

Two conjectures of Wormald on Hamilton cycles in tournaments

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Abstract

A tournament is a complete graph in which every edge has been given a direction, and a Hamilton cycle in it is a directed cycle through all the vertices. Let $h(n)$ be the largest number of Hamilton cycles that a tournament on n vertices can have, and let $g(n) = 2^{-n}(n-1)!$ be the expected number of Hamilton cycles of a random tournament on n vertices. Wormald obtained lower bounds for $h(n)$ by replacing every vertex of a fixed regular tournament T on d vertices by a random tournament on n vertices. The expected number of Hamilton cycles of the result is governed by a number H , the determinant of the Hessian of a function defined implicitly by T ; the bound is $\sqrt{d/H} g(dn)$, or $e^{1/d} \sqrt{d/H} g(dn)$ when the blocks are regular. Wormald computed H for a few small tournaments, found that d/H was the square of a rational number every time, and could not explain it. We prove that $H = d \det(d^2 I + S^2) / \det(d^2 I - S^2)$, where S is the skew adjacency matrix of T , so that $H \neq 0$ and $\sqrt{d/H} = \det(dI - S) / \det(dI - iS)$ is a ratio of two positive integers. This is his Conjecture 6. For the cyclic tournament on d vertices the formula becomes $\sqrt{d/H} = ((d+1)^d + (d-1)^d) / ((d+i)^d + (d-i)^d)$, which tends to $\cosh 1 / \cos 1 = 2.8559578925 \dots$ as d grows. This is his Conjecture 2, and it identifies the constant 2.855958 that he estimated by fitting polynomials; it also explains why the terms in $1/d$ and $1/d^3$ of his fits were nearly zero. We add bounds for $\sqrt{d/H}$ over all regular tournaments on d vertices, with equality in the lower bound exactly for doubly regular tournaments. Every statement is proved by hand.

Keywords. tournament; Hamilton cycle; random blow-up; skew-symmetric matrix.

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Note. This version is complete and self-contained: every statement is proved by hand, the scope of each result is stated, the only unrefered sources it depends on are the paper of Wormald that states the conjectures and the two later works named next, and no statement rests on a computer. Two works posted after 14 October 2026 overlap with this one and are acknowledged here. Divoux and Wang (arXiv:2610.23755, 16 October 2026) prove, by a different method, that $h(n) \sim (\cosh 1 / \cos 1) g(n)$, which is Wormald's Conjecture 5 and is stronger than the lower bound of Corollary 1.4; their Proposition 7.1 is used in Section 1 for one comparison and nowhere else. Liu and Ye (<https://github.com/LStar404/wormald-tournament-conjectures>, first posted 19 October 2026) treat Wormald's Conjectures 1, 2 and 6 independently. The source of this revision is in the repository https://github.com/creelie/Paper_8, release v1.0.0, archived on Zenodo as 10.5281/zenodo.23659586 (concept 10.5281/zenodo.23659585), together with programs that recompute the table and the numerical values of the paper.

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1 Introduction

Let $\text{hc}(T)$ be the number of Hamilton cycles of a tournament T , let $h(n)$ be the largest value of hc over tournaments on n vertices, and let $g(n) = 2^{-n}(n-1)!$ be the expected number of Hamilton cycles of a uniformly random tournament on n vertices.¹ The random tournament shows that $h(n) \geq g(n)$, an argument that goes back to Szele [10] for Hamilton paths. Alon [2] proved $h(n) = O(n^{3/2}g(n))$, and Friedgut and Kahn [7] lowered the exponent below $5/4$. On the side of constructions, Adler, Alon and Ross [1] found tournaments with $(e - o(1))g'(n)$ Hamilton paths, where $g'(n) = 2^{1-n}n!$ is the expected number of Hamilton paths of a random tournament. Divoux and Wang [6] have since found the global asymptotic $h(n) = (\cosh 1/\cos 1 + o(1))g(n)$ and the analogous result for Hamilton paths. The exact calculation of this paper is carried out directly from Wormald's definition and does not use that asymptotic.

Wormald [11] improved the constant e by averaging over random blow-ups of a fixed tournament. Let T be a tournament on $[d] = \{1, \dots, d\}$ with d odd. The random tournament $\mathcal{T}(T, n)$ on dn vertices is obtained by replacing each vertex j of T by an independent uniformly random tournament T_j on n vertices and each arc jk of T by all arcs from T_j to T_k ; in $\mathcal{R}(T, n)$ the T_j are uniformly random *regular* tournaments (n odd).² See Figure 1. When T is regular, Wormald's main results [11, Theorems 1 and 4] read as follows.

Theorem W. *Let T be a regular tournament on $d \geq 3$ vertices with adjacency matrix A . Put $M(y) = I - (\frac{1}{2}I + A) \text{diag}(y_1, \dots, y_d)$, $K(y) = \det M(y)$ and $z = 2/d$. Then $K(z, \dots, z) = 0$ and $\partial K/\partial y_d \neq 0$ there, so $K(y) = 0$ defines an analytic function $y_d = h(y_1, \dots, y_{d-1})$ near $(z, \dots, z)$. Let H be the determinant of the Hessian at $\theta = 0$ of*

$$\tilde{f}(\theta_1, \dots, \theta_{d-1}) = \log h(ze^{i\theta_1}, \dots, ze^{i\theta_{d-1}}).$$

If $H \neq 0$, then, as $n \rightarrow \infty$,

$$\mathbb{E} \text{hc}(\mathcal{T}(T, n)) \sim \rho_{\mathcal{T}}(T) g(dn) \quad \text{and} \quad \mathbb{E} \text{hc}(\mathcal{R}(T, n)) \sim \rho_{\mathcal{R}}(T) g(dn),$$

where $\rho_{\mathcal{T}}(T) = \sqrt{d/H}$ and $\rho_{\mathcal{R}}(T) = e^{1/d} \rho_{\mathcal{T}}(T)$.

