

Two conjectures of Wormald on Hamilton cycles in tournaments

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Abstract

For a regular tournament on d vertices with skew adjacency matrix S , the Hessian determinant H that Wormald attached to its random blow-ups equals $d \det(d^2 I + S^2) / \det(d^2 I - S^2)$. Hence $H \neq 0$ and d/H is the square of a rational number, as Wormald conjectured. For the cyclic tournament it gives $\sqrt{d/H} = ((d+1)^d + (d-1)^d) / ((d+i)^d + (d-i)^d)$, so the associated constants tend to $\cosh 1 / \cos 1 = 2.8559578925 \dots$; this proves a second conjecture of Wormald and identifies his numerical constant.

Keywords: tournament, Hamilton cycle, random blow-up, Hessian determinant, skew-symmetric matrix, formal verification

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1. Introduction

Let $\text{hc}(T)$ be the number of Hamilton cycles of a tournament T , let $h(n)$ be the largest value of hc over tournaments on n vertices, and let $g(n) = 2^{-n}(n-1)!$ be the expected number of Hamilton cycles of a uniformly random tournament on n vertices. The random tournament shows that $h(n) \geq g(n)$, an argument that goes back to Szele [8] for Hamilton paths. Alon [2] proved $h(n) = O(n^{3/2}g(n))$, and Friedgut and Kahn [3] lowered the exponent below $5/4$. On the side of constructions, Adler, Alon and Ross [1] found tournaments with $(e - o(1))g'(n)$ Hamilton paths, where

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$g'(n) = 2^{1-n}n!$ is the expected number of Hamilton paths of a random tournament. Whether $h(n) = \Theta(g(n))$ is open.

Wormald [9] improved the constant e by averaging over random blow-ups of a fixed tournament. Let T be a tournament on $[d] = \{1, \dots, d\}$ with d odd. The random tournament $\mathcal{T}(T, n)$ on dn vertices is obtained by replacing each vertex j of T by an independent uniformly random tournament T_j on n vertices and each arc jk of T by all arcs from T_j to T_k ; in $\mathcal{R}(T, n)$ the T_j are uniformly random *regular* tournaments (n odd). See Figure 1. When T is regular, Wormald's main results [9, Theorems 1 and 4] read as follows.

Theorem W. *Let T be a regular tournament on $d \geq 3$ vertices with adjacency matrix A . Put $M(y) = I - (\frac{1}{2}I + A) \text{diag}(y_1, \dots, y_d)$, $K(y) = \det M(y)$ and $z = 2/d$. Then $K(z, \dots, z) = 0$ and $\partial K / \partial y_d \neq 0$ there, so $K(y) = 0$ defines an analytic function $y_d = h(y_1, \dots, y_{d-1})$ near $(z, \dots, z)$. Let H be the determinant of the Hessian at $\theta = 0$ of*

$$\tilde{f}(\theta_1, \dots, \theta_{d-1}) = \log h(z e^{i\theta_1}, \dots, z e^{i\theta_{d-1}}).$$

If $H \neq 0$, then, as $n \rightarrow \infty$,

$$\mathbb{E} \text{hc}(\mathcal{T}(T, n)) \sim \rho_{\mathcal{T}}(T) g(dn) \quad \text{and} \quad \mathbb{E} \text{hc}(\mathcal{R}(T, n)) \sim \rho_{\mathcal{R}}(T) g(dn),$$

where $\rho_{\mathcal{T}}(T) = \sqrt{d/H}$ and $\rho_{\mathcal{R}}(T) = e^{1/d} \rho_{\mathcal{T}}(T)$.

Wormald computed H with a computer algebra system for several small tournaments, among them the cyclic tournament R_d , which has an arc from j to $j+r$ modulo d for $1 \leq r \leq (d-1)/2$, and he found $\rho_{\mathcal{R}}(R_{101}) = 2.855883\dots$. Since a random tournament from $\mathcal{R}(T, n)$ has at least its expected number of Hamilton cycles with positive probability, this gives $h(n) > 2.85588 g(n)$ for infinitely many n . He noted that it is unexplained why d/H was a perfect square in every example, and made the following conjectures [9, Conjectures 2 and 6].

Conjecture 2 (Wormald). *The limit $\lim_{d \rightarrow \infty} \rho_{\mathcal{R}}(R_d)$, over odd d , exists.*

Conjecture 6 (Wormald). *For every regular tournament on $d \geq 3$ vertices, $H \neq 0$ and d/H is the square of a rational number.*

By fitting polynomials in $1/d$ to his values for $d \leq 101$, Wormald estimated the limit in Conjecture 2 as 2.8559579, and he observed that the

coefficients of $1/d$ and of $1/d^3$ in the fit were nearly zero while that of $1/d^5$ was not. He also conjectured that this constant is $\lim h(n)/g(n)$ [9, Conjecture 5].

We prove both conjectures, by computing H in closed form. For a tournament with adjacency matrix A let $S = A - A^\top$ be its skew adjacency matrix: $S_{jk} = 1$ if jk is an arc, $S_{jk} = -1$ if kj is an arc, and $S_{jj} = 0$.

