hexagon:2610.00464v1

Ramsey's Theorem for Pairs in Pincus's Model XI

Frank Gilson
Independent Researcher

Version 1.2, 21 October 2026

Abstract

Howard and Rubin record Pincus's ordered colored permutation model as $N48$, but do not determine whether Ramsey's theorem for pairs holds there. We prove that $N48 \models \mathrm{RT}^2_2$. The proof uses least finite supports and the homogeneity of Pincus's ordered colored structure: after fixing all but one point of a support, the remaining point generates an infinite orbit whose unordered pairs form a single stabilizer orbit. Combining this with Pincus's results on the dense linear ordering principle (DO), the Boolean Prime Ideal Theorem (BPI), and the failure of Part_∞, together with Pincus's transfer theorems, yields $\mathrm{Con(ZF)} \Rightarrow \mathrm{Con(ZF} + \mathrm{RT}^2_2 + \mathrm{DO} + \mathrm{BPI} + \neg\mathrm{Part}_\infty)$, where Part_∞ states that every infinite set is the disjoint union of infinitely many infinite sets.

1 Introduction

Howard and Rubin (1998, Part III) list the ordered colored permutation model constructed by Pincus (1997, §2) as $N48$, Pincus's Model XI, in their catalog of consequences of the Axiom of Choice. Their entry records many principles as true or false in $N48$, but does not determine Form 17, Ramsey's theorem for pairs. We write RT^2_2 for the statement that, for every infinite set X and every coloring $c : [X]^2 \to 2$, there is an infinite $Y \subseteq X$ on which c is constant. Form 17 also does not follow from any form recorded true in $N48$ in the implication data distributed with the Choiceless Grapher (Dimitriou, 2025), whose matrix was supplied by Paul Howard. We checked this by computing the transitive closure of its positive implications.

Tachtsis (2016, Theorem 2.4) proved RT^2_2 in Mostowski's linearly ordered permutation model $N3$. We adapt that strategy to $N48$. After fixing all but one point of the least support of an object, the remaining point can be moved through a single color class inside one gap determined by the fixed support. Homogeneity of the ordered colored structure then produces an infinite orbit on whose unordered pairs the stabilizer acts transitively. Related stabilizer and homogeneity arguments in choiceless Ramsey theory appear in Blass (1977), Brot et al. (2021), and Banerjee et al. (2024).

Our main result is

$$N48 \models \mathrm{RT}^2_2.$$

Together with Pincus's results on $N48$ and his transfer theorems, this yields

$$\mathrm{Con(ZF)} \Longrightarrow \mathrm{Con}\Big(\mathrm{ZF} + \mathrm{RT}^2_2 + \mathrm{DO} + \mathrm{BPI} + \neg\mathrm{Part}_\infty\Big),$$

where DO is the dense linear ordering principle, BPI is the Boolean Prime Ideal Theorem, and Part_∞ asserts that every infinite set is the disjoint union of infinitely many infinite sets. Howard and Rubin give that last statement as Form 296. The contribution is the determination of Form 17 in $N48$ and the resulting ZF consistency statement.

2 Ramsey's theorem in $N48$

Let $\mathfrak{A} = (A, <, C_0, C_1, \ldots)$ be the countable universal homogeneous ordered colored structure used by Pincus. The order is dense without endpoints, and every color occurs between any two distinct atoms (Pincus, 1997, Proposition 2.1). Let $\Gamma = \mathrm{Aut}(\mathfrak{A})$, and for finite $E \subseteq A$ let $\mathrm{Fix}(E)$ be the subgroup fixing E pointwise. We work in the usual finite-support Fraenkel–Mostowski model obtained from this action, beginning in an ambient model of ZFA with Choice. Howard and Rubin (1998, Part III) record this model as $N48$, Pincus's Model XI.

Pincus's support-intersection lemma implies that every object in the model has a least finite support (Pincus, 1997, Lemma 2.2). Least supports transform equivariantly: if H is the least support of x, then the least support of πx is πH. Indeed, if F supports πx, then $\pi^{-1}F$ supports x, so $H \subseteq \pi^{-1}F$ and hence $\pi H \subseteq F$; conversely πH supports πx.

Lemma 2.1. *Let $G \subseteq A$ be finite and let x have a non-singleton $\mathrm{Fix}(G)$-orbit. Then there is a finite $K \supseteq G$ and an infinite K-supported set Y contained in that orbit such that $\mathrm{Fix}(K)$ acts transitively on $[Y]^2$.*

Proof. Let H be the least support of x. Since the $\mathrm{Fix}(G)$-orbit of x is not a singleton, $H \nsubseteq G$; otherwise $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(H)$ and every element of $\mathrm{Fix}(G)$ would fix x. Choose $h \in H \setminus G$ and put

$$K = G \cup (H \setminus \{h\}).$$

Let J be the maximal convex component of $A \setminus K$ containing h. If $h \in C_m$, set

$$I = J \cap C_m.$$

The set I is infinite. For if I were finite, then since J is an open interval in a dense order there would be a nonempty open subinterval $(u, v) \subseteq J$ avoiding I. Proposition 2.1 gives a point of C_m strictly between u and v, a contradiction.

