

EXCEPTIONAL ZEROS OF HECKE L -FUNCTIONS

CHRISTOPHER FREI AND WATARU KAI

ABSTRACT. We prove that Hecke L -functions have no exceptional zeros. We do so by generalising a variant due to Kerr of the proof by OpenAI in the case of Dirichlet L -functions.

CONTENTS

1. Introduction	1
2. Proof of Proposition 1.2	2
3. Setup	3
4. Vanishing polynomials have large degree	4
5. Construction of vanishing polynomial of small degree	5
6. Conclusion of the proof of Theorem 1.1	7
References	8

1. INTRODUCTION

Our work¹ is motivated by the recent proof by OpenAI [10] showing the non-existence of Landau-Siegel zeros of Dirichlet L -functions. This result is currently at the center of much activity and has been revisited in other contributions, making it more explicit [1], or providing alternative arguments [6, 4]. We contribute here the observation that the arguments extend to Hecke characters of arbitrary number fields.

Theorem 1.1. *For every number field K , there is a constant $\epsilon_K > 0$ such that the following holds. Let χ be a real Hecke character of K of conductor ideal $\mathfrak{q} \neq (1)$, and let $\beta \in (0, 1)$ a zero of $L(s, \chi)$. Then*

$$(1 - \beta) \log \mathfrak{N}\mathfrak{q} \geq \epsilon_K,$$

where $\mathfrak{N}\mathfrak{q}$ denotes the absolute norm of $\mathfrak{q}$.

The constant ϵ_K is in principle effective, but we have not attempted to make it explicit in order to keep the proofs short and clear. As for Dirichlet characters, it is well known that the only possible exceptional zeros are real zeros of L -functions of real Hecke characters, see [5, Theorem 5.35] or [7, Lemma 2.3], so it suffices to consider the case stated in Theorem 1.1. The case $\mathfrak{q} = (1)$ is excluded simply because we would have to replace $\log \mathfrak{N}\mathfrak{q}$ by something like $1 + \log \mathfrak{N}\mathfrak{q}$ to keep the statement valid; this exclusion is not essential because there are only finitely many χ with conductor ideal $\mathfrak{q} = (1)$. Of course, Theorem 1.1 is implied by the quasi-Riemann hypothesis for $L(s, \chi)$, a proof of which is claimed by Cureton and Thorner in [3]. We find it valuable to provide a rather short and more elementary argument. Theorem 1.1 was also obtained in independent work by Banks [2, Corollary 1.2], which appeared at the same time and uses similar methods.

We follow closely the proof of Kerr [6], who replaced the interpolation argument in [10] by an application of a theorem of Masser [9] providing zero-estimates for multivariate polynomials. The proofs over $\mathbb{Q}$ employ a concrete small basis of the quadratic field corresponding to χ . In

2020 *Mathematics Subject Classification.* 11M20, 11R42.

¹This draft is not intended for publication. We intend to publish its contents as part of another work in progress by the authors. Due to the recent interest in the OpenAI proof, we deemed it valuable to make this part available in advance.

order to set up our generalisation, we need a basis with analogous properties for relative quadratic extensions. This is provided by the following result. Write $\mathcal{O}_K$ for the ring of integers of K . Dependencies of implicit constants in $\ll$ -notation are signaled by subscripts.

Proposition 1.2. *Let K be a number field of degree $m = [K : \mathbb{Q}]$. Let F/K be an extension of degree $[F : K] = 2$ and discriminant ideal $\mathfrak{d}_{F/K}$. Then there is $\alpha \in \mathcal{O}_F$, such that $F = K(\alpha)$ and the following holds:*

- (1) $\alpha^2 \in \mathcal{O}_K$,
- (2) $[\mathcal{O}_F : \mathcal{O}_K[\alpha]] \ll_K 1$,
- (3) $|\alpha|_w \ll_K (\mathfrak{N} \mathfrak{d}_{F/K})^{1/(2m)}$ for all archimedean places w of F .

Once Proposition 1.2 is established in §2, it is straightforward to set up and execute the extension of Kerr’s proof to the number field setting. The details are given in §3–§6. There is one further place where the generalisation of [6] was not immediate: the proof of Lemma 4.2, showing a lower bound for the Dirichlet exponent of the subgroup $\Gamma \subseteq \mathbb{C}^3$ to which Masser’s theorem is applied. We would not have expected that it suffices here to consider only one of the complex embeddings of K .

AI use declaration: This draft and the mathematics in it were created without the use of artificial intelligence tools.

2. PROOF OF PROPOSITION 1.2

Choose a finite set S of primes of K which is also a complete set of representatives for the ideal class group $\text{Cl}(K)$, which exists by Chebotarev’s density theorem. We shall allow the implied constants in ‘ $\ll_K$ ’ to also depend on S .

We claim that there exists an element $a \in \mathcal{O}_K$ such that $F = K(\sqrt{a})$ and:

- (i) $v_{\mathfrak{r}}(a) \in \{0, 1\}$ for all primes $\mathfrak{r} \notin S$;
- (ii) $v_{\mathfrak{p}}(a) \ll_K 1$ for all $\mathfrak{p} \in S$.

