

ASYMPTOTICALLY OPTIMAL DEGREE FOR THE DOUBLE HALL PROPERTY

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ABSTRACT. A bipartite graph with parts X and Y has the double Hall property if every $A \subseteq X$ with $|A| \geq 2$ has at least $|A|$ vertices in Y with two or more neighbours in A . Chen, Lavrov, Ma, Su and Vandenbussche showed that such a graph with $|X| = n$ has maximum degree at least $(1 + o(1))\sqrt{2n}$, gave balanced regular constructions along infinitely many orders of degree $n^{\log_{79} 13 + o(1)} \approx n^{0.587}$, and random balanced constructions of degree $(1 + o(1))\sqrt{2n \log n}$. We show that the counting bound is asymptotically sharp: for every sufficiently large n there is a balanced D -regular bipartite graph with n vertices in each part, with the double Hall property on *both* sides, and

$$D \leq \sqrt{2n} + O(n^{21/80}).$$

The construction is deterministic. It repairs the empty class rectangles of the classical homology semiplane by rook-type local graphs, then pads and completes the result to any order. The error exponent is half of any admissible short-prime-interval exponent.

1. INTRODUCTION

Let G be a bipartite graph with parts X and Y . For $A \subseteq X$ write

$$\text{SN}(A) = \{y \in Y : |N(y) \cap A| \geq 2\}$$

for the *super-neighbours* of A . Following Chen, Lavrov, Ma, Su and Vandenbussche [3], G has the *double Hall property* (on X) if $|\text{SN}(A)| \geq |A|$ for every $A \subseteq X$ with $|A| \geq 2$. The property is motivated by cycle questions of Salia and others; see [3, 2].

For $n \geq 2$, let $f(n)$ be the least maximum degree of a double-Hall graph with $|X| = n$, and let $f_{\text{reg}}(n)$ be the least d for which a balanced d -regular graph with n vertices in each part has the property. Applying the definition to the pairs $\{x, x'\}$ shows that every two vertices of X have at least two common neighbours. Counting paths of length two from a fixed x gives

$$(1) \quad 2(n-1) \leq \Delta(\Delta-1), \quad \text{so} \quad f(n) \geq \frac{1}{2}(1 + \sqrt{8n-7})$$

[3, Theorem 1.10]. The same paper constructs balanced regular examples along infinitely many orders of degree $n^{\log_{79} 13 + o(1)}$ [3, Theorem 1.11] and shows that random balanced graphs of degree $(1 + o(1))\sqrt{2n \log n}$ work [3, Theorem 1.12].

Write $f_{\text{bi,reg}}(n)$ for the least degree of a balanced regular graph with the double Hall property on both X and Y . Clearly $f(n) \leq f_{\text{reg}}(n) \leq f_{\text{bi,reg}}(n)$.

Theorem 1. *As $n \rightarrow \infty$,*

$$f(n), f_{\text{reg}}(n), f_{\text{bi,reg}}(n) = (1 + o(1))\sqrt{2n}.$$

More precisely, $f_{\text{bi,reg}}(n) \leq \sqrt{2n} + O(n^{21/80})$ for all sufficiently large n .

The exponent comes from prime gaps. We prove the following transfer statement.

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Theorem 2. *Let $\frac{1}{2} \leq \theta < 1$. Suppose that for all sufficiently large real x there is a prime in $[x - x^\theta, x]$. Then $f_{\text{bi,reg}}(n) \leq \sqrt{2n} + O(n^{\theta/2})$, and a witnessing graph can be computed deterministically in time polynomial in n .*

Baker, Harman and Pintz [1] proved the hypothesis with $\theta = 21/40$, which gives Theorem 1. Their threshold is effectively determinable, although we do not compute a numerical value here. Any improvement in short-interval prime results improves the error term in the same way; for example, the preprint [4] claims $\theta = 0.52$.

Along the orders $N = (q^2 - 1)/2$ with q an odd prime power, no prime-gap input is needed.

Theorem 3. *Let $q \geq 1153$ be an odd prime power, $N = (q^2 - 1)/2$ and $g = (q - 1)/2$. There is a balanced regular bipartite graph with N vertices in each part, with the double Hall property on both sides, of degree*

$$q + 2\lfloor \sqrt{g} \rfloor + 4 = \sqrt{2N} + O(N^{1/4}).$$

The constant 1153 is the first odd q with $g = (q - 1)/2 \geq 576$ (Lemma 12). For smaller q one can use $K_{g,g}$ in the repair step, at the cost of a worse second-order term.

