\
& + 11021328\kappa^3 + 2304\kappa^2 v^6 + 72000\kappa^2 v^5 + 825984\kappa^2 v^4 + 4779264\kappa^2 v^3 \\
& + 14733480\kappa^2 v^2 + 21998196\kappa^2 v + 11432220\kappa^2 + 2112\kappa v^6 + 51696\kappa v^5 \\
& + 523668\kappa v^4 + 2739708\kappa v^3 + 7520550\kappa v^2 + 9936240\kappa v + 4871634\kappa + 744v^6 \\
& + 16050v^5 + 135534v^4 + 550403v^3 + 1095855v^2 + 992110v + 408222
\end{aligned}$$

Case 1: β_2 , in the variables (κ, v) with $s = v + 1$ (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6v^3 + 221184\kappa^6v^2 + 294912\kappa^6v + 122880\kappa^6 + 36864\kappa^5v^4 + 516096\kappa^5v^3 \\
& + 2052096\kappa^5v^2 + 2987520\kappa^5v + 1414656\kappa^5 + 9216\kappa^4v^5 + 253440\kappa^4v^4 \\
& + 2156544\kappa^4v^3 + 7777920\kappa^4v^2 + 12012096\kappa^4v + 6164096\kappa^4 + 768\kappa^3v^6 \\
& + 43776\kappa^3v^5 + 676608\kappa^3v^4 + 4564416\kappa^3v^3 + 15328368\kappa^3v^2 + 24086944\kappa^3v \\
& + 12840592\kappa^3 + 2304\kappa^2v^6 + 73728\kappa^2v^5 + 872064\kappa^2v^4 + 5164416\kappa^2v^3 \\
& + 16139664\kappa^2v^2 + 24372492\kappa^2v + 12999132\kappa^2 + 2112\kappa v^6 + 53568\kappa v^5 \\
& + 550608\kappa v^4 + 2884336\kappa v^3 + 7928610\kappa v^2 + 10657356\kappa v + 5545050\kappa + 768v^6 \\
& + 16104v^5 + 131492v^4 + 519037v^3 + 1027137v^2 + 1005718v + 543654
\end{aligned}$$

Case 1: β_3 , in the variables (κ, v) with $s = v + 1$ (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6v^3 + 239616\kappa^6v^2 + 353280\kappa^6v + 162816\kappa^6 + 36864\kappa^5v^4 + 534528\kappa^5v^3 \\
& + 2244864\kappa^5v^2 + 3510912\kappa^5v + 1771392\kappa^5 + 9216\kappa^4v^5 + 259200\kappa^4v^4 \\
& + 2282496\kappa^4v^3 + 8572704\kappa^4v^2 + 13849440\kappa^4v + 7404288\kappa^4 + 768\kappa^3v^6 \\
& + 44352\kappa^3v^5 + 703872\kappa^3v^4 + 4895088\kappa^3v^3 + 16893264\kappa^3v^2 + 27219696\kappa^3v \\
& + 14998992\kappa^3 + 2304\kappa^2v^6 + 75456\kappa^2v^5 + 916416\kappa^2v^4 + 5535072\kappa^2v^3 \\
& + 17548824\kappa^2v^2 + 26992356\kappa^2v + 15027708\kappa^2 + 2112\kappa v^6 + 54864\kappa v^5 \\
& + 569124\kappa v^4 + 2994180\kappa v^3 + 8333862\kappa v^2 + 11627448\kappa v + 6593874\kappa + 744v^6 \\
& + 15210v^5 + 121386v^4 + 476015v^3 + 984123v^2 + 1147306v + 812790
\end{aligned}$$