First, take any $a_0 \in \mathcal{O}_K$ such that $F = K(\sqrt{a_0})$. Suppose there is a prime $\mathfrak{r} \notin S$ such that $v_{\mathfrak{r}}(a_0) \geq 2$. Take $\mathfrak{p} \in S$ in the same ideal class as $\mathfrak{r}$, and $\mathfrak{p}' \in S$ in the opposite class. We have $\mathfrak{p}\mathfrak{p}' = (c)$ and $\mathfrak{r}\mathfrak{p}' = (d)$ for some $c, d \in \mathcal{O}_K \setminus \{0\}$. Then the ideal $a_0\mathfrak{p}^2\mathfrak{p}'^2 = (a_0c^2)$ is divisible by $\mathfrak{r}^2\mathfrak{p}'^2 = (d^2)$. Hence if we define $a_1 := a_0c^2/d^2 \in \mathcal{O}_K$ so that $F = K(\sqrt{a_1})$, we have $v_{\mathfrak{r}}(a_1) = v_{\mathfrak{r}}(a_0) - 2$ and the valuations at other primes outside S are unchanged. Repeating this process, we obtain an $a_{\infty} \in \mathcal{O}_K$ such that $F = K(\sqrt{a_{\infty}})$ and condition (i) is satisfied.

The $2|\text{Cl}(K)|$ -th power of any ideal is the square of a principal ideal. Hence we can divide a_{∞} by a square to obtain an element $a \in \mathcal{O}_K$ having the same valuations as a_{∞} at each $\mathfrak{r} \notin S$ and valuations $< 2|\text{Cl}(K)|$ at each prime in S . This a satisfies condition (ii) as well, verifying the claim.

Now, let $a \in \mathcal{O}_K$ satisfy the conditions in the claim. We show that its modification by a square in $\mathcal{O}_K^{\times}$ makes the assertions of the proposition true.

For primes $\mathfrak{r} \notin S$ coprime to 2, the condition that $\mathfrak{r}$ is ramified in $F = K(\sqrt{a})$ is equivalent to each of the following two conditions:

- $v_{\mathfrak{r}}(a) = 1$ (by condition (i) in the claim); and
- $\mathfrak{r} \mid \mathfrak{d}_{F/K}$, in which case $v_{\mathfrak{r}}(\mathfrak{d}_{F/K}) = 1$ by the assumption $\mathfrak{r} \nmid 2$.

Thus the ideals (a) and $\mathfrak{d}_{F/K}$ agree outside S and 2, and by condition (ii) the difference is also uniformly bounded. The product formula implies that $\prod_v |a|_v = |N_{K/\mathbb{Q}}(a)| \asymp_K \mathfrak{N} \mathfrak{q}$, where complex places are multiplied twice. Elementary geometry of numbers tells us that by the multiplication by a square in $\mathcal{O}_K^{\times}$, we can modify a so that the sizes $|a|_v$ for different archimedean places v are comparable, i.e., within bounded factors $O_K(1)$ of each other. This means that $|a|_v \asymp_K \mathfrak{N} \mathfrak{d}_{F/K}^{1/m}$ for each archimedean place v of K . Then any of the two square roots $\alpha = \sqrt{a}$ satisfies the desired properties (1) and (3).

To show property (2), we work in the absolute extension $F/\mathbb{Q}$. Let W be an integral basis of $F/\mathbb{Q}$, and consider the $\mathbb{Z}$ -basis A of $\mathcal{O}_K[\alpha]$ consisting of the elements $\eta_1, \dots, \eta_m, \eta_1\alpha, \dots, \eta_m\alpha$,

where $\eta_1, \dots, \eta_m$ is an integral basis of $K/\mathbb{Q}$. The discriminants of these bases satisfy

$$D(A) = [\mathcal{O}_F : \mathcal{O}_K[\alpha]]^2 D(W). \quad (2.1)$$

By the transitivity of discriminants of number fields, we have

$$D(W)\mathbb{Z} = \mathfrak{d}_{F/\mathbb{Q}} = \mathfrak{N} \mathfrak{d}_{F/K} \cdot \mathfrak{d}_{K/\mathbb{Q}}^2.$$

On the other hand, the transitivity of discriminants of bases (see, e.g. [8, Exercise 23 in §2]), yields

$$D(A) = N_{K/\mathbb{Q}}(D(1, \alpha)) D(\eta_1, \dots, \eta_m)^2 = N_{K/\mathbb{Q}}(4a) D(\eta_1, \dots, \eta_m)^2.$$

As $D(\eta_1, \dots, \eta_m)\mathbb{Z} = \mathfrak{d}_{K/\mathbb{Q}}$, inserting these formulas into (2.1) gives

$$[\mathcal{O}_F : \mathcal{O}_K[\alpha]]^2 = \mathfrak{N}(4a \mathfrak{d}_{F/K}^{-1}) \ll_K 1,$$

by the uniform bound of the ideal $a \mathfrak{d}_{F/K}^{-1}$. $\square$

3. SETUP

Here we set up the framework allowing us to generalise the arguments in [6] in order to prove Theorem 1.1. We write $m = [K : \mathbb{Q}]$, let $\overline{K}$ be an algebraic closure of K and fix an embedding $\overline{K} \subseteq \mathbb{C}$. Moreover, we fix an integral basis $\eta_1, \dots, \eta_m$ of K .

Let Ω_K denote the set infinite places of K , so for every $v \in \Omega_K$ we have an embedding $\sigma_v : K \rightarrow \mathbb{C}$ and a corresponding absolute value $|\cdot|_v = |\sigma_v(\cdot)|$.

