

# Nonexistence of $\lim$ Cohen–Macaulay modules following [Che26] and [Ang26]

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## Abstract

Using the inequality stated in [Che26, Main Claim 1.1], we prove nonexistence of  $\lim$  Cohen–Macaulay sequences for the completed section rings of smooth projective surfaces over  $\mathbf{C}$  with an ample globally generated divisor  $H$  satisfying  $K_Y \equiv 4H$  and  $15H^2/8 > \chi(\mathcal{O}_Y)$ . Such a pair is exhibited in [Ang26, Corollary 4.3], with  $H^2 = 1080$  and  $\chi(\mathcal{O}_Y) = 1926$ . For a complete local domain  $(R, \mathfrak{m})$  with  $\ell_R H_{\mathfrak{m}}^j(R) < \infty$  for  $0 \leq j < \dim R$ , we prove that a  $\lim$  Cohen–Macaulay sequence in the sense of Bhatt, Hochster, and Ma [BHM26] exists if and only if a  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence exists (Proposition 1.1). These may be different sequences. Chen’s geometric inequality [Che26, Main Claim 1.1] obstructs  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequences over the completed section rings above. The comparison therefore rules out  $\lim$  Cohen–Macaulay sequences over those rings.

## Introduction

For a Noetherian local ring  $(R, \mathfrak{m}, k)$ , Bhatt, Hochster, and Ma [BHM26, Definition 1.1] define  $\lim$  Cohen–Macaulay sequences of nonzero finite discrete modules  $M_n \in \text{Mod}_R$ . For a fixed system of parameters and each positive degree, the Koszul homology length divided by  $\mu_R(M_n)$  must tend to zero, where  $\mu_R(M_n)$  is the minimal number of generators of  $M_n$ . A  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence satisfies the corresponding condition for lower local cohomology; both definitions are given in Proposition 1.1.

For complete local domains  $(R, \mathfrak{m})$  satisfying  $\ell_R H_{\mathfrak{m}}^j(R) < \infty$  for  $0 \leq j < \dim R$ , Proposition 2.7 shows that the existence of these two different kinds of sequences is equivalent, up to changing the sequence. A given  $\lim$  Cohen–Macaulay sequence need not itself be  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay; see Proposition 2.9. Chen’s geometric inequality [Che26, Main Claim 1.1] obstructs  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequences for the completed section rings considered here. The comparison then excludes  $\lim$  Cohen–Macaulay sequences.

## Conventions

All rings are discrete commutative Noetherian rings. The inputs and witnesses in the statements are finite discrete modules. All tensor products, mapping objects, local cohomology, and fiber sequences are taken in the stable category  $\text{Mod}_R$ . The symbol  $\text{Hom}_R$  denotes the mapping spectrum with its natural  $R$ -module structure. Write  $H^i(X) = \pi_{-i}(X)$ ; a discrete module shifted by  $[s]$  lies in homotopy degree  $s$ . The truncations  $\tau_{\geq i}$  and  $\tau_{\leq i}$  use homotopy degrees. Free resolutions are indexed homologically, with differential from degree  $i$  to degree  $i - 1$ . Kernels and cokernels of maps between discrete modules are taken in the heart  $\text{Mod}_R^{\heartsuit}$ .

## 1 Definitions

In the definitions below,  $(R, \mathfrak{m}, k)$  is a complete local ring of dimension  $d \geq 1$ .

**Definition 1.1** (Koszul and local cohomology errors). Fix a system of parameters  $\mathbf{q} = (q_1, \dots, q_d)$ . For a finite discrete  $M \in \text{Mod}_R$ , let  $\mu_R(M)$  be its minimal number of generators and put

$$K(\mathbf{q}; M) = M \otimes_R \bigotimes_{i=1}^d \text{cofib}(q_i : R \longrightarrow R), \quad H_{\mathfrak{m}}^j(M) = \pi_{-j} \Gamma_{\mathfrak{m}}(M).$$

Here  $\Gamma_{\mathfrak{m}} : \text{Mod}_R \rightarrow \text{Mod}_R$  is the local cohomology functor. Define

$$\kappa_{\mathbf{q}}(M) = \max_{1 \leq i \leq d} \ell_R \pi_i K(\mathbf{q}; M), \quad \lambda_R(M) = \max_{0 \leq j < d} \ell_R H_{\mathfrak{m}}^j(M),$$

allowing infinite length in the second expression. A nonzero  $M$  is  $\varepsilon$ -CM if  $\kappa_{\mathbf{q}}(M) \leq \varepsilon \mu_R(M)$ , and is  $\varepsilon$ -local-cohomology-CM if  $\lambda_R(M) \leq \varepsilon \mu_R(M)$ . A sequence of nonzero finite discrete modules  $M_n \in \text{Mod}_R$  is a *lim Cohen–Macaulay sequence* [BHM26, Definition 1.1] if

