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Abstract: Motivated by the Smale mean value and the Dubinin–Sugawa dual mean value theorems, we formulate non-Archimedean Smale mean value and Dubinin–Sugawa dual mean value problems. We show that if a universal constant satisfying the non-Archimedean Smale mean value problem exists, it must be greater than or equal to one and if a universal constant satisfying the non-Archimedean Dubinin–Sugawa dual mean value problem exists, it must be less than or equal to one. We also derive other bounds whenever roots of unity exist in a non-Archimedean field.

Keywords: Smale mean value theorem, Dubinin–Sugawa dual mean value theorem, Non-Archimedean field.

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1. INTRODUCTION

In 1981, Steve Smale proved the following result using the Koebe 1/4-theorem [27].

Theorem 1.1. [27] (*Smale Mean Value Theorem*) Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P (i.e., $P'(z) \neq 0$), then

$$(1) \quad \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq 4|P'(z)|.$$

Note that Theorem 1.1 can also be stated as follows.

Theorem 1.2. [27] (*Smale Mean Value Theorem*) Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that

$$(2) \quad \left| \frac{P(z) - P(w)}{z - w} \right| \leq 4|P'(z)|.$$

The proof of Theorem 1.2 can also be found in books [19, 24]. Inequality (1) led Smale to ask the following problem. What is the best constant $4 > c > 0$ such that the Inequality (1) can be replaced by

$$(3) \quad \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq c|P'(z)|?$$

Smale made the following conjecture with regard to Inequality (3), which remains open today.

Conjecture 1.3. [25–30] (*Smale Mean Value Conjecture*) Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then

$$\min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq |P'(z)|$$

or

$$(4) \quad \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq \frac{n-1}{n} |P'(z)| = \frac{\deg(P)-1}{\deg(P)} |P'(z)|.$$

By taking polynomial $P(z) := z^n - nz \in \mathbb{C}[z]$, we see that the constant $(n-1)/n$ in inequality (4) cannot be improved further. Conjecture 1.3 also appears as Problem 1 in the Addenda after the list of 18 open problems of Smale in **Mathematical Problems for the Next Century** [30]. Now for a polynomial P of degree $n \geq 2$, define

$$S(P) := \sup_{z \in \mathbb{C}, P'(z) \neq 0} \frac{1}{|P'(z)|} \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|$$

and

$$M := \sup_{P \text{ is a non linear polynomial}} S(P).$$

Smale conjecture then asks whether $M = 1$ or $S(P) = \frac{\deg(P)-1}{\deg(P)}$? It is known that the Conjecture 1.3 holds for polynomials of degree 2, 3, 4, [19, 24, 31]. Through computer calculations, Smale conjecture is known to hold for polynomials of degree upto 10 [23].

Usually, the study of Smale mean value conjecture goes with the following conjecture which is known as normalization equivalence to Conjecture 1.3.

Conjecture 1.4. [19, 24, 27] (*Normalized Smale Mean Value Conjecture*) Let P be a degree $n \geq 2$ polynomial over $\mathbb{C}$ which is normalized, i.e., $P(0) = 0$ and $P'(0) = 1$. Then

$$S_0(P) := \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(w)}{w} \right| = \frac{\deg(P)-1}{\deg(P)}$$

or

$$M_0 := \sup_{P \text{ is a non linear polynomial}, P(0)=0, P'(0)=1} S_0(P) = 1.$$

In 1996, Andrievskii and Ruscheweyh showed that if critical points of a normalized polynomial P lie on the unit circle, then $S_0(P) \leq 1$ [1]. In 2009, Hinkkanen and Kayumov showed that if all the critical points of a normalized polynomial P are real, then $S_0(P) \leq \frac{2}{3}$ [12]. In 2010, Hinkkanen and Kayumov proved that if the critical points of normalized polynomial P lie on the sector $\{re^{i\theta} : r > 0, |\theta| \leq \pi/6\}$, then $S_0(P) \leq \frac{1}{2}$ and if the critical points of normalized polynomial P lie in the union of two rays $\{1 + re^{\pm i\theta} : r \geq 0\}$ for each $0 < \theta \leq \pi/2$, then $S_0(P) \leq \frac{2}{3}$ [13]. In [19] it is showed that if the critical points of a normalized polynomial are all real, then $S_0(P) \leq \frac{\deg(P)-2}{\deg(P)} \left[\left(\frac{\deg(P)-1}{\deg(P)-2} \right)^{\deg(P)-1} - 2 \right]$ and in [24] it is showed that $S_0(P) \leq e - 2$.

