

An explicit elementary proof of no Landau–Siegel zeros

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Abstract

We give an elementary proof of the absence of Landau–Siegel zeros with the explicit constant $c = 1/100$. For every primitive quadratic Dirichlet character χ of conductor $q \geq 3$, each real zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies $(1 - \beta) \log q > c$. The proof combines integer coefficient counting, polynomial congruences, a paired-tail hyperbola estimate, and a logarithmically weighted comparison of positive sums. All required constants are explicit, and one choice of parameters covers every conductor. The argument uses elementary arithmetic and real summation and integration, without separate zero computations for small conductors.

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1 Introduction

Our aim is to prove a Landau–Siegel zero gap by elementary methods while retaining a fully explicit numerical constant. We obtain $(1 - \beta) \log q > c$ with $c = 1/100$ in Theorem 1.1, using a single choice of parameters for every conductor $q \geq 3$. All estimates needed for this value of c have specified constants and remainders, including the estimates for small conductors. The proof uses integer and polynomial arithmetic, partial summation, and real integration.

The argument follows the Frobenius method of [7] and its subsequent variants [2, 5, 4, 14, 9]. For an exposition of the polynomial-method framework, see also Tao [12]. The arithmetic part combines polynomial congruences with integer coefficient counting; injectivity follows by counting relatively prime quadratic factors modulo a prime. The real-variable part pairs character tails before estimating them and then compares logarithmically weighted positive sums.

Throughout, χ denotes a primitive quadratic Dirichlet character of conductor $q \geq 3$, unless otherwise stated; see [1, Sections 6.5, 6.8, and 8.7–8.9].

The letter p denotes a prime, logarithms are natural, and

$$S_-(x, \chi) = \sum_{\substack{p \leq x \\ \chi(p) = -1}} \frac{\log p}{p}.$$

For $s > 0$, we use

$$L(s, \chi) = \lim_{x \rightarrow \infty} \sum_{n \leq x} \frac{\chi(n)}{n^s} \quad (s > 0).$$

The series converges by bounded character sums and partial summation [1, Theorem 6.17]. The condition $L(\beta, \chi) = 0$ below refers to this series.

Theorem 1.1 (An explicit real zero gap). *Let χ be a primitive quadratic Dirichlet character of conductor $q \geq 3$. Every real zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies*

$$(1 - \beta) \log q > \frac{1}{100}. \quad (1.1)$$

Remark 1.2 (Choice of the constant). The same elementary argument also yields $c = 1/64$; a verification is given in Appendix A. For $q \geq 7$, take $x = q^{46}$ and $y = q^4$ in Proposition 3.1 and Lemma 5.1. For $q = 3, 4, 5$, grouping the character series into positive blocks gives $L(s, \chi) > 0$ for $s > 0$, and no primitive character has conductor 6. We retain $c = 1/100$ so that a single choice of parameters covers all conductors and keeps the numerical verification short.

1.1 Earlier results and the scope of the theorem

Banks [2, Theorem 1.1] gives the constant 0.003 for $q \geq 10^{10}$, and for all $q \geq 3$ when combined with the small-conductor zero computations cited there. Kimura [6, Theorem 1 and its proof] gives 0.0005811 for all $q \geq 3$, 0.005965 for $\log q \geq 35.0131$, and 0.01015 for $\log q \geq 10000$. Thus $1/100$ is larger than their stated all-conductor constants, while Kimura's last bound is larger in its range. Our proof uses a single choice of parameters for all $q \geq 3$ and includes its numerical comparisons as rational inequalities.

The quasi-Riemann result of [8, Theorem 1.1] states that every Dirichlet L -function is zero-free in $\operatorname{Re} s > 7/8$, with the principal-character pole at $s = 1$ excepted. That result already implies the stronger inequality

$$(1 - \beta) \log q \geq \frac{1}{8} \log q \geq \frac{1}{8} \log 3 > \frac{1}{100}.$$

Theorem 1.1 quantifies the elementary Frobenius argument; the stronger half-plane theorem is not an input.

The arithmetic coefficients obtained here are not optimal. Green [4, Theorem 1.1] obtains a leading coefficient $1/2 + O(1/\alpha)$ at $x = q^\alpha$ for sufficiently large even integers α , when q is sufficiently large in terms of α . The stronger estimate attributed to Barreto in [9, Theorem 1], and also proved in [14, Theorem 1], is

$$S_-(x, \chi) \leq \frac{1}{2} \log x + \frac{1}{4} \log q + \log \log(3x) + O(1),$$

with an effective absolute implied constant. Our estimate below instead specifies every coefficient and the absolute remainder, and includes all small-parameter cases.

1.2 Proof outline

The arithmetic input is Proposition 3.1, which proves

$$S_-(x, \chi) \leq \frac{11}{20} \log x + \frac{29}{60} \log q + 6 \quad (x \geq 2). \quad (1.2)$$

For the parameters in (3.4) and an auxiliary prime p_0 with $\chi(p_0) = -1$, Lemma 3.3 proves that the evaluation map (3.5) is injective. If $0 \neq f$ were in its kernel, divide its coefficients by their largest common power of p_0 . The resulting reduction $g(t) = f(t, t^{p_0}) \pmod{p_0}$ is nonzero. The pairwise relatively prime quadratic factors then give the contradiction

$$\prod_{\substack{0 \leq u < m \\ 1 \leq v < m}} ((t - u)^2 - dv^2)^{s_0} \mid g(t) \pmod{p_0},$$

$$2s_0 m(m - 1) \leq \deg g < p_0 s_0 < 2s_0 m(m - 1).$$

The adapted integer bases (3.8) supply the prime-power divisibility (3.10). Lemma 3.2 then gives the upper bound (3.11); comparison with (3.15), followed by the remainder estimate (3.17) and the small cases in the proof of Proposition 3.1, yields (1.2).

For the real-variable part, put

$$a(n) = (1 * \chi)(n), \quad H_\beta(t) = \sum_{n \leq t} \frac{a(n)}{n^\beta}.$$

At a real zero $1/2 \leq \beta < 1$, Proposition 4.1 gives

$$H_\beta(t) = \frac{L(1, \chi)}{1 - \beta} t^{1-\beta} + E(t), \quad |E(t)| \leq \left(q + 1 + \frac{1}{2} \log t \right) t^{1/2-\beta}. \quad (1.3)$$

Estimating the two tails in (4.8) jointly gives (4.9), avoiding the factor $(1 - \beta)^{-1}$ that would arise from estimating them separately.