$$\frac{\kappa_{\mathbf{q}}(M_n)}{\mu_R(M_n)} \longrightarrow 0.$$

It is a *lim  $\Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence* if  $\lambda_R(M_n) < \infty$  for every  $n$ , and

$$\frac{\lambda_R(M_n)}{\mu_R(M_n)} \longrightarrow 0.$$

**Definition 1.2** (Duality). Fix a normalized dualizing object  $\omega_R \in \text{Mod}_R$ , so  $\text{Hom}_R(k, \omega_R) \simeq k$ , and set

$$D_R(X) = \text{Hom}_R(X, \omega_R).$$

For finite discrete  $M$ , one has  $\pi_i D_R(M) = 0$  for  $i \notin [0, d]$ , and biduality gives  $D_R(D_R(M)) \simeq M$ .

**Definition 1.3** (Reflexive modules). For a finite discrete  $V \in \text{Mod}_R$ , put  $V^{\vee} = \tau_{\geq 0} \text{Hom}_R(V, R)$ . We call  $V$  *reflexive* if the evaluation map  $V \rightarrow V^{\vee\vee}$  is an isomorphism.

## 2 From $\lim$ Cohen–Macaulay to $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequences

**Definition 2.1** (The replacement). Let  $(R, \mathfrak{m}, k)$  be complete local of dimension  $d \geq 1$ . Fix a normalized dualizing object  $\omega_R \in \text{Mod}_R$ , and let  $M \in \text{Mod}_R$  be finite and discrete. Choose a minimal free resolution  $P_{\bullet} \simeq D_R(M)$ , with each  $P_i$  finite free and  $P_i = 0$  for  $i < 0$ . Let  $\sigma_{< d} P_{\bullet}$  retain degrees  $0, \dots, d-1$ , and put

$$Q_M = \sigma_{\geq d} P_{\bullet} := P_{\bullet} / \sigma_{< d} P_{\bullet} \in \text{Mod}_R.$$

Thus the retained degrees are  $d, d+1, \dots$ , including degree  $d$ ; the differential out of degree  $d$  is replaced by zero. This is the brutal truncation of the chosen resolution. It has exactly one possibly nonzero homotopy module:

$$\pi_d Q_M = \text{coker}(P_{d+1} \longrightarrow P_d), \quad \pi_i Q_M = 0 \quad (i \neq d).$$

Consequently  $\pi_i D_R(Q_M) = 0$  for  $i \notin [-d, 0]$ . Define the finite discrete replacement and its map by

$$\boxed{N_M = \tau_{\geq 0} D_R(\sigma_{\geq d} P_\bullet) \xrightarrow{\rho_M} M.} \quad (1)$$

The map is the composite

$$N_M \longrightarrow D_R(Q_M) \longrightarrow D_R(D_R(M)) \simeq M,$$

obtained by dualizing the quotient  $D_R(M) \rightarrow Q_M$ . The connective cover retains  $\pi_0 D_R(Q_M)$ ; the negative homotopy groups of  $D_R(Q_M)$  need not vanish. The estimates below control the error caused by this final truncation.

**Remark 2.2** (Injective resolutions in the torsion category). Under the hypotheses of [Proposition 2.1](#), work in the presentable stable category

$$\mathcal{T} = \text{Mod}_{R, \mathfrak{m}} := \{X \in \text{Mod}_R : \Gamma_{\mathfrak{m}} X \simeq X\}$$

of modules supported at  $\mathfrak{m}$ . The standard  $t$ -structure on  $\text{Mod}_R$  restricts to  $\mathcal{T}$ ; its heart consists of discrete  $\mathfrak{m}$ -power torsion modules and has enough injectives. Let  $E \in \mathcal{T}^\heartsuit$  be the injective hull of  $k$ . Choose a minimal injective resolution in  $\mathcal{T}$ ,

$$\Gamma_{\mathfrak{m}} M \simeq (J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots), \quad J^j \simeq E^{\oplus t_j}, \quad t_j < \infty,$$

with  $J^j$  placed in homotopy degree  $-j$ . Here minimality means that  $\ker(J^j \rightarrow J^{j+1}) \rightarrow J^j$  is an injective hull in  $\mathcal{T}^\heartsuit$  for every  $j \geq 0$ . Retain the terms beginning with  $J^d$ , deleting its incoming differential. The resulting object

$$T_M = (0 \longrightarrow J^d \longrightarrow J^{d+1} \longrightarrow \cdots) \in \mathcal{T}$$

maps to  $\Gamma_{\mathfrak{m}} M$ . Since  $H_{\mathfrak{m}}^j(M) = 0$  for  $j > d$ , its only possibly nonzero homotopy module is

$$C_M := \pi_{-d} T_M = \ker(J^d \longrightarrow J^{d+1}) \in \mathcal{T}^\heartsuit, \quad T_M \simeq C_M[-d].$$

The module  $C_M$  need not be injective.