We fix an arbitrary integer $b \in \mathcal{O}_K$ that is not a square in K and a root $\beta \in \overline{K}$ of $x^2 - b$. Then for every infinite place w of the quadratic extension field $E := K(\beta)$ of K , lying above a place $v \in \Omega_K$, we have $|\beta|_w = |b|_v^{1/2} \ll_K 1$. Let $\mathfrak{d}_{E/K}$ denote the relative discriminant ideal of E/K .

Write $\mathbb{A}_K^\times$ for the idele group of K . The real Hecke character χ is a continuous homomorphism $\chi : \mathbb{A}_K^\times \rightarrow S^1$ that is trivial on $K^\times$ and satisfies $\chi^2 = 1$. As the Dedekind zeta function of K definitely has no zero at $s = 1$, the theorem's assertion holds true for the trivial character. Assume from now on that χ is non-trivial. Then the kernel of χ is an open subgroup of index two in $\mathbb{A}_K^\times$ containing $K^\times$, which corresponds by class field theory to a quadratic extension F_χ/K . Via the norm residue map, we may thus consider χ as a character of $\text{Gal}(F_\chi/K) = \mathbb{Z}/2\mathbb{Z}$. The conductor (ideal) $\mathfrak{q}$ of χ is equal to the conductor (ideal) of F_χ/K , and thus to the discriminant ideal $\mathfrak{d}_{F_\chi/K}$ of F_χ/K . Let $\alpha \in \mathcal{O}_{F_\chi}$ be a “small” generator, as supplied by Proposition 1.2 with $F = F_\chi$.

If $\mathfrak{N}\mathfrak{q}$ is sufficiently large in terms of K , the fields F_χ and E will be different from each other, and hence

$$L := F_\chi(\beta) = K(\alpha, \beta)$$

is a biquadratic extension of K . We consider the order $\mathcal{O} := \mathcal{O}_K[\alpha, \beta]$ of L , which is free over $\mathcal{O}_K$ with basis $1, \alpha, \beta, \alpha\beta$.

For $N > 0$, we define the finite subset

$$\mathcal{B}_K(N) := \{n_1\eta_1 + \dots + n_m\eta_m : n_i \in [0, N] \cap \mathbb{Z} \text{ for all } i\} \subseteq \mathcal{O}_K$$

and the finite subset

$$\mathcal{B}_L(N) := \{u_1 + u_2\alpha + u_3\beta + u_4\alpha\beta : u_j \in \mathcal{B}_K(N) \text{ for all } j\} \subseteq \mathcal{O}.$$

We denote by $\sigma, \tau \in \text{Gal}(L/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the generators that satisfy

$$\sigma(\alpha) = -\alpha, \quad \sigma(\beta) = \beta, \quad \text{and} \quad \tau(\alpha) = \alpha, \quad \tau(\beta) = -\beta.$$

By the properties of α from Proposition 1.2, there is a finite set S of prime ideals of $\mathcal{O}_K$, depending only on K , that contains all prime ideals $\mathfrak{p}$ that divide the discriminant $\mathfrak{d}_{E/K}$, the index $[\mathcal{O}_E : \mathcal{O}_K[\beta]]$ or the index $[\mathcal{O}_{F_\chi} : \mathcal{O}_K[\alpha]]$, or that satisfy $\mathfrak{N}\mathfrak{p} \leq 8$. We note that every prime ideal $\mathfrak{p} \notin S$ such that $\mathfrak{p} \nmid \mathfrak{q}$ will be unramified in L . With this setup, we have the following divisibility result, analogous to [10, (5.2)] and [6, (3)–(4)].

Lemma 3.1. *Let $\mathfrak{p} \notin S$ be a prime ideal of $\mathcal{O}_K$, such that $\mathfrak{p} \nmid \mathfrak{q}$ and $\chi(\mathfrak{p}) = -1$. Define the automorphism $g_{\mathfrak{p}} \in \text{Gal}(L/K)$ to be equal to σ if $\mathfrak{p}$ splits in E/K and equal to $\sigma\tau$ if $\mathfrak{p}$ is inert in E/K . Then every $\theta \in \mathcal{O}$ satisfies*

$$\theta^{\mathfrak{N}\mathfrak{p}} - g_{\mathfrak{p}}(\theta) \in \mathfrak{p}\mathcal{O}.$$

Proof. As $\chi(\mathfrak{p}) = -1$, the prime ideal $\mathfrak{p}$ is inert in F_{χ}/K . As $\mathfrak{p}$ does not divide $[\mathcal{O}_{F_{\chi}} : \mathcal{O}_K[\alpha]]$, this is equivalent to $X^2 - a$ not having a root in the residue field $\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$, which is by Euler's criterion equivalent to the congruence $a^{(\mathfrak{N}\mathfrak{p}-1)/2} \equiv -1 \pmod{\mathfrak{p}}$. We conclude that

$$\alpha^{\mathfrak{N}\mathfrak{p}} = \alpha a^{(\mathfrak{N}\mathfrak{p}-1)/2} \equiv -\alpha \pmod{\mathfrak{p}\mathcal{O}}.$$

Similarly, $\beta^{\mathfrak{N}\mathfrak{p}} \equiv \pm\beta \pmod{\mathfrak{p}\mathcal{O}}$, depending on whether $\mathfrak{p}$ splits or remains inert in E/K . As $\mathfrak{N}\mathfrak{p}$ is a power of the characteristic of $\mathcal{O}/\mathfrak{p}\mathcal{O}$, and $u^{\mathfrak{N}\mathfrak{p}} \equiv u \pmod{\mathfrak{p}}$ holds for all $u \in \mathcal{O}_K$, this implies for $\theta = u_1 + u_2\alpha + u_3\beta + u_4\alpha\beta \in \mathcal{O}$ that $\theta^{\mathfrak{N}\mathfrak{p}} \equiv u_1 - u_2\alpha \pm u_3\beta \mp u_4\alpha\beta = g_{\mathfrak{p}}(\theta) \pmod{\mathfrak{p}\mathcal{O}}$. $\square$