Finally, Lemma 5.1 converts this estimate into a prime-sum comparison. Writing $H_1(y) = \sum_{n \leq y} a(n)/n$ and $N(t) = \sum_{n \leq t} a(n) \log n/n$, the injection $(n, p) \mapsto np$ for $n \leq y < p \leq x$ gives (5.4), namely

$$H_1(y) \sum_{y < p \leq x} \frac{a(p) \log p}{p} \leq N(xy) - N(y).$$

The partial-summation identity (5.5) and positivity eliminate $L(1, \chi)/(1 - \beta)$ from the comparison, giving (5.1)–(5.2). With the fixed choices $y = q^6$ and $x = q^{70}$ in (5.6), the assumption $(1 - \beta) \log q \leq 1/100$ yields the incompatible bounds (5.13) and (5.14). The final numerical contradiction is (5.15).

1.3 Sources, refinements, and elementary scope

Coefficient counting follows Kerr [5, Lemmas 2–5]; the triangular basis and its derivative divisibility follow Green [4, Lemma 5.1]. The auxiliary-prime nonvanishing argument is the one in [14, proof of Theorem 1, Nonvanishing]; see also the Stepanov-type formulation in [9, Lemma 4]. We express the latter argument through relatively prime quadratic factors and polynomial division modulo a prime. Thus the entire arithmetic proof can be carried out with integer coefficients.

The positive-convolution method is classical; see [10, Exercise 58] and [11, Section 1, Lemma 5]. The harmonic normalization H_1 and the single-large-prime injection already appear in [13, proof of Proposition 3.5]. Bounds linear in $1 - \beta$ are also standard; for example, [5, Lemma 1] bounds the split-prime sum by a constant multiple of $\log q + (1 - \beta)(\log x)^2$. Our quantitative refinements are (1.2), the uniform remainder in Proposition 4.1, and the complete error term in Lemma 5.1. These give the stated constant without a preliminary lower bound for $L(1, \chi)$ or an estimate for $L'/L(1, \chi)$. The underlying counting, Frobenius, and positive-convolution mechanisms are established methods.

The prerequisites are the Chinese remainder theorem, quadratic reciprocity, polynomial arithmetic modulo a prime, and real summation and integration. Standard facts are cited in the relevant places. The proof uses no complex zero formula, analytic continuation, or functional equation. Proofs are retained for the specific estimates and constants used in (1.1).

2 Standard facts and explicit prime estimates

We use Euler summation, partial summation, and the Dirichlet hyperbola identity in their standard forms [1, Theorems 3.1, 4.2, and 3.17]. For the Chinese remainder theorem, polynomial congruences, and quadratic reciprocity, see [1, Sections 5.4–5.7 and Chapter 9].

Quadratic characters. The standard classification of primitive quadratic characters [3, Section 1.4, Theorem 1.1] gives a fundamental discriminant d_χ with $|d_\chi| = q$ and $\chi = (d_\chi/\cdot)$. In arithmetic terms, there is a squarefree integer $d \neq 1$ such that

$$d_\chi = \begin{cases} d, & d \equiv 1 \pmod{4}, \\ 4d, & d \equiv 2, 3 \pmod{4}. \end{cases}$$

Hence $|d| \leq q$ and

$$\chi(p) = \left(\frac{d}{p}\right) \quad \text{for every odd prime } p. \quad (2.1)$$

Only this elementary consequence of the classification is needed.

Numerical bounds. We will use the elementary inequalities

$$\begin{aligned} \frac{13}{12} < \log 3 < \frac{11}{10}, \quad \log 2 < \frac{7}{10}, \quad \log 5 < \frac{13}{8}, \\ \frac{35}{18} < \log 7 < \frac{39}{20}, \quad \log 100 < 5. \end{aligned} \quad (2.2)$$

For example, these bounds follow from the real power series

$$\log x = 2 \sum_{j=0}^{m-1} \frac{z^{2j+1}}{2j+1} + R_m, \quad 0 < R_m < \frac{2z^{2m+1}}{(2m+1)(1-z^2)}, \quad z = \frac{x-1}{x+1} \quad (x > 1).$$

The choices $(x, m) = (2, 1), (3, 2), (5, 3), (7, 9)$ suffice; $\log 100 = 2 \log 2 + 2 \log 5 < 5$. Let

$$\vartheta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{n \leq x} \Lambda(n), \quad M(x) = \sum_{p \leq x} \frac{\log p}{p},$$

where Λ is the von Mangoldt function. The standard Chebyshev and Mertens estimates are recorded with the constants needed below; compare [1, Section 4.5] and [6, Lemma 5(a),(b)]. We include only the calculations that fix the constants in the forms used here.

Lemma 2.1. *For $x \geq 2$,*

$$\vartheta(x) \leq \psi(x) \leq (\log 4)x, \quad (2.3)$$

$$M(x) \leq \log x + \log 4. \quad (2.4)$$

Moreover,

$$\sum_{\substack{p \text{ prime} \\ p \text{ odd}}} \frac{\log p}{p^2} < \frac{7}{20}, \quad (2.5)$$

and, for $x \geq y \geq 100$,

$$M(x) - M(y) > \log(x/y) - 2. \quad (2.6)$$

Proof. The binomial-coefficient argument for Chebyshev's estimate (cf. [1, Section 4.5]) gives

$$\begin{aligned} \psi(2m) - \psi(m) &\leq \log \binom{2m}{m} \leq 2m \log 2, \\ \psi(2m+1) - \psi(m+1) &\leq \log \binom{2m+1}{m} \leq 2m \log 2. \end{aligned}$$

Induction yields (2.3). For integers $n \geq 2$, factorial prime factorization [1, Theorem 3.14] then gives

$$nM(n) \leq \log(n!) + \vartheta(n) \leq n \log n + n \log 4.$$

Taking $n = \lfloor x \rfloor$ proves (2.4).

For the square-denominator sum, $f(t) = (\log t)/t^2$ is convex for $t \geq 3$, since $f''(t) = (6 \log t - 5)/t^4 > 0$. Every prime exceeding 7 belongs to one of the progressions $11 + 6j$ and $13 + 6j$, $j \geq 0$. Convexity bounds each term by the average over its centered interval of length 6. Summing these intervals gives

$$\begin{aligned} \sum_{\substack{p \text{ prime} \\ p \text{ odd}}} \frac{\log p}{p^2} &\leq \frac{\log 3}{9} + \frac{\log 5}{25} + \frac{\log 7}{49} + \frac{1}{6} \int_8^\infty \frac{\log t}{t^2} dt + \frac{1}{6} \int_{10}^\infty \frac{\log t}{t^2} dt \\ &= \frac{\log 3}{9} + \frac{\log 5}{25} + \frac{\log 7}{49} + \frac{1}{6} \left(\frac{\log 8 + 1}{8} + \frac{\log 10 + 1}{10} \right) \\ &< \frac{30607}{88200} < \frac{7}{20}, \end{aligned}$$

where the last line uses (2.2).