Write  $\Lambda_{\mathfrak{m}} : \text{Mod}_R \rightarrow \text{Mod}_R$  for completion, given by  $\Lambda_{\mathfrak{m}} X = \text{Hom}_R(\Gamma_{\mathfrak{m}} R, X)$  ([\[Sta26\]](#), [Tag 0A6V](#)). Then the replacement and its map are

$$\boxed{N_M \simeq \tau_{\geq 0} \Lambda_{\mathfrak{m}}(C_M[-d])}, \quad N_M \longrightarrow \Lambda_{\mathfrak{m}} T_M \longrightarrow \Lambda_{\mathfrak{m}} \Gamma_{\mathfrak{m}} M \simeq M.$$

Indeed, termwise Matlis duality gives a minimal free resolution  $P_i = \text{Hom}_R(J^i, E)$  of  $D_R(M)$ . For this choice of  $P_\bullet$ , the quotient  $D_R(M) \rightarrow Q_M$  is dual to the inclusion  $T_M \rightarrow \Gamma_{\mathfrak{m}} M$ . Local duality and completion therefore identify

$$T_M \simeq \text{Hom}_R(Q_M, E) \simeq \Gamma_{\mathfrak{m}} D_R(Q_M), \quad \Lambda_{\mathfrak{m}} T_M \simeq D_R(Q_M)$$

([\[Sta26\]](#), [Tag 0A81](#)). Thus this is the construction in [Proposition 2.1](#), including its map to  $M$ . The negative homotopy groups of  $\Lambda_{\mathfrak{m}} T_M$  can be nonzero; the final connective cover removes them and is finite and discrete.

**Proposition 2.3** (Discrete quantitative comparison). *Let  $(R, \mathfrak{m}, k)$  be a complete local domain of dimension  $d \geq 1$  such that*

$$h := \sum_{j=0}^{d-1} \ell_R H_{\mathfrak{m}}^j(R) < \infty. \quad (2)$$

Fix a system of parameters  $\mathbf{q}$ . There is a constant  $C = C(R, \mathbf{q}) \geq 1$  such that the replacement  $\rho_M : N_M \rightarrow M$  in (1) satisfies, for every finite discrete  $M \in \text{Mod}_R$ ,

$$\begin{aligned} \text{rank}_R N_M &\geq \text{rank}_R M, & H_{\mathfrak{m}}^j(N_M) &= 0 \quad (0 \leq j < \min\{2, d\}), \\ \ell_R \text{coker}(\rho_M) + \sum_{j=0}^{d-1} \ell_R H_{\mathfrak{m}}^j(N_M) &\leq C \kappa_{\mathbf{q}}(M). \end{aligned} \quad (3)$$

For every finite discrete  $M$  with  $\lambda_R(M) < \infty$ ,

$$\kappa_{\mathbf{q}}(M) \leq C \lambda_R(M). \quad (4)$$

In particular,

$$\begin{aligned} \mu_R(M) &> 0, & 0 \leq \varepsilon \leq \frac{1}{2}, & & \kappa_{\mathbf{q}}(M) &\leq \varepsilon \mu_R(M) \\ \implies N_M &\neq 0, & \lambda_R(N_M) &\leq C \varepsilon \mu_R(N_M). \end{aligned} \quad (5)$$

For the proof of Proposition 2.3, work under its hypotheses and choose a finite local inclusion  $(A, \mathfrak{n}, k) \subset (R, \mathfrak{m}, k)$ , with  $A$  complete regular of dimension  $d$ , and regular parameters  $\mathbf{x}$  of  $A$  ([Sta26], Tag 0323, Lemma 10.160.11). These are also parameters of  $R$ , and finite lengths agree over  $A$  and  $R$ . Put

$$\beta_i(M) = \dim_k \pi_i(k \otimes_A M), \quad b = \sum_{i=0}^d \beta_i(R).$$

The parameter comparison in [BHM26, Theorem 4.6] gives constants  $u, v \geq 1$ , independent of  $M$ , such that

$$\max_{i>0} \beta_i(M) \leq u \kappa_{\mathbf{q}}(M), \quad \kappa_{\mathbf{q}}(M) \leq v \max_{i>0} \beta_i(M). \quad (6)$$

Indeed,  $k \otimes_A M \simeq K(\mathbf{x}; M)$ , and that theorem compares the sums of positive Koszul homology lengths.