Method. The starting point is the classical homology semiplane over $\mathbb{F}_q$ [5]. Its signed incidence matrix has orthogonal rows, so by the strong-quadrangularity principle of Severini [6, 7] every point set that is not contained in one parallel class has enough super-neighbours (Section 4). The only failures are sets inside a single parallel class, and each such class faces an empty block class. We fill these $q + 1$ empty rectangles with a balanced regular double-Hall graph of degree $O(\sqrt{q})$, built from rook graphs (Section 5). This costs only a lower-order term in the degree. To reach every order n we duplicate whole parallel classes, which raises each degree by an explicitly bounded amount, and then complete to a regular graph using a sparse degree-completion lemma proved from Hall's theorem (Section 6).

What is not new. Homology semiplanes, strong quadrangularity, rook graphs, vertex twinning and degree completion via flows or matchings are classical; we credit each where it is used. The contribution is their combination for the double Hall degree problem, which closes the gap between the counting bound (1) and the best previous construction.

2. NOTATION AND TWO MONOTONICITY FACTS

All graphs are finite, simple and bipartite with a fixed bipartition (X, Y) . Adding edges cannot destroy a super-neighbour, so the double Hall property is preserved under adding edges. If X is partitioned into classes, a set $A \subseteq X$ with $|A| \geq 2$ either meets two classes or lies in one class.

Lemma 4 (class repair). *Let X be partitioned into classes. Let B and L be bipartite graphs on (X, Y) . Suppose $|\text{SN}_B(A)| \geq |A|$ whenever $|A| \geq 2$ and A meets two classes, and $|\text{SN}_L(A)| \geq |A|$ whenever $|A| \geq 2$ and A lies in one class. Then $B \cup L$ has the double Hall property on X . If B and L are edge-disjoint, degrees in $B \cup L$ are sums of degrees in B and in L .*

3. ORTHOGONAL SUPPORTS

Lemma 5 (strong quadrangularity [6, 7]). *Let W be a real matrix with rows indexed by X and columns by Y . Let $A \subseteq X$, and let $C(A)$ be the set of columns with at least two nonzero entries in the rows of A . Suppose the rows of A are pairwise orthogonal, and every row of A shares a nonzero column with another row of A . Then $|C(A)| \geq |A|$.*

Proof. Restrict the rows of A to the columns in $C(A)$. A column outside $C(A)$ has at most one nonzero entry in the rows of A , so deleting it changes no inner product between distinct rows of A . Hence the restricted rows are still pairwise orthogonal. Every row of A shares a nonzero column y with another row of A ; then $y \in C(A)$, so the restricted row is nonzero. Pairwise orthogonal nonzero vectors are linearly independent, so $|A| \leq \dim \mathbb{R}^{C(A)} = |C(A)|$. $\square$

4. THE HOMOLOGY SEMIBIPLANE

Let q be an odd prime power and $\mathbb{F} = \mathbb{F}_q$. Let P be the set of nonzero vectors of $\mathbb{F}^2$ modulo $u \sim -u$, so $|P| = (q^2 - 1)/2 =: N$. Take X and Y to be two disjoint copies of P and join $[u] \in X$ to $[v] \in Y$ when $u \cdot v \in \{1, -1\}$. This relation does not depend on the chosen representatives. This is the Desarguesian homology semiplane [5, §§1–2].

Lemma 6. (1) *The graph is q -regular.*

- (2) *Partition X into the $q+1$ parallel classes $C_\ell = \{[u] : u \in \ell\}$, one for each one-dimensional subspace ℓ ; each has $g = (q-1)/2$ elements. Partition Y into $E_\ell = \{[v] : v \in \ell^\perp\}$. There are no edges between C_ℓ and E_ℓ .*
- (3) *Every $y \in Y$ has at most one neighbour in each class C_ℓ , and every $x \in X$ has at most one neighbour in each class E_ℓ .*

Proof. (1) For $u \neq 0$, each affine line $\{v : u \cdot v = s\}$ has q points, and the lines for $s = 1$ and $s = -1$ are disjoint and swapped by $v \mapsto -v$. So exactly q classes $[v]$ are adjacent to $[u]$. The same holds with the roles of u and v exchanged. (2) A line through the origin has $q-1$ nonzero vectors, forming $(q-1)/2$ classes. If $u \in \ell$ and $v \in \ell^\perp$ then $u \cdot v = 0$. (3) If u and λu both satisfy $u \cdot v, \lambda u \cdot v \in \{\pm 1\}$, then $\lambda = \pm 1$, so $[\lambda u] = [u]$. The second statement is symmetric. $\square$

Proposition 7. *If $A \subseteq X$ meets two parallel classes, then $|\text{SN}(A)| \geq |A|$. The same holds on Y .*

Proof. Choose representatives and define $W_{[u],[v]} = 1$ if $u \cdot v = 1$, $W_{[u],[v]} = -1$ if $u \cdot v = -1$, and 0 otherwise. Its nonzero pattern is the incidence. Write $\sigma(t) \in \{1, -1, 0\}$ for this sign of $t \in \mathbb{F}$, so $\sigma(-t) = -\sigma(t)$.