For the interval estimate put $A(t) = \sum_{r \leq t} \Lambda(r)/r$. The factorial identity [1, Theorem 3.12, equation (17)] $\log(n!) = \sum_{r \leq n} \Lambda(r) \lfloor n/r \rfloor$, (2.3), and integral comparison for $\log(n!)$ give

$$\log n - 1 + \frac{1}{n} \leq A(n) \leq \log n - 1 + \log 4 + \frac{1 + \log n}{n}. \quad (2.7)$$

The higher prime powers satisfy

$$\begin{aligned} V &= \sum_p \frac{\log p}{p(p-1)} \leq \frac{\log 2}{2} + \frac{\log 3}{36} + \frac{5}{4} \sum_{\substack{p \text{ prime} \\ p \text{ odd}}} \frac{\log p}{p^2} \\ &< \frac{7}{20} + \frac{11}{360} + \frac{7}{16} = \frac{589}{720} < \frac{33}{40}. \end{aligned}$$

Here the first inequality isolates $p = 2, 3$ and uses $p/(p-1) \leq 5/4$ for $p \geq 5$. For real $x \geq y \geq 100$, (2.7) gives

$$A(x) - A(y) \geq \log(x/y) - \log 4 - \frac{1 + \log 100}{100}.$$

Indeed, the term $1/n$ absorbs $\log x - \log n$ for $n = \lfloor x \rfloor$, and $(1 + \log t)/t$ decreases for $t \geq 100$. Put $W(t) = A(t) - M(t)$. Then $W(x) \leq V$, whereas the terms $4, 9 \leq y$ give

$$W(y) \geq \frac{\log 2}{4} + \frac{\log 3}{9}.$$

Consequently $M(x) - M(y) > \log(x/y) - 2$, since the loss is less than

$$\frac{33}{40} + \frac{7}{4} \log 2 - \frac{1}{9} \log 3 + \frac{1 + \log 100}{100} < \frac{33}{40} + \frac{49}{40} - \frac{13}{108} + \frac{3}{50} = 2 - \frac{7}{675} < 2.$$

This proves (2.6). □

3 Integer counting and polynomial congruences

We prove a uniform prime-sum bound by coefficient counting and polynomial congruences. Every polynomial used in the construction has integer coefficients, and all polynomial congruences are coefficientwise. No hypothesis about zeros is imposed.

Proposition 3.1. *Let χ be a primitive quadratic Dirichlet character of conductor $q \geq 3$. For every real $x \geq 2$,*

$$S_-(x, \chi) \leq \frac{11}{20} \log x + \frac{29}{60} \log q + 6. \quad (3.1)$$

3.1 A counting lemma

The following lemma abstracts the local image counts and the coefficient-box argument in [5, Lemmas 3–5].

Lemma 3.2. *Let n, m be positive integers and let $\iota : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ be an injective additive map, and suppose that $\|\iota(e_i)\|_\infty \leq w_i$, where $e_1, \dots, e_n$ are the standard basis vectors and $w_i > 0$. Let $\mathcal{P}$ be a nonempty finite set of primes. Suppose that for each $p \in \mathcal{P}$ there is an integer basis $b_1^{(p)}, \dots, b_n^{(p)}$ of $\mathbb{Z}^n$ and nonnegative integers $h_i(p)$ such that*

$$\iota(b_i^{(p)}) \in p^{h_i(p)} \mathbb{Z}^m.$$

Then

$$H = \sum_{p \in \mathcal{P}} \sum_{i=1}^n h_i(p) \log p \leq n \log n + \sum_{i=1}^n \log w_i. \quad (3.2)$$

Proof. Choose an integer $\nu > \max_{p,i} h_i(p)$ and put $d_0 = \prod_{p \in \mathcal{P}} p^\nu$. In the basis belonging to p , changing the i th coefficient by a multiple of $p^{\nu-h_i(p)}$ leaves its image unchanged modulo p^ν . Thus

$$\#\text{im}(\iota \bmod p^\nu) \leq \prod_{i=1}^n p^{\nu-h_i(p)} = p^{\nu n - \sum_i h_i(p)}.$$

The Chinese remainder theorem gives

$$\#\text{im}(\iota \bmod d_0) \leq \prod_{p \in \mathcal{P}} p^{\nu n - \sum_i h_i(p)} = d_0^n e^{-H}.$$

Consider the half-open integer box

$$\mathcal{B} = \left\{ c \in \mathbb{Z}^n : 0 \leq c_i < \frac{d_0}{nw_i} \quad (1 \leq i \leq n) \right\}.$$

Since $[0, t)$ contains $\lceil t \rceil \geq t$ integers,

$$\#\mathcal{B} = \prod_{i=1}^n \left\lceil \frac{d_0}{nw_i} \right\rceil \geq \frac{d_0^n}{n^n \prod_i w_i}.$$

For $c, c' \in \mathcal{B}$ with $\iota(c) \equiv \iota(c') \pmod{d_0}$,

$$\iota(c - c') \in d_0 \mathbb{Z}^m, \quad \|\iota(c - c')\|_\infty \leq \sum_i w_i |c_i - c'_i| < d_0.$$

Hence $\iota(c - c') = 0$, and injectivity gives $c = c'$. Consequently

$$\frac{d_0^n}{n^n \prod_i w_i} \leq \#\mathcal{B} \leq \#\text{im}(\iota \bmod d_0) \leq d_0^n e^{-H},$$

which is equivalent to (3.2). □

3.2 Remainders and an integer evaluation map

Fix the squarefree integer $d \neq 1$ in (2.1), with $|d| \leq q$. For $f(t) \in \mathbb{Z}[t]$, let $\rho(f)$ be its remainder on division by the monic polynomial $t^2 - d$. Thus $\rho(f) = a + bt$ with $a, b \in \mathbb{Z}$. In particular,

$$\rho((a + bt)(c + et)) = (ac + dbe) + (ae + bc)t. \quad (3.3)$$

For $f, g \in \mathbb{Z}[t]$, remainder reduction gives

$$\rho(f + g) = \rho(f) + \rho(g), \quad \rho(fg) = \rho(\rho(f)\rho(g)).$$

For a linear remainder put

$$\|a + bt\| = |a| + \sqrt{|d|} |b|.$$

The multiplication formula shows that this size is submultiplicative:

$$\|\rho(fg)\| \leq \|\rho(f)\| \|\rho(g)\|.$$

Since $|d| \geq 1$, it bounds the absolute value of each of the two integer coefficients.