Finite duality identifies  $D_R(X)$  with  $\text{Hom}_A(X, A)[d]$  after restricting to  $\text{Mod}_A$ . For the discrete injective hull  $E = E_R(k) \in \text{Mod}_R$ , local duality gives

$$\Gamma_{\mathfrak{m}} D_R(X) \simeq \text{Hom}_R(X, E), \quad \sum_{r=0}^{d-1} \ell_R \pi_r \omega_R = h \quad (7)$$

for  $X \in \text{Mod}_R$  with finitely many nonzero homotopy modules, each finite ([Sta26], Tag 0A81). The regular subring  $A$  is used only in the estimates; the construction (1) does not use it.

**Lemma 2.4** (The ranks below degree  $d$ ). *For the chosen minimal resolution of  $D_R(M)$ , put  $t_i = \text{rank}_R P_i$ . Then*

$$\sum_{i=0}^{d-1} t_i \leq a \kappa_{\mathbf{q}}(M), \quad a = du(1+b)^{d-1}. \quad (8)$$

*Proof.* Construct the resolution from degree zero upwards. If  $Y_i \in \text{Mod}_R$  is the remaining object at the  $i$ -th step, minimality gives fiber sequences

$$P_i[i] \longrightarrow Y_i \longrightarrow Y_{i+1}, \quad Y_0 = D_R(M), \quad Y_i \in \text{Mod}_R^{\geq i}, \quad t_i = \mu_R(\pi_i Y_i).$$

Since the residue fields agree,  $t_i \leq \dim_k \pi_i(k \otimes_A Y_i)$ . The finite  $A$ -module  $M$  is perfect over  $A$ , so

$$k \otimes_A D_R(M) \simeq \text{Hom}_k(k \otimes_A M, k)[d].$$

Tensor the preceding fiber sequences with  $k$  over  $A$ . Their homotopy sequences give, for  $0 \leq i < d$ ,

$$t_i \leq \beta_{d-i}(M) + \sum_{j < i} \beta_{i-j-1}(R)t_j \leq \beta_{d-i}(M) + b \sum_{j < i} t_j.$$

Induction yields  $\sum_{i < d} t_i \leq (1+b)^{d-1} \sum_{q=1}^d \beta_q(M)$ . Now apply (6).  $\square$

**Lemma 2.5** (Dualizing the truncation). *The map  $\rho_M : N_M \rightarrow M$  satisfies*

$$\ell_R \operatorname{coker}(\rho_M) + \sum_{j=0}^{d-1} \ell_R H_{\mathfrak{m}}^j(N_M) \leq h \sum_{i=0}^{d-1} t_i.$$

Moreover,  $\operatorname{rank}_R N_M \geq \operatorname{rank}_R M$ , and  $H_{\mathfrak{m}}^j(N_M) = 0$  for  $0 \leq j < \min\{2, d\}$ .

*Proof.* Let  $G_M = \sigma_{<d} P_{\bullet} \in \operatorname{Mod}_R$ . It is perfect of Tor-amplitude  $[0, d-1]$ , and the quotient gives a fiber sequence

$$G_M \longrightarrow D_R(M) \longrightarrow Q_M. \quad (9)$$

Dualizing and using biduality gives

$$D_R(Q_M) \longrightarrow M \longrightarrow D_R(G_M). \quad (10)$$

The finite filtration of  $G_M$  has factors  $P_i[i]$ . Thus  $D_R(G_M)$  has a finite filtration with factors  $\omega_R^{t_i}[-i]$ , where  $0 \leq i < d$ . By (7),

$$\sum_{s \leq 0} \ell_R \pi_s D_R(G_M) \leq h \sum_{i < d} t_i. \quad (11)$$

Set  $W_M = \tau_{\leq -1} D_R(Q_M) \in \operatorname{Mod}_R$ . Because  $M$  is discrete, the homotopy sequence of (10) gives

$$0 \longrightarrow \operatorname{coker}(\rho_M) \longrightarrow \pi_0 D_R(G_M) \longrightarrow \pi_{-1} W_M \longrightarrow 0, \quad \pi_s W_M \simeq \pi_{s+1} D_R(G_M) \quad (s \leq -2).$$

Consequently all these modules have finite length, and

$$\ell_R \operatorname{coker}(\rho_M) + \sum_{s < 0} \ell_R \pi_s W_M \leq h \sum_{i < d} t_i. \quad (12)$$

In particular  $\Gamma_{\mathfrak{m}} W_M \simeq W_M$ , and  $\rho_M$  is surjective at the generic point. This proves the rank inequality.