4. VANISHING POLYNOMIALS HAVE LARGE DEGREE

The *generalised Dirichlet exponent* $\mu(\Gamma)$ of a non-trivial finitely generated additive subgroup $\Gamma \subseteq \mathbb{C}^n$ is given in [9] as

$$\mu(\Gamma) := \min_{1 \leq r \leq n} (l - l_r)/r,$$

where $l = \text{rk}_{\mathbb{Z}}(\Gamma)$ is the rank of Γ and l_r is the maximum of all ranks of $\Gamma \cap V$ for $V \subseteq \mathbb{C}^n$ a $\mathbb{C}$ -subspace of dimension $n - r$. Let $\gamma_1, \dots, \gamma_l$ be a basis of Γ and, for $N \geq 0$, let $\Gamma(N)$ denote the set of all linear combinations $n_1\gamma_1 + \dots + n_l\gamma_l$ with $n_i \in \mathbb{Z} \cap [0, N]$. As in [6], we will apply a theorem of Masser [9, Theorem 1], which states that a non-zero polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ that satisfies $F(\gamma) = 0$ for all $\gamma \in \Gamma(N)$ must have degree at least $(N/n)^{\mu(\Gamma)}$.

Recall that we fixed an embedding $K \subseteq L \subseteq \overline{K} \subseteq \mathbb{C}$. We consider the vectors

$$\gamma_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} \beta \\ \beta \\ -\beta \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \alpha\beta \\ -\alpha\beta \\ \alpha\beta \end{pmatrix} \in \mathbb{C}^3$$

and define $\Gamma \subseteq \mathbb{C}^3$ to be the subgroup generated by all the $\eta_i\gamma_j$, $1 \leq i \leq m$, $1 \leq j \leq 4$, or equivalently $\Gamma = \sum_{j=1}^4 \mathcal{O}_K\gamma_j$.

The following is an immediate generalisation of arguments given in [6, Lemma 7].

Lemma 4.1. *Let $\pi : \mathbb{C}^4 \rightarrow \mathbb{C}^3$ be a surjective $\mathbb{C}$ -linear map whose kernel is not contained in any proper subspace of $\mathbb{C}^4$ that is spanned by vectors in K^4 . Let $V := \pi(K^4)$. Then*

- (1) *V is a K -subspace of $\mathbb{C}^3$ of dimension $\dim_K(V) = 4$.*
- (2) *For every proper $\mathbb{C}$ -subspace $L \subseteq \mathbb{C}^3$, the intersection $V \cap L$ is a K -subspace of $\mathbb{C}^3$ of dimension $\dim_K(V \cap L) \leq \dim_{\mathbb{C}}(L)$.*

Proof. Note that $\dim_{\mathbb{C}} \ker(\pi) = 1$. Therefore, $\ker(\pi) \cap K^4 = \{0\}$, as otherwise it would generate a proper $\mathbb{C}$ -subspace containing $\ker(\pi)$. Hence, $V \simeq K^4$ as K -vector spaces.

For a proper $\mathbb{C}$ -subspace $L \subseteq \mathbb{C}^3$, let $W \subseteq \mathbb{C}^4$ be the $\mathbb{C}$ -subspace generated by $\pi^{-1}(L) \cap K^4$. By hypothesis, $\ker(\pi)$ is not contained in W , and thus $W \cap \ker(\pi) = \{0\}$. This shows that $\dim_K(V \cap L) = \dim_K(K^4 \cap \pi^{-1}(L)) = \dim_{\mathbb{C}}(W) = \dim_{\mathbb{C}}(\pi(W)) \leq \dim_{\mathbb{C}}(L)$. $\square$

Lemma 4.2. *The subgroup $\Gamma \subseteq \mathbb{C}^3$ defined above satisfies $\mu(\Gamma) \geq 4m/3$.*

Proof. Let $\pi : \mathbb{C}^4 \rightarrow \mathbb{C}^3$ be the map defined by $\pi(e_i) = \gamma_i$, with $e_1, \dots, e_4$ the canonical $\mathbb{C}$ -basis of $\mathbb{C}^4$. Then the extension of scalars $\Gamma_{\mathbb{Q}}$ of Γ from $\mathbb{Z}$ to $\mathbb{Q}$ consists of all K -linear combinations of $\gamma_1, \dots, \gamma_4$, hence $\Gamma_{\mathbb{Q}} = \pi(K^4) =: V$.

We observe that $\ker(\pi)$ is the $\mathbb{C}$ -subspace of $\mathbb{C}^4$ generated by the vector $(\alpha\beta, \beta, -\alpha, -1)$. As the coordinates of this vector are linearly independent over K , the map π satisfies the hypotheses of Lemma 4.1.

Hence, $\text{rk}_{\mathbb{Z}}(\Gamma) = \dim_{\mathbb{Q}}(V) = m \dim_K(V) = 4m$. Moreover, for any proper $\mathbb{C}$ -subspace $L \subseteq \mathbb{C}^3$, we see that $\text{rk}_{\mathbb{Z}}(\Gamma \cap L) = \dim_{\mathbb{Q}}(V \cap L) = m \dim_K(V \cap L) \leq m \dim_{\mathbb{C}}(L)$.