For now, let p_0 be an odd prime with $\chi(p_0) = -1$, $p_0 \geq 10000$, and $p_0 \geq 4q$. Set

$$s_0 = \left\lfloor \frac{p_0}{30} \right\rfloor, \quad n = p_0 s_0, \quad m = \lceil \sqrt{p_0} \rceil, \quad b_0 = 2\sqrt{p_0 q}. \quad (3.4)$$

We use every integer pair in the grid

$$0 \leq u < m, \quad 1 \leq v < m.$$

It contains $m(m-1)$ points. Notice that $2m(m-1) \geq m^2 > p_0$, since $m \geq 2$ and a prime is not a square. Also $m-1 < \sqrt{p_0}$ and $2(m-1) < p_0$.

Let $\mathcal{V}$ consist of the integer-coefficient polynomials

$$f(z, w) = \sum_{\substack{0 \leq r < p_0 \\ 0 \leq s < s_0}} c_{r,s} z^r w^s, \quad c_{r,s} \in \mathbb{Z},$$

identified with $\mathbb{Z}^n$ by their coefficient arrays. Define

$$\iota : \mathcal{V} \longrightarrow \mathbb{Z}^{2m(m-1)s_0} \quad (3.5)$$

by listing the two coefficients of every remainder

$$\rho((\partial_z^j f)(u + vt, u - vt)), \quad 0 \leq u < m, \quad 1 \leq v < m, \quad 0 \leq j < s_0.$$

All derivatives are ordinary formal derivatives, without division by factorials.

The next proof is a polynomial-remainder formulation of [14, proof of Theorem 1, Nonvanishing]; compare also [9, Lemma 4].

Lemma 3.3. *The integer map ι is injective.*

Proof. Suppose that $\iota(f) = 0$ for some nonzero $f \in \mathcal{V}$. Divide its coefficients by their largest common power of p_0 . The resulting polynomial, again called f , still satisfies $\iota(f) = 0$, and at least one coefficient is not divisible by p_0 .

For each grid point put

$$f_{u,v}(\xi) = (\xi - u)^2 - dv^2.$$

Modulo p_0 these are irreducible quadratics, because d is a nonsquare and v is nonzero modulo p_0 . They are pairwise different: equality of two such polynomials first gives $u = u'$ modulo p_0 and then $v^2 = (v')^2$ modulo p_0 . The specified ranges of u, v , together with $v + v' \leq 2(m-1) < p_0$, imply $u = u'$ and $v = v'$ as integers. Thus the polynomials $f_{u,v}$ are pairwise relatively prime modulo p_0 .

For $0 \leq j < s_0$, define the integer polynomial

$$h_{u,v,j}(\xi) = (\partial_z^j f)(\xi, 2u - \xi).$$

We first show that $f_{u,v}$ divides $h_{u,v,j}$ already in $\mathbb{Z}[\xi]$. Divide by the monic quadratic and write the remainder as $a\xi + b$. After substituting $\xi = u + vt$ and reducing modulo $t^2 - d$, the dividend has remainder zero by $\iota(f) = 0$, while the remainder of the divisor is zero. Hence

$$(au + b) + avt = 0.$$

Since $v > 0$, this forces $a = b = 0$ and proves the assertion.

Now reduce polynomial coefficients modulo p_0 , and put

$$g(\xi) = f(\xi, \xi^{p_0}) \pmod{p_0}.$$

The exponents $r + p_0s$, with $0 \leq r < p_0$ and $0 \leq s < s_0$, are distinct. Consequently $g \neq 0$ and $\deg g \leq p_0s_0 - 1$. Euler's criterion and Fermat's little theorem give the polynomial congruence

$$\xi^{p_0} \equiv u + (dv^2)^{(p_0-1)/2}(\xi - u) \equiv 2u - \xi \pmod{p_0, f_{u,v}(\xi)}. \quad (3.6)$$

Here the first congruence uses $(\xi - u)^2 \equiv dv^2$ modulo $f_{u,v}$. Every positive derivative of ξ^{p_0} is zero modulo p_0 , so

$$g^{(j)}(\xi) \equiv (\partial_z^j f)(\xi, \xi^{p_0}) \pmod{p_0}.$$

Together with (3.6) and the divisibility of $h_{u,v,j}$, this proves

$$f_{u,v} \mid g^{(j)} \pmod{p_0} \quad (0 \leq j < s_0).$$

These divisibilities imply $f_{u,v}^{s_0} \mid g \pmod{p_0}$ by polynomial division modulo p_0 : if $g \equiv f_{u,v}^k k_0 \pmod{p_0}$ and $0 \leq k < s_0$, the product rule gives

$$g^{(k)} \equiv k!(f'_{u,v})^k k_0 \pmod{p_0, f_{u,v}}.$$

The integer $k!$ is invertible modulo p_0 , since $k < s_0 < p_0$. Furthermore $f_{u,v}$ and $f'_{u,v} = 2(\xi - u)$ are relatively prime modulo p_0 , because $dv^2 \not\equiv 0 \pmod{p_0}$. The polynomial Euclidean algorithm therefore gives $f_{u,v} \mid k_0 \pmod{p_0}$. Starting with $k = 0$ and proceeding inductively proves the assertion.

Pairwise relative primality now gives

$$\prod_{\substack{0 \leq u < m \\ 1 \leq v < m}} f_{u,v}(\xi)^{s_0} \mid g(\xi) \pmod{p_0}.$$

The polynomial on the left is monic of degree $2m(m-1)s_0$. Since $g \neq 0$, we obtain the contradiction

$$2m(m-1)s_0 \leq \deg g \leq p_0s_0 - 1 < 2m(m-1)s_0.$$

□

3.3 Triangular bases and divisibility

Put

$$\mathcal{P} = \{p \leq p_0 : p \text{ is odd and } \chi(p) = -1\}, \quad h_{r,s}(p) = \min\left(\left\lfloor \frac{r}{p} \right\rfloor, s_0 - 1 - s\right),$$

and write

$$H_p = \sum_{\substack{0 \leq r < p_0 \\ 0 \leq s < s_0}} h_{r,s}(p), \quad H = \sum_{p \in \mathcal{P}} H_p \log p. \quad (3.7)$$

For each $p \in \mathcal{P}$, replace the monomials by the polynomials

$$g_{r,s}^{(p)}(z, w) = z^{r-ph_{r,s}(p)} w^s (z^p - w)^{h_{r,s}(p)}. \quad (3.8)$$

Writing $h = h_{r,s}(p)$, their expansion is

$$g_{r,s}^{(p)} = z^r w^s + \sum_{k=1}^h (-1)^k \binom{h}{k} z^{r-pk} w^{s+k}.$$

Every monomial stays in the original rectangle, and every additional term has first exponent smaller than r . Successive substitution in increasing order of r shows that these polynomials form an integer basis of $\mathcal{V}$. This is the triangular change of basis in Green [4, Lemma 5.1].