Case 1: β_4 , in the variables (κ, v) with $s = v + 1$ (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6v^3 + 258048\kappa^6v^2 + 417792\kappa^6v + 211968\kappa^6 + 36864\kappa^5v^4 + 552960\kappa^5v^3 \\
& + 2445312\kappa^5v^2 + 4088832\kappa^5v + 2210304\kappa^5 + 9216\kappa^4v^5 + 264960\kappa^4v^4 \\
& + 2411520\kappa^4v^3 + 9412800\kappa^4v^2 + 15897024\kappa^4v + 8968320\kappa^4 + 768\kappa^3v^6 \\
& + 44928\kappa^3v^5 + 731520\kappa^3v^4 + 5236320\kappa^3v^3 + 18572112\kappa^3v^2 + 30847872\kappa^3v \\
& + 17889360\kappa^3 + 2304\kappa^2v^6 + 77184\kappa^2v^5 + 960480\kappa^2v^4 + 5920608\kappa^2v^3 \\
& + 19157088\kappa^2v^2 + 30398652\kappa^2v + 18051900\kappa^2 + 2112\kappa v^6 + 55872\kappa v^5 \\
& + 586416\kappa v^4 + 3132360\kappa v^3 + 8994210\kappa v^2 + 13350660\kappa v + 8400426\kappa + 696v^6 \\
& + 14088v^5 + 113040v^4 + 463529v^3 + 1089285v^2 + 1600906v + 1328382
\end{aligned}$$

Case 2: β_0 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6s^3 + 184320\kappa^6s^2 + 196608\kappa^6s + 46080\kappa^6 + 36864\kappa^5s^4 + 479232\kappa^5s^3 \\
& + 1689600\kappa^5s^2 + 2039808\kappa^5s + 582144\kappa^5 + 9216\kappa^4s^5 + 241920\kappa^4s^4 \\
& + 1913856\kappa^4s^3 + 6250560\kappa^4s^2 + 8114112\kappa^4s + 2944896\kappa^4 + 768\kappa^3s^6 \\
& + 42624\kappa^3s^5 + 623232\kappa^3s^4 + 3887904\kappa^3s^3 + 11720976\kappa^3s^2 + 16040640\kappa^3s \\
& + 7616448\kappa^3 + 2304\kappa^2s^6 + 70272\kappa^2s^5 + 765216\kappa^2s^4 + 4066080\kappa^2s^3 \\
& + 11531712\kappa^2s^2 + 17030724\kappa^2s + 10388052\kappa^2 + 2112\kappa s^6 + 46656\kappa s^5 \\
& + 410064\kappa s^4 + 1963032\kappa s^3 + 5702838\kappa s^2 + 9554988\kappa s + 6878898\kappa + 456s^6 \\
& + 7224s^5 + 58920s^4 + 336431s^3 + 1216563s^2 + 2266246s + 1637982
\end{aligned}$$

Case 2: β_1 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 202752\kappa^6 s^2 + 242688\kappa^6 s + 67584\kappa^6 + 36864\kappa^5 s^4 + 497664\kappa^5 s^3 \\
& + 1867008\kappa^5 s^2 + 2447232\kappa^5 s + 824832\kappa^5 + 9216\kappa^4 s^5 + 247680\kappa^4 s^4 \\
& + 2033664\kappa^4 s^3 + 6940896\kappa^4 s^2 + 9611808\kappa^4 s + 4040256\kappa^4 + 768\kappa^3 s^6 \\
& + 43200\kappa^3 s^5 + 649728\kappa^3 s^4 + 4186512\kappa^3 s^3 + 13112880\kappa^3 s^2 + 19073040\kappa^3 s \\
& + 10116384\kappa^3 + 2304\kappa^2 s^6 + 72000\kappa^2 s^5 + 805824\kappa^2 s^4 + 4428384\kappa^2 s^3 \\
& + 13163016\kappa^2 s^2 + 20640780\kappa^2 s + 13396068\kappa^2 + 2112\kappa s^6 + 47664\kappa s^5 \\
& + 435948\kappa s^4 + 2216220\kappa s^3 + 6841842\kappa s^2 + 11900088\kappa s + 8652474\kappa + 408s^6 \\
& + 7278s^5 + 70098s^4 + 437285s^3 + 1586817s^2 + 2882530s + 2020302
\end{aligned}$$