Wormald computed H with a computer algebra system for several small tournaments, among them the cyclic tournament R_d , which has an arc from j to $j+r$ modulo d for $1 \leq r \leq (d-1)/2$, and he found $\rho_{\mathcal{R}}(R_{101}) = 2.855883\dots$. Since a random tournament from $\mathcal{R}(T, n)$ has at least its expected number of Hamilton cycles with positive probability, this gives $h(n) > 2.85588 g(n)$ for infinitely many n . He noted that it is unexplained why d/H was a perfect square in every example, and made the following conjectures [11, Conjectures 2 and 6].

Conjecture 2 (Wormald). *The limit $\lim_{d \rightarrow \infty} \rho_{\mathcal{R}}(R_d)$, over odd d , exists.*

Conjecture 6 (Wormald). *For every regular tournament on $d \geq 3$ vertices, $H \neq 0$ and d/H is the square of a rational number.*

By fitting polynomials in $1/d$ to his values for $d \leq 101$, Wormald estimated the limit in Conjecture 2 as 2.8559579, and he observed that the coefficients of $1/d$ and of $1/d^3$ in the

¹A tournament on n vertices has $\binom{n}{2}$ arcs, one for each pair of vertices, each of which may point either way; a uniformly random tournament chooses every direction independently with probability $1/2$. A given cyclic order of the n vertices is a Hamilton cycle with probability 2^{-n} , and there are $(n-1)!$ cyclic orders, which gives $g(n)$.

²A tournament is regular when every vertex has the same number of out-neighbours, which forces an odd number of vertices and out-degree $(d-1)/2$.

fit were nearly zero while that of $1/d^5$ was not. He also conjectured that this constant is $\lim h(n)/g(n)$ [11, Conjecture 5].

We prove both conjectures by computing H in closed form. For a tournament with adjacency matrix A let $S = A - A^\top$ be its skew adjacency matrix: $S_{jk} = 1$ if jk is an arc, $S_{jk} = -1$ if kj is an arc, and $S_{jj} = 0$.

Theorem 1.1. *Let T be a regular tournament on $d \geq 3$ vertices with skew adjacency matrix S , and let $0, \pm i\mu_1, \dots, \pm i\mu_m$ ($m = (d-1)/2$, $\mu_k \geq 0$) be the eigenvalues of S . Then*

$$H = d \frac{\det(d^2 I + S^2)}{\det(d^2 I - S^2)}. \quad (1.1)$$

Here $\det(d^2 I - S^2) = \det(dI - S)^2$ and $\det(d^2 I + S^2) = \det(dI - iS)^2$, where $\det(dI - S)$ and $\det(dI - iS)$ are positive integers, and

$$\rho_{\mathcal{T}}(T) = \sqrt{\frac{d}{H}} = \frac{\det(dI - S)}{\det(dI - iS)} = \prod_{k=1}^m \frac{d^2 + \mu_k^2}{d^2 - \mu_k^2}. \quad (1.2)$$

Corollary 1.2. *Conjecture 6 holds.*

For the cyclic tournament everything is explicit.

Theorem 1.3. *Let $d \geq 3$ be odd and let S be the skew adjacency matrix of R_d . Then $\det(xI - S) = \frac{1}{2}((x+1)^d + (x-1)^d)$, and*

$$\rho_{\mathcal{T}}(R_d) = \frac{(d+1)^d + (d-1)^d}{(d+i)^d + (d-i)^d} = \left(\frac{d^2 - 1}{d^2 + 1} \right)^{d/2} \frac{\cosh(d \operatorname{artanh} \frac{1}{d})}{\cos(d \arctan \frac{1}{d})}. \quad (1.3)$$

Consequently

$$\lim_{d \rightarrow \infty} \rho_{\mathcal{R}}(R_d) = \lim_{d \rightarrow \infty} \rho_{\mathcal{T}}(R_d) = \frac{\cosh 1}{\cos 1} = 2.85595789256511371\dots,$$

and $\rho_{\mathcal{R}}(R_d) = \frac{\cosh 1}{\cos 1} \left(1 - \frac{\tan 1 - \tanh 1}{3d^2} + O(d^{-4}) \right)$. In the expansion of $\rho_{\mathcal{R}}(R_d)$ in powers of $1/d$ the coefficients of d^{-1} and d^{-3} vanish and that of d^{-5} equals $-\cosh 1/(3 \cos 1)$.

So Conjecture 2 holds, its limit is the constant 2.855958 of Wormald's Conjecture 5, and the last statement accounts for the pattern he saw in his fits. The first expression in (1.3) reproduces every exact value in his table, from $\rho_{\mathcal{T}}(R_3) = 2$ and $\rho_{\mathcal{T}}(R_5) = 44/19$ to $\rho_{\mathcal{T}}(R_{13}) = 41637757888/15769379963$.³ Combined with Theorem W, Theorem 1.3 gives a lower bound for $h(n)$.

Corollary 1.4. $\limsup_{n \rightarrow \infty} h(n)/g(n) \geq \cosh 1 / \cos 1$. *The same holds for the largest number $h'(n)$ of Hamilton paths in a tournament on n vertices, with g' in place of g .*

Theorem 1.1 also turns questions about $\rho_{\mathcal{T}}$ into questions about the spectrum of S . Wormald's Conjecture 1 asks whether R_d maximises $\rho_{\mathcal{T}}$ among tournaments on d vertices; among regular ones we obtain the following bounds, and $\cosh 1 / \cos 1$ turns out to lie within 0.051% of the best value that blow-ups of regular tournaments can give.