Apply  $\Gamma_{\mathfrak{m}}$  to the fiber sequence  $N_M \rightarrow D_R(Q_M) \rightarrow W_M$ . Local duality identifies the result with

$$\Gamma_{\mathfrak{m}} N_M \longrightarrow \operatorname{Hom}_R(Q_M, E) \longrightarrow W_M.$$

The only possibly nonzero homotopy module of the middle object is in degree  $-d$ . Therefore

$$H_{\mathfrak{m}}^j(N_M) \simeq \pi_{1-j} W_M \quad (0 \leq j < d).$$

The asserted vanishing and length bound follow from  $W_M \in \operatorname{Mod}_R^{\leq -1}$  and (12).  $\square$

**Lemma 2.6** (Local cohomology controls Koszul homology). *For every finite discrete  $M$  with  $\lambda_R(M) < \infty$ ,*

$$\kappa_{\mathbf{q}}(M) \leq 2^d v \lambda_R(M).$$

*Proof.* The natural map  $k \otimes_A \Gamma_{\mathfrak{m}} M \rightarrow k \otimes_A M$  is an isomorphism. Filter  $\Gamma_{\mathfrak{m}} M$  by its homotopy truncations. The factor  $H_{\mathfrak{m}}^d(M)[-d]$  has no positive homotopy groups after tensoring with  $k$ , since  $\mathrm{pd}_A k = d$ . For a finite-length discrete  $A$ -module  $T$ , a composition series gives  $\dim_k \pi_p(k \otimes_A T) \leq \binom{d}{p} \ell_A(T)$ . It follows that, for  $i > 0$ ,

$$\beta_i(M) \leq \sum_{j=0}^{d-1} \binom{d}{i+j} \ell_R H_{\mathfrak{m}}^j(M) \leq 2^d \lambda_R(M),$$

with binomial coefficients outside  $[0, d]$  equal to zero. Use (6).  $\square$

*Proof of Proposition 2.3.* The three lemmas give (3) with constant  $ha$  and (4) with constant  $2^d v$ . For the normalization, put  $e_{\mathbf{q}} = e((\mathbf{q}), R)$ . The following rank bound holds for every finite discrete  $M$ :

$$\mu_R(M) \leq e_{\mathbf{q}} \mathrm{rank}_R M + \kappa_{\mathbf{q}}(M). \quad (13)$$

Indeed, write  $h_i = \ell_R \pi_i K(\mathbf{q}; M)$ . The multiplicity formula and nonnegativity of the second partial Euler characteristic give

$$h_0 - h_1 + \underbrace{(h_2 - h_3 + \cdots)}_{\chi_2 \geq 0} = e_{\mathbf{q}} \mathrm{rank}_R M.$$

The right side is zero when  $M$  has rank zero. The nonnegativity is Serre’s theorem; see [Put04]. Since  $\mu_R(M) \leq h_0$ , this proves (13). If  $M \neq 0$  and  $\kappa_{\mathbf{q}}(M) \leq \varepsilon \mu_R(M)$  with  $\varepsilon \leq 1/2$ , then

$$\mu_R(N_M) \geq \mathrm{rank}_R N_M \geq \mathrm{rank}_R M \geq \frac{\mu_R(M)}{2e_{\mathbf{q}}} > 0.$$

Hence  $\lambda_R(N_M) \leq 2e_{\mathbf{q}} ha \varepsilon \mu_R(N_M)$ . Take  $C = \max\{1, ha, 2^d v, 2e_{\mathbf{q}} ha\}$ .  $\square$

**Corollary 2.7.** *For a complete local domain satisfying (2), the following are equivalent:*

- (i) *For every  $\varepsilon > 0$ , there is a nonzero finite discrete  $\varepsilon$ -CM module for one fixed system of parameters.*
- (ii) *For every  $\varepsilon > 0$ , there is a nonzero finite discrete  $\varepsilon$ -local-cohomology-CM module.*

*The first condition is independent of the system of parameters. Equivalently,  $R$  admits a  $\lim$  Cohen–Macaulay sequence if and only if it admits a  $\lim$   $\Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence.*

*Proof.* Apply the proposition with input error  $\min\{1/2, \varepsilon/C\}$ . The converse uses (4) on the same module. Parameter independence follows from [BHM26, Theorem 4.6]. Choose errors tending to zero for the assertion about sequences.  $\square$

**Remark 2.8** (Cohen–Macaulay approximations). If  $M$  is maximal Cohen–Macaulay, then  $\pi_i D_R(M) = 0$  for  $i \neq d$ , so minimality gives  $P_i = 0$  for  $i < d$ . Thus  $Q_M \simeq D_R(M)$ , and  $\rho_M : N_M \rightarrow M$  is an isomorphism.