Euler's criterion and the binomial theorem give, coefficientwise,

$$\rho((u + vt)^p) \equiv u - vt \pmod{p}, \quad (3.9)$$

because $\rho(t^p) = d^{(p-1)/2} t \equiv -t \pmod{p}$. Thus the remainder of $z^p - w$ at every grid point has both coefficients divisible by p . For $\psi_p(z, w) = z^p - w$,

$$\partial_z^j \psi_p(z, w) = \begin{cases} z^p - w, & j = 0, \\ \frac{p!}{(p-j)!} z^{p-j}, & 1 \leq j \leq p, \\ 0, & j > p. \end{cases}$$

Thus every derivative has a remainder divisible by p at each grid point. In every term of the product rule applied to (3.8), each of its h factors $z^p - w$ therefore contributes a factor of p after evaluation and remainder reduction. Consequently

$$\iota(g_{r,s}^{(p)}) \in p^{h_{r,s}(p)} \mathbb{Z}^{2m(m-1)s_0}. \quad (3.10)$$

This argument includes all derivative orders used in ι , even when they are at least p .

On the grid,

$$\|u \pm vt\| \leq (m-1)(1 + \sqrt{|d|}) < \sqrt{p_0}(1 + \sqrt{q}) \leq b_0.$$

For $j \leq r$ the corresponding derivative of $z^r w^s$ therefore has remainder size at most $p_0^j b_0^{r+s-j}$; it is zero for $j > r$. Since $p_0 \geq 4q$ gives $b_0 \leq p_0$, we obtain the coordinate bound

$$\|\iota(z^r w^s)\|_\infty \leq (p_0/b_0)^{s_0-1} b_0^{r+s} = w_{r,s}.$$

Apply Lemma 3.2, using Lemma 3.3 and (3.10). The identity $\sum_{r,s} (r+s) = p_0 s_0 (p_0 + s_0 - 2)/2$ gives

$$H \leq p_0 s_0 \log(p_0 s_0) + p_0 s_0 (s_0 - 1) \log(p_0/b_0) + \frac{p_0 s_0 (p_0 + s_0 - 2)}{2} \log b_0.$$

Put $\lambda = (s_0 - 1)/(p_0 - 1) \leq 1/30$. Since $p_0 s_0 \leq p_0^2$,

$$\begin{aligned} \frac{2H}{p_0 s_0 (p_0 - 1)} &\leq \frac{2 \log(p_0 s_0)}{p_0 - 1} + \frac{2(s_0 - 1)}{p_0 - 1} \log p_0 + \frac{p_0 - s_0}{p_0 - 1} \log b_0 \\ &\leq \frac{4 \log p_0}{p_0 - 1} + \log b_0 + \lambda(2 \log p_0 - \log b_0). \end{aligned}$$

The coefficient of λ is positive. Substituting $\lambda \leq 1/30$ and $\log b_0 = \frac{1}{2} \log p_0 + \frac{1}{2} \log q + \log 2$ gives

$$\frac{2H}{p_0 s_0 (p_0 - 1)} \leq \frac{11}{20} \log p_0 + \frac{29}{60} \log q + \frac{29}{30} \log 2 + \frac{4 \log p_0}{p_0 - 1}. \quad (3.11)$$

3.4 Summing the exponents

Put $U = \sum_{p \in \mathcal{P}} (\log p)/p$. For every integer $k \geq 0$,

$$2 \sum_{s=0}^{s_0-1} \min(k, s_0 - 1 - s) \geq (2s_0 - 1)k - k^2. \quad (3.12)$$

For $k < s_0$ this is an equality. For $k \geq s_0$, the difference between the two sides is $(k - s_0)(k - s_0 + 1) \geq 0$. If $p_0 = ap + b$ with $0 \leq b < p$, summation in constant blocks gives

$$\begin{aligned} \sum_{r=0}^{p_0-1} \left\lfloor \frac{r}{p} \right\rfloor &= \frac{pa(a-1)}{2} + ab = \frac{p_0(p_0-p)}{2p} + \frac{b(p-b)}{2p} \\ &\geq \frac{p_0(p_0-1)}{2p} - \frac{p_0}{2}, \end{aligned} \quad (3.13)$$

$$\sum_{r=0}^{p_0-1} \left\lfloor \frac{r}{p} \right\rfloor^2 \leq \frac{p_0(p_0-1)(2p_0-1)}{6p^2}. \quad (3.14)$$

Apply (3.12) with $k = \lfloor r/p \rfloor$ and use (3.13)–(3.14). After discarding a nonnegative term $p_0/2$, this gives

$$2H_p \geq \frac{s_0 p_0 (p_0 - 1)}{p} - \frac{p_0 (p_0 - 1)}{2p} - s_0 p_0 - \frac{p_0 (p_0 - 1)(2p_0 - 1)}{6p^2}.$$

Multiply by $\log p$ and sum over $\mathcal{P}$. For the negative prime sums, use $\vartheta(p_0)$ and (2.5), respectively. We obtain

$$\frac{2H}{p_0 s_0 (p_0 - 1)} \geq U - \frac{U}{2s_0} - \frac{\vartheta(p_0)}{p_0 - 1} - \frac{7(2p_0 - 1)}{120s_0}. \quad (3.15)$$

3.5 The remainder and the small cases

Proof of Proposition 3.1. If an odd prime $p \leq x$ contributes to $S_-(x, \chi)$, let p_0 be the largest such prime. First suppose that $p_0 \geq 10000$ and $p_0 \geq 4q$. The odd-prime part of $S_-(x, \chi)$ equals U , and the possible contribution of 2 is at most $(\log 2)/2$. Combining (3.11) and (3.15) gives

$$S_-(x, \chi) \leq \frac{11}{20} \log p_0 + \frac{29}{60} \log q + \mathcal{R}(p_0),$$

where

$$\mathcal{R}(p_0) = \frac{\vartheta(p_0) + 4 \log p_0}{p_0 - 1} + \frac{U}{2s_0} + \frac{7(2p_0 - 1)}{120s_0} + \frac{22}{15} \log 2. \quad (3.16)$$

Here $s_0 \geq 333$ and $p_0 < 30(s_0 + 1)$, so $p_0/s_0 < 30 + 30/333 < 301/10$. By Lemma 2.1 and (2.2),

$$\vartheta(p_0) \leq (\log 4)p_0, \quad U \leq \log p_0 + \log 4, \quad \log 2 < \frac{7}{10}, \quad \log 10000 < 10.$$

The functions $t/(t-1)$, $(\log t)/(t-1)$ and $(\log t + \log 4)/t$ are decreasing for $t \geq 10000$. Thus the four terms in (3.16) satisfy

$$\begin{aligned} \frac{\vartheta(p_0) + 4 \log p_0}{p_0 - 1} &< \frac{10000(7/5) + 4 \cdot 10}{9999} = \frac{14040}{9999} < \frac{141}{100}, \\ \frac{U}{2s_0} &< \frac{(301/10)(10 + 7/5)}{20000} < \frac{1}{50}, \\ \frac{7(2p_0 - 1)}{120s_0} &< \frac{2107}{600} < \frac{53}{15}, \quad \frac{22}{15} \log 2 < \frac{77}{75}. \end{aligned}$$

Consequently

$$\mathcal{R}(p_0) < \frac{141}{100} + \frac{1}{50} + \frac{53}{15} + \frac{77}{75} = \frac{599}{100} < 6. \quad (3.17)$$

Since $p_0 \leq x$, this proves (3.1) in this case.