³Wormald remarks that extrapolating the exact values may be hard, citing the factorisation $53 \cdot 297535471$ of the denominator for $d = 11$. That number is the denominator for $d = 13$; for $d = 11$ it is $41833969 = 43 \cdot 307 \cdot 3169$.

Theorem 1.5. *Let T be a regular tournament on $d \geq 3$ vertices and $m = (d - 1)/2$. Then*

$$\left(\frac{d+1}{d-1}\right)^m \leq \rho_{\mathcal{T}}(T) \leq U_d,$$

with equality on the left if and only if T is doubly regular. Here, with $\sigma = \cot^2 \frac{\pi}{2d}$, $q = \lfloor d(d-1)/(2\sigma) \rfloor$ and $r = d(d-1)/2 - q\sigma$,

$$U_d = \left(\frac{d^2 + \sigma}{d^2 - \sigma}\right)^q \frac{d^2 + r}{d^2 - r},$$

and

$$\lim_{d \rightarrow \infty} U_d = \frac{(\pi^2 + 4)(3\pi^2 - 8)}{(\pi^2 - 4)(\pi^2 + 8)} = 2.8574011776 \dots$$

In particular $\rho_{\mathcal{R}}(T) > e$ for every regular tournament T , and every limit point of $\max_T \rho_{\mathcal{R}}(T)$, the maximum over regular tournaments on d vertices, lies in $[2.8559578, 2.8574012]$.

Divoux and Wang [6, Proposition 7.1] give a sharper universal bound for $Z(K) = \det(I - K)/\det(I - iK)$. For $K = S/d$ with odd d their bound is exactly the first expression in (1.3), so with (1.2) it shows that R_d maximises $\rho_{\mathcal{T}}$ among regular tournaments on d vertices. Theorem 1.5 is an elementary spectral estimate with a self-contained proof; it is not the sharpest bound known. Nothing proved here uses their asymptotic result.

A tournament is doubly regular if every two vertices have the same number, necessarily $(d-3)/4$, of common out-neighbours; such tournaments exist exactly when there is a skew Hadamard matrix of order $d+1$ [9], for instance for the quadratic residue tournaments of prime order $d \equiv 3 \pmod{4}$.⁴ For $d \leq 9$ an enumeration of all regular tournaments shows that R_d maximises $\rho_{\mathcal{T}}$ among regular tournaments (Table 1); the upper bound U_d is within 10^{-4} of $\rho_{\mathcal{T}}(R_d)$ there.

Remark 1.6. Wormald also lists values of $\rho_{\mathcal{T}}$ for seven 9-vertex circulants, the digraphs with an arc jk whenever $k - j \pmod{9}$ lies in a set D of size four, and from the smallest of them, 1.4582 for $D = \{1, 3, 4, 5\}$, he infers that some regular n -tournaments have at most about $1.4582 e^{1/9} g(n) \approx 1.63 g(n)$ Hamilton cycles. Only one of the seven sets, $D = \{1, 2, 3, 5\}$, defines a tournament: each of the other six contains both r and $9 - r$ for some r , so its circulant has 2-cycles and Theorem W does not apply to it. For $D = \{1, 2, 3, 5\}$ Theorem 1.1 gives $\rho_{\mathcal{T}} = 451696/183937 = 2.4557 \dots$, matching his value. By Theorem 1.5 every regular tournament on 9 vertices has $\rho_{\mathcal{T}} \geq (5/4)^4 = 2.4414 \dots$, and blow-ups of regular tournaments never give fewer than $(e - o(1))g(n)$ expected Hamilton cycles. The bound $1.63 g(n)$ is therefore not supported by these examples.

The proof of Theorem 1.1 has two parts. Proposition 3.1 identifies Wormald's analytic H , the Hessian determinant of an implicit function, with the determinant of an explicit matrix built from S ; this is done by a Schur complement and a second-order expansion. The rest is the algebra of skew-symmetric matrices. The determinantal and enumerative bookkeeping is of the same kind as in the author's earlier work on counts attached to group actions and finite complexes [3, 4, 5].

2 Preliminaries

Throughout, T is a regular tournament on $[d]$ with $d = 2m + 1 \geq 3$, A is its adjacency matrix and $S = A - A^{\top}$. We write J for the all-ones matrix, $\mathbf{1}$ for the all-ones vector, I for the

⁴The quadratic residue tournament on the field with d elements has an arc from a to b when $b - a$ is a nonzero square; for $d \equiv 3 \pmod{4}$ the element -1 is not a square, so exactly one of $b - a$ and $a - b$ is a square and this is a tournament.

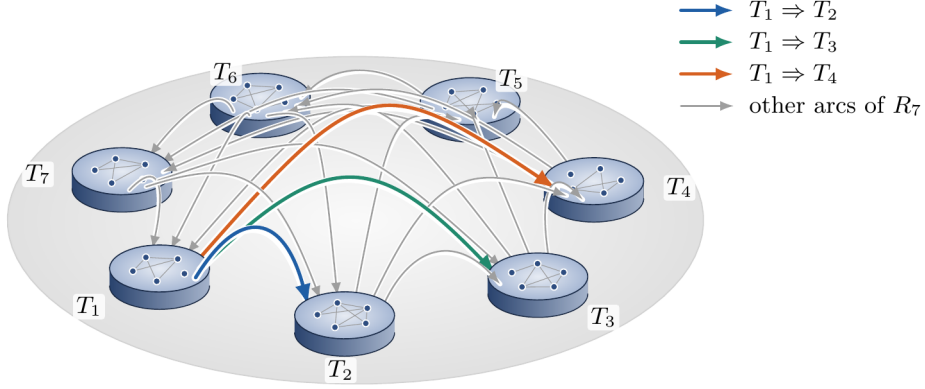


Figure 1: The random blow-up $\mathcal{T}(R_7, n)$. Each vertex j of the cyclic tournament R_7 becomes a random tournament T_j on n vertices, and every arc of R_7 becomes the complete set of arcs from one block to the next; the three arcs leaving T_1 are coloured and the other eighteen are grey.