These assertions together imply that

$$\mu(\Gamma) \geq \min_{1 \leq r \leq 3} \frac{4m - (3-r)m}{r} = \frac{4m}{3}. \quad \square$$

Together with [9, Theorem 1], the previous lemma implies the following.

Lemma 4.3. *Let $F \in \mathbb{C}[x_1, x_2, x_3] \setminus \{0\}$ and suppose that $F(\theta, \sigma(\theta), \sigma\tau(\theta)) = 0$ for all $\theta \in \mathcal{B}_L(N)$. Then $\deg F \gg_K N^{4m/3}$.*

Proof. We observe that for $\theta = u_1 + u_2\alpha + u_3\beta + u_4\alpha\beta \in \mathcal{O}$, we have

$$\begin{pmatrix} \theta \\ \sigma(\theta) \\ \sigma\tau(\theta) \end{pmatrix} = u_1\gamma_1 + u_2\gamma_2 + u_3\gamma_3 + u_4\gamma_4,$$

and hence our hypothesis implies that F vanishes on all elements of the subset

$$\Gamma(N) = \left\{ \sum_{i,j} n_{i,j} \eta_i \gamma_j : n_{i,j} \in \mathbb{Z} \cap [0, N] \right\} \subseteq \Gamma.$$

Hence, [9, Theorem 1] together with Lemma 4.2 implies that

$$\deg F \geq (N/3)^{\mu(\Gamma)} \gg_K N^{4m/3}. \quad \square$$

5. CONSTRUCTION OF VANISHING POLYNOMIAL OF SMALL DEGREE

For $D \in \mathbb{N}$, let $\mathcal{L}_D \subseteq L[x_1, x_2, x_3]$ be the free $\mathcal{O}$ -module generated by the monomials $x_1^i x_2^j x_3^k$ satisfying $i + 8(j + k) \leq D$. Hence, the elements of $\mathcal{L}_D$ are polynomials

$$f = \sum_{i+8(j+k) \leq D} c_{i,j,k} x_1^i x_2^j x_3^k, \quad \text{with } c_{i,j,k} \in \mathcal{O}. \quad (5.1)$$

The following lemma is essentially the same as [6, Lemma 2].

Lemma 5.1. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ with $\mathfrak{N}\mathfrak{p} > 8$ and $\ell \in \{2, 3\}$. For $i, j, k \in \mathbb{N}_0$ with $i + 8(j + k) \leq D$, define the polynomial*

$$G_{i,j,k} := x_1^v (x_1^{\mathfrak{N}\mathfrak{p}} - x_\ell)^e x_2^j x_3^k \quad \text{with } e := \lfloor i/\mathfrak{N}\mathfrak{p} \rfloor, \quad v := i - e\mathfrak{N}\mathfrak{p}.$$

These polynomials $G_{i,j,k}$ form a basis of $\mathcal{L}_D$.

Proof. The argument in [6, Lemma 2] does not use the assumption that p is prime. It works equally well with $\mathfrak{N}\mathfrak{p}$ instead of p : the polynomials $G_{i,j,k}$ lie in $\mathcal{L}_D$ and can be expressed in terms of the basis $\{x_1^i x_2^j x_3^k\}$ by a square triangular matrix with diagonal entries 1. $\square$

For any $\theta \in \mathcal{O}$, we write $\|\theta\| := \max_{w \in \Omega_L} \{\|\theta\|_w\}$, where Ω_L denotes the set of archimedean places of L . For $B \geq 1$ and $f \in \mathcal{L}_D$ as in (5.1), we let

$$\|f\|_B := \max_{i+8(j+k) \leq D} \{\|c_{i,j,k}\| B^{i+j+k}\}.$$

The following is our analogue of [6, Lemma 3].

Lemma 5.2. *There is a constant $C_K > 0$, depending only on K , such that for any $D, H, B \geq 1$,*

$$\#\{f \in \mathcal{L}_D : \|f\|_B \leq H\} \geq \prod_{i+8(j+k) \leq D} \left(\frac{H}{C_K \mathfrak{N}\mathfrak{q}^{1/(2m)} B^{i+j+k}} \right)^{4m}.$$

Proof. Let $N \geq 1$. Clearly, every $u \in \mathcal{B}_K(N)$ satisfies $\|u\| \ll_K N$. As $\|\alpha\| \ll_K \mathfrak{N}\mathfrak{q}^{1/(2m)}$ by Proposition 1.2 and $\|\beta\| \ll_K 1$, this shows that every $\theta \in \mathcal{B}_L(N)$ satisfies

$$\|\theta\| \ll_K \mathfrak{N}\mathfrak{q}^{1/(2m)} N. \quad (5.2)$$

Take $N_{i,j,k} := H/(C_K \mathfrak{N}\mathfrak{q}^{1/(2m)} B^{i+j+k})$, with sufficiently large C_K , then every f as in (5.1) with $c_{i,j,k} \in \mathcal{B}_L(N_{i,j,k})$ will satisfy $\|f\|_B \leq H$. There are $(\lfloor N_{i,j,k} \rfloor + 1)^{4m}$ choices for such $c_{i,j,k}$. $\square$

Here is our analogue of [6, Lemma 4]. We write $r := \text{rk}_{\mathcal{O}} \mathcal{L}_D$.