Case 2: β_2 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 221184\kappa^6 s^2 + 294912\kappa^6 s + 98304\kappa^6 + 36864\kappa^5 s^4 + 516096\kappa^5 s^3 \\
& + 2052096\kappa^5 s^2 + 2913792\kappa^5 s + 1158144\kappa^5 + 9216\kappa^4 s^5 + 253440\kappa^4 s^4 \\
& + 2156544\kappa^4 s^3 + 7685760\kappa^4 s^2 + 11370816\kappa^4 s + 5476864\kappa^4 + 768\kappa^3 s^6 \\
& + 43776\kappa^3 s^5 + 676608\kappa^3 s^4 + 4502976\kappa^3 s^3 + 14687088\kappa^3 s^2 + 22712480\kappa^3 s \\
& + 13239872\kappa^3 + 2304\kappa^2 s^6 + 73728\kappa^2 s^5 + 849024\kappa^2 s^4 + 4843776\kappa^2 s^3 \\
& + 15108816\kappa^2 s^2 + 24971412\kappa^2 s + 16961940\kappa^2 + 2112\kappa s^6 + 48960\kappa s^5 \\
& + 470448\kappa s^4 + 2540720\kappa s^3 + 8228070\kappa s^2 + 14620164\kappa s + 10623474\kappa + 384s^6 \\
& + 8088s^5 + 88540s^4 + 568947s^3 + 2017839s^2 + 3544930s + 2403702
\end{aligned}$$

Case 2: β_3 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 239616\kappa^6 s^2 + 353280\kappa^6 s + 141312\kappa^6 + 36864\kappa^5 s^4 + 534528\kappa^5 s^3 \\
& + 2244864\kappa^5 s^2 + 3446400\kappa^5 s + 1599744\kappa^5 + 9216\kappa^4 s^5 + 259200\kappa^4 s^4 \\
& + 2282496\kappa^4 s^3 + 8492064\kappa^4 s^2 + 13420320\kappa^4 s + 7280064\kappa^4 + 768\kappa^3 s^6 \\
& + 44352\kappa^3 s^5 + 703872\kappa^3 s^4 + 4841328\kappa^3 s^3 + 16464144\kappa^3 s^2 + 26971248\kappa^3 s \\
& + 16948896\kappa^3 + 2304\kappa^2 s^6 + 75456\kappa^2 s^5 + 896256\kappa^2 s^4 + 5320512\kappa^2 s^3 \\
& + 17362488\kappa^2 s^2 + 29917212\kappa^2 s + 20935332\kappa^2 + 2112\kappa s^6 + 50832\kappa s^5 \\
& + 515484\kappa s^4 + 2932068\kappa s^3 + 9796290\kappa s^2 + 17535072\kappa s + 12632490\kappa + 408s^6 \\
& + 9846s^5 + 113622s^4 + 719753s^3 + 2461029s^2 + 4166614s + 2729862
\end{aligned}$$

Case 2: β_4 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 258048\kappa^6 s^2 + 417792\kappa^6 s + 196608\kappa^6 + 36864\kappa^5 s^4 + 552960\kappa^5 s^3 \\
& + 2445312\kappa^5 s^2 + 4042752\kappa^5 s + 2138112\kappa^5 + 9216\kappa^4 s^5 + 264960\kappa^4 s^4 \\
& + 2411520\kappa^4 s^3 + 9355200\kappa^4 s^2 + 15716544\kappa^4 s + 9360000\kappa^4 + 768\kappa^3 s^6 \\
& + 44928\kappa^3 s^5 + 731520\kappa^3 s^4 + 5197920\kappa^3 s^3 + 18391632\kappa^3 s^2 + 31631232\kappa^3 s \\
& + 20963520\kappa^3 + 2304\kappa^2 s^6 + 77184\kappa^2 s^5 + 946080\kappa^2 s^4 + 5830368\kappa^2 s^3 \\
& + 19744608\kappa^2 s^2 + 35009892\kappa^2 s + 24880788\kappa^2 + 2112\kappa s^6 + 52992\kappa s^5 \\
& + 563856\kappa s^4 + 3328200\kappa s^3 + 11299830\kappa s^2 + 20179548\kappa s + 14341266\kappa + 456s^6 \\
& + 11832s^5 + 137520s^4 + 847799s^3 + 2796507s^2 + 4571326s + 2893806
\end{aligned}$$