identity and e_j for the standard basis vectors. Regularity means $A\mathbf{1} = A^\top\mathbf{1} = m\mathbf{1}$. Since $A + A^\top = J - I$,

$$I + 2A = J + S, \quad S\mathbf{1} = 0, \quad \mathbf{1}^\top S = 0. \quad (2.1)$$

Lemma 2.1. *Let $N = (I + 2A)/d$, $P = J/d$ and $Y = S/d$. Then $N = P + Y$, $N\mathbf{1} = \mathbf{1}$, $\mathbf{1}^\top N = \mathbf{1}^\top$, $P^2 = P$ and $PY = YP = 0$. The matrix Y is real and skew-symmetric, every eigenvalue of S has modulus at most $d - 1$, and $I \pm Y$ and $I - Y^2 = (I - Y)(I + Y)$ are invertible.*

Proof. Everything except the last two statements follows from (2.1). Each row of S has $d - 1$ entries ± 1 , so by Gershgorin's theorem the eigenvalues of S , which are purely imaginary, have modulus at most $d - 1$.⁵ The eigenvalues of $I \pm Y$ are $1 \pm i\mu/d$ with μ real, hence nonzero. $\square$

The second lemma restricts a determinant to the hyperplane $x_1 + \dots + x_d = 0$. Let E be the $d \times (d - 1)$ matrix whose j -th column is $e_j - e_d$.

Lemma 2.2. *If F is a $d \times d$ matrix over a field of characteristic 0 with $F\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top F = \mathbf{1}^\top$, then $\det(E^\top F E) = d \det F$.*

Proof. Let $Q = (E \mid \mathbf{1})$. Since $E^\top \mathbf{1} = 0$, $F\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top F = \mathbf{1}^\top$,

$$Q^\top F Q = \begin{pmatrix} E^\top F E & E^\top \mathbf{1} \\ \mathbf{1}^\top F E & \mathbf{1}^\top \mathbf{1} \end{pmatrix} = \begin{pmatrix} E^\top F E & 0 \\ 0 & d \end{pmatrix},$$

so $(\det Q)^2 \det F = d \det(E^\top F E)$. For $F = I$ this reads $(\det Q)^2 = d \det(E^\top E) = d \det(I + J_{d-1}) = d^2$, because $I + J_{d-1}$ has the eigenvalue d once and the eigenvalue 1 with multiplicity $d - 2$. Comparing the two identities gives the claim. $\square$

3 The Hessian determinant

Put $X = (I - Y)^{-1}Y$ and

$$F = I + X + X^\top. \quad (3.1)$$

⁵A real skew-symmetric matrix is normal with purely imaginary eigenvalues, and Gershgorin's theorem places every eigenvalue in a disc centred at a diagonal entry, here 0, with radius the sum of the absolute values of the other entries of its row.

Proposition 3.1. *Wormald's determinant is $H = \det(E^\top FE)$.*

Proof. Change of variables. For $u \in \mathbb{C}^d$ write $e^u = (e^{u_1}, \dots, e^{u_d})$ and $U = \text{diag}(u)$. Since $(\frac{1}{2}I + A)z = N$, the substitution $y = ze^u$ gives

$$K(ze^u) = G(u) := \det(I - Ne^U).$$

A Schur complement. Let $v = \mathbf{1}/\sqrt{d}$ and let W be a real $d \times (d-1)$ matrix whose columns form an orthonormal basis of $\mathbf{1}^\perp$, so that $(W \mid v)$ is orthogonal and $WW^\top = I - P$. Put $M(u) = I - Ne^U$. Since $PW = 0$ we have $NW = YW$, so $W^\top M(0)W = I - W^\top YW$, which is invertible because $W^\top YW$ is real and skew-symmetric. Hence $\Lambda(u) = W^\top M(u)W$ is invertible for u in a neighbourhood Ω of 0, and there

$$G(u) = \det \Lambda(u) \cdot \Phi(u), \quad \Phi(u) = v^\top Mv - v^\top MW \Lambda^{-1} W^\top Mv,$$

with $\det \Lambda(u) \neq 0$.⁶ Using $\mathbf{1}^\top N = \mathbf{1}^\top$, $N\mathbf{1} = \mathbf{1}$ and $W^\top \mathbf{1} = 0$ we find

$$\begin{aligned} v^\top M(u)v &= 1 - \frac{1}{d} \sum_j e^{u_j}, \\ v^\top M(u)W &= -\frac{1}{\sqrt{d}}(e^u - \mathbf{1})^\top W, \quad W^\top M(u)v = -\frac{1}{\sqrt{d}}W^\top N(e^u - \mathbf{1}). \end{aligned}$$

The last two are $O(|u|)$, so

$$\Phi(u) = -\frac{1}{d} \left(\mathbf{1}^\top u + \frac{1}{2} u^\top u + u^\top R^\# N u \right) + O(|u|^3), \quad (3.2)$$

where $R^\# = W(I - W^\top YW)^{-1} W^\top$.