Define

$$Y = \{\pi x : \pi \in \mathrm{Fix}(K)\}.$$

The set Y is K-supported, its elements are hereditarily symmetric, and hence Y belongs to the permutation model. Since $K \supseteq G$, it is contained in the $\mathrm{Fix}(G)$-orbit of x.

For $\pi \in \mathrm{Fix}(K)$, the least support of πx is

$$\pi H = (H \setminus \{h\}) \cup \{\pi h\},$$

and $\pi h \in I$. Conversely, for every $d \in I$, the finite map fixing K pointwise and sending h to d is an ordered-colored isomorphism between finite substructures. By homogeneity it extends to an element of $\mathrm{Fix}(K)$. Thus every $d \in I$ occurs as the moving support coordinate of some element of Y.

If $\pi h = \sigma h$, then $\sigma^{-1}\pi$ fixes H pointwise and hence fixes x, so $\pi x = \sigma x$. Conversely, if $\pi x = \sigma x$, equality of least supports gives $\pi H = \sigma H$, and since both automorphisms fix $H \setminus \{h\}$ pointwise, this forces $\pi h = \sigma h$. Thus distinct points of I give distinct elements of Y. Since I is infinite, Y is infinite; finiteness is absolute between the permutation model and the ambient universe.

Finally take two unordered pairs from Y. Choose representatives $\pi_1 x, \pi_2 x$ and $\sigma_1 x, \sigma_2 x$, and write $d_i = \pi_i h$ and $d'_i = \sigma_i h$. Distinct orbit elements give distinct moving coordinates, so after ordering we may assume $d_1 < d_2$ and $d'_1 < d'_2$. The finite order-preserving map fixing K and sending d_i to d'_i for $i = 1, 2$ preserves colors, and hence extends to some $\tau \in \mathrm{Fix}(K)$. Then $\sigma_i^{-1}\tau\pi_i$ fixes H pointwise, so $\tau(\pi_i x) = \sigma_i x$ for $i = 1, 2$. Therefore $\mathrm{Fix}(K)$ is transitive on $[Y]^2$. $\square$

Theorem 2.2. $N48 \models \mathrm{RT}^2_2$.

Proof. Let X be infinite in $N48$ and let $c : [X]^2 \to 2$ belong to $N48$. Choose one finite support G for both X and c.

Let

$$Z = \{z \in X : (\forall \pi \in \mathrm{Fix}(G))\, \pi z = z\}.$$

The set Z is G-supported and belongs to the permutation model. In the ambient model choose a well-order $\prec$ of Z. Every element of $\mathrm{Fix}(G)$ fixes every member of Z pointwise, and therefore fixes the graph of $\prec$. Hence $\prec$ is G-supported and belongs to the permutation model.

If Z is infinite, its internal well-order yields an injection $s : \omega \to Z$. Pull c back along s to a coloring of $[\omega]^2$. Ramsey's theorem on ω is provable in ZF, so there is an infinite homogeneous $W \subseteq \omega$. Then $s[W]$ is an infinite homogeneous subset of X.

Suppose instead that Z is finite. Since X is infinite, choose $x \in X \setminus Z$. Its $\mathrm{Fix}(G)$-orbit is non-singleton. By Lemma 2.1, there are a finite $K \supseteq G$ and an infinite K-supported Y in that orbit such that $\mathrm{Fix}(K)$ acts transitively on $[Y]^2$. Because G supports X, we have $Y \subseteq X$.

Since G supports c, every element of $\mathrm{Fix}(K)$ fixes c as a function. If $u \in [Y]^2$ and $\pi \in \mathrm{Fix}(K)$, then

$$c(\pi u) = \pi(c(u)) = c(u),$$

because the pure set 2 is fixed pointwise by the atom action. As $[Y]^2$ is a single $\mathrm{Fix}(K)$-orbit, c is constant on $[Y]^2$. Thus Y is an infinite homogeneous subset of X. □

3 A ZF consistency consequence

Pincus (1997, §§2–3) proves that this model satisfies DO and the Prime Ideal Theorem, equivalently BPI, while Part_∞ fails. Together with Theorem 2.2, this gives

$$N48 \models \mathrm{RT}^2_2 + \mathrm{DO} + \mathrm{BPI} + \neg\mathrm{Part}_\infty.$$