If the largest contributing odd prime satisfies $p_0 < 10000$, then the entire prime sum, including the possible contribution of 2, is at most $\log p_0 + \log 4$ by (2.4). Hence

$$S_-(x, \chi) - \frac{11}{20} \log x - \frac{29}{60} \log q \leq \frac{9}{20} \log p_0 + \log 4 < \frac{9}{20} \cdot 10 + \frac{7}{5} = \frac{59}{10} < 6.$$

If instead $p_0 < 4q$, the same elementary upper bound gives

$$S_-(x, \chi) - \frac{11}{20} \log x - \frac{29}{60} \log q < -\frac{1}{30} \log p_0 + \frac{89}{60} \log 4 < \frac{623}{300} < 6.$$

Finally, if there is no contributing odd prime, then $S_-(x, \chi) \leq (\log 2)/2$, and the same conclusion is immediate. $\square$

The construction retains the quadratic relation $t^2 = d$, expressed through a fixed polynomial remainder. The factor-count proof of Lemma 3.3 preserves the auxiliary-prime mechanism and expresses it entirely through polynomial division and degree counting.

4 A real-variable estimate at a real zero

The estimate below is an explicit refinement, at a real zero, of the standard weighted Dirichlet hyperbola argument; compare [10, Exercise 58] and [11, Section 1, Lemma 5]. We combine the two character tails before estimating them, preserving the cancellation between their kernels. This gives an explicit remainder bound uniform as β approaches 1, without an extra factor $(1 - \beta)^{-1}$ in the error. Throughout this section, χ is a nonprincipal quadratic character modulo $q \geq 3$; primitivity is not needed.

For $0 \leq u \leq v$, write

$$A_u(v) = \sum_{u < n \leq v} \chi(n).$$

The complete character sum vanishes [1, Section 6.9]. Write $\lfloor v \rfloor - \lfloor u \rfloor = kq + r$, where $0 \leq r < q$. Removing k complete periods and comparing the remaining block with its complement gives

$$|A_u(v)| \leq \min(r, q - r) \leq \frac{q}{2}.$$

Convergence for $s > 0$ and partial summation [1, Theorems 6.17 and 4.2] therefore give, for $u \geq 1$,

$$R_s(u) = \sum_{n > u} \frac{\chi(n)}{n^s} = s \int_u^\infty A_u(v) v^{-s-1} dv, \quad |R_s(u)| \leq \frac{q}{2} u^{-s}. \quad (4.1)$$

Put

$$a(n) = (1 * \chi)(n) = \sum_{d|n} \chi(d), \quad H_\beta(t) = \sum_{n \leq t} \frac{a(n)}{n^\beta} \quad (t \geq 1). \quad (4.2)$$

By the standard properties of Dirichlet convolution and real characters [1, Theorems 2.14 and 6.19], a is multiplicative, and

$$a(p^k) = \sum_{j=0}^k \chi(p)^j = \begin{cases} k+1, & \chi(p) = 1, \\ 1, & \chi(p) = 0, \\ \frac{1 + (-1)^k}{2}, & \chi(p) = -1. \end{cases}$$

Consequently $a(n) \geq 0$ for every n , and

$$a(1) = 1, \quad a(p) = 1 + \chi(p), \quad a(m^2) \geq 1, \quad H_\beta(t) \geq 1. \quad (4.3)$$

Proposition 4.1 (Paired tails). *Suppose that $1/2 \leq \beta < 1$ and $L(\beta, \chi) = 0$. Write $\varepsilon = 1 - \beta$. For every real $t \geq 1$,*

$$H_\beta(t) = \frac{L(1, \chi)}{\varepsilon} t^\varepsilon + E(t), \quad |E(t)| \leq \left(q + 1 + \frac{1}{2} \log t \right) t^{1/2-\beta}. \quad (4.4)$$

Proof. Set $u = \sqrt{t}$. The Dirichlet hyperbola identity [1, Theorem 3.17], applied to $dm \leq t$, gives

$$\begin{aligned} H_\beta(t) &= \sum_{d \leq u} d^{-\beta} \sum_{m \leq t/d} \frac{\chi(m)}{m^\beta} \\ &\quad + \sum_{m \leq u} \frac{\chi(m)}{m^\beta} \sum_{u < d \leq t/m} d^{-\beta}. \end{aligned} \quad (4.5)$$

The first line has absolute value at most

$$\sum_{d \leq u} d^{-\beta} \frac{q}{2} (t/d)^{-\beta} \leq \frac{q}{2} t^{1/2-\beta}, \quad (4.6)$$

by $L(\beta, \chi) = 0$ and (4.1).

Euler summation [1, Theorem 3.1], valid for real endpoints, gives

$$\sum_{u < n \leq v} n^{-\beta} = \frac{v^\varepsilon - u^\varepsilon}{\varepsilon} + \mathcal{R}_\beta(u, v), \quad |\mathcal{R}_\beta(u, v)| \leq u^{-\beta} \quad (v \geq u).$$

Indeed, with $\{w\} = w - \lfloor w \rfloor$, the remainder is

$$\mathcal{R}_\beta(u, v) = \{u\}u^{-\beta} - \{v\}v^{-\beta} - \beta \int_u^v \{w\}w^{-\beta-1} dw,$$

which lies between $-u^{-\beta}$ and $u^{-\beta}$. Applying this to the second line of (4.5) yields

$$\frac{t^\varepsilon}{\varepsilon} \sum_{m \leq u} \frac{\chi(m)}{m} - \frac{u^\varepsilon}{\varepsilon} \sum_{m \leq u} \frac{\chi(m)}{m^\beta} + E_0,$$

where

$$|E_0| \leq u^{-\beta} \sum_{m \leq u} m^{-\beta} \leq u^{\varepsilon-\beta} (1 + \log u) = \left(1 + \frac{1}{2} \log t \right) t^{1/2-\beta}. \quad (4.7)$$

Here $m^{-\beta} \leq u^\varepsilon/m$ for $m \leq u$. Since $L(\beta, \chi) = 0$, the two main sums equal

$$\frac{L(1, \chi)}{\varepsilon} t^\varepsilon - \frac{t^\varepsilon}{\varepsilon} R_1(u) + \frac{u^\varepsilon}{\varepsilon} R_\beta(u). \quad (4.8)$$