The matrix $R^\# N$. The skew matrix Y maps $\mathbf{1}^\perp$ into itself, so $YW = WW^\top YW$ and $(I - Y)W = W(I - W^\top YW)$. Therefore $W(I - W^\top YW)^{-1} = (I - Y)^{-1}W$ and $R^\# = (I - Y)^{-1}(I - P)$. As $(I - P)P = 0$ and $(I - P)Y = Y$, we get $R^\# N = (I - Y)^{-1}(I - P)(P + Y) = X$. So (3.2) becomes $\Phi(u) = -\frac{1}{d}(\mathbf{1}^\top u + u^\top (\frac{1}{2}I + X)u) + O(|u|^3)$.

The implicit function. Since $\partial \Phi / \partial u_d(0) = -1/d \neq 0$, there is an analytic function ψ near $0 \in \mathbb{C}^{d-1}$ with $\psi(0) = 0$ such that, near 0, $G(u) = 0$ exactly when $u_d = \psi(u_1, \dots, u_{d-1})$. By uniqueness in the implicit function theorem for K , $h(ze^{u'}) = ze^{\psi(u')}$ for u' near 0, and so $\tilde{f}(\theta) = \log z + \psi(i\theta)$ and $\text{Hess } \tilde{f}(0) = -\text{Hess } \psi(0)$. On the graph of ψ the expansion of Φ gives $\psi(u') = -\mathbf{1}^\top u' + O(|u'|^2)$, so that $u = Eu' + O(|u'|^2)$, and then

$$\psi(u') = -\mathbf{1}^\top u' - (Eu')^\top (\frac{1}{2}I + X)(Eu') + O(|u'|^3).$$

The Hessian of the quadratic term is $-E^\top(I + X + X^\top)E = -E^\top FE$. The same result comes from differentiating the equation $\Phi(u', \psi(u')) = 0$ twice. With all derivatives evaluated at 0, the first and second derivatives of Φ are

$$\Phi_j = -\frac{1}{d}, \quad \Phi_{jk} = -\frac{F_{jk}}{d}, \quad \text{and} \quad \psi_j = -1,$$

and the second derivative of $\Phi(u', \psi(u')) = 0$ reads

$$\Phi_{jk} + \Phi_{jd}\psi_k + \Phi_{dk}\psi_j + \Phi_{dd}\psi_j\psi_k + \Phi_d\psi_{jk} = 0,$$

which gives, for $1 \leq j, k < d$,

$$\psi_{jk} = -(F_{jk} - F_{jd} - F_{dk} + F_{dd}) = -(E^\top FE)_{jk}.$$

The logarithm is the holomorphic branch near $h(z, \dots, z) = z > 0$ with value $\log z$ there, and the substitution $u' = i\theta$ changes the sign of the second derivatives. Thus $\text{Hess } \tilde{f}(0) = E^\top FE$ entry by entry and $H = \det(E^\top FE)$. $\square$

⁶This is the determinant formula for a block matrix: $\det \begin{pmatrix} \Lambda & b \\ c^\top & \alpha \end{pmatrix} = \det \Lambda \cdot (\alpha - c^\top \Lambda^{-1} b)$ when Λ is invertible, applied to $(W \mid v)^\top M(W \mid v)$, whose determinant is $\det M$ since $(W \mid v)$ is orthogonal.

Lemma 3.2. $F = (I + Y^2)(I - Y^2)^{-1}$. In particular $F\mathbf{1} = \mathbf{1}$, $\mathbf{1}^\top F = \mathbf{1}^\top$ and

$$\det F = \frac{\det(d^2I + S^2)}{\det(d^2I - S^2)}.$$

Proof. As $Y^\top = -Y$, we have $X^\top = Y^\top(I - Y^\top)^{-1} = -Y(I + Y)^{-1}$. The matrices Y , $(I - Y)^{-1}$ and $(I + Y)^{-1}$ commute, so

$$\begin{aligned} F &= I + Y[(I + Y) - (I - Y)](I - Y^2)^{-1} \\ &= I + 2Y^2(I - Y^2)^{-1} = (I + Y^2)(I - Y^2)^{-1}. \end{aligned}$$

Since $Y\mathbf{1} = 0$ and $\mathbf{1}^\top Y = 0$, the row and column sums of F equal 1. Finally $I \pm Y^2 = d^{-2}(d^2I \pm S^2)$, so $\det(I \pm Y^2) = d^{-2d} \det(d^2I \pm S^2)$. $\square$

Proof of Theorem 1.1. By Proposition 3.1, Lemma 2.2 and Lemma 3.2,

$$H = \det(E^\top FE) = d \det F = d \frac{\det(d^2I + S^2)}{\det(d^2I - S^2)}.$$

Since S commutes with dI and $S^\top = -S$,

$$d^2I - S^2 = (dI - S)(dI - S)^\top, \quad d^2I + S^2 = (dI - iS)(dI - iS)^\top,$$

so $\det(d^2I - S^2) = \det(dI - S)^2$ and $\det(d^2I + S^2) = \det(dI - iS)^2$. The entries of $dI - iS$ are Gaussian integers, and the complex conjugate of $\det(dI - iS)$ is $\det(dI + iS) = \det((dI - iS)^\top) = \det(dI - iS)$, so $\det(dI - iS)$ is a rational integer. In terms of the eigenvalues,

$$\det(dI - S) = d \prod_{k=1}^m (d^2 + \mu_k^2), \quad \det(dI - iS) = d \prod_{k=1}^m (d^2 - \mu_k^2),$$

and both are positive because $\mu_k \leq d - 1$ by Lemma 2.1. This proves (1.2). $\square$

Corollary 3.3 (Quantitative nondegeneracy). *Let $\mathcal{H} = \text{Hess } \tilde{f}(0)$ and $a_d = (2d - 1)/(2d^2 - 2d + 1)$. Then $\mathcal{H}$ is real symmetric positive definite and, in the order on real symmetric matrices,*

$$a_d(I_{d-1} + J_{d-1}) \preceq \mathcal{H} \preceq I_{d-1} + J_{d-1}.$$

In particular every eigenvalue of $\mathcal{H}$ lies in $[a_d, d]$ and $da_d^{d-1} \leq H \leq d$.