Case 3: β_0 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 110592\kappa^6 s^2 + 49152\kappa^6 s + 3072\kappa^6 + 36864\kappa^5 s^4 + 405504\kappa^5 s^3 \\
& + 1026048\kappa^5 s^2 + 737280\kappa^5 s + 53760\kappa^5 + 9216\kappa^4 s^5 + 218880\kappa^4 s^4 + 1453056\kappa^4 s^3 \\
& + 3788736\kappa^4 s^2 + 3497664\kappa^4 s + 369408\kappa^4 + 768\kappa^3 s^6 + 40320\kappa^3 s^5 + 519552\kappa^3 s^4 \\
& + 2784480\kappa^3 s^3 + 7088208\kappa^3 s^2 + 7223808\kappa^3 s + 1269120\kappa^3 + 2304\kappa^2 s^6 \\
& + 63360\kappa^2 s^5 + 612576\kappa^2 s^4 + 2863968\kappa^2 s^3 + 6709728\kappa^2 s^2 + 6974148\kappa^2 s \\
& + 2124468\kappa^2 + 2112\kappa s^6 + 43200\kappa s^5 + 340944\kappa s^4 + 1325928\kappa s^3 + 2710374\kappa s^2 \\
& + 2965020\kappa s + 1639170\kappa + 696s^6 + 9816s^5 + 51528s^4 + 134399s^3 + 249555s^2 \\
& + 441334s + 465102
\end{aligned}$$

Case 3: β_1 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 129024\kappa^6 s^2 + 76800\kappa^6 s + 6144\kappa^6 + 36864\kappa^5 s^4 + 423936\kappa^5 s^3 \\
& + 1175808\kappa^5 s^2 + 981120\kappa^5 s + 99840\kappa^5 + 9216\kappa^4 s^5 + 224640\kappa^4 s^4 + 1561344\kappa^4 s^3 \\
& + 4308000\kappa^4 s^2 + 4304640\kappa^4 s + 638016\kappa^4 + 768\kappa^3 s^6 + 40896\kappa^3 s^5 + 544608\kappa^3 s^4 \\
& + 3031536\kappa^3 s^3 + 7983864\kappa^3 s^2 + 8603448\kappa^3 s + 2010720\kappa^3 + 2304\kappa^2 s^6 \\
& + 65088\kappa^2 s^5 + 648864\kappa^2 s^4 + 3105696\kappa^2 s^3 + 7502520\kappa^2 s^2 + 8373228\kappa^2 s \\
& + 3171468\kappa^2 + 2112\kappa s^6 + 44208\kappa s^5 + 353388\kappa s^4 + 1411932\kappa s^3 + 3091344\kappa s^2 \\
& + 3827370\kappa s + 2372670\kappa + 648s^6 + 8922s^5 + 47649s^4 + 143849s^3 + 354930s^2 \\
& + 696334s + 667602
\end{aligned}$$