Lemma 5.3. *Let T be a finite set of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ that are not in S , do not divide $\mathfrak{q}$ and satisfy $\chi(\mathfrak{p}) = -1$. For each $\mathfrak{p} \in T$, let*

$$k_{\mathfrak{p}} := \max\{1, \lfloor D/\mathfrak{N}\mathfrak{p} \rfloor\}, \quad E_{\mathfrak{p}} := \sum_{i+8(j+k) \leq D} \left\lfloor \frac{i}{\mathfrak{N}\mathfrak{p}} \right\rfloor, \quad \text{and let} \quad \mathfrak{P} := \prod_{\mathfrak{p} \in T} \mathfrak{p}^{k_{\mathfrak{p}}}.$$

For each $f \in \mathcal{L}_D$, consider the tuple of residue classes

$$v_f := (f(\theta, \sigma(\theta), \sigma\tau(\theta)) \bmod \mathfrak{P}\mathcal{O})_{\theta \in \mathcal{B}_L(N)} \in (\mathcal{O}/\mathfrak{P}\mathcal{O})^{\mathcal{B}_L(N)}.$$

Then

$$\#\{v_f : f \in \mathcal{L}_D\} \leq \mathfrak{N}\mathfrak{P}^{4r} \prod_{\mathfrak{p} \in T} \mathfrak{N}\mathfrak{p}^{-4E_{\mathfrak{p}}}.$$

Proof. Let $\mathfrak{p} \in T$ and take $\ell = 2$ if $\mathfrak{p}$ is split in F , and $\ell = 3$ if $\mathfrak{p}$ is inert in F . By Lemma 5.1, we can write

$$f = \sum_{i+8(j+k) \leq D} c_{i,j,k} G_{i,j,k},$$

with $c_{i,j,k} \in \mathcal{O}$. By Lemma 3.1, for each $\theta \in \mathcal{O}$, we have

$$G_{i,j,k}(\theta, \sigma(\theta), \sigma\tau(\theta)) \in \mathfrak{p}^{\lfloor i/\mathfrak{N}\mathfrak{p} \rfloor} \mathcal{O},$$

hence the value of the tuple $(f(\theta, \sigma(\theta), \sigma\tau(\theta)) \bmod \mathfrak{p}^{k_{\mathfrak{p}}} \mathcal{O})_{\theta \in \mathcal{B}_L(N)}$ is determined by the values of $c_{i,j,k} \bmod \mathfrak{p}^{k_{\mathfrak{p}} - \lfloor i/\mathfrak{N}\mathfrak{p} \rfloor} \mathcal{O}$.

Therefore, there are at most

$$\prod_{i+8(j+k) \leq D} \#\mathcal{O}/\mathfrak{p}^{k_{\mathfrak{p}} - \lfloor i/\mathfrak{N}\mathfrak{p} \rfloor} \mathcal{O} = \prod_{i+8(j+k) \leq D} \mathfrak{N}\mathfrak{p}^{4(k_{\mathfrak{p}} - \lfloor i/\mathfrak{N}\mathfrak{p} \rfloor)}$$

such tuples modulo $\mathfrak{p}^{k_{\mathfrak{p}}}$. By the Chinese remainder theorem, there are thus at most

$$\prod_{\mathfrak{p} \in T} \prod_{i+8(j+k) \leq D} \mathfrak{N}\mathfrak{p}^{4(k_{\mathfrak{p}} - \lfloor i/\mathfrak{N}\mathfrak{p} \rfloor)} = \mathfrak{N}\mathfrak{P}^{4r} \prod_{\mathfrak{p} \in T} \mathfrak{N}\mathfrak{p}^{-4E_{\mathfrak{p}}},$$

tuples v_f modulo $\mathfrak{P}$, as desired. $\square$

Our analogue of [6, Lemma 5] takes the following form.

Lemma 5.4. *There is a constant $c_K \geq 1$, depending only on K , such that the following holds. Assume that $\mathfrak{N}\mathfrak{q}$ is sufficiently large in terms of K and let $V > 0$. Let $\gamma \in \mathbb{N}$ be sufficiently large in terms of K and V , define*

$$N := \mathfrak{N}\mathfrak{q}^{\gamma}, \quad B := c_K N \mathfrak{N}\mathfrak{q}^{1/(2m)}, \quad D := V N^{4m/3}, \quad Q := \frac{D}{(\log N)^2},$$

and consider the following set of prime ideals of $\mathcal{O}_K$:

$$T := \{\mathfrak{p} : \mathfrak{N}\mathfrak{p} \leq Q, \mathfrak{p} \notin S, \mathfrak{p} \nmid \mathfrak{q}, \chi(\mathfrak{p}) = -1\}.$$

If

$$\sum_{\mathfrak{p} \in T} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} > \frac{13m}{10} \log B, \tag{5.3}$$

then there is a non-zero polynomial $F \in \mathcal{L}_D$ that satisfies $f(\theta, \sigma(\theta), \sigma\tau(\theta)) = 0$ for all $\theta \in \mathcal{B}_L(N)$.