Proof. The skew-symmetric matrix $Y = S/d$ has eigenvalues $i\mu/d$, where $|\mu| \leq d - 1$. Lemma 3.2 shows that the eigenvalues of the real symmetric matrix F are $(d^2 - \mu^2)/(d^2 + \mu^2)$, which lie in $[a_d, 1]$. Hence $a_d I \preceq F \preceq I$. Congruence by the full-rank matrix E , together with $E^\top E = I_{d-1} + J_{d-1}$ and Proposition 3.1, proves the matrix inequalities. The matrix $I_{d-1} + J_{d-1}$ has eigenvalues 1 and d and determinant d ; the bounds on the eigenvalues and on the determinant follow, since the determinant is monotone on positive definite matrices. $\square$

Proof of Corollary 1.2. By Theorem 1.1, H is a positive rational number and $d/H = (\det(dI - S)/\det(dI - iS))^2$. $\square$

4 The cyclic tournament

Let $d = 2m + 1 \geq 3$ and index the vertices of R_d by $\mathbb{Z}/d$. Let C be the permutation matrix with $C_{jk} = 1$ if and only if $k = j + 1$, and put $P_c = C^{m+1}$. Then $S = \sum_{r=1}^m (C^r - C^{-r})$ is the skew adjacency matrix of R_d .

Lemma 4.1. $S(I + P_c) = I - P_c$. Consequently, for all a, b in any commutative ring,

$$2 \det(aI + bS) = (a + b)^d + (a - b)^d. \quad (4.1)$$

Proof. Using $C^d = I$, so that $C^{r+m+1} = C^{r-m}$,

$$\begin{aligned} S(I + C^{m+1}) &= \sum_{r=1}^m (C^r - C^{-r}) + \sum_{s=1-m}^0 C^s - \sum_{s=1}^m C^s \\ &= \sum_{s=0}^{m-1} C^{-s} - \sum_{r=1}^m C^{-r} = I - C^{-m} = I - C^{m+1}. \end{aligned}$$

Hence $(aI + bS)(I + P_c) = (a + b)I + (a - b)P_c$. Since $m + 1$ is prime to d , P_c is the permutation matrix of a single cycle π of odd length d . In the expansion of $\det(\alpha I + \beta P_c)$ a permutation τ contributes only if $\tau(j) \in \{j, \pi(j)\}$ for all j ; if $\tau(j) = \pi(j)$ for one j then $\tau(\pi(j)) = \pi^2(j)$, and so on, so τ is the identity or π . As π is even, $\det(\alpha I + \beta P_c) = \alpha^d + \beta^d$. In particular $\det(I + P_c) = 2$, and taking determinants gives (4.1).⁷ $\square$

Proof of Theorem 1.3. With $a = x$ and $b = -1$, (4.1) is the characteristic polynomial. With $a = d$ and $b = -1$, and with $a = d$ and $b = -i$, it gives $\det(dI - S) = \frac{1}{2}((d + 1)^d + (d - 1)^d)$ and $\det(dI - iS) = \frac{1}{2}((d + i)^d + (d - i)^d)$, and (1.2) yields the first expression in (1.3). For the second, write $d \pm 1 = \sqrt{d^2 - 1} e^{\pm \operatorname{artanh}(1/d)}$ and $d \pm i = \sqrt{d^2 + 1} e^{\pm i \arctan(1/d)}$.

Dividing numerator and denominator by d^d ,

$$\rho_{\mathcal{T}}(R_d) = \frac{(1 + \frac{1}{d})^d + (1 - \frac{1}{d})^d}{2 \operatorname{Re}(1 + \frac{i}{d})^d} \rightarrow \frac{e + e^{-1}}{2 \operatorname{Re} e^i} = \frac{\cosh 1}{\cos 1},$$

and $\rho_{\mathcal{R}}(R_d) = e^{1/d} \rho_{\mathcal{T}}(R_d)$ has the same limit. For the expansion put $x = 1/d$. By the second expression in (1.3), $\rho_{\mathcal{R}}(R_d) = G(x)$ with

$$\log G(x) = x + \frac{1}{2x} \log \frac{1 - x^2}{1 + x^2} + \log \cosh \frac{\operatorname{artanh} x}{x} - \log \cos \frac{\arctan x}{x}.$$

The four terms are analytic near $x = 0$. The first two add up to $-\sum_{j \geq 1} x^{4j+1}/(2j + 1) = -\frac{x^5}{3} - \frac{x^9}{5} - \dots$, and the last two are even functions of x because $\operatorname{artanh}(x)/x = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots$ and $\arctan(x)/x = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots$. Hence $G(x) = e^{\mathcal{E}(x)}(1 - \frac{x^5}{3} + O(x^9))$ with $\mathcal{E}$ even, $e^{\mathcal{E}(0)} = \cosh 1 / \cos 1$ and $\mathcal{E}''(0)/2 = (\tanh 1 - \tan 1)/3$. The odd coefficients of G below x^5 vanish, the coefficient of x^5 is $-e^{\mathcal{E}(0)}/3$, and the coefficient of x^2 is $e^{\mathcal{E}(0)}(\tanh 1 - \tan 1)/3$. $\square$

Remark 4.2. By Theorem 1.3 the eigenvalues of S are the roots of $(x + 1)^d = -(x - 1)^d$, that is, $x = i \cot \frac{(2k-1)\pi}{2d}$ for $k = 1, \dots, d$. So $\mu_k = \cot \frac{(2k-1)\pi}{2d}$ in (1.2), and

$$\rho_{\mathcal{T}}(R_d) = \prod_{k=1}^m \frac{d^2 + \cot^2 \frac{(2k-1)\pi}{2d}}{d^2 - \cot^2 \frac{(2k-1)\pi}{2d}}.$$

⁷The identity is first proved over the polynomial ring $\mathbb{Z}[a, b]$, where 2 is not a zero divisor, and then specialised.