Case 3: β_2 (multiplied by 15120):

$$\begin{aligned}
& 12288\kappa^6 s^3 + 36864\kappa^6 s^2 + 27648\kappa^6 s + 3072\kappa^6 + 9216\kappa^5 s^4 + 110592\kappa^5 s^3 \\
& + 333312\kappa^5 s^2 + 316800\kappa^5 s + 46080\kappa^5 + 2304\kappa^4 s^5 + 57600\kappa^4 s^4 + 418176\kappa^4 s^3 \\
& + 1216416\kappa^4 s^2 + 1314816\kappa^4 s + 268608\kappa^4 + 192\kappa^3 s^6 + 10368\kappa^3 s^5 + 142512\kappa^3 s^4 \\
& + 822000\kappa^3 s^3 + 2243544\kappa^3 s^2 + 2577240\kappa^3 s + 777600\kappa^3 + 576\kappa^2 s^6 + 16704\kappa^2 s^5 \\
& + 171216\kappa^2 s^4 + 841152\kappa^2 s^3 + 2112504\kappa^2 s^2 + 2552232\kappa^2 s + 1156392\kappa^2 + 528\kappa s^6 \\
& + 11232\kappa s^5 + 91512\kappa s^4 + 381180\kappa s^3 + 905286\kappa s^2 + 1251030\kappa s + 835980\kappa + 144s^6 \\
& + 1986s^5 + 11433s^4 + 42686s^3 + 129795s^2 + 260896s + 231588
\end{aligned}$$

Case 3: β_3 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 165888\kappa^6 s^2 + 150528\kappa^6 s + 24576\kappa^6 + 36864\kappa^5 s^4 + 460800\kappa^5 s^3 \\
& + 1498368\kappa^5 s^2 + 1602432\kappa^5 s + 330240\kappa^5 + 9216\kappa^4 s^5 + 236160\kappa^4 s^4 \\
& + 1787136\kappa^4 s^3 + 5468640\kappa^4 s^2 + 6405888\kappa^4 s + 1750464\kappa^4 + 768\kappa^3 s^6 \\
& + 42048\kappa^3 s^5 + 595872\kappa^3 s^4 + 3557904\kappa^3 s^3 + 10095720\kappa^3 s^2 + 12453768\kappa^3 s \\
& + 4687200\kappa^3 + 2304\kappa^2 s^6 + 68544\kappa^2 s^5 + 722016\kappa^2 s^4 + 3657120\kappa^2 s^3 \\
& + 9624360\kappa^2 s^2 + 12625788\kappa^2 s + 6597468\kappa^2 + 2112\kappa s^6 + 45648\kappa s^5 + 382884\kappa s^4 \\
& + 1686372\kappa s^3 + 4362684\kappa s^2 + 6587070\kappa s + 4607670\kappa + 504s^6 + 7278s^5 + 48471s^4 \\
& + 224909s^3 + 762780s^2 + 1506334s + 1252602
\end{aligned}$$

Case 3: β_4 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 184320\kappa^6 s^2 + 196608\kappa^6 s + 46080\kappa^6 + 36864\kappa^5 s^4 + 479232\kappa^5 s^3 \\
& + 1671168\kappa^5 s^2 + 1993728\kappa^5 s + 560640\kappa^5 + 9216\kappa^4 s^5 + 241920\kappa^4 s^4 \\
& + 1904640\kappa^4 s^3 + 6123840\kappa^4 s^2 + 7788864\kappa^4 s + 2737920\kappa^4 + 768\kappa^3 s^6 \\
& + 42624\kappa^3 s^5 + 622080\kappa^3 s^4 + 3845280\kappa^3 s^3 + 11385072\kappa^3 s^2 + 15151296\kappa^3 s \\
& + 6860160\kappa^3 + 2304\kappa^2 s^6 + 70272\kappa^2 s^5 + 761760\kappa^2 s^4 + 3999648\kappa^2 s^3 \\
& + 11097696\kappa^2 s^2 + 15768348\kappa^2 s + 9197868\kappa^2 + 2112\kappa s^6 + 46656\kappa s^5 + 407856\kappa s^4 \\
& + 1918968\kappa s^3 + 5378874\kappa s^2 + 8668020\kappa s + 6218670\kappa + 456s^6 + 7320s^5 + 58560s^4 \\
& + 316169s^3 + 1106205s^2 + 2107834s + 1657602
\end{aligned}$$