The last two terms are estimated together. We combine the two convergent tails with a common finite cutoff before passing to the limit. Their sum is

$$E_1 = u^\varepsilon \sum_{m > u} \chi(m) g(m), \quad g(v) = \frac{v^{-\beta} - u^\varepsilon v^{-1}}{\varepsilon} = \frac{1}{v} \int_u^v w^{-\beta} dw.$$

For $v \geq u$, the kernel satisfies

$$g(v) \geq 0, \quad g(u) = \lim_{v \rightarrow \infty} g(v) = 0, \quad g'(v) = \frac{v^{-2}}{\varepsilon} (u^\varepsilon - \beta v^\varepsilon).$$

The derivative changes sign once, at $v = u\beta^{-1/\varepsilon}$. For $x = v/u \geq 1$, the assumption $\beta \geq 1/2$ gives

$$g(v) = \frac{u^{-\beta}}{x} \int_1^x w^{-\beta} dw \leq 2u^{-\beta} \frac{\sqrt{x} - 1}{x} \leq \frac{1}{2}u^{-\beta},$$

where the last inequality is $(\sqrt{x} - 2)^2 \geq 0$. Therefore

$$\int_u^\infty |g'(v)| dv = 2 \max_{v \geq u} g(v) \leq u^{-\beta}.$$

Another summation by parts, whose boundary terms vanish, yields

$$\sum_{m > u} \chi(m)g(m) = - \int_u^\infty A_u(v)g'(v) dv.$$

The bound $|A_u(v)| \leq q/2$ now gives

$$|E_1| \leq \frac{q}{2} u^{\varepsilon-\beta} = \frac{q}{2} t^{1/2-\beta}. \quad (4.9)$$

Combining the three errors gives

$$|E(t)| \leq \left(\frac{q}{2} + 1 + \frac{1}{2} \log t + \frac{q}{2} \right) t^{1/2-\beta},$$

as required. $\square$

Pairing the tails in (4.8) removes the factor ε^{-1} that their separate absolute estimates would introduce. The coefficient $L(1, \chi)/\varepsilon$ will be eliminated by comparing finite positive sums; no lower bound for it is needed. For comparison, the standard bound in [10, Exercise 58], specialized to a real zero, has remainder $O(q(1 - \beta)^{-1} t^{1/2-\beta})$. The estimate above retains the cancellation needed to avoid that additional dependence on the zero.

5 Positivity and the explicit zero gap

The use of $a = 1 * \chi$, the harmonic sum $H_1(y) = \sum_{n \leq y} a(n)/n$, and a single-large-prime comparison already appears in [13, proof of Proposition 3.5]. That proof uses a lower bound for $L(1, \chi)$ and an estimate for $L'/L(1, \chi)$; see also [11, Section 1, Lemma 5] for the positive-convolution approach to prime bias. The lemma below uses the logarithmic weight $\log p/p$ and retains the full error terms. Its Abel summation identity, combined with Proposition 4.1, supplies the explicit comparison without those preliminary estimates. This realizes the standard linear dependence on $1 - \beta$ noted in the introduction with the constants needed for a single choice of parameters for all conductors.

Lemma 5.1 (A logarithmically weighted comparison). *Under the hypotheses of Proposition 4.1, write $\varepsilon = 1 - \beta$ and put*

$$H_1(t) = \sum_{n \leq t} \frac{a(n)}{n}, \quad B(t) = \left(q + 1 + \frac{1}{2} \log t \right) t^{-1/2}.$$

Let $1 \leq y \leq x$ and suppose that $\varepsilon \log(xy) \leq 1$. Set $\kappa = \log x \log y + \frac{1}{2}(\log x)^2$. Then

$$J = \sum_{y < p \leq x} \frac{a(p) \log p}{p} \leq \varepsilon \kappa + \frac{\mathcal{E}(x, y)}{H_1(y)}, \quad (5.1)$$

$$\begin{aligned} \mathcal{E}(x, y) &= B(y)(\varepsilon \kappa + \log y) + B(xy) \log(xy) \\ &\quad + 2B(y) + 2y^{-1/2}. \end{aligned} \quad (5.2)$$

If also $y \geq \max(q, 100)$, then

$$S_-(x, \chi) \geq \log(x/y) - 2 - \frac{J}{2}. \quad (5.3)$$

Proof. Put $N(t) = \sum_{n \leq t} a(n)(\log n)/n$. For $n \leq y < p \leq x$, we have $(n, p) = 1$ and $y < np \leq xy$. The map $(n, p) \mapsto np$ is injective: if $np = n'p'$ with $p \neq p'$, then $p \mid n' \leq y < p$, a contradiction. Multiplicativity, nonnegativity, and $\log p \leq \log(np)$ give

$$\begin{aligned} H_1(y)J &= \sum_{\substack{n \leq y \\ y < p \leq x}} \frac{a(np) \log p}{np} \\ &\leq \sum_{y < m \leq xy} \frac{a(m) \log m}{m} = N(xy) - N(y). \end{aligned} \quad (5.4)$$

Write $D(t) = t^{-\varepsilon} E(t)$ and $\phi(t) = (\log t)t^{-\varepsilon}$. Partial summation gives

$$\begin{aligned} N(xy) - N(y) &= \phi(xy)H_\beta(xy) - \phi(y)H_\beta(y) \\ &\quad - \int_y^{xy} H_\beta(t)\phi'(t) dt. \end{aligned}$$

Substituting (4.4) yields exactly

$$N(xy) - N(y) = L(1, \chi)\kappa + R, \quad (5.5)$$

where

$$\begin{aligned} R &= D(xy) \log(xy) - D(y) \log y \\ &\quad + \int_y^{xy} (\varepsilon \log t - 1) \frac{D(t)}{t} dt. \end{aligned}$$

Since $|D(t)| \leq B(t)$ and $0 \leq \varepsilon \log t \leq 1$ on $[y, xy]$,

$$|R| \leq B(xy) \log(xy) + B(y) \log y + \int_y^\infty \frac{B(t)}{t} dt.$$

The integral is $2B(y) + 2y^{-1/2}$.

Positivity also gives $H_\beta(y) \leq y^\varepsilon H_1(y)$, so

$$\frac{L(1, \chi)}{\varepsilon} = y^{-\varepsilon} (H_\beta(y) - E(y)) \leq H_1(y) + B(y).$$

Since $\varepsilon > 0$ and $\kappa \geq 0$, it follows that

$$L(1, \chi)\kappa \leq \varepsilon \kappa (H_1(y) + B(y)).$$

Combining this with (5.4) and (5.5), and dividing by $H_1(y) \geq 1$, proves (5.1). No sign assumption or preliminary estimate for $L(1, \chi)$ has been used.