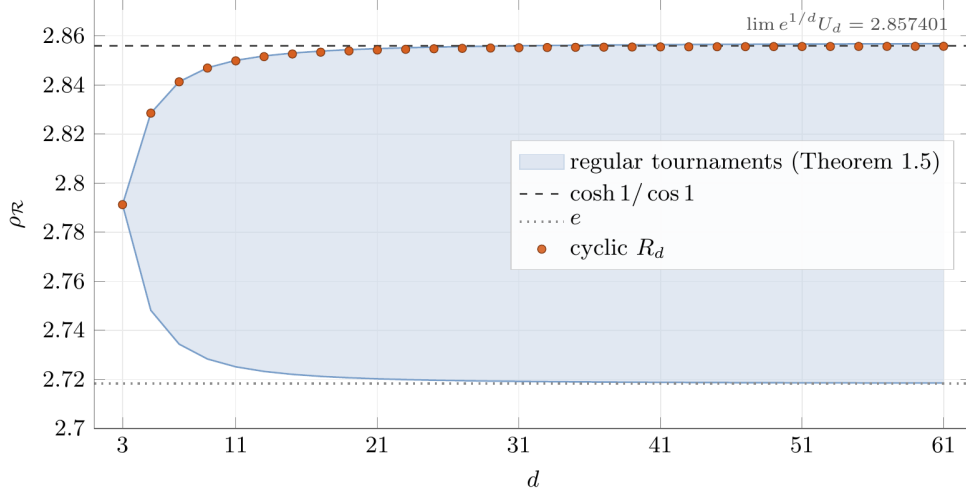


Figure 2: $\rho_{\mathcal{R}}(T)$ for regular tournaments on d vertices. The dots are the cyclic tournaments R_d ; the band is bounded by $e^{1/d}((d+1)/(d-1))^{(d-1)/2}$ and $e^{1/d}U_d$ (Theorem 1.5). The upper bound and $\rho_{\mathcal{R}}(R_d)$ both approach the region near $\cosh 1/\cos 1$, while the lower bound decreases to e .

Letting $d \rightarrow \infty$ term by term (justified by dominated convergence, since $\cot x < 1/x$) and using Euler's products

$$\cos x = \prod_{k \geq 1} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2}\right), \quad \cosh x = \prod_{k \geq 1} \left(1 + \frac{4x^2}{(2k-1)^2\pi^2}\right)$$

gives the limit a second time. The transitive tournament has the same characteristic polynomial (Lemma 5.1 below), although it is not regular.

Proof of Corollary 1.4. Let $\varepsilon > 0$ and choose d with $\rho_{\mathcal{R}}(R_d) > \cosh 1/\cos 1 - \varepsilon$. By Theorem W, for all large odd n some tournament in the support of $\mathcal{R}(R_d, n)$ has at least $(\cosh 1/\cos 1 - 2\varepsilon)g(dn)$ Hamilton cycles. Wormald's Theorem 5 [11] states that Theorem W holds for Hamilton paths with g' in place of g , and the same argument applies. $\square$

5 Bounds over all regular tournaments

Write $s_k = \mu_k^2$. Since $\text{tr}(SS^T) = d(d-1)$ and the eigenvalues of $SS^T = -S^2$ are 0 and $s_1, s_1, \dots, s_m, s_m$,

$$\sum_{k=1}^m s_k = \frac{d(d-1)}{2}, \quad \log \rho_{\mathcal{T}}(T) = \sum_{k=1}^m \varphi(s_k), \quad \varphi(s) = \log \frac{d^2 + s}{d^2 - s}. \quad (5.1)$$

The function φ is strictly convex on $[0, d^2)$, since $\varphi''(s) = (d^2 - s)^{-2} - (d^2 + s)^{-2} > 0$ for $s > 0$.

Lemma 5.1. *Let D_d be the skew-symmetric $d \times d$ matrix with all entries above the diagonal equal to 1. Its eigenvalues are $i \cot \frac{(2k-1)\pi}{2d}$, $k = 1, \dots, d$; in particular its spectral radius is $\cot \frac{\pi}{2d}$.*

Proof. Let $\omega^d = -1$ and $v = (1, \omega, \dots, \omega^{d-1})^\top$. Then

$$(D_d v)_j = \sum_{l>j} \omega^l - \sum_{l<j} \omega^l = \frac{\omega^{j+1} - \omega^d}{1 - \omega} - \frac{1 - \omega^j}{1 - \omega} = \frac{1 + \omega}{1 - \omega} \omega^j \quad (0 \leq j < d),$$

so $(1 + \omega)/(1 - \omega) = i \cot(\theta/2)$ is an eigenvalue for $\omega = e^{i\theta}$, $\theta = (2k - 1)\pi/d$. These d numbers are distinct. $\square$

Lemma 5.2 (Spectral radius bound). *If B is a real skew-symmetric $d \times d$ matrix with $|B_{jk}| \leq 1$, then its spectral radius is at most $\cot(\pi/(2d))$.*