Orthogonality. Let $[a] \neq [b]$. The product $\sigma(a \cdot v)\sigma(b \cdot v)$ is even in v , so the inner product of rows $[a]$ and $[b]$ is half the sum of this product over all $v \in \mathbb{F}^2$. If a, b are linearly independent, $v \mapsto (a \cdot v, b \cdot v)$ is a bijection of $\mathbb{F}^2$. The sum then equals $(\sum_{s \in \mathbb{F}} \sigma(s))^2 = 0$. If $b = \lambda a$, both factors are nonzero only when $\lambda = \pm 1$, which would give $[a] = [b]$. So every term vanishes.

Sharing. If A meets two classes, every $[a] \in A$ has some $[b] \in A$ in another class. Then a, b are independent, so some v has $a \cdot v = b \cdot v = 1$, and column $[v]$ is nonzero in both rows.

Lemma 5 now gives $|\text{SN}(A)| = |C(A)| \geq |A|$. The incidence is symmetric under exchanging X and Y , and the classes E_ℓ correspond to the classes $C_{\ell^\perp}$, so the same argument applies on Y . $\square$

5. ROOK GRAPHS AND MATCHED DUPLICATION

Lemma 8 (rook cross). *For $m \geq 2$ let $X = Y = [m]^2$ (disjoint copies), with $(a, b) \in X$ adjacent to $(i, j) \in Y$ when $a = i$ or $b = j$. This graph is $(2m-1)$ -regular, contains the diagonal matching $(a, b) \leftrightarrow (a, b)$, and has the double Hall property on both sides.*

Proof. Regard $A \subseteq X$ as a set of occupied cells. An occupied cell is itself a super-neighbour unless it is alone in both its row and its column; call such cells *isolated*. All other occupied cells are distinct super-neighbours. If there are $k \geq 2$ isolated cells $(a_1, b_1), \dots, (a_k, b_k)$, their rows and columns are distinct, and the unoccupied cells (a_i, b_{i+1}) (indices mod k) are k further distinct super-neighbours. If there is exactly one isolated cell (a, b) , take another occupied cell

(c, d) . The cell (a, d) is unoccupied, since (a, b) is alone in its row, and it sees both. So it is one more super-neighbour. In all cases $|\text{SN}(A)| \geq |A|$. The graph is symmetric. $\square$

Lemma 9 (matched duplication). *Let H be balanced, with $X = \{x_i\}$, $Y = \{y_i\}$ and every $x_i y_i$ an edge, and with the double Hall property on both sides. For a set T of indices, duplicate x_i and y_i for each $i \in T$, copying all incidences, including those between duplicates. The result has the double Hall property on both sides. If H is d -regular, the new degree of a copy of x_i is $d + |N_H(x_i) \cap \{y_j : j \in T\}|$, and symmetrically on Y .*

Proof. Let A be a set of new left vertices, B its set of original indices, and k the number of doubled indices with both copies in A , so $|A| = |B| + k$. If $|B| \geq 2$, every super-neighbour of B in H is a super-neighbour of A . Each doubled index i also gives the duplicate of y_i , which sees both copies of x_i ; these are k further distinct super-neighbours. If $|B| = 1$, then A is one doubled pair, which has two common neighbours y_i, y'_i . The other side is symmetric. $\square$

6. SPARSE REGULAR COMPLETION

Lemma 10 (completion). *Let H be a balanced bipartite graph with n vertices per side and maximum degree at most $D \leq n$. Let $\alpha_x = D - \deg_H x$ and $\beta_y = D - \deg_H y$, with total deficit $T_0 = \sum \alpha_x = \sum \beta_y$, largest deficits $a = \max \alpha$, $b = \max \beta$, and largest degrees $r = \max_X \deg_H$, $c = \max_Y \deg_H$. If*

$$T_0 \geq ac + br + 2ab,$$

then H is a subgraph of a simple D -regular bipartite graph on the same parts.

Proof. We need a subgraph F of the bipartite complement $\bar{H}$ with $\deg_F x = \alpha_x$ and $\deg_F y = \beta_y$.