If $y \geq q$, every prime $p > y$ is coprime to q , and $a(p) = 2$ exactly when $\chi(p) = 1$. Hence

$$S_-(x, \chi) - S_-(y, \chi) = M(x) - M(y) - \frac{J}{2}.$$

Now use $S_-(y, \chi) \geq 0$ and (2.6). □

Proof of Theorem 1.1. Suppose that $L(\beta, \chi) = 0$ for some $\beta \in (0, 1)$ with

$$\delta = (1 - \beta) \log q \leq \frac{1}{100}.$$

Take

$$y = q^6, \quad x = q^{70}. \quad (5.6)$$

Since $\log q \geq \log 3 > 1$, we have $\beta \geq 1 - 1/(100 \log q) > 1/2$. Moreover, $\varepsilon \log(xy) = 76\delta \leq 76/100 < 1$, $y \geq 729$, and $y > \max(q, 100)$. Thus Lemma 5.1 applies.

The squares supply a uniform normalization: by (4.3),

$$\begin{aligned} H_1(y) &\geq \sum_{m=1}^{27} \frac{1}{m^2} \geq \sum_{m=1}^7 \frac{1}{m^2} + \int_8^{28} \frac{dt}{t^2} \\ &= \frac{282431}{176400} > \frac{8}{5}. \end{aligned} \quad (5.7)$$

For $k \geq 2$, put

$$f_k(q) = \left(q + 1 + \frac{k}{2} \log q \right) q^{-k/2}.$$

Its derivative satisfies

$$f'_k(q) = q^{-k/2-1} \left[\left(1 - \frac{k}{2} \right) q - \frac{k^2}{4} \log q \right] < 0 \quad (q \geq 3).$$

Using (2.2) at $q = 3$ therefore gives

$$B(y) \leq (4 + 3 \log 3) 3^{-3} < \frac{73}{270}, \quad (5.8)$$

$$B(xy) \leq (4 + 38 \log 3) 3^{-38} < 50 \cdot 3^{-38} < \frac{1}{10000}, \quad (5.9)$$

$$\frac{2B(y) + 2y^{-1/2}}{\log q} < \left(\frac{73}{135} + \frac{2}{27} \right) \frac{12}{13} = \frac{332}{585} < \frac{4}{7}. \quad (5.10)$$

Here $\kappa = 2870(\log q)^2$, whence $\varepsilon \kappa / \log q = 2870\delta \leq 287/10$. Consequently, (5.2) satisfies the rational certificate

$$\begin{aligned} \frac{\mathcal{E}(x, y)}{\log q} &< \frac{73}{270} \left(\frac{287}{10} + 6 \right) + \frac{76}{10000} + \frac{4}{7} \\ &= \frac{1176629}{118125} = 10 - \frac{4621}{118125} < 10. \end{aligned} \quad (5.11)$$

Dividing by (5.7) now gives

$$\frac{J}{\log q} < \frac{287}{10} + \frac{5}{8} \cdot 10 = \frac{699}{20} < 35. \quad (5.12)$$

By (5.3),

$$S_-(q^{70}, \chi) > \left(70 - 6 - \frac{35}{2} \right) \log q - 2 = \frac{93}{2} \log q - 2. \quad (5.13)$$

On the other hand, Proposition 3.1 gives

$$S_-(q^{70}, \chi) \leq \left(\frac{11}{20} \cdot 70 + \frac{29}{60} \right) \log q + 6 = \frac{2339}{60} \log q + 6. \quad (5.14)$$

Combining these inequalities would imply $\frac{451}{60} \log q < 8$. This is impossible, since

$$\frac{451}{60} \log q \geq \frac{451}{60} \log 3 > \frac{451}{60} \cdot \frac{13}{12} = 8 + \frac{103}{720} > 8. \quad (5.15)$$

The theorem follows. $\square$

The same parameters apply to every conductor $q \geq 3$. The explicit arithmetic bound, the uniform paired-tail estimate, and the logarithmically weighted comparison together supply the stated constant without separate small-conductor zero computations. The constant records this elementary calculation; no numerical optimality is asserted.

A Verification of the constant $1/64$

Let χ be as in Theorem 1.1. Suppose that $L(\beta, \chi) = 0$ for some $\beta \in (0, 1)$, and put $\varepsilon = 1 - \beta$ and $\delta = \varepsilon \log q \leq 1/64$. For $q \geq 7$, take $x = q^{46}$ and $y = q^4$. Then $\beta > 1/2$, $\varepsilon \log(xy) = 50\delta \leq 25/32 < 1$, and $y \geq \max(q, 100)$. Since $\sqrt{y} = q^2 \geq 49$, the square normalization in (5.7) still gives $H_1(y) > 8/5$. The monotonicity calculation used for (5.8), with exponents 4 and 50, and (2.2) give

$$\begin{aligned} B(y) &\leq (8 + 2 \log 7) 7^{-2} < \frac{17}{70}, \\ B(xy) &\leq (8 + 25 \log 7) 7^{-25} < 57 \cdot 7^{-25} < \frac{1}{10000}, \\ y^{-1/2} &\leq \frac{1}{49}. \end{aligned}$$

Here $\kappa = 1242(\log q)^2$, so $\varepsilon\kappa/\log q \leq 621/32$. Consequently (5.2) gives

$$\begin{aligned} \frac{\mathcal{E}(x, y)}{\log q} &< \frac{17}{70} \left(\frac{621}{32} + 4 \right) + \frac{50}{10000} + \left(\frac{17}{35} + \frac{2}{49} \right) \frac{18}{35} \\ &= 6 - \frac{21863}{548800} < 6. \end{aligned} \tag{A.1}$$

Thus (5.1) yields $J/\log q < 621/32 + (5/8)6 = 741/32$. Combining Proposition 3.1 with (5.3) would give $(4453/960) \log q < 8$, whereas

$$\frac{4453}{960} \log q > \frac{4453}{960} \cdot \frac{35}{18} = 8 + \frac{3523}{3456} > 8.$$

For $q = 3, 4$, the period blocks are respectively $(3n+1)^{-s} - (3n+2)^{-s} > 0$ and $(4n+1)^{-s} - (4n+3)^{-s} > 0$ for $n \geq 0$ and $s > 0$. For $q = 5$, strict convexity of $f(t) = t^{-s}$ gives

$$f(5n+1) - f(5n+2) > f(5n+3) - f(5n+4),$$

so its period blocks are also positive. Since the character series converge for $s > 0$, this proves $L(s, \chi) > 0$ in these three cases. No primitive character has conductor 6. This proves the assertion in Remark 1.2.

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