Theorem 1.1. *Let T be a regular tournament on $d \geq 3$ vertices with skew adjacency matrix S , and let $0, \pm i\mu_1, \dots, \pm i\mu_m$ ($m = (d-1)/2$, $\mu_k \geq 0$) be the eigenvalues of S . Then*

$$H = d \frac{\det(d^2 I + S^2)}{\det(d^2 I - S^2)}. \quad (1.1)$$

Here $\det(d^2 I - S^2) = \det(dI - S)^2$ and $\det(d^2 I + S^2) = \det(dI - iS)^2$, where $\det(dI - S)$ and $\det(dI - iS)$ are positive integers, and

$$\rho_{\mathcal{T}}(T) = \sqrt{\frac{d}{H}} = \frac{\det(dI - S)}{\det(dI - iS)} = \prod_{k=1}^m \frac{d^2 + \mu_k^2}{d^2 - \mu_k^2}. \quad (1.2)$$

Corollary 1.2. *Conjecture 6 holds.*

For the cyclic tournament everything is explicit.

Theorem 1.3. *Let $d \geq 3$ be odd and let S be the skew adjacency matrix of R_d . Then $\det(xI - S) = \frac{1}{2}((x+1)^d + (x-1)^d)$, and*

$$\rho_{\mathcal{T}}(R_d) = \frac{(d+1)^d + (d-1)^d}{(d+i)^d + (d-i)^d} = \left(\frac{d^2 - 1}{d^2 + 1} \right)^{d/2} \frac{\cosh\left(d \operatorname{artanh} \frac{1}{d}\right)}{\cos\left(d \arctan \frac{1}{d}\right)}. \quad (1.3)$$

Consequently

$$\lim_{d \rightarrow \infty} \rho_{\mathcal{R}}(R_d) = \lim_{d \rightarrow \infty} \rho_{\mathcal{T}}(R_d) = \frac{\cosh 1}{\cos 1} = 2.85595789256511371 \dots,$$

and $\rho_{\mathcal{R}}(R_d) = \frac{\cosh 1}{\cos 1} \left(1 - \frac{\tan 1 - \tanh 1}{3d^2} + O(d^{-4}) \right)$. In the expansion of $\rho_{\mathcal{R}}(R_d)$ in powers of $1/d$ the coefficients of d^{-1} and d^{-3} vanish and that of d^{-5} equals $-\cosh 1/(3 \cos 1)$.

So Conjecture 2 holds, its limit is the constant 2.855958 of Wormald's Conjecture 5, and the last statement accounts for the pattern he saw in his fits. The first expression in (1.3) reproduces every exact value in his table, from $\rho_{\mathcal{T}}(R_3) = 2$ and $\rho_{\mathcal{T}}(R_5) = 44/19$ to $\rho_{\mathcal{T}}(R_{13}) = 41637757888/15769379963$.¹ Combined with Theorem W, Theorem 1.3 gives a lower bound for $h(n)$.

Corollary 1.4. $\limsup_{n \rightarrow \infty} h(n)/g(n) \geq \cosh 1 / \cos 1$. *The same holds for the largest number $h'(n)$ of Hamilton paths in a tournament on n vertices, with g' in place of g .*

Theorem 1.1 also turns questions about $\rho_{\mathcal{T}}$ into questions about the spectrum of S . Wormald's Conjecture 1 asks whether R_d maximizes $\rho_{\mathcal{T}}$ among tournaments on d vertices; among regular ones we obtain the following bounds, and $\cosh 1 / \cos 1$ turns out to lie within 0.051% of the best value that blow-ups of regular tournaments can give.

Theorem 1.5. *Let T be a regular tournament on $d \geq 3$ vertices and $m = (d-1)/2$. Then*

$$\left(\frac{d+1}{d-1}\right)^m \leq \rho_{\mathcal{T}}(T) \leq U_d,$$

with equality on the left if and only if T is doubly regular. Here, with $\sigma = \cot^2 \frac{\pi}{2d}$, $q = \lfloor d(d-1)/(2\sigma) \rfloor$ and $r = d(d-1)/2 - q\sigma$,

$$U_d = \left(\frac{d^2 + \sigma}{d^2 - \sigma}\right)^q \frac{d^2 + r}{d^2 - r},$$

and

$$\lim_{d \rightarrow \infty} U_d = \frac{(\pi^2 + 4)(3\pi^2 - 8)}{(\pi^2 - 4)(\pi^2 + 8)} = 2.8574011776 \dots$$

In particular $\rho_{\mathcal{R}}(T) > e$ for every regular tournament T , and every limit point of $\max_T \rho_{\mathcal{R}}(T)$, the maximum over regular tournaments on d vertices, lies in $[2.8559578, 2.8574012]$.