Case 4: β_0 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 258048\kappa^6 s^2 + 417792\kappa^6 s + 196608\kappa^6 + 36864\kappa^5 s^4 + 552960\kappa^5 s^3 \\
& + 2463744\kappa^5 s^2 + 4107264\kappa^5 s + 2193408\kappa^5 + 9216\kappa^4 s^5 + 264960\kappa^4 s^4 \\
& + 2420736\kappa^4 s^3 + 9495744\kappa^4 s^2 + 16190784\kappa^4 s + 9820032\kappa^4 + 768\kappa^3 s^6 \\
& + 44928\kappa^3 s^5 + 732672\kappa^3 s^4 + 5242848\kappa^3 s^3 + 18808560\kappa^3 s^2 + 33004224\kappa^3 s \\
& + 22382784\kappa^3 + 2304\kappa^2 s^6 + 77184\kappa^2 s^5 + 949536\kappa^2 s^4 + 5911392\kappa^2 s^3 \\
& + 20369856\kappa^2 s^2 + 36951228\kappa^2 s + 26844024\kappa^2 + 2112\kappa s^6 + 52992\kappa s^5 \\
& + 567984\kappa s^4 + 3405480\kappa s^3 + 11789898\kappa s^2 + 21449472\kappa s + 15456024\kappa + 456s^6 \\
& + 11928s^5 + 141240s^4 + 884657s^3 + 2943888s^2 + 4817647s + 3029670
\end{aligned}$$

Case 4: β_1 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 276480\kappa^6 s^2 + 482304\kappa^6 s + 251904\kappa^6 + 36864\kappa^5 s^4 + 571392\kappa^5 s^3 \\
& + 2668800\kappa^5 s^2 + 4724352\kappa^5 s + 2754816\kappa^5 + 9216\kappa^4 s^5 + 270720\kappa^4 s^4 \\
& + 2552064\kappa^4 s^3 + 10397472\kappa^4 s^2 + 18642432\kappa^4 s + 12074880\kappa^4 + 768\kappa^3 s^6 \\
& + 45504\kappa^3 s^5 + 760608\kappa^3 s^4 + 5611248\kappa^3 s^3 + 20859720\kappa^3 s^2 + 38111352\kappa^3 s \\
& + 26911056\kappa^3 + 2304\kappa^2 s^6 + 78912\kappa^2 s^5 + 1000224\kappa^2 s^4 + 6444672\kappa^2 s^3 \\
& + 22941768\kappa^2 s^2 + 42672900\kappa^2 s + 31510248\kappa^2 + 2112\kappa s^6 + 55152\kappa s^5 \\
& + 617748\kappa s^4 + 3824316\kappa s^3 + 13442640\kappa s^2 + 24534198\kappa s + 17638092\kappa + 504s^6 \\
& + 13974s^5 + 166293s^4 + 1023527s^3 + 3331581s^2 + 5344369s + 3304854
\end{aligned}$$

Case 4: β_2 (multiplied by 15120):