Proof. Recall that $r = \text{rk}_{\mathcal{O}}(\mathcal{L}_D)$, let $E_{\mathfrak{p}}, \mathfrak{P}$ be as in Lemma 5.3, and define

$$W := \sum_{i+8(j+k) \leq D} (i + j + k).$$

With C_K as in Lemma 5.2, let $H := \mathfrak{N}\mathfrak{P}^{1/m}/(4r)$. Then the number of $f \in \mathcal{L}_D$ with $\|f\|_B \leq H$ is at least

$$\prod_{i+8(j+k) \leq D} \left(\frac{H}{C_K \mathfrak{N}\mathfrak{q}^{1/(2m)} B^{i+j+k}} \right)^{4m} = \frac{\mathfrak{N}\mathfrak{P}^{4r}}{((4r)^m C_K^m \sqrt{\mathfrak{N}\mathfrak{q}})^{4r} B^{4mW}}.$$

Comparing this with the estimate from Lemma 5.3, we see that as soon as

$$\sum_{\mathfrak{p} \in T} E_{\mathfrak{p}} \log \mathfrak{N}\mathfrak{p} > mW \log B + r \log((4r)^m C_K^m \sqrt{\mathfrak{N}\mathfrak{q}}), \quad (5.4)$$

there are two polynomials $f, g \in \mathcal{L}_D$ with $\|f\|_B, \|g\|_B \leq H$ and $v_f = v_g$. Then the polynomial $F = f - g$ satisfies $F(\theta, \sigma(\theta), \sigma\tau(\theta)) \in \mathfrak{P}\mathcal{O}$ for all $\theta \in \mathcal{B}_L(N)$.

Let $\theta \in \mathcal{B}_L(N)$. By definition of N and B , we see as in (5.2) that $\|\theta\|, \|\sigma(\theta)\|, \|\sigma\tau(\theta)\| \leq B$ if only c_K is sufficiently large. As $\|F\|_B \leq 2H$, this implies that

$$\|F(\theta, \sigma(\theta), \sigma\tau(\theta))\| \leq 2rH \leq \frac{\mathfrak{N}\mathfrak{p}^{1/m}}{2},$$

and thus $|N_{L/\mathbb{Q}}(F(\theta, \sigma(\theta), \sigma\tau(\theta)))| \leq \mathfrak{N}\mathfrak{p}^4/2^{4m}$. On the other hand, as $F(\theta, \sigma(\theta), \sigma\tau(\theta)) \in \mathfrak{P}\mathcal{O} \subseteq \mathfrak{P}\mathcal{O}_L$, its norm in $\mathbb{Q}$ is divisible by $\mathfrak{N}(\mathfrak{P}\mathcal{O}_L) = \mathfrak{N}\mathfrak{p}^4$. In view of the above bound on its absolute value, this shows that $F(\theta, \sigma(\theta), \sigma\tau(\theta)) = 0$, as desired.

We still need to prove (5.4). As in the proof of [6, Lemma 5], we have the estimates

$$r = \frac{D^3}{384} + O(D^2), \quad W = \frac{5D^4}{6144} + O(D^3), \quad E_{\mathfrak{p}} = \frac{D^4}{1536\mathfrak{N}\mathfrak{p}} + O(D^3).$$

The last of these estimates implies, together with (5.3), that

$$\sum_{\mathfrak{p} \in T} E_{\mathfrak{p}} \log \mathfrak{N}\mathfrak{p} = \frac{D^4}{1536} \sum_{\mathfrak{p} \in T} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} + O_K(D^3Q) > \frac{13mD^4}{15360} \log B + O_K(D^3Q).$$

Together with the estimates of r and W above, this shows (5.4) if only γ is sufficiently large in terms of K and V . $\square$

6. CONCLUSION OF THE PROOF OF THEOREM 1.1

Analogously to [10, 6], the following input from analytic number theory will allow us to deduce the validity of hypothesis (5.3) in Lemma 5.4 if β sufficiently close to 1.

Lemma 6.1. *Assume that $\mathfrak{N}\mathfrak{q} \geq 3$. Let $\beta \in (0, 1)$ be a real zero of $L(s, \chi)$ and $X \geq 3$. Then*

$$\sum_{\substack{\mathfrak{N}\mathfrak{p} \leq X \\ \chi(\mathfrak{p})=1}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \ll_K \log \mathfrak{N}\mathfrak{q} + (1 - \beta)(\log X)^2.$$

Proof. This is a natural generalisation of the proof of [10, Lemma 2]. By [5, Theorem 5.6], we have

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2} \log q + \frac{\gamma'}{\gamma}(s, \chi) - b - \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right),$$

where ρ runs over the nontrivial zeros of $L(s, \chi)$ with multiplicities, $q \asymp_K \mathfrak{N}\mathfrak{q}$ and $\gamma(s, \chi)$ is the gamma factor given on [5, p.129]. By [5, Proposition 5.7], the constant b has real part

$$\Re(b) = - \sum_{\rho} \Re(\rho^{-1}).$$

For $s = 1 + 1/\log X \in (1, 2)$, the values $L'/L(s, \chi)$ and $\gamma'/\gamma(s, \chi)$ are real. Hence, taking real parts, we obtain

$$-\frac{L'}{L}(s, \chi) = \frac{1}{2} \log q + \frac{\gamma'}{\gamma}(s, \chi) - \sum_{\rho} \Re \left(\frac{1}{s - \rho} \right).$$

As $\Re((s - \rho)^{-1}) = (s - \Re(\rho))/|s - \rho|^2 > 0$ and the factor $\gamma'(\chi, x)/\gamma(\chi, s)$ is bounded for $s \in (1, 2)$ in terms of K alone, we get

$$-\frac{L'}{L}(s, \chi) \leq -\frac{1}{s - \beta} + O_K(\log \mathfrak{N}\mathfrak{q}).$$