A tournament is doubly regular if every two vertices have the same number, necessarily $(d-3)/4$, of common out-neighbours; such tournaments exist

¹Wormald remarks that extrapolating the exact values may be hard, citing the factorization $53 \cdot 297535471$ of the denominator for $d = 11$. That number is the denominator for $d = 13$; for $d = 11$ it is $41833969 = 43 \cdot 307 \cdot 3169$.

exactly when there is a skew Hadamard matrix of order $d+1$ [7], for instance for the quadratic residue tournaments of prime order $d \equiv 3 \pmod{4}$. For $d \leq 9$ an exhaustive computation shows that R_d maximizes $\rho_{\mathcal{T}}$ among regular tournaments (Table 1); the upper bound U_d is within 10^{-4} of $\rho_{\mathcal{T}}(R_d)$ there.

Remark 1.6. Wormald also lists values of $\rho_{\mathcal{T}}$ for seven 9-vertex circulants, the digraphs with an arc jk whenever $k-j \pmod{9}$ lies in a set D of size four, and from the smallest of them, 1.4582 for $D = \{1, 3, 4, 5\}$, he infers that some regular n -tournaments have at most about $1.4582 e^{1/9} g(n) \approx 1.63 g(n)$ Hamilton cycles. Only one of the seven sets, $D = \{1, 2, 3, 5\}$, defines a tournament: each of the other six contains both r and $9-r$ for some r , so its circulant has 2-cycles and Theorem W does not apply to it. For $D = \{1, 2, 3, 5\}$ Theorem 1.1 gives $\rho_{\mathcal{T}} = 451696/183937 = 2.4557\dots$, matching his value. By Theorem 1.5 every regular tournament on 9 vertices has $\rho_{\mathcal{T}} \geq (5/4)^4 = 2.4414\dots$, and blow-ups of regular tournaments never give fewer than $(e - o(1))g(n)$ expected Hamilton cycles. The bound $1.63 g(n)$ is therefore not supported by these examples.

The algebraic steps of the proofs are checked in Lean 4 with Mathlib [5, 6]: the identity behind (1.1), the integrality and squareness statements, the characteristic polynomial of R_d , the closed form (1.3), and the limit $\cosh 1 / \cos 1$. The one step that is not formalized is Proposition 3.1, which identifies Wormald's analytic definition of H with a determinant by the implicit function theorem. That step was checked by computing H exactly, directly from Wormald's definition, for one regular tournament of each skew spectrum with at most 9 vertices and for random ones with 11 and 13 vertices (Section 6).

2. Preliminaries

Throughout, T is a regular tournament on $[d]$ with $d = 2m + 1 \geq 3$, A is its adjacency matrix and $S = A - A^{\top}$. We write J for the all-ones matrix, $\mathbf{1}$ for the all-ones vector, I for the identity and e_j for the standard basis vectors. Regularity means $A\mathbf{1} = A^{\top}\mathbf{1} = m\mathbf{1}$. Since $A + A^{\top} = J - I$,

$$I + 2A = J + S, \quad S\mathbf{1} = 0, \quad \mathbf{1}^{\top}S = 0. \quad (2.1)$$

Lemma 2.1. *Let $N = (I + 2A)/d$, $P = J/d$ and $Y = S/d$. Then $N = P + Y$, $N\mathbf{1} = \mathbf{1}$, $\mathbf{1}^{\top}N = \mathbf{1}^{\top}$, $P^2 = P$ and $PY = YP = 0$. The matrix Y is real and skew-symmetric, every eigenvalue of S has modulus at most $d-1$, and $I \pm Y$ and $I - Y^2 = (I - Y)(I + Y)$ are invertible.*

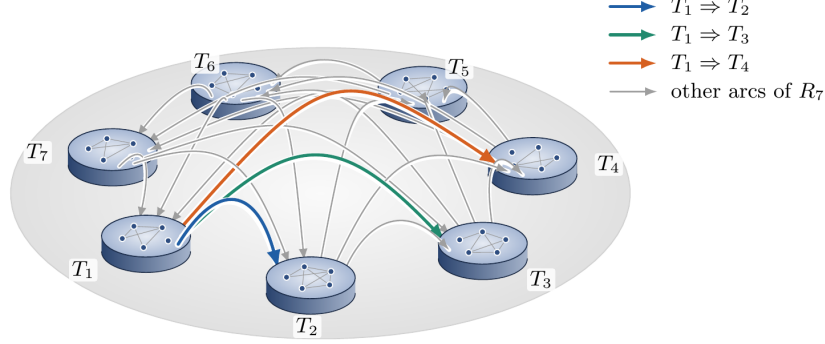


Figure 1: The random blow-up $\mathcal{T}(R_7, n)$. Each vertex j of the cyclic tournament R_7 becomes a random tournament T_j on n vertices, and every arc of R_7 becomes the complete set of arcs from one block to the next; the three arcs leaving T_1 are coloured and the other eighteen are grey.

Proof. Everything except the last two statements follows from (2.1). Each row of S has $d - 1$ entries ± 1 , so by Gershgorin's theorem the eigenvalues of S , which are purely imaginary, have modulus at most $d - 1$. The eigenvalues of $I \pm Y$ are $1 \pm i\mu/d$ with μ real, hence nonzero. $\square$

The second lemma restricts a determinant to the hyperplane $x_1 + \dots + x_d = 0$. Let E be the $d \times (d - 1)$ matrix whose j -th column is $e_j - e_d$.