In 2002, Beardon, Minda and Ng showed that $S(P) \leq 4^{\frac{\deg(P)-2}{\deg(P)-1}}$ [2]. In 2006, Fujikawa and Sugawa proved that $S(P) \leq 4^{\frac{1+(\deg(P)-2)4^{\frac{1}{\deg(P)-1}}}{\deg(P)+1}}$ [11]. In 2007, Conte, Fujikawa and Lakic showed that $S(P) \leq 4^{\frac{\deg(P)-1}{\deg(P)+1}}$ [3]. In 2007, Crane showed that there exists $n_0 \geq 2$ such that for all $n \geq n_0$, $S(P) \leq 4 - \frac{2}{\sqrt{\deg(P)}}$ [5] (also see [4]). In 2003, Ng proved that Conjecture 1.3 holds with $M = 2$ for all odd polynomials with

nonzero linear terms [16]. In 2006, Dubinin proved that if the critical points of a normalized polynomial P have equal modulus, then $S_0(P) \leq \frac{\deg(P)-1}{\deg(P)}$ [10].

At the other side, in 2002, Sendov formulated a generalization of Conjecture 1.3 [21]. In 2003, Sendov and Nikolov gave some variations of Conjecture 1.3 [22]. In 2005, Tyson showed that Tischler’s strong form of Smale mean value conjecture [31, 32] fails [33]. One can interpret Conjecture 1.3 in the form of finite Blaschke products [17, 18, 24]. It is also possible to connect Conjecture 1.3 with electrostatics [6]. In 1981, Smale also derived the following higher order mean value theorem [27].

Theorem 1.5. [27] (*Higher Order Smale Mean Value Theorem*) *Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that*

$$(5) \quad \frac{|P^{(k)}(z)|}{k!} \frac{|P(z) - P(w)|^{k-1}}{|P'(z)|^k} \leq 4^{k-1}, \quad \forall 2 \leq k \leq n.$$

In 2009, Dubinin and Sugawa derived the following striking result which is a dual result for Smale mean value theorem [9].

Theorem 1.6. [7, 9] (*Dubinin-Sugawa Dual Mean Value Theorem*) *Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that*

$$\frac{|P'(z)|}{n4^n} \leq \left| \frac{P(z) - P(w)}{z - w} \right|.$$

In other words,

$$\frac{|P'(z)|}{n4^n} \leq \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

Dubinin and Sugawa made the following conjecture in 2009 [9].

Conjecture 1.7. [7, 9] (*Dubinin-Sugawa Dual Mean Value Conjecture*) *Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then*

$$\frac{|P'(z)|}{n} \leq \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

For a polynomial P of degree $n \geq 2$, define

$$DS(P) := \inf_{z \in \mathbb{C}, P'(z) \neq 0} \frac{1}{|P'(z)|} \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

Similar to Conjecture 1.4, we have the following normalized version of Conjecture 1.7.

Conjecture 1.8. [7, 9] (*Normalized Smale Dubinin-Sugawa Dual Mean Value Conjecture*) *Let P be a degree $n \geq 2$ polynomial over $\mathbb{C}$ which is normalized, i.e., $P(0) = 0$ and $P'(0) = 1$. Then*

$$DS_0(P) := \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(w)}{w} \right| = \frac{1}{n} = \frac{1}{\deg(P)}.$$

In 2016, Ng and Zhang showed that if P is a normalized polynomial, then $DS_0(P) > \frac{1}{4^{\deg(P)}}$ [17]. In 2019 Dubinin showed that $DS(P) \geq \frac{1}{n} \tan\left(\frac{\pi}{4^{\deg(P)}}\right)$ [8]. In 2019, Hinkkanen, Kayumov and Khammatova showed that Conjecture 1.7 holds for polynomials of degree six [14]. In 2025, Hinkkanen, Kayumov and Khammatova showed that Conjecture 1.7 holds for polynomials of degree six [15].