Together with $-\zeta'_K/\zeta_K(s) = 1/(s-1) + O_K(1)$, this yields

$$-\frac{\zeta'_K}{\zeta_K}(s) - \frac{L'}{L}(s, \chi) \leq \frac{1}{s-1} - \frac{1}{s-\beta} + O_K(\log \mathfrak{N}\mathfrak{q}).$$

The main term on the right-hand side is

$$\frac{1}{s-1} - \frac{1}{s-\beta} = \frac{1-\beta}{(s-1)(s-\beta)} \leq (1-\beta)(\log X)^2.$$

We conclude our prove by observing that

$$\frac{2}{e} \sum_{\substack{\mathfrak{N}\mathfrak{p} \leq X \\ \chi(\mathfrak{p})=1}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} \leq 2 \sum_{\substack{\mathfrak{N}\mathfrak{p} \leq X \\ \chi(\mathfrak{p})=1}} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^s} \leq \sum_{\mathfrak{a}} \frac{\Lambda_K(\mathfrak{a})(1+\chi(\mathfrak{a}))}{\mathfrak{N}\mathfrak{a}^s} = -\frac{\zeta'_K}{\zeta_K}(s) - \frac{L'}{L}(s, \chi). \quad \square$$

6.1. Proof of Theorem 1.1. As there are $\ll_K 1$ quadratic Hecke characters with $\mathfrak{N}\mathfrak{q} \ll_K 1$ and $L(1, \chi) \neq 0$ holds for every Hecke character, we may assume that $\mathfrak{N}\mathfrak{q}$ is sufficiently large in terms of K . Suppose that $(1-\beta)\mathfrak{N}\mathfrak{q} < \epsilon$ for sufficiently small ϵ in terms of K . We will find a contradiction.

Let N, B, D, Q, T be as in Lemma 5.4, with parameters γ, V to be chosen soon, depending only on K . The number-field analog of Mertens' estimate is

$$\sum_{\mathfrak{N}\mathfrak{p} \leq X} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = \log X + O_K(1),$$

see [11, Lemma 2.3]. Together with Lemma 6.1, this implies that

$$\sum_{\mathfrak{p} \in T} \frac{\log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}} = \log Q + O_K(\log \mathfrak{N}\mathfrak{q} + (1-\beta)(\log Q)^2).$$

As $\log Q = (4m/3) \log N + O_K(\log \log N) + \log V$ and $\log B = \log N + 1/(2m) \log \mathfrak{N}\mathfrak{q} + O_K(1) = \left(1 + \frac{1}{2m\gamma}\right) \log N + O_K(1)$, this shows that (5.3) is satisfied as soon as γ is sufficiently large in terms of K and V , and ϵ is sufficiently small in terms of K, γ, V , hence in terms of K .

Hence, by Lemma 5.4, there is a non-zero polynomial $F \in \mathcal{L}_D$ that satisfies $F(\theta, \sigma(\theta), \sigma\tau(\theta)) = 0$ for all $\theta \in \mathcal{B}_L(N)$. Using Lemma 4.3, we see that $\deg F \gg_K N^{4m/3}$. However, as every $F \in \mathcal{L}_D$ satisfies $\deg F \leq D = VN^{4m/3}$, this yields a contradiction if V was chosen sufficiently small in terms of K . $\square$

REFERENCES

- [1] W.D. Banks. On the OpenAI proof that Landau-Siegel zeros do not exist, arXiv:2610.15483, 2026. [1](#)
- [2] W.D. Banks. Landau-siegel zeros over a base field and effective brauer-siegel, arXiv:2610.31402. [1](#)
- [3] A quasi-Riemann hypothesis for Dedekind zeta functions. Hexagon preprint hexagon:2610.00200v2, 2026. Contributors: Dean Cureton; Jesse Thorner. [1](#)
- [4] On Landau-Siegel zeros. Hexagon preprint hexagon:2610.00311v2, 2026. Contributors: Ben Green. [1](#)
- [5] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004. [1](#), [7](#)
- [6] B. Kerr. A simpler proof that there are no Landau-Siegel zeros, arXiv:2610.21099, 2026. [1](#), [2](#), [3](#), [4](#), [5](#), [6](#), [7](#)
- [7] J. C. Lagarias, H. L. Montgomery, and A. M. Odlyzko. A bound for the least prime ideal in the Chebotarev density theorem. *Invent. Math.*, 54(3):271–296, 1979. [1](#)
- [8] D. A. Marcus. *Number fields*. Universitext. Springer, Cham, second edition, 2018. With a foreword by Barry Mazur. [3](#)
- [9] D. W. Masser. On polynomials and exponential polynomials in several complex variables. *Invent. Math.*, 63(1):81–95, 1981. [1](#), [4](#), [5](#)
- [10] OpenAI. Uniform exclusion of Landau-Siegel zeros. OpenAI Math Release preprint [OAI:Uniform-exclusion-of-Landau-Siegel-zeros-October-1-2026](#), 2026. [1](#), [3](#), [7](#)
- [11] M. Rosen. A generalization of Mertens' theorem. *J. Ramanujan Math. Soc.*, 14(1):1–19, 1999. [8](#)

TECHNISCHE UNIVERSITÄT GRAZ, INSTITUT FÜR ANALYSIS UND ZAHLENTHEORIE, KOPERNIKUSGASSE 24/II, A-8010 GRAZ, AUSTRIA

Email address: `frei@math.tugraz.at`

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, ARAMAKI Aoba 6-3, 980-8578 SENDAI, JAPAN

Email address: `kaiw@tohoku.ac.jp`