Lemma 2.2. *If F is a $d \times d$ matrix over a field of characteristic 0 with $F\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top F = \mathbf{1}^\top$, then $\det(E^\top F E) = d \det F$.*

Proof. Let $Q = (E \mid \mathbf{1})$. Since $E^\top \mathbf{1} = 0$, $F\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top F = \mathbf{1}^\top$,

$$Q^\top F Q = \begin{pmatrix} E^\top F E & E^\top \mathbf{1} \\ \mathbf{1}^\top E & \mathbf{1}^\top \mathbf{1} \end{pmatrix} = \begin{pmatrix} E^\top F E & 0 \\ 0 & d \end{pmatrix},$$

so $(\det Q)^2 \det F = d \det(E^\top F E)$. For $F = I$ this reads $(\det Q)^2 = d \det(E^\top E) = d \det(I + J_{d-1}) = d^2$. Comparing the two identities gives the claim. $\square$

3. The Hessian determinant

Put $X = (I - Y)^{-1}Y$ and

$$F = I + X + X^\top. \quad (3.1)$$

Proposition 3.1. *Wormald's determinant is $H = \det(E^\top FE)$.*

Proof. Change of variables. For $u \in \mathbb{C}^d$ write $e^u = (e^{u_1}, \dots, e^{u_d})$ and $U = \text{diag}(u)$. Since $(\frac{1}{2}I + A)z = N$, the substitution $y = ze^u$ gives

$$K(ze^u) = G(u) := \det(I - Ne^U).$$

A Schur complement. Let $v = \mathbf{1}/\sqrt{d}$ and let W be a real $d \times (d-1)$ matrix whose columns form an orthonormal basis of $\mathbf{1}^\perp$, so that $(W \mid v)$ is orthogonal and $WW^\top = I - P$. Put $M(u) = I - Ne^U$. Since $PW = 0$ we have $NW = YW$, so $W^\top M(0)W = I - W^\top YW$, which is invertible because $W^\top YW$ is real and skew-symmetric. Hence $\Lambda(u) = W^\top M(u)W$ is invertible for u in a neighbourhood Ω of 0, and there

$$G(u) = \det \Lambda(u) \cdot \Phi(u), \quad \Phi(u) = v^\top Mv - v^\top MW \Lambda^{-1} W^\top Mv,$$

with $\det \Lambda(u) \neq 0$. Using $\mathbf{1}^\top N = \mathbf{1}^\top$, $N\mathbf{1} = \mathbf{1}$ and $W^\top \mathbf{1} = 0$ we find

$$\begin{aligned} v^\top M(u)v &= 1 - \frac{1}{d} \sum_j e^{u_j}, \\ v^\top M(u)W &= -\frac{1}{\sqrt{d}}(e^u - \mathbf{1})^\top W, \quad W^\top M(u)v = -\frac{1}{\sqrt{d}}W^\top N(e^u - \mathbf{1}). \end{aligned}$$

The last two are $O(|u|)$, so

$$\Phi(u) = -\frac{1}{d} \left(\mathbf{1}^\top u + \frac{1}{2} u^\top u + u^\top R^\# N u \right) + O(|u|^3), \quad (3.2)$$

where $R^\# = W(I - W^\top YW)^{-1}W^\top$. The matrix $R^\# N$. The skew matrix Y maps $\mathbf{1}^\perp$ into itself, so $YW = WW^\top YW$ and $(I - Y)W = W(I - W^\top YW)$. Therefore $W(I - W^\top YW)^{-1} = (I - Y)^{-1}W$ and $R^\# = (I - Y)^{-1}(I - P)$. As $(I - P)P = 0$ and $(I - P)Y = Y$, we get $R^\# N = (I - Y)^{-1}(I - P)(P + Y) = X$. So (3.2) becomes $\Phi(u) = -\frac{1}{d} \left(\mathbf{1}^\top u + u^\top (\frac{1}{2}I + X)u \right) + O(|u|^3)$.

The implicit function. Since $\partial \Phi / \partial u_d(0) = -1/d \neq 0$, there is an analytic function ψ near $0 \in \mathbb{C}^{d-1}$ with $\psi(0) = 0$ such that, near 0, $G(u) = 0$ exactly when $u_d = \psi(u_1, \dots, u_{d-1})$. By uniqueness in the implicit function theorem for K , $h(ze^{u'}) = ze^{\psi(u')}$ for u' near 0, and so $\tilde{f}(\theta) = \log z + \psi(i\theta)$ and $\text{Hess } \tilde{f}(0) = -\text{Hess } \psi(0)$. On the graph of ψ the expansion of Φ gives $\psi(u') = -\mathbf{1}^\top u' + O(|u'|^2)$, so that $u = Eu' + O(|u'|^2)$, and then

$$\psi(u') = -\mathbf{1}^\top u' - (Eu')^\top (\frac{1}{2}I + X)(Eu') + O(|u'|^3).$$