$$\begin{aligned}
& 12288\kappa^6 s^3 + 73728\kappa^6 s^2 + 138240\kappa^6 s + 79872\kappa^6 + 9216\kappa^5 s^4 + 147456\kappa^5 s^3 \\
& + 720384\kappa^5 s^2 + 1352064\kappa^5 s + 854784\kappa^5 + 2304\kappa^4 s^5 + 69120\kappa^4 s^4 + 671616\kappa^4 s^3 \\
& + 2839584\kappa^4 s^2 + 5341056\kappa^4 s + 3664512\kappa^4 + 192\kappa^3 s^6 + 11520\kappa^3 s^5 + 197232\kappa^3 s^4 \\
& + 1499568\kappa^3 s^3 + 5768136\kappa^3 s^2 + 10926552\kappa^3 s + 7977744\kappa^3 + 576\kappa^2 s^6 \\
& + 20160\kappa^2 s^5 + 263376\kappa^2 s^4 + 1753056\kappa^2 s^3 + 6418824\kappa^2 s^2 + 12178968\kappa^2 s \\
& + 9101328\kappa^2 + 528\kappa s^6 + 14400\kappa s^5 + 167712\kappa s^4 + 1063404\kappa s^3 + 3771354\kappa s^2 \\
& + 6881070\kappa s + 4927572\kappa + 144s^6 + 4062s^5 + 47685s^4 + 286694s^3 + 911109s^2 \\
& + 1430482s + 867888
\end{aligned}$$

Case 4: β_3 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 313344\kappa^6 s^2 + 629760\kappa^6 s + 399360\kappa^6 + 36864\kappa^5 s^4 + 608256\kappa^5 s^3 \\
& + 3101952\kappa^5 s^2 + 6156672\kappa^5 s + 4181760\kappa^5 + 9216\kappa^4 s^5 + 282240\kappa^4 s^4 \\
& + 2823936\kappa^4 s^3 + 12373728\kappa^4 s^2 + 24327936\kappa^4 s + 17531520\kappa^4 + 768\kappa^3 s^6 \\
& + 46656\kappa^3 s^5 + 817632\kappa^3 s^4 + 6400272\kappa^3 s^3 + 25409304\kappa^3 s^2 + 49665288\kappa^3 s \\
& + 37257264\kappa^3 + 2304\kappa^2 s^6 + 82368\kappa^2 s^5 + 1107936\kappa^2 s^4 + 7593024\kappa^2 s^3 \\
& + 28458840\kappa^2 s^2 + 54826452\kappa^2 s + 41323080\kappa^2 + 2112\kappa s^6 + 60048\kappa s^5 \\
& + 721884\kappa s^4 + 4653636\kappa s^3 + 16575252\kappa s^2 + 30169482\kappa s + 21503268\kappa + 648s^6 \\
& + 18210s^5 + 209547s^4 + 1229651s^3 + 3815043s^2 + 5854333s + 3474198
\end{aligned}$$

Case 4: β_4 (multiplied by 60480):

$$\begin{aligned}
& 49152\kappa^6 s^3 + 331776\kappa^6 s^2 + 712704\kappa^6 s + 491520\kappa^6 + 36864\kappa^5 s^4 + 626688\kappa^5 s^3 \\
& + 3330048\kappa^5 s^2 + 6967296\kappa^5 s + 5038080\kappa^5 + 9216\kappa^4 s^5 + 288000\kappa^4 s^4 \\
& + 2964480\kappa^4 s^3 + 13439040\kappa^4 s^2 + 27505344\kappa^4 s + 20657280\kappa^4 + 768\kappa^3 s^6 \\
& + 47232\kappa^3 s^5 + 846720\kappa^3 s^4 + 6813600\kappa^3 s^3 + 27832272\kappa^3 s^2 + 55865088\kappa^3 s \\
& + 42824640\kappa^3 + 2304\kappa^2 s^6 + 84096\kappa^2 s^5 + 1162080\kappa^2 s^4 + 8166048\kappa^2 s^3 \\
& + 31180800\kappa^2 s^2 + 60750948\kappa^2 s + 46057416\kappa^2 + 2112\kappa s^6 + 62208\kappa s^5 \\
& + 765456\kappa s^4 + 4984632\kappa s^3 + 17769174\kappa s^2 + 32219568\kappa s + 22847688\kappa + 696s^6 \\
& + 19320s^5 + 217680s^4 + 1247399s^3 + 3775992s^2 + 5645593s + 3257226
\end{aligned}$$

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