In this paper, we ask for non-Archimedean versions of Theorems 1.1 and 1.6. We show that if universal constants exist for the non-Archimedean Smale and the Dubinin–Sugawa dual mean value problems, the former is bounded below by unity, whereas the latter is bounded above by unity.

2. NON-ARCHIMEDEAN SMALE MEAN VALUE PROBLEM

Let $\mathbb{K}$ be a field. Recall that a map $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ is said to be a **non-Archimedean valuation** if the following conditions hold:

- (i) If $\lambda \in \mathbb{K}$ satisfies $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (iii) (Ultratriangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

We say $\mathbb{K}$ is non-Archimedean complete valued field if it is complete w.r.t. the ultrametric $d(\lambda, \mu) := |\lambda - \mu|$ for all $\lambda, \mu \in \mathbb{K}$ [20]. Throughout the article we assume that $\mathbb{K}$ is algebraically closed.

We ask for non-Archimedean version of Theorem 1.2 as follows.

Problem 2.1. (Non-Archimedean Smale Mean Value Problem) *Let $\mathbb{K}$ be a non-Archimedean (complete) valued field. Whether there is a universal constant $S_{\mathbb{K}}$ (which may depend upon $\mathbb{K}$) satisfying the following: Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{K}$. If $z \in \mathbb{K}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{K}$ of P such that*

$$\left| \frac{P(z) - P(w)}{z - w} \right| \leq S_{\mathbb{K}} |P'(z)|.$$

In the following result we derive a constraint for non-Archimedean Smale mean value constant, if exists.

Theorem 2.2. *Let $\mathbb{K}$ be a non-Archimedean valued field. The constant $S_{\mathbb{K}}$ in Problem 2.1, if exists, satisfies $S_{\mathbb{K}} \geq 1$.*

Proof. Let $P(z) := (z - a)(z - b) \in \mathbb{K}[z]$. Then $P'(z) = 2z - (a + b)$ for all $z \in \mathbb{K}$. Let $z \in \mathbb{K}$ be such that $P'(z) \neq 0$. Note that P' has only one zero, namely $c := \frac{a+b}{2}$. Define $x := z - a$ and $y := z - b$. Then

$$\begin{aligned} z - c &= z - \frac{a+b}{2} = \frac{z - a + z - b}{2} = \frac{x + y}{2}, \\ P'(z) &= z - a + z - b = x + y, \\ P(c) &= \left(\frac{a+b}{2} - a \right) \left(\frac{a+b}{2} - b \right) = \frac{-(b-a)^2}{4} = \frac{-(x-y)^2}{4} \end{aligned}$$

and

$$P(z) - P(c) = xy + \frac{(x-y)^2}{4} = \frac{(x+y)^2}{4}.$$

Hence

$$\left| \frac{P(z) - P(c)}{z - c} \right| = \left| \frac{x+y}{2} \right| = \frac{1}{|2|} |P'(z)|.$$

Therefore

$$S_{\mathbb{K}} \geq \frac{1}{|2|} \geq 1.$$

□

Theorem 2.3. *Let $\mathbb{K}$ be a non-Archimedean valued field. Assume that there is an $n \in \mathbb{N} \setminus \{1, 2\}$ such that $\mathbb{K}$ has $(n-1)$ -th roots of unity. The constant $S_{\mathbb{K}}$ in Problem 2.1, if exists, satisfies $S_{\mathbb{K}} \geq \frac{|n-1|}{|n|} = \left| 1 - \frac{1}{n} \right|$.*

Proof. We define the polynomial $P(z) := z^n - nz \in \mathbb{K}[z]$. The point $z = 0$ is not a critical point of P . Let $w \in \mathbb{K}$ be any critical point of P . Then w is a $(n-1)$ -th root of unity. We now see that

$$\begin{aligned} \frac{|P(0) - P(w)|}{|0 - w|} &= \frac{|P(w)|}{|w|} = |w^n - nw| = |w^{n-1} - n| \\ &= |1 - n| = \frac{|1 - n|}{|n|} |n| = \frac{|1 - n|}{|n|} |P'(0)|. \end{aligned}$$

Hence $S_{\mathbb{K}} \geq \frac{|n-1|}{|n|} = |1 - \frac{1}{n}|$. □

Based on Theorem 1.5 we formulate the following higher order problem.