The Hessian of the quadratic term is $-E^\top(I + X + X^\top)E = -E^\top FE$. Hence $\text{Hess } \tilde{f}(0) = E^\top FE$ and $H = \det(E^\top FE)$. $\square$

Lemma 3.2. $F = (I + Y^2)(I - Y^2)^{-1}$. In particular $F\mathbf{1} = \mathbf{1}$, $\mathbf{1}^\top F = \mathbf{1}^\top$ and

$$\det F = \frac{\det(d^2I + S^2)}{\det(d^2I - S^2)}.$$

Proof. As $Y^\top = -Y$, we have $X^\top = Y^\top(I - Y^\top)^{-1} = -Y(I + Y)^{-1}$. The matrices Y , $(I - Y)^{-1}$ and $(I + Y)^{-1}$ commute, so

$$\begin{aligned} F &= I + Y[(I + Y) - (I - Y)](I - Y^2)^{-1} \\ &= I + 2Y^2(I - Y^2)^{-1} = (I + Y^2)(I - Y^2)^{-1}. \end{aligned}$$

Since $Y\mathbf{1} = 0$ and $\mathbf{1}^\top Y = 0$, the row and column sums of F equal 1. Finally $\det(I \pm Y^2) = d^{-2d} \det(d^2I \pm S^2)$. $\square$

Proof of Theorem 1.1. By Proposition 3.1, Lemma 2.2 and Lemma 3.2,

$$H = \det(E^\top F E) = d \det F = d \frac{\det(d^2I + S^2)}{\det(d^2I - S^2)}.$$

Since S commutes with dI and $S^\top = -S$,

$$d^2I - S^2 = (dI - S)(dI - S)^\top, \quad d^2I + S^2 = (dI - iS)(dI - iS)^\top,$$

so $\det(d^2I - S^2) = \det(dI - S)^2$ and $\det(d^2I + S^2) = \det(dI - iS)^2$. The entries of $dI - iS$ are Gaussian integers, and the complex conjugate of $\det(dI - iS)$ is $\det(dI + iS) = \det((dI - iS)^\top) = \det(dI - iS)$, so $\det(dI - iS)$ is a rational integer. In terms of the eigenvalues,

$$\det(dI - S) = d \prod_{k=1}^m (d^2 + \mu_k^2), \quad \det(dI - iS) = d \prod_{k=1}^m (d^2 - \mu_k^2),$$

and both are positive because $\mu_k \leq d - 1$ by Lemma 2.1. This proves (1.2). $\square$

Proof of Corollary 1.2. By Theorem 1.1, H is a positive rational number and $d/H = (\det(dI - S)/\det(dI - iS))^2$. $\square$

4. The cyclic tournament

Let $d = 2m + 1 \geq 3$ and index the vertices of R_d by $\mathbb{Z}/d$. Let C be the permutation matrix with $C_{jk} = 1$ if and only if $k = j + 1$, and put $P_c = C^{m+1}$. Then $S = \sum_{r=1}^m (C^r - C^{-r})$ is the skew adjacency matrix of R_d .

Lemma 4.1. $S(I + P_c) = I - P_c$. Consequently, for all a, b in any commutative ring,

$$2 \det(aI + bS) = (a + b)^d + (a - b)^d. \quad (4.1)$$

Proof. Using $C^d = I$, so that $C^{r+m+1} = C^{r-m}$,

$$\begin{aligned} S(I + C^{m+1}) &= \sum_{r=1}^m (C^r - C^{-r}) + \sum_{s=1-m}^0 C^s - \sum_{s=1}^m C^s \\ &= \sum_{s=0}^{m-1} C^{-s} - \sum_{r=1}^m C^{-r} = I - C^{m+1}. \end{aligned}$$

Hence $(aI + bS)(I + P_c) = (a + b)I + (a - b)P_c$. Since $m + 1$ is prime to d , P_c is the permutation matrix of a single cycle π of odd length d . In the expansion of $\det(\alpha I + \beta P_c)$ a permutation τ contributes only if $\tau(j) \in \{j, \pi(j)\}$ for all j ; if $\tau(j) = \pi(j)$ for one j then $\tau(\pi(j)) = \pi^2(j)$, and so on, so τ is the identity or π . As π is even, $\det(\alpha I + \beta P_c) = \alpha^d + \beta^d$. In particular $\det(I + P_c) = 2$, and taking determinants gives (4.1). $\square$

Proof of Theorem 1.3. With $a = x$ and $b = -1$, (4.1) is the characteristic polynomial. With $a = d$ and $b = -1$, and with $a = d$ and $b = -i$, it gives $\det(dI - S) = \frac{1}{2}((d+1)^d + (d-1)^d)$ and $\det(dI - iS) = \frac{1}{2}((d+i)^d + (d-i)^d)$, and (1.2) yields the first expression in (1.3). For the second, write $d \pm 1 = \sqrt{d^2 - 1} e^{\pm \operatorname{artanh}(1/d)}$ and $d \pm i = \sqrt{d^2 + 1} e^{\pm i \arctan(1/d)}$.