If  $R$  is Cohen–Macaulay, then  $h = 0$ , and the construction gives an exact sequence of finite discrete modules

$$0 \longrightarrow L_M \longrightarrow N_M \xrightarrow{\rho_M} M \longrightarrow 0, \quad N_M \text{ maximal Cohen–Macaulay.}$$

The module  $L_M$  has finite injective dimension. Indeed,  $W_M = 0$  by (12), and  $D_R(G_M) \simeq L_M[1]$ . Its finite filtration has factors  $K_R^{t_i}[d-i]$ , where  $K_R = H^{-d}(\omega_R)$  is the canonical module, which has finite injective dimension. This is an Auslander–Buchweitz maximal Cohen–Macaulay approximation [AB89].

**Remark 2.9** (The witnesses may change). The proposition compares existence conditions; it does not bound the two errors on the same module. For example, let  $S = \mathbf{C}[[u, v, w]]$  and

$$P_n = \operatorname{coker}(S \xrightarrow{(u^n, v^n, w^n)} S^3), \quad M_n = S^n \oplus P_n.$$

The presentation and local duality give

$$\mu_S(M_n) = n + 3, \quad \kappa_{(u,v,w)}(M_n) = 1, \quad \ell_{SH_{(u,v,w)}^2}(M_n) = n^3.$$

Thus the normalized Koszul error tends to zero while the normalized local cohomology error tends to infinity. For these modules, (1) gives  $N_{M_n} \simeq S^{n+3}$ .

**Remark 2.10** (Extending lower bounds to arbitrary modules). Let  $(R, \mathfrak{m})$  be any three-dimensional Noetherian normal local domain. For every finite discrete  $M \in \operatorname{Mod}_R$ , the natural map to its reflexive hull induces a surjection

$$H_{\mathfrak{m}}^2(M) \longrightarrow H_{\mathfrak{m}}^2(M^{\vee\vee}) \longrightarrow 0, \quad \operatorname{rank}_R M^{\vee\vee} = \operatorname{rank}_R M. \quad (14)$$

To prove this, let  $T \subset M$  be its torsion submodule and put  $M' = M/T$ , with both regarded as discrete  $R$ -modules. Then  $\dim \operatorname{Supp} T \leq 2$ , so  $H_{\mathfrak{m}}^3(T) = 0$  and  $H_{\mathfrak{m}}^2(M) \rightarrow H_{\mathfrak{m}}^2(M')$  is surjective. The injection  $M' \rightarrow M^{\vee\vee}$  is an isomorphism at primes of height at most one ([Sta26], Tag 0AVC). Its cokernel has support of dimension at most one, so its second local cohomology vanishes. The second surjection follows, proving (14).

Consequently, a bound

$$\ell_R H_{\mathfrak{m}}^2(V) \geq \delta \operatorname{rank}_R V$$

for finite discrete reflexive  $V$  with finite-length  $H_{\mathfrak{m}}^2(V)$  implies the same bound for every finite discrete  $M$  with finite-length  $H_{\mathfrak{m}}^2(M)$ . For rank zero the bound is automatic. One may also allow infinite length, in which case the inequality is automatic whenever the left side is infinite. This assertion concerns the lower bound; it does not assert that  $H_{\mathfrak{m}}^2(M)$  has finite length for arbitrary  $M$ .

**Corollary 2.11** (A uniform obstruction). *Let  $R$  satisfy the hypotheses on the ring in Proposition 2.3, and fix a system of parameters  $\mathbf{q}$ . Suppose  $\delta > 0$  and*

$$\sum_{j=0}^{d-1} \ell_R H_{\mathfrak{m}}^j(N) \geq \delta \operatorname{rank}_R N$$

*for every finite discrete  $N \in \operatorname{Mod}_R$  satisfying  $\ell_R H_{\mathfrak{m}}^j(N) < \infty$  for  $0 \leq j < d$ . Then there is  $\eta > 0$  such that every nonzero finite discrete  $M \in \operatorname{Mod}_R$  satisfies*

$$\kappa_{\mathbf{q}}(M) \geq \eta \mu_R(M). \quad (15)$$

*In particular,  $R$  admits no lim Cohen–Macaulay sequence.*

*Proof.* Use Proposition 2.3, and let  $C$  be its constant. The module  $N$  in (3) gives

$$\delta \operatorname{rank}_R M \leq \delta \operatorname{rank}_R N \leq \sum_{j=0}^{d-1} \ell_R H_{\mathfrak{m}}^j(N) \leq C \kappa_{\mathbf{q}}(M).$$