Problem 2.4. (Higher Order Non-Archimedean Smale Mean Value Problem) *Let $\mathbb{K}$ be a non-Archimedean (complete) valued field. Whether there is a universal constant $R_{\mathbb{K}}$ (which may depend upon $\mathbb{K}$) satisfying the following: Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{K}$. If $z \in \mathbb{K}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{K}$ of P such that*

$$\frac{|P^{(k)}(z)|}{k!} \frac{|P(z) - P(w)|^{k-1}}{|P'(z)|^k} \leq R_{\mathbb{K}}^{k-1}, \quad \forall 2 \leq k \leq n.$$

Theorem 2.5. *Let $\mathbb{K}$ be a non-Archimedean valued field. The constant $R_{\mathbb{K}}$ in Problem 2.4, if exists, satisfies $R_{\mathbb{K}} \geq 1/(2|2|)$.*

Proof. We continue from the proof of Theorem 2.2. Let $z \in \mathbb{K}$ be not a zero of P' . We have $P''(z) = 2$. Hence

$$|P''(z)| = |2|, \quad |P'(z)| = |x + y|, \quad |P(z) - P(c)| = \frac{|x + y|^2}{|4|}.$$

We find

$$\frac{|P^{(2)}(z)|}{2!} \frac{|P(z) - P(c)|^{2-1}}{|P'(z)|^2} = \frac{|2| \left(\frac{|x+y|^2}{|4|} \right)}{2 |x+y|^2} = \frac{1}{2|2|}.$$

Therefore

$$R_{\mathbb{K}} \geq \frac{1}{2|2|}.$$

□

3. NON-ARCHIMEDEAN DUBININ-SUGAWA DUAL MEAN VALUE PROBLEM

We ask non-Archimedean version of Theorem 1.6 as follows.

Problem 3.1. (Non-Archimedean Dubinin-Sugawa Dual Mean Value Problem) *Let $\mathbb{K}$ be a non-Archimedean (complete) valued field. Whether there is a universal constant $D_{\mathbb{K}}$ (which may depend upon $\mathbb{K}$) satisfying the following: Let $P(z) := a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{K}$. If $z \in \mathbb{K}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{K}$ of P such that*

$$D_{\mathbb{K}} |P'(z)| \leq \frac{|P(z) - P(w)|}{|z - w|}.$$

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean valued field. The constant $D_{\mathbb{K}}$ in Problem 3.1, if exists, satisfies $D_{\mathbb{K}} \leq 1$.*

Proof. From the proof of Theorem 2.2, we get that $D_{\mathbb{K}} \leq |2| \leq 1$. □

Theorem 3.3. *Let $\mathbb{K}$ be a non-Archimedean valued field. Assume that there is an $n \in \mathbb{N} \setminus \{1, 2\}$ such that $\mathbb{K}$ has $(n - 1)$ -th roots of unity. The constant $D_{\mathbb{K}}$ in Problem 3.1, if exists, satisfies $D_{\mathbb{K}} \leq \frac{|n|}{|n-1|}$.*

Proof. By defining polynomial $P(z) := z^n - nz \in \mathbb{K}[z]$ we get that $D_{\mathbb{K}} \leq \frac{|n|}{|n-1|}$. $\square$

4. CONCLUSION

For 45 years, the Smale mean value theorem has remained a cornerstone result in the theory of polynomials, while the Dubinin–Sugawa dual mean value theorem has served a similar vital role for 17 years. In this article, we asked for non-Archimedean versions of both theorems. We established explicit bounds for these constants, conditioning on their existence.

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