Dividing numerator and denominator by d^d ,

$$\rho_{\mathcal{T}}(R_d) = \frac{(1 + \frac{1}{d})^d + (1 - \frac{1}{d})^d}{2 \operatorname{Re}(1 + \frac{i}{d})^d} \longrightarrow \frac{e + e^{-1}}{2 \operatorname{Re} e^i} = \frac{\cosh 1}{\cos 1},$$

and $\rho_{\mathcal{R}}(R_d) = e^{1/d} \rho_{\mathcal{T}}(R_d)$ has the same limit. For the expansion put $x = 1/d$. By the second expression in (1.3), $\rho_{\mathcal{R}}(R_d) = G(x)$ with

$$\log G(x) = x + \frac{1}{2x} \log \frac{1 - x^2}{1 + x^2} + \log \cosh \frac{\operatorname{artanh} x}{x} - \log \cos \frac{\arctan x}{x}.$$

The four terms are analytic near $x = 0$. The first two add up to $-\sum_{j \geq 1} x^{4j+1}/(2j+1) = -\frac{x^5}{3} - \frac{x^9}{5} - \dots$, and the last two are even functions of x because $\operatorname{artanh}(x)/x = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots$ and $\arctan(x)/x = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots$. Hence $G(x) = e^{\mathcal{E}(x)} \left(1 - \frac{x^5}{3} + O(x^9)\right)$ with $\mathcal{E}$ even, $e^{\mathcal{E}(0)} = \cosh 1 / \cos 1$ and $\mathcal{E}''(0)/2 = (\tanh 1 - \tan 1)/3$. The odd coefficients of G below x^5 vanish, the coefficient of x^5 is $-e^{\mathcal{E}(0)}/3$, and the coefficient of x^2 is $e^{\mathcal{E}(0)}(\tanh 1 - \tan 1)/3$. $\square$

Remark 4.2. By Theorem 1.3 the eigenvalues of S are the roots of $(x+1)^d = -(x-1)^d$, that is, $x = i \cot \frac{(2k-1)\pi}{2d}$ for $k = 1, \dots, d$. So $\mu_k = \cot \frac{(2k-1)\pi}{2d}$ in (1.2), and

$$\rho_{\mathcal{T}}(R_d) = \prod_{k=1}^m \frac{d^2 + \cot^2 \frac{(2k-1)\pi}{2d}}{d^2 - \cot^2 \frac{(2k-1)\pi}{2d}}.$$

Letting $d \rightarrow \infty$ term by term (justified by dominated convergence, since $\cot x < 1/x$) and using Euler's products

$$\cos x = \prod_{k \geq 1} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2}\right), \quad \cosh x = \prod_{k \geq 1} \left(1 + \frac{4x^2}{(2k-1)^2\pi^2}\right)$$

gives the limit a second time. The transitive tournament has the same characteristic polynomial (Lemma 5.1 below), although it is not regular.

Proof of Corollary 1.4. Let $\varepsilon > 0$ and choose d with $\rho_{\mathcal{R}}(R_d) > \cosh 1 / \cos 1 - \varepsilon$. By Theorem W, for all large odd n some tournament in the support of $\mathcal{R}(R_d, n)$ has at least $(\cosh 1 / \cos 1 - 2\varepsilon)g(dn)$ Hamilton cycles. Wormald's Theorem 5 [9] states that Theorem W holds for Hamilton paths with g' in place of g , and the same argument applies. $\square$

5. Bounds over all regular tournaments

Write $s_k = \mu_k^2$. Since $\operatorname{tr}(SS^{\top}) = d(d-1)$ and the eigenvalues of $SS^{\top} = -S^2$ are 0 and $s_1, s_1, \dots, s_m, s_m$,

$$\sum_{k=1}^m s_k = \frac{d(d-1)}{2}, \quad \log \rho_{\mathcal{T}}(T) = \sum_{k=1}^m \varphi(s_k), \quad \varphi(s) = \log \frac{d^2 + s}{d^2 - s}. \quad (5.1)$$

The function φ is strictly convex on $[0, d^2)$, since $\varphi''(s) = (d^2 - s)^{-2} - (d^2 + s)^{-2} > 0$ for $s > 0$.