With  $e_{\mathbf{q}} = e((\mathbf{q}), R)$ , (13) implies

$$\mu_R(M) \leq e_{\mathbf{q}} \operatorname{rank}_R M + \kappa_{\mathbf{q}}(M) \leq (e_{\mathbf{q}} C / \delta + 1) \kappa_{\mathbf{q}}(M).$$

Take  $\eta = (e_{\mathbf{q}} C / \delta + 1)^{-1}$ . This also covers rank zero.  $\square$

### 3 Nonexistence of $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequences from Chen’s inequality

Let  $Y$  be a smooth connected projective surface over  $\mathbf{C}$ , and let  $H$  be an ample globally generated Cartier divisor such that

$$K_Y \equiv 4H, \quad \delta = \frac{15}{8}H^2 - \chi(\mathcal{O}_Y) > 0.$$

In discrete commutative  $\mathbf{C}$ -algebras, put

$$B = \bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(nH)), \quad B_+ = \bigoplus_{n > 0} B_n, \quad R = \widehat{B_{B_+}}, \quad \mathfrak{m} = B_+R.$$

We use the geometric inequality stated in [Che26, Main Claim 1.1]:

$$\ell_R H_{\mathfrak{m}}^2(V) \geq \delta \operatorname{rank}_R V \tag{16}$$

for every nonzero finite discrete reflexive  $V \in \operatorname{Mod}_R$ . The length on the left is finite by [Che26, Lemma 2.5].

**Remark 3.1** (Reflexivity is unnecessary in the numerical lower bound). By Proposition 2.10, the reflexivity hypothesis in Chen’s numerical inequality (16) can be omitted. The same inequality holds for every finite discrete  $V \in \operatorname{Mod}_R$  with finite-length  $H_{\mathfrak{m}}^2(V)$ , and with the convention of infinite length it holds for every finite discrete  $V$ . This removes a hypothesis from the lower bound, not from Chen’s separate assertion of finite length. No reflexivity condition is therefore needed when applying the inequality to the quantitative criterion of Proposition 2.3.

**Corollary 3.2.** *For the pair  $(Y, H)$  above,  $R$  is a three-dimensional complete normal local domain with no  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence and hence no  $\lim$  Cohen–Macaulay sequence.*

*Proof.* The ring  $R$  is complete, normal, and three-dimensional by [Che26, Proposition 2.1]. Normality gives  $H_{\mathfrak{m}}^0(R) = H_{\mathfrak{m}}^1(R) = 0$ . Applying [Che26, Lemma 2.5] to  $V = R$  shows that  $\ell_R H_{\mathfrak{m}}^2(R) < \infty$ . Thus the ring hypotheses of Proposition 2.3 hold. By Proposition 3.1, the inequality supplied by [Che26, Main Claim 1.1] applies to every finite discrete module with finite-length second local cohomology. Fix parameters  $\mathbf{q}$ , put  $e_{\mathbf{q}} = e((\mathbf{q}), R)$ , and let  $C$  be the constant in Proposition 2.3. For every nonzero finite discrete  $M \in \operatorname{Mod}_R$  with  $\lambda_R(M) < \infty$ , Chen’s inequality gives  $\lambda_R(M) \geq \delta \operatorname{rank}_R M$ . Together with (13) and (4), this yields

$$\mu_R(M) \leq e_{\mathbf{q}} \operatorname{rank}_R M + \kappa_{\mathbf{q}}(M) \leq (e_{\mathbf{q}}/\delta + C)\lambda_R(M).$$

Thus no  $\lim \Gamma_{\mathfrak{m}}$ -Cohen–Macaulay sequence exists. By Proposition 2.7, no  $\lim$  Cohen–Macaulay sequence exists either.  $\square$

**The numerical condition and the example.** If  $\sigma(Y)$  denotes the signature of the intersection form on  $H^2(Y(\mathbf{C}), \mathbf{R})$ , the signature and Noether formulas give

$$\sigma(Y) = K_Y^2 - 8\chi(\mathcal{O}_Y), \quad \delta = \frac{\sigma(Y) - H^2}{8},$$

using  $K_Y \equiv 4H$ . Thus the numerical condition agrees with the degree–signature inequality in [Ang26].