Reduction to a perfect matching. Build a bipartite graph Γ (Tutte's gadget). Its left part is α_x copies of each x , together with one vertex e^Y for each edge e of $\bar{H}$. Its right part is β_y copies of each y , together with one vertex e^X for each e . Join every copy of x to e^X for each $e \ni x$. Join e^Y to e^X . Join e^Y to every copy of y for each $e \ni y$. Both parts have $T_0 + |E(\bar{H})|$ vertices. In a perfect matching, put $e = xy$ into F exactly when e^X is matched to a copy of x . This happens exactly when e^Y is matched to a copy of y . Then F has the required degrees.

Hall's condition. Let S be a set of left vertices. Let U be the set of x having a copy in S , and let S_2 be the set of gadget edges e^Y in S , with Y -endpoints V . Then $N(S)$ contains every e^X with e at U or $e \in S_2$, and every copy of every $y \in V$. Enlarging S_2 to all edges of $\bar{H}$ at V does not change V , so in checking $|N(S)| \geq |S|$ we may assume this. With $V' = Y \setminus V$ the requirement becomes

$$(2) \quad \alpha(U) + \beta(V') - T_0 \leq e_{\bar{H}}(U, V') \quad (U \subseteq X, V' \subseteq Y).$$

If $|V'| \geq r + a$ then $e_{\bar{H}}(U, V') \geq |U|(|V'| - r) \geq a|U| \geq \alpha(U)$, which dominates the left side. If $|U| \geq c + b$ the symmetric bound applies. Otherwise $\alpha(U) + \beta(V') \leq a(c + b) + b(r + a) \leq T_0$. So (2) holds, and Hall's theorem gives the perfect matching. $\square$

Remark 11. The inequality (2) is the cut condition of the integral flow formulation, which also gives a polynomial-time algorithm. The reduction through Hall's theorem is used here because it is elementary and is the route taken in the formalization.

Lemma 12 (rook interpolation). *For every $g \geq 576$ there is a balanced regular bipartite graph on $g + g$ vertices with the double Hall property on both sides, containing a perfect matching, of degree $2\lfloor\sqrt{g}\rfloor + 4$.*

Proof. Let $m = \lfloor\sqrt{g}\rfloor$ and $t = g - m^2 \leq 2m$. In the rook-cross graph on $[m]^2$ duplicate t distinct cells from the two cyclic diagonals $\{(i, i)\} \cup \{(i, i+1)\}$. Every row and every column of $[m]^2$ then contains at most two selected cells, so each rook neighbourhood contains at most four selected

cells. An original vertex and its duplicate have the same original neighbourhood, so each gains at most four edges from the duplication. By Lemma 9, all degrees lie in $[2m - 1, 2m + 3]$. Complete to $D = 2m + 4$ by Lemma 10. Here $T_0 \geq g$, $a, b \leq 5$ and $r, c \leq 2m + 3$, so the condition holds once $m^2 \geq 20m + 80$, that is, $m \geq 24$. Completion adds edges, so both double Hall properties and the diagonal matching survive by monotonicity. $\square$

7. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 3. Take the semiplane of Section 4. For each ℓ identify C_ℓ and E_ℓ with the two parts of the graph of Lemma 12 on g vertices, of degree $h = 2\lfloor\sqrt{g}\rfloor + 4$, and insert it into the empty rectangle $C_\ell \times E_\ell$. The inserted graphs are edge-disjoint from the base and from each other. So the result is $(q + h)$ -regular. Sets meeting two classes are handled by Proposition 7, sets inside one class by the local graph, and Lemma 4 applies on both sides. $\square$

Proof of Theorem 2. Let $Q = \sqrt{2n + 1}$. By the hypothesis there is an odd prime $q \leq Q$ with $Q - q = O(Q^\theta)$. Put $N = (q^2 - 1)/2 \leq n$, $g = (q - 1)/2$ and $t = n - N = (Q - q)(Q + q)/2 = O(q^{1+\theta})$. Let G be the d -regular graph of Theorem 3, $d = q + O(\sqrt{q})$. The local diagonals give a perfect matching $x \leftrightarrow \pi(x)$ with $\pi(C_\ell) = E_\ell$.

Class-block padding. Write $t = kg + s$ with $0 \leq s < g$, and order the parallel classes $\ell_1, \ell_2, \dots$. Let T consist of all indices in $C_{\ell_1}, \dots, C_{\ell_k}$ together with s indices of $C_{\ell_{k+1}}$. This needs $k + 1 \leq q + 1$, which holds for large n since $t = o(N)$. Duplicate T by Lemma 9. Consider any original $x \in C_\ell$. By Lemma 6(3) its base neighbours meet each block class in at most one vertex. Its local neighbours lie in E_ℓ . The duplicated columns lie in at most $k + 1$ block classes. Hence the load of x is at most

$$R := k + 1 + h,$$

and likewise on Y . All degrees now lie in $[d, d + R]$, and both sides have the double Hall property.