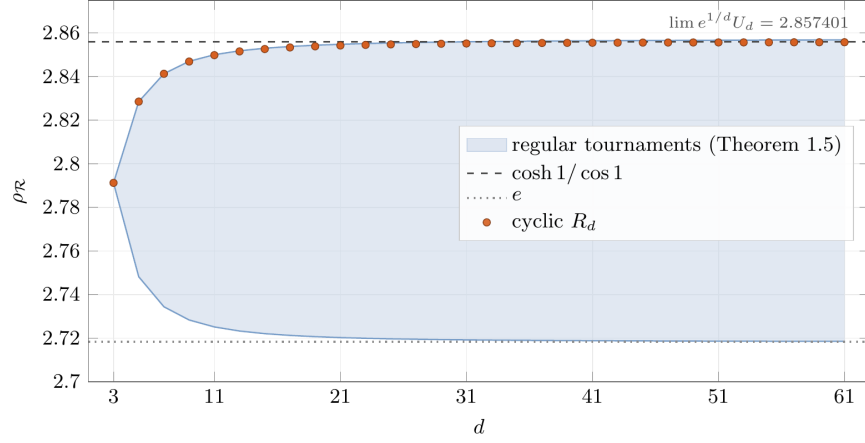


Figure 2: $\rho_{\mathcal{R}}(T)$ for regular tournaments on d vertices. The dots are the cyclic tournaments R_d ; the band is bounded by $e^{1/d}((d+1)/(d-1))^{(d-1)/2}$ and $e^{1/d}U_d$ (Theorem 1.5). The upper bound and $\rho_{\mathcal{R}}(R_d)$ both approach the region near $\cosh 1 / \cos 1$, while the lower bound decreases to e .

Lemma 5.1. *Let D_d be the skew-symmetric $d \times d$ matrix with all entries above the diagonal equal to 1. Its eigenvalues are $i \cot \frac{(2k-1)\pi}{2d}$, $k = 1, \dots, d$; in particular its spectral radius is $\cot \frac{\pi}{2d}$.*

Proof. Let $\omega^d = -1$ and $v = (1, \omega, \dots, \omega^{d-1})^\top$. Then

$$(D_d v)_j = \sum_{l>j} \omega^l - \sum_{l<j} \omega^l = \frac{\omega^{j+1} - \omega^d}{1 - \omega} - \frac{1 - \omega^j}{1 - \omega} = \frac{1 + \omega}{1 - \omega} \omega^j \quad (0 \leq j < d),$$

so $(1 + \omega)/(1 - \omega) = i \cot(\theta/2)$ is an eigenvalue for $\omega = e^{i\theta}$, $\theta = (2k-1)\pi/d$. These d numbers are distinct. $\square$

Proof of Theorem 1.5. Lower bound. By (5.1) and Jensen's inequality, $\log \rho_{\mathcal{T}}(T) \geq m \varphi(d) = m \log \frac{d+1}{d-1}$, with equality if and only if $s_1 = \dots = s_m = d$. As S is normal with $S\mathbf{1} = 0$, this happens exactly when $SS^\top = dI - J$. Expanding with $A + A^\top = J - I$ and $AJ = JA^\top = mJ$ gives

$$SS^\top = 4AA^\top + (2-d)J - I,$$

so the equality condition is $AA^\top = \frac{1}{4}((d+1)I + (d-3)J)$: every two vertices have $(d-3)/4$ common out-neighbours.

$\rho_{\mathcal{R}} > e$. With $x = 1/d$,

$$\log\left(e^{1/d}\left(\frac{d+1}{d-1}\right)^m\right) = x + \frac{1-x}{2x} \log \frac{1+x}{1-x} = 1 + \sum_{k \geq 1} \frac{x^{2k} - x^{2k+1}}{2k+1} > 1.$$

Upper bound. Grzesik, Král', Lovász and Volec [4, Lemma 12] proved that no skew-symmetric matrix with entries in $[-1, 1]$ has larger spectral radius than D_d , so $s_k \leq \sigma = \cot^2 \frac{\pi}{2d} < d^2$ by Lemma 5.1. Hence $(s_1, \dots, s_m)$ lies in the polytope $\{0 \leq s_k \leq \sigma, \sum s_k = d(d-1)/2\}$, on which the convex function $\sum \varphi(s_k)$ attains its maximum at a vertex. A point of the polytope with two coordinates in $(0, \sigma)$ is the midpoint of two other points of it, so at a vertex at most one coordinate lies strictly between 0 and σ ; this forces q coordinates equal to σ , one equal to r and the rest 0, and the value $\log U_d$. As $d \rightarrow \infty$, $\sigma/d^2 \rightarrow 4/\pi^2$ and $d(d-1)/(2d^2) \rightarrow \frac{1}{2}$; since $1 < \pi^2/8 < 2$ we get $q = 1$ for large d , $r/d^2 \rightarrow \frac{1}{2} - \frac{4}{\pi^2}$, and the stated limit. Finally, every limit point of $\max_T \rho_{\mathcal{R}}(T)$ is at least $\lim \rho_{\mathcal{R}}(R_d) = \cosh 1 / \cos 1$ and at most $\lim e^{1/d} U_d$. $\square$

6. Verification

All statements were checked by computer, independently of the proofs, and the algebraic ones were also checked in Lean.