In projective  $\mathbf{Q}$ -schemes, let

$$W_{\mathbf{Q}} = V_+(u_3^6 - u_0^6 + u_1^6, u_4^6 - u_0^6 + u_2^6, u_5^6 - u_1^6 + u_2^6) \subset \mathbf{P}_{\mathbf{Q}}^5.$$

Let  $\rho : Y_{\mathbf{Q}} \rightarrow W_{\mathbf{Q}}$  be the blow-up of the reduced singular locus, let  $E$  be its reduced exceptional divisor, and put  $H = 3\rho^*\mathcal{O}_{W_{\mathbf{Q}}}(1) - E$ . After extension to  $\mathbf{C}$ , the resulting pair satisfies

$$\begin{aligned} Y \text{ smooth,} \quad & H \text{ ample and globally generated,} \\ K_Y \sim 4H, \quad & H^2 = 1080, \quad \chi(\mathcal{O}_Y) = 1926. \end{aligned}$$

These are the surface and polarization in [Ang26, Remark 4.2 and Corollary 4.3] and [Che26, §7]. Hence  $\delta = 99$ . Applying Chen’s geometric inequality [Che26, Main Claim 1.1], Proposition 3.2 excludes lim Cohen–Macaulay sequences over the completed vertex of its full section ring.

**Remark 3.3** (Mixed characteristic). By Proposition 3.2, the rational example above gives, for every prime  $p$ , a four-dimensional complete normal local domain of mixed characteristic  $(0, p)$ , with residue field  $\mathbf{F}_p$ , admitting no lim Cohen–Macaulay sequence.

Here are the algebraic details. Let  $B_{\mathbf{Q}}$  be the full section ring of the rational pair above, let  $V = \mathbf{Z}_p$ ,  $K = \mathbf{Q}_p$ , and put  $B_K = B_{\mathbf{Q}} \otimes_{\mathbf{Q}} K$ . Choose homogeneous generators  $b_1, \dots, b_t$  of strictly positive degree for  $B_{\mathbf{Q}}$ . The normalization  $C$  of  $V[b_1, \dots, b_t]$  in  $\text{Frac}(B_K)$  is finite by excellence. It lies in  $B_K$ , is graded ([Sta26], Tag 077G), and satisfies  $C_0 = V$  and  $C[1/p] = B_K$ . Each graded piece is finite free over  $V$ , and  $\dim(C/pC) = 3$ . Lift homogeneous parameters of  $C/pC$  to  $s_0, s_1, s_2 \in C$ . Degree-wise Nakayama shows that  $C$  is finite over  $V[s_0, s_1, s_2]$ ; the parameter degrees may exceed one. Put

$$T = \widehat{C_{(p, C_+)}}, \quad A = V[[s_0, s_1, s_2]], \quad P = C_+T.$$

Then  $T$  is normal ([Sta26], Tag 0C23), finite over  $A$ , of dimension four, and  $T/P = V$ . The positive-degree ideal of the finite graded  $V$ -algebra  $C/(s_0, s_1, s_2)$  is nilpotent. Hence  $P$  is the unique prime over  $(s_0, s_1, s_2)A$ , and

$$\widehat{T_P} \cong T \otimes_A K[[s_0, s_1, s_2]] \cong (\widehat{B_K})_{(B_K)_+}. \quad (17)$$

For a hypothetical lim Cohen–Macaulay sequence of finite discrete  $M_n \in \text{Mod}_T$ , write

$$r_n = \text{rank}_T M_n, \quad e = \text{rank}_A T, \quad \varepsilon_n = \sum_{i=1}^4 \dim_{\mathbf{F}_p} \pi_i(\mathbf{F}_p \otimes_A M_n).$$

The Euler characteristic gives  $\mu_T(M_n) \leq er_n + \varepsilon_n$ . Since  $\varepsilon_n = o(\mu_T(M_n))$ , one has  $r_n > 0$  eventually and  $\varepsilon_n = o(r_n)$ . Localizing and completing a minimal finite  $A$ -free resolution bounds the positive Betti numbers of  $N_n = M_n \otimes_T \widehat{T_P}$  over  $K[[s_0, s_1, s_2]]$  by those of  $M_n$  over  $A$ . Flatness makes  $N_n$  discrete, and  $\text{rank}_{\widehat{T_P}} N_n = r_n$ . Since  $\mu_{\widehat{T_P}}(N_n) \geq r_n$ , these modules form a lim Cohen–Macaulay sequence over  $\widehat{T_P}$ .

Choose an abstract field embedding  $K \hookrightarrow \mathbf{C}$ . The map  $K[[s_0, s_1, s_2]] \rightarrow \mathbf{C}[[s_0, s_1, s_2]]$  is faithfully flat and local ([Sta26], Tag 00R4). Its base change in (17) is the completed section ring over  $\mathbf{C}$ . It preserves minimal generator numbers and lengths of the positive Koszul homotopy modules. The resulting sequence contradicts Proposition 3.2.

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