Corollary 3.1.

$$\mathrm{Con}(\mathrm{ZF}) \Longrightarrow \mathrm{Con}\Big(\mathrm{ZF} + \mathrm{RT}^2_2 + \mathrm{DO} + \mathrm{BPI} + \neg\mathrm{Part}_\infty\Big).$$

Proof. Let

$$\Theta = \mathrm{RT}^2_2 \wedge \mathrm{DO} \wedge \mathrm{BPI} \wedge \neg\mathrm{Part}_\infty.$$

We show that Θ belongs to Pincus's Class III as described in Pincus (1997, §2). Class III contains the boundable and injectively boundable statements (Pincus, 1997, pp. 447–448). Blass (1977, pp. 388–389) shows that RT^2_2 is injectively boundable, and Pincus (1997, p. 447) states that $\neg\mathrm{Part}_\infty$ is boundable. By Theorem 3 of Pincus (1997, p. 448), Class III contains both DO and the Prime Ideal Theorem. Pincus (1997, p. 448) also states that "all three classes are closed under conjunction". Hence Θ is a Class III statement.

Class III statements transfer from Fraenkel–Mostowski models to models of ZF. The underlying transfer construction is given in Pincus (1977a, §4), and its extension to the Prime Ideal Theorem is due to Pincus (1977b). Pincus (1997, §2) describes the resulting class in the form used here.

The transfer theorem, as stated in Pincus (1977a, pp. 136–137), assumes a Fraenkel–Mostowski model whose well-founded universe U satisfies ZFE + GCH, where Pincus (1977a, p. 145) defines E as the assertion that "the universe has a well ordering". This adds no consistency strength. The constructible universe of any model of ZF satisfies ZFC + GCH and has a definable global well-ordering, so it satisfies E in Pincus's sense. Taking the well-founded universe U to be L, one performs the usual Fraenkel–Mostowski construction with a countable set of atoms and forms $N48$. Since $N48 \models \Theta$, transfer yields a model of ZF satisfying Θ. □

AI assistance

OpenAI's ChatGPT was used for literature searching, proof development and auditing, source verification, and manuscript drafting. Anthropic's Claude was used for adversarial review of proofs, citations, and exposition. Meta AI, accessed through meta.ai, was used to evaluate the predecessor manuscript and to review portions of the $N48$ proof, transfer argument, and source support. Apodex, using its Deep Solve mode, was used for an additional audit of the predecessor manuscript and its source claims. The author reviewed the resulting material and is responsible for the claims and citations in the final manuscript.

References

A. Banerjee, A. Gopaulsingh, and Z. Molnár. On the Rainbow Ramsey theorem and the Canonical Ramsey Theorem for pairs without AC. arXiv:2401.02215v2 [math.LO], 2024.

A. Blass. Ramsey's theorem in the hierarchy of choice principles. *J. Symbolic Logic* **42** (1977), no. 3, 387–390. DOI: 10.2307/2272866.

J. Brot, M. Cao, and D. Fernández-Bretón. Finiteness classes arising from Ramsey-theoretic statements in set theory without choice. *Ann. Pure Appl. Logic* **172** (2021), no. 6, 102961. DOI: 10.1016/j.apal.2021.102961.

I. M. Dimitriou. *Choiceless Grapher (Jeffrey).* Software repository, GitHub, `ioannad/jeffrey`, commit `75d800d` (21 May 2025). Implication data supplied by P. Howard.

P. Howard and J. E. Rubin. *Consequences of the Axiom of Choice.* Mathematical Surveys and Monographs, vol. 59, American Mathematical Society, Providence, RI, 1998. DOI: 10.1090/surv/059.

D. Pincus. Adding dependent choice. *Ann. Math. Logic* **11** (1977a), no. 1, 105–145. DOI: 10.1016/0003-4843(77)90011-0.

D. Pincus. Adding dependent choice to the prime ideal theorem. In R. O. Gandy and J. M. E. Hyland (eds.), *Logic Colloquium '76*, Studies in Logic and the Foundations of Mathematics, vol. 87, North-Holland, Amsterdam, 1977b, pp. 547–565. DOI: 10.1016/S0049-237X(09)70445-8.

D. Pincus. The dense linear ordering principle. *J. Symbolic Logic* **62** (1997), no. 2, 438–456. DOI: 10.2307/2275540.

E. Tachtsis. On Ramsey's theorem and the existence of infinite chains or infinite anti-chains in infinite posets. *J. Symbolic Logic* **81** (2016), no. 1, 384–394. DOI: 10.1017/jsl.2015.47.