Wormald's definition. For a given regular tournament, H was computed in exact rational arithmetic directly from the definition in Theorem W: the first and second derivatives of the implicit function h at $(z, \dots, z)$ were obtained by implicit differentiation of K , with the derivatives of K evaluated as sums of determinants with replaced columns, and then converted to the Hessian of $\tilde{f}$. The result agreed with (1.1) for one regular tournament of each skew spectrum with at most 9 vertices (seventeen spectra, see Table 1), for R_{11} and R_{13} , for Wormald's 7-vertex example and for random regular tournaments with 11 and 13 vertices; an independent computation in PARI/GP gave the same values for nine of them, R_{11} included. The exact values of $\sqrt{d/H}$ in Wormald's table and his decimal values for $d = 21, 41, 81$ and 101 are reproduced by (1.3), except that his $\rho_{\mathcal{R}}(R_{81}) = 2.855842467\dots$ should read 2.855842468\dots

Exhaustive enumeration. A short C program lists the regular tournaments on $\mathbb{Z}/d$ in which 0 dominates $1, \dots, m$ and computes the characteristic polynomial of S for each; Table 1 and the claims about $d \leq 9$ in Section 5 come

Table 1: $\rho_{\mathcal{T}}(T)$ for the regular tournaments on $d \leq 9$ vertices, one row for each characteristic polynomial of S , from all regular tournaments on $\mathbb{Z}/d$ in which 0 dominates $1, \dots, m$; “labelled” counts these. Every regular tournament is isomorphic to one of them.

d	$\rho_{\mathcal{T}}(T)$	decimal	labelled	remark
5	44/19	2.315789	4	R_5
7	3032/1231	2.463038	36	R_7
	3088/1287	2.399378	84	Wormald’s example
	64/27	2.370370	12	doubly regular (Paley)
9	437584/171769	2.547514	576	R_9
	443488/177529	2.498116	5184	
	5488/2197	2.497952	64	R'_9 (blow-up of R_3)
	446656/180193	2.478764	6912	
	446512/180337	2.475987	5184	
	446368/180481	2.473213	1728	
	449152/182077	2.466824	6912	
	449008/182221	2.464085	1728	
	449824/182857	2.459977	10368	
	50080/20381	2.457191	5184	
	451696/183937	2.455710	576	circulant $\{1, 2, 3, 5\}$
	50176/20449	2.453714	1728	

from its output. The bounds of Theorem 1.5 were also checked on random regular tournaments with up to 21 vertices.

Lean. The formalization proves, from the standard axioms only, the identity $F = (I + Y^2)(I - Y^2)^{-1}$ and Lemma 2.2 for the matrix F of a regular tournament, the resulting formula $\det(E^{\top}FE) = d \det(d^2I + S^2) / \det(d^2I - S^2)$, the nonvanishing of both determinants, the fact that their ratio is the square of a nonzero rational, Lemma 4.1, the closed form (1.3) including $\operatorname{Re}(d+i)^d > 0$, and the limit $\cosh 1 / \cos 1$ for $e^{1/d} \sqrt{d / \det(E^{\top}FE)}$ along R_d . Proposition 3.1, the asymptotic expansion in Theorem 1.3 and Theorem 1.5 are proved in this paper and checked numerically, but not formalized.

Declaration of competing interest

The author has no competing interests to declare.

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Data availability

The Lean 4 development, the C, Python and PARI/GP checks and the data are available at <https://github.com/creelie/Breakthrough-problems-1>.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) to search the literature, to run the computer checks and the Lean formalization, and to draft the text. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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References

- [1] I. Adler, N. Alon and S. M. Ross, *On the maximum number of Hamiltonian paths in tournaments*, Random Structures Algorithms **18** (2001), no. 3, 291–296, [doi:10.1002/rsa.1010](https://doi.org/10.1002/rsa.1010).
- [2] N. Alon, *The maximum number of Hamiltonian paths in tournaments*, Combinatorica **10** (1990), no. 4, 319–324, [doi:10.1007/BF02128667](https://doi.org/10.1007/BF02128667).

- [3] E. Friedgut and J. Kahn, *On the number of Hamiltonian cycles in a tournament*, *Combin. Probab. Comput.* **14** (2005), no. 5–6, 769–781, [doi:10.1017/S0963548305006863](https://doi.org/10.1017/S0963548305006863).
- [4] A. Grzesik, D. Král', L. M. Lovász and J. Volec, *Cycles of a given length in tournaments*, *J. Combin. Theory Ser. B* **158** (2023), 117–145, [doi:10.1016/j.jctb.2022.07.007](https://doi.org/10.1016/j.jctb.2022.07.007).
- [5] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: *Automated Deduction – CADE 28, Lecture Notes in Comput. Sci.* **12699**, Springer, 2021, 625–635, [doi:10.1007/978-3-030-79876-5_37](https://doi.org/10.1007/978-3-030-79876-5_37).
- [6] The mathlib Community, *The Lean mathematical library*, in: *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, 367–381, [doi:10.1145/3372885.3373824](https://doi.org/10.1145/3372885.3373824).
- [7] K. B. Reid and E. Brown, *Doubly regular tournaments are equivalent to skew Hadamard matrices*, *J. Combin. Theory Ser. A* **12** (1972), no. 3, 332–338, [doi:10.1016/0097-3165\(72\)90098-2](https://doi.org/10.1016/0097-3165(72)90098-2).
- [8] T. Szele, *Kombinatorikai vizsgálatok az irányított teljes gráffal kapcsolatban*, *Mat. Fiz. Lapok* **50** (1943), 223–256.
- [9] N. C. Wormald, *Tournaments with many Hamilton cycles*, preprint (2004), [arXiv:2610.16245](https://arxiv.org/abs/2610.16245).