Completion. Set $D = d + R + 1$. Then $T_0 \geq n$, $a, b \leq R + 1$ and $r, c \leq d + R$, so Lemma 10 applies once $n \geq 2(R + 1)(d + R) + 2(R + 1)^2$. Since $k \leq t/g = O(q^\theta)$ and $h = O(\sqrt{q})$, we have $R = O(q^\theta)$ and $(R + 1)(d + R) = O(q^{1+\theta}) = o(n)$, so the condition holds for large n . Completion only adds edges, and therefore preserves the double Hall property on both sides. The final degree is

$$D = q + O(q^\theta) = \sqrt{2n} + O(n^{\theta/2}).$$

Every step is explicit: finding q , finite-field arithmetic, the rook graphs, duplication, and two completions computed by integral max-flow. $\square$

Proof of Theorem 1. Combine Theorem 2 with $\theta = 21/40$ [1] and the lower bound (1), which applies already to one side. $\square$

8. REMARKS AND OPEN PROBLEMS

- (1) The order of magnitude of the repair degree is optimal for this base. If edges are added to the homology semiplane so that every vertex gains at most r edges and double Hall holds on both sides, then $r(r + 1) \geq q - 3$. Indeed, same-class pairs have no common neighbour in the base, and each pair needs two. At each column, new-old common-neighbour witnesses contribute at most r , and new-new witnesses at most $\binom{r}{2}$. Thus $N(g - 1) \leq N(r + \binom{r}{2})$, giving the asserted bound. So the $N^{1/4}$ -order second term in Theorem 3 cannot be improved by local repair of this base.
- (2) Exact values of $f_{\text{reg}}(n)$ remain open. Equality in (1) forces a biplane-type structure. Infinite families of biplanes are not known.
- (3) Our graphs say nothing directly about the cycle conjectures that motivate the double Hall property [3, 2].

FORMAL VERIFICATION

The counting, construction and existence statements mapped below have a Lean 4 formalization with Mathlib (commit `0ed37c9258f1`). The prime-interval input [1] is an explicit hypothesis of the formal transfer theorem, not a result proved inside the companion. The algorithmic complexity remarks and the augmentation lower bound in the first item of the preceding section are written arguments, not claims of formal coverage by this companion. Every declaration in the recorded endpoint audit depends only on `propxt`, `Classical.choice` and `Quot.sound`.

Paper	Lean declaration (namespace <code>DoubleHall</code>)
Lower bound (1)	<code>pair_count_lower_bound</code> , <code>hasRegularBoth_lower</code>
Lemma 4	<code>doubleHallLeft_union_of_classes</code> , <code>localRel_hall</code>
Lemma 5	<code>card_le_card_multiCols</code>
Lemma 6, Prop. 7	<code>Semibiplane.degL_inc</code> , <code>two_mul_card_cls</code> , <code>not_inc_of_cls_eq_clsY</code> , <code>inc_cls_unique</code> , <code>W_orthogonal</code> , <code>card_le_superNbrs_of_not_inOneClass</code>
Lemma 8	<code>Rook.doubleHallLeft</code> , <code>Rook.degL_eq</code>
Lemma 9	<code>Dup.doubleHallLeft</code> , <code>Dup.degL_dup</code>
Lemma 10	<code>exists_factor</code> , <code>cutCondition_of_sparse</code> , <code>exists_regular_supergraph</code>
Lemma 12	<code>RookInterp.exists_local</code>
Theorem 3	<code>semibiplane_order_theorem</code>
Class-block padding	<code>Semibiplane.padding_theorem</code>
Theorem 2	<code>transfer_theorem</code> (explicit constant 33)

The formal statement of Theorem 2 is: if $\frac{1}{2} \leq \theta < 1$ and every sufficiently large real x has a prime in $[x - x^\theta, x]$, then for all large n there is a balanced D -regular graph with n vertices per side, double Hall on both sides, and $D \leq \sqrt{2n} + 33n^{\theta/2}$. The algorithmic remarks in the proofs are not formalized: the formal theorems are existence statements.

USE OF AI TOOLS

AI tools were used for exploration, drafting, computation and formalization. The author is responsible for the mathematical statements, proofs, source comparisons and final text.

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