Proof. The matrix iB is Hermitian, so it suffices to bound $|v^* iB v|$ for every unit vector $v \in \mathbb{C}^d$. Changing the signs of coordinates and permuting them preserves the hypotheses on B and the Rayleigh quotient. We may therefore write $v_j = r_j e^{i\theta_j}$ with $r_j \geq 0$ and $0 \leq \theta_1 \leq \dots \leq \theta_d < \pi$: choose the sign of each nonzero coordinate to put its argument in $[0, \pi)$ and then sort the arguments; zero coordinates may have argument zero. Then

$$|v^* iB v| = \left| -2 \sum_{j<k} B_{jk} r_j r_k \sin(\theta_k - \theta_j) \right| \leq 2 \sum_{j<k} r_j r_k \sin(\theta_k - \theta_j) = |v^* iD_d v|.$$

Every sine in the sum is nonnegative. By Lemma 5.1 and the Hermitian Rayleigh bound, the last expression is at most $\cot(\pi/(2d)) \|v\|^2$. This proves the claim. The bound is also recorded in [8, Lemma 12]; the proof above supplies the input used here. $\square$

Proof of Theorem 1.5. Lower bound. By (5.1) and Jensen's inequality, $\log \rho_{\mathcal{T}}(T) \geq m \varphi(d) = m \log \frac{d+1}{d-1}$, with equality if and only if $s_1 = \dots = s_m = d$. As S is normal with $S\mathbf{1} = 0$, this happens exactly when $SS^\top = dI - J$. Expanding with $A + A^\top = J - I$ and $AJ = JA^\top = mJ$ gives

$$SS^\top = 4AA^\top + (2 - d)J - I,$$

so the equality condition is $AA^\top = \frac{1}{4}((d+1)I + (d-3)J)$: every two vertices have $(d-3)/4$ common out-neighbours.

$\rho_{\mathcal{R}} > e$. With $x = 1/d$,

$$\log \left(e^{1/d} \left(\frac{d+1}{d-1} \right)^m \right) = x + \frac{1-x}{2x} \log \frac{1+x}{1-x} = 1 + \sum_{k \geq 1} \frac{x^{2k} - x^{2k+1}}{2k+1} > 1.$$

Upper bound. Lemma 5.2 gives $s_k \leq \sigma = \cot^2 \frac{\pi}{2d} < d^2$ by Lemma 5.1. Hence $(s_1, \dots, s_m)$ lies in the polytope $\{0 \leq s_k \leq \sigma, \sum s_k = d(d-1)/2\}$, on which the convex function $\sum \varphi(s_k)$ attains its maximum at a vertex. A point of the polytope with two coordinates in $(0, \sigma)$ is the midpoint of two other points of it, so at a vertex at most one coordinate lies strictly between 0 and σ ; this forces q coordinates equal to σ , one further coordinate equal to r when $r > 0$, and all remaining coordinates equal to 0, giving the value $\log U_d$ (also when $r = 0$). As $d \rightarrow \infty$, $\sigma/d^2 \rightarrow 4/\pi^2$ and $d(d-1)/(2d^2) \rightarrow \frac{1}{2}$; since $1 < \pi^2/8 < 2$ we get $q = 1$ for large d , $r/d^2 \rightarrow \frac{1}{2} - \frac{4}{\pi^2}$, and the stated limit. Finally, every limit point of $\max_T \rho_{\mathcal{R}}(T)$ is at least $\lim \rho_{\mathcal{R}}(R_d) = \cosh 1 / \cos 1$ and at most $\lim e^{1/d} U_d$. $\square$

Remark 5.3. Everything above is proved in full. Theorem 1.1 is the exact value of Wormald's H for every regular tournament, Theorem 1.3 its value and limit along the cyclic tournaments, and Theorem 1.5 two bounds whose proofs use only the trace identity (5.1), convexity and Lemma 5.2. The only inputs from outside the paper are Theorem W, which is used in Corollary 1.4 and nowhere else, and the existence of doubly regular tournaments from [9].

Table 1: $\rho_{\mathcal{T}}(T)$ for the regular tournaments on $d \leq 9$ vertices, one row for each characteristic polynomial of S , from all regular tournaments on $\mathbb{Z}/d$ in which 0 dominates $1, \dots, m$; “labelled” counts these. Every regular tournament is isomorphic to one of them. The rows come from an enumeration of these tournaments; each value in the second column is the determinant ratio of Theorem 1.1.

d	$\rho_{\mathcal{T}}(T)$	decimal	labelled	remark
5	44/19	2.315789	4	R_5
7	3032/1231	2.463038	36	R_7
	3088/1287	2.399378	84	Wormald’s example
	64/27	2.370370	12	doubly regular (Paley)
9	437584/171769	2.547514	576	R_9
	443488/177529	2.498116	5184	
	5488/2197	2.497952	64	R'_9 (blow-up of R_3)
	446656/180193	2.478764	6912	
	446512/180337	2.475987	5184	
	446368/180481	2.473213	1728	
	449152/182077	2.466824	6912	
	449008/182221	2.464085	1728	
	449824/182857	2.459977	10368	
	50080/20381	2.457191	5184	
	451696/183937	2.455710	576	circulant $\{1, 2, 3, 5\}$
	50176/20449	2.453714	1728	

Table 1 and the decimal values quoted in the text are computations, not theorems; the first expression in (1.3) and the exact values of Wormald’s table agree for every $d \leq 13$ that he lists, and his decimal value $\rho_{\mathcal{R}}(R_{81}) = 2.855842467\dots$ should read $2.855842468\dots$. The statement that R_d maximises $\rho_{\mathcal{T}}$ among regular tournaments on $d \leq 9$ vertices is read off from the table; for all d it follows from the bound of Divoux and Wang quoted in Section 1, and nothing else in this paper depends on it.

Declarations

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